Lecture 14: Locales and Topoi

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Recall that, if X is an object of a coherent category $\mathcal{C}$, then the poset $\mathrm{Sub}(X)$ is a distributive lattice. If $\mathcal{C}$ is a topos, we can say more.

Definition 1. A *locale* is a poset $\mathcal{U}$ with the following properties:

(a) Every subset $S \subseteq \mathcal{U}$ has a least upper bound $\bigvee S$.

It follows from (a) that every subset $S \subseteq \mathcal{U}$ also has a greatest lower bound $\bigwedge S$, given by the least upper bound of the set $\{U \in \mathcal{U} : (\forall V \in S) U \leq V\}$ of all lower bounds for S. In particular, every pair of elements $U, V \in \mathcal{U}$ have a meet $U \wedge V$.

(b) For each element $V \in \mathcal{U}$ and every set of elements $\{U_\alpha\}$, we have a distributive law

$$(\bigvee_\alpha U_\alpha) \wedge V = \bigvee_\alpha (U_\alpha \wedge V).$$

Remark 2. Every locale is a distributive lattice.

Exercise 3. Let $\mathcal{U}$ be a poset satisfying condition (a) of Definition 1. Show that $\mathcal{U}$ is a locale if and only if it is a *Heyting algebra*: that is, if and only if for every pair of elements $U, V \in \mathcal{U}$, there is an element $(U \Rightarrow V) \in \mathcal{U}$ such that $W \leq (U \Rightarrow V)$ if and only if $U \wedge W \leq V$.

Example 4. Let B be a complete Boolean algebra (that is, a Boolean algebra satisfying condition (a) of Definition 1). Then B is a locale.

Example 5. Let X be a topological space and let $\mathcal{U}(X)$ be the collection of open subsets of X, partially ordered with respect to inclusion. Then $\mathcal{U}(X)$ is a locale. Moreover, the join $\bigvee U_\alpha$ of a collection of elements $U_\alpha \in \mathcal{U}(X)$ coincides with the set-theoretic union $\bigcup U_\alpha$, and the meet of a pair $U, V \in \mathcal{U}(X)$ is given by the set-theoretic intersection $U \cap V$.

Beware that the meet of an *infinite* set of elements $U_\alpha \in \mathcal{U}(X)$ usually does not coincide with the intersection $\bigcap U_\alpha$, because the intersection $\bigcap U_\alpha$ need not be open; instead, $\bigwedge U_\alpha$ is given by the interior of $\bigcap U_\alpha$. In particular, we generally have

$$(\bigwedge_\alpha U_\alpha) \vee V \neq \bigwedge_\alpha (U_\alpha \vee V).$$

Proposition 6. *Let $\mathcal{X}$ be a topos and let X be an object of $\mathcal{X}$. Then the poset* $\mathrm{Sub}(X)$ *is a locale.*

Proof. Every collection of objects $\{U_i \subseteq X\}_{i \in I}$ has a join, given by the image of the map $\coprod_{i \in I} U_i \to X$. For

$V \subseteq X$, we compute

$$\begin{aligned}
(\bigvee U_i) \wedge V &= (\bigvee U_i) \times_X V \\
&= \mathrm{Im}(\coprod_{i \in I} U_i \to X) \times_X V \\
&= \mathrm{Im}((\coprod_{i \in I} U_i) \times_X V \to V) \\
&= \mathrm{Im}(\coprod_{i \in I} (U_i \times_X V) \to V) \\
&= \bigvee_{i \in I} U_i \wedge V.
\end{aligned}$$

□

Definition 7. Let $\mathcal{X}$ be a topos and let $\mathbf{1}$ be the final object of $\mathcal{X}$. Then $\mathrm{Sub}(\mathbf{1})$ is a locale. We will refer to $\mathrm{Sub}(\mathbf{1})$ as the *underlying locale of* $\mathcal{X}$.

In the situation of Definition 7, the poset $\mathrm{Sub}(\mathbf{1})$ can be regarded as a full subcategory of $\mathcal{X}$.

