Lecture 23X-Compatibility with Filtered Colimits

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Let $\mathcal{C}$ be a small pretopos, which we regard as fixed through this lecture. We have fully faithful embeddings

$$\mathcal{C} \hookrightarrow \mathrm{Shv}(\mathcal{C}) \hookrightarrow \mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \simeq \mathrm{Shv}(\mathrm{Stone}_{\mathcal{C}}) \simeq \mathrm{Shv}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}) \subseteq \mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr},\mathrm{op}}, \mathcal{S}\mathrm{et}).$$

Our goal in this lecture is to prove the following:

Theorem 1. *Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr},\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a functor. The $\mathscr{F}$ belongs to the essential image of $\mathcal{C}$ if and only if it satisfies the following pair of conditions:*

(a') *For every collection of models $\{M_i\}_{i \in I}$ of $\mathcal{C}$, the canonical map*

$$\mathscr{F}(\coprod_{i \in I}(\{i\}, M_i)) \to \prod_{i \in I} \mathscr{F}(\{i\}, M_i)$$

is a bijection.

(b') *For every object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$ and every point $x \in X$, the canonical map $\varinjlim_{x \in U} \mathscr{F}(U, \mathcal{O}_X|_U) \to \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$ is a bijection. Here U ranges over all clopen neighborhoods of x in X.*

We have already seen that conditions (a') and (b') are necessary. Moreover, we proved in Lecture 22X that conditions (a') and (b') imply that $\mathscr{F}$ is a sheaf on the category $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$, and therefore admits an essentially unique extension to a sheaf on the entire category $\mathrm{Stone}_{\mathcal{C}}$. Consequently, to show that $\mathscr{F}$ belongs to the essential image of $\mathrm{Shv}(\mathcal{C})$, it will suffice to show that this extension commutes with filtered colimits (Lecture 15X). In this case, we have seen that condition (a') guarantees that $\mathscr{F}$ also belongs to the essential image of $\mathcal{C}$ (Lecture 21X). We may therefore reformulate Theorem 1 as follows:

Theorem 2. *Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a sheaf and suppose that $\mathscr{F}|_{\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr},\mathrm{op}}}$ satisfies conditions (a') and (b') of Theorem 1. Then $\mathscr{F}$ preserves filtered colimits (that is, it carries filtered limits in $\mathrm{Stone}_{\mathcal{C}}$ to filtered colimits in $\mathcal{S}\mathrm{et}$). Using the criterion of Lecture 17X, we can state this more concretely as follows:*

(b) *For every object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$ and every point $x \in X$, the canonical map*

$$\varinjlim_{x \in U} \mathscr{F}(U, \mathcal{O}_X|_U) \to \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$$

is bijective; here the colimit is taken over all clopen neighborhoods $U \subseteq X$ of the point x.

(c) *The composite functor*

$$\mathrm{Mod}(\mathcal{C}) \hookrightarrow \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \xrightarrow{\mathscr{F}} \mathcal{S}\mathrm{et}$$

commutes with filtered colimits.

Proof. We begin by proving (c). Fix a diagram of models $\{M_\alpha\}_{\alpha \in I}$ indexed by a directed partially ordered set I. Set $M = \varinjlim_{\alpha \in I} M_\alpha$; we wish to show that the canonical map

$$\rho : \varinjlim_{\alpha} \mathscr{F}(M_\alpha) \to \mathscr{F}(M)$$

is bijective.

As in Lecture 22X, we can choose an ultrafilter $\mathcal{U}$ on the set I such that, for each $\beta \in I$, the subset $I_{\geq \beta} = \{\alpha \in I : \alpha \geq \beta\}$ is contained in $\mathcal{U}$. For each object $C \in \mathcal{C}$ and each index $\beta \in I$, the transition maps in the diagram $\{M_\alpha\}_{\alpha \in I}$ determine a canonical map

$$M_\beta(C) \to \prod_{\alpha \in I_{\geq \beta}} M_\alpha(C) \to \varinjlim_{J \in \mathcal{U}} \prod_{\alpha \in J} M_\alpha(C) = ((\prod_{\alpha \in I} M_\alpha)/\mathcal{U})(C).$$

Passing to the colimit over β, we obtain a map $M(C) \to ((\prod_{\alpha \in I} M_\alpha)/\mathcal{U})(C)$ depending functorially on C, which we can identify with a map of models $f : M \to (\prod_{\alpha \in I} M_\alpha)/\mathcal{U}$

Note that the canonical maps $M_\alpha \to M$ induce a map g from $(\prod_{\alpha \in I} M_\alpha)/\mathcal{U}$ to the ultrapower $M^I/\mathcal{U}$. By construction, the composition $M \xrightarrow{f} (\prod_{\alpha \in I} M_\alpha)/\mathcal{U} \xrightarrow{g} M^I/\mathcal{U}$ agrees with the diagonal map $\delta_M : M \to M^I/\mathcal{U}$.

