Lecture 5: Booleanization

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Recall the following definition from the previous lecture:

Definition 1. Let $\mathcal{C}$ be a category. We will say that $\mathcal{C}$ is a *coherent category* if it satisfies the following axioms:

($A1$) The category $\mathcal{C}$ admits finite limits.

($A2$) Every morphism $f : X \to Z$ in $\mathcal{C}$ admits a factorization $X \xrightarrow{g} Y \xrightarrow{h} Z$, where g is an effective epimorphism and h is a monomorphism.

($A3$) For every object $X \in \mathcal{C}$, the partially ordered set $\mathrm{Sub}(X)$ is an upper semilattice: that is, it has a least element, and every pair of subobjects $X_0, X_1 \subseteq X$ have a least upper bound $X_0 \vee X_1$.

($A4$) The collection of effective epimorphisms in $\mathcal{C}$ is stable under pullback.

($A5$) For every morphism $f : X \to Y$ in $\mathcal{C}$, the map $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ is a homomorphism of upper semilattices.

Definition 2. Let $\mathcal{C}$ and $\mathcal{C}'$ be coherent categories. A *morphism of coherent categories* from $\mathcal{C}$ to $\mathcal{C}'$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ with the following properties:

(1) The functor F is left exact: that is, it preserves finite limits.

(2) The functor F carries effective epimorphisms to effective epimorphisms.

(3) For every object $X \in \mathcal{C}$, the induced map $\mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$ is a homomorphism of upper semilattices: that is, it preserves smallest elements and joins.

Remark 3. Let $F : \mathcal{C} \to \mathcal{C}'$ satisfy condition (1) of Definition 2. Then F preserves monomorphisms. Consequently, condition (2) is equivalent to the requirement that for every morphism $f : X \to Z$, the functor F carries the canonical factorization $X \to \mathrm{Im}(f) \hookrightarrow Z$ to the factorization $F(X) \to \mathrm{Im}(F(f)) \hookrightarrow F(Z)$. Note also that in the situation of (3), the map $\mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$ is automatically a homomorphism of lower semilattices (that is, it preserves largest elements and meets).

Example 4. Let P and P' be distributive lattices. Then, when viewed as categories, P and P' are coherent categories. A morphism of coherent categories from P to P' is just a lattice homomorphism: that is, a map of partially ordered sets that preserves least upper bounds and greatest lower bounds for finite subsets.

Example 5. Let $\mathcal{C}$ be a category containing an object X. We can then form a new category $\mathcal{C}_{/X}$, whose objects are pairs (U, f), where $U \in \mathcal{C}$ is an object and $f : U \to X$ is a morphism. A morphism from (U, f) to (V, g) in $\mathcal{C}$ is a morphism $h : U \to V$ in $\mathcal{C}$ such that $f = g \circ h$. The construction $(U, f) \mapsto U$ determines a forgetful functor $\mathcal{C}_{/X} \to \mathcal{C}$, and we will generally abuse notation by identifying an object of $\mathcal{C}_{/X}$ with its image under this forgetful functor.

Exercise 6. Show that if $\mathcal{C}$ is a coherent category, then so is $\mathcal{C}_{/X}$. Moreover, the formation of fiber products, images, and unions of subobjects in $\mathcal{C}_{/X}$ can be computed in the underlying category $\mathcal{C}$.

Beware that the forgetful functor $\mathcal{C}_{/X} \to \mathcal{C}$ is not a morphism of coherent categories, because it does not preserve final objects. However, the forgetful functor has a right adjoint which *is* a morphism of coherent categories. More generally, suppose that $f : X \to Y$ is any morphism in $\mathcal{C}$. Then f determines a functor $f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$, given by the construction $U \mapsto U \times_X Y$.

Exercise 7. Show that if $f : X \to Y$ is a morphism in a coherent category $\mathcal{C}$, then the functor $f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ is a morphism of coherent categories. In fact, this is precisely the content of axioms $(A4)$ and $(A5)$.

Definition 8. Let $\mathcal{C}$ be a coherent category. A *model* of $\mathcal{C}$ is a morphism of coherent categories $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$. We let $\mathrm{Mod}(\mathcal{C})$ denote the full subcategory of the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ spanned by the models of $\mathcal{C}$; we refer to $\mathrm{Mod}(\mathcal{C})$ as the *category of models of* $\mathcal{C}$.