Definition 8. Let $\mathcal{X}$ be a topos. We say that $\mathcal{X}$ is *localic* if it is generated by $\mathrm{Sub}(\mathbf{1})$: that is, if every object $X \in \mathcal{X}$ admits a covering $\{U_i \to X\}$, where each U_i is a subobject of $\mathbf{1}$.

Example 9. Let $\mathcal{C}$ be a category which admits finite limits, equipped with a Grothendieck topology. Suppose that $\mathcal{C}$ is a poset (that is, every object of $\mathcal{C}$ can be identified with a subobject of the final object). Then the topos $\mathrm{Shv}(\mathcal{C})$ is localic: it is generated by objects of the form Lh_C, each of which is a subobject of the final object of $\mathrm{Shv}(\mathcal{C})$.

Example 10. Let X be a topological space. Then the topos $\mathrm{Shv}(X)$ is localic (this is a special case of Example 9).

We now prove a converse to Example 9.

Exercise 11. Let $\mathcal{U}$ be a locale. Show that $\mathcal{U}$ admits a Grothendieck topology, where a collection of maps $\{U_i \to X\}$ is a covering if $X = \bigvee U_i$.

Proposition 12. *Let $\mathcal{X}$ be a localic topos, and regard the underlying locale $\mathcal{U} = \mathrm{Sub}(\mathbf{1})$ as equipped with the Grothendieck topology of Exercise 11. Then we have a canonical equivalence $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{U})$.*

Proof. We can regard $\mathcal{U}$ as an essentially small full subcategory of $\mathcal{X}$ which is closed under finite limits. If $\mathcal{X}$ is localic, then $\mathcal{U}$ generates $\mathcal{X}$, so the desired result follows as in the proof of Giraud's theorem. □

We now proceed in the reverse direction.

Proposition 13. *Let $\mathcal{U}$ be a locale. Then the Yoneda embedding $h : \mathcal{U} \to \mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ induces an equivalence from $\mathcal{U}$ to the poset of subobjects of $\mathbf{1}$ in $\mathrm{Shv}(\mathcal{U})$.*

Proof. We first show that, for each $U \in \mathcal{U}$, the presheaf h_U is a sheaf. Suppose we are given a covering $\{V_i \to V\}_{i \in I}$ in $\mathcal{U}$; we wish to show that the canonical map

$$h_U(V) \to \prod_i h_U(V_i) \rightrightarrows \prod_{i,j} h_U(V_i \wedge V_j)$$

is an equalizer diagram. Equivalently, we wish to show that $V \le U$ if and only if each $V_i \le U$, which follows from the identity $V = \bigvee_{i \in I} V_i$.

It is clear that each h_U is a subobject of the final object of $\mathrm{Shv}(\mathcal{U})$ (note that $h_U(V)$ is a singleton for $V \leq U$, and empty otherwise). Conversely, let $\mathscr{F} \in \mathrm{Shv}(\mathcal{U})$ be a subobject of the final object, so that $\mathscr{F}(V)$ has at most one element for each $V \in \mathcal{U}$. Set $U = \bigvee_{\mathscr{F}(V) \neq \emptyset} V$. Then we have a covering $\{V \to U\}_{\mathscr{F}(V)\neq\emptyset}$. Invoking the assumption that $\mathscr{F}$ is a sheaf, we conclude that $\mathscr{F}(U) \neq \emptyset$. We therefore have $\mathscr{F}(V) = \begin{cases} * & \text{if } V \leq U \\ \emptyset & \text{otherwise.} \end{cases}$, so that $\mathscr{F} \simeq h_U$. □

We can summarize Propositions 12 and 13 more informally by saying that we have an equivalence

$$\{ \text{ Localic topoi } \} \simeq \{ \text{ Locales } \}.$$

To every localic topos $\mathcal{X}$, we can associate the locale $\mathrm{Sub}(\mathbf{1})$ of subobjects of the final object; to any locale $\mathcal{U}$, we can associate a topos $\mathrm{Shv}(\mathcal{U})$, and these constructions are mutually inverse (up to equivalence). In fact, we can be a bit more precise.