We now prove the surjectivity of ρ. Suppose we are given an element $x \in \mathscr{F}(M)$. Recall that conditions (a') and (b') imply that $\mathscr{F}$ commutes with the formation of ultraproducts (see Lecture 22X). We may therefore identify $\mathscr{F}(f)(x) \in \mathscr{F}((\prod_{\alpha \in I} M_\alpha)/\mathcal{U})$ with an element of $(\prod_{\alpha \in I} \mathscr{F}(M_\alpha))/\mathcal{U}$, which we can represent by a tuple of elements $\{x_\alpha \in \mathscr{F}(M_\alpha)\}_{\alpha \in J}$ for some subset $J \subseteq I$ which belong to the ultrafilter $\mathcal{U}$. Each x_α has some image y_α in $\mathscr{F}(M)$, and we can identify $\{y_\alpha\}_{\alpha \in J}$ with the image of x under the composite map

$$\mathscr{F}(M) \to \mathscr{F}((\prod_{\alpha \in I} M_\alpha)/\mathcal{U}) \to \mathscr{F}(M^I/\mathcal{U}).$$

Since this composite map agrees with the diagonal δ_M, the equality $y_\alpha = x$ must hold almost everywhere: that is, we can choose some $J' \subseteq J$ belonging to $\mathcal{U}$ such that $y_\alpha = x$ for $\alpha \in J'$. Then x belongs to the image of the map $\mathscr{F}(M_\alpha) \to \mathscr{F}(M)$ for each $\alpha \in J'$, and in particular belongs to the image of ρ.

We now show that ρ is injective. Fix an index $\beta \in I$ and a pair of elements $x, y \in \mathscr{F}(M_\beta)$ having the same image in $\mathscr{F}(M)$; we wish to show that x and y have the same image in $\mathscr{F}(M_\alpha)$ for some $\alpha \geq \beta$. Replacing I by the set $I_{\geq \beta}$, we can assume without loss of generality that β is a least element of I. Note that we have a commutative diagram of models

$$\begin{array}{ccc} M_\beta & \xrightarrow{\delta} & M_\beta^I/\mathcal{U} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & (\prod_{\alpha \in I} M_\alpha)/\mathcal{U}. \end{array}$$

It follows that x and y have the same image under the composite map

$$\mathscr{F}(M_\beta) \to \mathscr{F}(M_\beta^I/\mathcal{U}) \to \mathscr{F}((\prod_{\alpha \in I} M_\alpha)/\mathcal{U}) \simeq (\prod_{\alpha \in I} \mathscr{F}(M_\alpha))/\mathcal{U}.$$

In other words, x and y have the same image in $\mathscr{F}(M_\alpha)$ for almost every $\alpha \in I$; this completes the proof of (c).

We now prove (b). Fix an object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$ and a point $x \in X$; we wish to show that the canonical map

$$\phi : \varinjlim_{x \in U} \mathscr{F}(U, \mathcal{O}_X|_U) \to \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$$

is bijective. We first show that ϕ is injective. Suppose we are given a clopen subset $U \subseteq X$ containing x and a pair of elements $u, v \in \mathscr{F}(U, \mathcal{O}_X|_U)$ with the same image in $\mathscr{F}(\{x\}, \mathcal{O}_{X,x})$. We wish to show that u and v have the same image $\mathscr{F}(U', \mathcal{O}_X|_{U'})$ for some clopen subset $U' \subseteq U$ containing x. Choose a covering $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$, where $(Y, \mathcal{O}_Y)$ is free. Then u and v have the same image in $\mathscr{F}(\{y\}, \mathcal{O}_{Y,y})$ for each point $y \in Y_x = f^{-1}\{x\}$. Using condition (b'), we see that u and v have the same image in $\mathscr{F}(V_y, \mathcal{O}_Y|_{V_y})$ for some clopen neighborhood V_y of y. Covering the fiber Y_x by finitely many sets of the form V_y, we conclude that there is a clopen set $V \subseteq f^{-1}(U)$ containing Y_x such that u and v have the same image in $\mathscr{F}(V, \mathcal{O}_Y|_V)$. Since the map of topological spaces $f : Y \to X$ is closed, we can assume without loss of generality that $V = f^{-1}(U_0)$ for some clopen set $U_0 \subseteq U$ containing x. Note that the map $(V, \mathcal{O}_Y|_V) \to (U_0, \mathcal{O}_X|_{U_0})$ is a covering in $\mathrm{Stone}_{\mathcal{C}}$, so that the map of sets $\mathscr{F}(U_0, \mathcal{O}_X|_{U_0}) \to \mathscr{F}(V, \mathcal{O}_Y|_V)$ is injective. It follows that u and v have the same image in $\mathscr{F}(U_0, \mathcal{O}_X|_{U_0})$, as desired.

