The Loop Theorem: Reduction to a Special Case (Lecture 28)

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Let us continue our analysis of a general position map $f : S \to M$, where Σ is a compact surface and M is a 3-manifold. We will allow S and M to have boundary, but insist that f be proper: that is, f carries the boundary of S into the boundary of M. In the previous lecture, we saw that $f(S)$ is a smooth submanifold away from three types of points:

(1) Double points of the map f: that is, points $x \in M$ such that $f^{-1}\{x\}$ has exactly two elements, near each of which f is an immersion. These form a locally closed smooth submanifold of M having codimension 2 (which may intersect the boundary of M).

(2) Triple points of the map f: points $x \in M$ such that $f^{-1}\{x\}$ has exactly three points, near each of which f is an immersion. These form a submanifold of M having codimension 3: that is, a finite collection of points in M.

(3) Branch points of the map f: that is, points $x \in M$ such that $f^{-1}\{x\}$ has a single point over which f fails to be an immersion. The number of branch points of M is also finite, and every branch point lies in the closure of the set of double points of f.

We define the *singular locus* of f to be the subset of $f(M)$ consisting of these three types of points. Our analysis shows that the singular locus of f is a finite graph, which can be written as a union of the following constituents:

(i) *Double loops* of f: that is, closed loops in M contained in the locus of double points.

(ii) *Double arcs* of f: that is, closed arcs in M whose interior consists of double points, and whose endpoints are either triple points, branch points, or double points contained in ∂M.

We now turn to the proof of the loop theorem.

Theorem 1 (Loop Theorem)**.** *Let M be a connected 3-manifold with boundary, let X be a connected 2-manifold with boundary contained in ∂M, and let N be a normal subgroup of $\pi_1 X$. Suppose that N does not contain the kernel of the map $\pi_1 X \to \pi_1 M$. Then there exists an embedding $f : (D^2, S^1) \to (M, X)$ such that the underlying loop $S^1 \to X$ represents a class in $\pi_1 X$ which does not belong to N.*

By assumption, we can choose a loop $S^1 \to X$ representing a class which does not belong to N, but does belong to the kernel of the map $\pi_1 X \to \pi_1 M$. We can therefore extend the loop to a map $f_0 : (D^2, S^1) \to (M, X)$. This map f_0 need not be an embedding. However, we can assume without loss of generality that f_0 is piecewise linear and in general position (our discussion of general position maps above was in the smooth category, but there is a parallel discussion in the PL category; we will not worry about the details). The image $f_0(D^2)$ is a compact polyhedron in M. Let M_0 be a closed regular neighborhood of $f_0(D^2)$, and let $X_0 = M_0 \cap X$. Then M_0 is a *compact* 3-manifold with boundary which contains $f_0(D^2)$ as a deformation retract, and X_0 is a regular neighborhood of $f_0(S^1)$ (and therefore connected). Let N_0 be the inverse image of N in $\pi_1 X_0$. Note that $f_0|S^1$ represents a class in $\pi_1 X_0$ (well-defined up to conjugacy) which does not belong to N_0.

Suppose that M_0 admits a connected double cover $\widetilde{M}_0$. Since D^2 is simply connected, we can lift f_0 to a map $f_1 : D^2 \to \widetilde{M}_0$. Let M_1 be a regular neighborhood of $f_1(D^2)$, and X_1 its intersection with the inverse image of X_0 (so that X_1 is a regular neighborhood of $f_1(S^1)$), and let N_1 denote the inverse image of N_0 in $\pi_1 X_1$. Then f_1 can be regarded as a map $(D^2, S^1) \to (M_1, X_1)$, representing a class in $\pi_1 X_1$ which does not belong to N_1.

