Higher Algebra

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Contents

Let K denote the functor of *complex K-theory*, which associates to every compact Hausdorff space X the Grothendieck group $K(X)$ of isomorphism classes of complex vector bundles on X. The functor $X \mapsto K(X)$ is an example of a *cohomology theory*: that is, one can define more generally a sequence of abelian groups $\{K^n(X, Y)\}_{n \in \mathbf{Z}}$ for every inclusion of topological spaces $Y \subseteq X$, in such a way that the Eilenberg-Steenrod axioms are satisfied (see [49]). However, the functor K is endowed with even more structure: for every topological space X, the abelian group $K(X)$ has the structure of a commutative ring (when X is compact, the multiplication on $K(X)$ is induced by the operation of tensor product of complex vector bundles). One would like that the ring structure on $K(X)$ is a reflection of the fact that K itself has a ring structure, in a suitable setting.

To analyze the problem in greater detail, we observe that the functor $X \mapsto K(X)$ is *representable*. That is, there exists a topological space $Z = \mathbf{Z} \times BU$ and a universal class $\eta \in K(Z)$, such that for every sufficiently nice topological space X, the pullback of η induces a bijection $[X, Z] \to K(X)$; here $[X, Z]$ denotes the set of homotopy classes of maps from X into Z. According to Yoneda's lemma, this property determines the space Z up to homotopy equivalence. Moreover, since the functor $X \mapsto K(X)$ takes values in the category of commutative rings, the topological space Z is automatically a *commutative ring object* in the homotopy category $\mathcal{H}$ of topological spaces. That is, there exist addition and multiplication maps $Z \times Z \to Z$, such that all of the usual ring axioms are satisfied up to homotopy. Unfortunately, this observation is not very useful. We would like to have a robust generalization of classical algebra which includes a good theory of modules, constructions like localization and completion, and so forth. The homotopy category $\mathcal{H}$ is too poorly behaved to support such a theory.

An alternate possibility is to work with commutative ring objects in the category of topological spaces itself: that is, to require the ring axioms to hold "on the nose" and not just up to homotopy. Although this does lead to a reasonable generalization of classical commutative algebra, it not sufficiently general for many purposes. For example, if Z is a topological commutative ring, then one can always extend the functor $X \mapsto [X, Z]$ to a cohomology theory. However, this cohomology theory is not very interesting: in degree zero, it simply gives the following variant of classical cohomology:

$$\prod_{n \geq 0} \mathrm{H}^n(X; \pi_n Z).$$

In particular, complex K-theory cannot be obtained in this way. In other words, the $Z = \mathbf{Z} \times \mathrm{BU}$ for stable vector bundles cannot be equipped with the structure of a topological commutative ring. This reflects the fact that complex vector bundles on a space X form a *category*, rather than just a set. The direct sum and tensor product operation on complex vector bundles satisfy the ring axioms, such as the distributive law

$$\mathcal{E} \otimes (\mathcal{F} \oplus \mathcal{F}') \simeq (\mathcal{E} \otimes \mathcal{F}) \oplus (\mathcal{E} \otimes \mathcal{F}'),$$

but only up to isomorphism. However, although $\mathbf{Z} \times \mathrm{BU}$ has less structure than a commutative ring, it has more structure than simply a commutative ring object in the homotopy category $\mathcal{H}$, because the isomorphism displayed above is actually *canonical* and satisfies certain coherence conditions (see [91] for a discussion).

To describe the kind of structure which exists on the topological space $\mathbf{Z} \times \mathrm{BU}$, it is convenient to introduce the language of *commutative ring spectra*, or, as we will call them, $\mathbb{E}_\infty$-*rings*. Roughly speaking, an $\mathbb{E}_\infty$-ring can be thought of as a space Z which is equipped with an addition and a multiplication for which the axioms for a commutative ring hold not only up to homotopy, but up to *coherent* homotopy. The $\mathbb{E}_\infty$-rings play a role in stable homotopy theory analogous to the role played by commutative rings in ordinary algebra. As such, they are the fundamental building blocks of *derived algebraic geometry*.

One of our ultimate goals in this book is to give an exposition of the theory of $\mathbb{E}_\infty$-rings. Recall that ordinary commutative ring R can be viewed as a *commutative algebra object* in the category of abelian groups, which we view as endowed with a symmetric monoidal structure given by tensor product of abelian groups. To obtain the theory of $\mathbb{E}_\infty$-rings we will use the same definition, replacing abelian groups by *spectra* (certain algebro-topological objects which represent cohomology theories). To carry this out in detail, we need to say exactly what a spectrum *is*. There are many different definitions in the literature, having a

variety of technical advantages and disadvantages. Some modern approaches to stable homotopy theory have the feature that the collection of spectra is realized as a symmetric monoidal category (and one can define an $\mathbb{E}_\infty$-ring to be a commutative algebra object of this category): see, for example, [73].

We will take a different approach, using the framework of ∞-categories developed in [97]. The collection of all spectra can be organized into an ∞-category, which we will denote by Sp: it is an ∞-categorical counterpart of the ordinary category of abelian groups. The tensor product of abelian groups also has a counterpart: the *smash product* functor on spectra. In order to describe the situation systematically, we introduce the notion of a *symmetric monoidal ∞-category*: that is, an ∞-category $\mathcal{C}$ equipped with a tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is commutative and associative up to coherent homotopy. For any symmetric monoidal ∞-category $\mathcal{C}$, there is an associated theory of commutative algebra objects, which are themselves organized into an ∞-category CAlg($\mathcal{C}$). We can then define an $\mathbb{E}_\infty$-ring to be a commutative algebra object of the ∞-category of spectra, endowed with the symmetric monoidal structure given by smash products.

We now briefly outline the contents of this book (more detailed outlines can be found at the beginning of individual sections and chapters). Much of this book is devoted to developing an adequate language to make sense of the preceding paragraph. We will begin in Chapter 1 by introducing the notion of a *stable ∞-category*. Roughly speaking, the notion of stable ∞-category is obtained by axiomatizing the essential feature of stable homotopy theory: fiber sequences are the same as cofiber sequences. The ∞-category Sp of spectra is an example of a stable ∞-category. In fact, it is universal among stable ∞-categories: we will show that Sp is freely generated (as a stable ∞-category which admits small colimits) by a single object (see Corollary 1.4.4.6). However, there are a number of stable ∞-categories that are of interest in other contexts. For example, the derived category of an abelian category can be realized as the homotopy category of a stable ∞-category. We may therefore regard the theory of stable ∞-categories as a generalization of homological algebra, which has many applications in pure algebra and algebraic geometry.

We can think of an ∞-category $\mathcal{C}$ as comprised of a collection of objects $X, Y, Z, \ldots \in \mathcal{C}$, together with a *mapping space* $\mathrm{Map}_\mathcal{C}(X, Y)$ for every pair of objects $X, Y \in \mathcal{C}$ (which are equipped with coherently associative composition laws). In Chapter 2, we will study a variation on the notion of ∞-category, which we call an *∞-operad*. Roughly speaking, an ∞-operad $\mathcal{O}$ consists of a collection of objects together with a *space of operations* $\mathrm{Mul}_\mathcal{O}(\{X_i\}_{1 \leq i \leq n}, Y)\}$ for every finite collection of objects $X_1, \ldots, X_n, Y \in \mathcal{O}$ (again equipped with coherently associative multiplication laws). As a special case, we will obtain a theory of symmetric monoidal ∞-categories.

Given a pair of ∞-operads $\mathcal{O}$ and $\mathcal{C}$, the collection of maps from $\mathcal{O}$ to $\mathcal{C}$ is naturally organized into an ∞-category which we will denote by $\mathrm{Alg}_\mathcal{O}(\mathcal{C})$, and refer to as the *$\infty$-category of $\mathcal{O}$-algebra objects of* $\mathcal{C}$. An important special case is when $\mathcal{O}$ is the commutative ∞-operad and $\mathcal{C}$ is a symmetric monoidal ∞-category: in this case, we will refer to $\mathrm{Alg}_\mathcal{O}(\mathcal{C})$ as the *∞-category of commutative algebra objects of* $\mathcal{C}$ and denote it by CAlg($\mathcal{C}$). We will make a thorough study of algebra objects (commutative and otherwise) in Chapter 3.

In Chapter 4, we will specialize our general theory of algebras to the case where $\mathcal{O}$ is the *associative ∞-operad*. In this case, we will denote $\mathrm{Alg}_\mathcal{O}(\mathcal{C})$ by $\mathrm{Alg}(\mathcal{C})$ and refer to it the *∞-category of associative algebra objects of* $\mathcal{C}$. The ∞-categorical theory of associative algebra objects is an excellent formal parallel of the usual theory of associative algebras. For example, one can study left modules, right modules, and bimodules over associative algebras. This theory of modules has some nontrivial applications. For example, we will use it in Chapter 6 to prove a version of Deligne's conjecture (regarding the structure of the Hochschild cochain complex of an associative algebra) and an ∞-categorical analogue of the Barr-Beck theorem, which has many applications in higher category theory.

In ordinary algebra, there is a thin line dividing the theory of commutative rings from the theory of associative rings: a commutative ring R is just an associative ring whose elements satisfy the additional identity $xy = yx$. In the ∞-categorical setting, the situation is rather different. Between the theory of associative and commutative algebras is a whole hierarchy of intermediate notions of commutativity, which are described by the "little cubes" operads of Boardman and Vogt. In Chapter 5, we will introduce the notion of an $\mathbb{E}_k$-*algebra* for each $0 \leq k \leq \infty$. This definition reduces to the notion of an associative algebra in the case $k = 1$, and to the notion of a commutative algebra when $k = \infty$. The theory of $\mathbb{E}_k$-algebras has

many applications in intermediate cases $1 < k < \infty$, and is closely related to the topology of k-dimensional manifolds.

The theory of differential calculus provides techniques for analyizing a general (smooth) function $f : \mathbf{R} \to \mathbf{R}$ by studying linear functions which approximate f. A fundamental insight of Goodwillie is that the same ideas can be fruitfully applied to problems in homotopy theory. More precisely, we can sometimes reduce questions about general ∞-categories and general functors to questions about stable ∞-categories and exact functors, which are more amenable to attack by algebraic methods. In Chapter 7 we will develop Goodwillie's calculus of functors in the ∞-categorical setting. Moreover, we will apply our theory of ∞-operads to formulate and prove a Koszul dual version of the chain rule of Arone-Ching.

In Chapter 8, we will study $\mathbb{E}_k$-algebra objects in the symmetric monoidal ∞-category of spectra, which we refer to as $\mathbb{E}_k$-*rings*. This can be regarded as a robust generalization of ordinary noncommutative algebra (when $k = 1$) or commutative algebra (when $k \geq 2$). In particular, we will see that a great deal of classical commutative algebra can be extended to the setting of $\mathbb{E}_\infty$-rings.

We close the book with two appendices. Appendix A develops the theory of constructible sheaves on stratified topological spaces. Aside from its intrinsic interest, this theory has a close connection with some of the geometric ideas of Chapter 5 and should prove useful in facilitating the application of those ideas. Appendix B is devoted to some rather technical existence results for model category structures on (and Quillen functors between) certain categories of simplicial sets. We recommend that the reader refer to this material only as necessary.

Prerequisites

The following definition will play a central role in this book:

Definition 0.0.0.1. An ∞-*category* is a simplicial set $\mathcal{C}$ which satisfies the following extension condition:

($*$) Every map of simplicial sets $f_0 : \Lambda_i^n \to \mathcal{C}$ can be extended to an n-simplex $f : \Delta^n \to \mathcal{C}$, provided that $0 < i < n$.

Remark 0.0.0.2. The notion of ∞-category was introduced by Boardman and Vogt under the name *weak Kan complex* in [19]. They have been studied extensively by Joyal, and are often referred to as *quasicategories* in the literature.

If $\mathcal{E}$ is a category, then the nerve $\mathrm{N}(\mathcal{E})$ of $\mathcal{E}$ is an ∞-category. Consequently, we can think of the theory of ∞-categories as a generalization of category theory. It turns out to be a *robust* generalization: most of the important concepts from classical category theory (limits and colimits, adjoint functors, sheaves and presheaves, etcetera) can be generalized to the setting of ∞-categories. For a detailed exposition, we refer the reader to our book [97].

Remark 0.0.0.3. For a different treatment of the theory of ∞-categories, we refer the reader to Joyal's notes [78]. Other references include [19], [82], [79], [80], [115], [39], [40], [121], and [63].

Apart from [97], the formal prerequisites for reading this book are few. We will assume that the reader is familiar with the homotopy theory of simplicial sets (good references on this include [105] and [57]) and with a bit of homological algebra (for which we recommend [159]). Familiarity with other concepts from algebraic topology (spectra, cohomology theories, operads, etcetera) will be helpful, but not strictly necessary: one of the main goals of this book is to give a self-contained exposition of these topics from an ∞-categorical perspective.

Notation and Terminology

We now call the reader's attention to some of the terminology used in this book:

- We will make extensive use of definitions and notations from the book [97]. If the reader encounters something confusing or unfamiliar, we recommend looking there first. We adopt the convention that references to [97] will be indicated by use of the letter T. For example, Theorem T.6.1.0.6 refers to Theorem 6.1.0.6 of [97].

- We say that a category (or ∞-category) $\mathcal{C}$ is *presentable* if $\mathcal{C}$ admits small colimits and is generated under small colimits by a set of κ-compact objects, for some regular cardinal number κ. This is departure from the standard category-theoretic terminology, in which such categories are called *locally presentable* (see [1]).

- We let $\mathcal{S}et_\Delta$ denote the category of simplicial sets. If J is a linearly ordered set, we let Δ^J denote the simplicial set given by the nerve of J, so that the collection of n-simplices of Δ^J can be identified with the collection of all nondecreasing maps $\{0, \ldots, n\} \to J$. We will frequently apply this notation when J is a subset of $\{0, \ldots, n\}$; in this case, we can identify Δ^J with a subsimplex of the standard n-simplex Δ^n (at least if $J \neq \emptyset$; if $J = \emptyset$, then Δ^J is empty).

- We will often use the term *space* to refer to a Kan complex (that is, a simplicial set satisfying the Kan extension condition).

- Let $n \geq 0$. We will say that a space X is *n-connective* if it is nonempty and the homotopy sets $\pi_i(X, x)$ are trivial for $i < n$ and every vertex x of X (spaces with this property are more commonly referred to as $(n-1)$-*connected* in the literature). We say that X is *connected* if it is 1-connective. By convention, we say that every space X is (-1)-connective. We will say that a map of spaces $f : X \to Y$ is *n-connective* if the homotopy fibers of f are n-connective.

- Let $n \geq -1$. We say that a space X is *n-truncated* if the homotopy sets $\pi_i(X, x)$ are trivial for every $i > n$ and every vertex $x \in X$. We say that X is *discrete* if it is 0-truncated. By convention, we say that X is (-2)-truncated if and only if X is contractible. We will say that a map of spaces $f : X \to Y$ is *n-truncated* if the homotopy fibers of f are n-truncated.

- Throughout this book, we will use *homological* indexing conventions whenever we discuss homological algebra. For example, chain complexes of abelian groups will be denoted by

$$\cdots \to A_2 \to A_1 \to A_0 \to A_{-1} \to A_{-2} \to \cdots,$$

with the differential lowering the degree by 1.

- In Chapter 1, we will construct an ∞-category Sp, whose homotopy category hSp can be identified with the classical stable homotopy category. In Chapter 8, we will construct a symmetric monoidal structure on Sp, which gives (in particular) a tensor product functor $\mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$. At the level of the homotopy category hSp, this functor is given by the classical *smash product* of spectra, which is usually denoted by $(X, Y) \mapsto X \wedge Y$. We will adopt a different convention, and denote the smash product functor by $(X, Y) \mapsto X \otimes Y$.

- If $\mathbf{A}$ is a model category, we let $\mathbf{A}^{\circ}$ denote the full subcategory of $\mathbf{A}$ spanned by the fibrant-cofibrant objects.

- Let $\mathcal{C}$ be an ∞-category. We let $\mathcal{C}^{\simeq}$ denote the largest Kan complex contained in $\mathcal{C}$: that is, the ∞-category obtained from $\mathcal{C}$ by discarding all noninvertible morphisms.

- Let $\mathcal{C}$ be an ∞-category containing objects X and Y. We let $\mathcal{C}_{X/}$ and $\mathcal{C}_{/Y}$ denote the undercategory and overcategory defined in §T.1.2.9. We will generally abuse notation by identifying objects of these ∞-categories with their images in $\mathcal{C}$. If we are given a morphism $f : X \to Y$, we can identify X with an object of $\mathcal{C}_{/Y}$ and Y with an object of $\mathcal{C}_{X/}$, so that the ∞-categories

$$(\mathcal{C}_{X/})_{/Y} \qquad (\mathcal{C}_{/Y})_{X/}$$

are defined (and canonically isomorphic as simplicial sets). We will denote these ∞-categories by $\mathcal{C}_{X//Y}$ (beware that this notation is slightly abusive: the definition of $\mathcal{C}_{X//Y}$ depends not only on $\mathcal{C}$, X, and Y, but also on the morphism f).

- Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. We let $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which admit right adjoints, and $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D})$ the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which admit left adjoints. If $\mathcal{C}$ and $\mathcal{D}$ are presentable, then these subcategories admit a simpler characterization: a functor $F : \mathcal{C} \to \mathcal{D}$ belongs to $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ if and only if it preserves small colimits, and belongs to $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D})$ if and only if it preserves small limits and small κ-filtered colimits for a sufficiently large regular cardinal κ (see Corollary T.5.5.2.9).

- We will say that a map of simplicial sets $f : S \to T$ is *left cofinal* if, for every right fibration $X \to T$, the induced map of simplicial sets $\mathrm{Fun}_T(T, X) \to \mathrm{Fun}_T(S, X)$ is a homotopy equivalence of Kan complexes (in [97], we referred to a map with this property as *cofinal*). We will say that f is *right cofinal* if the induced map $S^{op} \to T^{op}$ is left cofinal: that is, if f induces a homotopy equivalence $\mathrm{Fun}_T(T, X) \to \mathrm{Fun}_T(S, X)$ for every *left* fibration $X \to T$. If S and T are ∞-categories, then f is left cofinal if and only if for every object $t \in T$, the fiber product $S \times_T T_{t/}$ is weakly contractible (Theorem T.4.1.3.1).

Acknowledgements

Chapter 1

Stable ∞-Categories

There is a very useful analogy between topological spaces and chain complexes with values in an abelian category. For example, it is customary to speak of *homotopies* between chain maps, *contractible* complexes, and so forth. The analogue of the homotopy category of topological spaces is the *derived category* of an abelian category $\mathcal{A}$, a triangulated category which provides a good setting for many constructions in homological algebra. However, it has long been recognized that for many purposes the derived category is too crude: it identifies homotopic morphisms of chain complexes without remembering *why* they are homotopic. It is possible to correct this defect by viewing the derived category as the homotopy category of an underlying ∞-category $\mathcal{D}(\mathcal{A})$. The ∞-categories which arise in this way have special features that reflect their "additive" origins: they are *stable*.

We will begin in §1.1 by giving the definition of stability and exploring some of its consequences. For example, we will show that if $\mathcal{C}$ is a stable ∞-category, then its homotopy category h$\mathcal{C}$ is triangulated (Theorem 1.1.2.14), and that stable ∞-categories admit finite limits and colimits (Proposition 1.1.3.4). The appropriate notion of *functor* between stable ∞-categories is an *exact* functor: that is, a functor which preserves finite colimits (or equivalently, finite limits: see Proposition 1.1.4.1). The collection of stable ∞-categories and exact functors between them can be organized into an ∞-category, which we will denote by $\mathrm{Cat}_\infty^{\mathrm{Ex}}$. In §1.1.4, we will establish some basic closure properties of the ∞-category $\mathrm{Cat}_\infty^{\mathrm{Ex}}$; in particular, we will show that it is closed under the formation of limits and filtered colimits in Cat_∞. The formation of limits in $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ provides a tool for addressing the classical problem of "gluing in the derived category".

In §1.2, we recall the definition of a *t-structure* on a triangulated category. If $\mathcal{C}$ is a stable ∞-category, we define a *t-structure* on $\mathcal{C}$ to be a t-structure on its homotopy category h$\mathcal{C}$. If $\mathcal{C}$ is equipped with a t-structure, we show that every filtered object of $\mathcal{C}$ gives rise to a spectral sequence taking values in the heart $\mathcal{C}^\heartsuit$ (Proposition 1.2.2.7). In particular, we show that every simplicial object of $\mathcal{C}$ determines a spectral sequence, using an ∞-categorical analogue of the Dold-Kan correspondence.

We will return to the setting of homological algebra in §1.3. To any abelian category $\mathcal{A}$ with enough projective objects, one can associate a stable ∞-category $\mathcal{D}^-(\mathcal{A})$, whose objects are (right-bounded) chain complexes of projective objects of $\mathcal{A}$. This ∞-category provides useful tools for organizing information in homological algebra. Our main result (Theorem 1.3.3.8) is a characterization of $\mathcal{D}^-(\mathcal{A})$ by a universal mapping property.

In §1.4, we will focus our attention on a particular stable ∞-category: the ∞-category Sp of *spectra*. The homotopy category of Sp can be identified with the classical *stable homotopy category*, which is the natural setting for a large portion of modern algebraic topology. Roughly speaking, a spectrum is a sequence of pointed spaces $\{X(n)\}_{n \in \mathbf{Z}}$ equipped with homotopy equivalences $X(n) \simeq \Omega X(n+1)$, where Ω denotes the functor given by passage to the loop space. More generally, one can obtain a stable ∞-category by considering sequences as above which take values in an arbitrary ∞-category $\mathcal{C}$ which admits finite limits; we denote this ∞-category by $\mathrm{Sp}(\mathcal{C})$ and refer to it as the *∞-category of spectrum objects of* $\mathcal{C}$.

1.1   Foundations

Our goal in this section is to introduce our main object of study for this chapter: the notion of a *stable ∞-category*. The theory of stable ∞-categories can be regarded as an axiomatization of the essential features of stable homotopy theory: most notably, that fiber sequences and cofiber sequences are the same. We will begin in §1.1.1 by reviewing some of the relevant notions (pointed ∞-categories, zero objects, fiber and cofiber sequences) and using them to define the class of stable ∞-categories.

In §1.1.2, we will review Verdier's definition of a *triangulated category*. We will show that if $\mathcal{C}$ is a stable ∞-category, then its homotopy category h$\mathcal{C}$ has the structure of a triangulated category (Theorem 1.1.2.14). The theory of triangulated categories can be regarded as an attempt to capture those features of stable ∞-categories which are easily visible at the level of homotopy categories. Triangulated categories which arise naturally in mathematics are usually given as the homotopy categories of stable ∞-categories, though it is possible to construct triangulated categories which are not of this form (see [113]).

Our next goal is to study the properties of stable ∞-categories in greater depth. In §1.1.3, we will show that a stable ∞-category $\mathcal{C}$ admits all finite limits and colimits, and that pullback squares and pushout squares in $\mathcal{C}$ are the same (Proposition 1.1.3.4). We will also show that the class of stable ∞-categories is closed under various natural operations. For example, we will show that if $\mathcal{C}$ is a stable ∞-category, then the ∞-category of Ind-objects Ind($\mathcal{C}$) is stable (Proposition 1.1.3.6), and that the ∞-category of diagrams Fun($K, \mathcal{C}$) is stable for any simplicial set K (Proposition 1.1.3.1).

In §1.1.4, we shift our focus somewhat. Rather than concerning ourselves with the properties of an individual stable ∞-category $\mathcal{C}$, we will study the collection of all stable ∞-categories. To this end, we introduce the notion of an *exact* functor between stable ∞-categories. We will show that the collection of all (small) stable ∞-categories and exact functors between them can itself be organized into an ∞-category $\mathrm{Cat}_\infty^{\mathrm{Ex}}$, and study some of the properties of $\mathrm{Cat}_\infty^{\mathrm{Ex}}$.

Remark 1.1.0.1. The theory of stable ∞-categories is not really new: most of the results presented here are well-known to experts. There exists a growing literature on the subject in the setting of *stable model categories*: see, for example, [37], [126], [128], and [72]. For a brief account in the more flexible setting of Segal categories, we refer the reader to [152].

Remark 1.1.0.2. Let k be a field. Recall that a *differential graded category* over k is a category enriched over the category of chain complexes of k-vector spaces. The theory of differential graded categories is closely related to the theory of stable ∞-categories. More precisely, one can show that the data of a (pretriangulated) differential graded category over k is equivalent to the data of a stable ∞-category $\mathcal{C}$ equipped with an enrichment over the monoidal ∞-category of k-module spectra. The theory of differential graded categories provides a convenient language for working with stable ∞-categories of algebraic origin (for example, those which arise from chain complexes of coherent sheaves on algebraic varieties), but is inadequate for treating examples which arise in stable homotopy theory. There is a voluminous literature on the subject; see, for example, [84], [101], [141], [35], and [147].

1.1.1   Stability

In this section, we introduce the definition of a stable ∞-category. We begin by reviewing some definitions from [97].

Definition 1.1.1.1. Let $\mathcal{C}$ be an ∞-category. A *zero object* of $\mathcal{C}$ is an object which is both initial and final. We will say that $\mathcal{C}$ is *pointed* if it contains a zero object.

In other words, an object $0 \in \mathcal{C}$ is zero if the spaces $\mathrm{Map}_{\mathcal{C}}(X, 0)$ and $\mathrm{Map}_{\mathcal{C}}(0, X)$ are both contractible for every object $X \in \mathcal{C}$. Note that if $\mathcal{C}$ contains a zero object, then that object is determined up to equivalence. More precisely, the full subcategory of $\mathcal{C}$ spanned by the zero objects is a contractible Kan complex (Proposition T.1.2.12.9).

Remark 1.1.1.2. Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is pointed if and only if the following conditions are satisfied:

(1) The ∞-category $\mathcal{C}$ has an initial object $\emptyset$.

(2) The ∞-category $\mathcal{C}$ has a final object 1.

(3) There exists a morphism $f : 1 \to \emptyset$ in $\mathcal{C}$.

The "only if" direction is obvious. For the converse, let us suppose that (1), (2), and (3) are satisfied. We invoke the assumption that $\emptyset$ is initial to deduce the existence of a morphism $g : \emptyset \to 1$. Because $\emptyset$ is initial, $f \circ g \simeq \mathrm{id}_\emptyset$, and because 1 is final, $g \circ f \simeq \mathrm{id}_1$. Thus g is a homotopy inverse to f, so that f is an equivalence. It follows that $\emptyset$ is also a final object of $\mathcal{C}$, so that $\mathcal{C}$ is pointed.

Remark 1.1.1.3. Let $\mathcal{C}$ be an ∞-category with a zero object 0. For any $X, Y \in \mathcal{C}$, the natural map

$$\mathrm{Map}_\mathcal{C}(X, 0) \times \mathrm{Map}_\mathcal{C}(0, Y) \to \mathrm{Map}_\mathcal{C}(X, Y)$$

has contractible domain. We therefore obtain a well defined morphism $X \to Y$ in the homotopy category $h\mathcal{C}$, which we will refer to as the *zero morphism* and also denote by 0.

Definition 1.1.1.4. Let $\mathcal{C}$ be a pointed ∞-category. A *triangle* in $\mathcal{C}$ is a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$, depicted as

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\downarrow & & \downarrow{\scriptstyle g} \\
0 & \longrightarrow & Z
\end{array}
$$

where 0 is a zero object of $\mathcal{C}$. We will say that a triangle in $\mathcal{C}$ is a *fiber sequence* if it is a pullback square, and a *cofiber sequence* if it is a pushout square.

Remark 1.1.1.5. Let $\mathcal{C}$ be a pointed ∞-category. A triangle in $\mathcal{C}$ consists of the following data:

(1) A pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$.

(2) A 2-simplex in $\mathcal{C}$ corresponding to a diagram

in $\mathcal{C}$, which identifies h with the composition $g \circ f$.

(3) A 2-simplex

in $\mathcal{C}$, which we may view as a *nullhomotopy* of h.

We will generally indicate a triangle by specifying only the pair of maps

$$X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z,$$

with the data of (2) and (3) being implicitly assumed.

Definition 1.1.1.6. Let $\mathcal{C}$ be a pointed ∞-category containing a morphism $g : X \to Y$. A *fiber* of g is a fiber sequence

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & Y. \end{array}$$

Dually, a *cofiber* of g is a cofiber sequence

$$\begin{array}{ccc} X & \xrightarrow{\ g\ } & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z. \end{array}$$

We will generally abuse terminology by simply referring to W and Z as the fiber and cofiber of g. We will also write $W = \mathrm{fib}(g)$ and $Z = \mathrm{cofib}(g)$.

Remark 1.1.1.7. Let $\mathcal{C}$ be a pointed ∞-category containing a morphism $f : X \to Y$. A cofiber of f, if it exists, is uniquely determined up to equivalence. More precisely, consider the full subcategory $\mathcal{E} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ spanned by the cofiber sequences. Let $\theta : \mathcal{E} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ be the forgetful functor, which associates to a diagram

$$\begin{array}{ccc} X & \xrightarrow{\ g\ } & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

the morphism $g : X \to Y$. Applying Proposition T.4.3.2.15 twice, we deduce that θ is a Kan fibration, whose fibers are either empty or contractible (depending on whether or not a morphism $g : X \to Y$ in $\mathcal{C}$ admits a cofiber). In particular, if every morphism in $\mathcal{C}$ admits a cofiber, then θ is a trivial Kan fibration, and therefore admits a section $\mathrm{cofib} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$, which is well defined up to a contractible space of choices. We will often abuse notation by also letting $\mathrm{cofib} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ denote the composition

$$\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}) \to \mathcal{C},$$

where the second map is given by evaluation at the final object of $\Delta^1 \times \Delta^1$.

