Lecture 30X-Higher Categorical Logic

April 24, 2018

In this final lecture, we sketch (without proof) how some of the ideas of this course can be extended to the setting of higher category theory. We assume that the reader is familiar with the language of ∞-categories (following the terminology of Higher Topos Theory).

Let $\mathcal{C}$ be an ∞-category which admits finite limits. To each morphism $f : X \to Y$ in $\mathcal{C}$, we can associate a simplicial object $X_\bullet$ of $\mathcal{C}$, where X_n is the $(n+1)$-fold fiber power of X over Y. We refer to $X_\bullet$ as the *Čech nerve* of f. It is an example of a *groupoid object* of $\mathcal{C}$. We will say that a morphism $f : X \to Y$ is an *effective epimorphism* if it exhibits Y as a geometric realization of the Čech nerve of f.

Definition 1. Let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ is an ∞*-pretopos* if it satisfies the following axioms:

($A1$) The ∞-category $\mathcal{C}$ admits finite limits.

($A2$) Every groupoid object $X_\bullet$ is *effective*: that is, it arises as the Čech nerve of an effective epimorphism $X_0 \to Y$ in $\mathcal{C}$ (in particular, $X_\bullet$ admits a geometric realization $Y \simeq |X_\bullet|$).

($A3$) The ∞-category $\mathcal{C}$ admits finite coproducts, and coproducts are disjoint (that is, the maps $X \hookrightarrow X \amalg Y \hookleftarrow Y$ are monomorphisms, and the fiber product $X \times_{X \amalg Y} Y$ is initial).

($A4$) The collection of effective epimorphisms in $\mathcal{C}$ is closed under pullbacks.

($A5$) The formation of finite coproducts in $\mathcal{C}$ is preserved by pullback.

Let $n \geq -1$ be an integer. We say that an object X of an ∞-category $\mathcal{C}$ is n*-truncated* if the mapping spaces $\mathrm{Map}_{\mathcal{C}}(Y, X)$ are n-truncated for each object $Y \in \mathcal{C}$: that is, if the homotopy groups $\pi_* \mathrm{Map}_{\mathcal{C}}(Y, X)$ vanish for $* > n$. We say that an object of $\mathcal{C}$ is *discrete* if it is 0-truncated.

Proposition 2. *Let $\mathcal{C}$ be an ∞-pretopos. For every integer n, let $\mathcal{C}_{\leq n}$ be the full subcategory of $\mathcal{C}$ spanned by the n-truncated objects. Then:*

- *The inclusion functor $\mathcal{C}_{\leq n} \hookrightarrow \mathcal{C}$ admits a left adjoint, which we will denote by $X \mapsto \tau_{\leq n} X$.*
- *The category $\mathcal{C}_{\leq 0}$ of discrete objects of $\mathcal{C}$ is a pretopos.*

Definition 3. Let $\mathcal{C}$ and $\mathcal{C}'$ be ∞-pretopoi. A *morphism of ∞-pretopoi* from $\mathcal{C}$ to $\mathcal{C}'$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ which preserves finite limits, finite coproducts, and effective epimorphisms. We let $\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{C}')$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ spanned by those morphisms of ∞-pretopoi.

If $\mathcal{C}$ is an ∞-pretopos, we define a *model* of $\mathcal{C}$ to be a morphism of ∞-pretopoi $\mathcal{C} \to \mathcal{S}$, where $\mathcal{S}$ denotes the ∞-category of spaces. The collection of models of $\mathcal{C}$ forms an ∞-category which we will denote by $\mathrm{Mod}(\mathcal{C})$.

Remark 4. It follows from Proposition 2 that the construction $\mathcal{C} \mapsto \mathcal{C}_{\leq 0}$ determines a forgetful functor

$$\{\infty\text{-pretopoi}\} \to \{\text{Pretopoi}\}.$$

One can show that this functor has a left adjoint, which carries an ordinary pretopos $\mathcal{C}$ to an ∞-pretopos that we will denote by $\mathcal{C}^+$. Roughly speaking, one can think of objects of $\mathcal{C}^+$ as "stacky" objects of $\mathcal{C}$.

Example 5. Let $\mathcal{S}$ be the ∞-category of spaces. Then $\mathcal{S}$ is an ∞-pretopos, whose underlying ordinary pretopos $\mathcal{S}_{\leq 0}$ can be identified with the category $\mathcal{S}\mathrm{et}$ of sets. The ∞-pretopos $\mathcal{S}\mathrm{et}^+$ can be identified with the union $\bigcup_{n \geq 0} \mathcal{S}_{\leq n}$, regarded as a full subcategory of $\mathcal{S}$.

Example 6. Let $\mathcal{S}\mathrm{et}_{\mathrm{fin}}$ be the pretopos of finite sets. Then $\mathcal{S}\mathrm{et}_{\mathrm{fin}}^+$ is the ∞-pretopos of *π-finite spaces*: that is, the full subcategory of $\mathcal{S}$ spanned by those spaces X for which the homotopy groups $\pi_n(X, x)$ are finite for each $n \geq 0$ (and each base point $x \in X$), and which vanish for $n \gg 0$. This is the initial object in the setting of ∞-pretopoi.

