Lecture 3: The Structure of $\mathrm{Syn}_0(T)$

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Throughout this lecture, we fix a first-order theory T. We will assume for simplicity of exposition that T is untyped, but all of our considerations extend to the case of typed theories without essential change. We begin with the following question from the previous lecture:

Question 1. Let T be a first-order theory and let $F : \mathrm{Syn}_0(T) \to \mathcal{S}\mathrm{et}$ be a functor. When does there exist a model M of T and an isomorphism of functors $F \simeq M[\bullet]$?

A satisfying answer to Question 1 would characterize the models of T as those functors $F : \mathrm{Syn}_0(T) \to \mathcal{S}\mathrm{et}$ which preserve some sort of structure which is present on both $\mathrm{Syn}_0(T)$ and $\mathcal{S}\mathrm{et}$. We therefore begin with the following:

Question 2. What are the important structural features of the category $\mathrm{Syn}_0(T)$, and how do they interact with the functors $M[\bullet] : \mathrm{Syn}_0(T) \to \mathcal{S}\mathrm{et}$ determined by models $M \vDash T$?

Proposition 3. *The category* $\mathrm{Syn}_0(T)$ *admits fiber products. Moreover, for every model* $M \vDash T$*, the functor*

$$\mathrm{Syn}_0(T) \to \mathcal{S}\mathrm{et} \qquad X \mapsto M[X]$$

preserves fiber products.

Proof. Suppose we are given a pair of morphisms $f : X \to Z$ and $g : Y \to Z$ in $\mathrm{Syn}_0(T)$; we would like to construct a fiber product $X \times_Z Y$. Write $X = [\alpha(\vec{x})]$, $Y = [\beta(\vec{y})$, and $Z = [\gamma(\vec{z})]$, and represent the morphisms f and g by formulae $\theta(\vec{x}, \vec{z})$ and $\theta'(\vec{y}, \vec{z})$. Introduce new sequences of variables $\vec{x}'$ and $\vec{y}'$ of the same length as $\vec{x}$ and $\vec{y}$, respectively, and let $\rho(\vec{x}', \vec{y}')$ be the formula $(\exists \vec{z})[\theta(\vec{x}', \vec{z}) \wedge \theta'(\vec{y}', \vec{z})]$. Note that for any model $M \vDash T$, there is a pullback square of sets

$$\begin{tikzcd} M[\rho] \arrow[r, "\pi_M"] \arrow[d, "\pi'_M"] & M[X] \arrow[d, "f_M"] \\ M[Y] \arrow[r, "g_M"] & M[Z]. \end{tikzcd}$$

Moreover, the collections of functions $\{\pi_M\}_{M \vDash T}$ and $\{\pi'_M\}_{M \vDash T}$ comprise morphisms $\pi : [\rho] \to X$ and $\pi' : [\rho] \to Y$ in the category $\mathrm{Syn}_0(T)$: their graphs are defined by the formulae $(\vec{x} = \vec{x'})$ and $(\vec{y} = \vec{y}')$, respectively. We claim that π and π' exhibit ρ as a fiber product of X and Y over Z (this fiber product is then obviously preserved by the functor $M[\bullet]$ for every $M \vDash T$).

Suppose that we are given some other object $W = [\varphi(\vec{w})]$ in $\mathrm{Syn}_0(T)$, together with maps $u : W \to X$ and $v : W \to Y$ satisfying $f \circ u = g \circ v$. Then, for each model $M \vDash T$, there is a unique map $h_M : M[W] \to M[\rho]$ satisfying $u_M = \pi_M \circ h_M$ and $v_M = \pi'_M \circ h_M$. To complete the proof, it will suffice to show that the functions $\{h_M\}_{M \vDash T}$ comprise a morphism $h : W \to [\rho]$ in the category $\mathrm{Syn}_0(T)$. To prove this, choose formulas $\psi(\vec{w}, \vec{x})$ and $\psi'(\vec{w}, \vec{y})$ satisfying

$$\Gamma(u_M) = M[\psi(\vec{w}, \vec{x})] \qquad \Gamma(v_M) = M[\psi(\vec{w}, \vec{y})]$$

for each $M \vDash T$. Then we have $\Gamma(h_M) = M[\mu(\vec{w}, \vec{x}', \vec{y}')]$, where $\mu(\vec{w}, \vec{x}', \vec{y}')$ is the formula $\psi(\vec{w}, \vec{x}') \wedge \psi(\vec{w}, \vec{y}')$. $\square$