Remark 1.1.1.8. The functor $\mathrm{cofib} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ can be identified with a left adjoint to the left Kan extension functor $\mathcal{C} \simeq \mathrm{Fun}(\{1\}, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$, which associates to each object $X \in \mathcal{C}$ a zero morphism $0 \to X$. It follows that cofib preserves all colimits which exist in $\mathrm{Fun}(\Delta^1, \mathcal{C})$ (Proposition T.5.2.3.5).

Definition 1.1.1.9. An ∞-category $\mathcal{C}$ is *stable* if it satisfies the following conditions:

(1) There exists a zero object $0 \in \mathcal{C}$.

(2) Every morphism in $\mathcal{C}$ admits a fiber and a cofiber.

(3) A triangle in $\mathcal{C}$ is a fiber sequence if and only if it a cofiber sequence.

Remark 1.1.1.10. Condition (3) of Definition 1.1.1.9 is analogous to the axiom for abelian categories which requires that the image of a morphism be isomorphic to its coimage.

Example 1.1.1.11. Recall that a *spectrum* consists of an infinite sequence of pointed topological spaces $\{X_i\}_{i \geq 0}$, together with homeomorphisms $X_i \simeq \Omega X_{i+1}$, where Ω denotes the loop space functor. The collection of spectra can be organized into a stable ∞-category Sp. Moreover, Sp is in some sense the universal example of a stable ∞-category. This motivates the terminology of Definition 1.1.1.9: an ∞-category $\mathcal{C}$ is stable if it resembles the ∞-category Sp, whose homotopy category hSp can be identified with the classical *stable homotopy category*. We will return to the theory of spectra (using a slightly different definition) in §1.4.3.

Example 1.1.1.12. Let $\mathcal{A}$ be an abelian category. Under mild hypotheses, we can construct a stable ∞-category $\mathcal{D}(\mathcal{A})$ whose homotopy category $h\mathcal{D}(\mathcal{A})$ can be identified with the *derived category of $\mathcal{A}$*, in the sense of classical homological algebra. We will outline the construction of $\mathcal{D}(\mathcal{A})$ in §1.3.2.

Remark 1.1.1.13. If $\mathcal{C}$ is a stable ∞-category, then the opposite ∞-category $\mathcal{C}^{op}$ is also stable.

Remark 1.1.1.14. One attractive feature of the theory of stable ∞-categories is that stability is a property of ∞-categories, rather than additional data. The situation for additive categories is similar. Although additive categories are often presented as categories equipped with additional structure (an abelian group structure on all Hom-sets), this additional structure is in fact determined by the underlying category: see Definition 1.1.2.1. The situation for stable ∞-categories is similar: we will see later that every stable ∞-category is canonically enriched over the ∞-category of spectra.

1.1.2 The Homotopy Category of a Stable ∞-Category

Let M be a module over a commutative ring R. Then M admits a resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

by projective R-modules. In fact, there are generally many choices for such a resolution. Two projective resolutions of M need not be isomorphic to one another. However, they are always *quasi-isomorphic*: that is, if we are given two projective resolutions $P_\bullet$ and $P'_\bullet$ of M, then there is a map of chain complexes $P_\bullet \to P'_\bullet$ which induces an isomorphism on homology groups. This phenomenon is ubiquitous in homological algebra: many constructions produce chain complexes which are not really well-defined up to isomorphism, but only up to quasi-isomorphism. In studying these constructions, it is often convenient to work in the *derived category* $\mathcal{D}(R)$ of the ring R: that is, the category obtained from the category of chain complexes of R-modules by formally inverting all quasi-isomorphisms.

The derived category $\mathcal{D}(R)$ of a commutative ring R is usually not an abelian category. For example, a morphism $f : X' \to X$ in $\mathcal{D}(R)$ usually does not have a cokernel in $\mathcal{D}(R)$. Instead, one can associate to f its *cofiber* (or *mapping cone*) X'', which is well-defined up to noncanonical isomorphism. In [154], Verdier introduced the notion of a *triangulated category* in order to axiomatize the structure on $\mathcal{D}(R)$ given by the formation of mapping cones. In this section, we will review Verdier's theory of triangulated categories (Definition 1.1.2.5) and show that the homotopy category of a stable ∞-category $\mathcal{C}$ is triangulated (Theorem 1.1.2.14).

We begin with some basic definitions.

Definition 1.1.2.1. Let $\mathcal{A}$ be a category. We will say that $\mathcal{A}$ is *additive* if it satisfies the following four conditions:

(1) The category $\mathcal{A}$ admits finite products and coproducts.

(2) The category $\mathcal{A}$ has a zero object, which we will denote by 0.

For any pair of objects $X, Y \in \mathcal{A}$, a *zero morphism* from X to Y is a map $f : X \to Y$ which factors as a composition $X \to 0 \to Y$. It follows from (2) that for every pair $X, Y \in \mathcal{A}$, there is a unique zero morphism from X to Y, which we will denote by 0.

(3) For every pair of objects X, Y, the map $X \coprod Y \to X \times Y$ described by the matrix

$$\begin{bmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_Y \end{bmatrix}$$

is an isomorphism; let $\phi_{X,Y}$ denote its inverse.

Assuming (3), we can define the sum of two morphisms $f, g : X \to Y$ to be the morphism $f + g$ given by the composition

$$X \to X \times X \xrightarrow{f,g} Y \times Y \xrightarrow{\phi_{Y,Y}} Y \coprod Y \to Y.$$

It is easy to see that this construction endows $\operatorname{Hom}_{\mathcal{A}}(X, Y)$ with the structure of a commutative monoid, whose identity is the unique zero morphism from X to Y.

(4) For every pair of objects $X, Y \in \mathcal{A}$, the addition defined above determines a group structure on $\operatorname{Hom}_{\mathcal{A}}(X, Y)$. In other words, for every morphism $f : X \to Y$, there exists another morphism $-f : X \to Y$ such that $f + (-f)$ is a zero morphism from X to Y.

Remark 1.1.2.2. An additive category $\mathcal{A}$ is said to be *abelian* if every morphism $f : X \to Y$ in $\mathcal{A}$ admits a kernel and a cokernel, and the canonical map $\operatorname{coker}(\ker(f) \to X) \to \ker(Y \to \operatorname{coker}(f))$ is an isomorphism.

Remark 1.1.2.3. Let $\mathcal{A}$ be an additive category. Then the composition law on $\mathcal{A}$ is bilinear: for pairs of morphisms $f, f' \in \operatorname{Hom}_{\mathcal{A}}(X, Y)$ and $g, g' \in \operatorname{Hom}_{\mathcal{A}}(Y, Z)$, we have

$$g \circ (f + f') = (g \circ f) + (g \circ f') \qquad (g + g') \circ f = (g \circ f) + (g' \circ f).$$

In other words, the composition law on $\mathcal{A}$ determines abelian group homomorphisms

$$\operatorname{Hom}_{\mathcal{A}}(X, Y) \otimes \operatorname{Hom}_{\mathcal{A}}(Y, Z) \to \operatorname{Hom}_{\mathcal{A}}(X, Z).$$

We can summarize the situation by saying that the category $\mathcal{A}$ is *enriched* over the category of abelian groups.

Remark 1.1.2.4. Let $\mathcal{A}$ be an additive category. It follows from condition (3) of Definition 1.1.2.1 that for every pair of objects $X, Y \in \mathcal{A}$, the product $X \times Y$ is canonically isomorphic to the coproduct $X \coprod Y$. It is customary to emphasize this identification by denoting both the product and the coproduct by $X \oplus Y$; we will refer to $X \oplus Y$ as the *direct sum* of X and Y.

Definition 1.1.2.5 (Verdier)**.** A *triangulated category* consists of the following data:

(1) An additive category $\mathcal{D}$.

(2) A translation functor $\mathcal{D} \to \mathcal{D}$ which is an equivalence of categories. We denote this functor by $X \mapsto X[1]$.

(3) A collection of *distinguished triangles*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1].$$

These data are required to satisfy the following axioms:

$(TR1)$ (a) Every morphism $f : X \to Y$ in $\mathcal{D}$ can be extended to a distinguished triangle in $\mathcal{D}$.

  (b) The collection of distinguished triangles is stable under isomorphism.

  (c) Given an object $X \in \mathcal{D}$, the diagram

$$X \xrightarrow{\operatorname{id}_X} X \to 0 \to X[1]$$

is a distinguished triangle.

$(TR2)$ A diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle if and only if the rotated diagram

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is a distinguished triangle.

($TR3$) Given a commutative diagram

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow{\scriptstyle f} & & \downarrow & & \vdots & & \downarrow{\scriptstyle f[1]} \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
\end{array}$$

in which both horizontal rows are distinguished triangles, there exists a dotted arrow rendering the entire diagram commutative.

($TR4$) Suppose given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{u} Y/X \xrightarrow{d} X[1]$$

$$Y \xrightarrow{g} Z \xrightarrow{v} Z/Y \xrightarrow{d'} Y[1]$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{w} Z/X \xrightarrow{d''} X[1]$$

in $\mathcal{D}$. There exists a fourth distinguished triangle

$$Y/X \xrightarrow{\phi} Z/X \xrightarrow{\psi} Z/Y \xrightarrow{\theta} Y/X[1]$$

such that the diagram

commutes.

We now consider the problem of constructing a triangulated structure on the homotopy category of an ∞-category $\mathcal{C}$. Let us begin by assuming only that $\mathcal{C}$ is a pointed ∞-category. We let $\mathcal{M}^{\Sigma}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ spanned by those diagrams

$$\begin{array}{ccc}
X & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0' & \longrightarrow & Y
\end{array}$$

which are pushout squares, and such that 0 and $0'$ are zero objects of $\mathcal{C}$. If $\mathcal{C}$ admits cofibers, then we can use Proposition T.4.3.2.15 (twice) to conclude that evaluation at the initial vertex induces a trivial fibration $\mathcal{M}^{\Sigma} \to \mathcal{C}$. Let $s : \mathcal{C} \to \mathcal{M}^{\Sigma}$ be a section of this trivial fibration, and let $e : \mathcal{M}^{\Sigma} \to \mathcal{C}$ be the functor given by evaluation at the final vertex. The composition $e \circ s$ is a functor from $\mathcal{C}$ to itself, which we will denote by $\Sigma : \mathcal{C} \to \mathcal{C}$ and refer to as the *suspension functor* on $\mathcal{C}$. Dually, we define $\mathcal{M}^{\Omega}$ to be the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ spanned by diagrams as above which are pullback squares with 0 and $0'$ zero objects of $\mathcal{C}$. If $\mathcal{C}$ admits fibers, then the same argument shows that evaluation at the final vertex induces a trivial fibration $\mathcal{M}^{\Omega} \to \mathcal{C}$. If we let s' denote a section to this trivial fibration, then the composition of s' with evaluation at the initial vertex induces a functor from $\mathcal{C}$ to itself, which we will refer to as the *loop functor* and denote by $\Omega : \mathcal{C} \to \mathcal{C}$. If $\mathcal{C}$ is stable, then $\mathcal{M}^{\Omega} = \mathcal{M}^{\Sigma}$. It follows that Σ and Ω are mutually inverse equivalences from $\mathcal{C}$ to itself.

Remark 1.1.2.6. If the ∞-category $\mathcal{C}$ is not clear from context, then we will denote the suspension and loop functors $\Sigma, \Omega : \mathcal{C} \to \mathcal{C}$ by $\Sigma_{\mathcal{C}}$ and $\Omega_{\mathcal{C}}$, respectively.

Notation 1.1.2.7. If $\mathcal{C}$ is a stable ∞-category and $n \geq 0$, we let

$$X \mapsto X[n]$$

denote the nth power of the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ constructed above (this functor is well-defined up to canonical equivalence). If $n \leq 0$, we let $X \mapsto X[n]$ denote the $(-n)$th power of the loop functor Ω. We will use the same notation to indicate the induced functors on the homotopy category $h\mathcal{C}$.

Remark 1.1.2.8. If the ∞-category $\mathcal{C}$ is pointed but not necessarily stable, the suspension and loop space functors need not be homotopy inverses but are nevertheless *adjoint* to one another (provided that both functors are defined).

If $\mathcal{C}$ is a pointed ∞-category containing a pair of objects X and Y, then the space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ has a natural base point, given by the zero map. Moreover, if $\mathcal{C}$ admits cofibers, then the suspension functor $\Sigma_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is essentially characterized by the existence of natural homotopy equivalences

$$\mathrm{Map}_{\mathcal{C}}(\Sigma(X), Y) \to \Omega \, \mathrm{Map}_{\mathcal{C}}(X, Y).$$

In particular, we conclude that $\pi_0 \, \mathrm{Map}_{\mathcal{C}}(\Sigma(X), Y) \simeq \pi_1 \, \mathrm{Map}_{\mathcal{C}}(X, Y)$, so that $\pi_0 \, \mathrm{Map}_{\mathcal{C}}(\Sigma(X), Y)$ has the structure of a group (here the fundamental group of $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is taken with base point given by the zero map). Similarly, $\pi_0 \, \mathrm{Map}_{\mathcal{C}}(\Sigma^2(X), Y) \simeq \pi_2 \, \mathrm{Map}_{\mathcal{C}}(X, Y)$ has the structure of an *abelian group*. If the suspension functor $X \mapsto \Sigma(X)$ is an equivalence of ∞-categories, then for every $Z \in \mathcal{C}$ we can choose X such that $\Sigma^2(X) \simeq Z$ to deduce the existence of an abelian group structure on $\mathrm{Map}_{\mathcal{C}}(Z, Y)$. It is easy to see that this group structure depends functorially on $Z, Y \in h\mathcal{C}$. We are therefore most of the way to proving the following result:

Lemma 1.1.2.9. *Let $\mathcal{C}$ be a pointed ∞-category which admits cofibers, and suppose that the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence. Then $h\mathcal{C}$ is an additive category.*

Proof. The argument sketched above shows that $h\mathcal{C}$ is (canonically) enriched over the category of abelian groups. It will therefore suffice to prove that $h\mathcal{C}$ admits finite coproducts. We will prove a slightly stronger statement: the ∞-category $\mathcal{C}$ itself admits finite coproducts. Since $\mathcal{C}$ has an initial object, it will suffice to treat the case of pairwise coproducts. Let $X, Y \in \mathcal{C}$, and let $\mathrm{cofib} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ denote the functor which assign to each morphism its cofiber, so that we have equivalences $X \simeq \mathrm{cofib}(X[-1] \overset{u}{\to} 0)$ and $Y \simeq \mathrm{cofib}(0 \overset{v}{\to} Y)$. Proposition T.5.1.2.2 implies that u and v admit a coproduct in $\mathrm{Fun}(\Delta^1, \mathcal{C})$ (namely, the zero map $X[-1] \overset{0}{\to} Y$). Since the functor cofib preserves coproducts (Remark 1.1.1.8), we conclude that X and Y admit a coproduct (which can be constructed as the cofiber of the zero map from $X[-1]$ to Y). $\qquad \square$

Let $\mathcal{C}$ be a pointed ∞-category which admits cofibers. By construction, any diagram

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & Y \end{array}$$

which belongs to $\mathcal{M}^{\Sigma}$ determines a canonical isomorphism $X[1] \to Y$ in the homotopy category $h\mathcal{C}$. We will need the following observation:

Lemma 1.1.2.10. *Let $\mathcal{C}$ be a pointed ∞-category which admits cofibers, and let*

$$\begin{array}{ccc} X & \overset{f}{\longrightarrow} & 0 \\ {\scriptstyle f'}\downarrow & & \downarrow \\ 0' & \longrightarrow & Y \end{array}$$

be a diagram in $\mathcal{C}$, *classifying a morphism* $\theta \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X[1], Y)$. (*Here* 0 *and* $0'$ *are zero objects of* $\mathcal{C}$.) *Then the transposed diagram*

$$
\begin{array}{ccc}
X & \xrightarrow{\ f'\ } & 0' \\
\downarrow{\scriptstyle f} & & \downarrow \\
0 & \longrightarrow & Y
\end{array}
$$

classifies the morphism $-\theta \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X[1], Y)$. *Here* $-\theta$ *denotes the inverse of* θ *with respect to the group structure on* $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X[1], Y) \simeq \pi_1 \mathrm{Map}_{\mathcal{C}}(X, Y)$.

Proof. Without loss of generality, we may suppose that $0 = 0'$ and $f = f'$. Let $\sigma : \Lambda_0^2 \to \mathcal{C}$ be the diagram

$$
0 \xleftarrow{\ f\ } X \xrightarrow{\ f\ } 0.
$$

For every diagram $p : K \to \mathcal{C}$, let $\mathcal{D}(p)$ denote the Kan complex $\mathcal{C}_{p/} \times_{\mathcal{C}} \{Y\}$. Then $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X[1], Y) \simeq \pi_0 \, \mathcal{D}(\sigma)$. We note that

$$
\mathcal{D}(\sigma) \simeq \mathcal{D}(f) \times_{\mathcal{D}(X)} \mathcal{D}(f).
$$

Since 0 is an initial object of $\mathcal{C}$, $\mathcal{D}(f)$ is contractible. In particular, there exists a point $q \in \mathcal{D}(f)$. Let

$$
\mathcal{D}' = \mathcal{D}(f) \times_{\mathrm{Fun}(\{0\}, \mathcal{D}(X))} \mathrm{Fun}(\Delta^1, \mathcal{D}(X)) \times_{\mathrm{Fun}(\{1\}, \mathcal{D}(X))} \mathcal{D}(f)
$$

$$
\mathcal{D}'' = \{q\} \times_{\mathrm{Fun}(\{0\}, \mathcal{D}(X))} \mathrm{Fun}(\Delta^1, \mathcal{D}(X)) \times_{\mathrm{Fun}(\{1\}, \mathcal{D}(X))} \{q\}
$$

so that we have canonical inclusions

$$
\mathcal{D}'' \hookrightarrow \mathcal{D}' \hookleftarrow \mathcal{D}(\sigma).
$$

The left map is a homotopy equivalence because $\mathcal{D}(f)$ is contractible, and the right map is a homotopy equivalence because the projection $\mathcal{D}(f) \to \mathcal{D}(X)$ is a Kan fibration. We observe that $\mathcal{D}''$ can be identified with the simplicial loop space of $\mathrm{Hom}_{\mathcal{C}}^{\mathrm{L}}(X, Y)$ (taken with the base point determined by q, which we can identify with the zero map from X to Y). Each of the Kan complexes $\mathcal{D}(\sigma)$, $\mathcal{D}'$, $\mathcal{D}''$ is equipped with a canonical involution. On $\mathcal{D}(\sigma)$, this involution corresponds to the transposition of diagrams as in the statement of the lemma. On $\mathcal{D}''$, this involution corresponds to reversal of loops. The desired conclusion now follows from the observation that these involutions are compatible with the inclusions $\mathcal{D}'', \mathcal{D}(\sigma) \subseteq \mathcal{D}'$. $\qquad\square$

Definition 1.1.2.11. Let $\mathcal{C}$ be a pointed ∞-category which admits cofibers. Suppose given a diagram

$$
X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \xrightarrow{\ h\ } X[1]
$$

in the homotopy category $\mathrm{h}\mathcal{C}$. We will say that this diagram is a *distinguished triangle* if there exists a diagram $\Delta^1 \times \Delta^2 \to \mathcal{C}$ as shown

$$
\begin{array}{ccccc}
X & \xrightarrow{\ \widetilde{f}\ } & Y & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \widetilde{g}} & & \downarrow \\
0' & \longrightarrow & Z & \xrightarrow{\ \widetilde{h}\ } & W,
\end{array}
$$

satisfying the following conditions:

(*i*) The objects $0, 0' \in \mathcal{C}$ are zero.

(*ii*) Both squares are pushout diagrams in $\mathcal{C}$.

(*iii*) The morphisms $\widetilde{f}$ and $\widetilde{g}$ represent f and g, respectively.

(*iv*) The map $h : Z \to X[1]$ is the composition of (the homotopy class of) $\widetilde{h}$ with the equivalence $W \simeq X[1]$ determined by the outer rectangle.

Remark 1.1.2.12. We will generally only use Definition 1.1.2.11 in the case where $\mathcal{C}$ is a stable ∞-category. However, it will be convenient to have the terminology available in the case where $\mathcal{C}$ is not yet known to be stable.

The following result is an immediate consequence of Lemma 1.1.2.10:

Lemma 1.1.2.13. *Let $\mathcal{C}$ be a stable ∞-category. Suppose given a diagram $\Delta^2 \times \Delta^1 \to \mathcal{C}$, depicted as*

$$
\begin{array}{ccc}
X & \longrightarrow & 0 \\
\downarrow{\scriptstyle f} & & \downarrow \\
Y & \xrightarrow{\ g\ } & Z \\
\downarrow & & \downarrow{\scriptstyle h} \\
0' & \longrightarrow & W,
\end{array}
$$

where both squares are pushouts and the objects $0, 0' \in \mathcal{C}$ are zero. Then the diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{-h'} X[1]$$

is a distinguished triangle in $\mathrm{h}\mathcal{C}$, where h' denotes the composition of h with the isomorphism $W \simeq X[1]$ determined by the outer square, and $-h'$ denotes the composition of h' with the map $-\operatorname{id} \in \operatorname{Hom}_{\mathrm{h}\mathcal{C}}(X[1], X[1]) \simeq \pi_1 \operatorname{Map}_{\mathcal{C}}(X, X[1])$.

We can now state the main result of this section:

Theorem 1.1.2.14. *Let $\mathcal{C}$ be a pointed ∞-category which admits cofibers, and suppose that the suspension functor Σ is an equivalence. Then the translation functor of Notation 1.1.2.7 and the class of distinguished triangles of Definition 1.1.2.11 endow $\mathrm{h}\mathcal{C}$ with the structure of a triangulated category.*

Remark 1.1.2.15. The hypotheses of Theorem 1.1.2.14 hold whenever $\mathcal{C}$ is stable. In fact, the hypotheses of Theorem 1.1.2.14 are *equivalent* to the stability of $\mathcal{C}$: see Corollary 1.4.2.27.

Proof. We must verify that Verdier's axioms $(TR1)$ through $(TR4)$ are satisfied.

$(TR1)$ Let $\mathcal{E} \subseteq \operatorname{Fun}(\Delta^1 \times \Delta^2, \mathcal{C})$ be the full subcategory spanned by those diagrams

$$
\begin{array}{ccccc}
X & \xrightarrow{\ f\ } & Y & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0' & \longrightarrow & Z & \longrightarrow & W
\end{array}
$$

of the form considered in Definition 1.1.2.11, and let $e : \mathcal{E} \to \operatorname{Fun}(\Delta^1, \mathcal{C})$ be the restriction to the upper left horizontal arrow. Repeated use of Proposition T.4.3.2.15 implies e is a trivial fibration. In particular, every morphism $f : X \to Y$ can be completed to a diagram belonging to $\mathcal{E}$. This proves (*a*). Part (*b*) is obvious, and (*c*) follows from the observation that if $f = \operatorname{id}_X$, then the object Z in the above diagram is a zero object of $\mathcal{C}$.

$(TR2)$ Suppose that

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle in h𝒞, corresponding to a diagram $\sigma \in \mathcal{E}$ as depicted above. Extend σ to a diagram

$$
\begin{array}{ccccc}
X & \longrightarrow & Y & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0' & \longrightarrow & Z & \longrightarrow & W \\
& & \downarrow & & \downarrow {\scriptstyle u} \\
& & 0'' & \longrightarrow & V
\end{array}
$$

where the lower right square is a pushout and $0''$ is a zero object of $\mathcal{C}$. We have a map between the squares

$$
\begin{array}{ccc}
X \longrightarrow 0 & \qquad & Y \longrightarrow 0 \\
\downarrow \qquad \downarrow & & \downarrow \qquad \downarrow \\
0' \longrightarrow W & & 0'' \longrightarrow V
\end{array}
$$

which induces a commutative diagram in the homotopy category h𝒞

$$
\begin{array}{ccc}
W & \longrightarrow & X[1] \\
\downarrow {\scriptstyle u} & & \downarrow {\scriptstyle f[1]} \\
V & \longrightarrow & Y[1]
\end{array}
$$

where the horizontal arrows are isomorphisms. Applying Lemma 1.1.2.13 to the rectangle on the right of the large diagram, we conclude that

$$
Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]
$$

is a distinguished triangle in h𝒞.

Conversely, suppose that

$$
Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]
$$

is a distinguished triangle in h𝒞. Since the functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence, we conclude that the triangle

$$
Y[-2] \xrightarrow{g[-2]} Z[-2] \xrightarrow{h[-2]} X[-1] \xrightarrow{-f[-1]} Y[-1]
$$

is distinguished. Applying the preceding argument five times, we conclude that the triangle

$$
X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]
$$

is distinguished, as desired.

$(TR3)$ Suppose we are given distinguished triangles

$$
X \xrightarrow{f} Y \to Z \to X[1]
$$

$$
X' \xrightarrow{f'} Y' \to Z' \to X'[1]
$$

in h𝒞. Without loss of generality, we may suppose that these triangles are induced by diagrams $\sigma, \sigma' \in \mathcal{E}$. Any commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X' & \xrightarrow{f'} & Y'
\end{array}
$$

in the homotopy category h$\mathcal{C}$ can be lifted (nonuniquely) to a square in $\mathcal{C}$, which we may identify with a morphism $\phi : e(\sigma) \to e(\sigma')$ in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$. Since e is a trivial fibration of simplicial sets, ϕ can be lifted to a morphism $\sigma \to \sigma'$ in $\mathcal{E}$, which determines a natural transformation of distinguished triangles

$$
\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1].
\end{array}
$$

$(TR4)$ Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathcal{C}$. In view of the fact that $e : \mathcal{E} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ is a trivial fibration, any distinguished triangle in h$\mathcal{C}$ beginning with f, g, or $g \circ f$ is uniquely determined up to (nonunique) isomorphism. Consequently, it will suffice to prove that there exist *some* triple of distinguished triangles which satisfies the conclusions of $(TR4)$. To prove this, we construct a diagram in $\mathcal{C}$

$$
\begin{array}{ccccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Y/X & \longrightarrow & Z/X & \longrightarrow & X' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & 0 & \longrightarrow & Z/Y & \longrightarrow & Y' & \longrightarrow & (Y/X)'
\end{array}
$$

where 0 is a zero object of $\mathcal{C}$, and each square in the diagram is a pushout (more precisely, we apply Proposition T.4.3.2.15 repeatedly to construct a map from the nerve of the appropriate partially ordered set into $\mathcal{C}$). Restricting to appropriate rectangles contained in the diagram, we obtain isomorphisms $X' \simeq X[1]$, $Y' \simeq Y[1]$, $(Y/X)' \simeq Y/X[1]$, and four distinguished triangles

$$
X \xrightarrow{f} Y \to Y/X \to X[1]
$$

$$
Y \xrightarrow{g} Z \to Z/Y \to Y[1]
$$

$$
X \xrightarrow{g \circ f} Z \to Z/X \to X[1]
$$

$$
Y/X \to Z/X \to Z/Y \to Y/X[1].
$$

The commutativity in the homotopy category h$\mathcal{C}$ required by $(TR4)$ follows from the (stronger) commutativity of the above diagram in $\mathcal{C}$ itself.

$\square$

Remark 1.1.2.16. The definition of a stable ∞-category is quite a bit simpler than that of a triangulated category. In particular, the octahedral axiom $(TR4)$ is a consequence of ∞-categorical principles which are basic and easily motivated.

Notation 1.1.2.17. Let $\mathcal{C}$ be a stable ∞-category containing a pair of objects X and Y. We let $\mathrm{Ext}^n_{\mathcal{C}}(X, Y)$ denote the abelian group $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X[-n], Y)$. If n is negative, this can be identified with the homotopy group $\pi_{-n} \mathrm{Map}_{\mathcal{C}}(X, Y)$. More generally, $\mathrm{Ext}^n_{\mathcal{C}}(X, Y)$ can be identified with the $(-n)$th homotopy group of an appropriate *spectrum* of maps from X to Y.

1.1.3 Closure Properties of Stable ∞-Categories

According to Definition 1.1.1.9, a pointed ∞-category $\mathcal{C}$ is stable if it admits certain pushout squares and certain pullback squares, which are required to coincide with one another. Our goal in this section is to prove that a stable ∞-category $\mathcal{C}$ admits *all* finite limits and colimits, and that the pushout squares in $\mathcal{C}$ coincide with the pullback squares in general (Proposition 1.1.3.4). To prove this, we will need the following easy observation (which is quite useful in its own right):

Proposition 1.1.3.1. *Let $\mathcal{C}$ be a stable ∞-category, and let K be a simplicial set. Then the ∞-category* $\mathrm{Fun}(K, \mathcal{C})$ *is stable.*

Proof. This follows immediately from the fact that fibers and cofibers in $\mathrm{Fun}(K, \mathcal{C})$ can be computed pointwise (Proposition T.5.1.2.2). □

Definition 1.1.3.2. If $\mathcal{C}$ is stable ∞-category, and $\mathcal{C}_0$ is a full subcategory containing a zero object and stable under the formation of fibers and cofibers, then $\mathcal{C}_0$ is itself stable. In this case, we will say that $\mathcal{C}_0$ is a *stable subcategory* of $\mathcal{C}$.

