Classification of Smooth Structures (Lecture 17)

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Recall that our goal is to prove the following result:

Theorem 1. *Let M be a PL manifold and $K \subseteq M$ a closed subpolyhedron. Then the above construction determines a homotopy equivalence from the simplicial set* $\mathrm{Smooth}(K)$ *of smooth structures on M to the simplicial set*

$$BO(m)^K \times_{BPL(m)^K} \{\chi | K\}$$

of liftings of $\chi | K$.

Lemma 2. *Theorem 1 is true when K consists of a single simplex.*

Proof. Choose a point $v \in K$. Restriction to v determines a commutative diagram

$$\begin{array}{ccc} \mathrm{Smooth}(K) & \longrightarrow & \mathrm{Smooth}(\{v\}) \\ \downarrow & & \downarrow \\ BO(m)^K \times_{BPL(m)^K} * & \longrightarrow & BO(m) \times_{BPL(m)} *. \end{array}$$

The right vertical map is an isomorphism of simplicial sets, and the bottom horizontal map is a homotopy equivalence because the inclusion $\{v\} \hookrightarrow K$ is a homotopy equivalence. Consequently, it will suffice to show that the restriction map $r : \mathrm{Smooth}(K) \to \mathrm{Smooth}(\{v\})$ is a trivial Kan fibration. In other words, we must show that every lifting problem of the form

$$\begin{array}{ccc} \partial\,\Delta^n & \xrightarrow{f} & \mathrm{Smooth}(K) \\ \downarrow & \overset{F}{\nearrow} & \downarrow \\ \Delta^n & \xrightarrow{f} & \mathrm{Smooth}(\{v\}) \end{array}$$

has a solution. The map f determines a smooth structure on $U \times \partial\,\Delta^n$ (fibered over $\partial\,\Delta^n$), where U is some neighrbood of K in M. Similarly, g determines a smooth structure on $V \times \Delta^n$, where V is a neighborhood of v in M; without loss of generality we may assume that $V \subseteq U$. Since r is a Kan fibration, we are free to replace g by any map which is homotopic (relative to the boundary $\partial\,\Delta^n$); we may therefore assume that the smooth structure is a product of the smooth structure determined by $g | \partial\,\Delta^n$ over a collar $C = \partial\,\Delta^n \times [0, 1)$ of $\partial\,\Delta^n$ in Δ^n. This smooth structure therefore extends over U, so we obtain a smooth structure S on $W = (U \times C) \coprod_{V \times C} (V \times \Delta^n)$.

Choose a PL isotopy h_t of M supported in U from the identity id_M to a map h_1 which carries Δ^n into V. Let $\chi : \Delta^n \to [0, 1]$ be the map which is equal to 1 on $\Delta^n - C$ and equal to the projection $C \to [0, 1)$ on C. The map $(x, z) \mapsto (h_{\chi(z)}(x), z)$ determines a PL map $H : M \times \Delta^n \to M \times \Delta^n$. Let $W' = H^{-1}(W)$. Our smooth structure on W determines a smooth structure on $H^{-1}(W)$, which contains $K \times \Delta^n$ and therefore determines a map $F : \Delta^n \to \mathrm{Smooth}(K)$. It is easy to see that this map has the desired properties. □

Now fix a triangulation S of the PL manifold M. We prove the following:

Lemma 3. *Let $K \subseteq M$ be a finite union of simplices of the triangulation S. Then Theorem 1 is true for K.*

Proof. We use induction on the number of simplices of S which belong to K. If K is empty, there is nothing to prove. Otherwise, choose a simplex σ belonging to K having maximal dimension, so we can write K as a pushout

$$\begin{array}{ccc} \partial\,\sigma & \longrightarrow & \sigma \\ \downarrow & & \downarrow \\ K_0 & \longrightarrow & K. \end{array}$$

The theorem holds for $\partial\,\sigma$ and K_0 by the inductive hypothesis, and it holds for σ by Lemma 2. We have diagrams

$$\begin{array}{ccc} \mathrm{Smooth}(K) & \longrightarrow & \mathrm{Smooth}(K_0) \\ \downarrow & & \downarrow \\ \mathrm{Smooth}(\sigma) & \longrightarrow & \mathrm{Smooth}(\partial\,\sigma) \end{array} \qquad \begin{array}{ccc} BO(m)^K \times_{BPL(m)^K} * & \longrightarrow & BO(m)^{K_0} \times_{BPL(m)^{K_0}} * \\ \downarrow & & \downarrow \\ BO(m)^\sigma \times_{BPL(m)^\sigma} * & \longrightarrow & BO(m)^{\partial\,\sigma} \times_{BPL(m)^{\partial\,\sigma}} *. \end{array}$$

