Lecture 16: Enumerations

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The contents of the course up to this point can be roughly summarized in the following diagram:

$$\begin{array}{ccccc}
\{\text{(Typed) First-Order Theories}\} & & & & \{\text{(Sober) Topological Spaces}\} \\
\sim \downarrow {\scriptstyle T \mapsto \mathrm{Syn}(T)} & & & & \sim \downarrow {\scriptstyle X \mapsto \mathcal{U}(X)} \\
\{\text{(Small) Boolean Pretopoi}\} & & & & \{\text{Spatial Locales}\} \\
\downarrow & & & & \downarrow \\
\{\text{(Small) Pretopoi}\} & & & & \{\text{Locales}\} \\
\sim \downarrow {\scriptstyle \mathcal{C} \mapsto \mathrm{Shv}(\mathcal{C})} & & & & \sim \downarrow {\scriptstyle \mathcal{X} \mapsto \mathrm{Shv}(\mathcal{X})} \\
\{\text{Coherent Topoi}\} & \hookrightarrow & \{\text{Topoi}\} & \hookleftarrow & \{\text{Localic Topoi}\}
\end{array}$$

Our ultimate goal in this course is to try to use the right side of this diagram to study the left side. To every first-order theory T, we can associate a *classifying topos* $\mathrm{Shv}(\mathrm{Syn}(T)) \simeq \mathrm{Shv}(\mathrm{Syn}_0(T))$. Our strategy will be to analyze T by comparing its classifying topos with localic topoi.

Question 1. Given an arbitrary topos $\mathcal{X}$, how well can $\mathcal{X}$ be approximated by a localic topos?

Our more immediate goal in this course is to prove the following result of Joyal and Tierney, which gives a precise answer to Question 1:

Theorem 2 (Joyal-Tierney)**.** *Let $\mathcal{X}$ be a topos. Then there exists a localic topos $\mathcal{U}$ and a geometric morphism $\mathcal{U} \to \mathcal{X}$ which is an effective epimorphism in the 2-categorical sense: that is, $\mathcal{X}$ can be identified with the colimit of the diagram*

$$\mathcal{U} \times_{\mathcal{X}} \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \;\substack{\longrightarrow\\\longrightarrow}\; \mathcal{U} \, ,$$

formed in the 2-category of topoi and geometric morphisms. Moreover, every topos appearing in this diagram is localic.

We will give a more precise formulation (and proof) of Theorem 2 slowly over the course of the next several lectures. Let us begin by outlining our strategy. Let $\mathcal{X}$ be a topos. If $\mathcal{X}$ is localic, then there is nothing to do (we can simply take $\mathcal{U} = \mathcal{X}$). Otherwise, there exists some object $X \in \mathcal{X}$ which cannot be covered by subobjects of the final object of $\mathcal{X}$. Let us proceed immediately to the "hardest" case, where we know nothing at all about the object X.

Definition 3. Let $\mathcal{S}\mathrm{et}_{\mathrm{fin}}$ denote the category of finite sets, and let $\mathcal{X}_{\mathcal{S}\mathrm{et}} = \mathrm{Fun}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}, \mathcal{S}\mathrm{et})$. We will refer to $\mathcal{X}_{\mathcal{S}\mathrm{et}}$ as the *classifying topos of sets*.

Remark 4. We can also describe $\mathcal{X}_{\mathcal{S}\mathrm{et}}$ as the category of functors from $\mathcal{S}\mathrm{et}$ to $\mathcal{S}\mathrm{et}$ which preserve filtered colimits.

The topos $\mathcal{X}_{\mathcal{S}\mathrm{et}}$ contains a distinguished object X_0, given by the inclusion functor $\mathcal{S}\mathrm{et}_{\mathrm{fin}} \hookrightarrow \mathcal{S}\mathrm{et}$ (under the alternate description of Remark 4, this corresponds to the identity functor $\mathrm{id} : \mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$).

Proposition 5. *For any topos $\mathcal{X}$, evaluation on the object $X_0 \in \mathcal{X}_{\mathcal{S}\mathrm{et}}$ induces an equivalence of categories*

$$\mathrm{Fun}^*(\mathcal{X}_{\mathcal{S}\mathrm{et}}, \mathcal{X}) \to \mathcal{X}\,.$$

In other words, we can identify geometric morphisms from $\mathcal{X}$ to $\mathcal{X}_{\mathcal{S}\mathrm{et}}$ with objects of the topos $\mathcal{X}$.

