Comparison of Smooth and PL Structures (Lecture 23)

April 3, 2009

In this lecture, we will attempt to prove that the theories of smooth and PL manifolds are equivalent. In view of the smoothing theory we have already developed, this is equivalent to the assertion that the spaces $PL(n)/O(n)$ are contractible for $n \geq 0$. We will attempt to prove this using induction on n. Of course, this attempt is doomed to failure, since there are PL manifolds which cannot be smoothed and PL manifolds which admit inequivalent smooth structures (such as Milnor's exotic spheres).

Let us assume that the space $PL(n-1)/O(n-1)$ is contractible, and attempt to prove that $PL(n)/O(n)$ is contractible. Consider the map

$$\phi : PL(n-1)/O(n-1) \to PL(n)/O(n).$$

The product smoothing theorem implies that all the homotopy fibers of ϕ are $(n-1)$-connected. In particular, they are connected, so that $PL(n)/O(n)$ is connected. Hence ϕ really only has one homotopy fiber up to equivalence, which can be identified with the loop space $\Omega PL(n)/O(n)$. Since this loop space is $(n-1)$-connected, we have proven the following:

Lemma 1. *If $PL(n-1)/O(n-1)$ is contractible, then $PL(n)/O(n)$ is n-connected.*

Consequently, $PL(n)/O(n)$ is contractible if and only if the loop space $\Omega^{n+1} PL(n)/O(n)$ is contractible. Let us try to understand this loop space.

First, consider the loop space $\Omega^n PL(n)/O(n)$. Let D^n be an n-dimensional disk in the PL setting, and equip the boundary ∂D^n with its standard smooth structure. Smoothing theory implies that the space of smoothings of D^n (compatible with our given smoothing on the boundary) can be identified the space of solutions to the lifting problem

$$\begin{array}{ccc} \partial D^n & \longrightarrow & BO(n) \\ \downarrow & \nearrow & \downarrow \\ D^n & \longrightarrow & BPL(n). \end{array}$$

Since the horizontal maps are constant (the disk D^n has trivial tangent microbundle in both the smooth and PL settings), this space of solutions can be identified with $\Omega^n PL(n)/O(n)$.

When we loop the space $\Omega^n PL(n)/O(n)$ one more time, we encounter not classifying spaces of smooth structures but classifying spaces for their automorphisms. More precisely, let $\mathrm{Diff}(D^n, \partial)$ denote the space of diffeomorphisms of the standard smooth disk D^n which are the identity near the boundary ∂D^n, and let $\mathrm{Homeo}_{PL}(D^n, \partial)$ be defined similarly. Then the spaces $B\,\mathrm{Diff}(D^n, \partial)$ and $B\,\mathrm{Homeo}_{PL}(D^n, \partial)$ can be identified with connected components of the classifying for smooth and PL manifolds which are bounded by the sphere S^{n-1}. As we have seen, there is a map (well-defined up to homotopy) $B\,\mathrm{Diff}(D^n, \partial) \to B\,\mathrm{Homeo}_{PL}(D^n, \partial)$. Denote the homotopy fiber of this map by $\mathrm{Homeo}_{PL}(D^n, \partial)/\,\mathrm{Diff}(D^n, \partial)$, so that $\mathrm{Homeo}_{PL}(D^n, \partial)/\,\mathrm{Diff}(D^n, \partial) \simeq \Omega^n PL(n)/O(n)$. We have a fibration sequence

$$\mathrm{Diff}(D^n, \partial) \to \mathrm{Homeo}_{PL}(D^n, \partial) \to \mathrm{Homeo}_{PL}(D^n, \partial)/\,\mathrm{Diff}(D^n, \partial).$$

Lemma 2. *The space $\mathrm{Homeo}_{PL}(D^n, \partial)$ is contractible.*

Lemma 2 is just an articulation of the Alexander trick, which we described in the last lecture: every PL homeomorphism of D^n which is the identity on the boundary is canonically isotopic to the identity.

It follows from Lemma 2 that we can identify $\mathrm{Diff}(D^n, \partial)$ with the loop space of $\mathrm{Homeo}_{PL}(D^n, \partial)/\mathrm{Diff}(D^n, \partial)$, and therefore with $\Omega^{n+1} PL(n)/O(n)$. We have proven:

Proposition 3. *Assume that $PL(n-1)/O(n-1)$ is contractible. Then $PL(n)/O(n)$ is contractible if and only if $\mathrm{Diff}(D^n, \partial)$ is contractible: in other words, if and only if the Alexander trick works in the smooth category.*

We can massage the criterion of Proposition 3 further. Let S^n denote the n-sphere, and choose a point $x \in S^n$. We can identify D^n with the submanifold obtained from S^n by removing the interior of a small disk around x. We have seen that, in the smooth category, this small disk is determined up to contractible ambiguity (this is not true in the PL category). Here is another way to articulate this idea: given a point $x \in S^n$, we can define a new smooth manifold M by forming the *real blow-up* of S^n at x. Namely, we let $M = (S^n - \{x\}) \coprod (T_{S^n,x} - \{0\})/\mathbb{R}_{>}$ be the space obtained from S^n by replacing the point x by the collection of all directed rays in the tangent space $T_{S^n,x}$. Then M has the structure of a smooth manifold, which depends functorially on the pair (S^n, x). This smooth manifold is simply a smooth n-disk D^n. Moreover, this construction determines an isomorphism of $\mathrm{Diff}(D^n, \partial)$ with the group $\mathrm{Diff}(S^n, \{x\})$ of diffeomorphisms of S^n which coincide with the identity near $\{x\}$. Thus:

