Lecture 8: Grothendieck Topologies

December 31, 2021

In the last lecture, we introduced the notion of a *pretopos*. Recall that a pretopos is a coherent category $\mathcal{C}$ with a few additional features: the category $\mathcal{C}$ is required to admit finite coproducts (which are required to be disjoint), and every equivalence relation in $\mathcal{C}$ is required to be effective.

Exercise 1. Let $\mathcal{C}$ and $\mathcal{D}$ be pretopoi, and let $\lambda : \mathcal{C} \to \mathcal{D}$ be a functor which preserves finite limits and effective epimorphisms. Show that the following conditions are equivalent:

(1) The functor λ is a morphism of coherent categories: that is, for every object $X \in \mathcal{C}$, the induced map $\mathrm{Sub}(X) \to \mathrm{Sub}(\lambda(X))$ is a homomorphism of upper semilattices.

(2) The functor λ preserves finite coproducts.

If these conditions are satisfied, we will say that λ is a *morphism of pretopoi*.

Definition 2. Let $\mathcal{C}$ be a pretopos. A *model of* $\mathcal{C}$ is a morphism of pretopoi $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$. We let $\mathrm{Mod}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ spanned by the models of $\mathcal{C}$.

Definition 3. Let T be a (typed) first-order theory. We say that T *eliminates imaginaries* if the weak syntactic category $\mathrm{Syn}_0(T)$ is a pretopos.

Every first-order theory T can be replaced by an essentially equivalent first-order theory which eliminates imaginaries. Suppose, for example, that $X = [\varphi(x)]$ is a formula in the language of T, having a single free variable x of type t, and that $R \subseteq X \times X$ is an equivalence relation defined by some formula $\psi(x, x')$. In this case, we can enlarge the language of T by adding a new type s (to be interpreted as "X modulo the equivalence relation R") and a new binary predicate P of type (t, s) (to be interpreted as the graph of the projection map $X \to X/R$), and adding the axioms

$$(\forall x, y)[P(x, y) \Rightarrow \varphi(x)]$$

$$(\forall x)[\varphi(x) \Rightarrow (\exists! y) P(x, y)]$$

$$(\forall x, x', y, y')[(P(x, y) \wedge P(x', y')) \Rightarrow (y = y' \Leftrightarrow \psi(x, x'))]$$

$$(\forall y)(\exists x)[P(x, y)].$$

Elaborating on this construction, one can produce a typed first-order theory T^{eq}, having the same models as T, for which T^{eq} eliminates imaginaries. This is a special case of a more general construction:

Theorem 4. *Let $\mathcal{C}$ be a small coherent category. Then there exists a small pretopos $\mathcal{C}^{\mathrm{eq}}$ and a morphism of coherent categories $\lambda : \mathcal{C} \to \mathcal{C}^{\mathrm{eq}}$ with the following universal property: if $\mathcal{D}$ is any pretopos, then composition with λ induces an equivalence of categories $\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}^{\mathrm{eq}}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{D})$. Here $\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the morphisms of coherent categories, and $\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}^{\mathrm{eq}}, \mathcal{D})$ is defined similarly.*

In particular, composition with λ induces an equivalence of categories $\mathrm{Mod}(\mathcal{C}^{\mathrm{eq}}) \to \mathrm{Mod}(\mathcal{C})$.

In the situation of Theorem 4, the pretopos $\mathcal{C}^{\mathrm{eq}}$ is determined (up to canonical equivalence) by the coherent category $\mathcal{C}$; we will refer to $\mathcal{C}^{\mathrm{eq}}$ as the *pretopos completion* of $\mathcal{C}$.

Definition 5. Let T be a (typed) first order theory. We let $\mathrm{Syn}(T)$ denote the pretopos completion of $\mathrm{Syn}_0(T)$. We will refer to $\mathrm{Syn}(T)$ as the *syntactic category* of T.