Remark 7. Let $\mathcal{C}$ be a pretopos and let $\mathcal{C}^+$ be the associated ∞-pretopos. Then we have a canonical equivalence

$$\mathrm{Mod}(\mathcal{C}^+) = \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}^+, \mathcal{S}) \simeq \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{S}_{\leq 0}) \simeq \mathrm{Mod}(\mathcal{C}).$$

In other words, the ∞-category of models of $\mathcal{C}^+$ can be identified with the ordinary category of models of $\mathcal{C}$. Consequently, the pretopos $\mathcal{C}$ and the ∞-pretopos $\mathcal{C}^+$ can be regarded as incarnations of the same object.

Remark 8. One can think of the relationship between ∞-pretopoi and ordinary pretopoi as analogous to the relationship between ordinary pretopoi and distributive lattices. Every ordinary pretopos $\mathcal{C}$ determines a distributive lattice, given by the poset $\mathrm{Sub}(\mathbf{1}_{\mathcal{C}})$. This construction has a left adjoint, given by the pretopos completion. The left adjoint is fully faithful (in both settings), but not essentially surjective: passing from an ∞-pretopos to an ordinary pretopos and passing from an ordinary pretopos to a distributive lattice typically loses a lot of information.

Definition 9. Let $\mathcal{C}$ be an ∞-pretopos. We will say that $\mathcal{C}$ is *bounded* if every object X of $\mathcal{C}$ is n-truncated, for some $n \gg 0$ (which might depend on X).

If $\mathcal{C}$ is an ∞-pretopos, then the full subcategory $\mathcal{C}_{<\infty} = \bigcup_{n \geq 0} \mathcal{C}_{\leq n}$ is a bounded ∞-pretopos. Moreover, the ∞-pretopoi $\mathcal{C}$ and $\mathcal{C}_{<\infty}$ have the same models (every model $M : \mathcal{C}_{<\infty} \to \mathcal{S}$ extends uniquely to a model of $\mathcal{C}$, given by the construction

$$(C \in \mathcal{C}) \mapsto (\varprojlim M(\tau_{\leq n} C) \in \mathcal{S}).$$

We will henceforth restrict our attention to the study of bounded ∞-pretopoi.

Remark 10. Remark 7 admits a converse. Let $\mathcal{C}$ be a small ∞-pretopos. Then $\mathcal{C}$ is of the form $\mathcal{C}_{\leq 0}^+$ if and only if it is bounded and the ∞-category $\mathrm{Mod}(\mathcal{C})$ is (equivalent to) an ordinary category.

Example 11. Let CRing denote the category of commutative rings, and let $\mathcal{X}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CRing}, \mathcal{S}\mathrm{et})$ spanned by those functors which preserve filtered colimits. Then $\mathcal{X}$ is a coherent topos. Let $\mathcal{X}^{\mathrm{coh}}$ denote the pretopos of coherent objects of $\mathcal{X}$. Then the category $\mathrm{Mod}(\mathcal{X}^{\mathrm{coh}})$ can be identified with the category CRing of commutative rings. Here we can replace the category CRing by an arbitrary compactly generated category.

This construction has an ∞-categorical analogue. Let $\mathcal{E}$ denote the ∞-category of connective E_∞-ring spectra (or any other compactly generated ∞-category), and let $\mathcal{X} \subseteq \mathrm{Fun}(\mathcal{E}, \mathcal{S})$ be the full subcategory spanned by those functors which preserve filtered colimits. Then $\mathcal{X}$ is an example of an *∞-topos*, and it has a full subcategory $\mathcal{X}_{<\infty}^{\mathrm{coh}} \subseteq \mathcal{X}$ of truncated coherent objects. This subcategory is a bounded ∞-pretopos, and the ∞-category $\mathrm{Mod}(\mathcal{X}_{<\infty}^{\mathrm{coh}})$ can be identified with $\mathcal{E}$.

The study of ∞-pretopoi can be regarded as a generalization of classical first order logic, suitable for studying the model theory of objects which behave in a homotopy-theoretic way (such as structured ring spectra).

Let $\mathcal{C}$ be a bounded ∞-pretopos. We let $\mathrm{Stone}_{\mathcal{C}}$ denote the ∞-category whose objects are pairs $(X, \mathcal{O}_X)$, where X is a Stone space and $\mathcal{O}_X : \mathcal{C} \to \mathrm{Shv}_{\mathcal{S}}(X)$ is a morphism of ∞-pretopoi (here $\mathrm{Shv}_{\mathcal{S}}(X)$ denotes the ∞-category of $\mathcal{S}$-valued sheaves on X). We can regard the ∞-category $\mathrm{Mod}(\mathcal{C})$ of models of $\mathcal{C}$ as a full subcategory of $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}$, spanned by those pairs $(X, \mathcal{O}_X)$ where X consists of a single point. As in the

