

# Classifying Spaces for Microbundles (Lecture 12)

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In this lecture, we will discuss construct a classifying space for microbundles of rank  $n$ . For simplicity, we will restrict our attention to piecewise linear microbundles (since this will be the principal case of interest later). Up to this point, we have only defined the notion of a microbundle on a polyhedron  $K$ . In discussing microbundles, it is convenient to have a slightly more general definition.

**Definition 1.** Let  $X_\bullet$  be a simplicial set. A *PL microbundle* (of rank  $n$ ) on  $X_\bullet$  consists of the following data:

- (1) For every  $n$ -simplex  $\sigma \in X_n$ , a PL microbundle  $E_\sigma \rightarrow \Delta^n$ .
- (2) For every nondecreasing map of linearly ordered sets  $f : \{0, \dots, m\} \rightarrow \{0, \dots, n\}$  inducing a map  $f^* : X_n \rightarrow X_m$  and every  $n$ -simplex  $\sigma$  in  $X_n$ , a PL isomorphism (not merely an equivalence) of microbundles  $E_\sigma \times_{\Delta^n} \Delta^m \simeq E_{f^*\sigma}$ .
- (3) Given a pair of nondecreasing maps

$$\{0, \dots, k\} \xrightarrow{g} \{0, \dots, m\} \xrightarrow{f} \{0, \dots, n\}$$

and an  $n$ -simplex  $\sigma \in \Delta^n$ , the associated diagram

$$\begin{array}{ccc} E_\sigma \times_{\Delta^n} \Delta^k & \xrightarrow{\quad} & E_{f^*\sigma} \times_{\Delta^m} \Delta^k \\ & \searrow & \swarrow \\ & E_{g^*f^*\sigma} & \end{array}$$

commutes.

There is a similar notion of an *equivalence* of microbundles on  $X$ : two microbundles on  $X$  are equivalent if they contain open submicrobundles with are isomorphic.

**Remark 2.** Let  $f : X_\bullet \rightarrow Y_\bullet$  be a map of simplicial sets. If  $E$  is a PL microbundle on  $Y_\bullet$ , then we obtain a PL microbundle  $f^*E$  on  $X_\bullet$ , defined by the formula  $(f^*E)_\sigma = E_{f(\sigma)}$ .

**Remark 3.** Let  $X_\bullet$  be a simplicial set with only finitely many nondegenerate simplices. Then the geometric realization  $|X_\bullet|$  has the structure of a finite polyhedron. Unwinding the definition, we see that giving a PL microbundle  $E$  on  $|X_\bullet|$  is equivalent to giving a PL microbundle on the simplicial set  $X_\bullet$ . Consequently, we can regard Definition 1 as a generalization of our earlier theory of PL microbundles (or at least a generalization of the theory of microbundles over finite polyhedra).

The main result of the previous lecture can be generalized to the present context: that is, every PL microbundle on  $X_\bullet \times \Delta^1$  is equivalent to the pullback of a microbundle on  $X_\bullet$ .

**Notation 4.** Let  $X_\bullet$  be a simplicial set. We let  $M(X_\bullet)$  denote the set of equivalence classes of microbundles on  $X_\bullet$ .

The main result of this lecture is the following:

**Theorem 5.** *The functor  $X_\bullet \rightarrow M(X_\bullet)$  is a representable functor on the homotopy category of simplicial sets. In other words, there exists a Kan complex  $K_\bullet$  and a PL microbundle  $\overline{E}$  on  $K_\bullet$  with the following universal property: for every simplicial set  $X_\bullet$ , the construction*

$$(f : X_\bullet \rightarrow K_\bullet) \mapsto f^*\overline{E}$$

*determines a bijection  $\theta : [X_\bullet, K_\bullet] \rightarrow M(X_\bullet)$ .*

We first prove Theorem 5 by means of a specific construction.

**Construction 6.** *For each  $n \geq 0$ , let  $K_n$  denote the set of subpolyhedra  $E \subseteq \Delta^n \times \mathbb{R}^\infty$  equipped with a map  $s : \Delta^n \rightarrow E$  such that the pair  $(E \rightarrow \Delta^n, s)$  is a PL microbundle over  $\Delta^n$ . (recall that a subpolyhedron of  $\Delta^n \times \mathbb{R}^\infty$  means a subpolyhedron of  $\Delta^n \times V$ , for some finite dimensional subspace  $V \subseteq \mathbb{R}^\infty$ ).*

Our first step is to show that  $K_\bullet$  is a Kan complex. In other words, we must show that every map  $f : \Lambda_i^n \rightarrow K_\bullet$  can be extended to an  $n$ -simplex of  $K_\bullet$ . The map  $f$  classifies a PL microbundle  $E \subseteq \Lambda_i^n \times \mathbb{R}^\infty$  over  $\Lambda_i^n$  (here we abuse notation by identifying a simplicial set with its geometric realization). We note that there is a PL retraction  $r$  from  $\Delta^n$  onto  $\Lambda_i^n$ . Then  $r^*E$  is a PL microbundle over  $\Delta^n$  equipped with an embedding  $r^*E \hookrightarrow \Delta^n \times \mathbb{R}^\infty$ .

