The Kister-Mazur Theorem (Lecture 14)

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Our first goal in this lecture is to finish off the proof of the Kister-Mazur theorem, which guarantees that the theory of microbundles is equivalent to the theory of $\mathbb{R}^n$-bundles. We will work in the PL setting (where the result is due to Kuiper and Lashof). More precisely, we will prove the following:

Theorem 1. *Let X be a polyhedron and let $E \to X$ be a PL microbundle. Then there exists an open subset $U \subseteq E$ (containing the zero section) such that the projection $U \to X$ is a PL fiber bundle, with fiber $\mathbb{R}^n$.*

Remark 2. This theorem can be refined in various ways: for example, the fiber bundle U Is unique up to isomorphism. This can be proven using essentially the same arguments and is left as an exercise.

We first need the following result:

Lemma 3. *Let $E \to S^k$ be a PL fiber bundle with fiber $\mathbb{R}^n$ over the k-sphere. Suppose that E is trivial as a microbundle. Then E is trivial as a fiber bundle.*

Proof. We can decompose S^k into hemispheres H_+ and H_-. These are contractible, so we can choose trivializations $E \times_{S^k} H_+ \simeq \mathbb{R}^n \times H_+$ and $E \times_{S^k} H_- \simeq \mathbb{R}^n \times H^-$. These trivializations determine a family of homeomorphism $\{f_v : \mathbb{R}^n \to \mathbb{R}^n\}_{v \in S^{k-1}}$ by restricting to the equator S^{k-1} (in other words, a single PL homeomorphism $f : \mathbb{R}^n \times S^{k-1} \to \mathbb{R}^n \times S^{k-1}$ which commutes with the projectino to S^{k-1}). To prove that E is trivial, we must show that the family $\{f_v\}$ is isotopic to a constant family.

By assumption, E is trivial, so there exists an equivalence of microbundles $E \simeq \mathbb{R}^n \times S^k$. This gives an isomorphism of an open subset U of E with an open subset V of $\mathbb{R}^n \times S^k$. Shrinking U and V, we can assume that V has the form $B(\epsilon) \times S^k$, where $B(\epsilon)$ denotes the open box $(-\epsilon, \epsilon)^n$. Identifying $B(\epsilon)$ with $\mathbb{R}^n$, we get an open embedding $\mathbb{R}^n \times S^k \hookrightarrow E$. Over H_+, this gives us a family of open embeddings $\{g_v^+ : \mathbb{R}^n \to \mathbb{R}^n\}$. Over H_-, we get another family of embeddings $\{g_v^- : \mathbb{R}^n \to \mathbb{R}^n\}$. Along the equator, we have $g_v^+ = f_v \circ g_v^-$.

Since the families of embeddings g_v^- and g_v^+ are defined on the contractible sets H^+ and H^-, they are isotopic to constant families. Since every PL open embedding $\mathbb{R}^n \to \mathbb{R}^n$ is isotopic to a PL homeomorphism, we can take the constant values to be homeomorphisms g^+ and g^-. It follows that f_v is isotopic (through open embeddings) to $g^+ \circ (g^-)^{-1}$. Applying again the main result of last time, we conclude that f_v is isotopic through homeomorphisms to $g^+ \circ (g^-)^{-1}$, so that E is constant as desired. □

We now turn to the proof of Theorem 1. For every closed subpolyhedron $X_0 \in X$, let us say that an open subset $U_0 \subseteq E$ is *good near* X_0 if there exists an open neighborhood $V \subseteq X$ of X_0 such that U_0 is an $\mathbb{R}^n$-bundle over V. Fix a triangulation of X and write X as the union of an increasing sequence of compact subpolyhedra

$$\emptyset = X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$$

such that each X_i is obtained from X_{i-1} by adjoining a simplex whose boundary already belongs to X_{i-1}. We will prove that there exists a collection of open subsets $U_0, U_1, \dots \subseteq E$ with the following properties:

(a) The open set U_i is good near X_i.

(b) The open set U_{i+1} coincides with U_i over a neighborhood of X_i.

We will then obtain a proof of Theorem 1 by setting $U = \bigcup(U_i \times_X X_i)$.

We start the induction by setting $U_0 = \emptyset$. Assume that U_i has been defined, and let X_{i+1} be obtained from X_i by adjoining a single k-simplex σ. Let U_i be an $\mathbb{R}^n$ bundle over the neighborhood V of X_i. In particular, U_i determines an $\mathbb{R}^n$ bundle over $\partial\,\sigma$. This $\mathbb{R}^n$-bundle extends to a microbundle over the contractible space σ, and is therefore trivial as a microbundle. By Lemma 3, it is also trivial as an $\mathbb{R}^n$ bundle. It follows that there exists a compact neighborhood Z of $\partial\,\sigma$ contained in V, such that $U_i \times_X Z \to Z$ can be identified with the trivial bundle $Z \times \mathbb{R}^n$.

