Lecture 25X-Ultracategories

April 13, 2018

We begin by recalling the following definition from Lecture 24X:

Definition 1. An *ultracategory fibration* is a category $\mathcal{E}$ together with a functor $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ with the following properties:

(1) The functor π is a local Grothendieck fibration.

(2) Let I be a set and let $f_i : \{i\} \hookrightarrow \beta I$ denote the inclusion map for each $i \in I$. Then the construction

$$(M \in \mathcal{E}_{\beta I}) \mapsto \{f_i^* M \in \mathcal{E}_{\{i\}}\}_{i \in I}$$

induces an equivalence of categories

$$\mathcal{E}_{\beta I} \to \prod_{i \in I} \mathcal{E}_{\{i\}}.$$

(3) Let $g : \beta I \to \beta J$ and $f : \beta J \to \beta K$ be maps in $\mathrm{Stone}^{\mathrm{fr}}$, and suppose that g carries I into J. Then the natural transformation $g^* \circ f^* \to (f \circ g)^*$ is an equivalence of functors from $\mathcal{M}_{\beta K}$ to $\mathcal{E}_{\beta I}$.

In this lecture, we will describe the structure of an arbitrary ultracategory fibration $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$.

Notation 2. Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ be an ultracategory fibration. We let $\mathcal{M}$ denote the fiber product $(\mathcal{E} \times_{\mathrm{Stone}^{\mathrm{fr}}} \{*\})^{\mathrm{op}}$. We will refer to $\mathcal{M}$ as the *underlying category* of the ultracategory fibration π.

Example 3. Let $\mathcal{C}$ be a small pretopos. Then the underlying category of the ultracategory fibration $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}} \to \mathrm{Stone}^{\mathrm{fr}}$ is the category $\mathrm{Mod}(\mathcal{C})$ of models of $\mathcal{C}$.

Let's now return to the general case. Let $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ be an ultracategory fibration with underlying category $\mathcal{M}$. For every set I, assumption (2) of Definition 1 supplies an equivalence of categories

$$\gamma_I : \mathcal{E}^{\mathrm{op}}_{\beta I} \xrightarrow{\sim} (\mathcal{M}^I).$$

Let $\mathcal{U}$ be an ultrafilter on I, which we can identify with a point of βI. Then $\mathcal{U}$ determines a map of spaces $* \to \beta I$, which gives rise to a pullback functor $\psi_{\mathcal{U}} : \mathcal{E}^{\mathrm{op}}_{\beta I} \to \mathcal{E}^{\mathrm{op}}_{\{\mathcal{U}\}} \simeq \mathcal{M}$. We let $P^{\mathcal{U}} : \mathcal{M}^I \to \mathcal{M}$ denote the functor given by the composition $\psi_{\mathcal{U}} \circ \gamma_I^{-1}$.

Example 4. Let $\mathcal{C}$ be a small pretopos and let $\pi : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}} \to \mathrm{Stone}^{\mathrm{fr}}$ be the forgetful functor. Then, for any set I and any ultrafilter $\mathcal{U}$ on I, the functor $P^{\mathcal{U}} : \mathrm{Mod}(\mathcal{C})^I \to \mathrm{Mod}(\mathcal{C})$ is given by

$$P^{\mathcal{U}}(\{M_i\}_{i \in I}) = (\prod_{i \in I} M_i)/\,\mathcal{U}.$$

We again return to the general case. Suppose we are given a continuous map $f : \beta I \to \beta J$, given by a collection of ultrafilters $\{\mathcal{U}_i\}_{i \in I}$ on the set J. Suppose we are given a pair of objects $E \in \mathcal{E}_{\beta I}$ and $E' \in \mathcal{E}_{\beta J}$, having images

$$\gamma_I(E) = \{M_i \in \mathcal{M}\}_{i \in I} \qquad \gamma_J(E') = \{N_j \in \mathcal{M}\}_{j \in J}.$$

