Lecture 12: Geometric Morphisms

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Definition 1. Let $\mathcal{X}$ and $\mathcal{Y}$ be topoi. A *geometric morphism* from $\mathcal{X}$ to $\mathcal{Y}$ is a functor $f^* : \mathcal{Y} \to \mathcal{X}$ which preserves finite limits, effective epimorphisms, and (small) coproducts. We let $\mathrm{Fun}^*(\mathcal{Y}, \mathcal{X})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{Y}, \mathcal{X})$ spanned by the geometric morphisms from $\mathcal{X}$ to $\mathcal{Y}$.

Remark 2. The terminology of Definition 1 is motivated by topology. If $f : X \to Y$ is a continuous map of topological spaces, then pullback of sheaves determines a functor $f^* : \mathrm{Shv}(Y) \to \mathrm{Shv}(X)$ satisfying the requirements of Definition 1. Under the mild assumption that Y is *sober*, this construction establishes a bijection between the set of continuous maps from X to Y and the set of isomorphism classes of geometric morphisms from the topos $\mathrm{Shv}(X)$ to the topos $\mathrm{Shv}(Y)$. We will return to this point later.

Definition 1 admits several reformulations:

Proposition 3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be topoi and let $f^* : \mathcal{Y} \to \mathcal{X}$ be a functor which preserves finite limits. The following conditions are equivalent:*

(1) *The functor f^* is a geometric morphism from $\mathcal{X}$ to $\mathcal{Y}$: that is, it preserves effective epimorphisms and coproducts.*

(2) *The functor f^* preserves small colimits.*

(3) *The functor f^* admits a right adjoint.*

Sketch. The equivalence (2) $\Leftrightarrow$ (3) follows from the adjoint functor theorem and the implication (2) $\Rightarrow$ (1) is clear. We complete the proof by showing that (1) $\Rightarrow$ (2). Assume that f^* is a geometric morphism; we wish to show that f^* preserves small colimits. By general nonsense, all small colimits can be constructed from coproducts and coequalizers. It will therefore suffice to show that for every pair of morphisms $u, v : U \to Y$ in $\mathcal{Y}$, the canonical map $\mathrm{Coeq}(f^*U \rightrightarrows f^*Y) \to f^* \mathrm{Coeq}(U \rightrightarrows Y)$ is an isomorphism in $\mathcal{X}$.

Note that the canonical map $Y \to \mathrm{Coeq}(U \rightrightarrows Y)$ cannot factor through any proper subobject of Y, and is therefore an effective epimorphism. We therefore have $\mathrm{Coeq}(U \rightrightarrows Y) \simeq \mathrm{Coeq}(R \rightrightarrows Y)$, where $R \subseteq Y \times Y$ is the equivalence relation $Y \times_{\mathrm{Coeq}(U \rightrightarrows Y)} Y$. In this case, we can characterize R as the equivalence relation *generated by* U: that is, it is the smallest equivalence relation on Y which contains the image of the map $(u, v) : U \to Y \times Y$. Similarly, we can identify $\mathrm{Coeq}(f^*U \rightrightarrows f^*Y)$ with the quotient of f^*Y by the equivalence relation $R' \subseteq f^*(Y) \times f^*(Y)$ generated by f^*U. It will therefore suffice to show that $R' = f^*R$ (as subobjects of $f^*Y \times f^*Y$.

We now describe passage from the object U to the equivalence relation R. Note that R depends only on the image of the map $(u, v) : U \to Y \times Y$. We may therefore replace U by $\mathrm{Im}(u, v)$ and thereby reduce to the case where U is a subobject of $Y \times Y$. Let U^{op} denote the image of U under the automorphism of $Y \times Y$ given by swapping the factors. Replacing U by the join $U \vee U^{\mathrm{op}}$, we can reduce to the case where U is a *symmetric* relation on Y. For each $n \geq 0$, let V_n denote the n-fold fiber product

$$U_f \times_{Y g} U_f \times_{Y g} U_f \times_{X g} \cdots_f \times_{Y g} U,$$

(so that $V_0 = Y$) and let U_n denote the image of V_n in the product $Y \times Y$ (under the maps from the outermost factors of U). The equivalence relation R can then be described as the join $\bigvee_{n \geq 0} U_n$ (Exercise: Check this.)

Note that the equivalence relation R was constructed from U by a combination of taking images of morphisms, fiber products, and joins of subobjects. By assumption, the functor $f^* : \mathcal{Y} \to \mathcal{X}$ is compatible with each of these operations. It follows that f^*R agrees with the equivalence relation R' generated by f^*U, as desired. □

Remark 4. Let $\mathcal{X}$ and $\mathcal{Y}$ be topoi and let $f^* : \mathcal{Y} \to \mathcal{X}$ be a geometric morphism from $\mathcal{X}$ to $\mathcal{Y}$. Proposition 3 implies that f^* admits a right adjoint, which we will denote by $f_* : \mathcal{X} \to \mathcal{Y}$. In this situation, the functors f^* and f_* are equivalent pieces of data: either can be recovered (up to canonical isomorphism) from the other. We will sometimes abuse terminology by saying that f_* is a *geometric morphism* from $\mathcal{X}$ to $\mathcal{Y}$.

