Lecture 14X: Pro-Objects

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Let $\mathcal{C}$ and $\mathcal{D}$ be categories which admit finite limits. We let $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which are *left exact*: that is, which preserve finite limits.

Definition 1. Let $\mathcal{C}$ be an essentially small category which admits finite limits. We let $\mathrm{Pro}(\mathcal{C})$ denote the category $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S}\mathrm{et})^{\mathrm{op}}$. We will refer to the objects of $\mathrm{Pro}(\mathcal{C})$ as *pro-objects* of $\mathcal{C}$, and to $\mathrm{Pro}(\mathcal{C})$ as *the category of pro-objects of* $\mathcal{C}$.

Remark 2. Let $\mathcal{C}$ be a category which admits finite limits. For each object $C \in \mathcal{C}$, the functor $D \mapsto \mathrm{Hom}_{\mathcal{C}}(C, D)$ preserves finite limits, and can therefore be regarded as an object of $\mathrm{Pro}(\mathcal{C})$. The Yoneda embedding $C \mapsto \mathrm{Hom}_{\mathcal{C}}(C, \bullet)$ induces a fully faithful functor $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$. In what follows, we will generally abuse notation by identifying $\mathcal{C}$ with its essential image in $\mathrm{Pro}(\mathcal{C})$.

Remark 3. In the category of sets, the formation of finite limits commutes with filtered colimits. It follows that the full subcategory $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S}\mathrm{et})$ is closed under filtered colimits in $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$. In particular, the category $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S}\mathrm{et})$ admits filtered colimits, so that $\mathrm{Pro}(\mathcal{C}) = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S}\mathrm{et})^{\mathrm{op}}$ admits filtered limits.

Example 4. Let $\mathcal{C}$ be a small category which admits finite limits. Suppose we are given a diagram $\{C_\alpha\}_{\alpha \in \mathcal{A}^{\mathrm{op}}}$ indexed by (the opposite of) a filtered category $\mathcal{A}$. Then we can also regard $\{C_\alpha\}$ as a diagram in the category $\mathrm{Pro}(\mathcal{C})$ (via the Yoneda embedding), where we can take the inverse limit. We will abuse notation by denoting this inverse limit also by $\{C_\alpha\}$. When viewed as a functor from $\mathcal{C}$ to the category of sets, it is given by the construction $D \mapsto \varinjlim_\alpha \mathrm{Hom}_{\mathcal{C}}(C_\alpha, D)$.

Remark 5. Let $\mathcal{C}$ be an essentially small category which admits finite limits, and let $F : \mathcal{C} \to \mathcal{S}\mathrm{et}$ be a functor. Then F has a canonical presentation

$$\varinjlim_{(C,\eta) \in \mathcal{A}} \mathrm{Hom}_{\mathcal{C}}(C, \bullet)$$

has a colimit of corepresentable functors, indexed by the category $\mathcal{A}$ whose objects are pairs (C, η) where $C \in \mathcal{C}$ and $\eta \in F(C)$, where

$$\mathrm{Hom}_{\mathcal{A}}((C, \eta), (C', \eta')) = \{f \in \mathrm{Hom}_{\mathcal{C}}(C', C) : F(f)(\eta') = \eta\}.$$

If the functor F preserves finite limits, then the category $\mathcal{A}$ is filtered. It follows that every object of $\mathrm{Pro}(\mathcal{C})$ has a (canonical) presentation as a filtered limit of objects of $\mathcal{C}$.

Remark 6. Let $\mathcal{C}$ be an essentially small category which admits finite limits. From the above discussion, we see that the category $\mathrm{Pro}(\mathcal{C})$ can be described more informally as follows:

- The objects of $\mathrm{Pro}(\mathcal{C})$ are diagrams $\{C_\alpha\}$ in $\mathcal{C}$, indexed by (the opposite of) a small filtered category.
- Given two such diagrams $\{C_\alpha\}$ and $\{D_\beta\}$, we have

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\{C_\alpha\}, \{D_\beta\}) = \varprojlim_\beta \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\{C_\alpha\}, D_\beta) = \varprojlim_\beta \varinjlim_\alpha \mathrm{Hom}_{\mathcal{C}}(C_\alpha, D_\beta).$$

Remark 7. The category $\mathrm{Pro}(\mathcal{C})$ can be characterized by a universal property. Let $\mathcal{D}$ be any category which admits small filtered limits, and let $\mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D})$ be the full subcategory of $\mathrm{Fun}(\mathrm{Pro}(\mathcal{C}), \mathcal{D})$ spanned by those functors which preserve small filtered limits. Then composition with the inclusion $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ induces an equivalence of categories $\mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. In other words, every functor $f : \mathcal{C} \to \mathcal{D}$ admits an essentially unique extension to a functor $F : \mathrm{Pro}(\mathcal{C}) \to \mathcal{D}$ which preserves small filtered limits.

