Lecture 18: Localic Morphisms

March 10, 2018

In Lecture 17, we introduced the notion of an open morphism (and of an open surjection) between topoi. Our first goal in this lecture is to provide a nontrivial example:

Proposition 1. *The geometric morphism $\varphi : \mathrm{Shv}(\mathrm{Equiv}(\mathbf{Z})) \to \mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq\emptyset}}$ constructed in Lecture 16 is an open surjection.*

Proof. Let us identify the classifying topos $\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq\emptyset}}$ of nonempty sets with the category of functors $\mathrm{Fun}(\mathcal{S}\mathrm{et}^{\neq\emptyset}_{\mathrm{fin}}, \mathcal{S}\mathrm{et})$. For every nonempty finite set S, let $h_S : \mathcal{S}\mathrm{et}^{\neq\emptyset}_{\mathrm{fin}} \to \mathcal{S}\mathrm{et}$ denote the functor corepresented by S, given by the formula $h_S(T) = \mathrm{Hom}(S, T) = T^S$. Equivalently, we can identify h_S with X_0^S, where $X_0 \in \mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq\emptyset}}$ is the "universal nonempty object" appearing in Lecture 16 (given by the inclusion map $\mathcal{S}\mathrm{et}^{\neq\emptyset}_{\mathrm{fin}} \hookrightarrow \mathcal{S}\mathrm{et}$).

The objects h_S generate the topos $\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq\emptyset}}$. We first show that for each S, the associated morphism of locales

$$\mathrm{Sub}(\varphi^* h_S) \to \mathrm{Sub}(h_S)$$

is an open surjection. Note that we can identify $\varphi^* h_S$ with $\mathscr{F}^S$, where $\mathscr{F}$ is the sheaf on $\mathrm{Equiv}(\mathbf{Z})$ whose stalk at a point $E \in \mathrm{Equiv}(\mathbf{Z})$ is given $\mathscr{F}_E = \mathbf{Z}/E$.

Let's begin by analyzing the poset $\mathrm{Sub}(h_S)$. By definition, we can identify the elements of h_S with *subfunctors* of h_S. Such a subobject is specified by giving a property P of maps between finite sets $S \to T$, having the property that for every map $f : S \to T$ with the property P and any map $g : T \to T'$, the composite map $(g \circ f) : S \to T'$ also has the property P. Note that each f factors canonically as a composition

$$S \to S/E_f \hookrightarrow T,$$

where E_f is the equivalence relation given by $(sE_f s') \Leftrightarrow (f(s) = f(s'))$. It follows that if the quotient map $S \to S/E_f$ has the property P, then the map f also has the property P. Conversely, if f has the property P, then the quotient map $S \to S/E_f$ must also have the property P, since there exists a retraction of T back onto the quotient S/E_f (here we use the fact that S is not empty).

Let $\mathrm{Equiv}(S)$ denote the collection of all equivalence relations on S. From the preceding discussion, we obtain an injective map

$$\{\text{Subobjects of } h_S\} \hookrightarrow \{\text{Subsets of } \mathrm{Equiv}(S)\}$$

which carries a subobject $F \subseteq h_S$ to the collection of all equivalence relations $E \in \mathrm{Equiv}(S)$ for which the quotient map $S \to S/E$ belongs to $F(S/E) \subseteq h_S(S/E) = \mathrm{Hom}(S, S/E)$. This map is not surjective: its image consists of the collection of all subsets $U \subseteq \mathrm{Equiv}(S)$ having the property that if any refinement of an equivalence relation E belongs to U, then E also belongs to U. Put another way, the image consists of the collection of all *open* subsets of $\mathrm{Equiv}(S)$, where we equip $\mathrm{Equiv}(S)$ with the topology generated by sub-basic open sets $U_{s,s'} = \{E \in \mathrm{Equiv}(S) : sEs'\}$.

