Lecture 21X-Characterization of $\mathcal{C}$

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Let $\mathcal{C}$ be an essentially small pretopos, which we regard as fixed throughout this lecture. We have fully faithful embeddings

$$\mathcal{C} \hookrightarrow \mathrm{Shv}(\mathcal{C}) \hookrightarrow \mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \simeq \mathrm{Shv}(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})) \simeq \mathrm{Shv}(\mathrm{Stone}_{\mathcal{C}}) \subseteq \mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}\mathrm{et}).$$

Moreover, in Lectures 17X and 19X we established the following:

Proposition 1. *Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a functor. Then $\mathscr{F}$ belongs to the essential image of the embedding $\mathrm{Shv}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ if and only if it satisfies the following conditions:*

- (a) *The functor $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ preserves finite products: that is, it carries finite coproducts in $\mathrm{Stone}_{\mathcal{C}}$ to finite products in the category of sets.*

- (b) *For every object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$ and every point $x \in X$, the canonical map*

$$\varinjlim_{x \in U} \mathscr{F}(U, \mathcal{O}_X|_U) \to \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$$

  is bijective; here the colimit is taken over all clopen neighborhoods $U \subseteq X$ of the point x.

- (c) *The composite functor*

$$\mathrm{Mod}(\mathcal{C}) \hookrightarrow \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \xrightarrow{\mathscr{F}} \mathcal{S}\mathrm{et}$$

  commutes with filtered colimits.

- (d) *For every elementary morphism $f : M \to N$ in $\mathrm{Mod}(\mathcal{C})$, we have an equalizer diagram*

$$\mathscr{F}(M) \to \mathscr{F}(N) \rightrightarrows \prod \mathscr{F}(P)$$

  where the product is taken over all commutative diagrams

$$M \xrightarrow{f} N \rightrightarrows P$$

  in $\mathrm{Mod}(\mathcal{C})$.

Our goal in this lecture is to explain what additional conditions need to be satisfied for the functor $\mathscr{F}$ to belong to the essential image of the embedding $\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$. This embedding is easy to describe: to an object $C \in \mathcal{C}$, it associates the functor

$$\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et} \qquad (X, \mathcal{O}_X) \mapsto \mathcal{O}_X^C(X),$$

which corresponds under the equivalence $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \simeq \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}} \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ to the evaluation functor $F \mapsto F(C)$. In the last lecture, we noted that the category $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ admits small coproducts, which are computed as (pointwise) products in the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$. It follows that if $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is given by evaluation at an object $C \in \mathcal{C}$, then it satisfies the following stronger version of condition (a):

(a^+) The functor $\mathscr{F}$ carries (possibly infinite) coproducts in $\mathrm{Stone}_{\mathcal{C}}$ to products in the category of sets.

We will show that, conversely, a functor $\mathscr{F}$ satisfying (a^+) together with conditions (b), (c), and (d) of Proposition 1 belongs to the essential image of $\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$. Moreover, it suffices to check (a^+) in a restricted class of examples.

Theorem 2. *Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a functor which satisfies the conditions of Proposition 1, so that $\mathscr{F}$ is isomorphic to the image of some object $\mathscr{F}_0 \in \mathrm{Shv}(\mathcal{C})$. The following conditions are equivalent:*

(1) *The sheaf $\mathscr{F}_0 \in \mathrm{Shv}(\mathcal{C})$ is representable by an object $C \in \mathcal{C}$.*

(2) *The functor $\mathscr{F}$ satisfies condition (a^+) above.*

(3) *The functor $\mathscr{F}$ satisfies the following weaker version of (a^+):*

  (a') *For every collection of models $\{M_i \in \mathrm{Mod}(\mathcal{C})\}_{i \in I}$, the canonical map*

$$\mathscr{F}(\coprod_{i \in I}(\{i\}, M_i)) \to \prod_{i \in I} \mathscr{F}(\{i\}, M_i)$$

  is a bijection.

The implication (1) ⇒ (2) was noted above, and the implication (2) ⇒ (3) is immediate. We will complete the proof by showing that (3) ⇒ (1). For this, we will need a variant of Deligne's completeness theorem.

Notation 3. Recall that every model $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$ admits an essentially unique extension to a functor $\mathrm{Shv}(\mathcal{C}) \to \mathcal{S}\mathrm{et}$ which preserves small colimits and finite limits (that is, to a *point* of the topos $\mathrm{Shv}(\mathcal{C})$). In what follows, we will denote this extension by $\widehat{M} : \mathrm{Shv}(\mathcal{C}) \to \mathcal{S}\mathrm{et}$.

