Lecture 28X-Ultrafunctors to $\mathcal{S}\mathrm{et}$

April 18, 2018

Let $\mathcal{S}\mathrm{et}$ denote the category of sets. Then $\mathcal{S}\mathrm{et}$ can be regarded as an ultracategory: for every ultrafilter $\mathcal{U}$ on a set I, we define $P^{\mathcal{U}} : \mathcal{S}\mathrm{et}^I \to \mathcal{S}\mathrm{et}$ by the formula

$$P^{\mathcal{U}}\{S_i\}_{i \in I} = (\prod_{i \in I} S_i)/\,\mathcal{U}\,.$$

Our goal in this lecture is to describe the category $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{S}\mathrm{et})$ of $\mathcal{S}\mathrm{et}$-valued ultrafunctors on $\mathcal{M}$, where $\mathcal{M}$ is an arbitrary ultracategory. For this, it will be more convenient to work at the level of ultracategory fibrations.

Construction 1. We define a category $\mathrm{Stone}_{\mathcal{S}\mathrm{et}}$ as follows:

- The objects of $\mathrm{Stone}_{\mathcal{S}\mathrm{et}}$ are pairs $(X, \mathcal{O}_X)$, where X is a Stone space and $\mathcal{O}_X$ is a sheaf of sets on X.
- A morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ in $\mathrm{Stone}_{\mathcal{S}\mathrm{et}}$ consists of a continuous map $f : X \to Y$ together with a map $f^* \mathcal{O}_Y \to \mathcal{O}_X$ of $\mathcal{S}\mathrm{et}$-valued sheaves on X.

We say that an object of $\mathrm{Stone}_{\mathcal{S}\mathrm{et}}$ is *free* if it can be written as a coproduct $\coprod_{i \in I}(\{i\}, S_i)$, for some family of sets $\{S_i\}_{i \in I}$ indexed by a set I. Note that this coproduct can be written as $(X, \mathcal{O}_X)$, where $X = \beta I$ is the Stone-Čech compactification of I and the stalk of $\mathcal{O}_X$ at an ultrafilter $\mathcal{U} \in \beta I$ is givenby

$$\mathcal{O}_{X,\mathcal{U}} = (\prod_{i \in I} S_i)/\,\mathcal{U}\,.$$

We let $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$ denote the full subcategory of $\mathrm{Stone}_{\mathcal{S}\mathrm{et}}$ spanned by the free objects.

Remark 2. Less explicitly, we can describe $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$ as the category $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}$, where $\mathcal{C}$ is the pretopos of coherent objects of the classifying topos $\mathcal{E}_{\mathcal{S}\mathrm{et}} = \mathrm{Fun}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}, \mathcal{S}\mathrm{et})$ (so that $\mathrm{Mod}(\mathcal{C})$ equivalent to the category of sets).

Note that we have an evident forgetful functor $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}} \to \mathrm{Stone}^{\mathrm{fr}}$, which is an ultracategory fibration. Consequently, if $\mathcal{M}$ is an ultracategory with associated ultracategory fibration $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$, then the category $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{S}\mathrm{et})$ of ultrafunctors from $\mathcal{M}$ to $\mathcal{S}\mathrm{et}$ can be identified with the opposite of the category $\mathrm{Mor}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}})$ of morphisms of ultracategory fibrations from $\mathcal{E}$ to $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$. Before we can describe the classification of such functors, we need a few preliminary remarks.

Lemma 3. *Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ be an ultracategory fibration. Suppose we are given an object $E \in \mathcal{E}$ with $\pi(E) = \beta I$ for some set I. For each element $i \in I$, choose a locally π-Cartesian morphism $f_i : E_i \to E$ lying over the inclusion map $\iota_i : \{i\} \hookrightarrow \beta I$ in $\mathrm{Stone}^{\mathrm{fr}}$. Then the morphisms f_i exhibit E as the coproduct of the objects E_i in $\mathcal{E}$.*

Proof. Let E' be any other object of $\mathcal{E}$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{E}}(E, E') & \xrightarrow{\circ f_\bullet} & \prod_{i \in I} \mathrm{Hom}_{\mathcal{E}}(E_i, E') \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Stone}^{\mathrm{fr}}}(\beta I, \pi(E')) & \longrightarrow & \prod_{i \in I} \mathrm{Hom}_{\mathrm{Stone}^{\mathrm{fr}}}(\{i\}, \pi(E')) \end{array}$$

and we wish to show that the upper horizontal map is bijective. Since the bottom horizontal map is bijective, it will suffice to show that the diagram is a pullback square. Equivalently, we wish to show that for every continuous map $g : \beta I \to \pi(E')$, the induced map

$$\mathrm{Hom}_{\mathcal{E}_{\beta I}}(E, g^* E') \to \prod_{i \in I} \mathrm{Hom}_{\mathcal{E}_{\{i\}}}(\iota_i^* E, (g \circ \iota_i)^* E')$$

is a bijection. This follows from our assumptions that the functor $(\prod \iota(i)^*) : \mathcal{E}_{\beta I} \to \prod_{i \in I} \mathcal{E}_{\{i\}}$ is an equivalence of categories and that the comparison maps $\iota_i^* \circ g^* \to (g \circ \iota_i)^*$ are isomorphisms. □

