Uniqueness of Prime Decompositions (Lecture 26)

April 13, 2009

In the last lecture, we introduced the notion of a *prime* 3-manifold, and showed that every 3-manifold can be obtained as a connected sum of prime factors. In this lecture, we will prove a theorem of Milnor which asserts that this decomposition is unique. We will assume for convenience that all of our 3-manifolds are connected and oriented.

Note that a prime 3-manifold need not have $\pi_2 M \simeq *$. For example, if $M = S^2 \times S^1$, then $\pi_2 M$ does not vanish, but M is prime (since $\pi_1 M$ cannot be factored nontrivially as a free product). However, this is essentially the only counterexample.

Definition 1. Let M be a 3-manifold which is not a 3-sphere. We will say that M Is *irreducible* if every embedded 2-sphere $S^2 \hookrightarrow M$ bounds a disk on one side or the other.

Remark 2. We say that an embedding $S^2 \hookrightarrow M$ is *separating* if $M - S^2$ is disconnected. Note that S^2 is separating if and only if its fundamental class $[S^2] \in \mathrm{H}_2(M; \mathbf{Z}/2\mathbf{Z})$ vanishes.

By definition, $M \neq S^3$ is prime if and only if every *separating* 2*-sphere* of M bounds a 3-disk. Consequently, every irreducible 3-manifold is prime. The product $S^2 \times S^1$ is an example of a prime 3-manifold which is not irreducible, but this example is unique (provided we stick to oriented 3-manifolds):

Proposition 3. *Let M be a compact, connected, oriented 3-manifold. Suppose that M contains a nonseparating 2-sphere S. Then M can be written as a connect sum $M_1 \# (S^2 \times S^1)$. In particular, if M is prime, then $M \simeq S^2 \times S^1$.*

Proof. Since S is nonseparating, there exists a loop L in M which intersects the 2-sphere S transversely in exactly one point. Let M_2' denote the union of a tubular neighborhood of L and a tubular neighborhood of S. Then the boundary of M_2' is equivalent to a connect sum of 2-spheres, so that $\partial M_2' \simeq S^2$. Let M_2 be the 3-manifold obtained by capping off this boundary 2-sphere with a disk. Then M_2 has the structure of an S^2-bundle over the loop L. Since M is orientable, this 2-sphere bundle must be trivial, so that $M_2 \simeq S^2 \times S^1$ and cutting along $\partial M_2'$ gives the desired connect sum decomposition of M. □

We now turn to the uniqueness of prime factorizations. Suppose that M is a compact connected 3-manifold and we have two prime decompositions

$$M_1 \# M_2 \# \cdots \# M_n \simeq M \simeq M_1' \# \cdots \# M_m'.$$

We will show that $m = n$ and that the diffeomorphism types of the prime factors agree up to a permutation. Our first step is to give a criterion which allows us to intrinsically detect if $S^2 \times S^1$ appears as a factor on one side. Note that if $M \simeq M' \# (S^2 \times S^1)$, then M contains a nonseparating 2-sphere. Conversely:

Proposition 4. *Let M be a compact, connected, oriented 3-manifold with a prime decomposition $M \simeq M_1 \# \ldots \# M_n$, and suppose that each M_i is irreducible. Then M contains no nonseparating 2-sphere.*

Proof. It will suffice to show that if M and N are 3-manifolds containing no nonseparating 2-spheres, then $M\#N$ likewise contains no nonseparating 2-sphere. Assume for a contradiction $M\#N$ contains a nonseparating 2-sphere S, and let T denote the separating 2-sphere given by the connect sum decomposition of $M\#N$. Without loss of generality we may assume that S and T meet transversely. Let k be the number of connected components of $S \cap T$, and assume that S has been chosen to minimize k. If $k = 0$, then without loss of generality we have $S \subseteq M$. Since $[S]$ is nontrivial in $\mathrm{H}_2(M\#N)$, it is nontrivial in $\mathrm{H}_2(M - D^3) \simeq \mathrm{H}_2(M)$ so that S is a separating 2-sphere of M, contrary to our assumption.

We may therefore assume that $k > 0$. Regard the intersection $S \cap T$ as a union of finitely many circles in $T \simeq S^2$. Choose an "innermost" circle $C \subseteq S \cap T$, so that C bounds a disk D in T whose interior does not intersect S. This circle also cuts S into 2-disks E_+ and E_-. Let $S_+ = D \cup E_+$ and $S_- = D \cup E_+$. Then $[S] = [S_+] + [S_-] \neq 0$, so that either S_+ or S_- is also a nonseparating 2-sphere in $M\#N$. Without loss of generality S_+ is nonseparating. Moving S_+ by a small isotopy, we can arrange that it intersects T in fewer than k components, contradicting the minimality of k. □

