Lecture 26X-Ultrasets

April 14, 2018

Our goal in this lecture is to unwind the definitions of the preceding lectures in a particularly simple case.

Definition 1 (Ultrasets-Version 1)**.** An *ultraset* is an ultracategory fibration $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ with the property that each fiber $\pi^{-1}\{\beta I\}$ is a category with only identity morphisms: that is, it is a set, regarded as a category.

Note that if $\pi : \mathcal{E} \to \mathcal{B}$ is any local Grothendieck fibration whose fibers are sets, then π is automatically a Grothendieck fibration (since every natural transformation between functors $\mathcal{E}_{B'} \to \mathcal{E}_B$ is automatically invertible). It is therefore a *fibration in sets*, which is classified by a functor $\mathcal{B}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, given by $B \mapsto \mathcal{E}_B$. In the case where $\mathcal{B} = \mathrm{Stone}^{\mathrm{fr}}$, such a fibration $\pi : \mathcal{E} \to \mathcal{B}$ is an ultracategory fibration if and only if the associated functor $\mathcal{B}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ preserves (possibly infinite) products. We can therefore reformulate Definition 1 as follows:

Definition 2 (Ultrasets-Version 2)**.** An *ultraset* is a functor $F : \mathrm{Stone}^{\mathrm{fr,op}} \to \mathcal{S}\mathrm{et}$ which carries coproducts in $\mathrm{Stone}^{\mathrm{fr}}$ to products in $\mathcal{S}\mathrm{et}$ (so that we have canonical isomorphisms $F(\beta I) = F(*)^I$).

We can obtain another equivalent formulation of Definition 2 using the notion of ultracategory introduced in Lecture 25X. Giving an ultraset (in the sense of either Definition 1 or 2) is equivalent to giving an ultracategory for which the underlying category $\mathcal{M}$ is a set. In this case, the definition of an ultracategory simplifies substantially:

Definition 3 (Ultrasets-Version 3)**.** An *ultraset* consists of the following data:

(1) A set M.

(2) For every set I and every ultrafilter $\mathcal{U}$ on I, a map of sets $P^{\mathcal{U}} : M^I \to M$.

This data is required to satisfy the following conditions:

(3) If $\mathcal{U}$ is the principal ultrafilter associated to an element $i \in I$, then $P^{\mathcal{U}} : M^I \to M$ is given by projection onto the ith factor.

(4) Given sets I and J, an ultrafilter $\mathcal{U}$ on I, and a collection $\{\mathcal{V}_i\}_{i \in I}$ of ultrafilters on J, we have an equality

$$P^{\mathcal{U} \wr \mathcal{V}_\bullet} = P^{\mathcal{U}} \circ \{P^{\mathcal{V}_i}\}_{i \in I}$$

of functions from M^J into M.

Suppose that M is an ultraset in the sense of Definition 3. Taking $I = M$ and $\mathrm{id}_M \in M^M$ to be the identity function, we see that the construction $\mathcal{U} \mapsto P^{\mathcal{U}}(\mathrm{id}_M)$ determines a function $r : \beta M \to M$. The structure of M as an ultraset is completely encoded by this function:

Lemma 4. *Let M be an ultraset (in the sense of Definition 3). Then, for any set I, any ultrafilter $\mathcal{U}$ on I, and any $f \in M^I$ (which we view as a function from I to M), we have*

$$P^{\mathcal{U}}(f) = u(f_* \mathcal{U}).$$

Here $f_* \mathcal{U}$ denotes the ultrafilter on M given by

$$(M_0 \in f_* \mathcal{U}) \Leftrightarrow (f^{-1} M_0 \in \mathcal{U}).$$

Proof. Set $J = M$. For each $i \in I$, let $\mathcal{V}_i$ be the principal ultrafilter on J associated to the element $f(i) \in M$. Axiom (3) then gives $P^{\mathcal{V}_i}(\mathrm{id}_M) = \mathrm{id}_M(f_i) = f(i)$. Note that the composite ultrafilter $\mathcal{U} \wr \mathcal{V}_\bullet$ can be identified with $f_* \mathcal{U}$. Applying axiom (4), we obtain

$$u(f_* \mathcal{U}) = P^{\mathcal{U} \wr \mathcal{V}_\bullet}(\mathrm{id}_M) = P^{\mathcal{U}}\{P^{\mathcal{V}_i}\{\mathrm{id}_M\}\}_{i \in I} = P^{\mathcal{U}}(f).$$

□

Proposition 5. *Let M be an ultraset (in the sense of Definition 3). Then the map $r : \beta M \to M$ defined above has the following properties:*

(*a*) *The restriction of r to M is the identity map id_M (where we identify M with a subset of βM).*