Let's try to describe the set

$$\mathrm{Hom}_{\mathcal{E}}(E, E') \times_{\mathrm{Hom}_{\mathrm{Stone}^{\mathrm{fr}}}(\beta I, \beta J)} \{f\}.$$

We then have canonical bijections

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{E}}(E, E') \times_{\mathrm{Hom}_{\mathrm{Stone}^{\mathrm{fr}}}(\beta I, \beta J)} \{f\} &\simeq \mathrm{Hom}_{\mathcal{E}_{\beta I}}(E, f^* E') \\
&\simeq \mathrm{Hom}_{\mathcal{M}^I}(\gamma_I(f^* E'), \gamma_I(E)) \\
&\simeq \prod_{i \in I} \mathrm{Hom}_{\mathcal{M}}(P^{\mathcal{U}_i}\{N_j\}_{j \in J}, M_i);
\end{aligned}$$

Here we are using condition (3) of Definition 1 to identify $\gamma_I(f^* E')$ with the tuple $\{P^{\mathcal{U}_i}\gamma_J(E')\}_{i \in I}$.

By virtue of this calculation, we can attempt to reconstruct the category $\mathcal{E}$ (up to equivalence) from the data of the category $\mathcal{M}$ and the functors $P^{\mathcal{U}} : \mathcal{M}^I \to \mathcal{M}$. Let's attempt to define a category $\overline{\mathcal{E}}$ as follows:

- The objects of $\overline{\mathcal{E}}$ are pairs $(I, \{M_i\}_{i \in I})$, where I is a set and $\{M_i\}_{i \in I}$ is a family of objects of $\mathcal{M}$ indexed by I.
- Given a pair of objects $(I, \{M_i\}_{i \in I})$ and $(J, \{N_j\})_{j \in J}$, we set

  $$\mathrm{Hom}_{\overline{\mathcal{E}}}((I, \{M_i\}_{i \in I}), (J, \{N_j\})_{j \in J}) = \coprod_{f : \beta I \to \beta J} \prod_{i \in I} \mathrm{Hom}_{\mathcal{M}}(P^{\mathcal{U}_i}\{N_j\}_{j \in J}, M_i)$$

  where the coproduct is taken over all continuous maps $f : \beta I \to \beta J$, which we identify with families of ultrafilters $\{\mathcal{U}_i\}_{i \in I}$ on the set J.

By virtue of the above discussion, choosing an inverse γ_I^{-1} to each of the functors γ_I gives a construction F

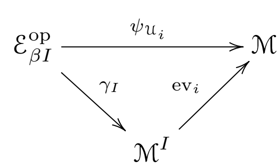
$$(I, \{M_i\}) \mapsto \gamma_I^{-1}\{M_i\}$$

which carries objects of $\overline{\mathcal{E}}$ to objects of $\mathcal{E}$, and we have canonical bijections

$$\mathrm{Hom}_{\overline{\mathcal{E}}}(\overline{E}, \overline{E}') = \mathrm{Hom}_{\mathcal{E}}(F(\overline{E}), F(\overline{E}'))$$

for every pair of objects $\overline{E}, \overline{E}' \in \overline{\mathcal{E}}$. It follows that there is a unique composition law on $\overline{\mathcal{E}}$ for which F is a functor (and therefore an equivalence of categories). We now give an explicit description of this structure in terms of the functors $P^{\mathcal{U}} : \mathcal{M}^I \to \mathcal{M}$.

Remark 5 (Identity Morphisms)**.** Let I be a set containing an element i, and let $\mathcal{U}_i$ denote the principal ultrafilter determined by the element i. By construction, the diagram of categories



commutes up to canoncial isomorphism, where ev_i is the functor given by evaluation on the ith coordinate. It follows that there is a canonical isomorphism

$$\epsilon_{I,i} : P^{\mathcal{U}_i} \simeq \mathrm{ev}_i$$

of functors from $\mathcal{M}^I$ to $\mathcal{M}$.