Lemma 1.1.3.3. *Let $\mathcal{C}$ be a stable ∞-category, and let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory which is stable under cofibers and under translations. Then $\mathcal{C}'$ is a stable subcategory of $\mathcal{C}$.*

Proof. It will suffice to show that $\mathcal{C}'$ is stable under fibers. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Theorem 1.1.2.14 shows that there is a canonical equivalence $\mathrm{fib}(f) \simeq \mathrm{cofib}(f)[-1]$. □

Proposition 1.1.3.4. *Let $\mathcal{C}$ be a pointed ∞-category. Then $\mathcal{C}$ is stable if and only if the following conditions are satisfied:*

(1) *The ∞-category $\mathcal{C}$ admits finite limits and colimits.*

(2) *A square*

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X' & \longrightarrow & Y'
\end{array}
$$

   in $\mathcal{C}$ is a pushout if and only if it is a pullback.

Proof. Condition (1) implies the existence of fibers and cofibers in $\mathcal{C}$, and condition (2) implies that a triangle in $\mathcal{C}$ is a fiber sequence if and only if it is a cofiber sequence. This proves the "if" direction.

Suppose now that $\mathcal{C}$ is stable. We begin by proving (1). It will suffice to show that $\mathcal{C}$ admits finite colimits; the dual argument will show that $\mathcal{C}$ admits finite limits as well. According to Proposition T.4.4.3.2, it will suffice to show that $\mathcal{C}$ admits coequalizers and finite coproducts. The existence of finite coproducts was established in Lemma 1.1.2.9. We now conclude by observing that a coequalizer for a diagram

$$
X \underset{f'}{\overset{f}{\rightrightarrows}} Y
$$

can be identified with $\mathrm{cofib}(f - f')$.

We now show that every pushout square in $\mathcal{C}$ is a pullback; the converse will follow by a dual argument. Let $\mathcal{D} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ be the full subcategory spanned by the pullback squares. Then $\mathcal{D}$ is stable under finite limits and under translations. It follows from Lemma 1.1.3.3 that $\mathcal{D}$ is a stable subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$.

Let $i : \Lambda_0^2 \hookrightarrow \Delta^1 \times \Delta^1$ be the inclusion, and let $i_! : \mathrm{Fun}(\Lambda_0^2, \mathcal{C}) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ be a functor of left Kan extension. Then $i_!$ preserves finite colimits, and is therefore exact (Proposition 1.1.4.1). Let $\mathcal{D}' = i_!^{-1} \mathcal{D}$.

Then $\mathcal{D}'$ is a stable subcategory of $\mathrm{Fun}(\Lambda_0^2, \mathcal{C})$; we wish to show that $\mathcal{D}' = \mathrm{Fun}(\Lambda_0^2, \mathcal{C})$. To prove this, we observe that any diagram

$$X' \leftarrow X \rightarrow X''$$

can be obtained as a (finite) colimit

$$e'_{X'} \coprod_{e'_X} e_X \coprod_{e''_X} e''_{X''}$$

where $e_X \in \mathrm{Fun}(\Lambda_0^2, \mathcal{C})$ denotes the diagram $X \leftarrow X \rightarrow X$, $e'_Z \in \mathrm{Fun}(\Lambda_0^2, \mathcal{C})$ denotes the diagram $Z \leftarrow 0 \rightarrow 0$, and $e''_Z \in \mathrm{Fun}(\Lambda_0^2, \mathcal{C})$ denotes the diagram $0 \leftarrow 0 \rightarrow Z$. It will therefore suffice to prove that a pushout of any of these five diagrams is also a pullback. This follows immediately from the following more general observation: any pushout square

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow{\scriptstyle f} & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

in an (arbitrary) ∞-category $\mathcal{C}$ is also a pullback square, provided that f is an equivalence. $\square$

Remark 1.1.3.5. Let $\mathcal{C}$ be a stable ∞-category. Then $\mathcal{C}$ admits finite products and finite coproducts (Proposition 1.1.3.4). Moreover, for any pair of objects $X, Y \in \mathcal{C}$, there is a canonical equivalence

$$X \amalg Y \rightarrow X \times Y,$$

given by the matrix

$$\begin{bmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_Y \end{bmatrix}.$$

Theorem 1.1.2.14 implies that this map is an equivalence. We will sometimes use the notation $X \oplus Y$ to denote a product or coproduct of X and Y in $\mathcal{C}$.

We conclude this section by establishing a few closure properties for the class of stable ∞-categories.

Proposition 1.1.3.6. *Let $\mathcal{C}$ be a (small) stable ∞-category, and let κ be a regular cardinal. Then the ∞-category $\mathrm{Ind}_\kappa(\mathcal{C})$ is stable.*

Proof. The functor j preserves finite limits and colimits (Propositions T.5.1.3.2 and T.5.3.5.14). It follows that $j(0)$ is a zero object of $\mathrm{Ind}_\kappa(\mathcal{C})$, so that $\mathrm{Ind}_\kappa(\mathcal{C})$ is pointed.

We next show that every morphism $f : X \rightarrow Y$ in $\mathrm{Ind}_\kappa(\mathcal{C})$ admits a fiber and a cofiber. According to Proposition T.5.3.5.15, we may assume that f is a κ-filtered colimit of morphisms $f_\alpha : X_\alpha \rightarrow Y_\alpha$ which belong to the essential image $\mathcal{C}'$ of j. Since j preserves fibers and cofibers, each of the maps f_α has a fiber and a cofiber in Ind_κ. It follows immediately that f has a cofiber (which can be written as a colimit of the cofibers of the maps f_α). The existence of $\mathrm{fib}(f)$ is slightly more difficult. Choose a κ-filtered diagram $p : \mathcal{I} \rightarrow \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}')$, where each $p(\alpha)$ is a pullback square

$$\begin{array}{ccc} Z_\alpha & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ X_\alpha & \xrightarrow{f_\alpha} & Y_\alpha. \end{array}$$

Let σ be a colimit of the diagram p; we wish to show that σ is a pullback diagram in $\mathrm{Ind}_\kappa(\mathcal{C})$. Since $\mathrm{Ind}_\kappa(\mathcal{C})$ is stable under κ-small limits in $\mathcal{P}(\mathcal{C})$, it will suffice to show that σ is a pullback square in $\mathcal{P}(\mathcal{C})$. Since $\mathcal{P}(\mathcal{C})$ is an ∞-topos, filtered colimits in $\mathcal{P}(\mathcal{C})$ are left exact (Example T.7.3.4.7); it will therefore suffice to show that each $p(\alpha)$ is a pullback diagram in $\mathcal{P}(\mathcal{C})$. This is obvious, since the inclusion $\mathcal{C}' \subseteq \mathcal{P}(\mathcal{C})$ preserves all limits which exist in $\mathcal{C}'$ (Proposition T.5.1.3.2).

To complete the proof, we must show that a triangle in $\mathrm{Ind}_\kappa(\mathcal{C})$ is a fiber sequence if and only if it is a cofiber sequence. Suppose we are given a fiber sequence

$$
\begin{array}{ccc}
Z & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
$$

in $\mathrm{Ind}_\kappa(\mathcal{C})$. The above argument shows that we can write this triangle as a filtered colimit of fiber sequences

$$
\begin{array}{ccc}
Z_\alpha & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
X_\alpha & \longrightarrow & Y_\alpha
\end{array}
$$

in $\mathcal{C}'$. Since $\mathcal{C}'$ is stable, we conclude that these triangles are also cofiber sequences. The original triangle is therefore a filtered colimit of cofiber sequences in $\mathcal{C}'$, hence a cofiber sequence. The converse follows by the same argument. $\square$

Corollary 1.1.3.7. *Let $\mathcal{C}$ be a stable ∞-category. Then the idempotent completion of $\mathcal{C}$ is also stable.*

Proof. According to Lemma T.5.4.2.4, we can identify the idempotent completion of $\mathcal{C}$ with a full subcategory of $\mathrm{Ind}(\mathcal{C})$ which is closed under shifts and finite colimits. $\square$

1.1.4 Exact Functors

Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between stable ∞-categories. Suppose that F carries zero objects into zero objects. It follows immediately that F carries triangles into triangles. If, in addition, F carries fiber sequences to fiber sequences, then we will say that F is *exact*. The exactness of a functor F admits the following alternative characterizations:

Proposition 1.1.4.1. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between stable ∞-categories. The following conditions are equivalent:*

(1) *The functor F is left exact. That is, F commutes with finite limits.*

(2) *The functor F is right exact. That is, F commutes with finite colimits.*

(3) *The functor F is exact.*

Proof. We will prove that (2) $\Leftrightarrow$ (3); the equivalence (1) $\Leftrightarrow$ (3) will follow by a dual argument. The implication (2) $\Rightarrow$ (3) is obvious. Conversely, suppose that F is exact. The proof of Proposition 1.1.3.4 shows that F preserves coequalizers, and the proof of Lemma 1.1.2.9 shows that F preserves finite coproducts. It follows that F preserves all finite colimits (see the proof of Proposition T.4.4.3.2). $\square$

The identity functor from any stable ∞-category to itself is exact, and a composition of exact functors is exact. Consequently, there exists a subcategory $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}} \subseteq \mathcal{C}\mathrm{at}_\infty$ in which the objects are stable ∞-categories and the morphisms are the exact functors. Our next few results concern the stability properties of this subcategory.

Proposition 1.1.4.2. *Suppose given a homotopy Cartesian diagram of ∞-categories*

$$
\begin{array}{ccc}
\mathcal{C}' & \xrightarrow{\ G'\ } & \mathcal{C} \\
{\scriptstyle F'}\downarrow & & \downarrow{\scriptstyle F} \\
\mathcal{D}' & \xrightarrow{\ G\ } & \mathcal{D}\,.
\end{array}
$$

Suppose further that $\mathcal{C}$, $\mathcal{D}'$, and $\mathcal{D}$ are stable, and that the functors F and G are exact. Then:

(1) *The ∞-category $\mathcal{C}'$ is stable.*

(2) *The functors F' and G' are exact.*

(3) *If $\mathcal{E}$ is a stable ∞-category, then a functor $H : \mathcal{E} \to \mathcal{C}'$ is exact if and only if the functors $F' \circ H$ and $G' \circ H$ are exact.*

Proof. Combine Proposition 1.1.3.4 with Lemma T.5.4.5.5. $\square$

Proposition 1.1.4.3. *Let $\{\mathcal{C}_\alpha\}_{\alpha \in A}$ be a collection of stable ∞-categories. Then the product*

$$\mathcal{C} = \prod_{\alpha \in A} \mathcal{C}_\alpha$$

is stable. Moreover, for any stable ∞-category $\mathcal{D}$, a functor $F : \mathcal{D} \to \mathcal{C}$ is exact if and only if each of the compositions

$$\mathcal{D} \xrightarrow{F} \mathcal{C} \to \mathcal{C}_\alpha$$

is an exact functor.

Proof. This follows immediately from the fact that limits and colimits in $\mathcal{C}$ are computed pointwise. $\square$

Theorem 1.1.4.4. *The ∞-category $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ admits small limits, and the inclusion*

$$\mathrm{Cat}_\infty^{\mathrm{Ex}} \subseteq \mathrm{Cat}_\infty$$

preserves small limits.

Proof. Using Propositions 1.1.4.2 and 1.1.4.3, one can repeat the argument used to prove Proposition T.5.4.7.3. $\square$

We have the following analogue of Theorem 1.1.4.4.

Proposition 1.1.4.5. *Let $p : X \to S$ be an inner fibration of simplicial sets. Suppose that:*

(i) *For each vertex s of S, the fiber $X_s = X \times_S \{s\}$ is a stable ∞-category.*

(ii) *For every edge $s \to s'$ in S, the restriction $X \times_S \Delta^1 \to \Delta^1$ is a coCartesian fibration, associated to an exact functor $X_s \to X_{s'}$.*

Then:

(1) *The ∞-category $\mathrm{Map}_S(S, X)$ of sections of p is stable.*

(2) *If $\mathcal{C}$ is an arbitrary stable ∞-category, and $f : \mathcal{C} \to \mathrm{Map}_S(S, X)$ induces an exact functor $\mathcal{C} \xrightarrow{f} \mathrm{Map}_S(S, X) \to X_s$ for every vertex s of S, then f is exact.*

(3) *For every set $\mathcal{E}$ of edges of S, let $Y(\mathcal{E}) \subseteq \mathrm{Map}_S(S, X)$ be the full subcategory spanned by those sections $f : S \to X$ of p with the following property:*

   (∗) *For every $e \in \mathcal{E}$, f carries e to a p_e-coCartesian edge of the fiber product $X \times_S \Delta^1$, where $p_e : X \times_S \Delta^1 \to \Delta^1$ denotes the projection.*

   Then each $Y(\mathcal{E})$ is a stable subcategory of $\mathrm{Map}_S(S, X)$.

Proof. Combine Proposition T.5.4.7.11, Theorem 1.1.4.4, and Proposition 1.1.3.1. $\square$

Proposition 1.1.4.6. *The ∞-category $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ admits small filtered colimits, and the inclusion $\mathrm{Cat}_\infty^{\mathrm{Ex}} \subseteq \mathrm{Cat}_\infty$ preserves small filtered colimits.*

Proof. Let $\mathcal{I}$ be a filtered ∞-category, $p : \mathcal{I} \to \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Ex}}$ a diagram, which we will indicate by $\{\mathcal{C}_I\}_{I \in \mathcal{I}}$, and $\mathcal{C}$ a colimit of the induced diagram $\mathcal{I} \to \mathcal{C}\mathrm{at}_{\infty}$. We must prove:

(i) The ∞-category $\mathcal{C}$ is stable.

(ii) Each of the canonical functors $\theta_I : \mathcal{C}_I \to \mathcal{C}$ is exact.

(iii) Given an arbitrary stable ∞-category $\mathcal{D}$, a functor $f : \mathcal{C} \to \mathcal{D}$ is exact if and only if each of the composite functors $\mathcal{C}_I \overset{\theta_I}{\to} \mathcal{C} \to \mathcal{D}$ is exact.

In view of Proposition 1.1.4.1, (ii) and (iii) follow immediately from Proposition T.5.5.7.11. The same result implies that $\mathcal{C}$ admits finite limits and colimits, and that each of the functors θ_I preserves finite limits and colimits.

To prove that $\mathcal{C}$ has a zero object, we select an object $I \in \mathcal{I}$. The functor $\mathcal{C}_I \to \mathcal{C}$ preserves initial and final objects. Since $\mathcal{C}_I$ has a zero object, so does $\mathcal{C}$.

We will complete the proof by showing that every fiber sequence in $\mathcal{C}$ is a cofiber sequence (the converse follows by the same argument). Fix a morphism $f : X \to Y$ in $\mathcal{C}$. Without loss of generality, we may suppose that there exists $I \in \mathcal{I}$ and a morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ in $\mathcal{C}_I$ such that $f = \theta_I(\tilde{f})$ (Proposition T.5.4.1.2). Form a pullback diagram $\tilde{\sigma}$

$$
\begin{array}{ccc}
\widetilde{W} & \longrightarrow & \widetilde{X} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \widetilde{Y}
\end{array}
$$

in $\mathcal{C}_I$. Since $\mathcal{C}_I$ is stable, this diagram is also a pushout. It follows that $\theta_I(\tilde{\sigma})$ is a triangle $W \to X \overset{f}{\to} Y$ which is both a fiber sequence and a cofiber sequence in $\mathcal{C}$. $\square$

1.2 Stable ∞-Categories and Homological Algebra

Let $\mathcal{A}$ be an abelian category with enough projective objects. In §1.3.2, we will explain how to associate to $\mathcal{A}$ a stable ∞-category $\mathcal{D}^-(\mathcal{A})$, whose objects are (right-bounded) chain complexes of projective objects of $\mathcal{A}$. The homotopy category $\mathcal{D}^-(\mathcal{A})$ is a triangulated category, which is usually called the *derived category* of $\mathcal{A}$. We can recover $\mathcal{A}$ as a full subcategory of the triangulated category $\mathrm{h}\mathcal{D}^-(\mathcal{A})$ (or even as a full subcategory of the ∞-category $\mathcal{D}^-(\mathcal{A})$): namely, $\mathcal{A}$ is equivalent to the full subcategory spanned by those chain complexes P_{ast} satisfying $\mathrm{H}_n(P_*) \simeq 0$ for $n \neq 0$. This subcategory can be described as the intersection

$$\mathcal{D}^-(\mathcal{A})_{\geq 0} \cap \mathcal{D}^-(\mathcal{A})_{\leq 0},$$

where $\mathcal{D}^-(\mathcal{A})_{\leq 0}$ is defined to be the full subcategory spanned by those chain complexes P_* with $\mathrm{H}_n(P_*) \simeq 0$ for $n > 0$, and $\mathcal{D}^-(\mathcal{A})_{\geq 0}$ is spanned by those chain complexes with $\mathrm{H}_n(P_*) \simeq 0$ for $n < 0$.

In §1.2.1, we will axiomatize the essence of the situation by reviewing the notion of a *t-structure* on a stable ∞-category $\mathcal{C}$. A t-structure on $\mathcal{C}$ is a pair of full subcategories $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ satisfying some axioms which reflect the idea that objects of $\mathcal{C}_{\geq 0}$ ($\mathcal{C}_{\leq 0}$) are "concentrated in nonnegative (nonpositive) degrees" (see Definition 1.2.1.1). In this case, one can show that the intersection $\mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$ is equivalent to the nerve of an abelian category, which we call the *heart* of $\mathcal{C}$ and denote by $\mathcal{C}^\heartsuit$. To any object $X \in \mathcal{C}$, we can associate homotopy objects $\pi_n X \in \mathcal{C}^\heartsuit$ (in the special case $\mathcal{C} = \mathcal{D}^-(\mathcal{A})$, the functor π_n associates to each chain complex P_* its nth homology $\mathrm{H}_n(P_*)$).

If $\mathcal{C}$ is a stable ∞-category equipped with a t-structure, then it is often possible to relate questions about $\mathcal{C}$ to homological algebra in the abelian category $\mathcal{C}^\heartsuit$. In §1.2.2, we give an illustration of this principle, by showing that every filtration on an object $X \in \mathcal{C}$ determines a spectral sequence $\{E_r^{p,q}, d_r\}_{r \geq 1}$ in the abelian category $\mathcal{C}^\heartsuit$, which (in good cases) converges to the homotopy objects $\pi_n X$ (Proposition 1.2.2.7). The first

page of this spectral sequence has a reasonably explicit description in terms of the homotopy objects of the successive quotients for the filtration of X. In practice, it is often difficult to describe $E_r^{p,q}$ when $r > 2$. However, there is a convenient description in the case $r = 2$, at least when X is given as the geometric realization of a simplicial object $X_\bullet$ of $\mathcal{C}$ (equipped with the corresponding skeletal filtration). In §1.2.4 we will show that this is essentially no loss of generality: if $\mathcal{C}$ is a stable ∞-category, then every nonnegatively filtered object X of $\mathcal{C}$ can be realized as the geometric realization of a simplicial object of $\mathcal{C}$, equipped with the skeletal filtration (Theorem 1.2.4.1). This assertion can be regarded as an ∞-categorical analogue of the classical *Dold-Kan correspondence* between simplicial objects and chain complexes in an abelian category, which we review in §1.2.3.

1.2.1   t-Structures on Stable ∞-Categories

Let $\mathcal{C}$ be an ∞-category. Recall that we say a full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is a *localization* of $\mathcal{C}$ if the inclusion functor $\mathcal{C}' \subseteq \mathcal{C}$ has a left adjoint (§T.5.2.7). In this section, we will introduce a special class of localizations, called *t-localizations*, in the case where $\mathcal{C}$ is stable. We will further show that there is a bijective correspondence between t-localizations of $\mathcal{C}$ and *t-structures* on the triangulated category $h\mathcal{C}$. We begin with a review of the classical theory of t-structures; for a more thorough introduction we refer the reader to [13].

Definition 1.2.1.1. Let $\mathcal{D}$ be a triangulated category. A *t-structure* on $\mathcal{D}$ is defined to be a pair of full subcategories $\mathcal{D}_{\geq 0}$, $\mathcal{D}_{\leq 0}$ (always assumed to be stable under isomorphism) having the following properties:

(1) For $X \in \mathcal{D}_{\geq 0}$ and $Y \in \mathcal{D}_{\leq 0}$, we have $\mathrm{Hom}_{\mathcal{D}}(X, Y[-1]) = 0$.

(2) We have inclusions $\mathcal{D}_{\geq 0}[1] \subseteq \mathcal{D}_{\geq 0}$, $\mathcal{D}_{\leq 0}[-1] \subseteq \mathcal{D}_{\leq 0}$.

(3) For any $X \in \mathcal{D}$, there exists a fiber sequence $X' \to X \to X''$ where $X' \in \mathcal{D}_{\geq 0}$ and $X'' \in \mathcal{D}_{\leq 0}[-1]$.

Notation 1.2.1.2. If $\mathcal{D}$ is a triangulated category equipped with a t-structure, we will write $\mathcal{D}_{\geq n}$ for $\mathcal{D}_{\geq 0}[n]$ and $\mathcal{D}_{\leq n}$ for $\mathcal{D}_{\leq 0}[n]$. Observe that we use a *homological* indexing convention.

Remark 1.2.1.3. In Definition 1.2.1.1, either of the full subcategories $\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0} \subseteq \mathcal{D}$ determines the other. For example, an object $X \in \mathcal{D}$ belongs to $\mathcal{D}_{\leq -1}$ if and only if $\mathrm{Hom}_{\mathcal{D}}(Y, X)$ vanishes for all $Y \in \mathcal{D}_{\geq 0}$.

Definition 1.2.1.4. Let $\mathcal{C}$ be a stable ∞-category. A *t-structure* on $\mathcal{C}$ is a t-structure on the homotopy category $h\mathcal{C}$. If $\mathcal{C}$ is equipped with a t-structure, we let $\mathcal{C}_{\geq n}$ and $\mathcal{C}_{\leq n}$ denote the full subcategories of $\mathcal{C}$ spanned by those objects which belong to $(h\mathcal{C})_{\geq n}$ and $(h\mathcal{C})_{\leq n}$, respectively.

Proposition 1.2.1.5. *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. For each $n \in \mathbf{Z}$, the full subcategory $\mathcal{C}_{\leq n}$ is a localization of $\mathcal{C}$.*

Proof. Without loss of generality, we may suppose $n = -1$. According to Proposition T.5.2.7.8, it will suffice to prove that for each $X \in \mathcal{C}$, there exists a map $f : X \to X''$, where $X'' \in \mathcal{C}_{\leq -1}$ and for each $Y \in \mathcal{C}_{\leq -1}$, the map

$$\mathrm{Map}_{\mathcal{C}}(X'', Y) \to \mathrm{Map}_{\mathcal{C}}(X, Y)$$

is a weak homotopy equivalence. Invoking part (3) of Definition 1.2.1.1, we can choose f to fit into a fiber sequence

$$X' \to X \xrightarrow{f} X''$$

where $X' \in \mathcal{C}_{\geq 0}$. According to Whitehead's theorem, we need to show that for every $k \leq 0$, the map

$$\mathrm{Ext}^k_{\mathcal{C}}(X'', Y) \to \mathrm{Ext}^k_{\mathcal{C}}(X, Y)$$

is an isomorphism of abelian groups. Using the long exact sequence associated to the fiber sequence above, we are reduced to proving that the groups $\mathrm{Ext}^k_{\mathcal{C}}(X', Y)$ vanish for $k \leq 0$. We now use condition (2) of Definition 1.2.1.1 to conclude that $X'[-k] \in \mathcal{C}_{\geq 0}$. Condition (1) of Definition 1.2.1.1 now implies that

$$\mathrm{Ext}^k_{\mathcal{C}}(X', Y) \simeq \mathrm{Hom}_{h\mathcal{C}}(X'[-k], Y) \simeq 0.$$

$\square$

Corollary 1.2.1.6. *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. The full subcategories $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$ are stable under all limits which exist in $\mathcal{C}$. Dually, the full subcategories $\mathcal{C}_{\geq n} \subseteq \mathcal{C}$ are stable under all colimits which exist in $\mathcal{C}$.*

Notation 1.2.1.7. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. We will let $\tau_{\leq n}$ denote a left adjoint to the inclusion $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$, and $\tau_{\geq n}$ a right adjoint to the inclusion $\mathcal{C}_{\geq n} \subseteq \mathcal{C}$.

Remark 1.2.1.8. Fix $n, m \in \mathbf{Z}$, and let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. Then the truncation functors $\tau_{\leq n}$, $\tau_{\geq n}$ map the full subcategory $\mathcal{C}_{\leq m}$ to itself. To prove this, we first observe that $\tau_{\leq n}$ is equivalent to the identity on $\mathcal{C}_{\leq m}$ if $m \leq n$, while if $m \geq n$ the essential image of $\tau_{\leq n}$ is contained in $\mathcal{C}_{\leq n} \subseteq \mathcal{C}_{\leq m}$. To prove the analogous result for $\tau_{\geq n}$, we observe that the proof of Proposition 1.2.1.5 implies that for each X, we have a fiber sequence

$$\tau_{\geq n}X \to X \xrightarrow{f} \tau_{\leq n-1}X.$$

If $X \in \mathcal{C}_{\leq m}$, then $\tau_{\leq n-1}X$ also belongs to $\mathcal{C}_{\leq m}$, so that $\tau_{\geq n}X \simeq \mathrm{fib}(f)$ belongs to $\mathcal{C}_{\leq m}$ since $\mathcal{C}_{\leq m}$ is stable under limits.

Warning 1.2.1.9. In §T.5.5.6, we introduced for every ∞-category $\mathcal{C}$ a full subcategory $\tau_{\leq n}\,\mathcal{C}$ of *n-truncated objects* of $\mathcal{C}$. In that context, we used the symbol $\tau_{\leq n}$ to denote a left adjoint to the inclusion $\tau_{\leq n}\,\mathcal{C} \subseteq \mathcal{C}$. This is *not* compatible with Notation 1.2.1.7. In fact, if $\mathcal{C}$ is a stable ∞-category, then it has no nonzero truncated objects at all: if $X \in \mathcal{C}$ is nonzero, then the identity map from X to itself determines a nontrivial homotopy class in $\pi_n \mathrm{Map}_{\mathcal{C}}(X[-n], X)$, for all $n \geq 0$. Nevertheless, the two notations are consistent when restricted to $\mathcal{C}_{\geq 0}$, in view of the following fact:

- Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. An object $X \in \mathcal{C}_{\geq 0}$ is k-truncated (as an object of $\mathcal{C}_{\geq 0}$) if and only if $X \in \mathcal{C}_{\leq k}$.

In fact, we have the following more general statement: for any $X \in \mathcal{C}$ and $k \geq -1$, X belongs to $\mathcal{C}_{\leq k}$ if and only if $\mathrm{Map}_{\mathcal{C}}(Y, X)$ is k-truncated for every $Y \in \mathcal{C}_{\geq 0}$. Because the latter condition is equivalent to the vanishing of $\mathrm{Ext}^n_{\mathcal{C}}(Y, X)$ for $n < -k$, we can use the shift functor to reduce to the case where $n = 0$ and $k = -1$, which is addressed by Remark 1.2.1.3.

Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure, and let $n, m \in \mathbf{Z}$. Remark 1.2.1.8 implies that we have a commutative diagram of simplicial sets

$$
\begin{array}{ccc}
\mathcal{C}_{\geq n} & \longrightarrow & \mathcal{C} \\
\downarrow{\scriptstyle \tau_{\leq m}} & & \downarrow{\scriptstyle \tau_{\leq m}} \\
\mathcal{C}_{\geq n} \cap \mathcal{C}_{\leq m} & \longrightarrow & \mathcal{C}_{\leq m} \, .
\end{array}
$$

As explained in §T.7.3.1, we get an induced transformation of functors

$$\theta : \tau_{\leq m} \circ \tau_{\geq n} \to \tau_{\geq n} \circ \tau_{\leq m}.$$

Proposition 1.2.1.10. *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. Then the natural transformation*

$$\theta : \tau_{\leq m} \circ \tau_{\geq n} \to \tau_{\geq n} \circ \tau_{\leq m}$$

is an equivalence of functors $\mathcal{C} \to \mathcal{C}_{\leq m} \cap \mathcal{C}_{\geq n}$.

