Lecture 4: Coherent Categories

January 31, 2018

Let T be a first-order theory and let $\mathrm{Syn}_0(T)$ denote the weak syntactic category of T. In the previous lecture, we proved that $\mathrm{Syn}_0(T)$ has the following properties:

$(A1)$ The category $\mathrm{Syn}_0(T)$ admits finite limits. In particular, it admits fiber products.

$(A2)$ Every morphism $f : X \to Z$ in $\mathrm{Syn}_0(T)$ admits a factorization $X \xrightarrow{g} Y \xrightarrow{h} Z$, where g is an effective epimorphism and h is a monomorphism.

$(A3)$ For every object $X \in \mathrm{Syn}_0(T)$, the partially ordered set $\mathrm{Sub}(X)$ is an upper semilattice: that is, it has a least element, and every pair of subobjects $X_0, X_1 \subseteq X$ have a least upper bound $X_0 \vee X_1$.

Remark 1. In fact, we actually proved the following stronger version of $(A3)$:

$(A3')$ For every object $X \in \mathrm{Syn}_0(T)$, the partially ordered set $\mathrm{Sub}(X)$ is a Boolean algebra.

We begin with a proof of the result promised in Lecture 3:

Theorem 2. *Let T be a first-order theory and let $F : \mathrm{Syn}_0(T) \to \mathcal{S}\mathrm{et}$ be a functor. Then F arises from a model $M \models T$ if and only if it satisfies the following three conditions:*

(1) *The functor F preserves finite limits.*

(2) *The functor F carries effective epimorphisms in $\mathrm{Syn}_0(T)$ to surjections of sets.*

(3) *For every object $X \in \mathrm{Syn}_0(T)$, the induced map $\mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$ is a homomorphism of upper semilattices: that is, it carries the least element of $\mathrm{Sub}(X)$ to the empty set, and carries joins $X_0 \vee X_1$ to unions of subsets of $F(X)$.*

Remark 3. In the statement of Theorem 2, we can replace (c) by the following *a priori* stronger statement:

$(3')$ For every object $X \in \mathrm{Syn}_0(T)$, the induced map $\mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$ is a homomorphism of Boolean algebras.

Note that condition (1) already guarantees that the map $\mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$ is a homomorphism of *lower* semilattices: that is, it carries X to $F(X)$, and carries meets $X_0 \wedge X_1$ to intersections of the corresponding subsets. It follows that it also preserves complements (if U is an element of a Boolean algebra with greatest element $\top$ and least element $\perp$, then the complement U^c is characterized by the identities $U \wedge U^c = \perp$ and $U \vee U^c = \top$, and is therefore preserved by all lattice homomorphisms).

The necessity of conditions (1) and (3) was noted in the previous lecture. The necessity of (2) is a consequence of the following:

Lemma 4. *Let $f : X \to Z$ be a morphism in $\mathrm{Syn}_0(T)$. Then f is an effective epimorphism if and only if, for every model $M \models T$, the induced map $f_M : M[X] \to M[Z]$ is surjective.*

Proof. Let $X \xrightarrow{g} Y \xrightarrow{h} Z$ be the canonical factorization of produced in the previous lecture. Since the factorization of f as an effective epimorphism followed by a monomorphism is unique (up to unique isomorphism), it follows that f is an effective epimorphism if and only if h is an isomorphism. We saw in the previous lecture that this is equivalent to the requirement that $h_M : M[Y] \to M[Z]$ is bijective for each $M \vDash T$, which is equivalent to the surjectivity of $g_M : M[X] \to M[Y]$. □

Proof Sketch of Theorem 2. The necessity of conditions (1), (2), and (3) has now been established; let us show the sufficiency. Let $F : \mathrm{Syn}_0(T) \to \mathcal{S}\mathrm{et}$ be a functor satisfying (1), (2), and (3); we wish to construct a model M of T and a collection of bijections $F(X) \simeq M[X]$ depending functorially on $X \in \mathrm{Syn}_0(T)$.