One of our goals over the next several lectures is construct the pretopos completion $\mathcal{C}^{\mathrm{eq}}$. To get a feeling for what we need to do, let us first suppose that we are given a triple of objects $X, Y, Z \in \mathcal{C}$. It could be that the objects Y and Z do not have a coproduct in $\mathcal{C}$ (they also might admit a coproduct in $\mathcal{C}$ which fails to be disjoint, in which case they will have a different coproduct in $\mathcal{C}^{\mathrm{eq}}$). We would like $\mathcal{C}^{\mathrm{eq}}$ to be an enlargement of the category $\mathcal{C}$ in which the coproduct $Y \amalg Z$ exists and is disjoint. Let us abuse notation by identifying objects of $\mathcal{C}$ with their image in $\mathcal{C}^{\mathrm{eq}}$ (our proof of Theorem 4 will show that the functor $\lambda : \mathcal{C} \to \mathcal{C}^{\mathrm{eq}}$ is fully faithful, so this abuse is mostly harmless). So, in order to make sense of the category $\mathcal{C}^{\mathrm{eq}}$, we must in particular define the set of maps $\mathrm{Hom}_{\mathcal{C}^{\mathrm{eq}}}(X, Y \amalg Z)$. Note that we have some obvious candidates for elements of this set: since Y and Z should be subobjects of $Y \amalg Z$, we should have monomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \hookrightarrow \mathrm{Hom}_{\mathcal{C}^{\mathrm{eq}}}(X, Y \amalg Z) \hookleftarrow \mathrm{Hom}_{\mathcal{C}}(X, Z).$$

However, it is not reasonable to expect to find *all* maps from X to $Y \amalg Z$ in this way. For example, if X can be written as the join of disjoint subobjects $X_0, X_1 \subseteq X$, then it is a disjoint coproduct of X_0 and X_1 in both $\mathcal{C}$ and $\mathcal{C}^{\mathrm{eq}}$. Consequently, any pair of maps $f_0 : X_0 \to Y$ and $f_1 : X_1 \to Z$ in $\mathcal{C}$ can be uniquely amalgamated to obtain a map $f : X \simeq X_0 \amalg X_1 \to Y \amalg Z$ in the pretopos $\mathcal{C}^{\mathrm{eq}}$. Roughly speaking, we would like to arrange that all morphisms in $\mathcal{C}^{\mathrm{eq}}$ are defined by a procedure similar to this: that is, they can be built by "gluing together" morphisms that are already present in the coherent category $\mathcal{C}$. To make this precise, it will be convenient to construct $\mathcal{C}^{\mathrm{eq}}$ as a subcategory of a much larger category of *sheaves* on the coherent category $\mathcal{C}$.

Definition 6. Let $\mathcal{C}$ be a category which admits finite limits. A *Grothendieck topology on* $\mathcal{C}$ is a specification of a collection of families of maps $\{f_i : U_i \to X\}_{i \in I}$, which we refer to as *coverings*, satisfying the following axioms:

- $(T1)$ If $\{f_i : U_i \to X\}$ is a covering and $g : Y \to X$ is any morphism in $\mathcal{C}$, then the collection of projection maps $\{U_i \times_X Y \to Y\}_{i \in I}$ is a covering.
- $(T2)$ Let $\{f_i : U_i \to X\}_{i \in I}$ is a covering and let $\{g_j : V_j \to X\}$ be a collection of maps. If, for each i, the projection maps $\{V_j \times_X U_i \to U_i\}$ form a covering, then $\{g_j : V_j \to X\}$ is a covering.
- $(T3)$ If $\{f_i : U_i \to X\}$ is a collection of morphisms such that some f_i admits a section (that is, a morphism $s : X \to U_i$ satisfying $f_i \circ s = \mathrm{id}_X$), then $\{f_i : U_i \to X\}$ is a covering.

Remark 7. One can also consider Grothendieck topologies on categories which do not admit finite limits. In this case, requirement $(T1)$ must be rephrased. In this course, we will not need to consider Grothendieck topologies on such categories.

Exercise 8. Let $\mathcal{C}$ be a category which admits fiber products which is equipped with a Grothendieck topology, and suppose that $\{U_i \to X\}$ is a covering. Show that any larger collection of morphisms (with the same target X) is also a covering.

Definition 9. A *Grothendieck site* is a category $\mathcal{C}$ together with a Grothendieck topology on $\mathcal{C}$.

Example 10. Let X be a topological space and let $\mathcal{U}$ be the collection of all open subsets of X, regarded as a partially ordered set with respect to inclusions. Then, when regarded as a category, the poset $\mathcal{U}$ carries a Grothendieck topology, where a collection of maps $\{U_i \to U\}_{i \in I}$ is a covering if $\bigcup_{i \in I} U_i = U$.

The original motivation for Definition 6 came from algebraic geometry:

Example 11. Let X be a scheme and let $\mathcal{C}$ be the category of schemes U equipped with an étale map $U \to X$. Then $\mathcal{C}$ can be equipped with a Grothendieck topology, where a collection of maps $\{U_i \to U\}$ is a covering if the induced map $\amalg_i U_i \to U$ is surjective. When endowed with this topology, $\mathcal{C}$ is referred to as the *small étale site* of the scheme X.