Proof. This is a classical fact concerning triangulated categories; we include a proof for completeness. Fix $X \in \mathcal{C}$; we wish to show that

$$\theta(X) : \tau_{\leq m}\tau_{\geq n}X \to \tau_{\geq n}\tau_{\leq m}X$$

is an isomorphism in the homotopy category of $\mathcal{C}_{\leq m} \cap \mathcal{C}_{\geq n}$. If $m < n$, then both sides are zero and there is nothing to prove; let us therefore assume that $m \geq n$. Fix $Y \in \mathcal{C}_{\leq m} \cap \mathcal{C}_{\geq n}$; it will suffice to show that composition with $\theta(X)$ induces an isomorphism

$$\mathrm{Ext}^0(\tau_{\geq n}\tau_{\leq m}X, Y) \to \mathrm{Ext}^0(\tau_{\leq m}\tau_{\geq n}X, Y) \simeq \mathrm{Ext}^0(\tau_{\geq n}X, Y).$$

We have a map of long exact sequences

$$
\begin{array}{ccc}
\mathrm{Ext}^0(\tau_{\leq n-1}\tau_{\leq m}X, Y) & \xrightarrow{f_0} & \mathrm{Ext}^0(\tau_{\leq n-1}X, Y) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^0(\tau_{\leq m}X, Y) & \xrightarrow{f_1} & \mathrm{Ext}^0(X, Y) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^0(\tau_{\geq n}\tau_{\leq m}X, Y) & \xrightarrow{f_2} & \mathrm{Ext}^0(\tau_{\geq n}X, Y) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1(\tau_{\leq n-1}\tau_{\leq m}X, Y) & \xrightarrow{f_3} & \mathrm{Ext}^1(\tau_{\leq n-1}X, Y) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1(\tau_{\leq m}X, Y) & \xrightarrow{f_4} & \mathrm{Ext}^1(X, Y).
\end{array}
$$

Since $m \geq n$, the natural transformation $\tau_{\leq n-1} \to \tau_{\leq n-1}\tau_{\leq m}$ is an equivalence of functors; this proves that f_0 and f_3 are bijective. Since $Y \in \mathcal{C}_{\leq m}$, f_1 is bijective and f_4 is injective. It follows from the "five lemma" that f_2 is bijective, as desired. □

Definition 1.2.1.11. Let $\mathcal{C}$ be a stable ∞-category equipped with a *t*-structure. The *heart* $\mathcal{C}^\heartsuit$ of $\mathcal{C}$ is the full subcategory $\mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0} \subseteq \mathcal{C}$. For each $n \in \mathbf{Z}$, we let $\pi_0 : \mathcal{C} \to \mathcal{C}^\heartsuit$ denote the functor $\tau_{\leq 0} \circ \tau_{\geq 0} \simeq \tau_{\geq 0} \circ \tau_{\leq 0}$, and we let $\pi_n : \mathcal{C} \to \mathcal{C}^\heartsuit$ denote the composition of π_0 with the shift functor $X \mapsto X[-n]$.

Remark 1.2.1.12. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure, and let $X, Y \in \mathcal{C}^\heartsuit$. The homotopy group $\pi_n \mathrm{Map}_{\mathcal{C}}(X, Y) \simeq \mathrm{Ext}_{\mathcal{C}}^{-n}(X, Y)$ vanishes for $n > 0$. It follows that $\mathcal{C}^\heartsuit$ is equivalent to (the nerve of) its homotopy category $h\mathcal{C}^\heartsuit$. The category $h\mathcal{C}^\heartsuit$ is abelian ([13]). We will often abuse terminology by identifying $\mathcal{C}^\heartsuit$ with the abelian category $h\mathcal{C}^\heartsuit$.

Warning 1.2.1.13. The definition of a t-structure on a triangulated category was introduced in [13]. However, the notation of [13] is slightly different from the notation employed here. We use homological rather than cohomological indexing conventions. Moreover, if $\mathcal{C}$ is a stable ∞-category equipped with a t-structure and $X \in \mathcal{C}$, then we denote the corresponding objects $\tau_{\leq 0}\tau_{\geq 0}X[-n]$ by $\pi_n X$, rather than $\mathrm{H}_n(X)$. This notation reflects our emphasis in this book: the stable ∞-categories of greatest interest to us are those which arise in stable homotopy theory (see §1.4), rather than those which arise in homological algebra.

Let $\mathcal{C}$ be a stable ∞-category. In view of Remark 1.2.1.3, t-structures on $\mathcal{C}$ are determined by the corresponding localizations $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$. However, not every localization of $\mathcal{C}$ arises in this way. Recall (see §T.5.5.4) that every localization of $\mathcal{C}$ has the form $S^{-1}\mathcal{C}$, where S is an appropriate collection of morphisms of $\mathcal{C}$. Here $S^{-1}\mathcal{C}$ denotes the full subcategory of $\mathcal{C}$ spanned by S-local objects, where an object $X \in \mathcal{C}$ is said to be S-local if and only if, for each $f : Y' \to Y$ in S, composition with f induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}}(Y, X) \to \mathrm{Map}_{\mathcal{C}}(Y', X).$$

If $\mathcal{C}$ is stable, then we extend the morphism f to a fiber sequence

$$Y' \to Y \to Y'',$$

and we have an associated long exact sequence

$$\ldots \to \mathrm{Ext}^i_{\mathcal{C}}(Y'', X) \to \mathrm{Ext}^i_{\mathcal{C}}(Y, X) \xrightarrow{\theta_i} \mathrm{Ext}^i_{\mathcal{C}}(Y', X) \to \mathrm{Ext}^{i+1}_{\mathcal{C}}(Y'', X) \to \ldots$$

The requirement that X be $\{f\}$-local amounts to the condition that θ_i be an isomorphism for $i \leq 0$. Using the long exact sequence, we see that if X is $\{f\}$-local, then $\mathrm{Ext}^i_{\mathcal{C}}(Y'', X) = 0$ for $i \leq 0$. Conversely, if $\mathrm{Ext}^i_{\mathcal{C}}(Y'', X) = 0$ for $i \leq 1$, then X is $\{f\}$-local. Experience suggests that it is usually more natural to require the vanishing of the groups $\mathrm{Ext}^i_{\mathcal{C}}(Y'', X)$ than it is to require that the maps θ_i to be isomorphisms. Of course, if Y' is a zero object of $\mathcal{C}$, then the distinction between these conditions disappears.

Definition 1.2.1.14. Let $\mathcal{C}$ be an ∞-category which admits pushouts. We will say that a collection S of morphisms of $\mathcal{C}$ is *quasisaturated* if it satisfies the following conditions:

(1) Every equivalence in $\mathcal{C}$ belongs to S.

(2) Given a 2-simplex $\Delta^2 \to \mathcal{C}$

if any two of f, g, and h belongs to S, then so does the third.

(3) Given a pushout diagram

if $f \in S$, then $f' \in S$.

Any intersection of quasisaturated collections of morphisms is quasisaturated. Consequently, for any collection of morphisms S there is a smallest quasisaturated collection $\overline{S}$ containing S. We will say that $\overline{S}$ is the *quasisaturated collection of morphisms generated by S*.

Definition 1.2.1.15. Let $\mathcal{C}$ be a stable ∞-category. A full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is *closed under extensions* if, for every fiber sequence triangle

$$X \to Y \to Z$$

such that X and Z belong to $\mathcal{C}'$, the object Y also belongs to $\mathcal{C}'$.

We observe that if $\mathcal{C}$ is as in Definition 1.2.1.14 and $L : \mathcal{C} \to \mathcal{C}$ is a localization functor, then the collection of all morphisms f of $\mathcal{C}$ such that $L(f)$ is an equivalence is quasisaturated.

Proposition 1.2.1.16. *Let $\mathcal{C}$ be a stable ∞-category, let $L : \mathcal{C} \to \mathcal{C}$ be a localization functor, and let S be the collection of morphisms f in $\mathcal{C}$ such that $L(f)$ is an equivalence. The following conditions are equivalent:*

(1) *There exists a collection of morphisms $\{f : 0 \to X\}$ which generates S (as a quasisaturated collection of morphisms).*

(2) *The collection of morphisms $\{0 \to X : L(X) \simeq 0\}$ generates S (as a quasisaturated collection of morphisms).*

(3) *The essential image of L is closed under extensions.*

(4) *For any $A \in \mathcal{C}$, $B \in L\,\mathcal{C}$, the natural map $\mathrm{Ext}^1(LA, B) \to \mathrm{Ext}^1(A, B)$ is injective.*

(5) *The full subcategories $\mathcal{C}_{\geq 0} = \{A : LA \simeq 0\}$ and $\mathcal{C}_{\leq -1} = \{A : LA \simeq A\}$ determine a t-structure on $\mathcal{C}$.*

Proof. The implication $(1) \Rightarrow (2)$ is obvious. We next prove that $(2) \Rightarrow (3)$. Suppose given a fiber sequence

$$X \to Y \to Z$$

where X and Z are both S-local. We wish to prove that Y is S-local. In view of assumption (2), it will suffice to show that $\mathrm{Map}_{\mathcal{C}}(A, Y)$ is contractible, provided that $L(A) \simeq 0$. In other words, we must show that $\mathrm{Ext}^i_{\mathcal{C}}(A, Y) \simeq 0$ for $i \leq 0$. We now observe that there is an exact sequence

$$\mathrm{Ext}^i_{\mathcal{C}}(A, X) \to \mathrm{Ext}^i_{\mathcal{C}}(A, Y) \to \mathrm{Ext}^i_{\mathcal{C}}(A, Z)$$

where the outer groups vanish, since X and Z are S-local and the map $0 \to A$ belongs to S.

We next show that $(3) \Rightarrow (4)$. Let $B \in L\,\mathcal{C}$, and let $\eta \in \mathrm{Ext}^1_{\mathcal{C}}(LA, B)$ classify a distinguished triangle

$$B \to C \xrightarrow{g} LA \xrightarrow{\eta} B[1].$$

Condition (3) implies that $C \in L\,\mathcal{C}$. If the image of η in $\mathrm{Ext}^1_{\mathcal{C}}(A, B)$ is trivial, then the localization map $A \to LA$ factors as a composition

$$A \xrightarrow{f} C \xrightarrow{g} LA.$$

Applying L to this diagram (and using the fact that C is local) we conclude that the map g admits a section, so that $\eta = 0$.

We now claim that $(4) \Rightarrow (5)$. Assume (4), and define $\mathcal{C}_{\geq 0}$, $\mathcal{C}_{\leq -1}$ as in (5). We will show that the axioms of Definition 1.2.1.1 are satisfied:

- If $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq -1}$, then $\mathrm{Ext}^0_{\mathcal{C}}(X, Y) \simeq \mathrm{Ext}^0_{\mathcal{C}}(LX, Y) \simeq \mathrm{Ext}^0_{\mathcal{C}}(0, Y) \simeq 0$.

- Since $\mathcal{C}_{\leq -1}$ is a localization of $\mathcal{C}$, it is stable under limits, so that $\mathcal{C}_{\leq -1}[-1] \subseteq \mathcal{C}_{\leq -1}$. Similarly, since the functor $L : \mathcal{C} \to \mathcal{C}_{\leq -1}$ preserves all colimits which exist in $\mathcal{C}$, the subcategory $\mathcal{C}_{\geq 0}$ is stable under finite colimits, so that $\mathcal{C}_{\geq 0}[1] \subseteq \mathcal{C}_{\geq 0}$.

- Let $X \in \mathcal{C}$, and form a fiber sequence
  $$X' \to X \to LX.$$
  We claim that $X' \in \mathcal{C}_{\geq 0}$; in other words, that $LX' = 0$. For this, it suffices to show that for all $Y \in L\,\mathcal{C}$, the morphism space
  $$\mathrm{Ext}^0_{\mathcal{C}}(LX', Y) = 0.$$
  Since Y is local, we have isomorphisms
  $$\mathrm{Ext}^0_{\mathcal{C}}(LX', Y) \simeq \mathrm{Ext}^0_{\mathcal{C}}(X', Y) \simeq \mathrm{Ext}^1_{\mathcal{C}}(X'[1], Y).$$
  We now observe that there is a long exact sequence
  $$\mathrm{Ext}^0(LX, Y) \xrightarrow{f} \mathrm{Ext}^0(X, Y) \to \mathrm{Ext}^1_{\mathcal{C}}(X'[1], Y) \to \mathrm{Ext}^1_{\mathcal{C}}(LX, Y) \xrightarrow{f'} \mathrm{Ext}^1_{\mathcal{C}}(X, Y).$$
  Here f is bijective (since Y is local) and f' is injective (in virtue of assumption (4)).

We conclude by showing that $(5) \Rightarrow (1)$. Let S' be the smallest quasisaturated collection of morphisms which contains the zero map $0 \to A$, for every $A \in \mathcal{C}_{\geq 0}$. We wish to prove that $S = S'$. For this, we choose an arbitrary morphism $u : X \to Y$ belonging to S. Then $Lu : LX \to LY$ is an equivalence, so we have a pushout diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{u'} & Y' \\
\downarrow & & \downarrow \\
X & \xrightarrow{u} & Y,
\end{array}
$$

where X' and Y' are fibers of the respective localization maps $X \to LX$, $Y \to LY$. Consequently, it will suffice to prove that $u' \in S'$. Since $X', Y' \in \mathcal{C}_{\geq 0}$, this follows from the two-out-of-three property, applied to the diagram

$$
\begin{array}{ccc}
 & X' & \\
\nearrow & & \searrow^{u'} \\
0 & \xrightarrow{\hspace{2cm}} & Y'.
\end{array}
$$

$\square$

Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. We let $\mathcal{C}^+ = \bigcup \mathcal{C}_{\leq n} \subseteq \mathcal{C}$, $\mathcal{C}^- = \bigcup \mathcal{C}_{\geq -n}$, and $\mathcal{C}^b = \mathcal{C}^+ \cap \mathcal{C}^-$. It is easy to see that $\mathcal{C}^-$, $\mathcal{C}^+$, and $\mathcal{C}^b$ are stable subcategories of $\mathcal{C}$. We will say that $\mathcal{C}$ is *left bounded* if $\mathcal{C} = \mathcal{C}^+$, *right bounded* if $\mathcal{C} = \mathcal{C}^-$, and *bounded* if $\mathcal{C} = \mathcal{C}^b$.

At the other extreme, given a stable ∞-category $\mathcal{C}$ equipped with a t-structure, we define the *left completion* $\widehat{\mathcal{C}}$ of $\mathcal{C}$ to be a homotopy limit of the tower

$$
\ldots \to \mathcal{C}_{\leq 2} \overset{\tau_{\leq 1}}{\to} \mathcal{C}_{\leq 1} \overset{\tau_{\leq 0}}{\to} \mathcal{C}_{\leq 0} \overset{\tau_{\leq -1}}{\to} \ldots
$$

Using the results of §T.3.3.3, we can obtain a very concrete description of this inverse limit: it is the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathbf{Z}), \mathcal{C})$ spanned by those functors $F : \mathrm{N}(\mathbf{Z}) \to \mathcal{C}$ with the following properties:

(1) For each $n \in \mathbf{Z}$, $F(n) \in \mathcal{C}_{\leq -n}$.

(2) For each $m \leq n \in \mathbf{Z}$, the associated map $F(m) \to F(n)$ induces an equivalence $\tau_{\leq -n} F(m) \to F(n)$.

We will denote this inverse limit by $\widehat{\mathcal{C}}$, and refer to it as the *left completion* of $\mathcal{C}$.

Proposition 1.2.1.17. *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. Then:*

(1) *The left completion $\widehat{\mathcal{C}}$ is also stable.*

(2) *Let $\widehat{\mathcal{C}}_{\leq 0}$ and $\widehat{\mathcal{C}}_{\geq 0}$ be the full subcategories of $\widehat{\mathcal{C}}$ spanned by those functors $F : \mathrm{N}(\mathbf{Z}) \to \mathcal{C}$ which factor through $\mathcal{C}_{\leq 0}$ and $\mathcal{C}_{\geq 0}$, respectively. Then these subcategories determine a t-structure on $\widehat{\mathcal{C}}$.*

(3) *There is a canonical functor $\mathcal{C} \to \widehat{\mathcal{C}}$. This functor is exact, and induces an equivalence $\mathcal{C}_{\leq 0} \to \widehat{\mathcal{C}}_{\leq 0}$.*

Proof. We observe that $\widehat{\mathcal{C}}$ can be identified with the homotopy inverse limit of the tower

$$
\ldots \to \mathcal{C}_{\leq 0} \overset{\tau_{\leq 0} \Sigma}{\to} \mathcal{C}_{\leq 0} \overset{\tau_{\leq 0} \Sigma}{\to} \mathcal{C}_{\leq 0}.
$$

In other words, $\widehat{\mathcal{C}}^{op}$ is equivalent to the ∞-category of spectrum objects $\mathrm{Sp}(\mathcal{C}_{\leq 0})$ (see Proposition 1.4.2.24 in §1.4.2), and assertion (1) is a special case of Corollary 1.4.2.17.

We next prove (2). We begin by observing that, if we identify $\widehat{\mathcal{C}}$ with a full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathbf{Z}), \mathcal{C})$, then the shift functors on $\widehat{\mathcal{C}}$ can be defined by the formula

$$
(F[n])(m) = F(m+n)[n].
$$

This proves immediately that $\widehat{\mathcal{C}}_{\geq 0}[1] \subseteq \widehat{\mathcal{C}}_{\geq 0}$ and $\widehat{\mathcal{C}}_{\leq 0}[-1] \subseteq \widehat{\mathcal{C}}_{\leq 0}$. Moreover, if $X \in \widehat{\mathcal{C}}_{\geq 0}$ and $Y \in \widehat{\mathcal{C}}_{\leq -1} = \widehat{\mathcal{C}}_{\leq 0}[-1]$, then $\mathrm{Map}_{\widehat{\mathcal{C}}}(X, Y)$ can be identified with a homotopy limit of a tower of spaces

$$\ldots \to \mathrm{Map}_{\mathcal{C}}(X(n), Y(n)) \to \mathrm{Map}_{\mathcal{C}}(X(n-1), Y(n-1)) \to \ldots$$

Since each of these spaces is contractible, we conclude that $\mathrm{Map}_{\widehat{\mathcal{C}}}(X, Y) \simeq *$; in particular, $\mathrm{Ext}^0_{\widehat{\mathcal{C}}}(X, Y) = 0$. Finally, we consider an arbitrary $X \in \widehat{\mathcal{C}}$. Let $X'' = \tau_{\leq -1} \circ X : \mathrm{N}(\mathbf{Z}) \to \mathcal{C}$, and let $u : X \to X''$ be the induced map. It is easy to check that $X'' \in \widehat{\mathcal{C}}_{\leq -1}$ and that $\mathrm{fib}(u) \in \widehat{\mathcal{C}}_{\geq 0}$. This completes the proof of (2).

To prove (3), we let $\mathcal{D}$ denote the full subcategory of $\mathrm{N}(\mathbf{Z}) \times \mathcal{C}$ spanned by pairs (n, C) such that $C \in \mathcal{C}_{\leq -n}$. Using Proposition T.5.2.7.8, we deduce that the inclusion $\mathcal{D} \subseteq \mathrm{N}(\mathbf{Z}) \times \mathcal{C}$ admits a left adjoint L. The composition

$$\mathrm{N}(\mathbf{Z}) \times \mathcal{C} \xrightarrow{L} \mathcal{D} \subseteq \mathrm{N}(\mathbf{Z}) \times \mathcal{C} \to \mathcal{C}$$

can be identified with a functor $\theta : \mathcal{C} \to \mathrm{Fun}(\mathrm{N}(\mathbf{Z}), \mathcal{C})$ which factors through $\widehat{\mathcal{C}}$. To prove that θ is exact, it suffices to show that θ is right exact (Proposition 1.1.4.1). Since the truncation functors $\tau_{\leq n} : \mathcal{C}_{\leq n+1} \to \mathcal{C}_{\leq n}$ are right exact, finite colimits in $\widehat{\mathcal{C}}$ are computed pointwise. Consequently, it suffices to prove that each of the compositions

$$\mathcal{C} \xrightarrow{\theta} \widehat{\mathcal{C}} \to \tau_{\leq n}\,\mathcal{C}$$

is right exact. But this composition can be identified with the functor $\tau_{\leq n}$.

Finally, we observe that $\widehat{\mathcal{C}}_{\leq 0}$ can be identified with a homotopy limit of the essentially constant tower

$$\ldots \mathcal{C}_{\leq 0} \xrightarrow{\mathrm{id}} \mathcal{C}_{\leq 0} \xrightarrow{\mathrm{id}} \mathcal{C}_{\leq 0} \xrightarrow{\tau_{\leq -1}} \mathcal{C}_{\leq -1} \to \ldots,$$

and that θ induces an identification of this homotopy limit with $\mathcal{C}_{\leq 0}$. $\square$

If $\mathcal{C}$ is a stable ∞-category equipped with a t-structure, then we will say that $\mathcal{C}$ is *left complete* if the functor $\mathcal{C} \to \widehat{\mathcal{C}}$ described in Proposition 1.2.1.17 is an equivalence.

Remark 1.2.1.18. Let $\mathcal{C}$ be as in Proposition 1.2.1.17. Then the inclusion $\mathcal{C}^+ \subseteq \mathcal{C}$ induces an equivalence $\widehat{\mathcal{C}^+} \to \widehat{\mathcal{C}}$, and the functor $\mathcal{C} \to \widehat{\mathcal{C}}$ induces an equivalence $\mathcal{C}^+ \to \widehat{\mathcal{C}}^+$. Consequently, the constructions

$$\mathcal{C} \mapsto \widehat{\mathcal{C}} \qquad \mathcal{C} \mapsto \mathcal{C}^+$$

furnish an equivalence between the theory of left bounded stable ∞-categories and the theory of left complete stable ∞-categories.

The following criterion is useful for establishing left completeness.

Proposition 1.2.1.19. *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. Suppose that $\mathcal{C}$ admits countable products, and that $\mathcal{C}_{\geq 0}$ is stable under countable products. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is left complete.*

(2) *The full subcategory $\mathcal{C}_{\geq \infty} = \bigcap \mathcal{C}_{\geq n} \subseteq \mathcal{C}$ consists only of zero objects of $\mathcal{C}$.*

Proof. We first observe every tower of objects

$$\ldots \to X_n \to X_{n-1} \to \ldots$$

in $\mathcal{C}$ admits a limit $\varprojlim\{X_n\}$: we can compute this limit as the fiber of an appropriate map

$$\prod X_n \to \prod X_n.$$

Moreover, if each X_n belongs to $\mathcal{C}_{\geq 0}$, then $\varprojlim\{X_n\}$ belongs to $\mathcal{C}_{\geq -1}$.

The functor $F : \mathcal{C} \to \widehat{\mathcal{C}}$ of Proposition 1.2.1.17 admits a right adjoint G, given by

$$f \in \widehat{\mathcal{C}} \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{Z}), \mathcal{C}) \mapsto \varprojlim(f).$$

Assertion (1) is equivalent to the statement that the unit and counit maps

$$u : F \circ G \to \mathrm{id}_{\widehat{\mathcal{C}}}$$

$$v : \mathrm{id}_{\mathcal{C}} \to G \circ F$$

are equivalences. If v is an equivalence, then any object $X \in \mathcal{C}$ can be recovered as the limit of the tower $\{\tau_{\leq n} X\}$. In particular, this implies that $X = 0$ if $X \in \mathcal{C}_{\geq \infty}$, so that $(1) \Rightarrow (2)$.

Now assume (2); we will prove that u and v are both equivalences. To prove that u is an equivalence, we must show that for every $f \in \widehat{\mathcal{C}}$, the natural map

$$\theta : \varprojlim(f) \to f(n)$$

induces an equivalence $\tau_{\leq -n} \varprojlim(f) \to f(n)$. In other words, we must show that the fiber of θ belongs to $\mathcal{C}_{\geq -n+1}$. To prove this, we first observe that θ factors as a composition

$$\varprojlim(f) \xrightarrow{\theta'} f(n-1) \xrightarrow{\theta''} f(n).$$

The octahedral axiom $((TR4)$ of Definition 1.1.2.5) implies the existence of a fiber sequence

$$\mathrm{fib}(\theta') \to \mathrm{fib}(\theta) \to \mathrm{fib}(\theta'').$$

Since $\mathrm{fib}(\theta'')$ clearly belongs to $\mathcal{C}_{\geq -n+1}$, it will suffice to show that $\mathrm{fib}(\theta')$ belongs to $\mathcal{C}_{\geq -n+1}$. We observe that $\mathrm{fib}(\theta')$ can be identified with the limit of a tower $\{\mathrm{fib}(f(m) \to f(n-1))\}_{m<n}$. It therefore suffices to show that each $\mathrm{fib}(f(m) \to f(n-1))$ belongs to $\mathcal{C}_{\geq -n+2}$, which is clear.

We now prove that v is an equivalence. Let X be an object of $\mathcal{C}$, and $v_X : X \to (G \circ F)(X)$ the associated map. Since u is an equivalence of functors, we conclude that $\tau_{\leq n}(v_X)$ is an equivalence for all $n \in \mathbf{Z}$. It follows that $\mathrm{cofib}(v_X) \in \mathcal{C}_{\geq n+1}$ for all $n \in \mathbf{Z}$. Invoking assumption (2), we conclude that $\mathrm{cofib}(v_X) \simeq 0$, so that v_X is an equivalence as desired. $\square$

Remark 1.2.1.20. The ideas introduced above can be dualized in an obvious way, so that we can speak of *right completions* and *right completeness* for a stable ∞-category equipped with a t-structure.

1.2.2 Filtered Objects and Spectral Sequences

Suppose given a sequence of objects

$$\ldots \to X(-1) \xrightarrow{f^0} X(0) \xrightarrow{f^1} X(1) \to \ldots.$$

in a stable ∞-category $\mathcal{C}$. Suppose further that $\mathcal{C}$ is equipped with a t-structure, and that the heart of $\mathcal{C}$ is equivalent to the nerve of an abelian category $\mathcal{A}$. In this section, we will construct a spectral sequence taking values in the abelian category $\mathcal{A}$, with the E_1-page described by the formula

$$E_1^{p,q} = \pi_{p+q} \mathrm{cofib}(f^p) \in \mathcal{A}.$$

Under appropriate hypotheses, we will show that this spectral sequence converges to the homotopy groups of the colimit $\varinjlim X(i)$.

Remark 1.2.2.1. The spectral sequence we construct in this section can be viewed as a generalization of the spectral sequence associated to a filtered chain complex in ordinary homological algebra. We refer the reader to [30] for a gentle account of this spectral sequence, and to [108] for a general introduction to spectral sequences.

Our first step is to construct some auxiliary objects in $\mathcal{C}$.

Definition 1.2.2.2. Let $\mathcal{C}$ be a pointed ∞-category, and let $\mathcal{I}$ be a linearly ordered set. We let $\mathcal{I}^{[1]}$ denote the partially ordered set of pairs of elements $i \leq j$ of $\mathcal{I}$, where $(i,j) \leq (i',j')$ if $i \leq i'$ and $j \leq j'$. An $\mathcal{I}$-*complex* in $\mathcal{C}$ is a functor $F : \mathrm{N}(\mathcal{I}^{[1]}) \to \mathcal{C}$ with the following properties:

(1) For each $i \in \mathcal{I}$, $F(i,i)$ is a zero object of $\mathcal{C}$.

(2) For every $i \leq j \leq k$, the associated diagram

$$
\begin{array}{ccc}
F(i,j) & \longrightarrow & F(i,k) \\
\downarrow & & \downarrow \\
F(j,j) & \longrightarrow & F(j,k)
\end{array}
$$

is a pushout square in $\mathcal{C}$.

We let $\mathrm{Gap}(\mathcal{I}, \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathcal{I}^{[1]}), \mathcal{C})$ spanned by the $\mathcal{I}$-complexes in $\mathcal{C}$.

Remark 1.2.2.3. Let $F \in \mathrm{Gap}(\mathbf{Z}, \mathcal{C})$ be a $\mathbf{Z}$-complex in a stable ∞-category $\mathcal{C}$. For each $n \in \mathbf{Z}$, the functor F determines pushout square

$$
\begin{array}{ccc}
F(n-1,n) & \longrightarrow & F(n-1,n+1) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & F(n,n+1),
\end{array}
$$

hence a boundary map $\delta : F(n,n+1) \to F(n-1,n)[1]$. If we set $C_n = F(n-1,n)[-n]$, then we obtain a sequence of maps

$$
\ldots \to C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \to \ldots
$$

in the homotopy category $h\mathcal{C}$. The commutative diagram

$$
\begin{array}{ccccc}
F(n,n+1) & \longrightarrow & F(n-2,n)[1] & \longrightarrow & F(n-2,n-2)[2] \\
& \searrow{\scriptstyle \delta} & \downarrow & & \downarrow \\
& & F(n-1,n)[1] & \xrightarrow{\delta} & F(n-2,n-1)[2]
\end{array}
$$

proves that $d_{n-1} \circ d_n \simeq 0$ (since $F(n-2,n-2) \simeq 0$), so that (C_*, d_*) can be viewed as a chain complex in the triangulated category $h\mathcal{C}$. This motivates the terminology of Definition 1.2.2.2.

Lemma 1.2.2.4. *Let $\mathcal{C}$ be a pointed ∞-category which admits pushouts. Let $\mathcal{I} = \mathcal{I}_0 \cup \{-\infty\}$ be a linearly ordered set containing a least element $-\infty$. We regard $\mathcal{I}_0$ as a linearly ordered subset of $\mathcal{I}^{[1]}$ via the embedding*

$$
i \mapsto (-\infty, i).
$$

Then the restriction map $\mathrm{Gap}(\mathcal{I}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}(\mathcal{I}_0), \mathcal{C})$ is an equivalence of ∞-categories.