Let $\mathcal{C}$ be an essentially small category which admits finite limits and let $\mathcal{I}$ be any small category. Since $\mathrm{Pro}(\mathcal{C})$ admits small filtered limit, the functor category $\mathrm{Fun}(\mathcal{I}, \mathrm{Pro}(\mathcal{C}))$ also admits small filtered limits (which are computed pointwise). Consequently, the inclusion functor $\mathrm{Fun}(\mathcal{I}, \mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{I}, \mathrm{Pro}(\mathcal{C}))$ admits an essentially unique extension to a functor

$$\mathrm{Pro}(\mathrm{Fun}(\mathcal{I}, \mathcal{C})) \to \mathrm{Fun}(\mathcal{I}, \mathrm{Pro}(\mathcal{C}))$$

which preserves small filtered limits. We will use the following standard result:

Proposition 8. *Let $\mathcal{C}$ be an essentially small category which admits finite limits and let I be a finite poset. Then the map*

$$\mathrm{Pro}(\mathrm{Fun}(I, \mathcal{C})) \to \mathrm{Fun}(I, \mathrm{Pro}(\mathcal{C}))$$

is an equivalence of categories. In particular, every diagram $I \to \mathrm{Pro}(\mathcal{C})$ can be written as a filtered limit of diagrams $I \to \mathcal{C}$.

Example 9. Applying Proposition 8 in the case $I = \{0 < 1\}$, we see that every morphism $f : C \to D$ in $\mathrm{Pro}(\mathcal{C})$ can be obtained as the limit of a filtered diagram of morphisms $\{f_\alpha : C_\alpha \to D_\alpha\}$ between objects of $\mathcal{C}$.

Corollary 10. *Let $\mathcal{C}$ be an essentially small category which admits finite limits. Then the category $\mathrm{Pro}(\mathcal{C})$ admits finite limits. Moreover, the inclusion $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ preserves finite limits.*

Proof. Let $\{C_i\}_{i \in I}$ be a finite diagram in $\mathcal{C}$ having a limit $C \in \mathcal{C}$, and let $\{D_\alpha\}$ be a filtered diagram in $\mathcal{C}$ which we identify with an object of $\mathrm{Pro}(\mathcal{C})$. Then

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\{D_\alpha\}, C) &\simeq \varinjlim_{\alpha} \mathrm{Hom}_{\mathcal{C}}(D_\alpha, C) \\ &\simeq \varinjlim_{\alpha} \varprojlim_{i} \mathrm{Hom}_{\mathcal{C}}(D_\alpha, C_i) \\ &\simeq \varprojlim_{i} \varinjlim_{\alpha} \mathrm{Hom}_{\mathcal{C}}(D_\alpha, C_i) \\ &\simeq \varprojlim_{i} \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\{D_\alpha\}, C). \end{aligned}$$

where we have invoked the fact that filtered colimits commute with finite limits in the category of sets. This proves that the inclusion $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ preserves finite limits. In particular, $\mathrm{Pro}(\mathcal{C})$ has an initial object. To complete the proof, it will suffice to show that every diagram $C \to D \leftarrow E$ in $\mathrm{Pro}(\mathcal{C})$ admits a fiber product. Using Proposition 8, we can realize our diagram as a filtered limit of diagrams $\{C_\alpha \to D_\alpha \leftarrow E_\alpha\}$ in $\mathcal{C}$. Then the filtered diagram $\{C_\alpha \times_{D_\alpha} E_\alpha\}$ represents a fiber product $C \times_D E$ in the category $\mathrm{Pro}(\mathcal{C})$. □

We will be particularly interested in studying $\mathrm{Pro}(\mathcal{C})$ in the case where $\mathcal{C}$ is a pretopos.

