Lecture 27X-Ultrafunctors

April 17, 2018

We now study functors between ultracategories.

Definition 1. Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ and $\pi' : \mathcal{E}' \to \mathrm{Stone}^{\mathrm{fr}}$ be ultracategory fibrations. A *morphism of ultracategory fibrations* from $\mathcal{E}$ to $\mathcal{E}'$ is a functor $F : \mathcal{E} \to \mathcal{E}'$ with the following properties:

(1) The diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{F} & \mathcal{E}' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & \mathrm{Stone}^{\mathrm{fr}} & \end{array}$$

commutes. In particular, for every $X \in \mathrm{Stone}^{\mathrm{fr}}$, F induces a functor $F_X : \mathcal{E}_X \to \mathcal{E}'_X$.

(2) The functor F carries locally π-Cartesian morphisms of $\mathcal{E}$ to locally π'-Cartesian morphisms of $\mathcal{E}'$.

We let $\mathrm{Mor}(\mathcal{E}, \mathcal{E}')$ denote the full subcategory of $\mathrm{Fun}(\mathcal{E}, \mathcal{E}') \times_{\mathrm{Fun}(\mathcal{E}, \mathrm{Stone}^{\mathrm{fr}})} \{\pi\}$ consisting of morphisms of ultracategory fibrations from $\mathcal{E}$ to $\mathcal{E}'$.

We will be primarily interested in the following example:

Proposition 2. *Let $\mathcal{C}$ and $\mathcal{C}'$ be small pretopoi, and let $f : \mathcal{C} \to \mathcal{C}'$ be a pretopos morphism (that is, a functor which preserves finite limits, finite coproducts, and effective epimorphisms). If X is a topological space and $\mathcal{O}_X : \mathcal{C}' \to \mathrm{Shv}(X)$ is an X-model of $\mathcal{C}'$, then $\mathcal{O}_X \circ f : \mathcal{C} \to \mathrm{Shv}(X)$ is an X-model of $\mathcal{C}$. It follows that the construction $(X, \mathcal{O}_X) \to (X, \mathcal{O}_X \circ f)$ determines a functor $\widetilde{F} : \mathrm{Top}_{\mathcal{C}'} \to \mathrm{Top}_{\mathcal{C}}$. Then:*

(1) *The functor $\widetilde{F}$ carries $\mathrm{Stone}_{\mathcal{C}'}$ into $\mathrm{Stone}_{\mathcal{C}}$ and $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}'}$ into $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}$.*

(2) *The induced map $F : \mathrm{Stone}_{\mathcal{C}'} \to \mathrm{Stone}_{\mathcal{C}}$ is a morphism of ultracategory fibrations.*

Proof. It is clear that $\widetilde{F}$ carries $\mathrm{Stone}_{\mathcal{C}'}$ into $\mathrm{Stone}_{\mathcal{C}}$. Let us identify $\mathrm{Stone}^{\mathrm{op}}_{\mathcal{C}}$ and $\mathrm{Stone}^{\mathrm{op}}_{\mathcal{C}'}$ with the full subcategories of $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ and $\mathrm{Fun}(\mathcal{C}', \mathcal{S}\mathrm{et})$ spanned by those functors which preserve finite limits and effective epimorphisms. Under these identifications, the functor $\widetilde{F}|_{\mathrm{Stone}_{\mathcal{C}'}}$ is given by precomposition with f. It follows that $\widetilde{F}|_{\mathrm{Stone}_{\mathcal{C}'}} : \mathrm{Stone}_{\mathcal{C}'} \to \mathrm{Stone}_{\mathcal{C}}$ commutes with coproducts (since these correspond to products in $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ and $\mathrm{Fun}(\mathcal{C}', \mathcal{S}\mathrm{et})$). Since $\widetilde{F}$ carries $\mathrm{Mod}(\mathcal{C}')^{\mathrm{op}} \subseteq \mathrm{Stone}_{\mathcal{C}'}$ into $\mathrm{Mod}(\mathcal{C}) \subseteq \mathrm{Stone}_{\mathcal{C}}$, it restricts to a functor $F : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}'} \to \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}$. By construction, this functor fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Stone}_{\mathcal{C}'} & \xrightarrow{F} & \mathrm{Stone}_{\mathcal{C}} \\ & {\scriptstyle\pi'}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \mathrm{Stone}^{\mathrm{fr}} . & \end{array}$$