For the moment, we will be primarily interested in Grothendieck sites which arise from coherent categories:

Proposition 12. *Let $\mathcal{C}$ be a coherent category. Then $\mathcal{C}$ admits a Grothendieck topology which can be described as follows: a collection of morphisms $\{f_i : U_i \to X\}_{i \in I}$ is a covering if there exists some finite subset $I_0 \subseteq I$ such that $\bigvee_{i \in I_0} \mathrm{Im}(f_i) = X$ (in the lattice $\mathrm{Sub}(X)$).*

Proof. Since the formation of images and joins of subobjects is compatible with pullback, it is clear that this notion of covering satisfies axiom $(T1)$.

Note that a collection of maps $\{f_i : U_i \to X\}_{i \in I}$ is a covering if, for some finite subset $I_0 \subseteq I$, there is no proper subobject $X' \subsetneq X$ such that f_i factors through X', for each $i \in I_0$. From this description, it is clear that $(T3)$ is satisfied.

To verify $(T2)$, suppose that we are given a covering $\{U_i \to X\}_{i \in I}$ and a collection of maps $\{V_j \to X\}_{j \in J}$ with the property that, for each $i \in I$, the collection $\{V_j \times_X U_i \to U_i\}_{j \in J}$ is a covering. We wish to show that $\{V_j \to X\}_{j \in J}$ is a covering. Without loss of generality, we may assume that I is finite and J are finite. Suppose that there exists a subobject $X' \subseteq X$ such that each of the maps $V_j \to X$ factors through X'. Then each $V_j \to_X U_i \to U_i$ factors through $X' \times_X U_i$. It follows that $X' \times_X U_i = U_i$ (as subobjects of U_i). In other words, the maps $U_i \to X$ factor through X'. Using our assumption that $\{U_i \to X\}$ is a covering, we conclude that $X' = X$. $\square$

Definition 13. Let $\mathcal{C}$ be a category. A *presheaf* (of sets) on $\mathcal{C}$ is a functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$.

Suppose that $\mathcal{C}$ admits fiber products and is equipped with a Grothendieck topology. We will say that a presheaf $\mathscr{F}$ is a *sheaf* if, for every covering $\{U_i \to X\}_{i \in I}$, the diagram of sets

$$\mathscr{F}(X) \longrightarrow \prod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathscr{F}(U_i \times_X U_j)$$

is an equalizer: that is, we can $\mathscr{F}(X)$ with the set of tuples $\{s_i \in \mathscr{F}(U_i)\}$ having the property that, for every pair $i, j \in I$, the elements s_i and s_j have the same image in $\mathscr{F}(U_i \times_X U_j)$. We let $\mathrm{Shv}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ spanned by the sheaves on $\mathcal{C}$.

Example 14. Let X be a topological space and let $\mathcal{U}$ be the partially ordered set of open subsets of X. A *sheaf on* X is defined to be a sheaf on the category $\mathcal{U}$, where we equip $\mathcal{U}$ with the Grothendieck topology of Example 10. We will denote the category of sheaves on X by $\mathrm{Shv}(X)$.

Example 15 (Representable Sheaves)**.** Let $\mathcal{C}$ be any category. For each object $Y \in \mathcal{C}$, let h_Y denote the functor represented by Y, given by the formula $h_Y(X) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$. We say that a Grothendieck topology on $\mathcal{C}$ is *subcanonical* if h_Y is a sheaf, for each $Y \in \mathcal{C}$. If this condition is satisfied, then the construction $Y \mapsto h_Y$ determines a fully faithful embedding $\mathcal{C} \hookrightarrow \mathrm{Shv}(\mathcal{C})$.

Proposition 16. *Let $\mathcal{C}$ be a coherent category. Then the Grothendieck topology of Proposition 12 is subcanonical. In particular, we can view $\mathcal{C}$ as a full subcategory of* $\mathrm{Shv}(\mathcal{C})$.

Proof. Fix an object $Y \in \mathcal{C}$; we wish to show that the functor h_Y is a sheaf. To prove this, suppose we are given a covering $\{U_i \to X\}$. We wish to show that the diagram of sets

