Surfaces and Complex Analysis (Lecture 34)

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Let Σ be a smooth surface. We have seen that Σ admits a conformal structure (which is unique up to a contractible space of choices). If Σ is oriented, then a conformal structure on Σ allows us to view Σ as a Riemann surface: that is, as a 1-dimensional complex manifold. In this lecture, we will exploit this fact together with the following important fact from complex analysis:

Theorem 1 (Riemann uniformization)**.** *Let Σ be a simply connected Riemann surface. Then Σ is biholomorphic to one of the following:*

(i) The Riemann sphere $\mathbf{C}\mathrm{P}^1$.

(ii) The complex plane $\mathbf{C}$.

(iii) The open unit disk $D = \{z \in \mathbf{C} : z < 1\}$.

If Σ is an arbitrary surface, then we can choose a conformal structure on Σ. The universal cover $\widehat{\Sigma}$ then inherits the structure of a simply connected Riemann surface, which falls into the classification of Theorem 1. We can then recover Σ as the quotient $\widehat{\Sigma}/\Gamma$, where $\Gamma \simeq \pi_1\Sigma$ is a group which acts freely on $\widehat{\Sigma}$ by holmorphic maps (if Σ is orientable) or holomorphic and antiholomorphic maps (if Σ is nonorientable). For simplicity, we will consider only the orientable case.

If $\widehat{\Sigma} \simeq \mathbf{C}\mathrm{P}^1$, then the group Γ must be trivial: every orientation preserving automorphism of S^2 has a fixed point (by the Lefschetz trace formula). Because Γ acts freely, we must have $\Gamma \simeq 0$, so that $\Sigma \simeq S^2$.

To see what happens in the other two cases, we need to understand the holomorphic automorphisms of $\mathbf{C}$ and D.

Theorem 2. *Let $f : \mathbf{C} \to \mathbf{C}$ be a holomorphic homeomorphism. Then f has the form $f(z) = az + b$.*

Proof. Since f is a homeomorphism, it extends continuously to the one-point compactification by setting $f(\infty) = \infty$. We can therefore regard f as a map from $\mathbf{C}\mathrm{P}^1$ to itself. We claim that this map is holomorphic. Without loss of generality, we may assume $f(0) = 0$. To prove this, consider the behavior of f in a neighborhood of ∞: we have a map $g : \mathbf{C} \to \mathbf{C}$ defined by

$$g(z) = \begin{cases} \frac{1}{f(\frac{1}{z})} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$

We claim that g is holomorphic. It is clearly holomorphic away from 0. The function

$$h(z) = \frac{1}{2\pi i} \int \frac{g(z)}{z} dz$$

is holomorphic and coincides with g away from the origin (by the Cauchy integral formula), and therefore coincides with g everywhere by continuity. The space of meromorphic functions on $\mathbf{C}\mathrm{P}^1$ having at most a simple pole at ∞ has dimension 2 (by the Riemann-Roch formula), and so consists of exactly those functions of the form $f(z) = az + b$. ∎

Note that a homeomorphicm of the form $f(z) = az + b$ has a fixed point $\frac{b}{1-a}$ if $a \neq 1$. Consequently, if Γ is a group acting freely on $\mathbf{C}$ by holomorphic homeomorphisms, then Γ must act by translations $z \mapsto z + b$. We can then identify Γ with a subgroup of $\mathbf{C}$ (regarded additively). The action of Γ on $\mathbf{C}$ is properly discontinuous if and only if Γ is discrete: then Γ is a lattice in $\mathbf{C}$, which has rank 2 if and only if $\mathbf{C}/\Gamma$ is compact. In this case, the quotient $\mathbf{C}/\Gamma$ is a torus. We have proven:

Proposition 3. *Let Σ be a surface equipped with a conformal structure whose universal cover is $\mathbf{C}$ (as a Riemann surface). Then Σ is a torus.*

(We will prove the converse in a moment.)

Let us now consider the most interesting case: the unit disk D.

Theorem 4 (The Schwarz Lemma)**.** *Let $f : D \to D$ be a biholomorphic map such that $f(0) = 0$. Then $f(z) = az$ for some unit complex number a.*

Proof. Since $f(0) = 0$, we can write $f(z) = zg(z)$ for some holomorphic function g. By the maximum principle, if $|z| \leq r < 1$, then

$$|g(z)| \leq |g(y)| = \frac{|f(y)|}{y} \leq \frac{1}{r}$$

for some y satisfying $|y| = r$. It follows that $|g(z)| \leq 1$, so that $|f(z)| \leq |z|$. Applying the same argument to f^{-1}, we deduce that $|f(z)| = |z|$, so that $|g(z)| = 1$ everywhere. Since g is holomorphic, it must be constant, so that f is a linear map given by multiplication by some unit complex number $a = g(0)$. □

Corollary 5. *Every biholomorphic map from D to itself has the form*

$$z \mapsto a\frac{z-b}{1-\bar{b}z}$$

where a is a unit complex number and $|b| < 1$.