We claim that F carries locally π-Cartesian morphisms in $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}'}$ to locally π-Cartesian morphisms in $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}$. Note that a morphism $g : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ in the category $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}'}$ is locally π'-Cartesian if and only if, for every *isolated* point $x \in X$, the induced map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an isomorphism in $\mathrm{Mod}(\mathcal{C}')$. It then follows that $\mathcal{O}_{Y,f(x)} \circ f \to \mathcal{O}_{X,x} \circ f$ is an isomorphism of models of $\mathcal{C}$, so that $F(g)$ is locally π-Cartesian. $\square$

We can now give a precise statement of Makkai's theorem:

Theorem 3 (Strong Conceptual Completeness)**.** *Let $\mathcal{C}$ and $\mathcal{C}'$ be small pretopoi, and let* $\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{C}')$ *denote the category of pretopos morphisms from $\mathcal{C}$ to $\mathcal{C}'$. Then the preceding construction induces an equivalence of categories*

$$\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{C}') \to \mathrm{Mor}(\mathrm{Stone}_{\mathcal{C}'}, \mathrm{Stone}_{\mathcal{C}})^{\mathrm{op}}.$$

We will give the proof of Theorem 3 over the next two lectures. First, let us try to describe more concretely what it is saying. Let's return to the case of a general pair of ultracategory fibrations

$$\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}} \qquad \pi' : \mathcal{E}' \to \mathrm{Stone}^{\mathrm{fr}}.$$

Suppose we are given a continuous map $f : \beta I \to \beta J$ in $\mathrm{Stone}^{\mathrm{fr}}$. Then pullback along f induces functors

$$\mathcal{E}_{\beta J} \to \mathcal{E}_{\beta I} \qquad \mathcal{E}'_{\beta J} \to \mathcal{E}'_{\beta I},$$

both of which we will denote by f^*. Condition (2) guarantees that the diagram

$$\begin{array}{ccc} \mathcal{E}_{\beta J} & \xrightarrow{f^*} & \mathcal{E}_{\beta I} \\ \downarrow{\scriptstyle F_{\beta J}} & & \downarrow{\scriptstyle F_{\beta I}} \\ \mathcal{E}'_{\beta J} & \xrightarrow{f^*} & \mathcal{E}'_{\beta I}. \end{array}$$

Let $\mathcal{M} = \mathcal{E}_*^{\mathrm{op}}$ and $\mathcal{M}' = \mathcal{E}'^{\mathrm{op}}_*$ denote the underlying categories of $\mathcal{E}$ and $\mathcal{E}'$, respectively. Then F induces a functor $F_0 : \mathcal{M} \to \mathcal{M}'$. For every set I, we have equivalences

$$\mathcal{E}_{\beta I}^{\mathrm{op}} \simeq \mathcal{M}^I \qquad \mathcal{E}'^{\mathrm{op}}_{\beta I} \simeq \mathcal{M}'^I$$

which fit into a commutative diagram

$$\begin{array}{ccc} \mathcal{E}_{\beta I}^{\mathrm{op}} & \xrightarrow{F_{\beta I}} & \mathcal{E}'^{\mathrm{op}}_{\beta I} \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ \mathcal{M}^I & \xrightarrow{F_0^I} & \mathcal{M}'^I. \end{array}$$

Consequently, a morphism of ultracategory fibrations $F : \mathcal{E} \to \mathcal{E}'$ is largely determined by the underlying functor $F_0 : \mathcal{M} \to \mathcal{M}'$. However, the functor F_0 is not arbitrary: applying the preceding paragraph in the case where $I = *$ is a single point and J is an arbitrary set, (so that $f : \beta I \to \beta J$ determines an ultrafilter $\mathcal{U}$ on J), we deduce that F_0 "commutes with ultraproducts indexed by $\mathcal{U}$", in the sense that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}^J & \xrightarrow{F_0^J} & \mathcal{M}'^J \\ \downarrow{\scriptstyle P^{\mathcal{U}}} & & \downarrow{\scriptstyle P'^{\mathcal{U}}} \\ \mathcal{M} & \xrightarrow{F_0} & \mathcal{M}', \end{array}$$

where $P^{\mathcal{U}} : \mathcal{M}^I \to \mathcal{M}$ and $P'^{\mathcal{U}} : \mathcal{M}'^I \to \mathcal{M}'$ are the maps given by f^*. This motivates the following:

Definition 4. Let $\mathcal{M}$ and $\mathcal{M}'$ be ultracategories, equipped with functors

$$P^{\mathcal{U}} : \mathcal{M}^I \to \mathcal{M} \qquad P'^{\mathcal{U}} : \mathcal{M}'^I \to \mathcal{M}'$$

where $\mathcal{U}$ is an ultrafilter on a set I, together with isomorphisms

$$\epsilon_{I,i} : P^{\mathcal{U}} \simeq \mathrm{ev}_i \qquad \epsilon'_{I,i} : P'^{\mathcal{U}} \simeq \mathrm{ev}_i$$

when $\mathcal{U}$ is the principal ultrafilter associated to an element $i \in I$, and "diagonal" maps

$$\mu_{\mathcal{U},\mathcal{V}_\bullet} : P^{\mathcal{U} \wr \mathcal{V}_\bullet} \to P^{\mathcal{U}} \circ \{P^{\mathcal{V}_i}\}_{i \in I} \qquad \mu'_{\mathcal{U},\mathcal{V}_\bullet} : P'^{\mathcal{U} \wr \mathcal{V}_\bullet} \to P'^{\mathcal{U}} \circ \{P'^{\mathcal{V}_i}\}_{i \in I}.$$

when $\mathcal{U}$ is an ultrafilter on a set I and $\{\mathcal{V}_i\}_{i \in I}$ is an I-indexed collection of ultrafilters on a set J.

An *ultrafunctor from* $\mathcal{M}$ *to* $\mathcal{M}'$ consists of the following data:

- A functor $F_0 : \mathcal{M} \to \mathcal{M}'$.
- For every set I and every ultrafilter $\mathcal{U}$ on a set I, an isomorphism $\gamma_{\mathcal{U}} : P'^{\mathcal{U}} \circ F_0^I \simeq F_0 \circ P^{\mathcal{U}}$ of functors from $\mathcal{M}^I$ to $\mathcal{M}'$.

These isomorphisms are required to satisfy the following conditions:

- If $\mathcal{U}$ is the principal ultrafilter associated to an element $i \in I$, then the diagram

$$\begin{array}{ccc} P'^{\mathcal{U}} \circ F_0^I & \xrightarrow{\gamma_{\mathcal{U}}} & F_0 \circ P^{\mathcal{U}} \\ \downarrow{\scriptstyle \epsilon'_{I,i}} & & \downarrow{\scriptstyle \epsilon_{I,i}} \\ \mathrm{ev}_i \circ F_0^I & \xrightarrow{=} & F_0 \circ \mathrm{ev}_i \end{array}$$

  commutes (in the category of functors from $\mathcal{M}^I$ to $\mathcal{M}'$).

- If $\mathcal{U}$ is an ultrafilter on a set I and $\{\mathcal{V}_i\}_{i \in I}$ is an I-indexed collection of ultrafilters on a set J, then the diagram

$$\begin{array}{ccccc} P'^{\mathcal{U} \wr \mathcal{V}_\bullet} \circ F_0^J & & \xrightarrow{\gamma_{\mathcal{U} \wr \mathcal{V}_\bullet}} & & F_0 \circ P^{\mathcal{U} \wr \mathcal{V}_\bullet} \\ \downarrow{\scriptstyle \mu'_{\mathcal{U},\mathcal{V}_\bullet}} & & & & \downarrow{\scriptstyle \mu_{\mathcal{U},\mathcal{V}_\bullet}} \\ P'^{\mathcal{U}} \circ \{P'^{\mathcal{V}_i}\}_{i \in I} \circ F_0^J & \xrightarrow{\gamma_{\mathcal{V}_\bullet}} & P'^{\mathcal{U}} \circ F_0^I \circ \{P^{\mathcal{V}_i}\}_{i \in I} & \xrightarrow{\gamma_{\mathcal{U}}} & F_0 \circ P^{\mathcal{U}} \circ \{P^{\mathcal{V}_i}\}_{i \in I} \end{array}$$

  commutes (in the category of functors from $\mathcal{M}^J$ to $\mathcal{M}'$).