Proposition 11. *Let $\mathcal{C}$ be a category which admits finite limits. Assume that every morphism $f : X \to Z$ in $\mathcal{C}$ factors as a composition $X \xrightarrow{g} Y \xrightarrow{h} Z$, where g is an effective epimorphism and h is a monomorphism. Then every morphism in $\mathrm{Pro}(\mathcal{C})$ factors as a composition $X \xrightarrow{g} Y \xrightarrow{h} Z$, where g is an effective epimorphism and h is a monomorphism.*

Proof. Let $f : X \to Z$ be a morphism in $\text{Pro}(\mathcal{C})$, which we can realize as a filtered limit of morphisms $\{f_\alpha : X_\alpha \to Z_\alpha\}$ in $\mathcal{C}$. Factor each f_α as a composition $X_\alpha \xrightarrow{g_\alpha} Y_\alpha \xrightarrow{h_\alpha} Z_\alpha$, where g_α is an effective epimorphism and h_α is a monomorphism. This factorization is functorial, so we can regard $Y = \{Y_\alpha\}$ as a pro-object of $\mathcal{C}$ equipped with morphisms $g : X \to Y$ and $h : Y \to Z$ with $f = h \circ g$. Note that $Y \times_Z Y \simeq \{Y_\alpha \times_{Z_\alpha} Y_\alpha\} \simeq \{Y_\alpha\} = Y$, so that h is a monomorphism in $\text{Pro}(\mathcal{C})$. We will complete the proof by showing that g is an effective epimorphism in $\text{Pro}(\mathcal{C})$. For this, we wish to show that for each object $C \in \text{Pro}(\mathcal{C})$, the diagram

$$\text{Hom}_{\text{Pro}(\mathcal{C})}(Y, C) \to \text{Hom}_{\text{Pro}(\mathcal{C})}(X, C) \rightrightarrows \text{Hom}_{\text{Pro}(\mathcal{C})}(X \times_Y X, C)$$

is an equalizer. Writing C as a filtered limit of objects of $\mathcal{C}$, we can assume that $C \in \mathcal{C}$. In this case, the diagram above is given by a filtered colimit of diagrams

$$\text{Hom}_{\mathcal{C}}(Y_\alpha, C) \to \text{Hom}_{\mathcal{C}}(X_\alpha, C) \rightrightarrows \text{Hom}_{\mathcal{C}}(X_\alpha \times_{Y_\alpha} X_\alpha, C).$$

We conclude by observing that each of these diagrams is an equalizer (since g_α is an effective epimorphism in $\mathcal{C}$), and the collection of equalizer diagrams in $\mathcal{S}\text{et}$ is closed under filtered colimits. □

Remark 12. The proof of Proposition 11 shows that a morphism $f : X \to Y$ in $\text{Pro}(\mathcal{C})$ is a monomorphism (effective epimorphism) if and only if it can be realized as a filtered limit of morphisms $\{f_\alpha : X_\alpha \to Y_\alpha\}$ which are monomorphisms (effective epimorphisms) in $\mathcal{C}$.

Remark 13. In the situation of Proposition 11, suppose that the formation of images in $\mathcal{C}$ is compatible with pullback (or equivalently, the collection of effective epimorphisms is stable under pullback). Then the category $\text{Pro}(\mathcal{C})$ has the same property: any diagram $X \xrightarrow{f} Y \xleftarrow{g} Z$ can be realized as a filtered limit of diagrams $\{X_\alpha \xrightarrow{f_\alpha} Y_\alpha \xleftarrow{g_\alpha} Z_\alpha\}$, in which case we have

$$\begin{aligned} \text{Im}(X \times_Y Z \to Z) &\simeq \{\text{Im}(X_\alpha \times_{Y_\alpha} Z_\alpha \to Z_\alpha)\} \\ &\simeq \{\text{Im}(X_\alpha \to Y_\alpha) \times_{Y_\alpha} Z_\alpha\} \\ &\simeq \text{Im}(X \to Y) \times_Y Z. \end{aligned}$$

Let $\mathcal{C}$ be an essentially small category which admits finite limits. Then $\text{Fun}^{\text{lex}}(\mathcal{C}, \mathcal{S}\text{et})$ is closed under limits in $\mathcal{S}\text{et}$, and therefore admits small limits. It follows that the category $\text{Pro}(\mathcal{C})$ admits small colimits. Moreover, the inclusion functor $\mathcal{C} \hookrightarrow \text{Pro}(\mathcal{C})$ preserves all colimits which exist in $\mathcal{C}$ (this is immediate from the definitions).

