Smoothing PL Fiber Bundles (Lecture 10)

February 25, 2009

Recall our assertion:

Theorem 1. *Suppose given a commutative diagram*

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "q"] & M \arrow[d, "p"] \\ L \arrow[r] & N \end{tikzcd}$$

where K and L are polyhedra, M and N are smooth manifolds, and the horizontal maps are PD homeomorphisms. Assume that p is a submersion of smooth manifolds (so that q is a submersion of PL manifolds). Then p is a smooth fiber bundle if and only if q is a PL fiber bundle.

In the last two lectures, we proved the "only if" direction. However, almost exactly the same argument can be used to prove the converse. The only step that really changes is the step in which we were forced to actually construct an isotopy. Consequently, Theorem 1 is a consequence of the following:

Lemma 2. *Let M be a compact PL manifold, and suppose that $M \times \mathbb{R}$ is equipped with a compatible smooth structure. Then, for every pair of integers $a \leq b$, there exists a smooth isotopy h_t of $M \times \mathbb{R}$ supported on a compact subset of $M \times (a-1, b+1)$ such that h_1 $M \times (-\infty, b]$ into $M \times (-\infty, a)$.*

Choose PL homeomorphism of $(a-1, b+1)$ with $\mathbb{R}$ which carries a to 0 and b to 1. Then we are reduced to proving the following:

Lemma 3. *Let M be a compact PL manifold and suppose that $M \times \mathbb{R}$ is equipped with a compatible smooth structure. Then there exists a compactly supported smooth isotopy h_t of $M \times \mathbb{R}$ such that h_1 carries $M \times (-\infty, 1]$ into $M \times (-\infty, 0)$.*

To prove Lemma 3, let us consider the following condition on a pair of closed subpolyhedra $K \subseteq L \subseteq M \times \mathbb{R}$:

($P_{K,L}$) For every open neighborhood U of K, there exists a compactly supported smooth isotopy h_t of $M \times \mathbb{R}$ such that $h_1(L) \subseteq U$.

Since isotopies can be concatenated, it is easy to see that conditions $P_{K,K'}$ and $P_{K',K''}$ imply $P_{K,K''}$. Moreover, Lemma 3 will follow if we can prove $P_{M\times(-\infty,-1],M\times(-\infty,1]}$. For this, we need to recall a bit of terminology from the theory of PL topology.

Definition 4. Let L be a polyhedron equipped with a triangulation. We say that a subpolyhedron $K \subseteq L$ is an *elementary collapse* of L if there exists a simplex σ of L with a face $\sigma_0 \subset \sigma$ having the following properties:

(i) The simplex σ is not contained as a face of any other simplex of L.

(*ii*) The simplex σ_0 is not contained as a face of any other simplex of L other than σ.

(*iii*) The polyhedron K is obtained from L by removing the interiors of σ and σ_0.

We say that K is a *collapse* of L if it can be obtained from L by a sequence of elementary collapses.

It turns out that the property that $K \subseteq L$ is a collapse does not depend strongly on the choice of a triangulation of L. More precisely, if K is a collapse of L with respect to one triangulation S of L, then K is also a collapse with respect to any sufficiently fine refinement of S. Consequently, we can define the notion of K being a collapse of L without mentioning a particular triangulation: it means that K is a collapse of L with respect to *some* triangulation of L.

The following assertions are not difficult to verify:

- The polyhedron $(-\infty, -1]$ is a collapse (in fact an elementary collapse) of $(-\infty, 1]$.
- If A is a collapse of B, then $M \times A$ is a collapse of $M \times B$, for any polyhedron M.

Combining these observations, we conclude that $M \times (-\infty, -1]$ is a collapse of $M \times (-\infty, 1]$. It follows that there exists a triangulation S of $M \times \mathbb{R}$ which contains $M \times (-\infty, 1]$ and $M \times (-\infty, -1]$ as subcomplexes such that each simplex of S is smoothly embedded in $M \times \mathbb{R}$, and such that $M \times (-\infty, -1]$ can be obtained from $M \times (-\infty, 1]$ by a finite sequence of elementary collapses. It will therefore suffice to prove the following:

Lemma 5. *Suppose that $K \subseteq L \subseteq M \times \mathbb{R}$, where K is obtained from L by an elementary collapse with respect to a simplex σ and a face σ_0 which are smoothly embedded in $M \times \mathbb{R}$. Then for every open set U containing K, there exists a smooth compactly supported isotopy h_t such that $h_1(L) \subseteq U$.*

The construction of this isotopy is now a local matter: we can choose it to be supported in a small tubular neighborhood of the smoothly embedded simplex σ (which is diffeomorphic to an open ball and therefore well-understood. We leave the details to the reader.