Fix a variable e and let $E \in \mathrm{Syn}_0(T)$ be the object corresponding to some tautology having e as a free variable (such as the formula $e = e$). For every finite set of variables $V_0 = \{x_1, \ldots, x_n\}$, let E^{V_0} denote the object of $\mathrm{Syn}_0(T)$ given by the formula $(x_1 = x_1) \wedge \cdots \wedge (x_n = x_n)$ (regarded as a formula with free variables in V_0). For every model $N \vDash T$, we have canonical bijections

$$N[E] \simeq N \qquad N[E^{V_0}] \simeq N^{V_0},$$

which exhibit E^{V_0} as a product $\prod_{x \in V_0} E$ in the category $\mathrm{Syn}_0(T)$.

Set $M = F(E)$. Since the functor F preserves products, we have canonical isomorphisms $F(E^{V_0}) \simeq M^{V_0}$. For every formula $\varphi(\vec{x})$ having free variables $V_0 = \{x_1, \ldots, x_n\}$, we have a special monomorphism $[\varphi(\vec{x})] \hookrightarrow E^{V_0}$ in $\mathrm{Syn}_0(T)$. Condition (1) guarantees that F preserves monomorphisms, so that we obtain an injective map of sets

$$\iota_{\varphi(\vec{x})} : F([\varphi(\vec{x})]) \hookrightarrow F(E^{V_0}) \simeq M^{V_0}.$$

Suppose we are given a map between finite sets of variables $V_0 \to V_1$, carrying each x_i to some element $y_i \in V_1$. Assuming that none of the variables in V_1 are bound in φ, we can then consider the formula $\varphi(\vec{y})$ with variables in V_1. In the category $\mathrm{Syn}_0(T)$, we have a commutative diagram

$$\begin{array}{ccc} [\varphi(\vec{y})] & \longrightarrow & E^{V_1} \\ \downarrow & & \downarrow \\ [\varphi(\vec{x})] & \longrightarrow & E^{V_0} \end{array}$$

where the horizontal maps are special monomorphisms. This diagram becomes a pullback square in every model of T, and is therefore a pullback square in $\mathrm{Syn}_0(T)$. Invoking assumption (1), we deduce:

($*$) In the situation above, we have a pullback square of sets

$$\begin{array}{ccc} F([\varphi(\vec{y})]) & \xrightarrow{\iota_{\varphi(\vec{y})}} & M^{V_1} \\ \downarrow & & \downarrow \\ F([\varphi(\vec{x})] & \xrightarrow{\iota_{\varphi(\vec{x})}} & M^{V_0}. \end{array}$$

Let P_i be a predicate of the language of T having arity n_i, so that we can regard $P_i(x_1, \ldots, x_{n_i})$ as a formula with n_i free variables. We regard M as a structure for the language of T by taking $M[P_i]$ to be the subset

$$F([P_i(x_1, \ldots, x_{n_i})]) \subseteq M^{\{x_1, \ldots, x_{n_i}\}} \simeq M^{n_i}.$$

We now prove the following:

($*'$) For every formula $\varphi(x_1, \ldots, x_n)$, the map $\iota_{\varphi(\vec{x})}$ induces a bijection $F([\varphi(\vec{x})]) \simeq M[\varphi(\vec{x})]$.

The proof proceeds by induction on the construction of φ; there are five cases to consider:

- (i) Suppose first that $\varphi(\vec{x})$ is a formula of the form $x = y$. By virtue of $(*)$, we can assume that x and y are the only free variables of φ (and that they are distinct). In this case, we note that $[\varphi]$ is equivalent, as a subobject of $E^{\{x,y\}} \simeq E \times E$, to the subobject given by the diagonal embedding $E \hookrightarrow E \times E$ (since this is true in every model of T). It follows that F carries $[\varphi]$ to the image of the diagonal map $M \simeq F(E) \to F(E) \times F(E) = M \times M$, which is $M[\varphi]$.