Remark 6. We can paraphrase Proposition 5 as follows: the classifying topos $\mathcal{X}_{\mathcal{S}\mathrm{et}}$ is freely generated, as a topos, by the object X_0.

Proof of Proposition 5. Note that $\mathcal{X}_{\mathcal{S}\mathrm{et}}$ can be described as the topos of presheaves on the category $\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}}$. Using Lecture 12, we deduce that composition with the Yoneda embedding induces an equivalence of categories

$$\mathrm{Fun}^*(\mathcal{X}_{\mathcal{S}\mathrm{et}}, \mathcal{X}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}}, \mathcal{X}),$$

where $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}}, \mathcal{X})$ denotes the category of functors $F : \mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}} \to \mathcal{X}$ which preserve finite limits.

For every finite set S, we can write S as a coproduct $\coprod_{s \in S} \{s\}$ in the category $\mathcal{S}\mathrm{et}_{\mathrm{fin}}$, or equivalently as a product of singletons in the category $\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}}$. It follows that for any functor F as above, we have canonical isomorphisms $F(S) = \prod_{s \in S} F(\{s\}) \simeq F(*)^S$, so that F is determined by its value on the one-point set $*$. Conversely, for any $X \in \mathcal{X}$, the construction $S \mapsto X^S$ determines a functor $\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}} \to \mathcal{X}$ preserving finite limits. It follows that evaluation at $*$ induces an equivalence of categories

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}}, \mathcal{X}) \to \mathcal{X}\,.$$

Note that the composite equivalence

$$\mathrm{Fun}^*(\mathcal{X}_{\mathcal{S}\mathrm{et}}, \mathcal{X}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}}, \mathcal{X}) \simeq \mathcal{X}$$

is given by evaluation on the representable presheaf $h_* \in \mathcal{X}_{\mathcal{S}\mathrm{et}} = \mathrm{Fun}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}, \mathcal{S}\mathrm{et})$, which is the object X_0. □

The topos $\mathcal{X}_{\mathcal{S}\mathrm{et}}$ is not localic. In some sense, it is as far from localic as possible: note that if there were to exist a covering $\{U_i \to X_0\}$ of the object $X_0 \in \mathcal{X}_{\mathcal{S}\mathrm{et}}$ by subobjects of the final object in $\mathcal{X}_{\mathcal{S}\mathrm{et}}$, then we would be able to find such a covering for *any* object X of *any* topos $\mathcal{X}$, since Proposition 5 supplies a geometric morphism $f : \mathcal{X} \to \mathcal{X}_{\mathcal{S}\mathrm{et}}$ satisfying $X \simeq f^* X_0$.

Remark 7. Let $\mathcal{X}$ be a topos containing an object X, so that we can write $X = \pi^* X_0$ for an essentially unique geometric morphism π from $\mathcal{X} \to \mathcal{X}_{\mathcal{S}\mathrm{et}}$. Then the right adjoint π_* of π^* is given by the formula

$$\begin{aligned}
\pi_*(Y)(S) &\simeq \mathrm{Hom}_{\mathcal{X}_{\mathcal{S}\mathrm{et}}}(h_S, \pi_*(Y)) \\
&\simeq \mathrm{Hom}_{\mathcal{X}_{\mathcal{S}\mathrm{et}}}(\prod_{s \in S} X_0, \pi_*(Y)) \\
&\simeq \mathrm{Hom}_{\mathcal{X}}(\pi^* \prod_{s \in S} X_0, Y) \\
&\simeq \mathrm{Hom}_{\mathcal{X}}(\prod_{s \in S} \pi^* X_0, Y) \\
&\simeq \mathrm{Hom}_{\mathcal{X}}(X^S, Y)
\end{aligned}$$

for $Y \in \mathcal{X}$ and $S \in \mathcal{S}\mathrm{et}_{\mathrm{fin}}$.

It will be convenient to introduce a slight variant of Definition 3.

