Lecture 19: Reconstruction of Localic Morphisms

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Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. In the previous lecture, we saw that there is an object $\Omega_{\mathcal{X}\,/\,\mathcal{Y}}$ of $\mathcal{Y}$ with the following universal property: for each object $Y \in \mathcal{Y}$, we have a canonical bijection

$$\mathrm{Hom}_{\mathcal{Y}}(Y, \Omega_{\mathcal{X}\,/\,\mathcal{Y}}) \simeq \mathrm{Sub}(f^* Y).$$

Our goal in this lecture is to explain that, if the morphism f is localic, then we can recover $\mathcal{X}$ from the topos $\mathcal{Y}$ and the object $\Omega_{\mathcal{X}\,/\,\mathcal{Y}}$. For this, we need to endow $\Omega_{\mathcal{X}\,/\,\mathcal{Y}}$ with a little bit of additional structure.

Exercise 1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. Show that the functor

$$(Y \in \mathcal{Y}) \mapsto \{U, V \in \mathrm{Sub}(f^* Y) : U \subseteq V\}$$

is representable by an object $\Omega^{\subseteq}_{\mathcal{X}\,/\,\mathcal{Y}}$ of $\mathcal{Y}$. Note that $\Omega^{\subseteq}_{\mathcal{X}\,/\,\mathcal{Y}}$ can then be viewed as a subobject of $\Omega_{\mathcal{X}\,/\,\mathcal{Y}} \times \Omega_{\mathcal{X}\,/\,\mathcal{Y}}$.

We now consider several (essentially equivalent) ways of looking at the object $\Omega_{\mathcal{X}\,/\,\mathcal{Y}}$:

(*a*) We can think of $\Omega_{\mathcal{X}\,/\,\mathcal{Y}}$ as a *partially ordered object of* $\mathcal{Y}$ (with partial order given by $\Omega^{\subseteq}_{\mathcal{X}\,/\,\mathcal{Y}} \subseteq \Omega_{\mathcal{X}\,/\,\mathcal{Y}} \times \Omega_{\mathcal{X}\,/\,\mathcal{Y}}$). Concretely, this means that the intersection

$$\Omega^{\subseteq}_{\mathcal{X}\,/\,\mathcal{Y}} \cap (\Omega^{\subseteq}_{\mathcal{X}\,/\,\mathcal{Y}})^{\mathrm{op}}$$

coincides with the image of the diagonal map $\Omega_{\mathcal{X}\,/\,\mathcal{Y}} \to \Omega_{\mathcal{X}\,/\,\mathcal{Y}} \times \Omega_{\mathcal{X}\,/\,\mathcal{Y}}$ (here $(\Omega^{\subseteq}_{\mathcal{X}\,/\,\mathcal{Y}})^{\mathrm{op}}$ denote the image of $\Omega^{\subseteq}_{\mathcal{X}\,/\,\mathcal{Y}}$ under the automorphism of $\Omega_{\mathcal{X}\,/\,\mathcal{Y}} \times \Omega_{\mathcal{X}\,/\,\mathcal{Y}}$ given by swapping the two factors), and that we have

$$\pi_1^* \Omega^{\subseteq}_{\mathcal{X}\,/\,\mathcal{Y}} \cap \pi_3^* \Omega^{\subseteq}_{\mathcal{X}\,/\,\mathcal{Y}} \subseteq \pi_2^* \Omega^{\subseteq}_{\mathcal{X}\,/\,\mathcal{Y}}$$

as subobjects of $\Omega_{\mathcal{X}\,/\,\mathcal{Y}} \times \Omega_{\mathcal{X}\,/\,\mathcal{Y}} \times \Omega_{\mathcal{X}\,/\,\mathcal{Y}}$, where

$$\pi_i : \Omega_{\mathcal{X}\,/\,\mathcal{Y}} \times \Omega_{\mathcal{X}\,/\,\mathcal{Y}} \times \Omega_{\mathcal{X}\,/\,\mathcal{Y}} \to \Omega_{\mathcal{X}\,/\,\mathcal{Y}} \times \Omega_{\mathcal{X}\,/\,\mathcal{Y}}$$

denotes the projection map which omits the ith factor.

The first demand encodes the requirement that the relation $\Omega^{\subseteq}_{\mathcal{X}\,/\,\mathcal{Y}}$ is reflexive and antisymmetric, and the second encodes transitivity.