We now show that ϕ is surjective. Fix an element $s \in \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$. For each point $y \in Y_x$, let s_y denote the image of s in $\mathscr{F}(\{y\}, \mathcal{O}_{Y,y})$. Using assumption (b'), we can lift s_y to an element $\widetilde{s}_y \in \mathscr{F}(V_y, \mathcal{O}_Y|_{V_y})$ for some clopen set $V_y \subseteq Y$ containing y. Applying the first part of the proof to the object $(Y_x, \mathcal{O}_Y|_{Y_x})$, we conclude that there is clopen set $T_y \subseteq Y_x$ containing y such that $\widetilde{s}_y$ and s have the same image in $\mathscr{F}(T_y, \mathcal{O}_Y|_{T_y})$. Shrinking V_y if necessary, we may assume that $T_y = V_y \cap Y_x$. Since Y_x is compact, it is contained in the union $V = V_{y_1} \cup \cdots \cup V_{y_n}$ for finitely many elements $y_1, \ldots, y_n \in Y_x$. By passing to a disjoint refinement of the covering of V by the sets V_{y_i}, we can amalgamate the sections $\{\widetilde{s}_y\}$ to a single element $\widetilde{s} \in \mathscr{F}(V, \mathcal{O}_Y|_V)$ such that s and $\widetilde{s}$ have the same image in $\mathscr{F}(Y_x, \mathcal{O}_Y|_{Y_x})$.

Let us abuse notation by identifying $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ with objects of $\mathrm{Pro}(\mathcal{C})$, and form the fiber product $(Y, \mathcal{O}_Y) \times_{(X, \mathcal{O}_X)} (Y, \mathcal{O}_Y)$ in $\mathrm{Pro}(\mathcal{C})$. Choose an effective epimorphism

$$(Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y) \times_{(X, \mathcal{O}_X)} (Y, \mathcal{O}_Y),$$

where $(Z, \mathcal{O}_Z) \in \mathrm{Stone}_{\mathcal{C}}$. We then have a pair of projection maps $\pi, \pi' : Z \to Y$. Let t and t' denote the images of $\widetilde{s}$ in $\mathscr{F}(\pi^{-1}(V), \mathcal{O}_Z|_{\pi^{-1}(V)})$ and $\mathscr{F}(\pi'^{-1}(V), \mathcal{O}_Z|_{\pi'^{-1}(V)})$, respectively. Note that t and t' have the same image in $\mathscr{F}(\{z\}, \mathcal{O}_{Z,z})$ for each $z \in Z \times_X \{x\}$. Using the first part of the proof, we see that there exists a clopen set $W \subseteq \pi^{-1}(V) \cap \pi'^{-1}(V)$ containing $Z \times_X \{x\}$ such that t and t' have the same image in $\mathscr{F}(W, \mathcal{O}_Z|_W)$. Since the projection maps $Z \to X \leftarrow Y$ are closed, we can choose a clopen subset $U \subseteq X$ containing x such that $U \times_X Y \subseteq V$ and $U \times_X Z \subseteq W$. Replacing V and W by the inverse images of U, we can invoke our assumption that $\mathscr{F}$ is a sheaf to deduce that the diagram of sets

$$\mathscr{F}(U, \mathcal{O}_X|_U) \to \mathscr{F}(V, \mathcal{O}_Y|_V) \rightrightarrows \mathscr{F}(Z, \mathcal{O}_Z|_W)$$

is an equalizer, so that $\widetilde{s} \in \mathscr{F}(V, \mathcal{O}_Y|_V)$ lifts uniquely to an element $s' \in \mathscr{F}(U, \mathcal{O}_X|_U)$. Using the commutativity of the diagram

$$\begin{array}{ccc} \mathscr{F}(U, \mathcal{O}_X|_U) & \longrightarrow & \mathscr{F}(\{x\}, \mathcal{O}_{X,x}) \\ \downarrow & & \downarrow \\ \mathscr{F}(V, \mathcal{O}_Y|_V) & \longrightarrow & \mathscr{F}(Y_x, \mathcal{O}_Y|_{Y_x}) \end{array}$$

(and the fact that the right vertical map is injective), we deduce that the upper horizontal map carries s' to s, so that s belongs to the image of ϕ. $\square$