Corollary 4. *Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ be an ultracategory fibration. Then the category $\mathcal{E}$ admits coproducts.*

Proof. Let $\mathcal{E}_* = \pi^{-1}\{*\}$. By virtue of Lemma 3, every object of $\mathcal{E}$ can be written as a coproduct of objects of $\mathcal{E}_*$. It will therefore suffice to show that every collection of objects $\{E_i \in \mathcal{E}_*\}_{i \in I}$ admits a coproduct. This follows from Lemma 3, since the functor $(\prod \iota(i)^*) : \mathcal{E}_{\beta I} \to \prod_{i \in I} \mathcal{E}_{\{i\}}$ is essentially surjective. □

Remark 5. It follows from the proof of Corollary 4 that if $\mathcal{D}$ is any category which admits coproducts, then a functor $F : \mathcal{E} \to \mathcal{D}$ preserves coproducts if and only if it preserves coproducts of families $\{E_i\}_{i \in I}$ where each E_i belongs to $\mathcal{E}_*$. In particular, the projection map $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ preserves coproducts.

Corollary 6. *Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ and $\pi' : \mathcal{E}' \to \mathrm{Stone}^{\mathrm{fr}}$ be ultracategory fibrations. Then any morphism of ultracategory fibrations $F : \mathcal{E} \to \mathcal{E}'$ preserves coproducts.*

Proof. By virtue of Remark 5, it will suffice to show that F preserves the coproduct of any family of objects $\{E_i\}_{i \in I}$ belonging to $\mathcal{E}_*$. Choose a collection of maps $f_i : E_i \to E$ in $\mathcal{E}$ which exhibit E as a coproduct of $\{E_i\}_{i \in I}$. It follows from Lemma 3 (and the uniqueness of coproducts) that each f_i is locally π-Cartesian. It follows that each $F(f_i)$ is locally π'-Cartesian, so that the morphisms $F(f_i)$ exhibit $F(E)$ as a coproduct of the family $\{F(E_i)\}_{i \in I}$ (by virtue of Lemma 3, applied to the ultracategory fibration π'). □

Notation 7. For every object $(X, \mathcal{O}_X) \in \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$, we let $\Gamma(X; \mathcal{O}_X)$ denote the set $\mathcal{O}_X(X)$. Then the construction $(X, \mathcal{O}_X) \mapsto \Gamma(X; \mathcal{O}_X)$ determines a functor $\Gamma : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}} \to \mathcal{S}\mathrm{et}^{\mathrm{op}}$. This functor is representable: we have canonical bijections

$$\Gamma(X; \mathcal{O}_X) \simeq \mathrm{Hom}_{\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}}((X, \mathcal{O}_X), (*, \mathbf{1})),$$

where $*$ denotes the one-point space and $\mathbf{1} \in \mathrm{Shv}(*)$ denotes the final object (corresponding to the one-element set). It follows that the functor Γ carries coproducts in $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$ to products in $\mathcal{S}\mathrm{et}$.

We can now state our main result.

Theorem 8. *Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ be an ultracategory fibration. Then composition with the functor $\Gamma : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}} \to \mathcal{S}\mathrm{et}^{\mathrm{op}}$ induces a fully faithful embedding*

$$\mathrm{Mor}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}) \to \mathrm{Fun}(\mathcal{E}, \mathcal{S}\mathrm{et}^{\mathrm{op}}),$$

whose essential image consists of those functors $T : \mathcal{E} \to \mathcal{S}\mathrm{et}^{\mathrm{op}}$ which satisfy the following pair of conditions:

(*a*) *The functor T carries coproducts in $\mathcal{E}$ to products of sets.*

(*b*) *For each object $E \in \mathcal{E}$ having image $X = \pi(E)$ and each point $x \in X$, the canonical map*

$$\varinjlim_{x \in U} T(E_U) \to T(E_{\{x\}})$$

is a bijection. Here the direct limit is taken over all clopen neighborhoods U of x, $E_U \in \mathcal{E}_U$ denotes the pullback of $E \in \mathcal{E}_X$ under the inclusion $U \hookrightarrow X$, and $E_{\{x\}} \in \mathcal{E}_{\{x\}}$ is defined similarly.

Proof. If $F : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$ is a morphism of ultracategory fibrations, then F preserves coproducts (Corollary 6), so that $\Gamma \circ F$ carries coproducts in $\mathcal{E}$ to products of sets (Notation 7). Moreover, if E and X are as in (*b*), then we can write $F(E) = (X, \mathcal{O}_X)$ for some sheaf of sets $\mathcal{O}_X$ on X. Our assumption that F is a morphism of ultractegory fibrations (and therefore preserves locally Cartesian morphisms) supplies a canonical isomorphisms $F(E_U) \simeq (U, \mathcal{O}_X|_U)$ for $U \subseteq X$ clopen and $F(E_{\{x\}}) \simeq (\{x\}, \mathcal{O}_{X,x})$, so that

$$(\Gamma \circ F)(E_{\{x\}} \simeq \mathcal{O}_{X,x} \simeq \varinjlim_{x \in U} \mathcal{O}_X(U) = \varinjlim_{x \in U} (\Gamma \circ F)(E_U).$$

Consequently, every functor belonging to the essential image of composition with Γ satisfies conditions (*a*) and (*b*).