Definition 14. Let $\mathcal{U}$ and $\mathcal{V}$ be locales. A *morphism of locales* from $\mathcal{V}$ to $\mathcal{U}$ is an order-preserving map $f^* : \mathcal{U} \to \mathcal{V}$ such that f^* preserves finite meets and arbitrary joins (equivalently, it preserves finite limits and small colimits, if we view $\mathcal{U}$ and $\mathcal{V}$ as categories). We let $\mathrm{Fun}^*(\mathcal{U}, \mathcal{V})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{U}, \mathcal{V})$ spanned by the morphisms of locales from $\mathcal{V}$ to $\mathcal{U}$ (note that $\mathrm{Fun}^*(\mathcal{U}, \mathcal{V})$ is a poset).

Proposition 15. *Let $\mathcal{U}$ be a locale and let $\mathcal{X}$ be a topos with underlying locale* $\mathrm{Sub}(\mathbf{1})$. *Then composition with the Yoneda embedding $h : \mathcal{U} \to \mathrm{Shv}(\mathcal{U})$ induces an equivalence of categories*

$$\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{U}), \mathcal{X}) \to \mathrm{Fun}^*(\mathcal{U}, \mathrm{Sub}(\mathbf{1})).$$

In other words, the category of geometric morphisms from $\mathcal{X}$ to $\mathrm{Shv}(\mathcal{U})$ *is equivalent to the poset of morphisms of locales from* $\mathrm{Sub}(\mathbf{1})$ *to $\mathcal{U}$.*

Proof. We proved in Lecture 12 that composition with h induces an equivalence of categories $\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{U}), \mathcal{X}) \to \mathrm{Fun}'(\mathcal{U}, \mathcal{X})$, where $\mathrm{Fun}'(\mathcal{U}, \mathcal{X})$ is the full subcategory of $\mathrm{Fun}(\mathcal{U}, \mathcal{X})$ spanned by those functors $f : \mathcal{U} \to \mathcal{X}$ which preserve finite limits and coverings. Since every object of $\mathcal{U}$ is a subobject of the final object, any functor $f : \mathcal{U} \to \mathcal{X}$ which preserves finite limits automatically carries each element of $\mathcal{U}$ to a subobject of the final object $\mathbf{1} \in \mathcal{X}$, and can therefore be identified with a map of posets $g : \mathcal{U} \to \mathrm{Sub}(\mathbf{1})$. In this case, the assumption that f preserves finite limits translates into the assumption that g preserves finite meets, and the assumption that f preserves coverings translates into the assumption that g preserves infinite joins. □

We can summarize the situation as follows: there are adjoint functors (of 2-categories)

$$\{\text{Topoi}\} \underset{\mathcal{U} \mapsto \mathrm{Shv}(\mathcal{U})}{\overset{\mathcal{X} \mapsto \mathrm{Sub}(\mathbf{1})}{\rightleftarrows}} \{\text{Locales}\}.$$

where the construction $\mathcal{U} \mapsto \mathrm{Shv}(\mathcal{U})$ is fully faithful by virtue of Proposition 13; its essential image is the 2-category of localic topoi. It follows that for every topos $\mathcal{X}$, there is a universal example of a localic topos which admits a geometric morphism from $\mathcal{X}$, given by $\mathrm{Shv}(\mathrm{Sub}(\mathbf{1}))$. We refer to this topos as the *localic reflection* of $\mathcal{X}$.

Example 16. Let X be a topological space equipped with an action of a (discrete) group G. Then the category $\mathrm{Shv}_G(X)$ of G-equivariant sheaves on X is a topos. The subobjects of the final object of $\mathrm{Shv}(X)$ can be identified with open subsets of X. It follows that subobjects of the final object of $\mathrm{Shv}_G(X)$ can be identified with G-equivariant open subsets of X, or equivalently with open subsets of the quotient X/G (where we endow X/G with the quotient topology). It follows that there is a canonical map $\mathrm{Shv}_G(X) \to \mathrm{Shv}(X/G)$ which exhibits $\mathrm{Shv}(X/G)$ as the localic reflection of $\mathrm{Shv}_G(X)$.