We can identify the sheaf $\mathscr{F}^S \in \mathrm{Shv}(\mathrm{Equiv}(\mathbf{Z}))$ with a topological space $\widetilde{\mathrm{Equiv}(\mathbf{Z})}_S$ equipped with a local homeomorphism $\widetilde{\mathrm{Equiv}(\mathbf{Z})}_S \to \mathrm{Equiv}(\mathbf{Z})$. The points of $\widetilde{\mathrm{Equiv}(\mathbf{Z})}_S$ are given by pairs (E, ρ), where E is an equivalence relation on $\mathbf{Z}$ and $\rho : S \to \mathbf{Z}/E$ is a map of sets. Under this identification, subobjects of

$\mathscr{F}^S$ correspond to open subsets of $\widetilde{\text{Equiv}(\mathbf{Z})}$, and the map of locales $\text{Sub}(\varphi^* h_S) \to \text{Sub}(h_S)$ arises from a continuous map of topological spaces

$$\pi_S : \widetilde{\text{Equiv}(\mathbf{Z})}_S \to \text{Equiv}(S)$$

which carries a pair $(E, \rho : S \to \mathbf{Z}/E)$ to the equivalence relation

$$E_\rho := \{(s, s') \in S^2 : \rho(s) = \rho(s')\} \in \text{Equiv}(S).$$

To complete the proof, it will suffice to show that π is an open surjection of topological spaces. It is clearly a surjection: every quotient of S can be embedded into a suitable quotient of $\mathbf{Z}$ (here we invoke the fact that $\mathbf{Z}$ is infinite). To show that it is open, let $U \subseteq \widetilde{\text{Equiv}(S)}_S$ be an open set containing a point (E, ρ). We wish to show that $\pi_S(U)$ contains an open neighborhood of E_ρ. In other words, we wish to show that if E_ρ is a refinement of some equivalence relation $\simeq$ on S, then we can find some other $(E', \rho') \in U$ such that $E'_{\rho'} = \simeq$. For this, we simply take E' to be the equivalence relation on $\mathbf{Z}$ corresponding to the quotient map $\mathbf{Z} \to (\mathbf{Z}/E) \amalg_{\text{Im}(\rho)} (S/\simeq)$, and ρ' to be the composite map $S \xrightarrow{\rho} \mathbf{Z}/E \to \mathbf{Z}/E'$.

To complete the proof, it will suffice to verify clause $(2')$ appearing in Proposition 14 of the previous lecture (for the set of generators $\{h_S\}_{S \in \mathcal{S}\text{et}^{\emptyset}_{\text{fin}}}$), it will suffice to show that for every map of nonempty finite sets $\alpha : S \to T$, the diagram of topological spaces

$$\begin{array}{ccc} \widetilde{\text{Equiv}(\mathbf{Z})}_T & \xrightarrow{\phi} & \widetilde{\text{Equiv}(\mathbf{Z})}_S \\ \downarrow{\scriptstyle \pi_T} & & \downarrow{\scriptstyle \pi_S} \\ \text{Equiv}(T) & \xrightarrow{\psi} & \text{Equiv}(S) \end{array}$$

has the property that $\psi^{-1}(\pi_S(U)) = \pi_T(\phi^{-1}U)$ for each open set $U \subseteq \widetilde{\text{Equiv}(\mathbf{Z})}_S$ as above. Suppose we are given a point (E, ρ) of U and that E_ρ is the restriction an equivalence relation E_T on the set T. We wish to show that, after replacing (E, ρ) by another point of U, we can arrange that ρ extends to a map $\widetilde{\rho} : T \to \mathbf{Z}/E$ satisfying $E_T = E_{\widetilde{\rho}}$. Equivalently, we wish to show that, possibly after changing (E, ρ), the monomorphism $S/E_\rho \hookrightarrow \mathbf{Z}/E$ can be extended to a monomorphism $T/E_T \hookrightarrow \mathbf{Z}/E$. This is always possible when the quotient $\mathbf{Z}/E$ is infinite, which can always be arranged by refining the equivalence relation E. $\square$

Let's now return to the picture of Lecture 16. Let $\mathcal{X}$ be a topos containing a set of generators $\{X_i\}_{i \in I}$. Set $X = \coprod_{i \in I}$, assume that $X \to \mathbf{1}$ is an effective epimorphism (which can always be arranged by including $\mathbf{1}$ as a generator), and form a pullback diagram

$$\begin{array}{ccc} \text{Enum}(X) & \xrightarrow{\pi} & \mathcal{X} \\ \downarrow & & \downarrow \\ \text{Shv}(\text{Equiv}(\mathbf{Z})) & \longrightarrow & \mathcal{X}_{\mathcal{S}\text{et}^{\neq\emptyset}} \end{array}$$

(we'll show later that such a thing exists). We would like to argue that $\text{Enum}(X)$ is localic. To prove this, it will be convenient to introduce a relative version of the condition of being localic.

