Lecture 6: Completeness Theorems

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We begin with the following classical statement:

Theorem 1 (Gödel's Completeness Theorem, Version 0)**.** *Let T be a first-order theory. If T is consistent, then there exists a model of T.*

To precisely formulate Theorem 1, we need a definition for what it means for a theory to be consistent, which we have not yet given. However, we can precisely formulate an analogous statement in the setting of categorical logic.

Definition 2. Let $\mathcal{C}$ be a coherent category with final object $\mathbf{1}$. We will say that $\mathcal{C}$ is *inconsistent* if $\mathrm{Sub}(\mathbf{1})$ has only a single element. Otherwise, we will say that $\mathcal{C}$ is *consistent*.

Exercise 3. Show that if a coherent category $\mathcal{C}$ is inconsistent, then $\mathcal{C}$ is equivalent to the category $\{*\}$ having only a single object and a single (identity) morphism.

Theorem 4 (Deligne's Completeness Theorem, Version 0)**.** *Let $\mathcal{C}$ be a small coherent category. If $\mathcal{C}$ is consistent, then there exists a model of $\mathcal{C}$.*

For our purposes, it will be useful to consider some variations on Theorems 1 and 4.

Theorem 5 (Gödel's Completeness Theorem, Version 1)**.** *Let T be a first order theory and let φ be a sentence in the language of T. If φ is true in every model of T, then φ is provable from the axioms of T.*

Equivalently, if φ is not *provable from T, then we can find a model of T in which φ is false.*

Remark 6. Theorem 1 follows immediately from Theorem 5 by taking φ to be a contradictory sentence, like $(\exists x)[\neg(x = x)]$. Conversely, we can deduce Theorem 5 by applying Theorem 1 to the theory $T \cup \{\neg\varphi\}$.

To translate Theorem 5 into the setting of categorical logic, we replace the notion of *sentence* in a first-order language by the notion of a *subobject of* $\mathbf{1}$ in a coherent category $\mathcal{C}$.

Theorem 7 (Deligne's Completeness Theorem, Version 1)**.** *Let $\mathcal{C}$ be a small coherent category and let $U \subseteq \mathbf{1}$ be a subobject of the final object. If $U \neq \mathbf{1}$, then there exists a model $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$ such that $M[U]$ is empty.*

Remark 8. Theorem 7 immediately implies Theorem 4 by taking U to be the least element of $\mathrm{Sub}(\mathbf{1})$ (which is carried to the empty set by any model of $\mathcal{C}$). In the non-Boolean case, it is a little bit harder (though still possible) to go in the other direction. We'll return to this point later.

We will prove Theorem 7 much later in this course.

Here is another equivalent form of Gödel's completeness theorem, which we mentioned in the first lecture:

Theorem 9 (Gödel's Completeness Theorem, Version 2)**.** *Let T be a first order theory and let $\varphi(\vec{x})$, $\psi(\vec{x})$ be two formulas of the language of T, using the same free variables. If $M[\psi(\vec{x})] \subseteq M[\varphi(\vec{x})]$ for every model M of T, then $T \vdash (\forall \vec{x})[\psi(\vec{x}) \Rightarrow \varphi(\vec{x})]$*

Equivalently, if the implication $\psi(\vec{x}) \Rightarrow \varphi(\vec{x})$ is not *provable from T, then we can find a model M such that $M[\psi(\vec{x})] \nsubseteq M[\varphi(\vec{x})]$.*

To prove Theorem 5 for a sentence φ, we can simply apply Theorem 9 (in the case of no free variables) by choosing ψ to be a tautological sentence, like $(\forall x)[x = x]$. Conversely, Theorem 9 can be deduced by applying Theorem 5 to a theory T' obtained by "adding new constants" $c_1, c_2, \ldots, c_n$, and adding an axiom demanding that they satisfy the formula ψ. We now translate this idea into categorical logic.