Example 9. Let $\mathcal{C} = \mathrm{Syn}_0(T)$ be the weak syntactic category of a (typed) first-order theory T. Then we can identify $\mathrm{Mod}(\mathcal{C})$ with the category of models of T (with morphisms given by elementary embeddings): that is the content of the theorem from the previous lecture.

We now return to a question from the previous lecture: given a category $\mathcal{C}$, can we construct a first-order theory T whose weak syntactic category is $\mathcal{C}$?

Construction 10. Let $\mathcal{C}$ be a small coherent category. We define a typed first-order theory $T(\mathcal{C})$ as follows:

- The types of $T(\mathcal{C})$ are the objects of $\mathcal{C}$. We use uppercase letters like X and Y to denote these types, and the corresponding lowercase letters x, y, etcetera to denote variables of those types.
- For every morphism $f : X \to Y$ in $\mathcal{C}$, the language of $T(\mathcal{C})$ has a single predicate P_f, of arity (X, Y).

By definition, a structure for the language $T(\mathcal{C})$ is a rule which associates to each object $X \in \mathcal{C}$ a set $M[X]$, and to each morphism $f : X \to Y$ a relation $M[P_f] \subseteq M[X] \times M[Y]$. We now list the axioms of $T(\mathcal{C})$, along with the constraints they place on a structure M:

- For every $f : X \to Y$, we have an axiom $(\forall x)(\exists! y)[P_f(x, y)]$. (So that $M[P_f]$ is the graph of a function $f_M : M[X] \to M[Y]$.)
- If $i : X \to X$ is the identity morphism, we have an axiom $(\forall x)[P_i(x, x)]$. (So that $i_M : M[X] \to M[X]$ is the identity map.)
- Given a pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$, we have an axiom $(\forall x, y, z)[(P_f(x, y) \wedge P_g(y, z)) \Rightarrow P_{gf}(x, z)]$. (So that $g_M \circ f_M = (g \circ f)_M$.)

These first axioms guarantee that a model M of $T(\mathcal{C})$ can be viewed as a functor from $\mathcal{C}$ to the category of sets. We now add additional axioms to guarantee that this functor has nice properties:

- If $\mathbf{1}$ is a final object of $\mathcal{C}$ and e is a variable of type $\mathbf{1}$, we have an axiom $(\exists! e)[e = e]$ (So that M preserves final objects.)
- For every pullback square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

  in $\mathcal{C}$, we have an axiom $(\forall x, y, y')[(P_f(x, y) \wedge P_g(y', y)) \Rightarrow (\exists! x')[P_{f'}(x', x) \wedge P_{g'}(x', y')]]$ (So that M preserves pullback squares.)
- If $f : X \to Y$ is an effective epimorphism in $\mathcal{C}$, we have an axiom $(\forall y)(\exists x)[P_f(x, y)]$ (so that M carries effective epimorphisms to surjections of sets).

- If X is an object of $\mathcal{C}$ and $f : X_0 \hookrightarrow X$ is a monomorphism which exhibits X_0 as the smallest element of $\mathrm{Sub}(X)$, then we have an axiom $\neg(\exists x_0)[x_0 = x_0]$ (so that M carries the smallest element of $\mathrm{Sub}(X)$ to the empty set).

- If X is an object of $\mathcal{C}$ which is given as the join of subobjects $f : Y \hookrightarrow X$ and $g : Z \hookrightarrow X$, then we have an axiom $(\forall x)[(\exists y)[P_f(y, x)] \vee (\exists z)[P_g(z, x)]]$ (so that M carries joins in $\mathrm{Sub}(W)$ to unions of subsets of $M[W]$).

The theory $T(\mathcal{C})$ has the property that models of $T(\mathcal{C})$ are the same as models of $\mathcal{C}$. Let us now make that idea more precise:

Construction 11. We define a functor $\lambda : \mathcal{C} \to \mathrm{Syn}_0(T(\mathcal{C}))$ as follows:

- For each object $X \in \mathcal{C}$, we set $\lambda(X) = [x = x]$, where x is some variable of the type X.

- For every morphism $f : X \to Y$ in $\mathcal{C}$, we let $\lambda(f) : \lambda(X) \to \lambda(Y)$ denote the morphism in $\mathcal{C}$ defined by the formula $P_f(x, y)$.