Corollary 4. *The category* $\mathrm{Syn}_0(T)$ *admits finite limits. Moreover, for every model* M *of* T*, the functor* $M[\bullet] : \mathrm{Syn}_0(T) \to \mathcal{S}\mathrm{et}$ *preserves finite limits.*

Proof. Since all finite limits can be built out of fiber products and final objects, it will suffice to show that $\mathrm{Syn}_0(T)$ has a final object which is preserved by each of the functors $M[\bullet]$. Fix a sentence φ such that $T \vDash \varphi$ (for example, φ could be an axiom of T, or it could be the sentence $(\forall x)[x = x]$), and set $\mathbf{1} = [\varphi] \in \mathrm{Syn}_0(T)$. Then, for every model M, the set $M[\mathbf{1}]$ has one element. It follows that, for any object $X = [\psi(\vec{x})] \in \mathrm{Syn}_0(T)$, there is a unique map $f_M : M[X] \to M[\mathbf{1}]$ for each $M \vDash T$. The functions $\{f_M\}_{M \vDash T}$ comprise a morphism f of $\mathrm{Syn}_0(T)$ (their graphs are defined by the formula $\psi(\vec{x})$ itself), which is evidently the unique map from X to $\mathbf{1}$ in $\mathrm{Syn}_0(T)$. It follows that $\mathbf{1}$ is a final object of $\mathrm{Syn}_0(T)$. □

Lemma 5. *Let* $f : X \to Y$ *be a morphism in* $\mathrm{Syn}_0(T)$*. Then* f *is an isomorphism if and only if, for every model* M *of* T*, the induced map* $f_M : M[X] \to M[Y]$ *is an isomorphism.*

Proof. The "only if" direction is obvious. To prove the converse, we must show that if each f_M is an isomorphism, then the collection of functions $\{f_M^{-1}\}_{M \vDash T}$ determines a morphism from Y to X in $\mathrm{Syn}_0(T)$. This is clear: any first-order definition of the graphs of each f_M is also a first-order definition of the graphs of f_M^{-1} (with the variables read in reverse order). □

Corollary 6. *Let* $f : X \to Y$ *be a morphism in* $\mathrm{Syn}_0(T)$*. Then* f *is a monomorphism if and only if, for every model* M *of* T*, the induced map* $f_M : M[X] \to M[Y]$ *is a monomorphism.*

Proof. Apply Lemma 5 to the diagonal map $X \to X \times_Y X$. □

Construction 7. Let $\varphi_0(\vec{x})$ and $\varphi(\vec{x})$ be two formulas in the same free variables such that $T \vDash (\forall \vec{x})[\varphi_0(\vec{x}) \Rightarrow \varphi(\vec{x})]$. Then, for every model M of T, we can identify $M[\varphi_0]$ with a subset of $M[\varphi]$. This identification determines a monomorphism from $X_0 = [\varphi_0(\vec{x})]$ to $X = [\varphi(\vec{x})]$ in the category $\mathrm{Syn}_0(T)$.

Let us call a monomorphism in $\mathrm{Syn}_0(T)$ *special* if it arises from Construction 7. We next show, up to isomorphism, every monomorphism in $\mathrm{Syn}_0(T)$ is special.

Proposition 8. *Let* $f : X \to Z$ *be a morphism in* $\mathrm{Syn}_0(T)$*. Then* f *admits a factorization* $X \xrightarrow{g} Y \xrightarrow{h} Z$*, where* g *is an epimorphism in* $\mathrm{Syn}_0(T)$ *and* h *is a special monomorphism in* $\mathrm{Syn}_0(T)$*. Moreover, we can arrange that this factorization is preserved by the functor* $M[\bullet]$ *for every model* $M \vDash T$ *(that is, each* g_M *is a surjection of sets, and each* h_M *is an injection of sets).*

