Classification of Surfaces (Lecture 33)

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In this lecture, we will (belatedly) discuss the classification of 2-manifolds, which we have frequently used in our discussion of 3-manifolds. We begin with the oriented case.

Theorem 1. *Let Σ be a connected compact oriented surface. Then Σ can be obtained as a connected sum $T\#T\#\cdots\#T$ of g copies of the torus T, for some $g \geq 0$.*

The integer g is called the *genus* of the surface Σ. It is a topological invariant of Σ: a simple calculation shows that $\chi(\Sigma) = 2 - 2g$.

The proof will require a few preliminaries.

Lemma 2. *Let Σ be a connected compact surface. Then $\chi(\Sigma) \leq 2$, and equality holds if and only if Σ is a 2-sphere.*

Proof. We have $\chi(\Sigma) = b_0 - b_1 + b_2$, where b_i denotes the ith Betti number of Σ. Since Σ is connected, we have $b_0 = 1$, and b_2 is either 1 or 0 depending on whether Σ is orientable or nonorientable. It follows that

$$\chi(\Sigma) = \begin{cases} 2 - b_1 & \text{if } \Sigma \text{ is orientable} \\ 1 - b_1 & \text{if } \Sigma \text{ is nonorientable.} \end{cases}$$

This proves the inequality. If equality holds, then Σ must be orientable, and therefore admits a complex structure. As we explained in a previous lecture, a Riemann surface with $\chi(\Sigma) = 2$ must be biholomorphic to the Riemann sphere, and in particular is a topological sphere. □

The following can be regarded as a baby version of the loop theorem:

Lemma 3. *Let Σ be a connected surface and let $N \subset \pi_1\Sigma$ be a proper normal subgroup. Then there is an embedded loop $f : S^1 \hookrightarrow \Sigma$ such that $[f] \notin N$.*

Proof. Since N is proper, we can choose a closed loop $f : S^1 \to \Sigma$ such that $[f]$ (which is well-defined up to conjugacy) does not belong to N. Without loss of generality, we may assume that f is in general position. Then f is an immersion with a finite number k of double points. We will assume that f has been chosen minimally. If $k = 0$, then f is an embedding and we are done. Otherwise, there exist $x, y \in S^1$ with $x \neq y$ but $f(x) = f(y)$. The points x and y partition S^1 into two intervals I_0 and I_1. The restrictions of f to I_0 and I_1 give two other loops $f_0, f_1 : S^1 \to \Sigma$. Since each of these loops has a smaller number of double points, the minimality of k guarantees that $[f_0], [f_1] \in N$. We now conclude by observing that $[f]$ belongs to the normal subgroup of $\pi_1\Sigma$ generated by $[f_0]$ and $[f_1]$, and therefore also belongs to N, which contradicts our assumption. □

We now prove Theorem 1. We proceed by descending induction on $\chi(\Sigma)$. If $\chi(\Sigma) \geq 2$, then Lemma 2 implies that $\chi(\Sigma) = 2$ and Σ is a 2-sphere. We may therefore assume that $\chi(\Sigma) = 2 - b_1 < 2$, so that $\mathrm{H}_1(\Sigma; \mathbf{Z}) \neq 0$. It follows that the commutator subgroup $[\pi_1\Sigma, \pi_1\Sigma]$ is a proper subgroup of $\pi_1\Sigma$. Using Lemma 3, we can choose an embedded loop $f : S^1 \hookrightarrow \Sigma$ which represents a nontrivial class in $\mathrm{H}_1(\Sigma; \mathbf{Z})$. It follows that f must be nonseparating, so that the surface Σ' obtained by cutting Σ along f is connected.

Let Σ'' be the closed surface obtained by capping off the boundary circles of Σ'. A simple calculation shows that

$$\chi(\Sigma'') = 2 + \chi(\Sigma') = 2 + \chi(\Sigma).$$

By the inductive hypothesis, Σ'' can be realized as a connected sum $T\#T\#\ldots\#T$.