Let W be a contractible neighborhood of σ (for example, the star of σ), so that the microbundle E is trivial over E. As in the proof of Lemma 3, this means we can choose an open embedding $j : \mathbb{R}^n \times W \hookrightarrow E$. Choose $\epsilon > 0$ such that j carries $B(\epsilon) \times \partial\,\sigma$ into U_i. Shrinking ϵ and Z if necessary, we can assume that $j(B(\epsilon) \times Z) \subseteq U_i$. We can therefore think of j as providing a family of open embeddings $\{j_z : B(\epsilon) \to \mathbb{R}^n\}_{z \in Z}$. The main result of last time shows that there exists a family of isotopies $\{h_{z,t} : B(\epsilon) \to \mathbb{R}^n\}_{z \in Z}$ where $h_{z,0} = j_z$ and each $h_{z,1}$ is a PL homeomorphism.

Choose open subsets $V_0 \subseteq V$, $W_0 \subseteq W$ with the following properties:

(1) The union $V_0 \cup W_0$ contains X_{i+1}.

(2) The intersection $V_0 \cap W_0$ is contained in Z.

(3) The set W_0 is disjoint from X_i.

Choose a map $\chi : V_0 \cup W_0 \to [0,1]$ such that $\chi = 1$ on a neighborhood of $(V_0 \cup W_0) - W_0$ and $\chi = 0$ on a neighborhood of $(V_0 \cup W_0) - V_0$. We now define U_{i+1} to be the open subset of E whose fiber over a point $x \in V_0 \cup W_0$ is defined as follows:

(a) If $\chi(x) = 1$, then the fiber of U_{i+1} over x is the fiber of U_i over x.

(b) If $\chi(x) = 0$, then the fiber of U_{i+1} over x is the image of j_x.

(c) If $x \in V_0 \cap W_0 \subseteq Z$, then the fiber of U_{i+1} over x is the image of $h_{x,\chi(x)}$.

It is not difficult to verify that this is a fiber bundle over $V_0 \cup W_0$ with the desired properties.

Remark 4. Using more elaborate reasoning of the same kind, we can show that the classifying space $BPL(n)$ for PL fiber bundles with fiber $\mathbb{R}^n$ is homotopy equivalent to the classifying space for PL microbundles constructed in Lecture 12. For this reason, the latter classifying space is typically denoted by $BPL(n)$. Analogous remarks apply in the smooth and topological setting. In the smooth case, microbundles are essentially the same as vector bundles, and the relevant classifying space is denoted by $BO(n)$.

Remark 5. Let E be a PL microbundle over a simplex Δ^n. Let us say that a *smoothing* of E is a smoothing of an open subset $U \subseteq E$ containing the zero section, so that the projection $U \to \Delta^n$ is submersive. We regard two smoothings as identical if they agree on a neighborhood of the zero section of E. Let $X_\bullet$ be the simplicial set whose n-simplices are pairs (σ, S), where σ is an n-simplex of $BPL(n)$ and S is a smoothing of the associated microbundle $E \to \Delta^n$. There is an evident forgetful map $f : X_\bullet \to BPL(n)$.

We also have a canonical vector bundle ζ over the simplicial set $X_\bullet$: it assigns to each simplex (σ, S) the vector bundle $\zeta_\sigma \to \Delta^n$ obtained by taking the vertical tangent space to E along the zero section (the tangent space is defined using the smoothing S). This vector bundle is classified by a map $X_\bullet \to BO(n)$. We will see in the next lecture that ζ is universal: that is, the classifying map χ is a homotopy equivalence. We can therefore identify $X_\bullet$ itself with a classifying space $BO(n)$ for vector bundles of rank n, and f with a map $BO(n) \to BPL(n)$. Informally, we think of this as coming from a group homomorphism $O(n) \to PL(n)$. (In fact, we do have an evident morphism from $O(n)$ to $PL(n)$ as discrete groups: every orthogonal transformation of $\mathbb{R}^n$ is in particular a piecewise linear homeomorphism.) The fiber of f is often denoted $PL(n)/O(n)$; it can be thought of as the space of all smoothings of the PL manifold $\mathbb{R}^n$.

The main result of the last lecture has another consequence:

Proposition 6. *Let $f : D^n \to \mathbb{R}^n$ be a tame embedding (in other words, an embedding such that $f(S^{n-1})$ admits a bicollar in $\mathbb{R}^n$). Then there is a homeomorphism of $\mathbb{R}^n$ to itself that carries $f(D^n)$ to the standard disk D^n.*

Proof. Let us identify D^n with the closure of the open box $B(1)$. Choosing an "outer collar" of $f(S^{n-1})$, we obtain an open embedding $f_0 : B(1+\epsilon) \to \mathbb{R}^n$. The main result of the last lecture shows that f_0 is isotopic to a homeomorphism f_1, via an isotopy fixed on $B(1+\frac{\epsilon}{2})$. Then f_1^{-1} carries $f_0(D^n) = f_1(D^n)$ to the standard disk. $\square$