Existence of Triangulations (Lecture 4)

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In the last lecture, we proved that if M is a smooth manifold, K a polyhedron, and $f : K \to M$ a piecewise differentiable homeomorphism (required to be an immersion on each simplex), then K is a piecewise linear manifold. The proof was based on two basic principles:

Proposition 1. *Let $f : K \to \mathbb{R}^n$ be a PD map and $K_0 \subseteq K$ a finite subpolyhedron. Then there exists another PD map $f' : K \to \mathbb{R}^n$ which is piecewise linear on K_0 and agrees with f outside a compact set. Moreover, we can arrange that f' is arbitrarily good approximation to f (in the C^1-sense).*

Proposition 2. *If $f, f' : K \to \mathbb{R}^n$ are PD maps which are sufficiently close to one another (in the C^1-sense) and f is a PD homeomorphism, then f' is a PD homeomorphism onto an open subset of $\mathbb{R}^n$.*

Our goal in this lecture is to apply these results to show that every smooth manifold M admits a Whitehead compatible triangulation. For simplicity, we will assume that M is compact; the noncompact case can be handled using same methods.

Definition 3. Let K be a finite polyhedron, M a smooth manifold, and $f : K \to M$ a map. We say that f is a *PD embedding* if f is injective and there exists a triangulation of K such that f is a smooth immersion on each simplex.

If $f : K \to M$ is a PD embedding, then we can identify K with its image $f(K)$. Any triangulation of K determines a triangulation of $f(K)$ by smooth embedded simplices in M.

Definition 4. Let $f : K \to M$ and $g : K' \to M$ be PD embeddings. We will say that f and g are *compatible* if the following conditions are satisfied:

(1) Let $X = f(K) \cap g(K') \subseteq M$. Then $f^{-1}(X) \subseteq K$ and $g^{-1}(X) \subseteq K'$ are polyhedral subsets of K and K'.

(2) The identification $f^{-1}(X) \simeq X \simeq g^{-1}(X)$ is a piecewise linear homeomorphism.

Suppose that f and g are compatible, and let X be as above. Then the coproduct $K \coprod_X K'$ can be endowed with the structure of a polyhedron, and the maps f and g can be amalgamated to give a PD embedding $f \cup g : K \coprod_X K'$ into M. Moreover, $f \cup g$ is compatible with another PD embedding $h : K'' \to M$ if and only if both f and g are compatible with h.

To prove that a compact smooth manifold M admits a Whitehead compatible triangulation, it will suffice to show that there exists a finite collection of PD embeddings $f_i : K_i \to M$ which are pairwise compatible and whose images cover M. (We can then iterate the amalgamation construction described above to produce a PD homeomorphism $K \to M$.)

For each point $x \in M$, choose a neighborhood W_x of x in M and a smooth identification $W_x \simeq \mathbb{R}^n$ which carries x to the origin in $\mathbb{R}^n$. Let $U_x \subseteq W_x$ denote the image of the unit ball in $\mathbb{R}^n$, and let f_x denote the composite map $[-2, 2]^n \hookrightarrow \mathbb{R}^n \hookrightarrow M$. Since M is compact, the covering $\{U_x\}_{x \in M}$ admits a finite subcovering by $\{U_x\}_{x \in \{x_1, \ldots, x_k\}}$. Let $W_i = W_{x_i}$, $U_i = U_{x_i}$, and $f_i = f_{x_i}$ for $1 \leq i \leq k$. The maps $f_i : [-2, 2]^n \to M$ are PD embeddings whose images cover M. However, the f_i are not necessarily pairwise compatible. To prove the existence of a Whitehead compatible triangulation of M, it will suffice to prove the following:

Proposition 5. *There exist PD embeddings $f'_i : [-2,2]^n \to M$ which are pairwise compatible, and can be chosen to be arbitrarily good approximations (in the C^1 sense) to the maps f_i.*

In fact, if f'_i is sufficiently close to f_i, then f'_i will factor through $W_i \simeq \mathbb{R}^n$ and will not carry the boundary of $[-2,2]^n$ into the closure $\overline{U}_i$, so that U_i is contained in the image of f'_i; thus the images of the f'_i will cover M and give us the desired triangulation of M.