By construction, the simplicial set  $K_\bullet$  comes equipped with a tautological microbundle  $\overline{E}$ . This microbundle  $\overline{E}$  gives a natural transformation of functors  $\theta : [X_\bullet, K_\bullet] \rightarrow M(X_\bullet)$ . Our next step is to show that  $\theta$  is surjective for every simplicial set  $X_\bullet$ . In other words, we claim that every microbundle  $E$  on  $X_\bullet$  is equivalent to  $f^*\overline{E}$  for some map  $X_\bullet \rightarrow K_\bullet$ . We will prove something slightly stronger: every microbundle  $E$  on  $X_\bullet$  is *isomorphic* to  $f^*\overline{E}$  for some map  $f : X_\bullet \rightarrow K_\bullet$ . To prove this, we construct  $f$  one simplex at a time. At each stage, we are given a microbundle  $E_\sigma$  over the  $n$ -simplex  $\Delta^n$ , and a PL embedding

$$i : E_\sigma \times_{\Delta^n} \partial \Delta^n \hookrightarrow \partial \Delta^n \times \mathbb{R}^\infty$$

(compatible with the projection to  $\mathbb{R}^\infty$ ). To extend  $f$  over the simplex  $\sigma$ , we need to extend  $i$  to a PL embedding  $E_\sigma \rightarrow \Delta^n \times \mathbb{R}^\infty$ . The existence of this extension follows from general position argument.

We now prove the injectivity of  $\theta$ . Suppose we are given two maps  $f, f' : X_\bullet \rightarrow K_\bullet$  and equivalence of microbundles  $f^*\overline{E} \simeq f'^*\overline{E}$ . Then there exists a microbundle  $U$  on  $X$  and open embeddings  $U \hookrightarrow f^*\overline{E}$  and  $U \hookrightarrow f'^*\overline{E}$ . We can then construct a microbundle  $E$  on  $X_\bullet \times \Delta^1$  as a pushout

$$(f^*E \times [0, \frac{1}{2})) \coprod_{U \times [0, \frac{1}{2}]} (U \times [0, 1]) \coprod_{U \times (\frac{1}{2}, 1]} (f'^*E \times (\frac{1}{2}, 1])$$

We now construct a homotopy  $h$  from  $f$  to  $f'$  such that  $h^*\overline{E}$  is isomorphic to the microbundle  $E$  on  $X_\bullet \times \Delta^1$ . The construction again proceeds one simplex  $\sigma$  at a time: at each stage, we are given a PL microbundle  $E_\sigma$  over  $\Delta^n \times \Delta^1$  and an embedding

$$i : E_\sigma \times_{\Delta^n \times \Delta^1} \partial(\Delta^n \times \Delta^1) \hookrightarrow \partial(\Delta^n \times \Delta^1) \times \mathbb{R}^\infty,$$

and we wish to extend  $i$  to an embedding  $E_\sigma \rightarrow (\Delta^n \times \Delta^1) \times \mathbb{R}^\infty$ . The existence of the desired extension again follows from general position arguments. This completes the proof of Theorem 5.

The simplicial set  $K_\bullet$  appearing in Theorem 5 is well-defined only up to homotopy equivalence. For some purposes it may be convenient to work with other models for the classifying space. It is therefore useful to have a criterion for determining whether or not a microbundle  $\overline{E}$  on a simplicial set  $K_\bullet$  satisfies the conclusions of Theorem 5.

**Definition 7.** We will say that a microbundle  $E$  on a Kan complex  $X_\bullet$  is *universal* if it satisfies the conclusions of Theorem 5.

Any microbundle  $E$  is classified by a map  $f : X_\bullet \rightarrow K_\bullet$ ; we note that  $E$  is universal if and only if  $f$  is a homotopy equivalence. We can therefore adopt the following more general definition, which makes sense even when  $X_\bullet$  is not a Kan complex: a microbundle  $E$  on  $X_\bullet$  is *universal* if it is classified by a weak homotopy equivalence  $f : X_\bullet \rightarrow K_\bullet$ .

**Remark 8.** The simplicial set  $K_\bullet$  is often denoted  $BPL(n)$ , for reasons which will become clear after the next lecture.

**Remark 9.** We can also consider classifying spaces for smooth or topological microbundles over simplicial sets. These admit classifying spaces  $BTop(n)$  and  $BSm(n)$ . Since the theory of smooth microbundles is equivalent to the theory of vector bundles, we can take  $BSm(n)$  to be a classifying space  $BO(n)$  for the orthogonal group  $O(n)$ .

**Remark 10.** We can also consider what might be called “PD microbundles”: that is, we can define a PD microbundle on a simplex  $\Delta^n$  to consist of a smooth microbundle  $E$  over  $\Delta^n$ , a PL microbundle  $E'$  over  $\Delta^n$ , and a PD homeomorphism  $E' \rightarrow E$  compatible with the projection to  $\Delta^n$ , and a PD microbundle on  $X_\bullet$  to be a compatible collection of PD microbundles over all simplices of  $X_\bullet$ . The above methods can be used to construct a classifying space for PD microbundles  $BPD(n)$ , equipped with forgetful maps

$$BO(n) \leftarrow BPD(n) \rightarrow BPL(n).$$

Our existence and uniqueness results for Whitehead compatible triangulations show that the left map is a homotopy equivalence (this is slightly easier than our results for manifolds, since we do not have guarantee that any maps are fiber bundles). This construction therefore yields a well-defined homotopy class of maps  $BO(n) \rightarrow BPL(n)$ .