Lecture 29X-Makkai Duality

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To every small pretopos $\mathcal{C}$, we have associated an ultracategory fibration $\pi : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}} \to \mathrm{Stone}^{\mathrm{fr}}$, with underlying ultracategory $\mathrm{Mod}(\mathcal{C})$. Here the assumption that $\mathcal{C}$ is small is not really essential: for any pretopos $\mathcal{C}$, we can consider the category $\mathrm{Mod}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ of pretopos morphisms from $\mathcal{C}$ to $\mathcal{S}\mathrm{et}$, and equip it with the structure of an ultracategory. In this lecture, we will study a right adjoint to this construction.

We begin with some preliminary remarks.

Proposition 1. *Let I be a set and let $X = \beta I$. Then the direct image functor*

$$\iota_* : \mathcal{S}\mathrm{et}^I \simeq \mathrm{Shv}(I) \to \mathrm{Shv}(X)$$

is a morphism of pretopoi: that is, it preserves finite limits, finite coproducts, and effective epimorphisms.

Proof. It is clear that ι_* preserves finite limits. The axiom of choice implies that every epimorphism in $\mathcal{S}\mathrm{et}^I$ admits a section, so that ι_* preserves effective epimorphisms. Consequently, to show that it is a morphism of pretopoi, it will suffice to show that for every finite set J, the functor ι_* carries the constant sheaf $\underline{J}_I$ on the discrete space I to the constant sheaf $\underline{J}_X$ on X (see Lemma 4 of Lecture 16X). This follows from the observation that every map $I \to J$ extends uniquely to a locally constant map $\beta I \to J$ (since J is compact), and that an analogous statement holds for each clopen subset of X. □

Corollary 2. *Let I be a set and let $\mathcal{U}$ be an ultrafilter on I. Then the ultraproduct functor*

$$P^{\mathcal{U}} : \mathcal{S}\mathrm{et}^I \to \mathcal{S}\mathrm{et} \qquad \{M_i\}_{i \in I} \mapsto (\prod_{i \in I} M_i)/\mathcal{U}$$

is a morphism of pretopoi.

Proof. The functor $P^{\mathcal{U}}$ is given by the composition

$$\mathcal{S}\mathrm{et}^I \xrightarrow{\iota_*} \mathrm{Shv}(\beta I) \xrightarrow{\mathcal{U}^*} \mathcal{S}\mathrm{et}$$

where $\mathcal{U}^*$ denotes the functor given by taking the stalk at the point of βI corresponding to the ultrafilter $\mathcal{U}$. □

Remark 3. Corollary 2 is a version of the Łos ultraproduct theorem: it implies that for any pretopos $\mathcal{C}$, composition with $P^{\mathcal{U}}$ induces a functor

$$\mathrm{Mod}(\mathcal{C})^I = \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{S}\mathrm{et}^I) \xrightarrow{P^{\mathcal{U}} \circ} \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{S}\mathrm{et}) = \mathrm{Mod}(\mathcal{C}),$$

where $\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{D})$ denotes the category of pretopos morphisms from $\mathcal{C}$ to $\mathcal{D}$.

Corollary 4. *Let $\pi' : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}} \to \mathrm{Stone}^{\mathrm{fr}}$ denote the forgetful functor. Then:*

- *For each $X \in \mathrm{Stone}^{\mathrm{fr}}$, the category $(\pi'^{-1}\{X\})^{\mathrm{op}}$ is a pretopos.*

- *For each morphism $f : Y \to X$ in $\mathrm{Stone}^{\mathrm{fr}}$, the pullback functor $f^* : (\pi'^{-1}\{X\})^{\mathrm{op}} \to (\pi'^{-1}\{Y\})^{\mathrm{op}}$ is a morphism of pretopoi.*