To prove Proposition 5, we will prove by induction on $j \leq k$ that we can choose maps $\{f_i^j\}_{1 \leq i \leq j}$ which are pairwise compatible PD embeddings where f_i^j is an arbitrarily close approximation to f_i (in the C^1-sense). The case $j = 1$ is obvious (take $f_1^1 = f_1$) and the case $j = k$ yields a proof of Proposition 5.

For the inductive step, let us suppose that the maps $\{f_i^{j-1}\}_{1 \leq i < j}$ have already been constructed. Since these maps are compatible, they can be amalgamated to produce a single PD embedding $f^{j-1} : K \to M$. We will replace $f^{j-1} : K \to M$ by a close approximation g which is compatible with f_j. We can then complete the proof by defining $f_j^j = f_j$ and f_i^j to be the composition

$$[-2,2]^n \hookrightarrow K \xrightarrow{g} M.$$

To prove the existence of g, we need the following:

Lemma 6. *Let M be a smooth manifold equipped with a smooth chart $\mathbb{R}^n \hookrightarrow M$, and let $f : K \to M$ be a PD embedding (where K is a finite polyhedron). Then there exist arbitrarily close approximations (in the C^1-sense) of f which are compatible with the embedding $[-2,2]^n \subset \mathbb{R}^n \hookrightarrow M$.*

Proof. Let L be the open subset of K corresponding to the inverse image of $\mathbb{R}^n$, and let L_0 be a finite subpolyhedron of L containing the inverse image of $[-3,3]^n$. According to Proposition 1, the map $f|L : L \to \mathbb{R}^n$ admits arbitrarily good approximations $f' : L \to \mathbb{R}^n$ which are piecewise linear on L_0 and which agree with $f|L$ outside a compact set. Provided that the approximation is sufficiently good, the inverse image $f'^{-1}[-2,2]^n$ will be contained in L_0. Since f' is piecewise linear on L_0, we deduce that f' is compatible with the embedding $[-2,2]^n \subset \mathbb{R}^n \hookrightarrow M$. Since $f' = f|L$ outside a compact set, the map $g : K \to M$ defined by the formula

$$g(x) = \begin{cases} f'(x) & \text{if } x \in L \\ f(x) & \text{if } x \notin L \end{cases}$$

is a well-defined PD embedding of K into M, which has the desired properties. □

Variant 7. *Suppose that M is a (compact) smooth manifold with boundary. Then we can modify the above proof to show that any PD homeomorphism $f_0 : K_0 \to \partial M$ can be extended to a PD homeomorphism $K \to M$ where K contains K_0 as a subpolyhedron. For example, we can first extend f_0 to a PD embedding $K_0 \times [0,1] \to M$ by choosing a smooth collar of ∂M. Then M can be covered by the image of $K_0 \times [0,1]$ together with finitely PD embeddings $[-2,2]^n \hookrightarrow \mathbb{R}^n \subseteq M$, and we can apply the above argument without essential change to make these embeddings compatible with one another.*

Variant 8. *Suppose that M is noncompact. The existence of Whitehead compatible triangulations of M can be established by adapting the above arguments: we cannot generally assume that the covering $\{U_i\}$ is finite, but we can use a paracompactness argument to guarantee that the covering is locally finite which is sufficient for the above constructions to go through.*

An alternative strategy uses Variant 7. Choose a smooth proper map $\chi : M \to \mathbb{R}$ with isolated critical points (for example, a Morse function). Then the critical values of χ are isolated, so we can choose a sequence of regular values

$$\{\ldots < r_{-1} < r_0 < r_1 < r_2 < \ldots\}$$

tending to infinity in both directions. We first apply the result in the compact case to find Whitehead compatible triangulations of the inverse images $\chi^{-1}\{r_i\}$, and then apply Variant 7 to extend these to Whitehead compatible triangulations of $\chi^{-1}[r_i, r_{i+1}]$; the result is a Whitehead compatible triangulation for the whole of M.