Our next goal is to understand how to compute the category $\mathrm{Fun}^*(\mathcal{Y}, \mathcal{X})$ of geometric morphisms in the case where $\mathcal{Y} = \mathrm{Shv}(\mathcal{C})$ is described as a category of sheaves.

Theorem 5. *Let $\mathcal{C}$ be a small category which admits finite limits which is equipped with a Grothendieck topology, and let $j : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ be the composition of the Yoneda embedding $h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ with the sheafification functor $L : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathcal{S}$. Then, for any topos $\mathcal{X}$, composition with j induces a fully faithful embedding*

$$\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X}),$$

whose essential image consists of those functors $f : \mathcal{C} \to \mathcal{X}$ which satisfy the following pair of conditions:

- (*a*) *The functor f is left exact: that is, it preserves finite limits.*
- (*b*) *For every covering $\{C_i \to C\}$ in the category $\mathcal{C}$, the maps $\{f(C_i) \to f(C)\}$ form a covering with respect to the canonical topology on $\mathcal{X}$.*

We begin by proving Theorem 5 in a special case, where we can ignore the topology on $\mathcal{C}$.

Exercise 6. Let $\mathcal{C}$ be a small category (which admits finite limits). Show that $\mathcal{C}$ admits a Grothendieck topology where the covering families $\{f_i : C_i \to C\}$ are those collections of maps where some f_i admits a section. We will refer to this Grothendieck topology as the *trivial topology* on $\mathcal{C}$.

Exercise 7. Let $\mathcal{C}$ be a small category which admits finite limits. Show that every presheaf on $\mathcal{C}$ is a sheaf with respect to the trivial topology. That is, if we equip $\mathcal{C}$ with the trivial topology, then we have $\mathrm{Shv}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$.

We first prove Theorem 5 in the case where the topology on $\mathcal{C}$ is trivial. We will use the following general fact about presheaf categories:

Proposition 8. *Let $\mathcal{C}$ be a small category and let $h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ be the Yoneda embedding. Then h exhibits the $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ as the category freely generated by $\mathcal{C}$ under small colimits. In other words, if $\mathcal{X}$ is any category which admits small colimits and $\mathrm{LFun}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}), \mathcal{X})$ is the category of functors from $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ to $\mathcal{X}$ which preserve small colimits, then composition with h induces an equivalence of categories*

$$\mathrm{LFun}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}), \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X}).$$

Remark 9. The inverse functor $\theta : \mathrm{Fun}(\mathcal{C}, \mathcal{X}) \to \mathrm{LFun}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}), \mathcal{X})$ can be described explicitly as follows. Let $f : \mathcal{C} \to \mathcal{X}$ be any functor. Then every object $X \in \mathcal{X}$ determines a presheaf $\mathscr{F}_X$ on $\mathcal{C}$, given by the formula $\mathscr{F}_X(C) = \mathrm{Hom}_{\mathcal{X}}(f(C), X)$. Then $\theta(f) : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathcal{X}$ can be descrbed as the left adjoint of the functor

$$\mathcal{X} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \qquad X \mapsto \mathscr{F}_X .$$

Proof of Theorem 5 for a trivial topology. Let $\mathcal{C}$ be a category which admits finite limits and let $\mathcal{X}$ be a topos. By virtue of Proposition 8, composition with h induces an equivalence

$$\{\text{Functors } F : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathcal{X} \text{ preserving colimits}\} \to \{\text{All functors } f : \mathcal{C} \to \mathcal{X}\}$$

We wish to show that this restricts to an equivalence of categories

$$\mathrm{Fun}^*(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}), \mathcal{X}) \to \{\text{Left exact functors } f : \mathcal{C} \to \mathcal{X}\}$$

(note that condition (b) is of Theorem 5 is automatic when the Grothendieck topology on $\mathcal{C}$ is trivial). In other words, we wish to show that if $F : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathcal{X}$ is a functor which preserves small colimits, then F preserves finite limits if and only if the composition $F \circ h : \mathcal{C} \to \mathcal{X}$ preserves finite limits. The "only if" direction is clear (and does not require the assumption that $\mathcal{X}$ is a topos), since the Yoneda embedding $C \mapsto h_C$ preserves finite limits.