Proof. If $X = \beta I$, then $(\pi'^{-1}\{X\})^{\mathrm{op}}$ is equivalent to $\mathcal{S}\mathrm{et}^I$. Moreover, if $Y = \beta J$ and $f : Y \to X$ is a map given by a family of ultrafilters $\{\mathcal{U}_j\}_{j \in J}$ on the set I, then the pullback functor f^* can be identified with the construction

$$(\{M_i\}_{i \in I} \in \mathcal{S}\mathrm{et}^I) \mapsto (\{\prod_{i \in I} M_i)/\mathcal{U}_j\}_{j \in J} \in \mathcal{S}\mathrm{et}^J),$$

and is therefore a morphism of pretopoi (Corollary 2). □

Corollary 5. *Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ be an ultracategory fibration and let $\mathrm{Mor}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}})$ denote the category of ultracategory fibration morphisms from $\mathcal{E}$ to $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$. Then the category $\mathrm{Mor}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}})^{\mathrm{op}}$ is a pretopos (not necessarily small).*

Proof. It follows from Corollary 4 that we can compute finite limits, finite coproducts, and quotients by equivalence relations in the category $\mathrm{Mor}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}})^{\mathrm{op}}$ "fiberwise." □

Or, stated in terms of ultracategories:

Corollary 6. *Let $\mathcal{M}$ be an ultracategory. Then the category $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{S}\mathrm{et})$ is a pretopos.*

Remark 7. In the situation of Corollary 6, the forgetful functor $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathcal{M}, \mathcal{S}\mathrm{et})$ is a morphism of pretopoi. Equivalently, for each $M \in \mathcal{M}$, the evaluation functor $F \mapsto F(M)$ determines a morphism of pretopoi $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{S}\mathrm{et}) \to \mathcal{S}\mathrm{et}$. In order words, finite limits, finite coproducts, and quotients by equivalence relations in $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{S}\mathrm{et})$ can be computed "pointwise".

Let $\mathcal{C}$ be a (small) pretopos. Then the category $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}$ can be identified with the full subcategory of

$$\mathrm{Fun}(\mathcal{C}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}) \times_{\mathrm{Fun}(\mathcal{C}, \mathrm{Stone}^{\mathrm{fr}})} \mathrm{Stone}^{\mathrm{fr}}$$

spanned by those pairs $(F : \mathcal{C} \to \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}, X)$ with the property that F determines a morphism of pretopoi $\mathcal{C} \to \mathrm{Shv}(X)$. Consequently, for any ultracategory fibration $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$, we have a canonical equivalence

$$\mathrm{Mor}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}})^{\mathrm{op}} \simeq \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathrm{Mor}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}})^{\mathrm{op}}).$$

Or, stated in terms of ultracategories:

Corollary 8. *Let $\mathcal{M}$ be an ultracategory and let $\mathcal{C}$ be a pretopos. Then there is a canonical equivalence of categories*

$$\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathrm{Mod}(\mathcal{C})) \simeq \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{S}\mathrm{et})).$$

We can regard the construction $\mathcal{C} \mapsto \mathrm{Mod}(\mathcal{C})$ as a contravariant functor from the 2-category of pretopoi (where morphisms are functors that preserve finite limits, finite coproducts, and effective epimorphisms) to the 2-category of ultracategories (where morphisms are ultrafunctor). We can interpret Corollary 8 as asserting that that this construction admits a right adjoint

$$\{\mathrm{Ultracategories}\}^{\mathrm{op}} \to \{\mathrm{Pretopoi}\} \qquad \mathcal{M} \mapsto \mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{S}\mathrm{et}).$$

In particular, for every pretopos $\mathcal{C}$, we have a unit map

$$u : \mathcal{C} \to \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathcal{S}\mathrm{et}),$$

which carries each object $Cin\, \mathcal{C}$ to the ultrafunctor given by evaluation at C. We saw in the previous lecture that this functor is an equivalence when $\mathcal{C}$ is small.

Corollary 9 (Strong Conceptual Completeness)**.** *Let $\mathcal{C}$ and $\mathcal{C}'$ be pretopoi. If $\mathcal{C}'$ is small, then the canonical map*

$$\theta : \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{C}') \to \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}'), \mathrm{Mod}(\mathcal{C}))$$

is an equivalence of categories.