The surface Σ can be obtained from Σ'' by removing small disks D_x and D_y around two points $x, y \in \Sigma''$ (to obtain Σ'), and then gluing the boundary of these disks together. Without loss of generality, we may assume that x and y are close to one another, so that D_x and D_y are contained in a larger disk D. Let K_0 be the surface with boundary obtained from Σ'' by removing the interior of D, and let K_1 be the surface obtained from D by removing the interiors of D_x and D_y and identifying their boundary. Then $\Sigma = K_0 \coprod_{S^1} K_1$, so we can identify Σ with the connected sum of two surfaces $\widehat{K}_0$ and $\widehat{K}_1$ obtained by capping off the boundary circles of K_0 and K_1. We note that $\widehat{K}_0 \simeq \Sigma''$, and a simple calculation shows that $\widehat{K} = T$ (if we like, we can take this to be a definition of the 2-manifold T). We then obtain

$$\Sigma \simeq \Sigma''\#T \simeq T\#T\#\ldots\#T$$

as desired.

We now treat the case of a nonorientable 2-manifold.

Theorem 4. *Let Σ be a closed connected nonorientable 2-manifold. Then Σ can be obtained as a connected sum $\mathbf{R}P^2 \simeq \mathbf{R}P^2\#\ldots\#\mathbf{R}P^2$ for some $k \geq 1$.*

Remark 5. In the situation of Theorem 4, the integer k is uniquely determined: a simple calculation of Euler characteristics shows that $\chi(\Sigma) = 2 - k$.

Warning 6. A priori, the connected sum $X\#Y$ of two surfaces X and Y is not well-defined: it depends on a choice of identification of the boundary circles of punctured copies of X and Y. This issue did not arise in the statement of Theorem 1, because in the orientable case there is a unique choice of identification which allows us to orient $X\#Y$ in a manner compatible with given orientations of X and Y (which we were implicitly using). It also does not matter in the case of Theorem 4, for a different reason: there exists an diffeomorphism of $\mathbf{R}P^2$ which fixes a point x and induces an orientation reversing automorphism of the tangent space at x. Namely, we observe that $\mathbf{R}P^2 = (\mathbb{R}^3 - \{0\})/\mathbb{R}^\times$ carries an action of the orthogonal group $O(3)$: any reflection in $O(3)$ will do the job.

We now prove Theorem 4. The proof proceeds by descending induction on $\chi(\Sigma)$ (which is at most 1, by virtue of Lemma 2). Since Σ is nonorientable, the 1st Stiefel-Whitney class $w_1 \in \mathrm{H}^1(\Sigma; \mathbf{Z}/2\mathbf{Z})$ induces a nontrivial map $\pi_1\Sigma \to \mathbf{Z}/2\mathbf{Z}$. Let N be the kernel of this map, so that N is a proper normal subgroup of $\pi_1\Sigma$. Using Lemma 3, we obtain an embedded loop $f : S^1 \to \Sigma$ such that $[f] \notin N$. Consequently, the restriction of w_1 to S^1 is nontrivial: this means that the normal bundle to the embedding $S^1 \hookrightarrow \Sigma$ is nontrivial, so that S^1 is a one-sided loop in Σ. Let K be a tubular neighborhood of S^1: then K is a Mobius band, whose boundary is another circle C. Let Σ' be the surface obtained from Σ by removing the interior of K, and let $\widehat{\Sigma}'$ and $\widehat{K}$ be the closed surfaces obtained by capping off the boundary circles of K and Σ'. Then $\widehat{K} = \mathbf{R}P^2$ (if you like, you can take this to be the definition of $\mathbf{R}P^2$, and we have $\Sigma \simeq \widehat{\Sigma}'\#\mathbf{R}P^2$. A simple calculation with Euler characteristics shows that $\chi(\Sigma) = \chi(\widehat{\Sigma}') + \chi(\mathbf{R}P^2) - 2 = \chi(\widehat{\Sigma}') - 1$.