1-categorical case, one can show that the category $\mathrm{Stone}_{\mathcal{C}}$ admits coproducts, so that every collection of models $\{M_i\}_{i \in I}$ admits a coproduct

$$\coprod_{i \in I} (\{i\}, M_i)$$

in the ∞-category $\mathrm{Stone}_{\mathcal{C}}$. This is a pair $(X, \mathcal{O}_X)$, where X can be identified with the Stone-Čech compactication βI, and the stalk of $\mathcal{O}_X$ at a point x corresponding to an ultrafilter $\mathcal{U}$ is given by the formula

$$\mathcal{O}_{X,x}^{C} = \varinjlim_{J \in \mathcal{U}} \prod_{i \in J} M_i(C)$$

(which we can think of as the ultraproduct of the models M_i indexed by $\mathcal{U}$). We let $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$ denote the full subcategory of $\mathrm{Stone}_{\mathcal{C}}$ spanned by the objects of this form, so that we have a forgetful functor $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}} \to \mathrm{Stone}^{\mathrm{fr}}$. The following definition encodes the essential properties of this fibration:

Definition 12. An ∞-*ultracategory* is a locally Cartesian fibration of ∞-categories $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ with the following properties:

(1) Let $\mathcal{M} = \mathcal{E}_*^{\mathrm{op}}$. Then, for every set I, the pullback functors associated to the inclusion maps $* \simeq \{i\} \hookrightarrow \beta I$ induce an equivalence of ∞-categories $\mathcal{E}_{\beta I}^{\mathrm{op}} \simeq \prod_{i \in I} \mathcal{M}$.

(2) If $f : E \to E'$ and $g : E' \to E''$ are locally π-Cartesian morphisms of $\mathcal{E}$, and $\pi(f) : \pi(E) \to \pi(E')$ carries isolated points of $\pi(E)$ to isolated points of $\pi(E')$, then $g \circ f$ is also locally π-Cartesian.

In this case, we will refer to $\mathcal{M}$ as the *underlying* ∞-*category* of the ultracategory $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ (and will abuse terminology by simply referring to $\mathcal{M}$ as an ultracategory).

In the case where $\mathcal{M}$ is an ordinary category, Definition 12 specializes to the notion of an ultracategory (or, more precisely, the notion of an ultracategory fibration) introduced earlier in this course. In this case, it was possible to rewrite Definition 12 explicitly in terms of structure on $\mathcal{M}$. This is already somewhat cumbersome in the 1-categorical case, and is much more so in the ∞-categorical setting: roughly speaking, the structure on $\mathcal{M}$ is given by a collection of "ultraproduct functors" $P^{\mathcal{U}} : \mathcal{M}^I \to \mathcal{M}$, together with some natural transformations between iterated ultraproducts which satisfy an infinite hierarchy of coherence conditions. These coherence conditions are encoded neatly in the structure of the locally Cartesian fibration $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$.

Example 13. Let $\mathcal{C}$ be an ∞-pretopos. Then the forgetful functor $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}} \to \mathrm{Stone}^{\mathrm{fr}}$ is an ∞-ultracategory, with underlying ∞-category $\mathrm{Mod}(\mathcal{C})$.

Definition 14. Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ and $\pi' : \mathcal{E}' \to \mathrm{Stone}^{\mathrm{fr}}$ be ∞-ultracategories, with underlying ∞-categories $\mathcal{M}$ and $\mathcal{M}'$, respectively. We let $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{M}')$ denote the full subcategory of

$$\mathrm{Fun}(\mathcal{E}, \mathcal{E}')^{\mathrm{op}} \times_{\mathrm{Fun}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}})^{\mathrm{op}}} \{\pi\}$$

spanned by those functors $F : \mathcal{E} \to \mathcal{E}'$ satisfying $\pi = \pi' \circ F$, for which F carries locally π-Cartesian morphisms to locally π'-Cartesian morphisms. We will refer to $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{M}')$ as the ∞-*category of ultrafunctors from* $\mathcal{M}$ *to* $\mathcal{M}'$.

Makkai duality then admits the following ∞-categorical generalization:

Theorem 15. *Let $\mathcal{C}$ and $\mathcal{C}'$ be small bounded ∞-pretopoi. Then the canonical map*

$$\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{C}') \to \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}'), \mathrm{Mod}(\mathcal{C}))$$

is an equivalence of ∞-categories. (In fact, it is enough to assume that $\mathcal{C}$ is bounded and $\mathcal{C}'$ is small.)

In other words, a small bounded ∞-pretopos $\mathcal{C}$ can be recovered from its ∞-category $\mathrm{Mod}(\mathcal{C})$ of models, regarded as an ∞-ultracategory. This can be proven using essentially the same strategy that we used to deduce the 1-categorical version of Makkai duality.

Remark 16. It is not so hard to classify ∞-ultracategories for which the underlying ∞-category $\mathcal{M}$ is a Kan complex. The ∞-ultracategories for which $\mathcal{M}$ is an n-truncated Kan complex can be identified with the n-truncated objects of $\mathrm{Top}_{\mathrm{ch}}^{+}$, where $\mathrm{Top}_{\mathrm{ch}}$ is the pretopos of compact Hausdorff spaces. (In other words, they are "stacky" versions of compact Hausdorff spaces: when $n = 0$, this recovers the description of ultrasets given in Lecture 26X.