Proposition 4. *Assume that $PL(n-1)/O(n-1)$ is contractible. Then $PL(n)/O(n)$ is contractible if and only if $\mathrm{Diff}(S^n, \{x\})$ is contractible.*

Let $\mathrm{Diff}_x(S^n)$ denote the group of diffeomorphisms ϕ of S^n which satisfy $\phi(x) = x$. We have a homotopy fiber sequence

$$\mathrm{Diff}(S^n, \{x\}) \to \mathrm{Diff}_x(S^n) \to G$$

where G denotes the monoid of equivalences from the smooth microbundle of S^n at x. Since a smooth microbundle is canonically determined by its tangent space along the zero section, this gives us a fiber sequence

$$\mathrm{Diff}(S^n, \{x\}) \to \mathrm{Diff}_x(S^n) \xrightarrow{\theta} GL_n(\mathbb{R})$$

where θ is given by differentiation at x. It follows that $\mathrm{Diff}(S^n, \{x\})$ is contractible if and only if θ is a homotopy equivalence.

Note that the group $O(n)$ acts on S^n by diffeomorphisms fixing the point x. We have a commutative diagram

$$\begin{array}{ccccc} & & \mathrm{Diff}_x(S^n) & & \\ & \overset{\theta'}{\nearrow} & & \overset{\theta}{\searrow} & \\ O(n) & & \xrightarrow{\quad\theta''\quad} & & GL_n(\mathbb{R}). \end{array}$$

Since θ'' is a homotopy equivalence, we deduce that θ is a homotopy equivalence if and only if θ' is a homotopy equivalence. In other words:

Proposition 5. *Assume that $PL(n-1)/O(n-1)$ is contractible. Then $PL(n)/O(n)$ is contractible if and only if the inclusion $O(n) \to \mathrm{Diff}_x(S^n)$ is a homotopy equivalence.*

The group $\mathrm{Diff}(S^n)$ acts on S^n. This gives rise to homotopy fiber sequences

$$\begin{array}{ccccc} O(n) & \longrightarrow & O(n+1) & \longrightarrow & S^n \\ \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \psi} & & \downarrow \\ \mathrm{Diff}_x(S^n) & \longrightarrow & \mathrm{Diff}(S^n) & \longrightarrow & S^n. \end{array}$$

It follows that θ' is a homotopy equivalence if and only if ψ is a homotopy equivalence. This proves the following:

Theorem 6. *Assume that $PL(n-1)/O(n-1)$ is contractible. Then $PL(n)/O(n)$ is contractible if and only if the inclusion $O(n+1) \to \mathrm{Diff}(S^n)$ is a homotopy equivalence.*

Example 7. The conditions of Theorem 6 are satisfied when $n = 1$: the space $PL(0)/O(0)$ is obviously contractible, while $O(2) \simeq \mathrm{Diff}(S^1)$ by the arguments given on the first day of class. This proves that the theory of smooth and PL manifolds are the same in dimension 1.

Example 8. To apply Theorem 6 when $n = 2$, we must show that $O(3) \simeq \mathrm{Diff}(S^2)$. This is a theorem of Smale, which we will prove in the next lecture.

Example 9. Theorem 6 also applies when $n = 3$. For this, we need to show that $O(4) \simeq \mathrm{Diff}(S^3)$. This assertion is known as the *Smale conjecture*. It was proven by Hatcher, but we will not present the details in class.

Example 10. It is unknown (at least by me) whether Theorem 6 applies when $n = 4$. This is equivalent to the question of whether the inclusion $O(5) \to \mathrm{Diff}(S^4)$ is a homotopy equivalence. Even the simplest consequence of this assertion is a difficult open question: is every orientation-preserving diffeomorphism of S^4 isotopic to the identity?

Remark 11. Even if the second hypothesis of Theorem 6 fails, the contractibility of $PL(n-1)/O(n-1)$ still has powerful consequences for the theory of n-manifolds. Namely, it implies that $PL(n)/O(n)$ is n-connected (Lemma 1). The smooth structures on a PL n-manifold M are classified by sections of a fibration over M with fiber $PL(n)/O(n)$. Since M is n-dimensional and these fibers are n-connected, we deduce that the space of sections is nonempty and connected: in other words, M admits a smooth structure which is unique up to PD isotopy. We can proceed further to argue that $PL(n+1)/O(n+1)$ must again be n-connected, so that every PL $(n+1)$-manifold M admits a compatible smooth structure (though we will not know that this smooth structure is unique).

For example, our present state of knowledge is enough to guarantee that every PL 2-manifold can be smoothed in an essentially unique way, and that every PL 3-manifold admits a smoothing. After we prove Smale's theorem in the next lecture, we will know that PL 3-manifolds admit essentially unique smoothings, and that PL 4-manifolds can be smoothed. Assuming the Smale conjecture, we can go further to say that PL 4-manifolds admit essentially unique smoothings, and that PL 5-manifolds can be smoothed.

These results are not optimal: as it turns out, PL manifolds of dimension ≤ 7 can be smoothed, and these smoothings are essentially unique in dimensions ≤ 6.