Proof. Let $\mathcal{J} = \{(i,j) \in \mathcal{I}^{[1]} : (i = -\infty) \vee (i = j)\}$. We now make the following observations:

(1) A functor $F : \mathrm{N}(\mathcal{I}^{[1]}) \to \mathcal{C}$ is a complex if and only if F is a left Kan extension of $F | \mathrm{N}(\mathcal{J})$, and $F(i,i)$ is a zero object of $\mathcal{C}$ for all $i \in \mathcal{I}$.

(2) Any functor $F_0 : \mathrm{N}(\mathcal{J}) \to \mathcal{C}$ admits a left Kan extension to $\mathrm{N}(\mathcal{I}^{[1]})$ (use Lemma T.4.3.2.13 and the fact that $\mathcal{C}$ admits pushouts).

(3) A functor $F_0 : N(\mathcal{J}) \to \mathcal{C}$ has the property that $F_0(i,i)$ is a zero object, for every $i \in \mathcal{I}$, if and only if F_0 is a right Kan extension of $F_0| N(\mathcal{J}_0)$.

(4) Any functor $F_0 : N(\mathcal{J}_0) \to \mathcal{C}$ admits a right Kan extension to $N(\mathcal{J})$ (use Lemma T.4.3.2.13 and the fact that $\mathcal{C}$ has a final object).

The desired conclusion now follows immediately from Proposition T.4.3.2.15. $\qquad \square$

Remark 1.2.2.5. Let $\mathcal{C}$ be a pointed ∞-category which admits pushouts (for example, a stable ∞-category). The assignment

$$[n] \mapsto \mathrm{Gap}([n], \mathcal{C})^{\simeq}$$

determines a simplicial object in the category $\mathcal{K}$an of Kan complexes. We can then define the *Waldhausen K-theory of* $\mathcal{C}$ to be a geometric realization of this bisimplicial set (for example, the associated diagonal simplicial set). In the special case where A is an A_{∞}-ring and $\mathcal{C}$ is the smallest stable subcategory of Mod_A which contains A, this definition recovers the usual K-theory of A. We refer the reader to [152] for a related construction.

Construction 1.2.2.6. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure, such that the heart of $\mathcal{C}$ is equivalent to the nerve of an abelian category $\mathcal{A}$. Let $X \in \mathrm{Gap}(\mathbf{Z}, \mathcal{C})$. We observe that for every triple of integers $i \leq j \leq k$, there is a long exact sequence

$$\ldots \to \pi_n X(i,j) \to \pi_n X(i,k) \to \pi_n X(j,k) \xrightarrow{\delta} \pi_{n-1} X(i,j) \to \ldots$$

in the abelian category $\mathcal{A}$. For every $p, q \in \mathbf{Z}$ and every $r \geq 1$, we define the object $E_r^{p,q} \in \mathcal{A}$ by the formula

$$E_r^{p,q} = \mathrm{im}(\pi_{p+q} X(p-r, p) \to \pi_{p+q} X(p-1, p+r-1)).$$

There is a differential $d_r : E_r^{p,q} \to E_r^{p-r,q+r-1}$, uniquely determined by the requirement that the diagram

$$
\begin{array}{ccccc}
\pi_{p+q}X(p-r,p) & \longrightarrow & E_r^{p,q} & \longrightarrow & \pi_{p+q}X(p-1, p+r-1) \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle d_r} & & \downarrow{\scriptstyle \delta} \\
\pi_{p+q-1}X(p-2r, p-r) & \longrightarrow & E_r^{p-r,q+r-1} & \longrightarrow & \pi_{p+q-1}X(p-r-1, p-1)
\end{array}
$$

be commutative.

Proposition 1.2.2.7. *Let $X \in \mathrm{Gap}(\mathbf{Z}, \mathcal{C})$ be as in Construction 1.2.2.6. Then:*

(1) *For each $r \geq 1$, the composition $d_r \circ d_r$ is zero.*

(2) *There are canonical isomorphisms*

$$E_{r+1}^{p,q} \simeq \ker(d_r : E_r^{p,q} \to E_r^{p-r,q+r-1}) / \mathrm{im}(d_r : E_r^{p+r,q-r+1} \to E_r^{p,q}).$$

Consequently, $\{E_r^{p,q}, d_r\}$ is a spectral sequence (with values in the abelian category $\mathcal{A}$).

Remark 1.2.2.8. For fixed $q \in \mathbf{Z}$, the complex $(E_1^{*,q}, d_1)$ in $\mathcal{A}$ can be obtained from the h$\mathcal{C}$-valued chain complex C_* described in Remark 1.2.2.3 by applying the homological functor π_q.

Proof. We have a commutative diagram

$$\pi_{p+q}X(p-r-1,p)$$

$$\pi_{p+q+1}X(p,p+r) \xrightarrow{\ \delta\ } \pi_{p+q}X(p-r,p) \xrightarrow{\ \delta\ } \pi_{p+q-1}X(p-2r,p-r)$$

$$E_r^{p+r,q-r+1} \xrightarrow{\ d_r\ } E_r^{p,q} \xrightarrow{\ d_r\ } E_r^{p-r,q+r-1}$$

$$\pi_{p+q+1}X(p+r-1,p+2r-1) \xrightarrow{\ \delta\ } \pi_{p+q}X(p-1,p+r-1) \xrightarrow{\ \delta\ } \pi_{p+q-1}X(p-r-1,p-1)$$

$$\pi_{p+q}X(p-1,p+r).$$

Since the upper left vertical map is an epimorphism, (1) will follow provided that we can show that the composition

$$\pi_{p+q+1}X(p,p+r) \xrightarrow{\delta} \pi_{p+q}X(p-r,p) \xrightarrow{\delta} \pi_{p+q-1}X(p-2r,p-r)$$

is zero. This follows immediately from the commutativity of the diagram

$$X(p,p+r) \xrightarrow{\ \delta\ } X(p-2r,p)[1] \longrightarrow X(p-r,p)[1]$$

$$0 \xrightarrow{\ \sim\ } X(p-r,p-r)[1] \longrightarrow X(p-2r,p-r)[2].$$

We next claim that the composite map

$$\phi : \pi_{p+q}X(p-r-1,p) \to E_r^{p,q} \xrightarrow{d_r} E_r^{p-r,q+r-1}$$

is zero. Because $E_r^{p-r,q+r-1} \to \pi_{p+q-1}X(p-r-1,p-1)$ is a monomorphism, this follows from the commutativity of the diagram

$$\pi_{p+q}X(p-r-1,p) \longrightarrow \pi_{p+q-1}X(p-2r,p-r-1)$$

$$\pi_{p+q}X(p-r,p) \longrightarrow \pi_{p+q-1}X(p-2r,p-r)$$

$$E_r^{p,q} \xrightarrow{\ d_r\ } E_r^{p-r,q+r-1}$$

$$\pi_{p+q-1}X(p-r-1,p-1),$$

since the composition of the left vertical line factors through $\pi_{p+q-1}X(p-r-1,p-r-1) \simeq 0$. A dual argument shows that the composition

$$E_r^{p+r,q+r-1} \xrightarrow{d_r} E_r^{p,q} \to \pi_{p+q}X(p-1,p+r)$$

is zero as well.

Let $Z = \ker(d_r : E_r^{p,q} \to E_r^{p-r,q+r-1})$ and $B = \operatorname{im}(d_r : E_r^{p+r,q-r+1} \to E_r^{p,q})$. The above arguments yield a sequence of morphisms

$$\pi_{p+q}X(p-r-1,p) \xrightarrow{\phi} Z \xrightarrow{\phi'} Z/B \xrightarrow{\psi'} E_r^{p,q}/B \xrightarrow{\psi} \pi_{p+q}X(p-1,p+r).$$

To complete the proof of (2), it will suffice to show that $\phi' \circ \phi$ is an epimorphism and that $\psi \circ \psi'$ is a monomorphism. By symmetry, it will suffice to prove the first assertion. Since ϕ' is evidently an epimorphism, we are reduced to showing that ϕ is an epimorphism.

Let K denote the kernel of the composite map

$$\pi_{p+q}X(p-r,p) \to E_r^{p,q} \xrightarrow{d_r} E_r^{p-r,q+r-1} \to \pi_{p+q-1}X(p-r-1,p-1),$$

so that the canonical map $K \to Z$ is an epimorphism. Choose a diagram

where the square on the left is a pullback. The exactness of the bottom row implies that f is an epimorphism. Let f' denote the composition

$$\widetilde{K} \xrightarrow{g} \pi_{p+q}X(p-r,p-1) \to \pi_{p+q}X(p-r,p).$$

The composition

$$\widetilde{K} \xrightarrow{f'} \pi_{p+q}X(p-r,p) \to \pi_{p+q}X(p-1,p+r)$$

factors through $\pi_{p+q}X(p-1,p-1) \simeq 0$. Since $E_r^{p,q} \to \pi_{p+q}X(p-1,p+r)$ is a monomorphism, we conclude that the composition $\widetilde{K} \xrightarrow{f'} \pi_{p+q}X(p-r,p) \to E_r^{p,q}$ is the zero map. It follows that the composition

$$\widetilde{K} \xrightarrow{f-f'} \pi_{p+q}X(p-r,p) \to Z$$

coincides with the composition $\widetilde{K} \xrightarrow{f} K \to Z$, and is therefore an epimorphism.

Form a diagram

where the left square is a pullback. Since the bottom line is exact, the map f'' is an epimorphism, so that the composition

$$\overline{K} \xrightarrow{f''} \widetilde{K} \xrightarrow{f-f'} \pi_{p+q}X(p-r,p) \to Z$$

is an epimorphism. This map coincides with the composition

$$\overline{K} \to \pi_{p+q}X(p-r-1,p) \xrightarrow{\phi} Z,$$

so that ϕ is an epimorphism as well. $\qquad\square$

Definition 1.2.2.9. Let $\mathcal{C}$ be a stable ∞-category. A *filtered object* of $\mathcal{C}$ is a functor $X : \mathrm{N}(\mathbf{Z}) \to \mathcal{C}$.

Suppose that $\mathcal{C}$ is equipped with a t-structure, and let $X : \mathrm{N}(\mathbf{Z}) \to \mathcal{C}$ be a filtered object of $\mathcal{C}$. According to Lemma 1.2.2.4, we can extend X to a complex in $\mathrm{Gap}(\mathbf{Z} \cup \{-\infty\}, \mathcal{C})$. Let $\overline{X}$ be the associated object of $\mathrm{Gap}(\mathbf{Z}, \mathcal{C})$, and let $\{E_r^{p,q}, d_r\}_{r \geq 1}$ be the spectral sequence described in Construction 1.2.2.6 and Proposition 1.2.2.7. We will refer to $\{E_r^{p,q}, d_r\}_{r \geq 1}$ as the *spectral sequence associated to the filtered object X*.

Remark 1.2.2.10. In the situation of Definition 1.2.2.9, Lemma 1.2.2.4 implies that $\overline{X}$ is determined up to contractible ambiguity by X. It follows that the spectral sequence $\{E_r^{p,q}, d_r\}_{r \geq 1}$ is independent of the choice of $\overline{X}$, up to canonical isomorphism.

Example 1.2.2.11. Let $\mathcal{A}$ be a sufficiently nice abelian category, and let $\mathcal{C}$ be the derived ∞-category of $\mathcal{A}$ (see §1.3.2). Let $\mathrm{Fun}(\mathrm{N}(\mathbf{Z}), \mathcal{C})$ be the ∞-category of filtered objects of $\mathcal{C}$. Then the homotopy category $\mathrm{hFun}(\mathrm{N}(\mathbf{Z}), \mathcal{C})$ can be identified with the classical *filtered derived category* of $\mathcal{A}$, obtained from the category of filtered complexes of objects of $\mathcal{A}$ by inverting all filtered quasi-isomorphisms. In this case, Definition 1.2.2.9 recovers the usual spectral sequence associated to a filtered complex.

Our next goal is to study the convergence of the spectral sequence of Definition 1.2.2.9. We will treat only the simplest case, which will be sufficient for our applications in this book.

Definition 1.2.2.12. Let $\mathcal{C}$ be an ∞-category. We will say that $\mathcal{C}$ *admits sequential colimits* if every diagram $\mathrm{N}(\mathbf{Z}_{\geq 0}) \to \mathcal{C}$ has a colimit in $\mathcal{C}$.

If $\mathcal{C}$ is stable and admits sequential colimits, we will say that a t-structure on $\mathcal{C}$ is *compatible with sequential colimits* if the full subcategory $\mathcal{C}_{\leq 0}$ is stable under the colimits of diagrams indexed by $\mathrm{N}(\mathbf{Z}_{\geq 0})$.

Remark 1.2.2.13. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure, so that the heart of $\mathcal{C}$ is equivalent to (the nerve of) an abelian category $\mathcal{A}$. Suppose that $\mathcal{C}$ admits sequential colimits. Then $\mathcal{C}_{\geq 0}$ admits sequential colimits, so that $\mathrm{N}(\mathcal{A})$, being a localization of $\mathcal{C}_{\geq 0}$, also admits sequential colimits. If the t-structure on $\mathcal{C}$ is compatible with sequential colimits, then the inclusion $\mathrm{N}(\mathcal{A}) \subseteq \mathcal{C}$ and the homological functors $\{\pi_n : \mathcal{C} \to \mathrm{N}(\mathcal{A})\}_{n \in \mathbf{Z}}$ preserve sequential colimits. It follows that sequential colimits in the abelian category $\mathcal{A}$ are exact: in other words, the colimit of a sequence of monomorphisms in $\mathcal{A}$ is again a monomorphism.

Proposition 1.2.2.14. *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure, and let $X : \mathrm{N}(\mathbf{Z}) \to \mathcal{C}$ be a filtered object of $\mathcal{C}$. Assume that $\mathcal{C}$ admits sequential colimits, and that the t-structure on $\mathcal{C}$ is compatible with sequential colimits. Suppose furthermore that $X(n) \simeq 0$ for $n \ll 0$. Then the associated spectral sequence (Definition 1.2.2.9) converges*

$$E_r^{p,q} \Rightarrow \pi_{p+q} \varinjlim(X).$$

Proof. Let $\mathcal{A}$ be an abelian category such that the heart of $\mathcal{C}$ is equivalent to (the nerve of) $\mathcal{A}$. The convergence assertion of the Proposition has the following meaning:

(*i*) For fixed p and q, the differentials $d_r : E_r^{p,q} \to E_r^{p-r,q+r-1}$ vanish for $r \gg 0$.

Consequently, for sufficiently large r we obtain a sequence of epimorphisms

$$E_r^{p,q} \to E_{r+1}^{p,q} \to E_{r+2}^{p,q} \to \cdots$$

Let $E_\infty^{p,q}$ denote the colimit of this sequence (in the abelian category $\mathcal{A}$).

(*ii*) Let $n \in \mathbf{Z}$, and let $A_n = \pi_n \varinjlim_m X(m)$. Then there exists a filtration

$$\cdots \subseteq F^{-1} A_n \subseteq F^0 A_n \subseteq F^1 A_n \subseteq \cdots$$

of A_n, with $F^m A_n \simeq 0$ for $m \ll 0$, and $\varinjlim_m F^m A_n \simeq A_n$.

(*iii*) For every $p, q \in \mathbf{Z}$, there exists an isomorphism $E_\infty^{p,q} \simeq F^p A_{p+q} / F^{p-1} A_{p+q}$ in the abelian category $\mathcal{A}$.

To prove (i), (ii), and (iii), we first extend X to an object $\overline{X} \in \mathrm{Gap}(\mathbf{Z} \cup \{-\infty\}, \mathcal{C})$, so that for each $n \in \mathbf{Z}$ we have $X(n) = \overline{X}(-\infty, n)$. Without loss of generality, we may suppose that $X(n) \simeq *$ for $n < 0$. This implies that $\overline{X}(i, j) \simeq *$ for $i, j < 0$. It follows that $E_r^{p-r, q+r-1}$, as a quotient $\pi_{p+q}\overline{X}(p - 2r, p - r)$, is zero for $r > p$. This proves (i).

To satisfy (ii), we set $F^p A_n = \mathrm{im}(\pi_n X(p) \to \pi_n \varinjlim(X))$. It is clear that $F^p A_n \simeq *$ for $p < 0$, and the isomorphism $\varinjlim F^p A_n \simeq A_n$ follows from the compatibility of the homological functor π_n with sequential colimits (Remark 1.2.2.13).

To prove (iii), we note that for $r > p$, the object $E_r^{p,q}$ can be identified with the image of the map $\pi_{p+q}X(p) \simeq \pi_{p+q}\overline{X}(p - r, p) \to \pi_{p+q}\overline{X}(p - 1, p + r)$. Let $Y = \varinjlim_r \overline{X}(p - 1, p + r)$. It follows that $E_\infty^{p,q}$ can be identified with the image of the map $\pi_{p+q}X(p) \xrightarrow{f} \pi_{p+q}Y$. We have a fiber sequence

$$X(p - 1) \to \varinjlim(X) \to Y,$$

which induces an exact sequence

$$0 \to F^{p-1}A_{p+q} \to A_{p+q} \xrightarrow{f'} \pi_{p+q}Y.$$

We have a commutative triangle

$$
\begin{array}{ccc}
 & A_{p+q} & \\
 \overset{g}{\nearrow} & & \overset{f'}{\searrow} \\
 \pi_{p+q}X(p) & \xrightarrow{\quad f \quad} & \pi_{p+q}Y.
\end{array}
$$

Since the image of g is $F^p A_{p+q}$, we obtain canonical isomorphisms

$$E_\infty^{p,q} \simeq \mathrm{im}(f) \simeq \mathrm{im}(f' | F^p A_{p+q}) \simeq F^p A_{p+q} / \ker(f') \simeq F^p A_{p+q} / F^{p-1} A_{p+q}.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

1.2.3 The Dold-Kan Correspondence

Our goal in this section is to review some classical results in homological algebra: most importantly, the Dold-Kan correspondence, which establishes an equivalence between the category of simplicial objects of an abelian category $\mathcal{A}$ with the category of nonnegatively graded chain complexes over $\mathcal{A}$ (Theorem 1.2.3.7). This material will be used in studying an ∞-categorical version of the Dold-Kan correspondence in §1.2.4, and in our construction of derived ∞-categories in §1.3.

We begin by reviewing some basic definitions from homological algebra.

Definition 1.2.3.1. Let $\mathcal{A}$ be an additive category. A *chain complex* with values in $\mathcal{A}$ is a composable sequence of morphisms

$$\cdots \to A_2 \xrightarrow{d(2)} A_1 \xrightarrow{d(1)} A_0 \xrightarrow{d(0)} A_{-1} \to \cdots$$

in $\mathcal{A}$ such that $d(n-1) \circ d(n) = 0$ for every integer n. The collection of chain complexes with values in $\mathcal{A}$ is itself an additive category, which we will denote by $\mathrm{Ch}(\mathcal{A})$.

For each integer n, we let $\mathrm{Ch}(\mathcal{A})_{\geq n}$ denote the full subcategory of $\mathrm{Ch}(\mathcal{A})$ spanned by those chain complexes A_* where $A_k \simeq 0$ for $k < n$. Similarly, we let $\mathrm{Ch}(\mathcal{A})_{\leq n}$ denote the full subcategory of $\mathrm{Ch}(\mathcal{A})$ spanned by those complexes A_* such that $A_k \simeq 0$ for $k > n$.

Remark 1.2.3.2. Throughout this book, we will always use *homological* indexing conventions for our chain complexes. In particular, the differential on a chain complex always lowers degrees by 1.

Remark 1.2.3.3. Let

$$\cdots \to A_2 \stackrel{d(2)}{\to} A_1 \stackrel{d(1)}{\to} A_0 \stackrel{d(0)}{\to} A_{-1} \to \cdots$$

be a chain complex with values in an additive category $\mathcal{A}$. We will typically denote this chain complex by (A_*, d), where A_* is the underlying $\mathbf{Z}$-graded object of $\mathcal{A}$ and d is the map of degree -1 from A_* to itself given by $d(n)$ in degree n (so that $d^2 = 0$). Often we will further abuse notation and simply denote this chain complex by A_* or simply by A, implicitly assuming that a suitable differential has also been supplied.

Remark 1.2.3.4. Let $\mathcal{A}$ be an abelian category. Then the inclusion $\mathrm{Ch}(\mathcal{A})_{\geq 0} \hookrightarrow \mathrm{Ch}(\mathcal{A})$ admits a right adjoint, which carries a chain complex

$$\cdots \to M_2 \stackrel{d(2)}{\to} M_1 \stackrel{d(1)}{\to} M_0 \stackrel{d(0)}{\to} M_{-1} \to \cdots$$

to the truncated chain complex

$$\cdots \to M_2 \stackrel{d(2)}{\to} M_1 \stackrel{d(1)}{\to} \ker(d(0)) \to 0 \to \cdots$$

We will denote this functor by $\tau_{\geq 0} : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})_{\geq 0}$. Similarly, the inclusion $\mathrm{Ch}(\mathcal{A})_{\leq 0} \hookrightarrow \mathrm{Ch}(\mathcal{A})$ admits a left adjoint, which we will denote by $\tau_{\leq 0} : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})_{\leq 0}$.

Construction 1.2.3.5. Let $\mathcal{A}$ be an additive category and let $A = (A_*, d)$ be a nonnegatively graded chain complex with values in $\mathcal{A}$. We define a simplicial object $\mathrm{DK}_\bullet(A)$ of $\mathcal{A}$ as follows:

(1) For each $n \geq 0$, the object $\mathrm{DK}_n(A)$ is given by the direct sum $\bigoplus_{\alpha:[n] \to [k]} A_k$; here the sum is taken over all surjective maps $[n] \to [k]$ in $\mathbf{\Delta}$.

(2) Let $\beta : [n'] \to [n]$ be a morphism in $\mathbf{\Delta}$. The induced map

$$\beta^* : \mathrm{DK}_n(A) \simeq \bigoplus_{\alpha:[n] \to [k]} A_k \to \bigoplus_{\alpha':[n'] \to [k']} A_{k'} \simeq \mathrm{DK}_{n'}(A)$$

is given by the matrix of morphisms $\{f_{\alpha,\alpha'} : A_k \to A_{k'}\}$, where the map $f_{\alpha,\alpha'}$ is the identity if $k = k'$ and the diagram

$$
\begin{array}{ccc}
[n'] & \stackrel{\beta}{\longrightarrow} & [n] \\
\downarrow{\alpha'} & & \downarrow{\alpha} \\
[k'] & \stackrel{\mathrm{id}}{\longrightarrow} & [k]
\end{array}
$$

commutes, the map $f_{\alpha,\alpha'}$ is given by the differential d if $k' = k - 1$ and the diagram

$$
\begin{array}{ccc}
[n'] & \longrightarrow & [n] \\
\downarrow{\alpha'} & & \downarrow{\alpha} \\
[k'] & \stackrel{\sim}{\longrightarrow} \{1,\ldots,k\} \longrightarrow & [k]
\end{array}
$$

commutes, and $f_{\alpha,\alpha'}$ is zero otherwise.

The construction $A \mapsto \mathrm{DK}_\bullet(A)$ determines a functor from the category $\mathrm{Ch}(\mathcal{A})_{\geq 0}$ to the category $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A})$ of simplicial objects of $\mathcal{A}$. We will denote this functor by DK, and refer to it as the *Dold-Kan construction*.

Example 1.2.3.6. For every simplicial set $K_\bullet$, let $\mathbf{Z}K_\bullet$ denote the free simplicial abelian group generated by $K_\bullet$ (so that $(\mathbf{Z}K)_n$ is the free abelian group generated by the set K_n for each $n \geq 0$). Let $\mathbf{Z}[n]_*$ denote the chain complex of abelian groups given by

$$\mathbf{Z}[n]_k = \begin{cases} \mathbf{Z} & \text{if } k = n \\ 0 & \text{if } k \neq n. \end{cases}$$

Then there is a canonical isomorphism of simplicial abelian groups

$$\mathrm{DK}_\bullet(\mathbf{Z}[n]) \simeq \mathbf{Z}\Delta^n / \mathbf{Z}\,\partial\,\Delta^n.$$

Our main goal in this section is to prove the following:

Theorem 1.2.3.7 (Dold-Kan Correspondence). *Let $\mathcal{A}$ be an additive category. The functor*

$$\mathrm{DK} : \mathrm{Ch}(\mathcal{A})_{\geq 0} \to \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{A})$$

is fully faithful. If $\mathcal{A}$ is idempotent complete, then $\mathcal{A}$ is an equivalence of categories.

The proof of Theorem 1.2.3.7 will proceed by reducing to the case where $\mathcal{A}$ is the category of abelian groups. In this case, we can explicitly describe an inverse to the functor DK: it is given by assigning to each simplicial abelian group $A_\bullet$ the associated *normalized chain complex $N_*(A)$*.

Definition 1.2.3.8. Let $\mathcal{A}$ be an additive category and let $A_\bullet$ be a semisimplicial object of $\mathcal{A}$. Fix $n > 0$. For each $0 \leq i \leq n$, we let $d_i : A_n \to A_{n-1}$ denote the associated face map (determined by the unique injective map $[n-1] \to [n]$ whose image does not contain $i \in [n]$). Let $d(n) : A_n \to A_{n-1}$ denote the alternating sum $\sum_{0 \leq i \leq n}(-1)^i d_i$. An easy calculation shows that $d(n-1) \circ d(n) \simeq 0$ for $n > 0$, so that

$$\cdots \to A_2 \overset{d(2)}{\to} A_1 \overset{d(1)}{\to} A_0 \to 0 \to \cdots$$

is a chain complex with values in $\mathcal{A}$. We will denote this chain complex by $C_*(A)$, and refer to it as the *unnormalized chain complex* associated to $A_\bullet$.

If $A_\bullet$ is a simplicial object of $\mathcal{A}$, we let $C_*(A)$ denote the unnormalized chain complex of the underlying semisimplicial object of $A_\bullet$.

Definition 1.2.3.9. Let $\mathcal{A}$ be an abelian category, and let $A_\bullet$ be a simplicial object of $\mathcal{A}$. For each $n \geq 0$, we let $N_n(A)$ denote the subobject of A_n given by the intersection $\bigcap_{1 \leq i \leq n} \ker(d_i)$ (more formally: $N_n(A)$ is defined to be a kernel of the map $A_n \to \bigoplus_{1 \leq i \leq n} A_{n-1}$ given by $\{d_i\}_{1 \leq i \leq n}$). If $n > 0$, the map d_0 carries $N_n(A)$ into $N_{n-1}(A)$; we therefore obtain a chain complex

$$\cdots \to N_2(A) \to N_1(A) \to N_0(A) \to 0 \to \cdots$$

which we will denote by $N_*(A)$. We will refer to $N_*(A)$ as the *normalized chain complex of $A_\bullet$*. The construction $A_\bullet \mapsto N_*(A)$ determines a functor $N : \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{A}) \to \mathrm{Ch}(\mathcal{A})_{\geq 0}$, which we will refer to as the *normalized chain complex functor*.

Notation 1.2.3.10. If $K_\bullet$ is a simplicial set, we define

$$N_*(K) = N_*(\mathbf{Z}K) \qquad C_*(K) = C_*(\mathbf{Z}K),$$

so that $N_*(K) \subseteq C_*(K)$ are chain complexes of abelian groups. By definition, the homology of K is given by the homology of the chain complex $C_*(K)$ (which is the same as the homology of $N_*(K)$, by Proposition 1.2.3.17).

Remark 1.2.3.11. Let $\mathcal{A}$ be an abelian category, let $A = (A_*, d)$ be a nonnegatively graded chain complex with values in $\mathcal{A}$, and let $\mathrm{DK}_\bullet(A)$ be the associated simplicial object of $\mathcal{A}$. If $\alpha : [n] \to [k]$ is a surjective morphism in $\boldsymbol{\Delta}$ for $k < n$, then there exists $1 \leq i \leq n$ such that the composite map

$$[n-1] \overset{\beta}{\to} [n] \overset{\alpha}{\to} [k]$$

is also surjective; here β denotes the unique injective map whose image does not contain $i \in [n]$. It follows that the subobject $N_n(\mathrm{DK}(A)) \subseteq \mathrm{DK}_n(A) \simeq \bigoplus_{\alpha:[n]\to[k]} A_k$ can be identified with the summand A_n corresponding to the identity map $\alpha : [n] \to [n]$. The isomorphisms $N_n(\mathrm{DK}(A)) \simeq A_n$ are compatible with differentials, giving a canonical isomorphism of chain complexes

$$A \simeq N_*(\mathrm{DK}(A)).$$

Lemma 1.2.3.12. *Let $\mathcal{A}$ be an abelian category. The isomorphism of functors $u : \mathrm{id}_{\mathrm{Ch}_{\geq 0}(\mathcal{A})} \simeq N_* \circ \mathrm{DK}$ constructed in Remark 1.2.3.11 exhibits N_* as a right adjoint to DK.*

Proof. Let $A = (A_*, d)$ be a nonnegatively graded chain complex with values in $\mathcal{A}$, and let $B_\bullet$ be a simplicial object of $\mathcal{A}$. We wish to show that the canonical map

$$\theta : \mathrm{Hom}_{\mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{A})}(\mathrm{DK}_\bullet(A), B_\bullet) \to \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(N_*(\mathrm{DK}(A)), N_*(B)) \overset{u}{\to} \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A_*, N_*(B))$$

is bijective. To this end, suppose we are given a morphism $\phi : A_* \to N_*(B)$ in $\mathrm{Ch}(\mathcal{A})$, given by a collection of maps $\phi_n : A_n \to N_n(B) \subseteq B_n$. We define a map $\Phi_n : \mathrm{DK}_n(A) \simeq \bigoplus_{\alpha:[n] \to [k]} A_k \to B_n$ to be the sum of the maps $f_\alpha : A_k \overset{\phi_k}{\to} B_k \overset{\alpha^*}{\to} B_n$, where $\alpha^* : B_k \to B_n$ is the map associated to α by the simplicial object $B_\bullet$. It is easy to see that the maps Φ_n together determine a map of simplicial objects $\Phi : \mathrm{DK}_\bullet(A) \to B_\bullet$, and that Φ is the unique preimage of ϕ under θ. $\square$

Lemma 1.2.3.13. *Let $\mathcal{A}b$ denote the category of abelian groups. Then the functor $\mathrm{DK} : \mathrm{Ch}(\mathcal{A}b)_{\geq 0} \to \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{A}b)$ is an equivalence of categories.*

Proof. Let $N_* : \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{A}b) \to \mathrm{Ch}(\mathcal{A}b)_{\geq 0}$ be the normalized chain complex functor (Definition 1.2.3.9), so that N_* is right adjoint to DK (Lemma 1.2.3.12) and the unit map $u : \mathrm{id} \to N_* \circ \mathrm{DK}$ is an isomorphism of functors (Remark 1.2.3.11). It will therefore suffice to show that the counit map $v : \mathrm{DK} \circ N_* \to \mathrm{id}$ is an isomorphism of functors. In other words, we must show that for every simplicial abelian group $A_\bullet$, the canonical map

$$\theta : \mathrm{DK}_\bullet(N_*(A)) \to A_\bullet$$

is an isomorphism of simplicial abelian groups.