Proposition 14. *Let $\mathcal{C}$ be an essentially small category which admits finite limits and finite coproducts. Then the category $\text{Pro}(\mathcal{C})$ admits finite coproducts, given by the formula*

$$\{C_\alpha\} \coprod \{D_\beta\} = \{C_\alpha \coprod D_\beta\}.$$

Proof. It suffices to observe that for any object $E \in \mathcal{C}$, we have

$$\varinjlim_{\alpha,\beta} \text{Hom}_{\mathcal{C}}(C_\alpha \coprod D_\beta, E) \simeq (\varinjlim_{\alpha} \text{Hom}_{\mathcal{C}}(C_\alpha, E)) \times (\varinjlim_{\beta} \text{Hom}_{\mathcal{C}}(D_\beta, E)).$$

□

Given objects $C, D \in \text{Pro}(\mathcal{C})$, we can use Proposition 8 to write $C = \{C_\alpha\}$ and $D = \{D_\alpha\}$ as limits of diagrams indexed by the same category. In this case, the coproduct $C \amalg D$ is given by $\{C_\alpha \amalg D_\alpha\}$.

Remark 15. In the situation of Proposition 14, suppose that the formation of coproducts in $\mathcal{C}$ is preserved by pullback. Then the same is true in $\mathrm{Pro}(\mathcal{C})$. Given morphisms $f : C \to X$, $g : D \to X$, and $h : Y \to X$ in $\mathrm{Pro}(\mathcal{C})$, we can apply Proposition 8 to realize f, g, and h as filtered limits of maps $f_\alpha : C_\alpha \to X_\alpha$, $g_\alpha : D_\alpha \to X_\alpha$, and $h_\alpha : Y_\alpha \to X_\alpha$ (indexed by the same category), so that both $(C \amalg D) \times_X Y$ and $(C \times_X Y) \amalg (D \times_X Y)$ are represented by the diagram

$$\{(C_\alpha \amalg D_\alpha) \times_{X_\alpha} Y_\alpha\} \simeq \{(C_\alpha \times_{X_\alpha} Y_\alpha) \amalg (D_\alpha \times_{X_\alpha} Y_\alpha)\}.$$

Remark 16. In the situation of Proposition 14, suppose that coproducts in $\mathcal{C}$ are disjoint. Then, for every pair of objects $C = \{C_\alpha\}$ and $D = \{D_\alpha\}$ in $\mathcal{C}$, we deduce that

$$C \times_{C \amalg D} D \simeq \{C_\alpha \amalg_{C_\alpha \amalg D_\alpha} D_\alpha\} = \{\emptyset\}$$

is an initial object of $\mathrm{Pro}(\mathcal{C})$: that is, coproducts are disjoint in $\mathrm{Pro}(\mathcal{C})$.

Combining the above results, we obtain the following:

Proposition 17. *Let $\mathcal{C}$ be an essentially small coherent category with disjoint coproducts (for example, a pretopos). Then* $\mathrm{Pro}(\mathcal{C})$ *is also a coherent category with disjoint coproducts.*

Warning 18. It is not true that if $\mathcal{C}$ is a pretopos, then $\mathrm{Pro}(\mathcal{C})$ is also a pretopos. For example, let $\mathcal{C}$ be the category of finite sets. Then the category $\mathrm{Pro}(\mathcal{C})$ of profinite sets can be identified with the category of Stone spaces: that is, the category whose objects are totally disconnected compact Hausdorff spaces, and whose morphisms are continuous maps. Let $C \in \mathrm{Pro}(\mathcal{C})$ be the Cantor set, which we identify with the collection of infinite sequences $(n_1, n_2, n_3, \ldots)$ where $n_i \in \{0, 1\}$. The construction

$$(n_1, n_2, n_3, \ldots) \mapsto \sum \frac{n_i}{2^i}$$

defines a continuous surjection $C \to [0,1]$, and the fiber product $R = C \times_{[0,1]} C$ can be regarded as an equivalence relation on C in the category of Stone spaces. However, this equivalence relation is *not* effective: given any Stone space X, a continuous map $C \to X$ which equalizes the two projection maps $R \rightrightarrows C$ must factor through a continuous map $[0,1] \to X$. Such a map is automatically constant (since X is totally disconnected), so that $C \times_X C = C \times C$ is larger than R.