Returning to our decomposition

$$M_1\#M_2\#\cdots\#M_n \simeq M \simeq M_1'\#\cdots\#M_m',$$

we deduce that if some $M_i \simeq S^2 \times S^1$, then also some $M_j' \simeq S^2 \times S^1$. Reordering the decompositions, we may assume $i = j = 1$. We would like to assert that the complementary summands $M_2\#\ldots\#M_n$ and $M_2'\#\ldots\#M_m'$ are diffeomorphic. These complementary summands can be obtained by cutting M along nonseparating 2-spheres in the factors $S^2 \times S^1$, and then capping of the resulting boundary spheres by disks. To prove that the resulting manifold is unique up to diffeomorphism, it suffices to prove the following:

Proposition 5. *Let M be a compact, connected, oriented 3-manifold containing a pair of nonseparating 2-spheres S and T. Then there is an (orientation preserving) diffeomorphism of M with itself that carries S to T.*

Proof. Moving S by an isotopy, we can assume that S and T meet transversely. We work by induction on the number k of connected components of $S \cap T$. If $k > 0$, then we can write $[S] = [S_+] + [S_-]$ as before, so that either S_+ or S_- is a nonseparating k-sphere in M; without loss of generality, S_+ is nonseparating. Moving S_+ by a small isotopy, we can arrange that it is disjoint from S and intersects T in fewer than k components. Applying the inductive hypothesis, we obtain diffeomorphisms of M carrying S to S_+ and S_+ to T; the composition of these diffeomorphisms then does the job.

If $k = 0$, then S and T are disjoint. Since $M - S$ is connected, $M - (S \cup T)$ has at most 2 components. Assume first that $M - (S \cup T) = N \coprod N'$, and let $\overline{N}$ and $\overline{N'}$ be the 3-manifolds obtained by capping off the boundary 2-spheres of N and N'. Since the orientation-preserving diffeomorphism group $\mathrm{Diff}^+(\overline{N})$ acts transitively on pairs of distinct points of $\overline{N}$, we can find a diffeomorphism which restricts to a diffeomorphism of N which exchanges the two boundary components. Similarly, we can find such a diffeomorphism of N. Modifying them by an isotopy if necessary (using the connectedness of $\mathrm{Diff}^+(S^2)$), we can assume that they glue to give a diffeomorphism of M which exchanges S and T.

The proof when $M - (S \cup T)$ is similar: we let $\overline{M}$ denote the 3-manifold obtained by capping off the boundary 3-spheres in $M - (S \cup T)$, and use the fact that $\mathrm{Diff}^+(\overline{M})$ acts transitively on quadruples of distinct points in $\overline{M}$. □

By repeatedly applying the above result, we are reduced to proving the uniqueness of prime decompositions

$$M_1\#M_2\#\cdots\#M_n \simeq M \simeq M_1'\#\cdots\#M_m'$$

in which each factor (on either side) is irreducible. Without loss of generality $n, m > 1$ (otherwise $M = S^3$ or is irreducible, and there is nothing to prove). Let T be the separating 2-sphere of M corresponding to the decomposition $M_1'\#(M_2'\#\cdots\#M_m')$. Similarly, we can choose nonintersecting 2-spheres $S_1, \ldots, S_{n-1}$ giving rise to the first decomposition. Without loss of generality, T meets $\bigcup S_i$ transversely in k circles. We assume that the system of spheres $\{S_i\}$ has been chosen to minimize k. If $k = 0$, then T is contained in some M_i.

Since M_i is irreducible, T bounds a 3-disk in M_i. Let $\{M_{j_1}, \ldots, M_{j_k}\}$ denote the collection of those M_j which are attached to M_i via spheres contained in this 3-disk. Reindexing, we can assume that $j_1 = 1, \ldots, j_k = k$. Then T separates M into pieces $M_1 \# \ldots \# M_k$ and $M_{k+1} \# \ldots \# M_n$. It follows either that $k = 1$, $M_1 \simeq M'_1$, and $M_2 \# \ldots \# M_n \simeq M'_2 \# \ldots \# M'_m$, or that $k = n-1$, $M_n \simeq M'_1$, and $M_1 \# \ldots \# M_{n-1} \simeq M'_2 \# \ldots \# M'_m$. In either case, we can conclude by induction that the prime factors agree up to reindexing.

If $k > 0$, then as before we can choose an innermost circle C in the intersection $(\bigcup S_i) \cap T$, so that C bounds a disk D in T which does not intersect any S_i; this disk is then contained in $M_j - B^3 \subseteq M$. Let $S = S_i$ be the the boundary sphere of $M_j - B^3$ containing C, so that C cuts S into two disks E_- and E_+. Let $S_+ = E_+ \cup D$. Then S_+ is a 2-sphere in M_j; since M_j is irreducible we conclude that S_+ bounds a 3-disk X. Replacing E_+ by E_- if necessary, we can assume that $B^3 \subseteq X$. By a small isotopy, we can arrange that S_+ does not intersect T along C. Replacing S_i by S_+, we obtain a system of spheres which intersects T in fewer than k circles, and can conclude by the inductive hypothesis.