Proof. It is an easy exercise to see that the collection of such transformations forms a group G which maps D to itself. It is therefore a subgroup of the group G' of holomorphic automorphisms of D. We claim that $G = G'$. Fix $g' \in G'$, we wish to show that $g' \in G$. Composing g' with a transformation of the form $z \mapsto \frac{z-b}{1-\bar{b}z}$, we can assume that $g'(0) = 0$. Theorem 4 now implies that $g' \in G$. □

The group G has another description: it is precisely the group of orientation-preserving isometries of the unit disk D with respect to the hyperbolic metric $|ds|_{hyp} = \frac{2|ds|}{1-|z|^2}$. Consequently, if Σ is a conformal surface whose universal cover is D, then Σ is the quotient D/Γ, where Γ is a subgroup of G acting by hyperbolic isometries of D. It follows that Σ admits a hyperbolic metric: that is, a Riemannian metric of constant curvature -1. More precisely, Σ admits a unique hyperbolic metric compatible with the given conformal structureo n Σ.

Proposition 6. *Let Σ be as above. Then $\widetilde{\Sigma} \simeq D$ if and only if Σ has genus at least 2: that is, if and only if $\chi(\Sigma) < 0$.*

Proof. If Σ has genus ≥ 2, then we have already seen that $\widetilde{\Sigma}$ cannot be S^2 or $\mathbf{C}$, so the desired result follows from the Riemann uniformization theorem. Conversely, suppose that $\widetilde{\Sigma} = D$. Then Σ admits a hyperbolic metric. The Gauss-Bonnet theorem allows us to compute $\chi(\Sigma)$ as an integral of the curvature of this metric: since the curvature is everywhere negative, we get $\chi(\Sigma) < 0$. □

In the next few lectures, we will exploit the existence of hyperbolic metrics to understand the diffeomorphism groups of surfaces of genus $g \geq 2$. We conclude this lecture by explaining how these ideas carry over to the case of surfaces with boundary and nonorientable surfaces. There are two rather different ways to use hyperbolic metrics in this case.

Suppose first that Σ is a surface with boundary. Each boundary component of Σ is a circle. We can therefore view Σ as the real blow-up of a closed surface Σ' obtained by collapsing each boundary circle of Σ. Choose a conformal structure on Σ', and identify $\Sigma - \partial\,\Sigma$ with the punctured Riemann surface obtained by removing finitely many points from Σ. Then $\Sigma - \partial\,\Sigma$ has a universal cover X, which is the unit disk if and only if $\chi(\Sigma) < 0$. The arguments sketched above show that $\Sigma - \partial\,\Sigma$ inherits the structure of a hyperbolic manifold. This manifold is not compact, but there is a good replacement: it is a hyperbolic surface of finite area. In fact, one can show that this construction establishes an equivalence of categories between *punctured* Riemann surfaces of negative Euler characteristic and (oriented) hyperbolic surfaces finite volume.

There is another very different way to apply these ideas to a compact connected surface Σ, which we need not assume to be closed or oriented. Let $\overline{\Sigma}$ be the orientation double cover of Σ, and let σ be its canonical involution. The *double* $d(\Sigma)$ of Σ is the quotient of $\overline{\Sigma}$ obtained by identifying x with $\sigma(x)$ for $x \in \partial\,\Sigma$. Then $d(\Sigma)$ is a compact closed oriented surface (which is connected if and only if Σ is either nonorientable or has boundary). It is equipped with an orientation reversing involution, which we will continue to denote by σ. We can recover the original surface σ by forming the quotient $d(\Sigma)/\sigma$, and we can recover $\partial\,\Sigma$ as the fixed point locus of σ.

Remark 7. The preceding construction establishes a correspondence between compact surfaces with boundary and compact closed oriented surface with an orientation-reversing involution.

We can now apply all of our preceding methods to the double $d(\Sigma)$, but keeping track of the orientation reversing involution σ. First, choose an arbitrary Riemannian metric on $d(\Sigma)$. Averaging under σ, we can assume that this metric is σ-equivariant. The metric determines a complex structure on $d(\Sigma)$, with respect to which σ is an antiholomorphic involution. This lets us think of $d(\Sigma)$ as a Riemann surface with a real structure: in other words, as an algebraic curve over $\mathbb{R}$. The original surface Σ can be recovered as the set of closed points of the underlying $\mathbb{R}$-scheme, while the boundary $\partial\,\Sigma$ can be identified with the set of $\mathbb{R}$-points of this scheme.

Assuming $d(\Sigma)$ is connected (for simplicity), the universal cover $\widetilde{d(\Sigma)}$ is biholomorphic to either $\mathbf{C}\mathrm{P}^1$, $\mathbf{C}$, or the unit disk D. Note that $\chi(d(\Sigma)) = 2\chi(\Sigma)$, so this universal cover is the unit disk D if and only if $\chi(\Sigma) < 0$. In this case, the surface $d(\Sigma)$ inherits a canonical hyperbolic structure, and the map σ is an orientation-reversing isometry. It follows that $\Sigma = d(\Sigma)/\sigma$ again inherits a hyperbolic metric, this time as a manifold with boundary. Moreover, we understand what happens to the metric as we approach the boundary: namely, the boundary consists of geodesics.

We can summarize the discussion as follows: let Σ be a compact surface such that each connected component of Σ has negative Euler characteristic. The above construction determines a bijection between conformal structures on Σ which behave well at the boundary (these form a contractible space) and hyperbolic metrics on Σ with respect to which the boundary is geodesic.