Proof. Write $X = [\varphi(\vec{x})]$ and $Z = [\psi(\vec{z})]$, so that f is given by a formula $\theta(\vec{x}, \vec{z})$. Set $Y = [(\exists \vec{X})[\theta(\vec{x}, \vec{z})]]$. Note that, for every model $M \vDash T$, we can identify $M[Y]$ with the image of the map $f_M : M[X] \to M[Z]$. In particular, the map f_M factors canonically as a composition

$$M[X] \xrightarrow{g_M} M[Y] \xrightarrow{h_M} M[Z].$$

Note that $T \vDash (\forall \vec{z})[(\exists \vec{x} \theta(\vec{x}, \vec{z})) \Rightarrow \psi(\vec{z})]$, so that the collection of maps $\{h_M\}_{M \vDash T}$ define a special monomorphism $h : Y \to Z$ in $\mathrm{Syn}_0(T)$. Moreover, the graphs of the maps $\{g_M\}_{M \vDash T}$ are defined by the formula $\theta(\vec{x}, \vec{z})$, and therefore determine a morphism $g : X \to Y$ in $\mathrm{Syn}_0(T)$. To complete the proof, it will suffice to show that g is an epimorphism; we will prove a stronger assertion below. □

Warning 9. The factorization of Proposition 8 is not completely determined (even up to isomorphism) by saying that g is an epimorphism and h is a monomorphism. For example, if T is a propositional theory, then the category $\mathrm{Syn}_0(T)$ is equivalent to a poset: in this case, *every* morphism of $\mathrm{Syn}_0(T)$ is both an epimorphism and a monomorphism.

To address the uniqueness of the factorization appearing in Proposition 8, it is convenient to introduce some category-theoretic terminology.

Definition 10. Let $\mathcal{C}$ be a category which admits fiber products, and suppose we are given a morphism $g : X \to Y$ in $\mathcal{C}$. Let $X \times_Y X$ denote the fiber product of X with itself over Y, and let $\pi, \pi' : X \times_Y X \to X$ denote the projection maps onto the two factors. We will say that g is an *effective epimorphism* if g exhibits Y as a coequalizer of the maps $\pi, \pi' : X \times_Y X \to X$. In other words, g is an effective epimorphism if, for every object $W \in \mathcal{C}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(Y, W) \simeq \{u \in \mathrm{Hom}_{\mathcal{C}}(X, W) : u \circ \pi = u \circ \pi'\}.$$

Remark 11. Let $\mathcal{C}$ be a category which admits fiber products. Then every effective epimorphism is an epimorphism. In the category of sets, the converse is true: if $g : X \to Y$ is a surjective map of sets, then we can recover Y as the quotient of X by the equivalence relation $R = X \times_Y X = \{(x, x') : g(x) = g(x')\}$. However, this is not true in a general category.

Example 12. Let $\mathcal{C}$ be the category of commutative rings. Then a ring homomorphism $f : R \to S$ is an effective epimorphism in $\mathcal{C}$ if and only if f is surjective. However, there are plenty of non-surjective ring homomorphisms which are epimorphisms in $\mathcal{C}$, such as localization maps $R \mapsto R[1/t]$.

Example 13. The map $g : X \to Y$ constructed in the proof of Proposition 8 is actually an effective epimorphism. To prove this, suppose we are given an object W and a map $u : X \to W$ in $\mathrm{Syn}_0(T)$ satisfying $u \circ \pi = u \circ \pi'$, where $\pi, \pi' : X \times_Y X \to X$ are the projection maps. For every model $M \vDash T$, we have $(u \circ \pi)_M = (u \circ \pi')_M$. Since g_M is a surjection of sets, it is an effective epimorphism; it follows that there is a unique map $\overline{u}_M : M[Y] \to M[W]$ such that $u_M = \overline{u}_M \circ g_M$. We claim that $\{\overline{u}_M\}_{M \in \vDash T}$ determines a morphism $\overline{u} : Y \to W$ in $\mathrm{Syn}_0(T)$ (which is automatically the unique solution to $u = g \circ \overline{u}$). Writing $W = [\alpha(\vec{w})]$ (and retaining the notation of Proposition 8), we see that u can be described by the formula $\overline{\beta}(\vec{z}, \vec{w})$ given by

$$(\exists \vec{x})[\theta(\vec{x}, \vec{z}) \wedge \beta(\vec{z}, \vec{w})],$$

where $\beta(\vec{x}, \vec{w})$ is any formula defining the morphism u.