Lemma 4. *Let $u : \mathscr{G} \to \mathscr{F}$ be a morphism in the topos $\mathrm{Shv}(\mathcal{C})$. If u is not an effective epimorphism, then there exists a model M of $\mathcal{C}$ for which the map $\widehat{M}(\mathscr{G}) \to \widehat{M}(\mathscr{F})$ is not surjective.*

Proof. Since $\mathscr{F}$ admits a covering by representable functors, our assumption that u is not an effective epimorphism guarantees that we can choose an object $C \in \mathcal{C}$ and a morphism $h_C \to \mathscr{F}$ for which the projection map

$$h_C \times_{\mathscr{F}} \mathscr{G} \to h_C$$

is not an effective epimorphism. For any model $M \in \mathrm{Mod}(\mathcal{C})$, we have a pullback diagram of sets

$$\begin{array}{ccc} \widehat{M}(h_C \times_{\mathscr{F}} \mathscr{G}) & \longrightarrow & M(C) \\ \downarrow & & \downarrow \\ \widehat{M}(\mathscr{G}) & \longrightarrow & \widehat{M}(\mathscr{F}). \end{array}$$

Consequently, if the upper horizontal map is not surjective, then the lower horizontal map is also not surjective. We may therefore replace $\mathscr{F}$ by h_C (and $\mathscr{G}$ by the fiber product $h_C \times_{\mathscr{F}} \mathscr{G}$) and thereby reduce to the case where $\mathscr{F}$ is representable by an object $C \in \mathcal{C}$.

Choose an effective epimorphism $u : P \to C$ in $\mathrm{Pro}(\mathcal{C})$, where P is weakly projective. Under the equivalence $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \simeq \mathrm{Stone}_{\mathcal{C}}$, we can identify P with an object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$. Moreover, the map u determines a global section s of $\mathcal{O}_X^C(X)$. For each point $x \in X$, let us regard $\mathcal{O}_{X,x}$ as a model of $\mathcal{C}$, so that s determines an element $s_x \in \mathcal{O}_{X,x}^C$. Assume, for a contradiction, that each of the maps

$$\widehat{\mathcal{O}}_{X,x}(\mathscr{G}) \to \widehat{\mathcal{O}}_{X,x}(h_C) = \mathcal{O}_{X,x}^C$$

is surjective. Then each s_x can be lifted to an element $\widetilde{s}_x \in \widehat{\mathcal{O}}_{X,x}(\mathscr{G})$. Choose a covering $\{h_{C_i} \to \mathscr{G}\}_{i \in I}$ in the topos $\mathrm{Shv}(\mathcal{C})$. Then, for each point $x \in X$, we can choose an index $i(x) \in I$ such that $\widetilde{s}_x$ lifts to a point $t_x \in \widehat{\mathcal{O}}_{X,x}(h_{C_{i(x)}}) \simeq \mathcal{O}_{X,x}^{C_{i(x)}}$. Choose an open set $U(x) \subseteq X$ containing x such that $\overline{s}_x$ can be lifted to $t \in \mathcal{O}_X^{C_{i(x)}}(U(x))$. Shrinking $U(x)$ if necessary, we may assume that the image of t in $\mathcal{O}_X^C(U(x))$ agrees with the restriction $s|_{U(x)}$.

Note that the open sets $\{U(x)\}_{x \in X}$ cover the topological space X. Since X is compact, we can choose a finite collection of points $x_1, x_2, \ldots, x_n \in X$ for which the open sets $U(x_1), \ldots, U(x_n)$ cover X. By construction, each restriction $s|_{U(x_j)}$ can be lifted to a section of $\mathcal{O}_X^{C_{i(x_j)}}$ over the open set $U(x_j)$. It follows that s is a global section of the subsheaf $\mathcal{O}_X^{C_0} \subseteq \mathcal{O}_X^C$, where $C_0 = \mathrm{Im}(\coprod C(x_j) \to C)$. Our assumption that u is an effective epimorphism then shows that we must have $C_0 = C$, contradicting our assumption that the map $\mathscr{G} \to h_C$ is not an effective epimorphism. □

Proof of Theorem 2. Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a functor which satisfies the conditions of Proposition 1, so that $\mathscr{F}$ arises from a sheaf $\mathscr{F}_0 \in \mathrm{Shv}(\mathcal{C})$. Assume further that $\mathscr{F}$ satisfies condition (a'). We wish to prove that $\mathscr{F}_0$ belongs to the essential image of the Yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{Shv}(\mathcal{C})$. Since $\mathcal{C}$ is a pretopos, the sheaf $\mathscr{F}_0 \in \mathrm{Shv}(\mathcal{C})$ is representable by an object of $\mathcal{C}$ if and only if it is quasi-compact and quasi-separated.

