Lecture 17: Open Morphisms

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Let $f : X \to Y$ be a continuous map of topological spaces. We say that f is *open* if, for every open subset $U \subseteq X$, the image $f(U)$ is an open subset of Y. In this case, the construction $U \mapsto f(U)$ determines a map of posets

$$f_! : \mathcal{U}(X) \to \mathcal{U}(Y).$$

We observe that $f_!$ can be characterized as the *left adjoint* of the inverse image map

$$f^* : \mathcal{U}(Y) \to \mathcal{U}(X) \qquad V \mapsto f^{-1}(V).$$

In other words, for open subsets $U \subseteq X$ and $V \subseteq Y$, we have $(f(U) \subseteq V) \Leftrightarrow (U \subseteq f^{-1}(V))$. This motivates the following:

Definition 1. Let $\mathcal{U}$ and $\mathcal{V}$ be locales, and let $f : \mathcal{U} \to \mathcal{V}$ be a morphism of locales, which we identify with a map of posets $f^* : \mathcal{V} \to \mathcal{U}$ which preserves finite meets and arbitrary joins. We say that f is *open* if it satisfies the following conditions:

(1) The map f^* admits a left adjoint $f_! : \mathcal{U} \to \mathcal{V}$. That is, for each $U \in \mathcal{U}$, there exists an object $f_!(U) \in \mathcal{V}$ such that $(f_!(U) \leq V) \Leftrightarrow (U \leq f^*(V))$, for each $V \in \mathcal{V}$.

(2) The map $f_!$ satisfies the projection formula $f_!(U \wedge f^*V) = f_!(U) \wedge V$ for $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

Example 2. Let $f : X \to Y$ be an open map of topological spaces. Then the induced map of locales $\mathcal{U}(X) \to \mathcal{U}(Y)$ is open. The projection formula of Definition 1 follows from the identity $f(U \cap f^{-1}(V)) = f(U) \cap V$ for $U \subseteq X$ and $V \subseteq Y$.

Warning 3. The converse of Example 2 need not hold, even if X and Y are assumed to be sober.

Proposition 4. *Let $f : X \to Y$ be a map of topological spaces, and suppose that every point of Y is closed (this condition is satisfied, for example, if Y is Hausdorff). If the map of posets $f^* : \mathcal{U}(Y) \to \mathcal{U}(X)$ admits a left adjoint $f_!$, then f is open. In particular, if the map of locales $\mathcal{U}(X) \to \mathcal{U}(Y)$ is open, then f is open.*

Proof. Let $U \subseteq X$ be an open set; we wish to show that $f(U) \subseteq Y$ is open. We will complete the proof by showing that $f(U) = f_!(U)$. By definition, $f_!(U)$ is the smallest open subset of Y which contains $f(U)$. Conversely, suppose that $y \in Y$ is a point which does not belong to $f(U)$. Since y is closed, the complement $Y - \{y\}$ is an open subset of Y which contains $f(U)$, so we have $f_!(U) \subseteq Y - \{y\}$. It follows that y does not belong to $f_!(U)$. $\square$

Example 5. Let Y be the Sierpinski space, consisting of a closed point x and an open point η. Let $X = \{x\}$ be the subset consisting only of the closed point. Then:

- The inclusion map $X \hookrightarrow Y$ is not open (as a map of topological spaces).
- The map of posets $f^{-1} : \mathcal{U}(Y) \to \mathcal{U}(X)$ has a left adjoint $f_! : \mathcal{U}(X) \to \mathcal{U}(Y)$.

- The map of locales $\mathcal{U}(X) \to \mathcal{U}(Y)$ is not open, because the map $f_!$ does not satisfy the projection formula: we have
$$\{\eta\} = f_!(X) \cap \{\eta\} \neq f_!(X \cap f^{-1}\{\eta\}) = f_!(\emptyset) = \emptyset.$$

Definition 6. Let $f : \mathcal{U} \to \mathcal{V}$ be a morphism of locales. We will say that f is an *open surjection* if it is open and $f_!(\mathbf{1}_{\mathcal{U}}) = \mathbf{1}_{\mathcal{V}}$, where $\mathbf{1}_{\mathcal{U}}$ and $\mathbf{1}_{\mathcal{V}}$ denote the largest elements of $\mathcal{U}$ and $\mathcal{V}$, respectively.