(*b*) Rather than viewing $\Omega_{\mathcal{X}\,/\,\mathcal{Y}}$ as an object of $\mathcal{Y}$, we can identify it with the functor that it represents, given by

$$Y \mapsto \mathrm{Hom}_{\mathcal{Y}}(Y, \Omega_{\mathcal{X}\,/\,\mathcal{Y}}) \simeq \mathrm{Sub}(f^* Y).$$

The axioms of (*a*) translate to the requirement that, for each $Y \in \mathcal{Y}$, the image of the inclusion map

$$\mathrm{Hom}_{\mathcal{Y}}(Y, \Omega^{\subseteq}_{\mathcal{X}\,/\,\mathcal{Y}}) \hookrightarrow \mathrm{Hom}_{\mathcal{Y}}(Y, \Omega_{\mathcal{X}\,/\,\mathcal{Y}} \times \Omega_{\mathcal{X}\,/\,\mathcal{Y}}) \simeq \mathrm{Sub}(f^* Y) \times \mathrm{Sub}(f^* Y)$$

determines a partial ordering of $\mathrm{Sub}(f^* Y)$. Of course, this is why those axioms are satisfied in the first place (by construction, this is just the partial order on $\mathrm{Sub}(f^* Y)$ given by inclusions of subobjects).

In other word, we can think of $\Omega_{\mathcal{X}\,/\,\mathcal{Y}}$ as encoding a functor

$$\mathcal{Y}^{\mathrm{op}} \to \{\text{Partially Ordered Sets}\}.$$

Such a functor is representable by a partially ordered object of $\mathcal{Y}$ (in the sense of (a)) if and only if it is a sheaf with respect to the canonical topology on $\mathcal{Y}$.

(b') In the situation of (b), we can be more specific. Each the partially ordered sets $\mathrm{Sub}(f^*Y)$ is a locale, and each $Y \to Y'$ in $\mathcal{Y}$ induces a map of posets $\mathrm{Sub}(f^*Y') \to \mathrm{Sub}(f^*Y)$ which preserves finite meets and arbitrary joins; that is, it can be regarded as a locale morphism from $\mathrm{Sub}(f^*Y)$ to $\mathrm{Sub}(f^*Y')$. We can therefore identify $\Omega_{\mathcal{X}\,/\,\mathcal{Y}}$ with a functor

$$\mathcal{Y} \to \{\text{Locales}\}.$$

We can do even better: in Lecture 17, we saw that each of the locale morphisms $\mathrm{Sub}(f^*Y) \to \mathrm{Sub}(f^*Y')$ is open. We therefore obtain a functor

$$\mathcal{Y} \to \{\text{Locales, Open Morphisms of Locales}\}.$$

(c) Given any category $\mathcal{C}$ and a functor $P : \mathcal{C}^{\mathrm{op}} \to \{\text{Partially Ordered Sets}\}$, we can form a new category $\widetilde{\mathcal{C}}$ described as follows:

- The objects of $\widetilde{\mathcal{C}}$ are pairs (C, U), where $C \in \mathcal{C}$ and $U \in P(C)$.
- A morphism from (C, U) to (C', U') in $\widetilde{\mathcal{C}}$ is a morphism $g : C \to C'$ with the property that $U \leq P(f)(U')$ (in the poset $P(C)$).

Note that the category $\widetilde{\mathcal{C}}$ is equipped with a forgetful functor $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$, given on objects by $\pi(C, U) = C$. Moreover, we can recover the original functor P from $\widetilde{\mathcal{C}}$ and the functor π. For each $C \in \mathcal{C}$, we can identify the poset $P(C)$ with the fiber $\pi^{-1}\{C\}$. If $g : C \to C'$ is a morphism in $\mathcal{C}$ and U' is an element of $P(C')$, identified with an object $\widetilde{C}' \in \widetilde{\mathcal{C}}$ satisfying $\pi(\widetilde{C}') = C'$, then we can identify $P(f)(U') \in P(C)$ with the largest element of the poset $\{\widetilde{C} \in \pi^{-1}\{C\} : (\exists \widetilde{g} : \widetilde{C} \to \widetilde{C}')[\pi(\widetilde{g}) = g]\}$. In fact, this construction determines an equivalence

$$\{\text{Functors } P : \mathcal{C}^{\mathrm{op}} \to \{\text{Posets}\}\} \simeq \{\text{Functors } \pi : \widetilde{\mathcal{C}} \to \mathcal{C} \text{ which are fibered in posets}\}.$$

In the case of interest, we take $\mathcal{C} = \mathcal{Y}$ and $P : \mathcal{Y}^{\mathrm{op}} \to \{\text{Posets}\}$ to be the functor $Y \mapsto \mathrm{Sub}(f^*\,\mathcal{Y})$. In this case, we will denote the category $\widetilde{\mathcal{C}}$ by $\mathrm{Loc}(f)$; it can be described concretely as follows:

- The objects of $\mathrm{Loc}(f)$ are pairs (Y, U) where Y is an object of $\mathcal{Y}$ and $U \subseteq f^*Y$ is a subobject in $\mathcal{X}$. As a mnemonic aide, we will denote such an object by $(U \subseteq f^*Y)$.
- A morphism from $(U \subseteq f^*Y)$ to $(U' \subseteq f^*Y')$ is a morphism $g : Y \to Y'$ in the topos $\mathcal{Y}$ satisfying $U \subseteq U' \times_{f^*Y'} f^*Y$ (as subobjects of f^*Y).

