

Product Structure Theorem: Inductive Step (Lecture 21)

March 30, 2009

Recall that we are in the process of proving the product structure theorem for smooth structures on PL manifolds, which (by virtue of smoothing theory) is equivalent to the following connectivity estimate:

Theorem 1. *Let $m \geq 0$. Then all homotopy fibers of the map $PL(m)/O(m) \rightarrow PL(m+1)/O(m+1)$ are m -connected.*

We have reduced the proof to the following statement:

Proposition 2. *Let $K \subseteq \mathbb{R}^m \times \mathbb{R}$ be a polyhedron which is the closed star of the origin 0 with respect to some PL triangulation of \mathbb{R}^{m+1} (so that K is the cone on ∂K , with the origin as the cone point), let $\pi : K \rightarrow \mathbb{R}$ denote the projection onto the last factor. Let $f : K \rightarrow \mathbb{R}^{m+1}$ be a PD embedding satisfying the following conditions:*

- (1) *The image of f is the unit ball $B(1) \subseteq \mathbb{R}^{m+1}$.*
- (2) *For $\frac{1}{2} \leq t \leq 1$ and $x \in \partial K$, we have $f(tx) = tf(x)$.*
- (3) *The projection π is injective when restricted to the vertices of K (with respect to some PL triangulation), so that $\pi \circ f^{-1}$ is regular on the interior of the unit ball except possibly at the origin.*

Then, after modifying f by a PD isotopy which is trivial on ∂K , we can arrange that $\pi \circ f^{-1}$ is regular on the interior of the unit ball.

Let $S^m \subseteq B(1)$ denote the unit sphere. Condition (1) implies that f restricts to a PD homeomorphism $f_0 : \partial K \rightarrow S^m$. Since π is injective on vertices, the composition $\pi \circ f_0^{-1} : S^m \rightarrow \mathbb{R}$ is regular except possibly at the images of the vertices of ∂K . In particular, it is regular in a neighborhood of $\partial K \cap \pi^{-1}\{0\}$. Using the arguments of Lecture 19, we deduce that there is a PD isotopy $\{g_t : \partial K \rightarrow S^m\}_{t \in [0,1]}$ such that $g_0 = f_0$ and $\pi \circ g_1^{-1} : S^m \rightarrow \mathbb{R}$ has 0 as a regular value. This map decomposes the sphere S^m into two smooth submanifolds

$$D_- = \{g_1 \pi^{-1} \mathbb{R}_{\leq 0}\} \quad D_+ = \{g_1 \pi^{-1} \mathbb{R}_{\geq 0}\}.$$

The map g_1 provides PD homeomorphisms of D_- and D_+ with PL m -disks.

We will now use the product smoothing theorem for $(m-1)$ -manifolds (which we may assume as an inductive hypothesis) to verify the following:

Lemma 3. *Let $X = [0, 1]^m$ be a PL m -disk. Then, up to PD isotopy and X has a unique smooth structure (in other words, there are no exotic smooth structures on PL m -disks).*

Proof. Smoothing theory tells us that smooth structures on X are classified by the following homotopy-theoretic data:

- (a) Solutions to the lifting problem

$$\begin{array}{ccc} & & BO(m) \\ & \nearrow & \downarrow \\ X & \longrightarrow & BPL(m). \end{array}$$

(b) Solutions to the induced lifting problem

$$\begin{array}{ccc}
 & & BO(m-1) \\
 & \nearrow & \downarrow \\
 \partial X & \dashrightarrow & BO(m) \times_{BPL(m)} BPL(m-1).
 \end{array}$$

Using Theorem 1 in dimensions $< m$, we deduce that $PL(m)/O(m)$ is connected. Since X is contractible, problem (a) has a unique solution up to homotopy. Solutions to problem (b) can be described as sections of a fibration $\phi : \widehat{\partial X} \rightarrow \partial X$ whose fibers are homotopy fibers of the map $PL(m-1)/O(m-1) \rightarrow PL(m)/O(m)$. Invoking Theorem 1 again (in dimension $m-1$), we deduce that these fibers are $(m-1)$ -connected. Since ∂X has dimension $(m-1)$, the fibration ϕ has a unique section up to homotopy. \square

Returning to our problem, we deduce that the smooth submanifolds $D_- \cdot D_+ \subseteq S^m$ are diffeomorphic to smooth disks. We now need the following:

Lemma 4. *Let $B(1)$ denote the open unit ball in \mathbb{R}^m , and suppose we are given a smooth orientation-preserving embedding $i : B(1) \rightarrow S^m$. Then i is isotopic to the standard embedding.*

We can identify S^m with the one-point compactification of \mathbb{R}^m . Without loss of generality, we may assume that the image of i does not contain the point at infinity (since i is not surjective, we can always reduce to this situation by applying a rotation of the sphere S^m). Then Lemma 4 is an immediate consequence of the following:

Lemma 5. *Let $B(1)$ denote the open unit ball in \mathbb{R}^m , and let $i : B(1) \rightarrow \mathbb{R}^m$ be a smooth orientation-preserving embedding. Then i is isotopic to the standard embedding.*

Proof. Applying a translation of \mathbb{R}^m , we can arrange that $i(0) = 0$. Acting by a linear map, we can arrange that the derivative of i is equal to zero near the origin (since i is orientation preserving, this linear map can be chosen to lie in the identity component of $GL(n, \mathbb{R})$). Define a smooth homotopy $\{i_t : B(1) \rightarrow \mathbb{R}^m\}$ from $i_0 = i$ to the standard inclusion by the formula $i_t(x) = (1-t)i(x) + tx$. This map is generally not an isotopy. However, it is an isotopy near 0, and therefore on a ball $B(\epsilon)$ for ϵ sufficiently small. Let $j : B(1) \rightarrow \mathbb{R}^m$ be the map given by $j(x) = \frac{i(\epsilon x)}{\epsilon}$. Then i_t determines an isotopy from j to the standard embedding. Moreover, i is isotopic to j , since we have a smooth family of maps

$$\left\{ j_t(x) = \frac{i(tx)}{t} \right\}_{t \in [\epsilon, 1]}.$$

\square

Remark 6. We can carry out a version of the proof of Lemma 5 with parameters, given an appropriate generalization of the condition that i be orientation-preserving (we need to be able to arrange that the derivative of i is the identity near the origin). This argument can be used to prove the following fact: any smooth microbundle contains an (essentially unique) smooth disk bundle. This is the key difference between the smooth and PL categories: a PL microbundle always contains a PL \mathbb{R}^n -bundle, but this generally cannot be refined to a PL disk bundle.

We now return to the proof of Proposition 2. Lemma 4 implies that we can adjust the PD isotopy $\{g_t\}$ by a smooth isotopy of S^m to arrange that D_- (and therefore D_+) can be identified with the standard disks in S^m . It follows that $D_- \cap D_+$ is the standard equator $S^{m-1} \subseteq S^m$, given by the zero locus of the projection $\pi : S^m \hookrightarrow \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$ onto the last factor. In other words, we can assume that $\pi \circ g_1^{-1}$ coincides with π on S^{m-1} . Using the uniqueness of smooth collars, we may further adjust our isotopy so that $\pi \circ g_1^{-1}$ coincides with π on a neighborhood $\pi^{-1}(-\epsilon, \epsilon)$ of S^{m-1} .

We now define a PD isotopy $\{f_t : K \rightarrow B(1)\}_{t \in [0,1]}$ by the formula

$$f_t(sx) = \begin{cases} f(sx) & \text{if } s \leq \frac{1}{2} \\ sg_{tt'}(x) & \text{if } s = \frac{1}{2} + \frac{t'}{6}, 0 \leq t' \leq 1 \\ sg_1(x) & \text{if } \frac{4}{6} \leq s \leq \frac{5}{6} \\ sg_{tt'}(x) & \text{if } s = 1 - \frac{t'}{6}, 0 \leq t' \leq 1. \end{cases}$$

We can then replace $f = f_0$ by f_1 in the statement of Proposition 2. Replacing K by $\frac{5}{6}K$ and multiplying f by $\frac{6}{5}$, we are reduced to proving the following analogue of Proposition 2:

Proposition 7. *Let $K \subseteq \mathbb{R}^m \times \mathbb{R}$ be a polyhedron which is the closed star of the origin 0 with respect to some PL triangulation of \mathbb{R}^{m+1} (so that K is the cone on ∂K , with the origin as the cone point), let $\pi : K \rightarrow \mathbb{R}$ denote the projection onto the last factor. Let $f : K \rightarrow \mathbb{R}^{m+1}$ be a PD embedding satisfying the following conditions:*

- (1) *The image of f is the unit ball $B(1) \subseteq \mathbb{R}^{m+1}$.*
- (2) *For $\frac{4}{5} \leq t \leq 1$ and $x \in \partial K$, we have $f(tx) = tf(x)$.*
- (3) *The projection π is injective when restricted to the vertices of K (with respect to some PL triangulation), so that $\pi \circ f^{-1}$ is regular on the interior of the unit ball except possibly at the origin.*
- (4) *The maps π and $\pi \circ f^{-1}$ coincide on $S^m \cap \pi^{-1}(-\epsilon, \epsilon) \subseteq B(1)$ for some $\epsilon > 0$.*

We will prove Proposition 7 in the next lecture, thereby completing the proof of the product structure theorem.