We begin by showing that θ is injective in each degree. Fix $n \geq 0$, and let $x \in \mathrm{DK}_n(N_*(A))$, so that x corresponds to a collection of elements $x_\alpha \in N_k(A)$ indexed by surjective maps $\alpha : [n] \to [k]$ in $\boldsymbol{\Delta}$. Assume that $x \neq 0$; we wish to prove that $\theta(x) \neq 0$. Let S be the collection consisting of those surjective maps $\alpha : [n] \to [k]$ such that $x_\alpha \neq 0$. Since $x \neq 0$, the set S is nonempty. Let k be the smallest nonnegative integer such that there exists a map $\alpha : [n] \to [k]$ in S. Given any such map, we let m_α^i be the least element of $\alpha^{-1}\{i\}$ for $0 \leq i \leq k$. Assume that $\alpha : [n] \to [k]$ has been chosen such that $x_\alpha \neq 0$ and the sum $m_\alpha^0 + \cdots + m_\alpha^k$ is as small as possible. The assignment $i \mapsto m_\alpha^i$ determines a map $\beta : [k] \to [n]$, which is right inverse to α. We will prove that $\theta(x) \neq 0$ by showing that $\beta^* \theta(x) = x_\alpha \in A_k$. To prove this, it will suffice to show that for every surjective map $\alpha' : [n] \to [k']$ different from α, we have $\beta^* \alpha'^* x_{\alpha'} = 0$ in A_k. If $\alpha' \notin S$, then $x_{\alpha'} = 0$ and the result is obvious. Assume therefore that $\alpha' \in S$, and let γ denote the composite map

$$[k] \overset{\beta}{\to} [n] \overset{\alpha'}{\to} [k'].$$

Since α is surjective, we have $m_\alpha^0 = 0$ so that $\gamma(0) = 0$. Since $x_{\alpha'} \in N_{k'}(A)$, we have $\gamma^* x_{\alpha'} = 0$ unless the image of γ contains every nonzero element of k'. We may therefore assume that γ is surjective. The minimality of k implies that $k' \geq k$, so that $k = k'$ and γ is the identity map. Thus $\alpha'(m_\alpha^i) = i$ for $0 \leq i \leq k$, so that $m_\alpha^i \geq m_{\alpha'}^i$. On the other hand, our minimality assumption on α guarantees that $m_\alpha^0 + \cdots + m_\alpha^k \leq m_{\alpha'}^0 + \cdots + m_{\alpha'}^k$. It follows that $m_\alpha^i = m_{\alpha'}^i$ for $0 \leq i \leq k$, so that $\alpha = \alpha'$ contrary to our assumption.

We now prove that θ induces a surjection $\theta_n : \mathrm{DK}_n(N_*(A)) \to A_n$ using induction on n. For $0 \leq i \leq n$, let $d_i : A_n \to A_{n-1}$ denote the ith face map, and let $A(i)_n = \bigcap_{j>i} \ker(d_j) \subseteq A_n$. We will prove by induction on i that the image of θ_n contains $A(i)_n$. When $i = 0$, we have $A(i)_n = N_n(A)$ and the result is obvious. Assume therefore that $0 < i \leq n$, and that the image of θ_n contains $A(i-1)_n$. Let $x \in A(i)_n$; we wish to prove that x belongs to the image of θ_n. Let $\alpha : [n] \to [n]$ be given by the formula

$$\alpha(j) = \begin{cases} j & \text{if } j \neq i \\ i-1 & \text{if } j = i, \end{cases}$$

and let $x' = \alpha^*(x) \in A_n$. Since α factors through $[n-1]$, x' belongs to the image of a degeneracy map $A_{n-1} \to A_n$ and therefore to the image of θ_n (since θ_{n-1} is surjective by the inductive hypothesis). It will therefore suffice to show that $x - x'$ belongs to the image of θ_n. This follows from the inductive hypothesis, since $x - x' \in A(i-1)_n$. □

Remark 1.2.3.14. Let $A_\bullet$ be a simplicial abelian group. Then the underlying simplicial set of $A_\bullet$ is a Kan complex with a canonical base point (given by $0 \in A_0$), so that we can define homotopy sets $\pi_n A$ for each $n \geq 0$. The abelian group structure on $A_\bullet$ determines an abelian group structure on each $\pi_n A$, which agrees with the usual group structure on $\pi_n A$ for $n > 0$. Unwinding the definitions, we see that $\pi_n A$ can be identified with the nth homology group of the normalized chain complex $N_*(A)$.

Proof of Theorem 1.2.3.7. Enlarging the universe if necessary, we may assume that the additive category $\mathcal{A}$ is small. Define $j : \mathcal{A} \to \mathrm{Fun}(\mathcal{A}^{op}, \mathcal{A}b)$ by the formula $j(A)(B) = \mathrm{Hom}_{\mathcal{A}}(B, A)$. We first claim that j is fully faithful. To prove this, we let $j' : \mathcal{A} \to \mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Set})$ denote the usual Yoneda embedding, so that j' is given by composing j with the forgetful functor $U : \mathcal{A}b \to \mathrm{Set}$. Yoneda's lemma implies that for any pair of objects $A, B \in \mathcal{A}$, the composite map

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \xrightarrow{\theta} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{A}^{op}, \mathcal{A}b)}(j(A), j(B)) \xrightarrow{\theta'} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Set})}(j'(A), j'(B))$$

is bijective. This implies that θ' is surjective. Since the functor U is faithful, the map θ' is also injective and therefore an isomorphism. By the two-out-of-three property, we conclude that θ is bijective as desired.

Note that $\mathcal{A}' = \mathrm{Fun}(\mathcal{A}^{op}, \mathcal{A}b)$ is itself an additive category (in fact, an abelian category) and that the functor j preserves finite sums and products. It follows that the diagram

$$
\begin{array}{ccc}
\mathrm{Ch}(\mathcal{A})_{\geq 0} & \xrightarrow{\mathrm{DK}} & \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{A}) \\
\downarrow & & \downarrow \\
\mathrm{Ch}(\mathcal{A}')_{\geq 0} & \xrightarrow{\mathrm{DK}} & \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{A}')
\end{array}
$$

commutes up to canonical isomorphism. Here the vertical maps are fully faithful embeddings, and Lemma 1.2.3.13 implies that the bottom horizontal map is an equivalence of categories. It follows that DK : $\mathrm{Ch}(\mathcal{A})_{\geq 0} \to \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{A})$ is a fully faithful embedding. Moreover, we obtain the following characterization of its essential image: a simplicial object $A_\bullet$ of $\mathcal{A}$ belongs to the essential image of DK if and only if the chain complex $N_*(j(A))$ belongs to the essential image of the fully faithful embedding $\mathrm{Ch}(\mathcal{A})_{\geq 0} \to \mathrm{Ch}(\mathcal{A}')_{\geq 0}$. This is equivalent to the requirement that each $N_n(j(A))$ belongs to the essential image of j. Note that $N_n(j(A))$ is a direct summand of $\mathrm{DK}_n(N_*(j(A))) \simeq j(A_n)$. If $\mathcal{A}$ is idempotent complete, it follows automatically that $N_n(j(A))$ belongs to the essential image of j, so that DK : $\mathrm{Ch}(\mathcal{A})_{\geq 0} \to \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{A})$ is an equivalence of categories. □

Remark 1.2.3.15. Let $\mathcal{A}$ be an idempotent complete additive category. Theorem 1.2.3.7 guarantees that the functor DK : $\mathrm{Ch}(\mathcal{A})_{\geq 0} \to \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{A})$ is an equivalence of categories. We let $N_* : \mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{A}) \to \mathrm{Ch}(\mathcal{A})_{\geq 0}$ denote a homotopy inverse to this equivalence. We will refer to N_* as the *normalized chain complex* functor. It follows from Lemma 1.2.3.12 that this definition agrees with Definition 1.2.3.9 in the case where $\mathcal{A}$ is abelian. In fact, we can say more: for each $n \geq 0$ and every simplicial object $A_\bullet$ in $\mathcal{A}$, the object $N_n(A)$ can be identified with a kernel of the map $A_n \to \bigoplus_{1 \leq i \leq n} A_{n-1}$ given by the face maps $\{d_i\}_{1 \leq i \leq n}$ (note that since $\mathcal{A}$ is not assumed to be abelian, the existence of this kernel is not immediately obvious). More precisely, we claim that the canonical map $u_n : N_n(A) \to \mathrm{DK}_n(N_*(A)) \simeq A_n$ induces an injective map $\mathrm{Hom}_{\mathcal{A}}(X, N_n(A)) \to \mathrm{Hom}_{\mathcal{A}}(X, A_n)$, whose image consists of those maps $\phi : X \to A_n$ such that the composite map $X \xrightarrow{\phi} A_n \xrightarrow{d_i} A_{n-1}$ is zero for $1 \leq i \leq n$. To prove this, we invoke Theorem 1.2.3.7 to reduce to the case where $A_\bullet = \mathrm{DK}_\bullet(B)$ for some $B_* \in \mathrm{Ch}(\mathcal{A})_{\geq 0}$, and note that for every surjective map $\alpha : [n] \to [k]$ which is not an isomorphism, there exists an injective map $\beta : [n-1] \to [n]$ with $\beta(0) = 0$ such that $\alpha \circ \beta$ is again surjective.

We observe that, as in Definition 1.2.3.9, the maps $u_n : N_n(A) \to A_n$ determine a monomorphism of chain complexes $u : N_*(A) \to C_*(A)$.

Remark 1.2.3.16. Let $\mathcal{A}$ be an idempotent complete additive category. If $A_\bullet$ is a simplicial object of $\mathcal{A}$ and $0 \leq i \leq n-1$, we let $s_i : A_{n-1} \to A_n$ denote the ith degeneracy map: that is, the map obtained from the unique surjective morphism $\alpha : [n] \to [n-1]$ in $\boldsymbol{\Delta}$ satisfying $\alpha(i) = \alpha(i+1) = i$. The normalized chain complex $N_*(A)$ described in Remark 1.2.3.15 admits a dual description: for each $n \geq 0$, the object $N_n(A)$ can be identified with a cokernel of the map

$$\bigoplus_{0 \leq i < n} A_{n-1} \to A_n$$

given by $\{s_i\}_{0 \leq i < n}$ (note that the existence of this cokernel is not immediately obvious when $\mathcal{A}$ is not an abelian category). Indeed, Theorem 1.2.3.7 allows us to assume that $A_\bullet = \mathrm{DK}_\bullet(B)$ for some chain complex $B_* \in \mathrm{Ch}(\mathcal{A})_{\geq 0}$, in which case the result is obvious. The identifications above give maps $v_n : A_n \to N_n(A)$ for $n \geq 0$, which determine an epimorphism of chain complexes $v : C_*(A) \to N_*(A)$.

In the situation of Remark 1.2.3.16, it is easy to see that the map $v : C_*(A) \to N_*(A)$ is a left inverse to the monomorphism $u : N_*(A) \to C_*(A)$: that is, the composition

$$N_*(A) \xrightarrow{u} C_*(A) \xrightarrow{v} N_*(A)$$

is the identity. Though u and v are not inverse to one another, one has the following closely related result:

Proposition 1.2.3.17. *Let $\mathcal{A}$ be an abelian category and let $A_\bullet$ be a simplicial object of $\mathcal{A}$. Then the canonical maps*

$$u : N_*(A) \to C_*(A) \qquad v : C_*(A) \to N_*(A)$$

are quasi-isomorphisms of chain complexes.

Proof. Since u is right inverse to v, it will suffice to show that v is a quasi-isomorphism. This is equivalent to the assertion that the chain complex $\ker(v)$ is acyclic (since v is an epimorphism). Using Theorem 1.2.3.7, we may assume that $A_\bullet = \mathrm{DK}_\bullet(B)$ for some chain complex $B_* \in \mathrm{Ch}(\mathcal{A})_{\geq 0}$. For each n, we have a canonical isomorphism $C_n(A) \simeq \bigoplus_{\alpha:[n] \to [k]} B_k$, where the sum is taken over all surjective maps $\alpha : [n] \to [k]$. For every integer i, let $C_n^{\leq i}(A)$ be the subobject of $C_n(A)$ generated by the summands B_k such that $\alpha(j) = \alpha(j-1)$ for some j satisfying $n - i \leq j \leq n$. Then $C_*^{\leq i}(A)$ is a subcomplex of $\ker(v)$, which coincides with $\ker(v)$ in degrees $\leq i+1$. It follows that the inclusion $C_*^{\leq i}(A) \to \ker(v)$ induces an isomorphism on homology in degrees $\leq i$ (and an epimorphism in degree $i+1$). It will therefore suffice to show that each of the chain complexes $C_*^{\leq i}(A)$ is acyclic. For this, we proceed by induction on i. If $i < 0$, then $C_*^{\leq i}(A) \simeq 0$ and the result is obvious. We may therefore assume that $i \geq 0$. We have a short exact sequence of chain complexes

$$0 \to C_*^{\leq i-1}(A) \to C_*^{\leq i}(A) \to K_* \to 0$$

for some $K_* \in \mathrm{Ch}(\mathcal{A})_{\geq 0}$. Using the associated long exact sequence together with the inductive hypothesis, we are reduced to proving that the chain complex K_* is acyclic. To prove this, consider the maps $\{(-1)^{n-i-1} s_{n-i-1} : A_n \to A_{n+1}\}$. It is not difficult to see that these maps preserve the subcomplexes $C_*^{\leq i-1}(A), C_*^{\leq i}(A) \subseteq C_*(A)$, and therefore determine a map $h : K_* \to K_{*+1}$. A simple calculation shows that $dh + hd = \mathrm{id}_K$, so that id_K is chain homotopic to the zero map and K_* is acyclic as desired. $\square$

Remark 1.2.3.18. Let $\mathcal{A}$ be an idempotent complete additive category. The restriction functor

$$\mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{A}) \to \mathrm{Fun}(\boldsymbol{\Delta}_s^{op}, \mathcal{A})$$

admits a left adjoint F, which carries each semisimplicial object $A_\bullet$ of $\mathcal{A}$ to the simplicial object $F(A)_\bullet$ given by left Kan extension along the inclusion $\mathbf{\Delta}_s^{op} \hookrightarrow \mathbf{\Delta}^{op}$. Unwinding the definitions, we see that $F(A)_\bullet$ is described by the formula

$$F(A)_n = \bigoplus_{\alpha:[n]\to[k]} A_k,$$

where α ranges over all surjective maps in $\mathbf{\Delta}$. We observe that for $k < n$, the corresponding summand of $F(A)_n$ lies in the image of some degeneracy map $A_{n-1} \to A_n$. Let $v : C_*(F(A)) \to N_*(F(A))$ be as in Remark 1.2.3.16; we conclude that the composite map

$$\theta_A : C_*(A) \to C_*(F(A)) \to N_*(F(A))$$

is an isomorphism in $\mathrm{Ch}(\mathcal{A})_{\geq 0}$.

Suppose now that $A_\bullet$ can be extended to a simplicial object of $\mathcal{A}$, which we will denote also by $A_\bullet$. We then obtain a map of simplicial objects $F(A)_\bullet \to A_\bullet$, which induces a map of chain complexes $C_*(A) \xrightarrow{\theta_A} N_*(F(A)) \to N_*(A_\bullet)$ which agrees with the map v of Remark 1.2.3.16.

We now discuss the behavior of the Dold-Kan correspondence with respect to tensor products.

Definition 1.2.3.19. Suppose we are given additive categories $\mathcal{A}^1, \mathcal{A}^2, \ldots, \mathcal{A}^n$, and $\mathcal{B}$. We will say that a functor $F : \mathcal{A}^1 \times \cdots \times \mathcal{A}^n \to \mathcal{B}$ is *multi-additive* if the functor F preserves direct sums separately in each variable.

Remark 1.2.3.20. Let $F : \mathcal{A}^1 \times \cdots \times \mathcal{A}^n \to \mathcal{B}$ be a multi-additive functor, and suppose we choose objects $A^i, A'^i \in \mathcal{A}^i$ for $1 \leq i \leq n$. The induced map

$$\prod_{1\leq i\leq n} \mathrm{Hom}_{\mathcal{A}_i}(A^i, A'^i) \to \mathrm{Hom}_{\mathcal{B}}(F(\{A^i\}), F(\{A'^i\}))$$

is additive in each variable: that is, it induces a map of abelian groups

$$\bigotimes_{1\leq i\leq n} \mathrm{Hom}_{\mathcal{A}^i}(A^i, A'^i) \to \mathrm{Hom}_{\mathcal{B}}(F(\{A^i\}), F(\{A'^i\})).$$

Remark 1.2.3.21. Let $F : \mathcal{A}^1 \times \cdots \times \mathcal{A}^n \to \mathcal{B}$ be a multi-additive functor. Then F induces a multi-additive functor

$$\mathrm{Ch}(F)_{\geq 0} : \mathrm{Ch}(\mathcal{A}^1)_{\geq 0} \times \cdots \times \mathrm{Ch}(\mathcal{A}^n)_{\geq 0} \to \mathrm{Ch}(\mathcal{B})_{\geq 0},$$

which is given on objects by the formula

$$\mathrm{Ch}(F)_{\geq 0}((A^1, d^1), \ldots, (A^n, d^n))_p = \bigoplus_{p=p_1+\cdots+p_n} F(A^1_{p_1}, \ldots, A^n_{p_n}),$$

where the differential d is given on the summand $F(A^1_{p_1}, \ldots, A^n_{p_n})$ by the sum

$$\sum_{1\leq i\leq n} (-1)^{p_1+\cdots+p_{i-1}} F(\mathrm{id}_{A^1_{p_1}}, \ldots, \mathrm{id}_{A^{i-1}_{p_{i-1}}}, d^i, \mathrm{id}_{A^{i+1}_{p_{i+1}}}, \ldots, \mathrm{id}_{A^n_{p_n}}).$$

Note that the direct sum is essentially finite, since the summand corresponding to a decomposition $p = p_1 + \cdots + p_n$ is zero unless $p_1, \ldots, p_n \geq 0$. If the additive category $\mathcal{B}$ admits countable coproducts, then the same formula defines a multi-additive functor $\mathrm{Ch}(F) : \mathrm{Ch}(\mathcal{A}^1) \times \cdots \times \mathrm{Ch}(\mathcal{A}^n) \to \mathrm{Ch}(\mathcal{B})$.

If $F : \mathcal{A}^1 \times \cdots \times \mathcal{A}^n \to \mathcal{B}$ is a multi-additive functor, then F induces a functor

$$\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A}^1) \times \cdots \times \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A}^n) \to \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{B}),$$

given by pointwise composition with F. In what follows, we will abuse notation by denoting this functor also by F.

Construction 1.2.3.22. Let $F : \mathcal{A}^1 \times \cdots \times \mathcal{A}^n \to \mathcal{B}$ be a multi-additive functor between idempotent complete additive categories. Suppose we are given simplicial objects $A^i_\bullet$ of $\mathcal{A}^i$ for $1 \le i \le n$. For each $p \ge 0$, let

$$\overline{AW}_p : F(A^1_p, \ldots, A^n_p) \to \mathrm{Ch}(F)_{\ge 0}(C_*(A^1), \ldots, C_*(A^n))_p$$

be the map given by the sum of the maps

$$F(\alpha_1^*, \ldots, \alpha_n^*) : F(A^1_p, \ldots, A^n_p) \to F(A^1_{p_1}, \ldots, A^n_{p_n}),$$

where α_i denotes the map $[p_i] \simeq \{p_1 + \ldots + p_{i-1}, p_1 + \ldots + p_{i-1} + 1, \ldots, p_1 + \ldots + p_{i-1} + p_i\} \hookrightarrow [p]$ in $\mathbf{\Delta}$. Using Remark 1.2.3.16, we deduce the existence of a unique map $\mathrm{AW}_p : N_p F(A^1, \ldots, A^n) \to \mathrm{Ch}(F)_{\ge 0}(N_*(A^1), \ldots, N_*(A^n))_p$ such that the diagram

$$
\begin{array}{ccc}
F(A^1_p, \ldots, A^n_p) & \xrightarrow{\ \overline{AW}_p\ } & \mathrm{Ch}(F)_{\ge 0}(C_*(A^1), \ldots, C_*(A^n))_p \\
\downarrow & & \downarrow \\
N_p F(A^1, \ldots, A^n) & \xrightarrow{\ \mathrm{AW}_p\ } & \mathrm{Ch}(F)_{\ge 0}(N_*(A^1), \ldots, N_*(A^n))_p.
\end{array}
$$

commutes. Both of these maps commute with the differential and determine maps of chain complexes

$$\overline{\mathrm{AW}} : C_*(F(A^1, \ldots, A^n)) \to \mathrm{Ch}(F)_{\ge 0}(C_*(A^1), \ldots, C_*(A^n))$$

$$\mathrm{AW} : N_*(F(A^1, \ldots, A^n)) \to \mathrm{Ch}(F)_{\ge 0}(N_*(A^1), \ldots, N_*(A^n)).$$

We will refer to these maps as the *Alexander-Whitney* maps associated to F.

Remark 1.2.3.23. Let $F : \mathcal{A}^1 \times \cdots \times \mathcal{A}^n \to \mathcal{B}$ be a multi-additive functor between idempotent complete additive categories. The Alexander-Whitney maps associated to F depend on the choice of ordering of the categories $\mathcal{A}^1, \ldots, \mathcal{A}^n$.

Remark 1.2.3.24. Let $F : \mathcal{A}^1 \times \cdots \times \mathcal{A}^n \to \mathcal{B}$ be a multi-additive functor. The Alexander-Whitney maps

$$\overline{\mathrm{AW}} : C_*(F(A^1, \ldots, A^n)) \to \mathrm{Ch}(F)_{\ge 0}(C_*(A^1), \ldots, C_*(A^n))$$

$$\mathrm{AW} : N_*(F(A^1, \ldots, A^n)) \to \mathrm{Ch}(F)_{\ge 0}(N_*(A^1), \ldots, N_*(A^n))$$

depend functorially on the sequence of simplicial objects $\{A^i_\bullet \in \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A}^i)\}_{1 \le i \le n}$, and can therefore be viewed as natural transformations of functors.

Remark 1.2.3.25. Let $\alpha : \{1, \ldots, n\} \to \{1, \ldots, m\}$ be an order-preserving map, and suppose we are given multi-additive functors

$$F : \prod_{1 \le i \le m} \mathcal{B}^i \to \mathcal{C} \qquad G^i : \prod_{\alpha(j)=i} \mathcal{A}^j \to \mathcal{B}^i$$

between idempotent complete additive categories, and let H denote the composite functor

$$\prod_{1 \le j \le n} \mathcal{A}^j \xrightarrow{\ \prod_i G^i\ } \prod_{1 \le i \le m} \mathcal{B}^i \xrightarrow{\ F\ } \mathcal{C}.$$

Then $\mathrm{Ch}(H)_{\ge 0}$ is equivalent to the composition

$$\prod_{1 \le j \le n} \mathrm{Ch}(\mathcal{A}^j)_{\ge 0} \xrightarrow{\ \prod_i \mathrm{Ch}(G^i)_{\ge 0}\ } \prod_{1 \le i \le m} \mathrm{Ch}(\mathcal{B}^i)_{\ge 0} \xrightarrow{\ \mathrm{Ch}(F)_{\ge 0}\ } \mathrm{Ch}(\mathcal{C})_{\ge 0}.$$

Let AW_F, AW_{G^i}, and AW_H be the Alexander-Whitney natural transformations associated to F, G^i, and H respectively. For any sequence of simplicial objects $A^i_\bullet \in \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A}^i)$, the map

$$\mathrm{AW}_H : N_*(H(A^1, \ldots, A^n)) \to \mathrm{Ch}(H)_{\geq 0}(N_*(A^1), \ldots, N_*(A^n))$$

is given by the composition

$$
\begin{array}{rcl}
N_*(H(A^1, \ldots, A^n)) & \simeq & N_* F(G^1(\{A^j\}_{\alpha(j)=1}), \ldots, G^m(\{A^j\}_{\alpha(j)=m})) \\[4pt]
& \xrightarrow{\mathrm{AW}_F} & \mathrm{Ch}(F)_{\geq 0}\{N_* G^i \{A^j\}_{\alpha(j)=i}\} \\[4pt]
& \xrightarrow{\{\mathrm{AW}_{G^i}\}_{1 \leq i \leq m}} & \mathrm{Ch}(F)_{\geq 0}\{\mathrm{Ch}(G^i)_{\geq 0}\{N_*(A^j)_{\alpha(j)=i}\}\} \\[4pt]
& \simeq & \mathrm{Ch}(H)_{\geq 0}(N_*(A^1), \ldots, N_*(A^n)).
\end{array}
$$

The analogous assertion is also true at the level of unnormalized chain complexes: the Alexander-Whitney maps $\overline{\mathrm{AW}}_H$ associated to H can be obtained by composing the Alexander-Whitney maps $\overline{\mathrm{AW}}_F$ and $\overline{\mathrm{AW}}_{G^i}$ associated to F and G^i, respectively.

Example 1.2.3.26. Let $\mathcal{A}$ be a monoidal category. Assume that $\mathcal{A}$ is additive, idempotent complete, and that the tensor product $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ preserves finite products separately in each variable. Then the category of chain complexes $\mathrm{Ch}(\mathcal{A})_{\geq 0}$ inherits a monoidal structure (from the construction described in Remark 1.2.3.21). The Alexander-Whitney construction supplies maps

$$N_*(A_\bullet \otimes B_\bullet) \to N_*(A_\bullet) \otimes N_*(B_\bullet)$$

$$C_*(A_\bullet \otimes B_\bullet) \to C_*(A_\bullet) \otimes C_*(B_\bullet).$$

It follows from Remark 1.2.3.25 that these maps are compatible with the associativity constraints for the tensor product operations on $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A})$ and $\mathrm{Ch}(\mathcal{A})_{\geq 0}$. In other words, the functors $N_*, C_* : \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A}) \to \mathrm{Ch}(\mathcal{A})_{\geq 0}$ can be promoted to left-lax monoidal functors, as defined in §T.A.1.3. It follows that the Dold-Kan equivalence $\mathrm{DK} : \mathrm{Ch}(\mathcal{A})_{\geq 0} \to \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A})$ is right-lax monoidal (Definition T.A.1.3.5).

Example 1.2.3.27. Let **Latt** denote the category of *lattices*: that is, the full subcategory of the category of abelian groups spanned by those abelian groups which are isomorphic to $\mathbf{Z}^n$ for some $n \geq 0$. For any additive category $\mathcal{A}$, there is an essentially unique multi-additive functor $\otimes : \mathbf{Latt} \times \mathcal{A} \to \mathcal{A}$, which is given on objects by $\mathbf{Z}^n \otimes A \simeq A^n$. If $\mathcal{A}$ is idempotent complete, the Alexander-Whitney construction determines a map

$$N_*(L \otimes A) \to N_*(L) \otimes N_*(A),$$

for any simplicial objects $L_\bullet$ and $A_\bullet$ in **Latt** and $\mathcal{A}$, respectively. It follows from Remark 1.2.3.25 that if $L'_\bullet$ is another simplicial object of **Latt**, then the diagram

$$
\begin{CD}
N_*(L' \otimes L \otimes A) @>>> N_*(L') \otimes N_*(L \otimes A) \\
@VVV @VVV \\
N_*(L' \otimes L) \otimes N_*(A) @>>> N_*(L') \otimes N_*(L) \otimes N_*(A)
\end{CD}
$$

is commutative.