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(U_i, Y) \rightrightarrows \prod_{i,j \in I} \mathrm{Hom}_{\mathcal{C}}(U_i \times_X U_j, Y)$$

is an equalizer. In other words, we wish to show every collection of morphisms $f_i : U_i \to Y$ which is compatible, in the sense that f_i and f_j define the same morphism from $U_i \times_X U_j$ into Y, define an essentially unique map $f : X \to Y$. Without loss of generality, we can assume that I is finite (exercise: justify this). For each $i \in I$, let X_i denote the image of the map $U_i \to X$. Then we have an effective epimorphism $U_i \to X_i$, and the map coequalizes the projection maps

$$\pi, \pi' : U_i \times_{X_i} U_i \simeq U_i \times_X U_i \to U_i.$$

It follows that each f_i factors uniquely as a composition $U_i \to X_i \xrightarrow{\overline{f}_i} Y$. Moreover, for $i \neq j$, the maps $\overline{f}_i$ and $\overline{f}_j$ have the same restriction to $X_i \wedge X_j$ (since this can be checked after composition with the effective epimorphism

$$U_i \times_X U_j \to X_i \times_X X_j = X_i \wedge X_j.$$

It will therefore suffice to show that we can (uniquely) amalgamate the maps $\overline{f}_i$ to a single map $f : X \to Y$.

We now proceed as in the previous lecture. Each $\overline{f}_i$ has a graph $\Gamma(\overline{f}_i)$, which we can view as a subobject of $X \times Y$. Set $Z = \bigwedge \Gamma(\overline{f}_i)$. We will complete the proof by showing that Z is the graph of a morphism from X to Y. To prove this, it will suffice to show that the composition

$$Z \hookrightarrow X \times Y \to X$$

is a monomorphism (it will then necessarily be an isomorphism, since its image contains $\bigvee X_i$ which is equal to X by virtue of our assumption that $\{U_i \to X\}$ is covering). In other words, it will suffice to show that the diagonal map $Z \hookrightarrow Z \times_X Z$ is an isomorphism. Note that, as subobjects of $X \times Y \times Y$, Z can be identified with the join of the graphs of the maps $(\overline{f}_i, \overline{f}_i) : X_i \to Y \times Y$, while $Z \times_X Z$ can be identified with the join of the graphs of the morphisms

$$(\overline{f}_i|_{X_i \wedge X_j}, \overline{f}_j|_{X_i \wedge X_j}) : X_i \wedge X_j \to Y \times Y.$$

We complete the proof by observing that we have inclusions

$$\Gamma(\overline{f}_i|_{X_i \wedge X_j}, \overline{f}_j|_{X_i \wedge X_j}) \subseteq \Gamma(\overline{f}_i, \overline{f}_i),$$

since $\overline{f}_i$ and $\overline{f}_j$ have the same restriction to $X_i \wedge X_j$. $\square$

We can now describe our approach to the proof of Theorem 4. Given a coherent category $\mathcal{C}$, we can view it as a subcategory of the sheaf category $\mathrm{Shv}(\mathcal{C})$. We will realize the pretopos completion $\mathcal{C}^{\mathrm{eq}}$ as a slightly larger subcategory of $\mathrm{Shv}(\mathcal{C})$. In order to show that this works, we will need to study the structural features of the category $\mathrm{Shv}(\mathcal{C})$, and other categories like it.

Definition 17. Let $\mathcal{X}$ be a category. We say that $\mathcal{X}$ is a *topos* if there is an equivalence of categories $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{C})$, where $\mathcal{C}$ is a small Grothendieck site which admits finite limits.

Example 18. For every topological space X, the category $\mathrm{Shv}(X)$ is a topos.

Example 19. Let T be a first-order theory and let $\mathrm{Syn}_0(T)$ be the weak syntactic category of T, equipped with the Grothendieck topology of Proposition 12. Then $\mathrm{Shv}(\mathrm{Syn}_0(T))$ is a topos. We will refer to $\mathrm{Shv}(\mathrm{Syn}_0(T))$ as the *classifying topos of T*.

A useful heuristic is that a topos is a kind of generalized topological space; the classifying topos of a first-order theory T can be viewed as a kind of "space of all models of T."

Warning 20. In the situation of Definition 17, the category $\mathcal{C}$ is not uniquely determined by $\mathcal{X}$. In general, many different Grothendieck sites can give rise to the same topos. For example, we will see that if $\mathcal{C}$ is a small coherent category and $\mathcal{C}^{\mathrm{eq}}$ is its pretopos completion, then the topoi $\mathrm{Shv}(\mathcal{C})$ and $\mathrm{Shv}(\mathcal{C}^{\mathrm{eq}})$ are equivalent.

Remark 21. In the situation of Definition 17, the requirement that $\mathcal{C}$ admits finite limits is not necessary: we impose it only because we have not defined the notion of Grothendieck topology in full generality. However, it is a harmless assumption (if a category has the form $\mathrm{Shv}(\mathcal{C})$ for an arbitrary small Grothendieck site $\mathcal{C}$, then it also has the form $\mathrm{Shv}(\mathcal{C}')$ where $\mathcal{C}'$ is a small Grothendieck site which admits finite limits).