Lecture 20X-Ultraproducts

March 31, 2018

In this lecture, we review the theory of ultrafilters and ultraproducts.

Definition 1. Let I be a set. An *ultrafilter on* I is a collection $\mathcal{U}$ of subsets of I satisfying the following conditions:

(a) The set $\mathcal{U}$ is closed under finite intersections. That is, the set I belongs to $\mathcal{U}$, and for every $J, J' \in \mathcal{U}$, the intersection $J \cap J'$ also belongs to $\mathcal{U}$.

(b) The set $\mathcal{U}$ is closed upwards: that is, if $J \subseteq J'$ and J is contained in $\mathcal{U}$, then J' is also contained in $\mathcal{U}$.

(c) For every subset $J \subseteq I$, exactly one of the sets J and $I - J$ belongs to $\mathcal{U}$.

Exercise 2. In Definition 1, show that (b) can be deduced from (a) and (c).

Remark 3. Let I be a set. Then the datum of an ultrafilter $\mathcal{U}$ on I is equivalent to the datum of a finitely additive measure

$$\mu : \{\text{Subsets of } I\} \rightarrow \{0, 1\};$$

the equivalence is implemented by taking $\mathcal{U} = \{J \subseteq I : \mu(J) = 1\}$.

Example 4 (Principal Ultrafilters)**.** Let I be a set containing an element i, and let $\mathcal{U}_i$ be the collection of all subsets of I which contain i. Then $\mathcal{U}_i$ is an ultrafilter on I. We refer to $\mathcal{U}_i$ as the *principal ultrafilter associated to* i.

Exercise 5. Let $\mathcal{U}$ be a collection of subsets of a set I. We say that $\mathcal{U}$ is a *filter on* I if it satisfies conditions (a) and (b) of Definition 1. Show that if $\mathcal{U}$ is a filter on I such that $\emptyset \notin \mathcal{U}$, then $\mathcal{U}$ can be enlarged to an ultrafilter on I.

Construction 6 (Ultraproducts)**.** Let $\{M_i\}_{i \in I}$ be a collection of sets indexed by a set I, and let $\mathcal{U}$ be an ultrafilter on I. We let $(\prod_{i \in I} \mathcal{M}_i)/\,\mathcal{U}$ denote the direct limit

$$\varinjlim_{J \in \mathcal{U}} \prod_{i \in J} M_i.$$

We will refer to $(\prod_{i \in I} \mathcal{M}_i)/\,\mathcal{U}$ as the *ultraproduct of the sets* M_i *with respect to the ultrafilter* $\mathcal{U}$.

Exercise 7. In the situation of Construction 6, suppose that each of the sets M_i is nonempty. Show that the ultraproduct $(\prod_{i \in I} M_i)/\,\mathcal{U}$ can be identified with the quotient of $\prod_{i \in I} M_i$ by an equivalence relation $\sim$, where $\{x_i\}_{i \in I} \simeq \{y_i\}_{i \in I}$ if $\{i \in I : x_i = y_i\}$ belongs to the ultrafilter $\mathcal{U}$ (in this case, we say that the sequences $\{x_i\}_{i \in I}$ and $\{y_i\}_{i \in I}$ agree *almost everywhere* with respect to $\mathcal{U}$).

Beware that this is not necessarily true if some M_j is empty. In this case, the product $\prod_{i \in I} M_i$ is also empty. However, the ultraproduct $(\prod_{i \in I} M_i)/\,\mathcal{U}$ will be nonempty if the set $\{i \in I : M_i \neq \emptyset\}$ belongs to the ultrafilter $\mathcal{U}$.

Example 8. In the situation of Construction 6, suppose that $\mathcal{U} = \mathcal{U}_j$ is the principal ultrafilter associated to an element $j \in I$. Then the ultraproduct $(\prod_{i \in I} M_i)/\,\mathcal{U}$ can be identified with M_j.

Ultraproducts appear in mathematical logic because they behave well with respect to the truth of first-order formulas.

Theorem 9 (Łos's Ultraproduct Theorem, Pretopos Version)**.** *Let $\mathcal{C}$ be a pretopos, let $\{M_i\}_{i \in I}$ be a collection of models of $\mathcal{C}$ indexed by a set I, and let $\mathcal{U}$ be an ultrafilter on I. Then the construction*

$$(C \in \mathcal{C}) \mapsto (\prod_{i \in I} M_i(C))/\mathcal{U}$$

is also a model of $\mathcal{C}$

Corollary 10 (Łos's Ultraproduct Theorem, Classical Version)**.** *Let T be a first-order theory in a language $\{P_j\}_{j \in J}$. Let $\{M_i\}_{i \in I}$ be a collection of models of T, and assume for simplicity that each M_i is nonempty. Suppose we are given an ultrafilter $\mathcal{U}$ on the set I, and set $M = (\prod_{i \in I} M_i)/\mathcal{U}$. Regard M as a structure for the language L by declaring*

$$(M \vDash P_j(\{\vec{c}_i\}_{i \in I})) \Leftrightarrow \{i \in I : M_i \vDash P_j(\vec{c}_i)\} \in \mathcal{U}\,.$$