By construction, a model M of $T(\mathcal{C})$ can be viewed as a morphism of coherent categories $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$, so that $M[P_f(x, y)]$ is the graph of a function f_M from $M[X]$ to $M[Y]$. Moreover, since we have $(g \circ f)_M = g_M \circ f_M$ for each M, it follows that $\lambda(g \circ f) = \lambda(g) \circ \lambda(f)$. Similarly, $\lambda(\mathrm{id}_X) = \mathrm{id}_{\lambda(X)}$, so that λ is a functor from $\mathcal{C}$ to $\mathrm{Syn}_0(T(\mathcal{C}))$. Moreover, this functor preserves finite limits, effective epimorphisms, and joins of subobjects (since these properties can be tested in every model of $T(\mathcal{C})$). In other words, λ is a morphism of coherent categories.

Note that the identification
$$\{\text{Models of } T(\mathcal{C})\} \simeq \{ \text{ Models of } \mathcal{C} \text{ }\}$$
is simply given by composition with the functor F of Construction 11. This composition determines a functor
$$\mathrm{Mod}(T(\mathcal{C})) \simeq \mathrm{Mod}(\mathrm{Syn}_0(T(\mathcal{C}))) \to \mathrm{Mod}(\mathcal{C}).$$
By construction, the composite functor is *bijective* on objects. Beware that it is not necessarily an equivalence of categories. Our next goal is to discuss the following:

Theorem 12. *Let $\mathcal{C}$ be a small coherent category. Then the functor $\lambda : \mathcal{C} \to \mathrm{Syn}_0(T(\mathcal{C}))$ of Construction 11 is an equivalence of categories if and only if $\mathcal{C}$ is Boolean.*

Remark 13. The "only if" direction is obvious, since the weak syntactic category $\mathrm{Syn}_0(T(\mathcal{C}))$ is Boolean.

Remark 14. In the situation of Theorem 12, we can think of the weak syntactic category $\mathrm{Syn}_0(T(\mathcal{C}))$ as a "Booleanization" of $\mathrm{Syn}_0(\mathcal{C})$. If $f : \mathcal{C} \to \mathcal{D}$ is any morphism of coherent categories, then f can be completed to a diagram
$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \downarrow{\scriptstyle \lambda_{\mathcal{C}}} & & \downarrow{\scriptstyle \lambda_{\mathcal{D}}} \\ \mathrm{Syn}_0(T(\mathcal{C})) & \xrightarrow{\mathrm{Syn}_0(T(f))} & \mathrm{Syn}_0(T(\mathcal{D})). \end{array}$$
which commutes up to canonical isomorphism. If $\mathcal{D}$ is Boolean, then Theorem 12 guarantees that $\lambda_{\mathcal{D}}$ is an equivalence, so that f is isomorphic to the composition $\mathcal{C} \xrightarrow{\lambda_{\mathcal{C}}} \mathrm{Syn}_0(T(\mathcal{C}))\,\mathrm{Syn}_0(T(\mathcal{C})) \xrightarrow{g} \mathcal{D}$ where g is the composition $\lambda_{\mathcal{D}}^{-1} \circ \mathrm{Syn}_0(T(f))$. It is possible to show that this factorization is essentially unique (but this requires additional input).

Example 15 (The Theory of Groups)**.** We will see later that there is a natural way to choose a coherent category $\mathcal{C}$ for which the category $\mathrm{Mod}(\mathcal{C})$ is equivalent to the category whose objects are groups and whose morphisms are group homomorphisms. In this case, $T(\mathcal{C})$ would be equivalent to the category whose objects are groups and whose morphisms are elementary embeddings of groups. Here we could replace groups by other mathematical structures of a similar flavor (abelian groups, rings, Lie algebras, etcetera).

Example 16. A topological space X is said to be *spectral* if it satisfies the following conditions:

- The quasi-compact open subsets of X form a basis for the topology of X.
- The space X is quasi-compact, and the intersection $U \cap V$ is quasi-compact whenever $U, V \subseteq X$ are quasi-compact open sets.
- Every irreducible closed subset of X has a unique generic point.

For example, the underlying topological space of any quasi-compact and quasi-separated scheme is spectral (and conversely, due to a theorem of Hochster).

Let X be a spectral space. We say that a subset $K \subseteq X$ is *constructible* if it belongs to the Boolean algebra generated by the quasi-compact open subsets of X. We can then equip X with a new topology, called the *constructible topology*, by taking the constructible subsets of X as a basis. Let us denote the resulting topological space by X^c; one can show that it is a *Stone space* (that is, it is compact, Hausdorff, and totally disconnected).