(*b*) *Let $\rho : \beta(\beta M) \to \beta M$ be the unique continuous map whose restriction to βM is the identity (where we identify βM with an open subset of $\beta(\beta M)$). Then we have a commutative diagram*

$$\begin{array}{ccc} \beta(\beta M) & \xrightarrow{\rho} & \beta M \\ \downarrow{\scriptstyle \beta(u)} & & \downarrow{\scriptstyle u} \\ \beta M & \xrightarrow{u} & M. \end{array}$$

Proof. Assertion (a) is an immediate consequence of axiom (3) of Definition 3. To prove (b) let us apply axiom (4) in the case where $I = \beta M$, $J = M$, the construction $i \mapsto \mathcal{V}_i$ is the identity (that is, it associates to each element of βM the corresponding ultrafilter on M). Let $\mathcal{U}$ be an ultrafilter on βM, which we can identify with a point of $\beta(\beta M)$. Then axiom (4) gives an equality

$$P^{\mathcal{U} \wr \mathcal{V}_\bullet}(\mathrm{id}_M) = P^{\mathcal{U}}\{P^{\mathcal{V}}(\mathrm{id}_M)\}_{\mathcal{V} \in \beta M} = P^{\mathcal{U}}(u)$$

Unwinding the definitions, we see that $\mathcal{U} \wr \mathcal{V}_\bullet$ coincides with $\rho(\mathcal{U})$ as an ultrafilter on M, so the left hand side is given by $r(\rho(\mathcal{U}))$. On the other hand, Lemma 4 allows us to rewrite the right hand side as $r(r_* \mathcal{U})$. □

Proposition **??** has a converse:

Proposition 6. *Let M be a set equipped with a map $r : \beta M \to M$ satisfying conditions (a) and (b) of Proposition 5. Then M inherits the structure of an ultraset, given by*

$$P^{\mathcal{U}}(f) = u(f_* \mathcal{U})$$

for $f \in M^I$ and $\mathcal{U} \in \beta I$.

Proof. We must show that $P^{\mathcal{U}}$ satisfies conditions (3) and (4) of Definition 3. Suppose first that $\mathcal{U}$ is a principal ultrafilter on a set I and that $f \in M^I$. Then $f_* \mathcal{U}$ is the principal ultrafilter on $f(i) \in M$, so that $\mathcal{P}^{\mathcal{U}}(f) = u(f_*(\mathcal{U})) = u(f(i)) = f(i)$ by virtue of (a).

Now suppose we are given sets I and J and a collection of ultrafilters $\{\mathcal{V}_i\}_{i \in I}$ on the set J, so that the construction $i \mapsto \mathcal{V}_i$ determines a map of sets $g : I \to \beta J$. We wish to show that we have an equality

$$P^{\mathcal{U} \wr \mathcal{V}_\bullet}(f) = P^{\mathcal{U}}\{P^{\mathcal{V}_i}(f)\}_{i \in I}.$$

for each ultrafilter $\mathcal{U}$ on the set I. Unwinding the definitions, we see that the left hand side is given (as a function of $\mathcal{U}$) by the composition

$$\beta I \xrightarrow{\beta g} \beta\beta J \xrightarrow{\beta\beta f} \beta\beta M \xrightarrow{\rho} \beta M \xrightarrow{u} M$$

while the right hand side is given by

$$\beta I \xrightarrow{\beta g} \beta\beta J \xrightarrow{\beta f_*} \beta\beta M \xrightarrow{\beta(u)} \beta M \xrightarrow{u} M;$$

these maps coincide by virtue of property (b). □

We leave it to the reader to check that for a fixed set M, these constructions establish mutually inverse bijections

$$\{\text{Functions } P^{\mathcal{U}} : M^I \to M \text{ satisfying Definition 3}\} \simeq \{\text{Functions } r : \beta M \to M \text{ satisfying } (a) \text{ and } (b)\}.$$

This leads to another formulation:

Definition 7 (Ultrasets-Version 4)**.** An *ultraset* is a set M equipped with a map $r : \beta M \to M$ satisfying conditions (a) and (b) of Proposition 5.