Remark 7. Let $f : \mathcal{U} \to \mathcal{V}$ be an open surjection. Then, for each $V \in \mathcal{V}$, we have

$$V = \mathbf{1}_{\mathcal{V}} \wedge V = f_!(\mathbf{1}_{\mathcal{U}}) \wedge V = f_!(\mathbf{1}_{\mathcal{U}} \wedge f^* V) = f_! f^* V.$$

It follows that an open morphism f is an open surjection if and only if the counit map $f_! f^* \to \mathrm{id}_{\mathcal{V}}$ is an isomorphism.

Example 8. Let $f : X \to Y$ be a map of topological spaces. If f is an open surjection, then the induced map of locales $\mathcal{U}(X) \to \mathcal{U}(Y)$ is an open surjection. The converse holds if every point of Y is closed.

Remark 9. Let $f : \mathcal{U} \to \mathcal{V}$ and $g : \mathcal{V} \to \mathcal{W}$ be morphisms of locales. If f and g are open (respectively open surjections), then the composition $g \circ f$ is also open (respectively an open surjection). The map $(g \circ f)_!$ is given by $g_! \circ f_!$, and the projection formula follows from the calculation

$$(g \circ f)_!(U) \wedge W = g_!(f_!(U)) \wedge W = g_!(f_!(U) \wedge g^* W) = g_!(f_!(U \wedge f^* g^* W)) = (g \circ f)_!(U \wedge (g \circ f)^* W).$$

Example 10. Let $\mathcal{X}$ be a topos and let $f : X \to Y$ be a morphism between objects of $\mathcal{X}$. Then the construction
$$(V \subseteq Y) \mapsto (V \times_Y X \subseteq X)$$
determines a map of locales $\mathrm{Sub}(X) \to \mathrm{Sub}(Y)$. This map is automatically open: it has a left adjoint given by the construction
$$(U \subseteq X) \mapsto (\mathrm{Im}(U \to Y) \subseteq Y),$$
and the projection formula follows from the calculation

$$\begin{aligned} \mathrm{Im}(U \to Y) \times_Y V &= \mathrm{Im}(U \times_Y V \to V) \\ &= \mathrm{Im}(U \times_X (X \times_Y V) \to Y). \end{aligned}$$

Remark 11. In the situation of Example 10, the map of locales $\mathrm{Sub}(X) \to \mathrm{Sub}(Y)$ is an open surjection if and only if f is an effective epimorphism.

Definition 12. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. We say that f is *open* if it satisfies the following conditions: Then, for every object $Y \in \mathcal{Y}$, the pullback functor f^* determines a map from $\mathrm{Sub}(Y)$ to $\mathrm{Sub}(f^* Y)$ which preserves arbitrary joins and finite meets; that is, it determines a map of locales $f_Y : \mathrm{Sub}(f^*(Y)) \to \mathrm{Sub}(Y)$.

We say that f is *open* if it satisfies the following:

(1) For each $Y \in \mathcal{Y}$, the map of posets $f_Y^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(f^*))$ admits a left adjoint $f_{Y!}$.

(2) For every morphism $Y' \to Y$ in $\mathcal{Y}$ and each subobject $U \subseteq Y$, we have $Y' \times_Y (f_{Y!} U) = f_{Y'!}(f^* Y' \times_{f^* Y} U)$ (as subobjects of Y').

Note that (1) and (2) imply that each $f_Y : \mathrm{Sub}(f^*(Y)) \to \mathrm{Sub}(Y)$ is an open morphism of locales (the projection formula follows by applying (2) in the case where Y' is a subobject of Y). We say that f is an *open surjection* if each f_Y is an open surjection of locales.