Definition 2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. We will say that f is *localic* (or that $\mathcal{X}$ is *localic relative to* $\mathcal{Y}$) if there exist generators $\{X_i\}$ for $\mathcal{X}$, each of which appears as a subobject of $f^* Y_i$ for some $Y_i \in \mathcal{Y}$.

Example 3. Let $\mathcal{X}$ be a topos with a set of generators $\{X_i\}_{i \in I}$, and set $X = \coprod_{i \in I} X_i$. Then X is classified by a geometric morphism $\rho : \mathcal{X} \to \mathcal{X}_{\mathcal{S}\text{et}}$, characterized by the requirement that $X \simeq \rho^* X_0$ where $X_0 \in \mathcal{X}_{\mathcal{S}\text{et}}$ is the "universal" object. The geometric morphism ρ is localic: by construction, $\mathcal{X}$ is generated by subobjects of $\rho^* X_0$. (If $X \to \mathbf{1}$ is an effective epimorphism, then the classifying map $\mathcal{X} \to \mathcal{X}_{\mathcal{S}\text{et}^{\neq\emptyset}}$ is likewise localic).

Proposition 4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. Then:*

(1) *If $\mathcal{X}$ is localic, f is localic.*

(2) *If $\mathcal{Y}$ is localic and f is localic, then $\mathcal{X}$ is localic.*

Proof. If $\mathcal{X}$ is localic, then it is generated by subobjects of $\mathbf{1}_{\mathcal{X}} \simeq f^*\mathbf{1}_{\mathcal{Y}}$, so that f is localic. This proves (1). To prove (2), assume that $\mathcal{Y}$ is localic and that f is localic. Since f is localic, every object $X \in \mathcal{X}$ admits a covering $\{U_i \to X\}$, where each U_i appears as a subobject of f^*Y_i for some $Y_i \in \mathcal{Y}$. Since $\mathcal{Y}$ is localic, each Y_i admits a covering $\{V_{i,j} \to Y_i\}$, where each $V_{i,j}$ is a subobject of $\mathbf{1}_{\mathcal{Y}}$. Then X admits a covering $\{(f^*V_{i,j}) \times_{f^*Y_i} U_i \to X\}$, where each $(f^*V_{i,j}) \times_{f^*Y_i} U_i$ is a subobject of $f^*\mathbf{1}_{\mathcal{Y}} \simeq \mathbf{1}_{\mathcal{X}}$. □

Corollary 5. *Let $\mathcal{X}$ be a topos, so that there is an essentially unique geometric morphism $f : \mathcal{X} \to \mathcal{S}\mathrm{et}$. Then $\mathcal{X}$ is localic if and only if f is localic.*

We now consider a relative version of Proposition 4:

Proposition 6. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be geometric morphisms of topoi. Then:*

(1) *If $g \circ f$ is localic, then f is localic.*

(2) *If f and g are localic, then $g \circ f$ is localic.*

Proof. Assertion (1) follows from the observation that every subobject of $(g \circ f)^*(Z)$ is a subobject of an object of the form f^*Y, by taking $Y = g^*Z$. To prove (2), assume that f and g are localic. For each object $X \in \mathcal{X}$, we can find a cover $\{U_i \to X\}$, where each U_i is a subobject of some f^*Y_i. Then each Y_i admits a cover $\{V_{i,j} \to Y_i\}$, where each $V_{i,j}$ is a subobject of some $g^*Z_{i,j} \in \mathcal{Z}$. It follows that X admits a covering $\{f^*(V_{i,j}) \times_{f^*Y_i} U_i \to X\}$ where each $f^*(V_{i,j}) \times_{f^*Y_i} U_i$ can be regarded as a subobject of $(g \circ f)^*Z_{i,j}$. □

In Lecture 14, we showed that the datum of a localic topos $\mathcal{X}$ is determined by the datum of its underlying localic $\mathrm{Sub}(\mathbf{1}_{\mathcal{X}})$. Our next goal will be to establish a relative version of this observation, where we encode the datum of a localic geometric morphism $f : \mathcal{X} \to \mathcal{Y}$ in terms of a single "partially ordered" object of $\mathcal{Y}$. First, we need the following general fact:

Proposition 7. *Let $\mathcal{X}$ be a topos, and regard the construction $X \mapsto \mathrm{Sub}(X)$ as a contravariant functor from $\mathcal{X}$ to the category of sets (carrying each morphism $f : X \to Y$ to the inverse image map $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$). Then the functor $X \mapsto \mathrm{Sub}(X)$ is representable. In other words, there exists an object $\Omega_{\mathcal{X}}$ and bijections $\mathrm{Sub}(X) \simeq \mathrm{Hom}_{\mathcal{X}}(X, \Omega_{\mathcal{X}})$ depending functorially on X.*

Remark 8. The object $\Omega_{\mathcal{X}}$ appearing in the statement of Proposition 7 is called a *subobject classifier* of $\mathcal{X}$.