Theorem 10 (Deligne's Completeness Theorem, Version 2)**.** *Let $\mathcal{C}$ be a small coherent category, let $X \in \mathcal{C}$ be an object, and let $U \subseteq X$ be a subobject. If $U \neq X$, then there exists a model $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$ such that $M[U] \neq M[X]$.*

Theorem 7 is just a special case of Theorem 10, where we take X to be the final object. But we can go in the reverse direction as well:

Proof of Theorem 10 from Theorem 7. We saw in the previous lecture that $\mathcal{C}_{/X}$ is also a coherent category. Moreover, we can regard X as the final object of $\mathcal{C}_{/X}$, and U as a subobject of the final object of $\mathcal{C}_{/X}$. Consequently, if $U \neq X$, then we can apply Theorem 7 to construct a model $N : \mathcal{C}_{/X} \to \mathcal{S}\mathrm{et}$ such that $N[U] = \emptyset$. Define $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$ by the formula $M[Y] = N[X \times Y]$. From an exercise of the previous lecture, the functor

$$\mathcal{C} \to \mathcal{C}_{/X} \qquad Y \mapsto X \times Y$$

is a morphism of coherent categories, so that M is a model of $\mathcal{C}$. We will complete the proof by showing that the inclusion $M[U] \hookrightarrow M[X]$ is not bijective. Equivalently, we wish to show that $N[X \times U] \hookrightarrow N[X \times X]$ is not bijective (where we regard $X \times U$ and $X \times X$ as objects of $\mathcal{C}_{/X}$ via projection onto the first factor). Suppose otherwise. Then the pullback diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ X \times U \arrow[r] & X \times X \end{tikzcd}$$

in $\mathcal{C}_{/X}$ would yield a pullback diagram of sets

$$\begin{tikzcd} N[U] \arrow[r] \arrow[d] & N[X] \arrow[d] \\ N[X \times U] \arrow[r] & N[X \times X] \end{tikzcd}$$

where the bottom horizontal map is bijective, so the upper horizontal map would be bijective as well. This contradicts our choice of N (since $N[U] = \emptyset$ and $N[X]$ is a singleton). $\square$

Exercise 11. Let $\mathcal{C}$ be a coherent category and let $X \in \mathcal{C}$ be an object. Show that the following data are equivalent:

- Models of the coherent category $\mathcal{C}_{/X}$.
- Models M of the coherent category $\mathcal{C}$, together with a chosen element of the set $M[X]$.

In the case where $\mathcal{C} = \mathrm{Syn}_0(T)$ is the weak syntactic category of a first-order theory T and $X = [\varphi(x_1, \ldots, x_n)]$, this supports the idea that $\mathcal{C}_{/X}$ corresponds to a theory obtained by "adding constants" $c_1, c_2, \ldots, c_n$ that are required to satisfy the formula $\varphi(c_1, \ldots, c_n)$.

In the statement of Theorem 10, we do not need to assume that U is a subobject of X:

Theorem 12 (Deligne's Completeness Theorem, Version 3)**.** *Let $\mathcal{C}$ be a small coherent category, and let $f : U \to X$ be a morphism in $\mathcal{C}$. If f is not an isomorphism, then there exists a model $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$ such that the induced map $M[U] \to M[X]$ is not bijective.*

Proof. Assume that, for every model M, the induced map $f_M : M[U] \to M[X]$ is bijective. Factor f as a composition $U \xrightarrow{g} \mathrm{Im}(f) \xrightarrow{h} X$, where g is an effective epimorphism and h is a monomorphism. In every model M, the induced factorization $M[U] \xrightarrow{g_M} M[\mathrm{Im}(f)] \xrightarrow{h_M} M[X]$ exhibits $M[\mathrm{Im}(f)]$ as the image of f_M. Since f_M is bijective, it follows that g_M and h_M are bijective. Applying Theorem 10 to the morphism h, we deduce that h is an isomorphism.