- (ii) Suppose that $\varphi(\vec{x}) = P_i(x_{j_1}, \ldots, x_{j_{n_i}})$. In this case, the desired result follows from the definition of $M[P_i]$ (together with $(*)$).

- (iii) Suppose that $\varphi(\vec{x})$ has the form $\varphi_0(\vec{x}) \vee \varphi_1(\vec{x})$. In this case, the desired result follows from our inductive hypothesis together with (3).

- (iv) Suppose that $\varphi(\vec{x})$ has the form $\neg\psi(\vec{x})$. In this case, the desired result follows from our inductive hypothesis together condition ($3'$) of Remark 3.

- (v) Suppose that $\varphi(\vec{x})$ has the form $(\exists y)[\psi(\vec{x}, y)]$. In this case, we have a natural map $f : [\psi(\vec{x}, y)] \to [\varphi(\vec{x})]$ in $\mathrm{Syn}_0(T)$. The realization of f in every model is surjective, so f is an effective epimorphism (Lemma 4). Applying (2), we conclude that the induced map $F([\psi(\vec{x}, y)]) \to F([\varphi(\vec{x})])$ is surjective, so that $\iota_{\varphi(\vec{x})}(F([\varphi(\vec{x})]))$ can be identified with the image of the composite map

$$F([\psi(\vec{x}, y)]) \xrightarrow{\iota_{\psi(\vec{x},y)}} M^n \times M \to M^n.$$

  Applying our inductive hypothesis, we conclude that this is the set

$$\{(c_1, \ldots, c_n) \in M^n : (\exists d \in M)[M \models \psi(c_1, \ldots, c_n, d)]\} = \{(c_1, \ldots, c_n) \in M^n : M \models \varphi(c_1, \ldots, c_n)\}.$$

Applying $(*')$ in the case where φ is an axiom of T, we deduce that

$$(M \models \varphi) \Leftrightarrow (F([\varphi]) \neq \emptyset).$$

However, $[\varphi]$ is a final object of $\mathrm{Syn}_0(T)$ and F preserves final objects by (a), so that $F([\varphi])$ is a singleton and therefore φ is true in M. It follows that M is a model of T, and assertion $(*')$ supplies bijections $F([\varphi(\vec{x})]) \simeq M[\varphi(\vec{x})]$ for each formula $\varphi(\vec{x})$. We leave it to the reader to verify that these bijections are natural in $[\varphi(\vec{x})]$ (as an object of $\mathrm{Syn}_0(T)$). □

To complete the transition from the language of first-order logic to the language of category theory, we need to address the following:

Question 5. Let $\mathcal{C}$ be a small category. Under what conditions does there exist a typed first-order theory T such that $\mathcal{C}$ is equivalent to $\mathrm{Syn}_0(T)$?

We have already noted that $\mathcal{C}$ must satisfy conditions ($A1$) through ($A3$) above. Roughly speaking, we can think of ($A1$) through ($A3$) as describing *operations* that can be performed in the category $\mathcal{C}$: that is, procedures for combining various objects of $\mathcal{C}$ to produce new objects. We now formulate two additional axioms which describe compatibilities among these operations.

Proposition 6. *Let T be a typed first order theory and let $\mathrm{Syn}_0(T)$ be the syntactic category of T. Then weak syntactic category $\mathrm{Syn}_0(T)$ satisfies the following:*

($A4$) *For every pullback diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

in $\mathrm{Syn}_0(T)$, if f is an effective epimorphism, then f' is also an effective epimorphism.