Proof of Proposition 7. The proof of Giraud's theorem shows that the Yoneda embedding $h : \mathcal{X} \to \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ induces an equivalence of $\mathcal{X}$ with the category of sheaves on itself, where we equip $\mathcal{X}$ with the topology given by the coverings. It will therefore suffice to show that the construction $X \mapsto \mathrm{Shv}(X)$ is a sheaf. In other words, we must show that for every covering $\{U_i \to X\}_{i \in I}$, the diagram of sets

$$\mathrm{Sub}(X) \to \prod_{i \in I} \mathrm{Sub}(U_i) \rightrightarrows \prod_{i,j \in I} \mathrm{Sub}(U_i \times_X U_j)$$

is an equalizer. Note that a subobject $V \subseteq X$ be recovered as the join $\bigvee_{i \in I} \mathrm{Im}(U_i \times_X V \to X)$, so the map $\mathrm{Sub}(X) \to \prod_{i \in I} \mathrm{Sub}(U_i)$ is injective. Conversely, suppose we are given an element of $\prod_{i \in I} \mathrm{Sub}(U_i)$, given by a collection of subobjects $V_i \subseteq U_i$. Set $V = \bigvee_{i \in I} \mathrm{Im}(V_i \to X)$. Then, for each $j \in I$, we have

$$\begin{aligned} V \times_X U_j &= (\bigvee_{i \in I} \mathrm{Im}(V_i \to X)) \times_X U_j \\ &= \bigvee_{i \in I} (\mathrm{Im}(V_i \to X) \times_X U_j) \\ &= \bigvee_{i \in I} (\mathrm{Im}(V_i \times_X U_j \to U_j). \end{aligned}$$

If $\{V_i\}_{i \in I}$ belongs to the equalizer, then we can rewrite this as

$$\begin{aligned}
\bigvee_{i \in I} \mathrm{Im}(U_i \times_X V_j \to U_j) &= \bigvee_{i \in I} \mathrm{Im}(U_i \times_X V_j \to V_j) \\
&= \bigvee_{i \in I} (\mathrm{Im}(U_i \to X) \times_X V_j) \\
&= (\bigvee_{i \in I} \mathrm{Im}(U_i \to X)) \times_X V_j \\
&= X \times_X V_j \\
&= V_j.
\end{aligned}$$

□

Corollary 9. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. Then the functor*

$$\mathcal{Y} \to \mathcal{S}\mathrm{et} \qquad Y \mapsto \mathrm{Sub}(f^* Y)$$

is representable by an object $\Omega_{\mathcal{X}/\mathcal{Y}} \in \mathcal{Y}$.

Proof. Take $\Omega_{\mathcal{X}/\mathcal{Y}} = f_* \Omega_{\mathcal{X}}$, where $\Omega_{\mathcal{X}}$ is a subobject classifier of $\mathcal{X}$ and f_* is right adjoint to f^*. □

Example 10. In the situation of Corollary 9, suppose that $\mathcal{Y} = \mathcal{S}\mathrm{et}$ is the topos of sets. Then we have

$$\begin{aligned}
\Omega_{\mathcal{X}/\mathcal{Y}} &= \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(\mathbf{1}_{\mathcal{S}\mathrm{et}}, \Omega_{\mathcal{X}/\mathcal{Y}}) \\
&\simeq \mathrm{Sub}(f^* \mathbf{1}_{\mathcal{S}\mathrm{et}}) \\
&\simeq \mathrm{Sub}(\mathbf{1}_{\mathcal{X}}).
\end{aligned}$$

In other words, we can identify $\Omega_{\mathcal{X}/\mathcal{Y}}$ with the underlying locale of the topos $\mathcal{X}$.

We will see in the next lecture that a localic morphism $f : \mathcal{X} \to \mathcal{Y}$ can be recovered from the object $\Omega_{\mathcal{X}/\mathcal{Y}} \in \mathcal{Y}$ (together with a suitable partial ordering of it).