The construction $X \mapsto X^c$ can be regarded as a special case of the Booleanization procedure of Construction 11. If X is a spectral space, then the collection of quasi-compact open subsets of X forms a distributive lattice P, which we can regard as a coherent category. Then the Booleanization $\mathrm{Syn}_0(T(P))$ is a Boolean coherent category in which every object admits a monomorphism to the final object, and is therefore equivalent to a Boolean algebra. This turns out to be the Boolean algebra of constructible subsets of X, or equivalently of quasi-compact open subsets of X^c.

In the situation above, we can identify X with the set of equivalence classes of models of P, and X^c with the set of equivalence classes of models of $\mathrm{Syn}_0(T(P))$. The fact that the topological spaces X and X^c have the same points is an illustration of the general fact that a coherent category $\mathcal{C}$ and its Booleanization $\mathrm{Syn}_0(T(\mathcal{C}))$ have "the same" models. However, the *categories* of models need not be equivalent. In the example of a spectral space X, this corresponds to the observation that in general there can be closure relations between points of X (that is, it is possible for a point $x \in X$ to lie in the closure of a different point $y \in X$), but not in X^c (since X^c is a Hausdorff space).

We now begin the proof of Theorem 12 (we will continue in the next lecture).

Proposition 17. *Let $\mathcal{C}$ be a small Boolean coherent category. Let X be an object of $\mathcal{C}$ which is given as a product $\prod_{1 \leq i \leq n} X_i$, and suppose that $\varphi(x_1, \ldots, x_n)$ is a formula in the language of $T(\mathcal{C})$ whose variables x_i have type X_i. Then there exists a subobject $Y \subseteq X$ such that $\lambda(Y)$ and $[\varphi(x_1, \ldots, x_n)]$ coincide as subobjects of $\lambda(X_1) \times \cdots \times \lambda(X_n) \simeq \lambda(X)$.*

Proof. We proceed by induction on the construction of the formula φ. There are five cases:

- (i) Suppose $\varphi(\vec{x})$ has the form $x_i = x_j$, for some pair i, j with $X_i = X_j$. In this case, we can take Y to be the fiber product $X \times_{X_i \times X_j} X_i$.
- (ii) Suppose that $\varphi(\vec{x})$ has the form $P_f(x_i, x_j)$, where $f : X_i \to X_j$ is a morphism in $\mathcal{C}$. In this case, we take Y to be the fiber product $X \times_{X_i \times X_j} X_i$ (where X_i is embedded in the product $X_i \times X_j$ as the graph of f).
- (iii) Suppose that $\varphi(\vec{x})$ has the form $\varphi_0(\vec{x}) \vee \varphi_1(\vec{x})$. By our inductive hypothesis we can assume that there are subobjects $Y_0, Y_1 \subseteq X$ satisfying $\lambda(Y_0) = [\varphi_0(\vec{x})]$ and $\lambda(Y_1) = [\varphi_1(\vec{x})]$ (as subobjects of $\lambda(X)$). We then take $Y = Y_0 \vee Y_1$.

- (iv) Suppose that $\varphi(\vec{x})$ has the form $\neg\psi(\vec{x})$. By the inductive hypothesis, we can choose a subobject $Y' \subseteq X$ such that $\lambda(Y') = [\psi(\vec{x})]$ as subobjects of $\lambda(X)$. Our assumption that $\mathcal{C}$ is Boolean guarantees that Y' has a complement $Y \in \mathrm{Sub}(X)$. Since F induces a Boolean algebra homomorphism $\mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$, it follows that $\lambda(Y) = [\varphi(\vec{x})]$ (as subobjects of $\lambda(X)$).

- (v) Suppose that $\varphi(\vec{x})$ has the form $(\exists z)[\psi(\vec{x}, z)]$, where z is a variable of type Z. In this case, our inductive hypothesis guarantees that there exists a subobject $\overline{Y} \subseteq X \times Z$ such that $\lambda(\overline{Y}) = [\psi(\vec{x}, z)]$ as subobjects of $\lambda(X \times Z) \simeq \lambda(X) \times \lambda(Z)$. Let Y denote the image of the composite map $\overline{Y} \hookrightarrow X \times Z \to X$. Since F preserves images, it follows that $\lambda(Y)$ is the image of the map $[\psi(\vec{x}, z)] \to F(X)$, which coincides with $[\varphi(\vec{x})]$.

$\square$