Proof. Using the adjunction of Corollary 8, we can identify θ with the map

$$\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{C}') \to \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}'), \mathcal{S}\mathrm{et}))$$

given by composition with the unit map $u : \mathcal{C}' \to \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}'), \mathcal{S}\mathrm{et})$. □

Remark 10. Makkai's strong conceptual completeness theorem can be interpeted as a kind of duality between the theory of pretopoi (having to do with the syntax of first-order logic) and the theory of ultracategories (which is a way of capturing the semantics of first-order logic). Concretely, this duality is implemented by the category $\mathcal{S}\mathrm{et}$ of sets, which simultaneously has the structure of a pretopos and an ultracategory (moreover, these structures are "compatible" by virtue of Corollary 2). It follows that for every pretopos $\mathcal{C}$, the category $\mathrm{Mod}(\mathcal{C})$ of pretopos morphisms from $\mathcal{C}$ to $\mathcal{S}\mathrm{et}$ has the structure of an ultracategory, and that for every ultracategory $\mathcal{M}$ the category $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{S}\mathrm{et})$ of ultrafunctors from $\mathcal{M}$ to $\mathcal{S}\mathrm{et}$ has the structure of a pretopos.

Corollary 11. *The construction $\mathcal{C} \mapsto \mathrm{Mod}(\mathcal{C})$ determines a fully faithful embedding of 2-categories*

$$\{\textit{Small pretopoi}\}^{\mathrm{op}} \hookrightarrow \{\textit{Ultracategories}\}.$$

It follows immediatey from the definitions that an ultrafunctor $F : \mathcal{M} \to \mathcal{M}'$ is invertible (in the 2-category of ultracategories) if and only if it is an equivalence (in particular if F is an equivalence of categories, then a homotopy inverse F^{-1} to F can also be regarded as an ultrafunctor). We therefore obtain the following:

Corollary 12 (Makkai-Reyes Conceptual Completeness Theorem)**.** *Let $f : \mathcal{C} \to \mathcal{C}'$ be a morphism of small pretopoi. If the induced map $\mathrm{Mod}(\mathcal{C}') \to \mathrm{Mod}(\mathcal{C})$ is an equivalence of categories, then f is an equivalence of categories.*

Remark 13. Note that the analogue Corollary **??** would *not* hold if we were to allow $\mathcal{C}$ and $\mathcal{C}'$ to be arbitrary coherent categories: the canonical map from a coherent category $\mathcal{C}$ to its pretopos completion $\mathcal{C}_{\mathrm{eq}}$ induces an equivalence of categories of models $\mathrm{Mod}(\mathcal{C}_{\mathrm{eq}}) \to \mathrm{Mod}(\mathcal{C})$, but is generally not an equivalence. Consequently, we can regard Corollary **??** as evidence that pretopoi are a good class of objects to work with. For example, if T is a first-order theory, then the syntactic category $\mathrm{Syn}(T)$ is the "maximal" enlargement of the weak syntactic category $\mathrm{Syn}_0(T)$ with the same semantics.

Corollary 11 can be considered as a categorified version of Stone duality, which establishes a fully faithful embedding

$$\{\text{Boolean algebras}\}^{\mathrm{op}} \hookrightarrow \{\text{Compact Hausdorff spaces}\}$$

(whose essential image is the category Stone of Stone spaces). In fact, it is even a generalization of Stone duality: there is a commutative diagram

$$\begin{array}{ccc} \{\text{Boolean algebras}\}^{\mathrm{op}} & \longrightarrow & \{\text{Compact Hausdorff Spaces}\} \\ \downarrow & & \downarrow \\ \{\text{Small pretopoi}\}^{\mathrm{op}} & \xrightarrow{\mathrm{Mod}} & \{\text{Ultracategories}\} \end{array}$$

where the left vertical map is given by pretopos completion and the right vertical map associates to each compact Hausdorff space the associated ultraset (see Lecture 26X).

Warning 14. The fully faithful embedding of Corollary 11 is not essentially surjective. For example, if X is a compact Hausdorff space which is not a Stone space, then X (regarded as an ultracategory with only identity morphisms) does not belong to the essential image.