Iterating this procedure, we can produce a sequence of maps $f_i : (D^2, S^1) \to (M_i, X_i)$. We claim that this process must eventually terminate, meaning that eventually the compact 3-manifold M_i does not admit any connected double covers. To see this, we observe that if $f_{i+1} D^2 \to f_i D^2$ is a homeomorphism, then the map $M_{i+1} \to M_i$ must be a homotopy equivalence (since each M_j contains $f_j(D^2)$ as a deformation retract), so that $\phi : \pi_1 M_{i+1} \to \pi_1 M_i$ is surjective; this is a contradiction, since by construction M_{i+1} is a subset of a connected double cover of M_i so that the image of ϕ has index at least 2. Consequently, each of the maps $f_{i+1} D^2 \to f_i D^2$ must fail to be a homeomorphism. This implies that some double point of f_i fails to be a double point of f_{i+1}. Thus f_{i+1} has fewer double curves than f_i. Since the number of double curves of f_0 is finite, our process must halt after finitely many steps.

Suppose therefore that we have constructed a map $f_m : (D^2, S^1) \to (M_m, X_m)$ where M_m does not admit any connected double covers. In other words, the cohomology group $\mathrm{H}^1(M_m)$ vanishes (here and in what follows, we will assume that all cohomology is taken with coefficients in $\mathbf{Z}/2\mathbf{Z}$). Then $\mathrm{H}_1(M_m) \simeq 0$ and, by Poincare duality, $\mathrm{H}_2(M_m, \partial M_m) \simeq 0$. Using the long exact sequence

$$\mathrm{H}_2(M_m, \partial M_m) \to \mathrm{H}_1(\partial M_m) \to \mathrm{H}_1(M_m),$$

we deduce that $\mathrm{H}_1(\partial M_m) \simeq 0$. Consequently, every boundary component of M_m must be a sphere.

The space X_m is a connected compact surface with (nonempty) boundary equipped with an embedding into some boundary component of X_m, which must be a 2-sphere S^2. It follows that the group $\pi_1 X_m$ is generated by the conjugacy classes of elements which are represented by loops in the boundary ∂X_m. Since N_m does not contain the class of the loop $f_m | S^1$, we have $N_m \neq \pi_1 X_m$. It follows that N_m does not contain the class of some embedded loop $S^1 \hookrightarrow \partial X_m$. This embedded loop bounds an embedded disk in the 2-sphere $S^2 \subseteq \partial M_m$, and therefore bounds an embedded disk in M_m. We can therefore choose an embedding $g_m : (D^2, S^1) \to (M_m, X_m)$ such that the underlying map $S^1 \to X_m$ represents a class in $\pi_1 X_m$ not belonging to N_m.

Of course, the composition of g_m with the projection $M_m \to M$ need not be an embedding. However, we will "descend" g_m to a sequence of embeddings $g_i : (D^2, S^1) \to (M_i, X_i)$ (representing a loop in $\pi_1 X_i$ not belonging to N_i) using descending induction on i. Assuming that g_{i+1} has been constructed, consider the composite map

$$g'_i : D^2 \stackrel{g_{i+1}}{\to} M_{i+1} \subseteq \widetilde{M}_i \to M_i.$$

This map need not be an embedding. However, it is the composition of an embedding with a 2-fold covering map. It follows that g'_i is an immersion and that g'_i has no triple points. Moving g'_i by a small isotopy, we can assume that g'_i is a general position map with the same properties. To construct g'_i from g_i, it suffices to prove the following special case of the Loop Theorem:

Theorem 2. *Let M be a connected 3-manifold with boundary, let X be a connected 2-manifold with boundary contained in ∂M, and let N be a normal subgroup of $\pi_1 X$. Suppose we are given a map $g' : (D^2, S^1) \to (M, X)$ with the following properties:*

(1) *The map g' is an immersion without triple points (consequently, the singular locus of g' consists of closed double loops and double arcs which join double points belonging to X).*

(2) *The restriction $g' | S^1$ represents a class in $\pi_1 X$ which does not belong to N.*

Then there exists an embedding $g : (D^2, S^1) \to (M, X)$ which satisfies (2).

We will prove Theorem 2 in the next lecture.