Proposition 1.2.3.28. *Let $F : \mathcal{A}^1 \times \ldots \times \mathcal{A}^n \to \mathcal{B}$ be a multi-additive functor between idempotent complete additive categories, and assume that $\mathcal{B}$ is abelian. For every collection of simplicial objects $A^i_\bullet$ of $\mathcal{A}^i$, the Alexander-Whitney maps*

$$\mathrm{AW} : N_*(F(A^1, \ldots, A^n)) \to \mathrm{Ch}(F)_{\geq 0}(N_*(A^1), \ldots, N_*(A^n))$$

are quasi-isomorphisms.

Proof. Assume that $n > 0$ (otherwise the result is obvious) and fix $m \geq 0$; we will show that AW induces an isomorphism on homology in degrees $\leq m$. Using Theorem 1.2.3.7, we can assume that $A_\bullet^i = \mathrm{DK}_\bullet(X^i)$, where X_*^i is a chain complex in $\mathcal{A}^i$ for $1 \leq i \leq n$. For $k \in \mathbf{Z}$, let $X(k)_*^1$ denote the quotient chain complex of X_j^1, given by

$$X(k)_j^1 = \begin{cases} X_j^i & \text{if } j \leq i \\ 0 & \text{otherwise.} \end{cases}$$

We have commutative diagrams

$$
\begin{array}{ccc}
N_*(F(\mathrm{DK}(X^1), \ldots, \mathrm{DK}(X^n))) & \xrightarrow{\ \ \mathrm{AW}\ \ } & \mathrm{Ch}(F)_{\geq 0}(X^1, \ldots, X^n) \\
\downarrow & & \downarrow \\
N_*(F(\mathrm{DK}(X(k)^1), \mathrm{DK}(X^2), \ldots, \mathrm{DK}(X^n))) & \longrightarrow & \mathrm{Ch}(F)_{\geq 0}(X(k)^1, X^2, \ldots, X^n)
\end{array}
$$

where the vertical maps are isomorphisms in degrees $< k$. It will therefore suffice to show that the lower horizontal map is a quasi-isomorphism for some $k > m$. We prove that the bottom horizontal map is a quasi-isomorphism for all k, using induction on k. If $k < 0$, then both sides vanish and there is nothing to prove. If $k \geq 0$, then the inductive hypothesis allows us to reduce to proving Proposition 1.2.3.28 after replacing X_*^1 by $X(k)_*^1 / X(k-1)_*^1$. In other words, we may assume that the chain complex $X_*^1 = M^1[p_1]$ consists of a single object $M^1 \in \mathcal{A}^1$, concentrated in degree $p_1 = k$. Using the same argument, we may assume that each $X_*^i \simeq M^i[p_i]$ for some $M^i \in \mathcal{A}^i$, $p_i \geq 0$. Then $\mathrm{Ch}(F)_{\geq 0}(X^1, \ldots, X^n) \simeq F(M^1, \ldots, M^n)[p]$, where $p = p_1 + \ldots + p_n$.

Let us view $\mathcal{B}$ as tensored over the category of lattices **Latt**, as in Example 1.2.3.27. A mild variant of Example 1.2.3.6 shows that the simplicial object $F(\mathrm{DK}(X^1), \ldots, \mathrm{DK}(X^n))$ can be identified with $F(\{(\mathbf{Z}\Delta^{p_i}/\mathbf{Z}\,\partial\,\Delta^{p_i}) \otimes M^i\}) \simeq \mathbf{Z}(\prod_i \Delta^{p_i})/\mathbf{Z}(\partial \prod_i \Delta^{p_i}) \otimes F(M^1, \ldots, M^n)$. Let $K = \prod_{1 \leq i \leq n} \Delta^{p_i}$ and let ∂K denote the simplicial subset of K consisting of those simplices σ such that for some $1 \leq i \leq n$, the composite map $\sigma \to K \to \Delta^{p_i}$ is not surjective. Unwinding the definitions, we see that the Alexander-Whitney map AW is obtained by tensoring $F(M^1, \ldots, M^n) \in \mathcal{B}$ with a map

$$\theta : N_*(K)/N_*(\partial K) \to \mathbf{Z}[p]$$

of chain complexes of abelian groups. It will therefore suffice to show that θ admits a chain homotopy inverse. Since the domain and codomain of θ are finite chain complexes of free abelian groups, it will suffice to show that θ is a quasi-isomorphism. In other words, we must show that the relative homology $\mathrm{H}_q(K, \partial K)$ vanishes for $q \neq p$, and that θ induces an isomorphism $\mathrm{H}_p(K, \partial K) \simeq \mathbf{Z}$. This follows from a straightforward calculation. $\square$

1.2.4   The ∞-Categorical Dold-Kan Correspondence

Let $\mathcal{A}$ be an abelian category. Then the classical Dold-Kan correspondence (see [159]) asserts that the category $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A})$ of simplicial objects of $\mathcal{A}$ is equivalent to the category $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ of (homologically) nonnegatively graded chain complexes

$$\ldots \xrightarrow{d} A_1 \xrightarrow{d} A_0 \to 0.$$

In this section, we will prove an analogue of this result when the abelian category $\mathcal{A}$ is replaced by a stable ∞-category.

We begin by observing that if $X_\bullet$ is a simplicial object in a stable ∞-category $\mathcal{C}$, then $X_\bullet$ determines a simplicial object of the homotopy category $h\mathcal{C}$. The category $h\mathcal{C}$ is not abelian, but it is additive and has the following additional property (which follows easily from the fact that $h\mathcal{C}$ admits a triangulated structure):

($*$) If $i : X \to Y$ is a morphism in $h\mathcal{C}$ which admits a left inverse, then there is an isomorphism $Y \simeq X \oplus X'$ such that i is identified with the map $(\mathrm{id}, 0)$.

These conditions are sufficient to construct a Dold-Kan correspondence in $h\mathcal{C}$. Consequently, every simplicial object $X_\bullet$ of $\mathcal{C}$ determines a chain complex

$$\ldots \to C_1 \to C_0 \to 0$$

in the homotopy category $h\mathcal{C}$. In §1.2.2, we saw another construction which gives rise to the same type of data. Namely, Lemma 1.2.2.4 and Remark 1.2.2.3 show that every $\mathbf{Z}_{\geq 0}$-filtered object

$$Y(0) \xrightarrow{f_1} Y(1) \xrightarrow{f_2} \ldots$$

determines a chain complex C_* with values in $h\mathcal{C}$, where $C_n = \mathrm{cofib}(f_n)[-n]$. Thus, every $\mathbf{Z}_{\geq 0}$-filtered object of $\mathcal{C}$ determines a simplicial object of the homotopy category $h\mathcal{C}$. Our goal in this section is to prove the following more precise result, whose proof will be given at the end of this section:

Theorem 1.2.4.1 (∞-Categorical Dold-Kan Correspondence). *Let $\mathcal{C}$ be a stable ∞-category. Then the ∞-categories $\mathrm{Fun}(\mathrm{N}(\mathbf{Z}_{\geq 0}), \mathcal{C})$ and $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{C})$ are (canonically) equivalent to one another.*

Remark 1.2.4.2. Let $\mathcal{C}$ be a stable ∞-category. We may informally describe the equivalence of Theorem 1.2.4.1 as follows. To a simplicial object $X_\bullet$ of $\mathcal{C}$, we assign the filtered object

$$D(0) \to D(1) \to D(2) \to \ldots$$

where $D(k)$ is the colimit of the k-skeleton of $X_\bullet$. In particular, the colimit $\varinjlim D(j)$ can be identified with geometric realizations of the simplicial object $X_\bullet$.

Remark 1.2.4.3. Let $\mathcal{C}$ be a stable ∞-category, and let X be a simplicial object of $\mathcal{C}$. Using the Dold-Kan correspondence, we can associate to X a chain complex

$$\ldots \to C_2 \to C_1 \to C_0 \to 0$$

in the triangulated category $h\mathcal{C}$. More precisely, for each $n \geq 0$, let $L_n \in \mathcal{C}$ denote the nth *latching object* of X (see §T.A.2.9), so that X determines a canonical map $\alpha : L_n \to X_n$. Then $C_n \simeq \mathrm{cofib}(\alpha)$, where the cofiber can be formed either in the ∞-category $\mathcal{C}$ or in its homotopy category $h\mathcal{C}$ (since L_n is actually a direct summand of X_n).

Using Theorem 1.2.4.1, we can also associate to X a filtered object

$$D(0) \to D(1) \to D(2) \to \ldots$$

of $\mathcal{C}$. Using Lemma 1.2.2.4 and Remark 1.2.2.3, we can associate to this filtered object another chain complex

$$\ldots \to C_1' \to C_0' \to 0$$

with values in $h\mathcal{C}$. For each $n \geq 0$, let $X(n)$ denote the restriction of X to $\mathrm{N}(\mathbf{\Delta}_{\leq n}^{op})$, and let $X'(n)$ be a left Kan extension of $X(n-1)$ to $\mathrm{N}(\mathbf{\Delta}_{\leq n}^{op})$. Then we have a canonical map $\beta : X'(n) \to X(n)$, which induces an equivalence $X'(n)_m \to X(n)_m$ for $m < n$, while $X'(n)_n$ can be identified with the latching object L_n. Let $X''(n) = \mathrm{cofib}(\beta)$. Then $X''(n)_m = 0$ for $m < n$, while $X''(n)_n \simeq C_n$. Corollary 1.2.4.18 determines a canonical isomorphism $\varinjlim X''(n) \simeq C_n[n]$ in the homotopy category $h\mathcal{C}$. The map $D(n-1) \to D(n)$ can be identified with the composition

$$D(n-1) \simeq \varinjlim X(n-1) \simeq \varinjlim X'(n) \to \varinjlim X(n) \simeq D(n).$$

It follows that $C_n' \simeq \mathrm{cofib}(D(n-1) \to D(n))[-n] \simeq X''(n)_n[-n]$ is canonically isomorphic to C_n. It is not difficult to show that these isomorphisms are compatible with the differentials, so that we obtain an isomorphism of chain complexes $C_* \simeq C_*'$ with values in the triangulated category $h\mathcal{C}$.

Remark 1.2.4.4. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure, whose heart is equivalent to (the nerve of) an abelian category $\mathcal{A}$. Let $X_\bullet$ be a simplicial object of $\mathcal{C}$, and let

$$D(0) \to D(1) \to D(2) \to \ldots$$

be the associated filtered object (Theorem 1.2.4.1). Using Definition 1.2.2.9 (and Lemma 1.2.2.4), we can associate to this filtered object a spectral sequence $\{E_r^{p,q}, d_r\}_{r \geq 1}$ in the abelian category $\mathcal{A}$. In view of Remarks 1.2.2.8 and 1.2.4.3, for each $q \in \mathbf{Z}$ we can identify the complex $(E_1^{*,q}, d_1)$ with the normalized chain complex associated to the simplicial object $\pi_q X_\bullet$ of $\mathcal{A}$.

In the situation of Remark 1.2.4.4, suppose that the ∞-category $\mathcal{C}$ admits small colimits and that the t-structure on $\mathcal{C}$ is compatible with filtered colimits, so that the geometric realization $|X_\bullet| \simeq \varinjlim D(n) \in \mathcal{C}$ is defined. Proposition 1.2.2.14 implies that the spectral sequence converges to a filtration on the homotopy groups $\pi_{p+q} \varinjlim(D(n)) \simeq \pi_{p+q}|X_\bullet|$. If we assume that $X_\bullet$ is a simplicial object of $\mathcal{C}_{\geq 0}$, then we get a much stronger notion of convergence (which requires weaker assumptions on $\mathcal{C}$):

Proposition 1.2.4.5. *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure whose heart is equivalent to the nerve of an abelian category $\mathcal{A}$. Let $X_\bullet$ be a simplicial object of $\mathcal{C}_{\geq 0}$, let $D(0) \to D(1) \to D(2) \to \ldots$ be the associated filtered object of $\mathcal{C}$ (Theorem 1.2.4.1), and let $\{E_r^{p,q}, d_r\}_{r \geq 1}$ be the associated spectral sequence in $\mathcal{A}$. Then:*

(1) *The objects $E_1^{p,q} \in \mathcal{A}$ vanish unless $p, q \geq 0$.*

(2) *For each $r \geq 1$, the objects $E_r^{p,q}$ vanish unless $p, q \geq 0$.*

(3) *Fix $p, q \geq 0$. For $r > p, q + 1$, we have canonical isomorphisms*

$$E_r^{p,q} \simeq E_{r+1}^{p,q} \simeq E_{r+2}^{p,q} \simeq \cdots$$

in the abelian category $\mathcal{A}$. We let $E_\infty^{p,q} \in \mathcal{A}$ denote the colimit of this sequence of isomorphisms, so that $E_\infty^{p,q} \simeq E_{r'}^{p,q}$ for all $r' \geq r$.

(4) *For $0 \leq m \leq n$, we have $\operatorname{cofib}(D(m) \to D(n)) \in \mathcal{C}_{\geq m+1}$.*

(5) *Fix an integer n. The map $\pi_n D(k) \to \pi_n D(k+1)$ is an epimorphism for $k = n$ and an isomorphism for $k > n$. In particular, we have isomorphisms*

$$\pi_n D(n+1) \simeq \pi_n D(n+2) \simeq \cdots$$

We let A_n denote the colimit of this sequence of isomorphisms, so that we have isomorphisms $A_n \simeq \pi_n D(k)$ for $k > n$.

(6) *For each integer $n \geq 0$, the object $A_n \in \mathcal{A}$ admits a finite filtration*

$$0 = F^{-1} A_n \subseteq F^0 A_n \subseteq \cdots \subseteq F^n A_n = A_n,$$

where $F^p A_n$ is the image of the map $\pi_n D(p) \to \pi_n D(n+1) \simeq A_n$. We have canonical isomorphisms $F^p A_{p+q}/F^{p-1} A_{p+q} \simeq E_\infty^{p,q}$.

(7) *Suppose either that $\mathcal{C}$ admits countable colimits, or that $\mathcal{C}$ is left complete. Then the simplicial object $X_\bullet$ of $\mathcal{C}$ has a geometric realization $X \in \mathcal{C}_{\geq 0}$. Moreover, we have canonical isomorphisms $\pi_n X \simeq A_n$ in the abelian category $\mathcal{A}$.*

Before giving the proof of Proposition 1.2.4.8, we embark on a brief digression concerning idempotent morphisms in an ∞-category $\mathcal{C}$.

Proposition 1.2.4.6. *Let $\mathcal{C}$ be an ∞-category, let X be an object of $\mathcal{C}$, and let $e : X \to X$ be a morphism. The following conditions are equivalent:*

(1) *The morphism e is idempotent in the ∞-category $\mathcal{C}$. That is, the map $\Delta^1 \to \mathcal{C}$ determined by e extends to a map $\mathrm{Idem} \to \mathcal{C}$, where Idem is the ∞-category of Definition T.4.4.5.2.*

(2) *The morphism e is idempotent in the homotopy category $h\mathcal{C}$. That is, e^2 is homotopic to e.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious. We will prove that (2) $\Rightarrow$ (1). Enlarging $\mathcal{C}$ if necessary, we may suppose that $\mathcal{C}$ admits sequential colimits. Regard the set $\mathbf{Z}_{\geq 0}$ of nonnegative integers as linearly ordered, and let K denote the simplicial subset of $\mathrm{N}(\mathbf{Z}_{\geq 0})$ consisting of all vertices of $\mathrm{N}(\mathbf{Z}_{\geq 0})$, together with those edges given by pairs $i \leq j$ where $j \leq i+1$. Let $f : K \to \mathcal{C}$ be the map which carries each vertex of K to the object $X \in \mathcal{C}$, and each nondegenerate edge of K to the morphism $e : X \to X$. Since $\mathcal{C}$ admits sequential colimits, the diagram f admits a colimit Y. In particular, we obtain a canonical map $X = f(0) \to Y$, which we will denote by r.

Let K' denote the 1-skeleton of $K^{\triangleleft}$, and let $f' : K' \to \mathcal{C}$ be the map which carries each vertex of K' to the object $X \in \mathcal{C}$, and each nondegenerate edge of K' to the morphism $e : X \to X$. Using the fact that e^2 is homotopic to e, we deduce that f' can be extended to a map $\overline{f} : K^{\triangleleft} \to \mathcal{C}$. The extension $\overline{f}$ classifies a map $i : Y = \varinjlim(f) \to X$ such that $i \circ r$ is homotopic to e. Assertion (1) now follows from Corollary T.4.4.5.7. $\qquad\square$

Corollary 1.2.4.7. *Let $\mathcal{C}$ be an ∞-category which is idempotent-complete. Then the homotopy category $h\mathcal{C}$ is also idempotent-complete.*

Remark 1.2.4.8. Let $\mathcal{C}$ be a stable ∞-category, let $X_\bullet$ be a simplicial object of $\mathcal{C}$, let

$$D(0) \to D(1) \to \dots$$

be the associated filtered object. Let $\mathcal{C}^{\vee}$ denote the idempotent completion of $\mathcal{C}$. It follows from Corollary 1.2.4.7 that the homotopy category $h\mathcal{C}^{\vee}$ is an idempotent complete additive category. We may therefore apply Theorem 1.2.4.1 to conclude that each X_n can be written as a finite coproduct of objects of the form $\mathrm{cofib}(D(m-1) \to D(m))[-m]$, where $0 \leq m \leq n$ (here $D(-1) \simeq 0$ by convention).

Proof of Proposition 1.2.4.5. To prove (1), we to observe that $E_1^{p,q}$ is the pth term of the normalized chain complex associated to the simplicial object $\pi_q X_\bullet$ in $\mathcal{A}$. This homotopy group vanishes for $p < 0$ because the chain complex is nonnegatively graded and for $q < 0$ because we have assumed that each X_n belongs to $\mathcal{C}_{\geq 0}$. Assertion (2) follows immediately from (1) using induction on r, since $E_{r+1}^{p,q}$ can be identified with the homology of the complex

$$E_r^{p+r,q-r+1} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{p-r,q+r-1}.$$

If $r > p$ and $r > q+1$, then assertion (2) implies that the outer terms vanish, so that we have isomorphisms $E_r^{p,q} \simeq E_{r+1}^{p,q}$ which proves (3). Note that Remark 1.2.4.8 shows that $\mathrm{cofib}(D(m) \to D(m+1)) \in \mathcal{C}_{\geq m+1}$ for each $m \geq 0$. This proves (4) in the case $n = m+1$; the general case follows by induction on $n-m$. Assertion (5) follows immediately from (4). If we define $F^p A_n$ to be the image of the map $\pi_n D(p) \to \pi_n D(n+1) \simeq A_n$, then we clearly have inclusions

$$\dots \subseteq F^{-1} A_n \subseteq F^0 A_n \subseteq \dots \subseteq F^n A_n \subseteq F^{n+1} A_n \subseteq \dots$$

Since $D(p) \simeq 0$ for $p < 0$, we deduce that $F^p A_n = 0$ for $p < 0$. Similarly, the surjectivity of the map $\pi_n D(n) \to \pi_n D(n+1)$ shows that $F^n A_n = A_n$. To complete the proof of (6), we note that $E_\infty^{p,q}$ can be described as the image of the morphism

$$\theta : \pi_{p+q} \mathrm{cofib}(D(p-r) \to D(p)) \to \pi_{p+q} \mathrm{cofib}(D(p-1) \to D(p+r-1))$$

for $r \gg 0$. If $r > p+q+1$, then $D(p-r) \simeq 0$ and $D(p+r-1) \simeq A_{p+q}$, and we can describe $E_r^{p,q}$ as the image of $\pi_{p+q} D(p)$ in the quotient $\mathrm{coker}(\pi_{p+q} D(p-1) \to A_{p+q}) \simeq A_{p+q}/F^{p-1} A_{p+q}$, which is isomorphic to the quotient $F^p A_{p+q}/F^{p-1} A_{p+q}$.

It remains to prove (7). Assume first that $\mathcal{C}$ admits countable colimits. Then the existence of $X \simeq |X_\bullet| \simeq \varinjlim D(n)$ is obvious. Moreover, for each $m \geq 0$, we deduce that

$$\mathrm{cofib}(D(m) \to X) \simeq \varinjlim_n \mathrm{cofib}(D(m) \to D(n)) \in \mathcal{C}_{\geq m+1}$$

(using (4)), so that $\pi_k X \simeq \pi_k D(m) \simeq A_k$ for $k < m$. If we assume instead that $\mathcal{C}$ is left complete, then we must work a bit harder. We first show that the sequence

$$D(0) \to D(1) \to D(2) \to \cdots$$

has a colimit in $\mathcal{C}$. Since $\mathcal{C}$ is a homotopy limit of the tower of ∞-categories

$$\cdots \to \mathcal{C}_{\leq 2} \to \mathcal{C}_{\leq 1} \to \mathcal{C}_{\leq 0}$$

under (colimit-preserving) truncation functors, it will suffice to show that for each $n \geq 0$, the sequence

$$\tau_{\leq n} D(0) \to \tau_{\leq n} D(1) \to \cdots \to \tau_{\leq n} D(k) \to \cdots$$

has a colimit in $\mathcal{C}_{\leq n}$. This is clear, since assertion (4) implies that this sequence is eventually constant. This proves the existence of $X \simeq \varinjlim D(n)$. Moreover, for every integer n, we have $\pi_n X \simeq \pi_n(\tau_{\leq n} X) \simeq \pi_n(\tau_{\leq n} D(k)) \simeq \pi_n D(k)$ for $k \gg 0$, which provides the desired isomorphisms $\pi_n X \simeq A_n$. $\qquad\square$

Variant 1.2.4.9. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure, and let $X_\bullet : \mathrm{N}(\boldsymbol{\Delta}_s)^{op} \to \mathcal{C}$ be a semisimplicial object of $\mathcal{C}$. Let $Y_\bullet : \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathcal{C}$ be the simplicial object of $\mathcal{C}$ obtained by the process of left Kan extension, and let $\{E_r^{p,q}, d_r\}_{r \geq 1}$ be the spectral sequence associated to $Y_\bullet$ by the construction of Remark 1.2.4.4. Unwinding the definitions, we see that for every integer q, $\{E_1^{*,q}\}$ is the *unnormalized* chain complex associated to the semisimplicial object $\pi_q X_\bullet$ of the heart of $\mathcal{C}$. More precisely, we have canonical isomorphisms $E_1^{p,q} \simeq \pi_q X_p$, and the differential $d_1 : E_1^{p,q} \to E_1^{p-1,q}$ is the alternating sum of the face maps $\pi_q X_p \to \pi_q X_{p-1}$ induced by the inclusions $[p-1] \hookrightarrow [p]$.

Example 1.2.4.10. Let $\mathcal{C}$ be a stable ∞-category with a left complete t-structure, and let $X_\bullet$ be a semisimplicial object of the heart of $\mathcal{C}$. Proposition 1.2.4.5 implies that $X_\bullet$ admits a colimit $X \in \mathcal{C}_{\geq 0}$, whose homotopy groups $\pi_q X$ are given by the homologies of the unnormalized chain complex

$$\cdots \to \pi_0 X_2 \to \pi_0 X_1 \to \pi_0 X_0.$$

Corollary 1.2.4.11. *Let $\mathcal{C}$ be a stable ∞-category equipped with a left complete t-structure, whose heart is equivalent to (the nerve of) an abelian category $\mathcal{A}$. Let $X_\bullet$ be a semisimplicial object of $\mathcal{C}_{\geq 0}$, and assume that for every integer q the unnormalized chain complex*

$$\cdots \to \pi_q X_2 \to \pi_q X_1 \xrightarrow{\theta_q} \pi_q X_0$$

is an acyclic resolution of the object $A_q = \mathrm{coker}(\theta_q) \in \mathcal{A}$. Then:

(1) *There exists a geometric realization $X = |X_\bullet|$ in $\mathcal{C}$.*

(2) *The object X belongs to $\mathcal{C}_{\geq 0}$, and for $q \geq 0$ the canonical map $\pi_q X_0 \to \pi_q X$ induces an isomorphism $A_q \simeq \pi_q X$.*

Proof. Combine Proposition 1.2.4.5 with Variant 1.2.4.9. $\qquad\square$

Corollary 1.2.4.12. *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure which is both right and left complete, whose heart is equivalent to (the nerve of) an abelian category $\mathcal{A}$. Let $X_\bullet$ be a semisimplicial object of $\mathcal{C}$, and assume that for every integer $q \geq 0$ the unnormalized chain complex*

$$\cdots \to \pi_q X_2 \to \pi_q X_1 \xrightarrow{\theta_q} \pi_q X_0$$

is an acyclic resolution of the object $A_q = \mathrm{coker}(\theta_q) \in \mathcal{A}$. Then:

(1) *There exists a geometric realization $X = |X_\bullet|$ in $\mathcal{C}$.*

(2) *For every integer q, the canonical map $\pi_q X_0 \to \pi_q X$ induces an isomorphism $A_q \simeq \pi_q X$.*

Proof. Since $\mathcal{C}$ is right complete, we can write $X_\bullet$ as the colimit of a sequence of semisimplicial objects $\tau_{\geq -n} X_\bullet$. Using Corollary 1.2.4.11, we deduce that each of the semisimplicial objects $\tau_{\geq -n} X_\bullet$ admits a geometric realization $X(-n) \in \mathcal{C}_{\geq -n}$, whose homotopy group objects are given by

$$\pi_q X(-n) \simeq \begin{cases} A_q & \text{if } q \geq -n \\ 0 & \text{if } q < -n. \end{cases}$$

The right completeness of $\mathcal{C}$ shows that the sequence

$$X(0) \to X(-1) \to X(-2) \to \cdots$$

has a colimit $X \in \mathcal{C}$ such that, for each $n \geq 0$, the map $X(-n) \to X$ induces an equivalence $X(-n) \simeq \tau_{\geq -n} X$; in particular, we canonical isomorphisms $\pi_q X \simeq \pi_q X(-n) \simeq A_q$ for any $n \geq -q$. It follows from Lemma T.5.5.2.3 that we can identify X with a geometric realization $|X_\bullet|$. $\qquad\square$

We now turn to the proof of Theorem 1.2.4.1. Recall that if $\mathcal{C}$ is a stable ∞-category, then a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ is a pushout square if and only if it is a pullback square (Proposition 1.1.3.4). The main step in the proof of Theorem 1.2.4.1 is the following generalization of Proposition 1.1.3.4 to cubical diagrams of higher dimension:

Proposition 1.2.4.13. *Let $\mathcal{C}$ be a stable ∞-category, and let $\sigma : (\Delta^1)^n \to \mathcal{C}$ be a diagram. Then σ is a colimit diagram if and only if σ is a limit diagram.*

The proof will require a few preliminaries.

Lemma 1.2.4.14. *Let $\mathcal{C}$ be a stable ∞-category. A square*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow {\scriptstyle f'} & & \downarrow {\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

in $\mathcal{C}$ is a pullback if and only if the induced map $\alpha : \mathrm{cofib}(f') \to \mathrm{cofib}(f)$ is an equivalence.

Proof. Form an expanded diagram

$$\begin{array}{ccccc} X' & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow {\scriptstyle f'} & & \downarrow {\scriptstyle f} & & \downarrow \\ Y' & \longrightarrow & Y & \longrightarrow & \mathrm{cofib}(f) \end{array}$$

where the right square is a pushout. Since $\mathcal{C}$ is stable, the right square is also a pullback. Lemma T.4.4.2.1 implies that the left square is a pullback if and only if the outer square is a pullback, which is in turn equivalent to the assertion that α is an equivalence. $\qquad\square$

Lemma 1.2.4.15. *Let $\mathcal{C}$ be a stable ∞-category, let K be a simplicial set, and suppose that $\mathcal{C}$ admits K-indexed colimits. Let $\overline{\alpha} : K^\triangleright \times \Delta^1 \to \mathcal{C}$ be a natural transformation between a pair of diagrams $\overline{p}, \overline{q} : K^\triangleright \to \mathcal{C}$. Then $\overline{\alpha}$ is a colimit diagram if and only if $\mathrm{cofib}(\overline{\alpha}) : K^\triangleright \to \mathcal{C}$ is a colimit diagram.*

Proof. Let $p = \overline{p}|K$, $q = \overline{q}|K$, and $\alpha = \overline{\alpha}|K \times \Delta^1$. Since $\mathcal{C}$ admits K-indexed colimits, there exist colimit diagrams $\overline{p}', \overline{q}' : K^{\triangleright} \to \mathcal{C}$ extending p and q, respectively. We obtain a square

$$
\begin{array}{ccc}
\overline{p}' & \longrightarrow & \overline{p} \\
\downarrow & & \downarrow \\
\overline{q}' & \longrightarrow & \overline{q}
\end{array}
$$

in the ∞-category $\mathrm{Fun}(K^{\triangleright}, \mathcal{C})$. Let ∞ denote the cone point of $K^{\triangleright}$. Using Corollary T.4.2.3.10, we deduce that $\overline{\alpha}$ is a colimit diagram if and only if the induced square

$$
\begin{array}{ccc}
\overline{p}'(\infty) & \longrightarrow & \overline{p}(\infty) \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
\overline{q}'(\infty) & \longrightarrow & \overline{q}(\infty)
\end{array}
$$

is a pushout. According to Lemma 1.2.4.14, this is equivalent to the assertion that the induced map $\beta : \mathrm{cofib}(f') \to \mathrm{cofib}(f)$ is an equivalence. We conclude by observing that β can be identified with the natural map

$$\varinjlim(\mathrm{cofib}(\alpha)) \to \mathrm{cofib}(\overline{\alpha})(\infty),$$

which is an equivalence if and only if $\mathrm{cofib}(\overline{\alpha})$ is a colimit diagram. $\square$

Proof of Proposition 1.2.4.13. By symmetry, it will suffice to show that if σ is a colimit diagram, then σ is also a limit diagram. We work by induction on n. If $n = 0$, then we must show that every initial object of $\mathcal{C}$ is also final, which follows from the assumption that $\mathcal{C}$ has a zero object. If $n > 0$, then we may identify σ with a natural transformation $\alpha : \sigma' \to \sigma''$ in the ∞-category $\mathrm{Fun}((\Delta^1)^{n-1}, \mathcal{C})$. Assume that σ is a colimit diagram. Using Lemma 1.2.4.15, we deduce that $\mathrm{cofib}(\alpha)$ is a colimit diagram. Since $\mathrm{cofib}(\alpha) \simeq \mathrm{fib}(\alpha)[1]$, we conclude that $\mathrm{fib}(\alpha)$ is a colimit diagram. Applying the inductive hypothesis, we deduce that $\mathrm{fib}(\alpha)$ is a limit diagram. The dual of Lemma 1.2.4.15 now implies that σ is a limit diagram, as desired. $\square$

We now turn to the proof of Theorem 1.2.4.1 itself.