Proof. Let M be a model of T. Then the diagram

$$\begin{array}{ccc} M[X'] & \longrightarrow & M[X] \\ \downarrow{\scriptstyle f'_M} & & \downarrow{\scriptstyle f_M} \\ M[Y'] & \longrightarrow & M[Y] \end{array}$$

is a pullback square of sets. Since f is an effective epimorphism, the map f_M is surjective (Lemma 4). It follows that f'_M is also surjective. Since this is true for every model M of T, Lemma 4 implies that f'_M is an effective epimorphism. □

Note that if $\mathcal{C}$ is any category which admits fiber products, then any morphism $f : X \to Y$ in $\mathcal{C}$ induces a map of posets $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$, given by $f^{-1}(Y_0) = X \times_Y Y_0$.

Proposition 7. *For every typed first-order theory T, the weak syntactic category $\mathrm{Syn}_0(T)$ satisfies the following:*

($A5$) *For every morphism $f : X \to Y$ in $\mathrm{Syn}_0(T)$, the map $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ is a homomorphism of upper semilattices. That is, it preserves smallest elements, and we have $f^{-1}(Y_0 \vee Y_1) = f^{-1}(Y_0) \vee f^{-1}(Y_1)$ for $Y_0, Y_1 \subseteq Y$.*

Proof. Let $f : X \to Y$ be a morphism in $\mathrm{Syn}_0(T)$ and suppose we are given a pair of subobjects $Y_0, Y_1 \subseteq Y$. Then Y_0 and Y_1 are contained in $Y_0 \vee Y_1$, so $f^{-1}(Y_0)$ and $f^{-1}(Y_1)$ are contained in $f^{-1}(Y_0 \vee Y_1)$. It follows that we have $f^{-1}(Y_0) \vee f^{-1}(Y_1) \subseteq f^{-1}(Y_0 \vee Y_1)$ (as subobjects of X), To show that this inclusion is an equality, it will suffice to show that in every model $M \models T$, we have $M[f^{-1}(Y_0) \vee f^{-1}(Y_1)] = M[f^{-1}(Y_0 \vee Y_1)]$ (as subsets of $M[X]$). Since M is compatible with pullbacks and with joins of subobjects, this reduces to the equality $f_M^{-1} M[Y_0] \cup f_M^{-1} M[Y_1] = f_M^{-1}(M[Y_0] \cup M[Y_1])$. □

Definition 8. Let $\mathcal{C}$ be a category. We will say that $\mathcal{C}$ is *coherent* if it satisfies the following axioms:

($A1$) The category $\mathcal{C}$ admits finite limits.

($A2$) Every morphism $f : X \to Z$ in $\mathcal{C}$ admits a factorization $X \xrightarrow{g} Y \xrightarrow{h} Z$, where g is an effective epimorphism and h is a monomorphism.

($A3$) For every object $X \in \mathcal{C}$, the partially ordered set $\mathrm{Sub}(X)$ is an upper semilattice: that is, it has a least element, and every pair of subobjects $X_0, X_1 \subseteq X$ have a least upper bound $X_0 \vee X_1$.

($A4$) The collection of effective epimorphisms in $\mathcal{C}$ is stable under pullback.

($A5$) For every morphism $f : X \to Y$ in $\mathcal{C}$, the map $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ is a homomorphism of upper semilattices.

Example 9. For every typed first-order theory T, the weak syntactic category $\mathrm{Syn}_0(T)$ is a coherent category.

Example 10. The category $\mathcal{S}et$ of sets is a coherent category.

Example 11. Let P be a partially ordered set, considered as a category. Then axioms ($A1$) through ($A5$) can be restated as follows:

($A1$) The partially ordered set P is a lower semilattice: that is, it has a largest element $\top$ and pairwise meets $p \wedge q$.

($A2$) Automatically satisfied; for each $p \leq q$ in P, we associate the factorization $p \leq p \leq q$.

($A3$) The partially ordered set P is also an upper semilattice: that is, it has a smallest element $\perp$ and pairwise joins $p \vee q$.

($A4$) Automatically satisfied, since the effective epimorphisms in P are isomorphisms.