Remark 13. Let $f : \mathcal{X} \to \mathcal{Y}$ be an open geometric morphism (respectively open surjection) of topoi. Then the induced map of locales $\mathrm{Sub}(\mathbf{1}_{\mathcal{X}}) \to \mathrm{Sub}(\mathbf{1}_{\mathcal{Y}})$ is open (respectively an open surjection).

In the situation of Definition 12, we do not need to consider *every* object $Y \in \mathcal{Y}$; it suffices to check on a set of generators.

Proposition 14. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. Suppose that there exists a collection of generators $\mathcal{Y}_0$ satisfying the following:*

$(1')$ *For each $Y \in \mathcal{Y}_0$, the map of posets $f^*_Y : \mathrm{Sub}(Y) \to \mathrm{Sub}(f^*))$ admits a left adjoint $f_{Y!}$.*

$(2')$ *For every morphism $Y' \to Y$ between objects of $\mathcal{Y}_0$ and each subobject $U \subseteq Y$, we have $Y' \times_Y (f_{Y!}U) = f_{Y'!}(f^*Y' \times_{f^*Y} U)$ (as subobjects of Y').*

Then f is open. Moreover, if the map of locales $f_Y : \mathrm{Sub}(f^*Y) \to \mathrm{Sub}(Y)$ is an open surjection for each $Y \in \mathcal{Y}_0$, then f is an open surjection.

Proof. Let Y be an arbitrary object of $\mathcal{Y}$, and choose a covering $\{Y_i \to Y\}_{i \in I}$ where each Y_i belongs to $\mathcal{Y}_0$. Then $\{f^*Y_i \to f^*Y\}_{i \in I}$ is a covering in the topos X. Then then ahve

$$\begin{aligned}
(U \subseteq f^*V) \quad &\Leftrightarrow \quad (\forall i \in I)(U \times_{f^*Y} f^*Y_i \subseteq f^*V \times_{f^*Y} f^*Y_i) \\
&\Leftrightarrow \quad (\forall i \in I)(U \times_{f^*Y} f^*Y_i \subseteq f^*(V \times_Y Y_i) \\
&\Leftrightarrow \quad (\forall i \in I) f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \subseteq V \times_Y Y_i \\
&\Leftrightarrow \quad (\forall i \in I) \operatorname{Im}(f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \to Y) \subseteq V \\
&\Leftrightarrow \quad \bigvee_{i \in I} \operatorname{Im}(f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \to Y) \subseteq V.
\end{aligned}$$

It follows that the map of posets f^*_Y has a left adjoint $f_{Y!}$, given by the formula

$$f_{Y!}(U) = \bigvee_{i \in I} \operatorname{Im}(f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \to Y).$$

That is, f satisfies condition (1) of Definition 12.

We now verify condition (2). Let $\{Y_i \to Y\}$ be as above, and suppose we are given a morphism $Y' \to Y$ in $\mathcal{Y}$. For each $i \in I$, choose a covering $\{Y'_{i,j} \to Y_i \times_Y Y'\}$ where each $Y'_{i,j}$ belongs to $\mathcal{Y}_0$, so that the $Y'_{i,j}$ comprise a covering of Y'. For $U \in \mathrm{Sub}(f^*Y)$, we compute

$$\begin{aligned}
f_{Y'!}(U \times_{f^*Y} f^*Y') \quad &= \quad \bigvee_{i,j} \operatorname{Im}(f_{Y'_{i,j}!}(U \times_{f^*Y} f^*Y' \times_{f^*Y'} f^*Y'_{i,j}) \to Y') \\
&= \quad \bigvee_{i,j} \operatorname{Im}(f_{Y'_{i,j}!}(U \times_{f^*Y} f^*Y'_{i,j}) \to Y') \\
&= \quad \bigvee_{i,j} \operatorname{Im}(f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \times_{Y_i} Y'_{i,j} \to Y') \\
&= \quad \bigvee_{i} \operatorname{Im}(f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \times_{Y_i} (Y_i \times_Y Y') \to Y') \\
&= \quad \bigvee_{i} \operatorname{Im}(f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \times_Y Y' \to Y') \\
&= \quad \bigvee_{i} (\operatorname{Im}(f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \to Y) \times_Y Y') \\
&= \quad (\bigvee_{i} \operatorname{Im}(f_{Y_i!}(U \times_{f^*Y} f^*Y_i) \to Y)) \times_Y Y' \\
&= \quad (f_{Y!}U) \times_Y Y'.
\end{aligned}$$