We first show that $\mathscr{F}_0$ is quasi-compact. Choose a collection $\{u_i : h_{C_i} \to \mathscr{F}_0\}_{i \in I}$ of representatives for all maps from representable sheaves to $\mathscr{F}_0$. Since $\mathscr{F}_0$ is not quasi-compact, none of these maps is an effective epimorphism. For each index $i \in I$, we can use Lemma 4 to choose a model M_i and a point $\eta_i \in \mathscr{F}(\{i\}, M_i)$ which does not belong to the image of the map $M_i(C_i) \to \widehat{M}_i(\mathscr{F}_0) = \mathscr{F}(\{i\}, M_i)$. Set $(X, \mathcal{O}_X) = \amalg_{i \in I}(\{i\}, M_i)$, where the coproduct is formed in the category $\mathrm{Stone}_{\mathcal{C}}$. Using condition (a'), we see that the system $\{\eta_i\}_{i \in I}$ can be lifted (uniquely) to a point $\eta \in \mathscr{F}(X, \mathcal{O}_X)$ under the bijection $\mathscr{F}(X, \mathcal{O}_X) \to \prod_{i \in I} \mathscr{F}(\{i\}, M_i)$.

For each point $x \in X$, let η_x denote the image of η in $\mathscr{F}(\{x\}, \mathcal{O}_{X,x}) \simeq \widehat{\mathcal{O}}_{X,x}(\mathscr{F}_0)$. Then there exists some $i(x) \in I$ such that η_x can be lifted to an element $\widetilde{\eta}_x \in \widehat{\mathcal{O}}_{X,x}(h_{C_{i(x)}}) \simeq \mathcal{O}_{X,x}^{C_{i(x)}}$. Choose a clopen open set $U(x)$ containing x and lift of $\widetilde{\eta}_x$ to some $s_x \in \mathcal{O}_X^{C_{i(x)}}(U(x))$. Let $\overline{s}_x$ denote the image of s_x in $\mathscr{F}(U(x), \mathcal{O}_X|_{U(x)})$. By construction, $\overline{s}_x$ and η have the same image in $\mathscr{F}(\{x\}, \mathcal{O}_{X,x})$. It follows from (b) that we can assume, after shrinking $U(x)$ if necessary, that $\overline{s}_x = \eta|_{U(x)}$.

Since X is compact, we can choose finitely many points $x_1, \ldots, x_n$ for which the open sets $U(x_1), \ldots, U(x_n)$ cover X. Then the map

$$(u_{i(x_1)}, \ldots, u_{i(x_n)}) : (h_{C_{i(x_1)}} \amalg \cdots \amalg h_{C_{i(x_n)}} \to \mathscr{F}_0$$

can be identified with $u_j : h_{C_j} \to \mathscr{F}_0$ for some $j \in I$. Let y denote the image of j in $X = \beta I$ (corresponding to the principal ultrafilter associated to j). Then we have $y \in U(x)$ for some $x \in \{x_1, \ldots, x_n\}$. By construction, it follows that $\eta|_{U(x)}$ can be lifted to the point $s_x \in \mathcal{O}_X^{C_{i(x)}}(U(x))$, so that the stalk η_y belongs to the image of the map

$$\mathcal{O}_{X,y}^{C_{i(x)}} \simeq \widehat{\mathcal{O}}_{X,y}(h_{C_{i(x)}}) \to \widehat{\mathcal{O}}_{X,y}(\mathscr{F}_0)$$

determined by the map $u_{i(x)} : h_{C_{i(x)}} \to \mathscr{F}_0$. However, the map $u_{i(x)}$ factors through $u_j : h_{C_j} \to \mathscr{F}_0$, so that η_y also belongs to the image of the map

$$\mathcal{O}_{X,y}^{C_j} \simeq M_j(C_j) \to \widehat{\mathcal{O}}_{X,y}(\mathscr{F}_0),$$

contradicting our choice of M_j. This completes the proof that $\mathscr{F}_0$ is quasi-compact.

We now complete the proof by showing that $\mathscr{F}_0$ is quasi-separated. Choose a pair of quasi-compact objects $\mathscr{G}_0, \mathscr{H}_0 \in \mathrm{Shv}(\mathcal{C})_{/\mathscr{F}_0}$; we wish to show that the fiber product $\mathscr{G}_0 \times_{\mathscr{F}_0} \mathscr{H}_0$ is quasi-compact. Covering $\mathscr{G}$ and $\mathscr{H}$ by representable sheaves, we may assume that $\mathscr{G}_0$ and $\mathscr{H}_0$ are representable by objects of $\mathcal{C}$. Let $\mathscr{G}$ and $\mathscr{H}$ denote the images of $\mathscr{G}_0$ and $\mathscr{H}_0$ in the category $\mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$. Then $\mathscr{G}$ and $\mathscr{H}$ satisfy condition (a') (even condition (a^+)), so that $\mathscr{G} \times_{\mathscr{F}} \mathscr{H}$ also satisfies condition (a'). The preceding argument then shows that $\mathscr{G}_0 \times_{\mathscr{F}_0} \mathscr{H}_0$ is quasi-compact, as desired. □