($A5$) P satisfies the distributive law $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$.

A partially ordered set satisfying these conditions is called a *distributive lattice.*

Remark 12. Let $\mathcal{C}$ be any coherent category and let X be an object of $\mathcal{C}$. Then the poset of subobjects $\mathrm{Sub}(X)$ is a distributive lattice.

Remark 13. Let $\mathcal{C}$ be any category which admits fiber products. Then, for every object $X \in \mathcal{C}$, the poset $\mathrm{Sub}(X)$ is automatically a lower semilattice: it has a largest element given by X itself, and every pair of subobjects $X_0, X_1 \subseteq X$ has a greatest lower bound given by the fiber product $X_0 \times_X X_1$. Moreover, for any map $f : X \to Y$ in $\mathcal{C}$, the pullback map $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ is automatically a homomorphism of lower semilattices: that is, it preserves largest elements and intersections.

Remark 14. Let $\mathcal{C}$ be a category satisfying ($A2$). We saw in the previous lecture that if a morphism $f : X \to Z$ admits a factorization $X \xrightarrow{g} Y \xrightarrow{h} Z$, where g is an effective epimorphism and h is a monomorphism, then the factorization is unique (up to unique isomorphism). We will emphasize that uniqueness by writing $Y = \mathrm{Im}(f)$ and referring to it as the *image* of f.

Remark 15. Let $\mathcal{C}$ be a category satisfying ($A1$) and ($A2$). Then ($A4$) is equivalent to the following:

($A4'$) For any pullback diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Z' & \longrightarrow & Z \end{array}$$

in $\mathcal{C}$, we have $\mathrm{Im}(f') = \mathrm{Im}(f) \times_Z Z'$ (as subobjects of Z').

To see that ($A4'$) $\Rightarrow$ ($A4$), consider as a diagram as above and suppose that f is an effective epimorphism. Then $\mathrm{Im}(f) = Z$, so ($A4'$) implies that $\mathrm{Im}(f') = Z'$. It follows that f' is also an effective epimorphism. Conversely, suppose that ($A4$) is satisfied and consider a diagram as above, which we extend to a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ \mathrm{Im}(f) \times_{Z'} Z & \longrightarrow & \mathrm{Im}(f) \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ Z' & \longrightarrow & Z \end{array}$$

where g is an effective epimorphism, h is a monomorphism, and the lower square is a pullback. Since the outer rectangle is also a pullback, the upper square is a pullback. Applying ($A4$), we deduce that g' is an effective epimorphism. Moreover, h' is a pullback of h, and therefore a monomorphism. It follows that $\mathrm{Im}(f) \times_{Z'} Z = \mathrm{Im}(f')$ as subobjects of Z', as desired.

We now return to Question 5.

Definition 16. Let $\mathcal{C}$ be a coherent category. We will say that $\mathcal{C}$ is *Boolean* if it satisfies the following stronger version of ($A3$):

($A3'$) For every object $X \in \mathcal{C}$, the partially ordered set $\mathrm{Sub}(X)$ is a Boolean algebra.

Theorem 17. *Let $\mathcal{C}$ be a small category. The following conditions are equivalent:*

(a) *The category* $\mathcal{C}$ *is a Boolean coherent category.*

(b) *There exists a typed first-order theory* T *and an equivalence of categories* $\mathcal{C} \simeq \mathrm{Syn}_0(T)$.

We will discuss Theorem 17 in the next lecture.

Remark 18. In the situation of Theorem 17, suppose that we want to guarantee that $\mathcal{C}$ is equivalent to $\mathrm{Syn}_0(T)$, where T is an *untyped* first-order theory. In this case, we must add the following axiom to our list:

($A6$) There exists a single object $X \in \mathcal{C}$ with the property that every object of $\mathcal{C}$ can be realized as a subobject of X^n for some $n \gg 0$.