Here the first equality follows from the construction of $f_{Y'!}$, the third from assumption $(2')$, the fourth from the fact that $\{Y'_{i,j} \to Y_i \times_Y Y'\}$ is a covering for each $i \in I$, the sixth from the fact that the formation of images commutes with pullback, the seventh from the fact that joins commute with pullback, and the eighth from the construction of $f_{Y!}$.

We conclude by observing that if f_{Y_i} is an open surjection, then we have

$$f_{Y!}(f^*(Y)) = \bigvee_{i \in I} \mathrm{Im}(f_{Y_i!}(f^*Y \times_{f^*Y} f^*Y_i) \to Y) = \bigvee_{i \in I} \mathrm{Im}(Y_i \to Y) = Y,$$

so that f_Y is also an open surjection. $\square$

Corollary 15. *Let $\mathcal{X}$ be a topos and let $\mathcal{U}$ be a locale. Suppose we are given a geometric morphism $f : \mathcal{X} \to \mathrm{Shv}(\mathcal{U})$, which we can identify with a morphism of locales $g : \mathrm{Sub}(\mathbf{1}_{\mathcal{X}}) \to \mathcal{U}$. Then f is an open morphism of topoi (in the sense of Definition 12) if and only if g is an open morphism of locales (in the sense of Definition 1). Similarly, f is an open surjection if and only if g is an open surjection.*

Proof. The "only if" direction follows from Remark 13. Conversely suppose that g is an open morphism (open surjection) of locales; we wish to show that f is open (an open surjection). Let $h : \mathcal{U} \to \mathrm{Shv}(\mathcal{U})$ be the Yoneda embedding, so that $\mathcal{U}$ is generated by objects of the form h_U. For each $U \in \mathcal{U}$, we can identify $\mathrm{Sub}(h_U)$ with the subset $\{V \in \mathcal{U} : V \leq U\}$, and $\mathrm{Sub}(f^* h_U)$ with the subset $\{W \in \mathrm{Sub}(\mathbf{1}_{\mathcal{X}}) : W \subseteq g^* U\}$. Consequently, we can identify $f^*_{h_U}$ with the map

$$\{V \in \mathcal{U} : V \leq U\} \to \{W \in \mathrm{Sub}(\mathbf{1}_{\mathcal{X}}) : W \subseteq g^* U\}$$

given by restriction of g^*. Since g is open, this map admits a left adjoint $f_{h_U!}$, which is given by the restriction of $g_! : \mathcal{U} \to \mathrm{Sub}(\mathbf{1}_{\mathcal{X}})$. Consequently, f satisfies condition $(1')$ of Proposition 14. Condition $(2')$ follows from the projection formula for $g_!$. It follows that f is open. If g is an open surjection, then we have $f_{h_U!}(f^*_{h_U} U) = g_! g^* U = U$ by virtue of Remark 7. $\square$

Corollary 16. *Let $f : \mathcal{U} \to \mathcal{V}$ be a morphism of locales. Then f is open (respectively an open surjection) if and only if the induced map $\mathrm{Shv}(\mathcal{U}) \to \mathrm{Shv}(\mathcal{V})$ is an open geometric morphism of topoi.*

Corollary 17. *Let $\mathcal{X}$ be any topos and let $\mathrm{Sub}(\mathbf{1}_{\mathcal{X}})$ be its underlying locale. Then the unit map $\mathcal{X} \to \mathrm{Shv}(\mathrm{Sub}(\mathbf{1}_{\mathcal{X}}))$ is an open surjection of topoi.*