For any collection of objects $\{M_i\}_{i \in I}$, the identity morphism from $(I, \{M_i\}_{i \in I})$ to itself in $\overline{\mathcal{E}}$ is encoded by the family of maps

$$\{\epsilon_{I,i}(\{M_j\}_{j \in I} : P^{\mathcal{U}_i}\{M_j\}_{j \in I} \simeq M_i\}_{i \in I}.$$

Remark 6 (Composition)**.** Let $f : \beta J \to \beta K$ be a morphism in $\mathrm{Stone}_{\mathcal{C}}$, given by a collection $\{\mathcal{V}_j\}_{j\in J}$ of ultrafilters on K. Suppose we are given an ultrafilter $\mathcal{U}$ on J, which we can identify with a map $g : * \to \beta J$. Let $\mathcal{U} \wr \mathcal{V}_\bullet$ denote the ultrafilter on K given by the composition $f \circ g$, so that

$$(K_0 \in \mathcal{U} \wr \mathcal{V}_\bullet) \Leftrightarrow (\{j \in J : K_0 \in \mathcal{V}_j\} \in \mathcal{U})$$

Then we have a natural transformation of functors $g^* \circ f^* \to (f \circ g)^*$ from $\mathcal{E}_{\beta K}$ to $\mathcal{E}_*$. Passing to opposite categories and composing with the equivalence γ_K, we obtain a map

$$P^{\mathcal{U} \wr \mathcal{V}_\bullet}\{M_k\}_{k\in K} \to P^{\mathcal{U}}\{P^{\mathcal{V}_j}\{M_k\}_{k\in K}\}_{j\in J}$$

depending functorially on $\{M_k\}_{k\in K} \in \mathcal{M}^K$; we will write this as a natural transformation

$$\mu_{\mathcal{U},\mathcal{V}_\bullet} : P^{\mathcal{U} \wr \mathcal{V}_\bullet} \to P^{\mathcal{U}} \circ \{P^{\mathcal{V}_j}\}_{j\in J}.$$

Using the natural transformations $\mu_{\mathcal{U},\mathcal{V}_\bullet}$, we can describe the composition of morphisms in the category $\overline{\mathcal{E}}$. Let $f : \beta J \to \beta K$ be as above, and suppose we are given another map $g : \beta I \to \beta J$, given by a collection of ultrafilters $\{\mathcal{U}_i\}_{i\in I}$ on the set J. For each $i \in I$, let $\mathcal{U}_i \wr \mathcal{V}_\bullet \in \beta K$ be the image of $\mathcal{U}_i$ under the map f. Suppose that we lift f and g to morphisms

$$\overline{g} : (I, \{M_i\}_{i\in I}) \to (J, \{M'_j\}_{j\in J})$$

$$\overline{f} : (J, \{M'_j\}_{j\in J}) \to (K, \{M''_k\}_{k\in K})$$

in the category $\overline{\mathcal{E}}$. Then $\overline{f}$ is given by specifying a collection of maps

$$\{\overline{f}_j : P^{\mathcal{V}_j}\{M''_k\}_{k\in K} \to M'_j\}_{j\in J},$$

in the category $\mathcal{M}$, and $\overline{g}$ is given by specifying a family of maps

$$\{\overline{g}_i : P^{\mathcal{U}_i}\{M'_j\}_{j\in J} \to M_i\}_{i\in I}$$

in the category $\mathcal{M}$. Unwinding the definitions, we see that the composition $\overline{f} \circ \overline{g}$ in $\mathcal{E}$ is encoded by the family of composite maps

$$P^{\mathcal{U}_i \wr \mathcal{V}_\bullet}\{M''_k\}_{k\in K} \xrightarrow{\mu_{\mathcal{U}_i,\mathcal{V}_\bullet}} P^{\mathcal{U}_i}\{P^{\mathcal{V}_j}\{M''_k\}_{k\in K}\}_{j\in J} \xrightarrow{\overline{f}_j} P^{\mathcal{U}_i}\{M'_j\}_{j\in J} \xrightarrow{\overline{g}_i} M_i.$$