Let $R = U \times_{\mathrm{Im}(f)} U$ be the equivalence relation determined by g, which we regard as a subobject of $U \times U$. Let us also regard U as a subobject of $U \times U$ via the diagonal inclusion, so that U is a subobject of R. For every model M, the assumption that g_M is bijective guarantees that $M[U] = M[R]$ (as subsets of $M[U \times U]$. Applying Theorem 10, we conclude that $U = R$ as subobjects of $U \times U$. Since g is an effective epimorphism, we conclude $\mathrm{Im}(f)$ can be identified with the coequalizer of the projection maps $\pi_0, \pi_1 : U \times_{\mathrm{Im}(f)} U \to U$, which is equivalent to the coequalizer of the identity maps $\mathrm{id}_U, \mathrm{id}_U : U \to U$. It follows that g is also an isomorphism, so that $f = h \circ g$ is an isomorphism. □

Corollary 13. *Let $\lambda : \mathcal{C} \to \mathcal{D}$ be a morphism of small coherent categories. Assume that f satisfies the following conditions:*

(1) *As a functor, λ is essentially surjective: that is, every object of $\mathcal{D}$ is isomorphic to $\lambda(X)$, for some $X \in \mathcal{C}$.*

(2) *For every object $X \in \mathcal{C}$, the functor λ induces a surjection* $\mathrm{Sub}(X) \to \mathrm{Sub}(\lambda(X))$.

(3) *Every model of $\mathcal{C}$ can be extended to a model of $\mathcal{D}$.*

Then λ is an equivalence of categories.

Proof. By virtue of (1), it will suffice to show that the functor λ is fully faithful. We first show that it is faithful. Let $f, g : X \to Y$ be two morphisms in $\mathcal{C}$ having the same source and target, and suppose that $\lambda(f) = \lambda(g)$. We wish to show that $f = g$. Let $E \subseteq X$ be the equalizer of f and g; we wish to show that $E = X$. If not, we can apply Theorem 10 to choose a model $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$ such that $M[E] \neq M[X]$. Using (3), we can assume that $M = N \circ \lambda$, where $N : \mathcal{D} \to \mathcal{S}\mathrm{et}$ is a model of $\mathcal{D}$. Then $N[\lambda(E)] \neq N[\lambda(X)]$, so that $\lambda(E) \neq \lambda(X)$ (as a subobject of $\lambda(X)$). Since λ preserves finite limits, we can identify $\lambda(E)$ with the equalizer of $\lambda(f)$ and $\lambda(g)$. It follows that $\lambda(f) \neq \lambda(g)$, contradicting our assumption.

We now show that the functor λ is full. Let X and Y be objects of $\mathcal{C}$, and suppose we are given some morphism $\overline{f} : \lambda(X) \to \lambda(Y)$. We wish to show that $\overline{f} = \lambda(f)$, for some morphism $f : X \to Y$ in $\mathcal{C}$. Let $\Gamma(\overline{f})$ denote the graph of $\overline{f}$, regarded as a subobject of $\lambda(X) \times \lambda(Y) \simeq \lambda(X \times Y)$. Using assumption (2), we can assume that $\Gamma(\overline{f}) = \lambda(U)$ for some subobject $U \subseteq X \times Y$. To complete the proof, it will suffice to show that U is the graph of a morphism from X to Y: that is, that the composite map $U \hookrightarrow X \times Y \xrightarrow{\pi_X} X$ is an isomorphism in $\mathcal{C}$. Using Theorem 12, we are reduced to proving that the map $M[U] \to M[X]$ is bijective for every model M of $\mathcal{C}$. Using assumption (3), we can assume that $M = N \circ \lambda$, where $N : \mathcal{D} \to \mathcal{S}\mathrm{et}$ is a model of $\mathcal{D}$. In this case, the result is clear, because the map $\lambda(U) \to \lambda(X)$ is an isomorphism (it is the projection from the graph $\Gamma(\overline{f}) \subseteq \lambda(X) \times \lambda(Y)$ onto the first factor). □

We can now prove the Theorem from the previous lecture:

Theorem 14. *Let $\mathcal{C}$ be a small Boolean coherent category. Then the map $\lambda : \mathcal{C} \to \mathrm{Syn}_0(T(\mathcal{C}))$ of the previous lecture is an equivalence of categories.*

Proof. The functor λ satisfies conditions (1) and (3) of Corollary 13 by construction. We saw in the previous lecture that it also satisfies condition (2) when $\mathcal{C}$ is Boolean. □