Definition 8. Let us regard the category $\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}}$ as equipped with a Grothendieck topology, where a collection of maps of finite sets $\{S \to T_i\}_{i \in I}$ is a covering if $S \to T_i$ is injective for some i. We will denote the topos $\mathrm{Shv}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}}) \subseteq \mathcal{X}_{\mathcal{S}\mathrm{et}}$ by $\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq \emptyset}}$, and refer to it as the *classifying topos for nonempty sets.*

Exercise 9. Show that a functor $F : \mathcal{S}\mathrm{et}_{\mathrm{fin}} \to \mathcal{S}\mathrm{et}$ is a sheaf for the topology of Definition 8 if and only if the diagram of sets

$$F(\emptyset) \to F(*) \rightrightarrows F(* \amalg *)$$

is an equalizer. In particular, the object $X_0 \in \mathcal{X}_{\mathcal{S}\mathrm{et}}$ belongs to the subcategory $\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq\emptyset}}$.

Exercise 10. Let $\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\neq\emptyset}$ denote the category of nonempty finite sets. Show that composition with the inclusion functor $\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\neq\emptyset}$ induces an equivalence of categories

$$\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq\emptyset}} \subseteq \mathrm{Fun}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\neq\emptyset}, \mathcal{S}\mathrm{et}).$$

We have the following analogue of Proposition 5:

Proposition 11. *For any topos $\mathcal{X}$, evaluation on the object $X_0 \in \mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq\emptyset}}$ induces a fully faithful embedding*

$$\mathrm{Fun}^*(\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq\emptyset}}, \mathcal{X}) \to \mathcal{X}$$

whose essential image consists of those objects $X \in \mathcal{X}$ for which the map $X \to \mathbf{1}$ is an effective epimorphism in $\mathcal{X}$.

Proof. Using Lecture 12, we deduce that composition with the Yoneda embedding induces a fully faithful embedding

$$\mathrm{Fun}^*(\mathcal{X}_{\mathcal{S}\mathrm{et}}, \mathcal{X}) \to \mathrm{Fun}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}}, \mathcal{X})$$

whose essential image is spanned by those functors $F : \mathcal{S}\mathrm{et}_{\mathrm{fin}}^{\mathrm{op}} \to \mathcal{X}$ which preserve finite limits and coverings. The first condition guarantees that we can write $F(S) = X^S$ for some object $X \in \mathcal{X}$ (as in the proof of Proposition 5). In this case, preservation of coverings translates to the condition that for any injective map $S \to T$ of finite set, the induced map $X^T \to X^S$ is an effective epimorphism of sets. If $S \neq \emptyset$, then this is automatic (since the map $X^T \to X^S$ admits a section). Consequently, preservation of coverings translates to the condition that the projection map $X^T \to \mathbf{1}$ is an effective epimorphism in $\mathcal{X}$. We conclude by observing that if this condition is satisfied when T is a singleton, then it is satisfied for all T (since the collection of effective epimorphisms in $\mathcal{X}$ is closed under products). □

Construction 12. Let $\mathrm{Equiv}(\mathbf{Z})$ denote the set of all equivalence relations on the set $\mathbf{Z}$ of integers. We will regard $\mathrm{Equiv}(\mathbf{Z})$ as a topological space, where the collection of subsets

$$U_{i,j} = \{E \in \mathrm{Equiv}(\mathbf{Z}) : iEj\}$$

forms a sub-basis of open sets. (In other words, we equip $\mathrm{Equiv}(\mathbf{Z})$ with the coarsest topology for which each $U_{i,j}$ is an open set).

Let $\mathscr{F}$ denote the sheaf of sets on $\mathrm{Equiv}(\mathbf{Z})$ whose stalk at a point $E \in \mathrm{Equiv}(\mathbf{Z})$ is given by the quotient $\mathbf{Z}/E$. More formally, we can describe $\mathscr{F}$ as the quotient $\underline{\mathbf{Z}}/\mathscr{R}$, where $\underline{\mathbf{Z}}$ denotes the constant sheaf with value $\mathbf{Z}$, and $\mathscr{R} \subseteq \underline{\mathbf{Z}} \times \underline{\mathbf{Z}}$ is the equivalence relation given by $\amalg_{i,j \in \mathbf{Z}} : h_{U_{i,j}}$.

Note that the canonical map $\mathscr{F} \to \mathbf{1}$ is an effective epimorphism in the topos $\mathrm{Shv}(\mathrm{Equiv}(\mathbf{Z}))$ (by construction, it comes with many sections). Applying Proposition 11, we see that there is an essentially unique geometric morphism

$$\varphi : \mathrm{Shv}(\mathrm{Equiv}(\mathbf{Z})) \to \mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq\emptyset}}$$

satisfying $\varphi^* X_0 = \mathscr{F}$. This gives a localic "approximation" to the topos $\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq\emptyset}}$.