It follows from the above discussion that all of the data needed to construct the category $\overline{\mathcal{E}}$ is encoded by the functors $P^{\mathcal{U}} : \mathcal{M}^I \to \mathcal{M}$, the natural transformations $\epsilon_{I,i}$ (which encode identity morphisms in $\overline{\mathcal{E}}$), and the natural transformations $\mu_{\mathcal{U},\mathcal{V}_\bullet}$. This motivates the following definition, which is a variant of a notion introduced by Makkai:

Definition 7 (Ultracategories)**.** An *ultracategory* consists of the following data:

(1) A category $\mathcal{M}$.

(2) For every set I and every ultrafilter $\mathcal{U}$ on I, a functor $P^{\mathcal{U}} : \mathcal{M}^I \to \mathcal{M}$.

(3) For every set I and every element $i \in I$, an isomorphism of functors $\epsilon_{I,i} : P^{\mathcal{U}_i} \simeq \mathrm{ev}_i$, where $\mathcal{U}_i$ denotes the principal ultrafilter associated to i (and $\mathrm{ev}_i : \mathcal{M}^I \to \mathcal{M}$ is given by projection onto the ith factor).

(4) For every pair of sets I and J, every ultrafilter $\mathcal{U}$ on I, and every family $\{\mathcal{V}_i\}_{i\in I}$ of ultrafilters on J, a natural transformation
$$\mu_{\mathcal{U},\mathcal{V}_\bullet} : P^{\mathcal{U} \wr \mathcal{V}_\bullet} \to P^{\mathcal{U}} \circ \{P^{\mathcal{V}_i}\}_{i\in I}$$
of functors from $\mathcal{M}^J$ to $\mathcal{M}$.

These maps are required to satisfy the following axioms:

(A) In the situation of (4), suppose that $\mathcal{U}$ is the principal ultrafilter associated to some element $i_0 \in I$, so that $\mathcal{U} \wr \mathcal{V}_\bullet = \mathcal{V}_{i_0}$. Then, for any collection of objects $\{M_j\}_{j\in J}$, we have a commutative diagram

$$\begin{array}{ccc} P^{\mathcal{U}\wr\mathcal{V}_\bullet}\{M_j\}_{j\in J} & \xrightarrow{\mu_{\mathcal{U},\mathcal{V}_\bullet}} & P^{\mathcal{U}}\{P^{\mathcal{V}_j}\{M_j\}_{j\in J}\}_{i\in I} \\ & \searrow^{=} \quad \swarrow_{\epsilon_{I,i_0}} & \\ & P^{\mathcal{V}_{i_0}}\{M_j\}_{j\in J} & \end{array}$$

(B) In the situation of (4), suppose that $I = J$ and that each $\mathcal{V}_i$ is the principal ultrafilter associated to i, so that $\mathcal{U} \wr \mathcal{V}_\bullet = \mathcal{U}$. Then, for any collection of objects $\{M_j\}_{j\in J}$, we have a commutative diagram

$$\begin{array}{ccc} P^{\mathcal{U}\wr\mathcal{V}_\bullet}\{M_j\}_{j\in J} & \xrightarrow{\mu_{\mathcal{U},\mathcal{V}_\bullet}} & P^{\mathcal{U}}\{P^{\mathcal{V}_j}\{M_j\}_{j\in J}\}_{i\in I} \\ & \searrow^{=} \quad \swarrow_{\prod \epsilon_{J,\bullet}} & \\ & P^{\mathcal{U}}\{M_i\}_{i\in I}. & \end{array}$$