In what follows, it will be convenient to adopt perspective (c).

Remark 2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. Then the category $\mathrm{Loc}(f)$ has a forgetful functor to $\mathcal{X}$, given by the construction $(U \subseteq f^*Y) \mapsto U$.

Remark 3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. Then the category $\mathrm{Loc}(f)$ admits finite limits. For example, given a pair of morphisms

$$(U_0 \subseteq f^*Y_0) \to (U_{01} \subseteq f^*Y_{01}) \leftarrow (U_1 \subseteq f^*Y_1),$$

the fiber product is given by $(U_0 \times_{U_{01}} U_1 \subseteq f^*(Y_0 \times_{Y_{01}} Y_1))$.

Exercise 4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. Let us say that a collection of morphisms $\{(U_i \subseteq f^* Y_i) \to (U \subseteq f^* Y)\}_{i \in I}$ in the category $\mathrm{Loc}(f)$ is a *covering* if the diagram $\{U_i \to U\}$ is a covering in $\mathcal{X}$. Show that this determines a Grothendieck topology on the category $\mathrm{Sub}(f)$. Note that we do *not* require that the objects Y_i cover Y.

Our next goal is to prove the following:

Theorem 5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a localic geometric morphism. Then there is a canonical equivalence $\mathcal{X} \simeq \mathrm{Shv}(\mathrm{Loc}(f))$. In particular, the topos $\mathcal{X}$ can be recovered from the fibration $\mathrm{Loc}(f) \to \mathcal{Y}$ (or, equivalently, from $\mathcal{Y}$ together with the partially ordered object $\Omega_{\mathcal{X}\,/\,\mathcal{Y}}$).*

The proof of Theorem 5 will require some preliminaries.

Notation 6. Let $\mathcal{X}_0 \subseteq \mathcal{X}$ be the full subcategory spanned by those objects $X \in \mathcal{X}$ for which there exists a monomorphism $X \hookrightarrow f^* Y$, for some Y in $\mathcal{Y}$. Note that the forgetful functor

$$\mathrm{Loc}(f) \to \mathcal{X} \qquad (U \subseteq f^* Y) \mapsto U$$

factors through $\mathcal{X}_0$ (in fact, $\mathcal{X}_0$ is defined as the essential image of this forgetful functor).

Remark 7. The subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ is closed under finite limits. For example, if we are given a fiber product $X_0 \times_{X_{01}} X_1$ where X_0, X_{01}, and X_1 belong to $\mathcal{X}_0$, then we can choose monomorphisms $u : X_0 \hookrightarrow f^* Y_0$ and $v : X_1 \hookrightarrow f^* Y_1$ for some $Y_0, Y_1 \in \mathcal{Y}$. In this case, u and v induce a monomorphism $X_0 \times_{X_{01}} X_1 \hookrightarrow f^*(Y_0 \times Y_1)$.

Lemma 8. *Let $\mathcal{Z}$ be a topos and let $\mathcal{Z}_0 \subseteq \mathcal{Z}$ be a full subcategory satisfying the following conditions:*

(1) *The full subcategory $\mathcal{Z}_0 \subseteq \mathcal{Z}$ contains a set of generators for $\mathcal{Z}$.*

(2) *The full subcategory $\mathcal{Z}_0 \subseteq \mathcal{Z}$ is closed under finite limits.*

Then the Yoneda embedding induces an equivalence of categories $\mathcal{Z} \simeq \mathrm{Shv}(\mathcal{Z}_0)$, where we equip $\mathcal{Z}_0$ with the Grothendieck topology given by the covering families in $\mathcal{Z}$.

Proof. When $\mathcal{Z}_0$ is small, we proved this in Lecture 10. The general case follows by writing $\mathcal{Z}_0$ as a union of small subcategories satisfying (1) and (2). □

Corollary 9. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a localic geometric morphism of topoi, and let $\mathcal{X}_0 \subseteq \mathcal{X}$ be as in Notation* **??**. *Then the Yoneda embedding induces an equivalence $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{X}_0)$, where $\mathcal{X}_0$ is equipped with the Grothendieck topology given by the covering families in $\mathcal{X}$.*

In the situation of Theorem 5, we have a forgetful functor

$$\pi : \mathrm{Loc}(f) \to \mathcal{X}_0 \qquad \pi(U \subseteq f^* Y) = U.$$

This functor preserves finite limits (Remark 3) and coverings, so that composition with π induces a functor $\mathrm{Shv}(\mathcal{X}_0) \to \mathrm{Shv}(\mathrm{Loc}(f))$. We will complete the proof by establishing the following (which does not require the assumption that f is localic):