Proposition 14. *Let $\mathcal{C}$ be a category which admits fiber products and let $f : X \to Z$ be a morphism in $\mathcal{C}$. If f can be factored as a composition $X \xrightarrow{g} Y \xrightarrow{h} Z$ where h is a monomorphism and g is an effective epimorphism, then that factorization is unique (up to unique isomorphism).*

Proof. Suppose we are given another factorization $X \xrightarrow{g'} Y' \xrightarrow{h'} Z$, where h' is a monomorphism and g' is an effective epimorphism. We claim that there is a unique morphism $u : Y \to Y'$ for which the diagram

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "\mathrm{id}"] & Y \arrow[r, "g"] \arrow[d, "u"] & Z \arrow[d, "\mathrm{id}"] \\
X \arrow[r, "g'"] & Y' \arrow[r, "h'"] & Z
\end{tikzcd}$$

commutes. Since g is an effective epimorphism, it will suffice to show that $g' \circ \pi = g' \circ \pi'$, where $\pi, \pi' : X \times_Y X \to X$ are the projection maps. Since h' is a monomorphism, we are reduced to proving that $h' \circ g' \circ \pi = h' \circ g' \circ \pi'$. But we can rewrite this equality as $h \circ g \circ \pi = h \circ g \circ \pi'$, which follows from the identity $g \circ \pi = g \circ \pi'$.

Applying the same argument with the roles of Y and Y' reversed, we will obtain a morphism $v : Y' \to Y$; it follows from the uniqueness of the factorization above that u and v are mutually inverse isomorphisms. $\square$

Corollary 15. *Let $f : X \to Z$ be a morphism in the syntactic category $\mathrm{Syn}_0(T)$. Then the factorization $X \xrightarrow{g} Y \xrightarrow{h} Z$ is characterized uniquely (up to unique isomorphism) by the fact that g is an effective epimorphism and h is a monomorphism.*

Corollary 16. *Let $X = [\varphi(\vec{x})]$ be an object of $\mathrm{Syn}_0(T)$, and let $f : X_0 \hookrightarrow X$ be a monomorphism in $\mathrm{Syn}_0(T)$. Then f is isomorphic to a special monomorphism $[\varphi_0(\vec{x})] \hookrightarrow [\varphi(\vec{x})]$ (by an isomorphism which is the identity on X).*

Proof. Proposition 8 supplies a factorization of f as a composition $X_0 \xrightarrow{g} Y \xrightarrow{h} X$ where g is an effective epimorphism and h is a special monomorphism. However, if f is already a monomorphism, then we also have the factorization $X_0 \xrightarrow{\mathrm{id}} X_0 \xrightarrow{f} X$. Invoking the uniqueness of Proposition 14, we deduce that there is a commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & X_0 \arrow[r, "f"] \arrow[d] & X \arrow[d, "\mathrm{id}"] \\ X_0 \arrow[r, "g"] & Y \arrow[r, "h"] & X \end{tikzcd}$$

where the vertical maps are isomorphisms. □

Notation 17. Let $\mathcal{C}$ be any category and let X be an object of $\mathcal{C}$. We let $\mathrm{Sub}(X)$ denote the set of equivalence classes of monomorphisms $i_0 : X_0 \to X$, where two monomorphisms $i_0 : X_0 \to X$ and $i_1 : X_1 \to X$ are considered to be equivalent if there is an isomorphism $e : X_0 \simeq X_1$ for which the diagram

$$\begin{tikzcd} X_0 \arrow[rr, "e"] \arrow[dr, "i_0"] & & X_1 \arrow[dl, "i_1"'] \\ & X & \end{tikzcd}$$

commutes. We will refer to $\mathrm{Sub}(X)$ as the *set of subobjects of* X.

We will generally abuse notation by simply identifying elements of $\mathrm{Sub}(X)$ with the objects X_0 representing them (in this case, we implicitly assume that a monomorphism $X_0 \hookrightarrow X$ has been supplied). Given a pair of subobjects $X_0, X_1 \in \mathrm{Sub}(X)$, we write $X_0 \subseteq X_1$ if there exists a commutative diagram

$$\begin{tikzcd} X_0 \arrow[rr, "e"] \arrow[dr, "i_0"] & & X_1 \arrow[dl, "i_1"'] \\ & X; & \end{tikzcd}$$

in this case e is automatically unique (and is also a monomorphism).