(C) Suppose we are given a diagram $* \xrightarrow{f} \beta I \xrightarrow{g} \beta J \xrightarrow{h} \beta K$ in $\mathrm{Stone}^{\mathrm{fr}}$, corresponding to an ultrafilter $\mathcal{U}$ on I, a collection of ultrafilters $\{\mathcal{V}_i\}_{i\in I}$ on J and a collection of ultrafilters $\{\mathcal{W}_j\}_{j\in J}$ on K. Then, for every collection $\{M_k\}_{k\in K}$, we have a commutative diagram

$$\begin{array}{ccc} P^{\mathcal{U}\wr\mathcal{V}_\bullet\wr\mathcal{W}_\bullet}\{M_k\}_{k\in K} & \xrightarrow{\mu_{\mathcal{U},\mathcal{V}_\bullet\wr\mathcal{W}_\bullet}} & P^{\mathcal{U}}\{P^{\mathcal{V}_i\wr\mathcal{W}_\bullet}\{M_k\}_{k\in K}\}_{i\in I} \\ \downarrow{\scriptstyle \mu_{\mathcal{U}\wr\mathcal{V}_\bullet,\mathcal{W}_\bullet}} & & \downarrow{\scriptstyle \mu_{\mathcal{V}_i,\mathcal{W}_\bullet}} \\ P^{\mathcal{U}\wr\mathcal{V}_\bullet}\{P^{\mathcal{W}_j}\{M_k\}_{k\in K}\}_{j\in J} & \xrightarrow{\mu_{\mathcal{U},\mathcal{V}_\bullet}} & P^{\mathcal{U}}\{P^{\mathcal{V}_i}\{P^{\mathcal{W}_j}\{M_k\}_{k\in K}\}_{j\in J}\}_{i\in I}. \end{array}$$

Remark 8. Roughly speaking, we can think of an ultracategory as a category $\mathcal{M}$ equipped with a notion of "how to take ultraproducts of objects in $\mathcal{M}$": that is, given a collection of objects $\{M_i\}_{i\in I}$ and an ultrafilter $\mathcal{U}$ on the index category I, we can form a new object $P^{\mathcal{U}}\{M_i\}_{i\in I} \in \mathcal{M}$ which we think of as the ultraproduct of the objects M_i with respect to the ultrafilter $\mathcal{U}$. Datum (3) asserts that an ultraproduct indexed by a *principal* ultrafilter returns one of the objects that we started with, and datum (4) encodes natural comparison maps from ultraproducts to iterated ultraproducts (such as a "diagonal" map from any object M to the ultrapower $P^{\mathcal{U}}\{M\}_{i\in I}$).

Given an ultracategory $\mathcal{M}$, we can define a category $\overline{\mathcal{E}}$ with objects and morphisms defined above, where the identity morphisms are determined by the natural isomorphisms $\epsilon_{I,i}$ and the composition is determined by the morphisms $\mu_{\mathcal{U},\{\mathcal{V}_i\}}$. Conditions ($A$), ($B$), and ($C$) are exactly what is needed to guarantee that the resulting composition law is unital (on both sides) and associative. Moreover, the construction

$$(I, \{M_i\}_{i\in I}) \mapsto \beta I$$

determines a forgetful functor $\pi : \overline{\mathcal{E}} \to \mathrm{Stone}^{\mathrm{fr}}$, which is *essentially* a local Grothendieck fibration. More precisely, it defines a local Grothendieck fibration from $\overline{\mathcal{E}}$ to the full subcategory of $\mathrm{Stone}^{\mathrm{fr}}$ spanned by spaces which are identical to (rather than merely homeomorphic to) βI, for some set I. Moreover, the functor $\overline{\pi}$ satisfies the obvious analogues of conditions (2) and (3) of Definition 1. We can summarize the situation informally as follows:

Proposition 9. *The constructions of this lecture establish an equivalence between the following data:*

- *Ultracategory fibrations* $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ *(in the sense of Definition 1).*
- *Ultracategories* $\mathcal{M}$ *(in the sense of Definition 7).*