To prove the converse, assume that $f = F \circ h$ preserves finite limits. Let $\mathcal{D} \subseteq \mathcal{X}$ be a small full subcategory which contains a set of generators for $\mathcal{X}$, contains the image of f, and is closed under finite limits. Then the canonical topology on $\mathcal{X}$ determines a Grothendieck topology on $\mathcal{D}$, and we proved in Lecture 10 that $\mathcal{X}$ can be identified with the category of sheaves $\mathrm{Shv}(\mathcal{D})$. Let $h' : \mathcal{D} \to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ be the Yoneda embedding of $\mathcal{D}$ and let $L : \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathcal{X}$ be the sheafification functor. Using Proposition 8, we see that the functor $h' \circ f : \mathcal{C} \to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ admits an essentially unique extension to a functor $F' : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ which preserves small colimits. Moreover, the functor $L \circ F' : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathcal{X}$ preserves small colimits, and $L \circ F' \circ h \simeq L \circ h' \circ f \simeq f$. Using Proposition 8 again, we can identify $L \circ F'$ with F. We wish to show that F preserves finite limits. Since the sheafification functor L preserves finite limits, we are reduced to proving that F' preserves finite limits. Concretely, the functor F' is given by left Kan extension along the functor $f^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$. In particular, if $\mathscr{F}$ is a presheaf on $\mathcal{C}$, then $F'(\mathscr{F})$ is the presheaf on $\mathcal{D}$ given by the formula

$$F'(\mathscr{F})(D) = \varinjlim_{D \to f(C)} \mathscr{F}(C).$$

We wish to show that, for each object $D \in \mathcal{D}$, the construction $\mathscr{F} \mapsto F'(\mathscr{F})(D)$ preserves finite limits. This follows from the observation that the colimit $\varinjlim_{D \to f(C)} \mathscr{F}(C)$ is indexed by a *filtered* diagram (in fact, the category $\mathcal{C}^{\mathrm{op}} \times_{\mathcal{D}^{\mathrm{op}}} (\mathcal{D}_{D/})^{\mathrm{op}}$ admits finite colimits, since $\mathcal{C}$ has finite limits which are preserved by the functor f). ∎

Proof of Theorem 5 in general. Now suppose that $\mathcal{C}$ is a category which admits finite limits which is equipped with an arbitrary Grothendieck topology. Let $L : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathrm{Shv}(\mathcal{C})$ be the sheafification functor and let $h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ be the Yoneda embedding. For any topos $\mathcal{X}$, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathcal{X}) & \xrightarrow{\circ j} & \mathrm{Fun}(\mathcal{C}, \mathcal{X}) \\
\downarrow{\scriptstyle \circ L} & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{Fun}^*(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}), \mathcal{X}) & \xrightarrow{\circ h} & \mathrm{Fun}(\mathcal{C}, \mathcal{X})
\end{array}$$

The first part of the proof shows that the lower horizontal map is a fully faithful embedding, whose essential image consists of those functors $f : \mathcal{C} \to \mathcal{X}$ which preserve finite limits. Moreover, the left vertical map is also fully faithful; its essential image consists of those geometric morphisms $F^* : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathcal{X}$ having the property that the right adjoint $F_* : \mathcal{X} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ factors through the full subcategory $\mathrm{Shv}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$. Using the description of F_* given in Remark 9, we see that the upper vertical map induces an equivalence from $\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathcal{X})$ to the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{X})$ spanned by those functors $f : \mathcal{C} \to \mathcal{S}\mathrm{et}$ which satisfy the following pair of conditions:

- (a) The functor f preserves finite limits.
- (b') For each object $X \in \mathcal{X}$, the functor $\mathscr{F}_X : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ given by $\mathscr{F}_X(C) = \mathrm{Hom}_{\mathcal{X}}(f(C), X)$ is a sheaf.

Unwinding the definitions, we can rephrase (b') as follows:

(b'') For each object $X \in \mathcal{X}$ and each covering $\{C_i \to C\}$ in $\mathcal{C}$, the diagram

$$\mathrm{Hom}_{\mathcal{X}}(f(C), X) \to \prod_i \mathrm{Hom}_{\mathcal{X}}(f(C_i), X) \rightrightarrows \prod_{i,j} \mathrm{Hom}_{\mathcal{X}}(f(C_i \times_C C_j), X)$$

is an equalizer.

Allowing X to vary, we can rephrase this as:

(b''') For each covering $\{C_i \to C\}$ in $\mathcal{C}$, the diagram

$$\coprod_{i,j} f(C_i \times_C C_j) \rightrightarrows \coprod_i f(C_i) \to f(C)$$

is a coequalizer in $\mathcal{X}$.

Assuming (a), we can rewrite the diagram of (b''') as

$$\coprod_{i,j} f(C_i) \times_{f(C)} f(C_j) \rightrightarrows \coprod_i f(C_i) \to f(C),$$

so that (b''') is equivalent to condition (b) of Theorem 5. $\square$