Construction 13. Let $\mathcal{X}$ be a topos and let $X \in \mathcal{X}$ be an object for which $X \to \mathbf{1}$ is an effective epimorphism, so we can write $X = \varphi^* X_0$ for an essentially unique geometric morphism $\psi : \mathcal{X} \to \mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq\emptyset}}$. We let $\mathrm{Enum}(X)$ denote the fiber product $\mathcal{X} \times_{\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq\emptyset}}} \mathrm{Shv}(\mathrm{Equiv}(\mathbf{Z}))$, formed in the 2-category of topoi. We will refer to $\mathrm{Enum}(X)$ as the *topos of enumerations of X*.

Remark 14. In the situation of Construction 13, we have a commutative diagram of topoi

$$\begin{array}{ccc} \mathrm{Enum}(X) & \xrightarrow{\pi} & \mathcal{X} \\ \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi} \\ \mathrm{Shv}(\mathrm{Equiv}(\mathbf{Z})) & \xrightarrow{\varphi} & \mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq\emptyset}} \end{array} ,$$

hence canonical isomorphisms

$$\pi^* X \simeq \pi^* \psi^* X_0 \simeq \psi'^* \varphi^* X_0 \simeq \psi'^* \mathscr{F} \simeq \psi'^* (\underline{\mathbf{Z}}) / \psi'^* \mathscr{R} .$$

It follows that, after pulling back along the geometric morphism π', the object X can be equipped with an *enumeration*: that is, a cover by countably many copies of the final object.

The proof of Theorem 2 will proceed via Construction 13. Given a topos $\mathcal{X}$, we can choose a set of generators $\{X_i\}_{i \in I}$. Enlarging this set if necessary (say by adding the final object $\mathbf{1}$), we can assume that the coproduct $X = \coprod_{i \in I} X_i$ has the property that the projection $X \to \mathbf{1}$ is an effective epimorphism. In this case, we will show (following Joyal and Tierney) that the projection map

$$\pi : \mathrm{Enum}(X) \to \mathcal{X}$$

satisfies the requirements of Theorem 2.

Warning 15. Recall that a *point* of a topos $\mathcal{X}$ is a geometric morphism to $\mathcal{X}$ from the topos $\mathcal{S}\mathrm{et}$ of sets. Then:

- Points of the topos $\mathcal{X}_{\mathcal{S}\mathrm{et}}$ can be identified with sets (Proposition 5), and points of the topos $\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq\emptyset}}$ can be identified with nonempty sets (Proposition 11).
- One can show that the space $\mathrm{Equiv}(\mathbf{Z})$ is sober, so that points of the topos $\mathrm{Shv}(\mathrm{Equiv}(\mathbf{Z}))$ can be identified with points of $\mathrm{Equiv}(\mathbf{Z})$: that is, with equivalence relations E on $\mathbf{Z}$. Put differently, we can think of a point of $\mathrm{Shv}(\mathrm{Equiv}(\mathbf{Z}))$ as a set S equipped with a surjection $\mathbf{Z} \to S$.

Suppose now that we are in the situation of Construction 13 and that we are given a point $x : \mathcal{S}\mathrm{et} \to \mathcal{X}$. Then lifting x to a point of $\mathrm{Enum}(X) = \mathcal{X} \times_{\mathcal{X}_{\mathcal{S}\mathrm{et}^{\neq\emptyset}}} \mathrm{Shv}(\mathrm{Equiv}(\mathbf{Z}))$ is equivalent to choosing a surjection $\mathbf{Z} \to x^*(X)$.

It follows that Construction 13 can easily produce examples of topoi that have no points at all. For example, we could take $\mathcal{X} = \mathcal{S}\mathrm{et}$ to be the topos of sets, and $X \in \mathcal{X}$ to be any uncountable set. In this case, the enumeration topos $\mathrm{Enum}(X)$ will not have any points (because there are no surjections from $\mathbf{Z}$ to X). Nevertheless, we will see that the projection map $\pi : \mathrm{Enum}(X) \to \mathcal{X}$ still satisfies the requirements of Theorem 2. (In particular, $\mathrm{Enum}(X)$ is a localic topos whose associated locale is *non-spatial*.)