Given ultrafunctors $(F_0, \{\gamma_{\mathcal{U}}\})$ and $(F'_0, \{\gamma'_{\mathcal{U}}\})$ from $\mathcal{M}$ to $\mathcal{M}'$, we will say that a natural transformation $\rho : F_0 \to F'_0$ is a *morphism of ultrafunctors* if, for every ultrafilter $\mathcal{U}$ on a set I, the diagram of natural transformations

$$\begin{array}{ccc} P'^{\mathcal{U}} \circ F_0^I & \xrightarrow{\rho^I} & P'^{\mathcal{U}} \circ F_0'^I \\ \downarrow{\scriptstyle \gamma_{\mathcal{U}}} & & \downarrow{\scriptstyle \gamma'_{\mathcal{U}}} \\ F_0 \circ P^{\mathcal{U}} & \xrightarrow{\rho} & F'_0 \circ P^{\mathcal{U}}. \end{array}$$

commutes. We let $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{M}')$ denote the category whose objects are ultrafunctors from $\mathcal{M}$ to $\mathcal{M}'$ and whose morphisms are morphisms are ultrafunctors.

Warning 5. Makkai introduced a notion of ultrafunctor between ultracategories which is *a priori* more restrictive than our Definition 4: that is, Makkai requires a larger collection of diagrams to commute. However, the difference turns out to be irrelevant in the primary case of interest to us (where $\mathcal{M}$ is the category of models of a small pretopos $\mathcal{C}$), by virtue of Theorem 8 below.

Example 6. Let X and X' be compact Hausdorff spaces. We saw in the previous lecture that we can think of X and X' as *ultrasets*: that is, as ultracategories with only trivial morphisms. In this situation, Definition 4 simplifies considerably: an ultrafunctor from X to X' is simply a map of sets $F_0 : X \to X'$ with the following property: for every map of sets $f : I \to X$ and every ultrafilter $\mathcal{U}$ on I, we have $F_0(P^{\mathcal{U}}(f)) = P'^{\mathcal{U}}(F_0 \circ f)$

in X'. By general nonsense, it suffices to check this equality in the case where $I = X$ and f is the identity. It follows that F_0 is an ultrafunctor if and only if it is a morphism of β-algebras: that is, the diagram

$$\begin{array}{ccc} \beta X & \xrightarrow{\beta(F_0)} & \beta X' \\ \downarrow & & \downarrow \\ X & \xrightarrow{F_0} & X' \end{array}$$

commutes, where the vertical maps are the continuous extensions of id_X and $\mathrm{id}_{X'}$, respectively. This is equivalent to the requirement that F_0 is continuous.

Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ and $\pi' : \mathcal{E}' \to \mathrm{Stone}^{\mathrm{fr}}$ be ultracategory fibrations, with underlying ultracategories $\mathcal{M}$ and $\mathcal{M}'$, respectively. If $F : \mathcal{E} \to \mathcal{E}'$ is a morphism of ultracategory fibrations (in the sense of Definition 1), then it is not difficult to see that the underlying functor $F_0 : \mathcal{M} \to \mathcal{M}'$ has the structure of an ultrafunctor (with isomorphisms $\gamma_{\mathcal{U}}$ defined as in the discussion preceding Definition 4). Passage from F to F_0 determines a functor

$$\mathrm{Mor}(\mathcal{E}, \mathcal{E}') \to \mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{M}')^{\mathrm{op}}.$$

Moreover, just as an ultracategory fibration can be recovered (up to equivalence) from its underlying ultracategory, a morphism of ultracategory fibrations can be recovered (up to isomorphism) from its underlying ultrafunctor. More precisely, we can apply the analysis of Lecture 25X to obtain the following:

Proposition 7. *Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ and $\pi' : \mathcal{E}' \to \mathrm{Stone}^{\mathrm{fr}}$ be ultracategory fibrations with underlying ultracategories $\mathcal{M} = \mathcal{E}_*^{\mathrm{op}}$ and $\mathcal{M}' = \mathcal{E}'^{\mathrm{op}}_*$. Then the preceding construction induces an equivalence of categories*

$$\mathrm{Mor}(\mathcal{E}, \mathcal{E}')^{\mathrm{op}} \to \mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{M}').$$

Combining this result with Theorem 3, we obtain the following reformulation of Theorem 3:

Theorem 8 (Strong Conceptual Completeness)**.** *Let $\mathcal{C}$ and $\mathcal{C}'$ be small pretopoi and let $\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{C}')$ denote the category of pretopos morphisms from $\mathcal{C}$ to $\mathcal{C}'$. Then there is a canonical equivalence of categories*

$$\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{C}') \to \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}'), \mathrm{Mod}(\mathcal{C})).$$