Proposition 10. *Composition with π induces an equivalence of categories $\mathrm{Shv}(\mathcal{X}_0) \to \mathrm{Shv}(\mathrm{Loc}(f))$.*

Sketch. Let $\mathscr{F} : \mathrm{Loc}(f)^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a sheaf. Essentially, we need to show that for an object $(U \subseteq f^* Y) \in \mathrm{Loc}(f)$, the set $\mathscr{F}(U \subseteq f^* Y)$ depends only on $U \in \mathcal{X}_0$, and not on the particular realization of U as a subobject of $f^* Y$. To this end, let us regard the object $U \in \mathcal{X}_0$ as fixed, and suppose we are given two different monomorphisms $U \hookrightarrow f^* Y$ and $U \hookrightarrow f^* Y'$. The product then defines a monomorphism $U \hookrightarrow f^*(Y \times Y')$. We therefore have a diagram of sets

$$\mathscr{F}(U \subseteq f^* Y) \to \mathscr{F}(U \subseteq f^*(Y \times Y')) \leftarrow \mathscr{F}(U \subseteq f^* Y').$$

We claim that these maps are bijective. This is a special case the following:

($*$) Let U be a subobject of f^*Y, and suppose we are given a morphism $g : Y \to Z$ in $\mathcal{Y}$ such that the composite map $U \hookrightarrow f^*Y \to f^*Z$ is still a monomorphism. Then the induced map $\mathscr{F}(U \subseteq f^*Z) \to \mathscr{F}(U \subseteq f^*Y)$ is bijective.

In the situation of ($*$), the map $(U \subseteq f^*Y) \to (U \subseteq f^*Z)$ is a covering (for the Grothendieck topology of Exercise 4). We therefore have an equalizer diagram of sets

$$\mathscr{F}(U \subseteq f^*Z) \to \mathscr{F}(U \subseteq f^*Y) \rightrightarrows \mathscr{F}(U \subseteq f^*(Y \times_Z Y)).$$

Consequently, to prove ($*$), it will suffice to show that two different projection maps $Y \times_Z Y \to Y$ induce the same map from $\mathscr{F}(U \subseteq f^*Y)$ to $\mathscr{F}(U \subseteq f^*(Y \times_Z Y))$. It is clear that these maps agree after composition with the map

$$\mathscr{F}(U \subseteq f^*(Y \times_Z Y)) \to \mathscr{F}(U \subseteq f^*Y)$$

given by composition with the diagonal map $\delta : Y \to Y \times_Z Y$. Consequently, to prove assertion ($*$) for the map $g : Y \to Z$, it will suffice to prove ($*$) for the map $\delta : Y \to Y \times_Z Y$. In particular, assertion ($*$) is true whenever g is a monomorphism (since in this case δ is an isomorphism). However, the map δ is always a monomorphism, and therefore satisfies ($*$); it follows that ($*$) is true in general.

Using ($*$) (and the discussion which precedes it), we see that we can identify $\mathscr{F}(U \subseteq f^*Y)$ with $\mathscr{G}(U)$, for some set $\mathscr{G}(U)$ which is independent of the embedding $U \hookrightarrow f^*Y$. We leave it to the reader to verify that the construction $U \mapsto \mathscr{G}(U)$ is functorial (hint: realize $\mathscr{G}$ as the left Kan extension along the projection $\mathrm{Loc}(f)^{\mathrm{op}} \to \mathcal{X}_0^{\mathrm{op}}$). We claim that $\mathscr{G}$ is a sheaf on $\mathcal{X}_0$. To prove this, suppose we are given a covering $\{U_i \to U\}$ in $\mathcal{X}_0$. Choose monomorphisms

$$U \hookrightarrow f^*Y \qquad U_i \hookrightarrow f^*Y_i.$$

Then the diagram

$$\{(U_i \subseteq f^*(Y_i \times Y)) \to (U \subseteq f^*Y)\}$$

is a covering in $\mathrm{Loc}(f)$. Our assumption that $\mathscr{F}$ is a sheaf then supplies an equalizer diagram

$$\mathscr{F}(U \subseteq f^*Y) \to \prod_i \mathscr{F}(U_i \subseteq f^*(Y_i \times Y)) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \times_U U_j \subseteq f^*(Y_i \times Y_j \times Y)),$$

which we can rewrite as

$$\mathscr{G}(U) \to \prod_{i \in I} \mathscr{G}(U_i) \rightrightarrows \prod_{i,j} \mathscr{G}(U_i \times_U U_j).$$

It is now easy to verify that the construction $\mathscr{F} \mapsto \mathscr{G}$ determines an inverse to the functor $\mathrm{Shv}(\mathcal{X}_0) \to \mathrm{Shv}(\mathrm{Loc}(f))$ given by composition with π. □