There are now two cases to consider. If $\widehat{\Sigma}'$ is nonorientable, then the inductive hypothesis implies that $\widehat{\Sigma}'$ is a connected sum of finitely many copies of $\mathbf{R}P^2$: it then follows that Σ is a connected sum of finitely many copies of $\mathbf{R}P^2$. If $\widehat{\Sigma'}$ is orientable, then we apply Theorem 1 to deduce that $\widehat{\Sigma'}$ is a connected sum of g copies of the torus T, for some $g \geq 0$. If $g = 0$, then $\widehat{\Sigma}' \simeq S^2$, so that $\Sigma \simeq S^2\#\mathbf{R}P^2 \simeq \mathbf{R}P^2$. The case $g > 0$ is handled through repeated application of the following Lemma:

Lemma 7. *There is a diffeomorphism*

$$\mathbf{R}P^2\#\mathbf{R}P^2\#\mathbf{R}P^2 \simeq T\#\mathbf{R}P^2.$$

Proof. Choose a pair of embedded circles $C, C' \subset T$ which meet transversely in one point x. Let us identify $T\#\mathbf{R}P^2$ with the 2-manifold obtained from T by removing a small disk D around x, and gluing on a Mobius band K along the boundary ∂D. Then $C - C \cap D$ and $C' - C' \cap D$ can be extended to *nonintersecting* embedded loops $\overline{C}$ and $\overline{C}'$ on $T\#\mathbf{R}P^2$, both of which are one-sided. Using the preceding arguments, we deduce that there exists a decomposition

$$T\#\mathbf{R}P^2 \simeq (\mathbf{R}P^2\#\mathbf{R}P^2)\#\Sigma,$$

where Σ is the surface obtained by removing tubular neighborhoods of $\overline{C}$ and $\overline{C}'$ and capping of their boundary components. A simple calculation shows that $\chi(\Sigma) = 1$, so that Σ must be nonorientable: we therefore have $\Sigma \simeq \mathbf{R}P^2\#\Sigma'$. Then $\chi(\Sigma') = 2$, so that Σ' is a 2-sphere (Lemma 2). It follows that $\Sigma \simeq \mathbf{R}P^2$ so that

$$T\#\mathbf{R}P^2 \simeq \mathbf{R}P^2\#\mathbf{R}P^2\#\mathbf{R}P^2$$

as desired. □

Remark 8. In the next few lectures, we will need to understand not only closed 2-manifolds, but also 2-manifolds with boundary. However, it is easy to extend the above classification: the boundary of a (compact) 2-manifold Σ is a compact 1-manifold, hence a union of finitely many circles. If we let Σ' be the 2-manifold obtained by capping off these boundary circles, then Σ' is diffeomorphic to a 2-manifold of the form

$$T\#T\#\ldots\#T \qquad \mathbf{R}P^2\#\mathbf{R}P^2\#\ldots\#\mathbf{R}P^2,$$

and Σ is obtained from Σ' by removing small disks around finitely many points.

Remark 9. Let Σ be a compact connected 2-manifold (possibly nonorientable or with boundary). The properties of Σ depend strongly on the sign of the Euler characteristic $\chi(\Sigma)$. It is therefore convenient to list the possibilities for Σ when χ is nonnegative:

- If $\chi(\Sigma) = 2$, then $\Sigma \simeq S^2$ (Lemma 2).
- If $\chi(\Sigma) = 1$, then either $\Sigma \simeq \mathbf{R}P^2$ or $\Sigma \simeq D^2$.
- If $\chi(\Sigma) = 0$, there are several possibilities. If Σ is orientable, then either $\Sigma \simeq T$ or Σ is a twice-punctured sphere (an annulus $S^1 \times [0,1]$). Each of these possibilties has a nonorientable analogue: if Σ is nonorientable and has boundary, then it is diffeomorphic to a punctured copy of $\mathbf{R}P^2$: this is a Mobius band, given by a nonorientable $[0,1]$-bundle over S^1. If Σ is nonorientable and closed, then it is diffeomorphic to the Klein bottle $\mathbf{R}P^2\#\mathbf{R}P^2$. This 2-manifold can be viewed as obtained by gluing together two Mobius bands along their boundary, which realizes it as a nonorientable S^1-bundle over S^1 (alternatively, one can start with the surface Σ which is a nonorientable S^1-bundle over S^1; then $\chi(\Sigma) = 0$ so that Theorem 4 guarantees a diffeomorphism $\Sigma \simeq \mathbf{R}P^2\#\mathbf{R}P^2$.
- If $\chi < 0$, then we are in the "generic case".