Then M is also a model of T. Moreover, for any formula $\varphi(\vec{x})$ in the language L, we have

$$(M \vDash \varphi(\{\vec{c}_i\}_{i \in I})) \Leftrightarrow \{i \in I : M_i \vDash \varphi(\vec{c}_i)\} \in \mathcal{U}\,.$$

Proof. Apply Theorem 9 to the syntactic category $\mathrm{Syn}(T)$. (Note that the desired conclusion can be restated as $M[\varphi] \simeq (\prod_{i \in I} M_i[\varphi])/\mathcal{U}$.) □

It is not difficult to give a direct proof of Theorem 9 (or Corollary 10): the essential point is that the formation of ultraproducts commutes with the formation of finite limits, finite coproducts, and images. However, we will give a different explanation of Theorem 9, which connects up with the material of the last few lectures.

For the remainder of this lecture, let $\mathcal{C}$ be a small pretopos. Recall that the category $\mathrm{Pro}(\mathcal{C})$ has small limits and colimits.

Proposition 11. (1) *The subcategory $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C})$ of weakly projective pro-objects of $\mathcal{C}$ has (possibly infinite) coproducts, which are preserved by the inclusion $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \hookrightarrow \mathrm{Pro}(\mathcal{C})$.*

(2) *For every object $C \in \mathcal{C}$, the construction $M \mapsto M(C)$ determines a functor $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ which preserves (possibly infinite) products: that is, it carries coproducts in $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ to products of sets.*

(3) *The category $\mathrm{Stone}_{\mathcal{C}}$ has (possibly infinite) coproducts. Moreover, for each object $C \in \mathcal{C}$, the functor $(X, \mathcal{O}_X) \mapsto \mathcal{O}_X^C(X)$ carries coproducts in $\mathrm{Stone}_{\mathcal{C}}$ to products of sets.*

Proof. Recall that $\mathrm{Pro}(\mathcal{C})$ can be defined as the opposite of the category $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S}\mathrm{et})$ of left exact functors from $\mathcal{C}$ to $\mathcal{S}\mathrm{et}$. Since the class of left exact functors is closed under inverse limits, it follows that colimits in $\mathrm{Pro}(\mathcal{C})$ are computed pointwise. In particular, given a collection of pro-objects $\{M_i\}_{i \in I}$, the coproduct $M = \amalg_{i \in I} M_i$ in the category $\mathrm{Pro}(\mathcal{C})$ is given by the formula $M(C) = \prod_{i \in I} M_i(C)$. From this description, it is clear that that if each M_i is weakly projective, then so is M (note that a product of surjections in the category of sets is again a surjection). This proves (1) and (2), and assertion (3) is just a restatement. □

Example 12 (Ultrafilters)**.** Let $\mathcal{C} = \mathcal{S}\mathrm{et}_{\mathrm{fin}}$ be the category of finite sets, so that $\mathrm{Stone}_{\mathcal{C}} \simeq \mathrm{Stone}$ is the category of Stone spaces. Proposition 11 implies that the category Stone admits coproducts. Beware that the inclusion $\mathrm{Stone} \hookrightarrow \mathrm{Top}$ does *not* preserve coproducts: a coproduct of Stone spaces is Hausdorff and totally disconnected, but usually not compact.

For example, let I be a set, and consider the coproduct $\amalg_{i \in I}\{i\}$, formed in the category Stone. We denote this coproduct by βI and refer to it as the *Stone-Čech compactification of I*. It is characterized by the following universal property: there is a map $\rho : I \to \beta I$ such that composition with ρ induces a bijection

$$\mathrm{Hom}_{\mathrm{Top}}(\beta I, X) \to \prod_{i \in I} X$$

for any Stone space X (or, more generally, any compact Hausdorff space X). In particular, taking X to be a two-point space, we obtain a bijection

$$\{\text{Clopen subsets of } \beta I\} \simeq \{\text{Arbitrary subsets of } I\}.$$

In other words, we can describe βI as the spectrum of the Boolean algebra $P(I)$ of subsets of I. It follows that βI can be identified with the set of Boolean algebra homomorphisms $\mu : P(I) \to \{0,1\}$: that is, with the collection of all ultrafilters on I (see Remark 3). The topology on βI is generated by open (and closed) sets of the form

$$U_J := \{\mathcal{U} \in \beta I : J \in \mathcal{U}\},$$

where J ranges over all subsets of I (in fact, the construction $J \mapsto U_J$ implements the isomorphism of $P(I)$ with the Boolean algebra of clopen subsets of βI).

Remark 13. In the situation of Example 12, the canonical map $\rho : I \to \beta I$ carries each element $i \in I$ to the principal ultrafilter $\mathcal{U}_i$ of Example 4.