Proposition 18. *Let $X = [\varphi(\vec{x})]$ be an object of $\mathrm{Syn}_0(T)$. Then:*

(1) *Every subobject of X is has the form $[\varphi_0(\vec{x})]$, where $\varphi_0(\vec{x})$ satisfies $T \vDash (\forall \vec{z})[\varphi_0(\vec{x}) \Rightarrow \varphi(\vec{x})]$ (equipped with the special monomorphism to X described in Construction 7.*

(2) *Given a pair of subobjects $X_0 = [\varphi_0(\vec{x})]$ and $X_1 = [\varphi_1(\vec{x})]$, we have $X_0 \subseteq X_1$ if and only if $T \vDash (\forall \vec{x})[\varphi_0(\vec{x}) \Rightarrow \varphi_1(\vec{x})]$.*

(3) *Given a pair of subobjects $X_0 = [\varphi_0(\vec{x})]$ and $X_1 = [\varphi_1(\vec{x})]$, we have $X_0 = X_1$ (in $\mathrm{Sub}(X)$) if and only if $T \vDash (\forall \vec{x})[\varphi_0(\vec{x}) \Leftrightarrow \varphi_1(\vec{x})]$*

Proof. Assertion (1) is Corollary 16 and (3) follows from (2). For assertion (2), it is clear that if $X_0 \subseteq X_1$, then for every model $M \vDash T$ we must have $M[X_0] \subseteq M[X_1]$ (where we identify both with subsets of $M[X]$), so that $T \vDash (\forall \vec{x})[\varphi_0(\vec{x}) \Rightarrow \varphi_1(\vec{x})]$. Conversely, if each $M[X_0]$ is a subset of $M[X_1]$, then the inclusion maps $e_M : M[X_0] \hookrightarrow M[X_1]$ define a morphism $e : X_0 \to X_1$ (as in Construction 7), for which the diagram

$$\begin{tikzcd} X_0 \arrow[rr, "e"] \arrow[dr] & & X_1 \arrow[dl] \\ & X & \end{tikzcd}$$

commutes in $\mathrm{Syn}_0(T)$. □

Corollary 19. *Let Z be an object of* $\mathrm{Syn}_0(T)$*. Then the partially ordered set* $\mathrm{Sub}(Z)$ *is a Boolean algebra. Moreover, for every model M of T, the construction $(M[Z_0] \in \mathrm{Sub}(Z)) \mapsto (M[Z_0] \subseteq M[Z])$ induces a Boolean algebra homomorphism from* $\mathrm{Sub}(Z)$ *to the lattice of all subsets of $M[Z]$.*

Proof. Write $Z = [\varphi(\vec{z})]$, and let us identify $\mathrm{Sub}(Z)$ with the collection of equivalence classes of formulae $\varphi_0(\vec{z})$ in the free variables $\vec{z}$ satisfying $T \vDash (\forall \vec{z})[\varphi_0(\vec{z}) \Rightarrow \varphi(\vec{z})]$. This poset has a largest element (given by $\varphi(\vec{z})$), a smallest element (given by $(\exists x)\neg[x = x]$), meets (given by $\varphi_0(\vec{z}) \wedge \varphi_1(\vec{z})$), joins (given by $\varphi_0(\vec{z}) \vee \varphi_1(\vec{z})$), and complements (given by $\varphi_0(\vec{z}) \mapsto (\neg\varphi_0(\vec{z}) \wedge \varphi(\vec{z}))$). $\square$

We are now ready to answer Question 1:

Theorem 20. *Let $F : \mathrm{Syn}_0(T) \to \mathcal{S}\mathrm{et}$ be a functor. Then F arises from a model $M \vDash T$ if and only if it satisfies the following three conditions:*

(*a*) *The functor F preserves finite limits.*

(*b*) *The functor F carries effective epimorphisms in* $\mathrm{Syn}_0(T)$ *to surjections of sets.*

(*c*) *For every object $X \in \mathrm{Syn}_0(T)$, the induced map $\mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$ is a homomorphism of upper semilattices: that is, it preserves least upper bounds of finite subsets.*

We will prove Theorem 20 in the next lecture.