Let us give another interpretation of this definition. Let $\mathrm{Top}_{\mathrm{ch}}$ denote the category of compact Hausdorff topological spaces and continuous maps. This category is equipped with a forgetful functor $G : \mathrm{Top}_{\mathrm{ch}} \to \mathcal{S}\mathrm{et}$. The forgetful functor G has a left adjoint F, which associates to each set I its Stone-Čech compactification βI, regarded as a topological space. The composition $\beta = G \circ F$ is the functor which associates to each set I the Stone-Čech compactification βI, regarded merely as a set. This realization endows β with the structure of a *monad*: that is, as an associative algebra in the category $\mathrm{Fun}(\mathcal{S}\mathrm{et}, \mathcal{S}\mathrm{et})$ of endofunctors of $\mathcal{S}\mathrm{et}$. Explicitly, this algebra structure is encoded by a unit map

$$u : \mathrm{id}_{\mathcal{S}\mathrm{et}} \to \beta \qquad \rho : \beta \circ \beta \to \beta.$$

Unwinding the definitions, we see that u associates to each set M the canonical inclusion $M \hookrightarrow \beta M$, while ρ associates to each set M the map $\beta\beta M \to \beta M$ appearing in Proposition 5. We can therefore restate Definition 7 in the language of monads:

Definition 8 (Ultrasets-Version 5)**.** An *ultraset* is an algebra over the monad $\beta : \mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$.

By general nonsense, the forgetful functor $G : \mathrm{Top}_{\mathrm{ch}} \to \mathcal{S}\mathrm{et}$ factors through the category of algebras for the monad β. In other words, every compact Hausdorff space X can be regarded as an ultraset, taking $r : \beta X \to X$ to be the unique map which agrees with the identity on X and is continuous (where βX is regarded as the Stone-Čech compactifiction of X as a discrete set, and the right hand side uses the compact Hausdorff topology on X).

Theorem 9. *The adjunction* $\mathcal{S}\mathrm{et} \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Top}_{\mathrm{ch}}$ *is monadic. In other words, it induces an equivalence of categories*

$$\{\textit{Compact Hausdorff spaces}\} \xrightarrow{\sim} \{\textit{Ultrasets}\}.$$

By virtue of Theorem 9, we obtain yet another formulation of the notion of an ultraset:

Definition 10 (Ultrasets-Version 6)**.** An *ultraset* is a compact Hausdorff space.

Remark 11. We can make the equivalence of Definitions 10 and 2 explicit, without passing through the theory of ultracategories. Note that every topological space X represents a functor

$$F_X : \mathrm{Stone}^{\mathrm{fr,op}} \to \mathcal{S}\mathrm{et},$$

given by $F_X(\beta I) = \mathrm{Hom}_{\mathrm{Top}}(\beta I, X)$. If X is compact Hausdorff, then the functor F_X carries coproducts in $\mathrm{Stone}^{\mathrm{fr}}$ to products of sets, and is therefore an ultraset in the sense of Definition 2.

Proof Sketch of Theorem 9. We will explicitly describe an inverse to the functor

$$\{\text{Compact Hausdorff spaces}\} \to \{\text{Ultrasets}\}.$$

Let X be an ultraset: that is, a set equipped with a map $r : \beta X \to X$ satisfying the requirements of Proposition 5. We wish to show that X can be equipped with the structure of a compact Hausdorff space for which the map r is continuous (in which case r will be the unique continuous extension of the identity map $\mathrm{id}_X : X \to X$, by the universal property of βX). Note that r determines an equivalence relation $R \subseteq (\beta X) \times_X (\beta X)$, given by

$$((x, y) \in R) \Leftrightarrow (r(x) = r(y)).$$

Since r is surjective, we can identify X with the quotient $(\beta X)/R$. Let us endow X with the quotient topology, so that X is automatically compact and r is automatically continuous. The only nontrivial point is to verify that this topology is Hausdorff: that is, that the equivalence relation R is closed.

Consider the projection maps $\pi, \pi' : R \to \beta X$, and set $Y = \beta R$. Note that π and π' extend uniquely to continuous maps $\overline{\pi}, \overline{\pi}' : \beta R \to \beta X$. The pair

$$(\overline{\pi}, \overline{\pi}') : \beta R \to \beta X \times \beta X$$

is a continuous map of compact Hausdorff spaces, and therefore has closed image. Clearly, this image contains R. We will complete the proof by showing that the image is exactly R: that is, that the diagram

$$\begin{array}{ccc} \beta R & \xrightarrow{\overline{\pi}} & \beta X \\ \downarrow{\scriptstyle \overline{\pi'}} & & \downarrow{\scriptstyle r} \\ \beta X & \xrightarrow{r} & X \end{array}$$

commutes. Note that the maps $r \circ \overline{\pi}$ and $r \circ \overline{\pi}'$ are maps from βR to X in the category of algebras for the monad β. Consequently, to show that they coincide, it will suffice to show that they coincide when restricted to R (since βR is the free β-algebra generated by R). We are therefore reduced to proving the commutativity of the diagram

$$\begin{array}{ccc} R & \xrightarrow{\pi} & \beta X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle r} \\ \beta X & \xrightarrow{r} & X, \end{array}$$

which is immediate from the definition of R. $\square$