We close this lecture by describing another application of Corollary 13. Let T be a first-order theory. Suppose that we wanted to deduce Gödel's completeness theorem (in any of the three incarnations above) from Deligne's completeness theorem (in the corresponding incarnation). In this case, we cannot really proceed by applying Deligne's theorem to the weak syntactic category $\mathrm{Syn}_0(T)$ as we have defined it, because our definition already has Gödel's theorem "built in." Recall that we can identify morphisms from $X = [\varphi(\vec{x})]$ to $Y = [\psi(\vec{y})]$ with equivalence classes of formulae

$$T \vDash (\forall \vec{x}, \vec{y})[\theta(\vec{x}, \vec{y}) \Rightarrow \varphi(\vec{x}) \wedge \psi(\vec{y})] \wedge (\forall \vec{x})[\varphi(\vec{x}) \Rightarrow (\exists ! \vec{y})[\theta(\vec{x}, \vec{y})]],$$

where $\theta(\vec{x}, \vec{y})$ is equivalent to $\theta'(\vec{x}, \vec{y})$ if

$$T \vDash (\forall \vec{x}, \vec{y})[\theta(\vec{x}, \vec{y}) \Leftrightarrow \theta'(\vec{x}, \vec{y})].$$

This definition was not really syntactic in nature: it refers to the truth of formulae, rather than to their provability. What we should really do is to define morphisms from X to Y to be equivalence classes of formulae $\theta(\vec{x}, \vec{y})$ satisfying

$$T \vdash (\forall \vec{x}, \vec{y})[\theta(\vec{x}, \vec{y}) \Rightarrow \varphi(\vec{x}) \wedge \psi(\vec{y})] \wedge (\forall \vec{x})[\varphi(\vec{x}) \Rightarrow (\exists ! \vec{y})[\theta(\vec{x}, \vec{y})]],$$

modulo the equivalence relation given by

$$T \vdash (\forall \vec{x}, \vec{y})[\theta(\vec{x}, \vec{y}) \Leftrightarrow \theta'(\vec{x}, \vec{y})].$$

Let us denote the category obtained from this procedure by $\mathrm{Syn}'_0(T)$. Of course, this definition depends *a priori* on what it means to be provable. However, suppose that we give a definition which is sufficiently robust that we can justify analogues for $\mathrm{Syn}'_0(T)$ for all of the main facts about $\mathrm{Syn}_0(T)$; namely:

(a) The constructions of fiber products, images, and unions of subobjects that we have given in $\mathrm{Syn}_0(T)$ also work in $\mathrm{Syn}'_0(T)$. For example, if $\theta(\vec{x}, \vec{y})$ is a formula defining a morphism from $[\varphi(\vec{x})]$ to $[\psi(\vec{y})]$, then that morphism factors as a composition

$$[\varphi(\vec{x})] \xrightarrow{\theta} [(\exists \vec{x})\theta(\vec{x}, \vec{y'})] \xrightarrow{\vec{y} = \vec{y'}} [\psi(\vec{y})],$$

where the map on the left is an effective epimorphism and the map on the right is a monomorphism.

(b) Every morphism of coherent categories $\mathrm{Syn}'_0(T) \to \mathcal{S}\mathrm{et}$ arises from a model of the theory T.

In this case, (a) will imply that the construction $[\varphi(\vec{x})] \mapsto [\varphi(\vec{x})]$ determines a morphism of coherent categories $\mathrm{Syn}'_0(T) \to \mathrm{Syn}_0(T)$. This functor will satisfy conditions (1) and (2) of Corollary 13, and assumption (b) guarantees that it will also satisfy (3). Applying Corollary 13, we deduce that $\mathrm{Syn}'_0(T)$ and $\mathrm{Syn}_0(T)$ are equivalent. We can also deduce Gödel's completeness theorem by applying Theorem 4 to the coherent category $\mathrm{Syn}'_0(T)$: if T is consistent as a first-order theory, then $\mathrm{Syn}'_0(T)$ will be consistent as a coherent category (in the sense of Definition 2), and therefore has a model $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$; ($b$) allows us to interpret this as a model of the theory T.