We have seen that the "only if" direction of Theorem 1 is a crucial step toward our understanding of the classification of PL structures on a given smooth manifold. Similarly, the "if" direction of Theorem 1 plays a vital role in understanding smooth structures on a given PL manifold. It is to this topic that we now turn.

The main result that we are heading toward is that the problem of smoothing a PL manifold is governed by an *h-principle*: that is, it can be reduced to a problem of homotopy theory. Roughly speaking, we would like to say that there is a space of smooth structures on M, which can be described as the space of sections of a fibration $E \to M$ such that the fiber E_x over a point $x \in M$ describes smooth structures on M near the point x.

To make this more precise, we would like to have a good understanding of a small neighborhoods of x in M, and how they depend on the choice of x. In the case where M is smooth, the theory of vector bundles provides such an understanding. Namely, there exists a vector bundle T_M on M (the tangent bundle) whose fiber at a point $x \in M$ is diffeomorphic to a small neighborhood of x in M. This diffeomorphism can be chosen canonically, for example, if a Riemannian metric on M has been specified. We would like to have a replacement for the theory of vector bundles in the piecewise linear setting. Milnor's theory of *microbundles* provides such a replacement.

Definition 6 (Milnor)**.** Let X be a topological space. An *topological microbundle* on X (of rank n) is a map $p : E \to X$ equipped with a section $s : X \to E$ satisfying the following condition:

($*$) For every point $x \in X$, there exists a neighborhood of $U \subseteq X$ containing x and an open subset of E homeomorphic to $U \times \mathbb{R}^n$, such that the section s can be identified with the zero section $U \simeq U \times \{0\} \hookrightarrow U \times \mathbb{R}^n$.

An *equivalence* of microbundles E and E' over X is a homeomorphism $h : U \simeq U'$ fitting into a commutative diagram

$$\begin{tikzcd} U \arrow[rr, "h"] \arrow[dr] & & U' \arrow[dl] \\ & X, & \end{tikzcd}$$

where U is an open subset of E containing the image of the section $s : X \to E$, U' is an open subset of E' containing the image of $s' : X \to E'$, and the map $h \circ s = s'$.

Remark 7. There are similar definitions in the smooth and PL categories. For example, in the PL case we modify Definition 6 by requiring E and X to be polyhedra and all of the relevant maps to be piecewise linear. In the smooth case, we require E and X to be smooth manifolds and all of the homeomorphisms to be diffeomorphisms.

Example 8. Let M be a topological (PL, smooth) manifold. The *tangent microbundle* T_M is defined to be the product $M \times M$, mapping to M via the projection $\pi_1 : M \times M \to M$, with section $s : M \to M \times M$ given by the diagonal map.

Example 9. Let ζ be a (smooth) vector bundle over a (smooth) manifold M. Then the map $\zeta \to M$ is a (smooth) microbundle.

In the smooth case, the converse is true as well. Namely, suppose that $p : E \to M$ is a smooth microbundle. Replacing E by a small open neighborhood of $s(M)$, we can assume that p is a submersion of smooth manifolds, so that p has a relative tangent bundle $T_{E/M}$. The pullback $s^* T_{E/M}$ is then a smooth vector bundle over M, which can itself be regarded as a microbundle over M. In fact, this microbundle is equivalent to E: choosing a Riemannian metric on E allows us to define an "exponential spray" which identifies an open subset of $s^* T_{E/M}$ with an open subset of E containing $s(M)$.

This construction shows that the theory of microbundles is equivalent to the theory of vector bundles in the setting of smooth manifolds.

We will take up the theory of microbundles again in the next lecture.