Example 14 (Ultraproducts)**.** Let us now return to the situation where $\mathcal{C}$ is an arbitrary small pretopos. Suppose we are given a collection of models $\{M_i \in \mathrm{Mod}(\mathcal{C})\}_{i \in I}$. We can then regard each pair $(\{i\}, M_i)$ as an object of $\mathrm{Stone}_{\mathcal{C}}$, and form the coproduct

$$(X, \mathcal{O}_X) = \amalg_{i \in I}(\{i\}, M_i)$$

in $\mathrm{Stone}_{\mathcal{C}}$.

Note that the forgetful functor $\mathrm{Stone}_{\mathcal{C}} \to \mathrm{Stone}$ preserves coproducts: it is given by the composition

$$\mathrm{Stone}_{\mathcal{C}} \simeq \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \hookrightarrow \mathrm{Pro}(\mathcal{C}) = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S}\mathrm{et})^{\mathrm{op}} \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}, \mathcal{C})^{\mathrm{op}} = \mathrm{Pro}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}) = \mathrm{Stone}$$

induced by the morphism of pretopoi $\mathcal{S}\mathrm{et}_{\mathrm{fin}} \to \mathcal{C}$. It follows that we can identify the Stone space X with the Stone-Čech compactification βI. In particular, the construction

$$(J \subseteq I) \mapsto U_J = \{\mathcal{U} \in \beta I : J \in \mathcal{U}\}$$

induces a bijection from the collection $P(I)$ of subsets of I to the collection of clopen subsets of X. Unwinding the definitions, we see that $\mathcal{O}_X$ is given by the formula

$$\mathcal{O}_X^C(U_J) = \prod_{i \in J} M_i(C).$$

In particular, given a point $x \in X$ corresponding to an ultrafilter $\mathcal{U}$ on the set I, we have

$$\begin{aligned} \mathcal{O}^C_{X,x} &= \varinjlim_{x \in U_J} \mathcal{O}_X^C(U_J) \\ &= \varinjlim_{J \in \mathcal{U}} \prod_{i \in J} M_i(C) \\ &= (\prod_{i \in I} M_i(C))/\,\mathcal{U}. \end{aligned}$$

Proof of Theorem 9. Let $\mathcal{C}$ be a pretopos, let $\{M_i\}_{i \in I}$ be a collection of models of $\mathcal{C}$ indexed by a set I, and let $\mathcal{U}$ be an ultrafilter on I. Forming the coproduct $(X, \mathcal{O}_X) = \amalg_{i \in I}(\{i\}, M_i)$ in $\mathrm{Stone}_{\mathcal{C}}$, we observe that $\mathcal{U}$ can be identified with a point $x \in X \simeq \beta I$, and that the stalk $\mathcal{O}_{X,x}$ is a model of $\mathcal{C}$ given by the formula $C \mapsto (\prod_{i \in I} M_i(C))/\,\mathcal{U}$. □

We can summarize the situation informally as follows: given a collection of models $\{M_i\}_{i \in I}$ of a pretopos $\mathcal{C}$, we can construct a larger family of models parametrized by the Stone-Čech compactification βI, which assigns to each ultrafilter $\mathcal{U} \in \beta I$ the corresponding ultraproduct $(\prod_{i \in I} M_i)/\,\mathcal{U}$.

Definition 15. We will say that an object $M \in \mathrm{Pro}(\mathcal{C})$ is *free* if it can be written as a coproduct $\coprod_{i \in I} M_i$ in $\mathrm{Pro}(\mathcal{C})$, where each M_i is a model of $\mathcal{C}$. Note that in this case, M is automatically weakly projective.

We say that an object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$ is *free* if it corresponds to a free object of $\mathrm{Pro}(\mathcal{C})$ under the equivalence $\mathrm{Stone}_{\mathcal{C}} \simeq \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$: that is, if it can be written as a coproduct

$$\coprod_{i \in I} (\{i\}, M_i)$$

in the category $\mathrm{Stone}_{\mathcal{C}}$.

Proposition 16. (1) *For every object $Z \in \mathrm{Pro}(\mathcal{C})$, there exists an effective epimorphism $M \to Z$, where M is free.*

(2) *For every object $(X, \mathcal{O}_X)$, there exists a covering $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ in $\mathrm{Stone}_{\mathcal{C}}$, where $(Y, \mathcal{O}_Y)$ is free.*

Proof. To prove (1) we may assume without loss of generality that Z is weakly projective. In this case, (1) and (2) are equivalent. Let us therefore consider (2). Fix an object $(X, \mathcal{O}_X)$ in $\mathrm{Stone}_{\mathcal{C}}$, and form the coproduct

$$(Y, \mathcal{O}_Y) = \coprod_{x \in X} (\{x\}, \mathcal{O}_{X,x}).$$

We claim that the tautological map $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a covering. Using the criterion of Lecture 18X, we are reduced to showing that for each point $x \in X$, we can choose a point $y \in Y$ lying over x for which the induced map of models $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is an isomorphism. Identifying Y with the set βX of ultrafilters on X, it suffices to choose y to correspond to the principal ultrafilter $\mathcal{U}_x$; in this case, the canonical map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is an isomorphism (Example 8). $\square$