The square on the right is a homotopy pullback square since the diagram above is a homotopy pushout square of polyhedra. The square on the left is a homotopy pullback square since it is a pullback square in which each of the morphisms is a Kan fibration (by the main result of last time). We therefore have a map of homotopy pullback squares which induces a homotopy equivalence everywhere except perhaps in the upper left hand corner. It follows that it induces a homotopy equivalence in the upper left hand corner as well: that is, the map $\mathrm{Smooth}(K) \to BO(m)^K \times_{BPL(m)^K} *$ is a homotopy equivalence as desired. □

We can now prove Proposition 1 in general. Let K be an arbitrary closed subpolyhedron of M (for example, M itself). We can choose a filtration of K

$$K_0 \subseteq K_1 \subseteq \dots$$

with $K = \bigcup_i K_i$, where each K_i is a finite subpolyhedron. We have a homotopy equivalence of towers $\{\mathrm{Smooth}(K_i)\} \to \{BO(m)^{K_i} \times_{BPL(m)^{K_i}} *\}$. All of the transition maps in these towers are Kan fibrations (for the left tower, this follows from the main result of last time; for the right tower, it follows from the observation that each map of PL singular complexes $\mathrm{Sing}^{PL}_\bullet X_i \to \mathrm{Sing}^{PL}_\bullet X_{i+1}$ is a monomorphism of simplicial sets). It follows that the homotopy inverse limits of these towers can be identified with the ordinary inverse limits, so we get a homotopy equivalence

$$\mathrm{Smooth}(K) \simeq \varprojlim \mathrm{Smooth}(K_i) \simeq \varprojlim BO(m)^{K_i} \times_{BPL(m)^{K_i}} * \simeq BO(m)^K \times_{BPL(m)^K} *.$$

This completes the proof of Theorem 1.

We can informally summarize Theorem 1 by saying that smooth structures on a PL manifold M can be identified with liftings of the canonical map $\chi : M \to BPL(m)$ to a map $\widetilde{\chi} : M \to BO(m)$. More precisely, we get a bijection of the set of homotopy classes of such liftings with the set $\pi_0 \,\mathrm{Smooth}(M)$. It is therefore of interest to describe the latter set more explicitly. In other words, we ask the following question: given two smooth structures s_0 and s_1 (compatible with the given PL structure) on M, when do they belong to the same connected component of $\mathrm{Smooth}(M)$? This is true if and only if s_0 and s_1 can be joined by an edge in $\mathrm{Smooth}(M)$. In other words, if and only if there exists a PD homeomorphism $M \times [0,1] \to N$ (compatible with the projection to $[0,1]$), where $p : N \to [0,1]$ is a fiber bundle of smooth manifolds. In this case, we can identify N with the trivial fiber bundle $N_0 \times [0,1]$, where $N_0 = p^{-1}\{0\}$ is the smooth manifold determined by the smoothing s_0. We can summarize the situation as follows:

Claim 4. *Let M be a PL manifold equipped with a Whitehead compatible smooth structure s_0. Then another smooth structure s_1 is equivalent to s_0 (in other words, it belongs to the same connected component of* $\mathrm{Smooth}(M)$*) if and only if there exists a PD isotopy $h : M \times [0,1] \to M$, where $h_0 = \mathrm{id}_M$ and s_1 is the smooth structure obtained by pulling back s_0 along the homeomorphism h_1.*

Variant 5. *Suppose that M is a PL manifold, K a closed subset, and the smooth structures s_0 and s_1 coincide in a neighborhood of K. Then s_0 and s_1 belong to the same connected component of the fiber* $\mathrm{Smooth}(M) \times_{\mathrm{Smooth}(K)} *$ *if and only if there exists a PD isotopy h_t as above, which is constant in a neighborhood of K. This can be proven by essentially the same argument, together with the smooth version of the isotopy extension theorem.*