Conversely, suppose we are given a functor $T : \mathcal{E} \to \mathcal{S}\mathrm{et}^{\mathrm{op}}$ satisfying (*a*) and (*b*). We define a functor $F_T : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$ by the formula $F_T(E) = (X, \mathcal{O}_X)$, where $X = \pi(E)$ and $\mathcal{O}_X(U) = T(E_U)$ for $U \subseteq X$ clopen (condition (*a*) guarantees that the construction $U \mapsto T(E_U)$ carries coproducts of disjoint clopen subsets of X to products of sets, so this formula determines a sheaf $\mathcal{O}_X$ on X which is unique up to canonical isomorphism). For each clopen set $U \subseteq X$, we can identify E_U with the coproduct of the objects $E_{\{x\}}$ where x ranges over the isolated points of U (Lemma 3), so that condition (*a*) gives

$$\mathcal{O}_X(U) \simeq \prod_{x \in U, x \text{ isolated}} \mathcal{O}_{X,x}\,.$$

It follows that $(X, \mathcal{O}_X)$, is given by the coproduct of $(\{x\}, \mathcal{O}_{X,x})$ as x ranges over the isolated points of X, so that $(X, \mathcal{O}_X)$ belongs to $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$.

By construction, F_T is a functor from $\mathcal{E}$ to $\mathrm{Stone}_{\mathcal{S}\mathrm{et}}$ which fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{F_T} & \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}} \\ & {}_{\pi}\searrow \quad \swarrow_{\pi'} & \\ & \mathrm{Stone}^{\mathrm{fr}}. & \end{array}$$

We claim that F_T is a morphism of ultracategory fibrations. In other words, we wish to show that for each locally π-Cartesian morphism $g : E \to E'$ in $\mathcal{E}$, the image $F_T(g)$ is a locally π'-Cartesian morphism in $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$. Write $F_T(E) = (X, \mathcal{O}_X)$ and $F_T(E') = (X', \mathcal{O}_{X'})$, so that $\pi(g)$ is a continuous map from X to X'. We then have a canonical map $\pi(g)^* \mathcal{O}_{X'} \to \mathcal{O}_X$, and we wish to show that this map of sheaves is an isomorphism after taking the stalk at each *isolated* point $x \in X$. Equivalently, we must show that the canonical map

$$\varinjlim_{\pi(g)(x) \in U} T(E'_U) \to T(E_{\{x\}})$$

is bijective, where U ranges over the collection of clopen subsets of X' containing $\pi(g)(x)$. This follows immediately from (*b*).

We leave it to the reader to verify that the constructions $F \mapsto \Gamma \circ F$ and $T \mapsto F_T$ are mutually inverse (up to canonical isomorphism). □

If $\mathcal{C}$ is a small pretopos, then each object $C \in \mathcal{C}$ determines a morphism of ultracategory fibrations $F^C : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}} \to \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}}$, given by $(X, \mathcal{O}_X) \mapsto (X, \mathcal{O}_X^C)$. The construction $C \mapsto F^C$ then determines a functor $\mathcal{C} \to \mathrm{Mor}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}})^{\mathrm{op}}$. In Lecture 23X, we showed that the composition

$$\mathcal{C} \to \mathrm{Mor}(\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}})^{\mathrm{op}} \xrightarrow{\Gamma \circ} \mathrm{Fun}(\mathrm{Stone}^{\mathrm{fr,op}}_{\mathcal{C}}, \mathcal{S}\mathrm{et})$$

$$C \mapsto ((X, \mathcal{O}_X) \mapsto \Gamma(X; \mathcal{O}_X^C))$$

is a fully faithful embedding, whose essential image is spanned by those functors $\mathrm{Stone}^{\mathrm{fr,op}}_{\mathcal{C}} \to \mathcal{S}\mathrm{et}$ satisfying conditions (a) and (b) of Theorem 8. Combining this with Theorem 8, we obtain the following:

Corollary 9. *Let $\mathcal{C}$ be a small pretopos. Then the construction $C \mapsto F^C$ induces an equivalence of categories*

$$\mathcal{C} \to \mathrm{Mor}(\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}, \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{S}\mathrm{et}})^{\mathrm{op}}.$$

Or, stated in terms of ultrafunctors:

Corollary 10 (Makkai)**.** *Let $\mathcal{C}$ be a small pretopos. Then there is an equivalence of categories*

$$\mathcal{C} \to \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathcal{S}\mathrm{et}),$$

which carries an object C to the ultrafunctor $\mathrm{Mod}(\mathcal{C}) \to \mathcal{S}\mathrm{et}$ given by evaluation at C.