Lemma 1.2.4.16. *Fix $n \geq 0$, and let S be a subset of the open interval $(0, 1)$ of cardinality $\leq n$. Let Y be the set of all sequences of real numbers $0 \leq y_1 \leq \ldots \leq y_n \leq 1$ such that $S \subseteq \{y_1, \ldots, y_n\}$. Then Y is a contractible topological space.*

Proof. Let S have cardinality $m \leq n$, and let Z denote the set of sequences of real numbers $0 \leq z_1 \leq \ldots \leq z_{n-m} \leq 1$. Then Z is homeomorphic to a topological $(n-m)$-simplex. Moreover, there is a homeomorphism $f : Z \to Y$, which carries a sequence $\{z_i\}$ to a suitable reordering of the sequence $\{z_i\} \cup S$. $\square$

Lemma 1.2.4.17. *Let $n \geq 0$, let $\boldsymbol{\Delta}_{\leq n}$ denote the full subcategory of $\boldsymbol{\Delta}$ spanned by the objects $\{[m]\}_{0 \leq m \leq n}$, and let $\mathfrak{I}$ denote the full subcategory of $(\boldsymbol{\Delta}_{\leq n})_{/[n]}$ spanned by the injective maps $[m] \to [n]$. Then the induced map*

$$\mathrm{N}(\mathfrak{I}) \to \mathrm{N}(\boldsymbol{\Delta}_{\leq n})$$

is right cofinal.

Proof. Fix $m \leq n$, and let $\mathfrak{J}$ denote the category of diagrams

$$[m] \leftarrow [k] \xrightarrow{i} [n]$$

where i is injective. According to Theorem T.4.1.3.1, it will suffice to show that the simplicial set $\mathrm{N}(\mathfrak{J})$ is weakly contractible (for every $m \leq n$).

Let X denote the simplicial subset of $\Delta^m \times \Delta^n$ spanned by those nondegenerate simplices whose projection to Δ^n is also nondegenerate. Then $\mathrm{N}(\mathcal{J})$ can be identified with the barycentric subdivision of X. Consequently, it will suffice to show that the topological space $|X|$ is contractible. For this, we will show that the fibers of the map $\phi : |X| \to |\Delta^m|$ are contractible.

We will identify the topological m-simplex $|\Delta^m|$ with the set of all sequences of real numbers $0 \leq x_1 \leq \ldots \leq x_m \leq 1$. Similarly, we may identify points of $|\Delta^n|$ with sequences $0 \leq y_1 \leq \ldots \leq y_n \leq 1$. A pair of such sequences determines a point of X if and only if each x_i belongs to the set $\{0, y_1, \ldots, y_n, 1\}$. Consequently, the fiber of ϕ over the point $(0 \leq x_1 \leq \ldots \leq x_m \leq 1)$ can be identified with the set

$$Y = \{0 \leq y_1 \leq \ldots \leq y_n \leq 1 : \{x_1, \ldots, x_m\} \subseteq \{0, y_1, \ldots, y_n, 1\}\} \subseteq |\Delta^n|,$$

which is contractible (Lemma 1.2.4.16). □

Corollary 1.2.4.18. *Let $\mathcal{C}$ be a stable ∞-category, and let $F : \mathrm{N}(\boldsymbol{\Delta}_{\leq n})^{op} \to \mathcal{C}$ be a functor such that $F([m]) \simeq 0$ for all $m < n$. Then there is a canonical isomorphism $\varinjlim(F) \simeq X[n]$ in the homotopy category $h\mathcal{C}$, where $X = F([n])$.*

Proof. Let $\mathcal{J}$ be as in Lemma 1.2.4.17, let G'' denote the composition $\mathrm{N}(\mathcal{J})^{op} \to \mathrm{N}(\boldsymbol{\Delta}_{\leq n})^{op} \xrightarrow{F} \mathcal{C}$, and let G denote the constant map $\mathrm{N}(\mathcal{J})^{op} \to \mathcal{C}$ taking the value X. Let $\mathcal{J}_0$ denote the full subcategory of $\mathcal{J}$ obtained by deleting the final object. There is a canonical map $\alpha : G \to G''$, and $G' = \mathrm{fib}(\alpha)$ is a left Kan extension of $G' | \mathrm{N}(\mathcal{J}_0)^{op}$. We obtain a fiber sequence

$$\varinjlim(G') \to \varinjlim(G) \to \varinjlim(G'')$$

in the homotopy category $h\mathcal{C}$. Lemma 1.2.4.17 yields an equivalence $\varinjlim(F) \simeq \varinjlim(G'')$, and Lemma T.4.3.2.7 implies the existence of an equivalence $\varinjlim(G') \simeq \varinjlim(G' | \mathrm{N}(\mathcal{J}_0)^{op})$.

We now observe that the simplicial set $\mathrm{N}(\mathcal{J})^{op}$ can be identified with the barycentric subdivision of the standard n-simplex Δ^n, and that $\mathrm{N}(\mathcal{J}_0)^{op}$ can be identified with the barycentric subdivision of its boundary $\partial \Delta^n$. It follows (see §T.4.4.4) that we may identify the map $\varinjlim(G') \to \varinjlim(G)$ with the map $\beta : X \otimes (\partial \Delta^n) \to X \otimes \Delta^n$. The cofiber of β is canonically isomorphic (in $h\mathcal{C}$) to the n-fold suspension $X[n]$ of X. □

Lemma 1.2.4.19. *Let $\mathcal{C}$ be a stable ∞-category, let $n \geq 0$, and let $F : \mathrm{N}(\boldsymbol{\Delta}_{+,\leq n})^{op} \to \mathcal{C}$ be a functor (here $\boldsymbol{\Delta}_{+,\leq n}$ denotes the full subcategory of $\boldsymbol{\Delta}_+$ spanned by the objects $\{[k]\}_{-1 \leq k \leq n}$). The following conditions are equivalent:*

(i) *The functor F is a left Kan extension of $F | \mathrm{N}(\boldsymbol{\Delta}_{\leq n})^{op}$.*

(ii) *The functor F is a right Kan extension of $F | \mathrm{N}(\boldsymbol{\Delta}_{+,\leq n-1})^{op}$.*

Proof. Condition (ii) is equivalent to the assertion that the composition

$$F' : \mathrm{N}(\boldsymbol{\Delta}^{op}_{+,\leq n-1})^{\triangleleft}_{[n]/} \to \mathrm{N}(\boldsymbol{\Delta}^{op}_{+,\leq n})_{[n]/} \xrightarrow{F} \mathcal{C}$$

is a limit diagram. Let $\mathcal{J}$ denote the full subcategory of $\mathrm{N}(\boldsymbol{\Delta}^{op}_{+,\leq n-1})_{[n]/}$ spanned by those maps $[i] \to [n]$ which are *injective*. The inclusion $\mathcal{J} \hookrightarrow \mathrm{N}(\boldsymbol{\Delta}^{op}_{+,\leq n})_{[n]/}$ admits a right adjoint, and is therefore right cofinal. Consequently, condition (ii) is equivalent to the requirement that the restriction $F' | \mathcal{J}^{\triangleleft}$ is a limit diagram. Since $\mathcal{J}^{\triangleleft}$ is isomorphic to $(\Delta^1)^{n+1}$, Proposition 1.2.4.13 asserts that $F' | \mathcal{J}^{\triangleleft}$ is a limit diagram if and only if $F' | \mathcal{J}^{\triangleleft}$ is a colimit diagram. In view of Lemma 1.2.4.17, $F' | \mathcal{J}^{\triangleleft}$ is a colimit diagram if and only if F is a colimit diagram, which is equivalent to (i). □

Proof of Theorem 1.2.4.1. Our first step is to describe the desired equivalence in more precise terms. Let $\mathcal{J}_+$ denote the full subcategory of $\mathrm{N}(\mathbf{Z}_{\geq 0}) \times \mathrm{N}(\boldsymbol{\Delta}_+)^{op}$ spanned by those pairs $(n, [m])$, where $m \leq n$, and let $\mathcal{J}$ be the full subcategory of $\mathcal{J}_+$ spanned by those pairs $(n, [m])$ where $0 \leq m \leq n$. We observe that there is

a natural projection $p : \mathcal{I} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$, and a natural embedding $i : \mathrm{N}(\mathbf{Z}_{\geq 0}) \to \mathcal{I}_+$, which carries $n \geq 0$ to the object $(n, [-1])$.

Let $\mathrm{Fun}^0(\mathcal{I}, \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ spanned by those functors $F : \mathcal{I} \to \mathcal{C}$ such that, for every $s \leq m \leq n$, the image under F of the natural map $(m, [s]) \to (n, [s])$ is an equivalence in $\mathcal{C}$. Let $\mathrm{Fun}^0(\mathcal{I}_+, \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{I}_+, \mathcal{C})$ spanned by functors $F_+ : \mathcal{I}_+ \to \mathcal{C}$ such that $F = F_+ | \mathcal{I}$ belongs to $\mathrm{Fun}^0(\mathcal{I}, \mathcal{C})$, and F_+ is a left Kan extension of F. Composition with p, composition with i, and restriction from $\mathcal{I}_+$ to $\mathcal{I}$ yields a diagram of ∞-categories

$$\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{C}) \xrightarrow{G} \mathrm{Fun}^0(\mathcal{I}, \mathcal{C}) \xleftarrow{G'} \mathrm{Fun}^0(\mathcal{I}_+, \mathcal{C}) \xrightarrow{G''} \mathrm{Fun}(\mathrm{N}(\mathbf{Z}_{\geq 0}), \mathcal{C}).$$

We will prove that G, G', and G'' are equivalences of ∞-categories.

To show that G is an equivalence of ∞-categories, we let $\mathcal{I}^{\leq k}$ denote the full subcategory of $\mathcal{I}$ spanned by pairs $(n, [m])$ where $m \leq n \leq k$, and let $\mathcal{I}^k$ denote the full subcategory of $\mathcal{I}$ spanned by those pairs $(n, [m])$ where $m \leq n = k$. Then the projection p restricts to an equivalence $\mathcal{I}^k \to \mathrm{N}(\boldsymbol{\Delta}_{\leq k})^{op}$. Let $\mathrm{Fun}^0(\mathcal{I}^{\leq k}, \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{I}^{\leq k}, \mathcal{C})$ spanned by those functors $F : \mathcal{I}^{\leq k} \to \mathcal{C}$ such that, for every $s \leq m \leq n \leq k$, the image under F of the natural map $(m, [s]) \to (n, [s])$ is an equivalence in $\mathcal{C}$. We observe that this is equivalent to the condition that F be a right Kan extension of $F | \mathcal{I}^k$. Using Proposition T.4.3.2.15, we deduce that the restriction map $r : \mathrm{Fun}^0(\mathcal{I}^{\leq k}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{I}^k, \mathcal{C})$ is an equivalence of ∞-categories. Composition with p induces a functor $G_k : \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_{\leq k})^{op}, \mathcal{C}) \to \mathrm{Fun}^0(\mathcal{I}^{\leq k}, \mathcal{C})$ which is a section of r. It follows that G_k is an equivalence of ∞-categories. We can identify G with the homotopy inverse limit of the functors $\varprojlim(G_k)$, so that G is also an equivalence of ∞-categories.

The fact that G' is an equivalence of ∞-categories follows immediately from Proposition T.4.3.2.15, since for each $n \geq 0$ the simplicial set $\mathcal{I}_{/(n,[-1])}$ is finite and $\mathcal{C}$ admits finite colimits.

We now show that G'' is an equivalence of ∞-categories. Let $\mathcal{I}_+^{\leq k}$ denote the full subcategory of $\mathcal{I}_+$ spanned by pairs $(n, [m])$ where either $m \leq n \leq k$ or $m = -1$. We let $\mathcal{D}(k)$ denote the full subcategory of $\mathrm{Fun}(\mathcal{I}_+^{\leq k}, \mathcal{C})$ spanned by those functors $F : \mathcal{I}_+^{\leq k} \to \mathcal{C}$ with the following pair of properties:

(i) For every $0 \leq s \leq m \leq n \leq k$, the image under F of the natural map $(m, [s]) \to (n, [s])$ is an equivalence in $\mathcal{C}$.

(ii) For every $n \leq k$, F is a left Kan extension of $F | \mathcal{I}^{\leq k}$ at $(n, [-1])$.

Then $\mathrm{Fun}^0(\mathcal{I}_+, \mathcal{C})$ is the inverse limit of the tower of restriction maps

$$\ldots \to \mathcal{D}(1) \to \mathcal{D}(0) \to \mathcal{D}(-1) = \mathrm{Fun}(\mathrm{N}(\mathbf{Z}_{\geq 0}), \mathcal{C}).$$

To complete the proof, we will show that for each $k \geq 0$, the restriction map $\mathcal{D}(k) \to \mathcal{D}(k-1)$ is a trivial Kan fibration.

Let $\mathcal{I}_0^{\leq k}$ be the full subcategory of $\mathcal{I}_+^{\leq k}$ obtained by removing the object $(k, [k])$, and let $\mathcal{D}'(k)$ be the full subcategory of $\mathrm{Fun}(\mathcal{I}_0^{\leq k}, \mathcal{C})$ spanned by those functors F which satisfy condition (i) and satisfy (ii) for $n < k$. We have restriction maps

$$\mathcal{D}(k) \xrightarrow{\theta} \mathcal{D}'(k) \xrightarrow{\theta'} \mathcal{D}(k-1).$$

We observe that a functor $F : \mathcal{I}_0^{\leq k} \to \mathcal{C}$ belongs to $\mathcal{D}'(k)$ if and only if $F | \mathcal{I}_+^{\leq k-1}$ belongs to $\mathcal{D}(k-1)$ and F is a left Kan extension of $F | \mathcal{I}_+^{\leq k-1}$. Using Proposition T.4.3.2.15, we conclude that θ' is a trivial Kan fibration.

We will prove that θ is a trivial Kan fibration by a similar argument. According to Proposition T.4.3.2.15, it will suffice to show that a functor $F : \mathcal{I}_+^{\leq k} \to \mathcal{C}$ belongs to $\mathcal{D}(k)$ if and only if $F | \mathcal{I}_0^{\leq k}$ belongs to $\mathcal{D}'(k)$ and F is a right Kan extension of $F | \mathcal{I}_0^{\leq k}$. This follows immediately from Lemma 1.2.4.19 and the observation that the inclusion $\mathcal{I}^k \subseteq \mathcal{I}^{\leq k}$ is left cofinal. $\square$

1.3 Homological Algebra and Derived Categories

Homological algebra provides a rich supply of examples of stable ∞-categories. Suppose that $\mathcal{A}$ is an abelian category with enough projective objects. In §1.3.2, we will explain how to associate to $\mathcal{A}$ an ∞-category $\mathcal{D}^-(\mathcal{A})$, which we call the *derived ∞-category of $\mathcal{A}$*, whose objects can be identified with (right-bounded) chain complexes with values in $\mathcal{A}$. The ∞-category $\mathcal{D}^-(\mathcal{A})$ is stable, and its homotopy category $\mathrm{h}\mathcal{D}^-(\mathcal{A})$ can be identified (as a triangulated category) with the usual derived category of $\mathcal{A}$ (as defined, for example, in [159]). Our construction of $\mathcal{D}^-(\mathcal{A})$ uses a variant of the homotopy coherent nerve which is defined for differential graded categories, which we describe in §1.3.1.

As we mentioned in §1.2, the stable ∞-category $\mathcal{D}^-(\mathcal{A})$ is equipped with a t-structure, and there is a canonical equivalence of abelian categories $\mathcal{A} \to \mathcal{D}^-(\mathcal{A})^\heartsuit$. In §1.3.3, we will show that $\mathcal{D}^-(\mathcal{A})$ is universal with respect to these properties. More precisely, if $\mathcal{C}$ is any stable ∞-category equipped with a left-complete t-structure, then any right exact functor $\mathcal{A} \to \mathcal{C}^\heartsuit$ extends (in an essentially unique way) to an exact functor $\mathcal{D}^-(\mathcal{A}) \to \mathcal{C}$ (Proposition 1.3.3.12). This observation can be regarded as providing an abstract approach to the theory of derived functors (see Example 1.3.3.4).

By an entirely parallel discussion, if $\mathcal{A}$ is abelian category with enough *injective* objects, we can associate to $\mathcal{A}$ a left-bounded derived ∞-category $\mathcal{D}^+(\mathcal{A})$. This case is in some sense more fundamental: a theorem of Grothendieck asserts that if $\mathcal{A}$ is a presentable abelian category in which filtered colimits are exact, then $\mathcal{A}$ has enough injective objects (Corollary 1.3.5.7). In §1.3.5, we will explain how to associate to such an abelian category an *unbounded* derived ∞-category $\mathcal{D}(\mathcal{A})$, which contains $\mathcal{D}^+(\mathcal{A})$ as a full subcategory (as well as $\mathcal{D}^-(\mathcal{A})$, in case $\mathcal{A}$ has enough projective objects). The ∞-category $\mathcal{D}(\mathcal{A})$ can be realized as the underlying ∞-category of a combinatorial model category $\mathbf{A}$ (whose underlying category is the category of unbounded chain complexes in $\mathcal{A}$). Here some words of caution are in order: $\mathbf{A}$ is not a simplicial model category in an obvious way, so that the results of [97] do not apply to $\mathbf{A}$ directly. We therefore devote §1.3.4 to a general discussion of ∞-categories obtained from arbitrary model categories (or, more generally, categories equipped with a distinguished class of *weak equivalences* or *quasi-isomorphisms*) which are not assumed to be simplicial.

In §1.3.6, we specialize to the case where $\mathcal{A}$ is a *locally Noetherian* abelian category. In this case, there is an enlargement of the ∞-category $\mathcal{D}(\mathcal{A})$ which is useful in many applications, whose objects are arbitrary chain complexes Q_* of injective objects of $\mathcal{A}$. Following Krause, we will show that this ∞-category is compactly generated, and that its compact objects can be identified with bounded chain complexes whose homologies are Noetherian objects of $\mathcal{A}$ (Theorem 1.3.6.7).

Remark 1.3.0.1. The derived category of an abelian category was introduced in Verdier's thesis ([154]). A good introductory reference is [159].

1.3.1 Nerves of Differential Graded Categories

Let $\mathcal{A}$ be an additive category. Then the collection of chain complexes with values in $\mathcal{A}$ can be organized into an ∞-category $\mathcal{C}$, which may described informally as follows:

- The objects of $\mathcal{C}$ are chain complexes

$$\cdots \to M_1 \to M_0 \to M_{-1} \to \cdots$$

  with values in $\mathcal{C}$.

- Given objects $M_*, N_* \in \mathcal{C}$, morphisms from M_* to N_* are given by chain complex homomorphisms $f : M_* \to N_*$.

- Given a pair of morphisms $f, g : M_* \to N_*$ in $\mathcal{C}$, 2-morphisms from f to g are given by chain homotopies: that is, collections of maps $h_n : M_n \to N_{n+1}$ satisfying $d \circ h_n + h_{n-1} \circ d = f - g$.

- $\cdots$

To make this description precise, we could proceed in several steps:

(1) Let $\mathrm{Ch}(\mathcal{A})$ denote the ordinary category introduced in Definition 1.2.3.1: the objects of $\mathrm{Ch}(\mathcal{A})$ are chain complexes with values in $\mathcal{A}$, and the morphisms in $\mathrm{Ch}(\mathcal{A})$ are morphisms of chain complexes.

(2) To every pair of objects $M_*, N_* \in \mathrm{Ch}(\mathcal{A})$, we can associate a chain complex of abelian groups $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, N_*)$, whose homology groups $\mathrm{H}_m(\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, N_*))$ are isomorphic to the group of chain homotopy classes of maps from M_* into the shifted chain complex $N_*[m]$ (see Definition 1.3.2.1). By means of this observation, we can regard $\mathrm{Ch}(\mathcal{A})$ as enriched over the category $\mathrm{Ch}(\mathcal{A}b)$ of chain complexes of abelian groups.

(3) The truncation functor $\tau_{\geq 0}$ is a right-lax monoidal functor from the category $\mathrm{Ch}(\mathcal{A}b)$ to the category $\mathrm{Ch}(\mathcal{A}b)_{\geq 0}$ of nonnegative graded chain complexes of abelian groups. Applying $\tau_{\geq 0}$ objectwise to the morphism objects in $\mathrm{Ch}(\mathcal{A})$, we can regard $\mathrm{Ch}(\mathcal{A})$ as enriched over the category $\mathrm{Ch}(\mathcal{A}b)_{\geq 0}$ of nonnegatively graded chain complexes.

(4) The Dold-Kan correspondence supplies an equivalence of $\mathrm{Ch}(\mathcal{A}b)_{\geq 0}$ with the category of simplicial abelian groups (Theorem 1.2.3.7). By means of the Alexander-Whitney construction, we can regard DK as a right-lax monoidal functor. We may therefore regard the category $\mathrm{Ch}(\mathcal{A})$ as enriched over the category of $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A}b)$ of simplicial abelian groups.

(5) Using the forgetful functor from simplicial abelian groups to simplicial sets, we can regard $\mathrm{Ch}(\mathcal{A})$ as a simplicial category. Since every simplicial abelian group is automatically a Kan complex (Corollary 1.3.2.12), $\mathrm{Ch}(\mathcal{A})$ is automatically fibrant when viewed as a simplicial category.

(6) Applying the homotopy coherent nerve construction (Definition T.1.1.5.5) to the simplicial category $\mathrm{Ch}(\mathcal{A})$, we obtain an ∞-category $\mathrm{N}(\mathrm{Ch}(\mathcal{A}))$.

However, this turns out to be unnecessarily complicated. In this section, we will explain how to eliminate steps (3), (4), and (5). That is, we describe how to proceed directly from a category $\mathcal{E}$ enriched over $\mathrm{Ch}(\mathcal{A}b)$ to an ∞-category $\mathrm{N}_{\mathrm{dg}}(\mathcal{E})$, which we call the *differential graded nerve* of $\mathcal{E}$. Our main result is that the result of this procedure is canonically equivalent (though not isomorphic) to the homotopy coherent nerve of simplicial category obtained by applying steps (3), (4), and (5) (Proposition 1.3.1.17).

We begin with some general definitions.

Definition 1.3.1.1. Let k be a commutative ring. A *differential graded category* $\mathcal{C}$ over k consists of the following data:

- A collection $\{X, Y, \ldots\}$, whose elements are called the *objects* of $\mathcal{C}$.

- For every pair of objects X and Y, a chain complex of k-modules

$$\cdots \to \mathrm{Map}_{\mathcal{C}}(X, Y)_1 \to \mathrm{Map}_{\mathcal{C}}(X, Y)_0 \to \mathrm{Map}_{\mathcal{C}}(X, Y)_{-1} \to \cdots,$$

which we will denote by $\mathrm{Map}_{\mathcal{C}}(X, Y)_*$.

- For every triple of objects X, Y, and Z, a composition map

$$\mathrm{Map}_{\mathcal{C}}(Y, Z)_* \otimes_k \mathrm{Map}_{\mathcal{C}}(X, Y)_* \to \mathrm{Map}_{\mathcal{C}}(X, Z)_*,$$

which we can identify with a collection of k-bilinear maps

$$\circ : \mathrm{Map}_{\mathcal{C}}(Y, Z)_p \times \mathrm{Map}_{\mathcal{C}}(X, Y)_q \to \mathrm{Map}_{\mathcal{C}}(X, Z)_{p+q}$$

satisfying the Leibniz rule $d(g \circ f) = dg \circ f + (-1)^p g \circ df$.

- For each object $X \in \mathcal{C}$, an *identity morphism* $\mathrm{id}_X \in \mathrm{Map}_{\mathcal{C}}(X, X)_0$ such that

$$g \circ \mathrm{id}_X = f \qquad \mathrm{id}_X \circ f = f$$

for all $f \in \mathrm{Map}_{\mathcal{C}}(Y, X)_p$, $g \in \mathrm{Map}_{\mathcal{C}}(X, Y)_q$.

The composition law is required to be associative in the following sense: for every triple $f \in \mathrm{Map}_{\mathcal{C}}(W, X)_p$, $g \in \mathrm{Map}_{\mathcal{C}}(X, Y)_q$, and $h \in \mathrm{Map}_{\mathcal{C}}(Y, Z)_r$, we have

$$(h \circ g) \circ f = h \circ (g \circ f).$$

In the special case where $k = \mathbf{Z}$ is the ring of integers, we will refer to a differential graded category over k simply as a *differential graded category*.

Remark 1.3.1.2. Let $\mathcal{C}$ be a differential graded category. For every object X of $\mathcal{C}$, the identity morphism id_X is a cycle: that is, $d\,\mathrm{id}_X = 0$. This follows from the Leibniz rule

$$d\,\mathrm{id}_X = d(\mathrm{id}_X \circ \mathrm{id}_X) = (d\,\mathrm{id}_X) \circ \mathrm{id}_X + \mathrm{id}_X \circ(d\,\mathrm{id}_X) = 2d\,\mathrm{id}_X \,.$$

Remark 1.3.1.3. Let $\phi : k \to k'$ be a homomorphism of commutative rings. Then every differential graded category over k' can be regarded as a differential graded category over k by neglect of structure. In particular, every differential graded category over a commutative ring k can be regarded as a differential graded category over the ring of integers $\mathbf{Z}$.

Remark 1.3.1.4. If k is a commutative ring, we can identify differential graded categories over k with categories enriched over the category $\mathrm{Ch}(k)$ of chain complexes of k-modules. In particular, every differential graded category $\mathcal{C}$ can be regarded as an ordinary category, with morphisms given by $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \{f \in \mathrm{Map}_{\mathcal{C}}(X, Y)_0 : df = 0\}$.

Remark 1.3.1.5. Let $\mathcal{C}$ be a differential graded category. There is another category canonically associated to $\mathcal{C}$, called the *homotopy category* of $\mathcal{C}$ and denoted by $\mathrm{h}\mathcal{C}$. It may be defined precisely as follows:

- The objects of $\mathrm{h}\mathcal{C}$ are the objects of $\mathcal{C}$.

- The morphisms in $\mathrm{h}\mathcal{C}$ are given by the formula

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) = \mathrm{H}_0(\mathrm{Map}_{\mathcal{C}}(X, Y)_*) = \mathrm{coker}(d : \mathrm{Map}_{\mathcal{C}}(X, Y)_1 \to \mathrm{Hom}_{\mathcal{C}}(X, Y)).$$

That is, a morphism in $\mathrm{h}\mathcal{C}$ is given by an equivalence class $[f]$ of 0-cycles $f \in \mathrm{Map}_{\mathcal{C}}(X, Y)_0$, where two 0-cycles $f, g \in \mathrm{Map}_{\mathcal{C}}(X, Y)_0$ are regarded as equivalent if there exists an element $z \in \mathrm{Map}_{\mathcal{C}}(X, Y)_1$ with $dz = f - g$.

- Composition of morphisms in $\mathrm{h}\mathcal{C}$ is characterized by the formula $[f \circ g] = [f] \circ [g]$.

Construction 1.3.1.6. Let $\mathcal{C}$ be a differential graded category. We will associate to $\mathcal{C}$ a simplicial set $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})$, which we call the *differential graded nerve* of $\mathcal{C}$. For each $n \geq 0$, we define $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})_n \simeq \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^n, \mathrm{N}_{\mathrm{dg}}(\mathcal{C}))$ to be the set of all ordered pairs $(\{X_i\}_{0 \leq i \leq n}, \{f_I\})$, where:

(a) For $0 \leq i \leq n$, X_i denotes an object of the differential graded category $\mathcal{C}$.

(b) For every subset $I = \{i_- < i_m < i_{m-1} < \cdots < i_1 < i_+\}$ with $m \geq 0$, f_I is an element of the abelian group $\mathrm{Map}_{\mathcal{C}}(X_{i_-}, X_{i_+})_m$, satisfying the equation

$$df_I = \sum_{1 \leq j \leq m} (-1)^j (f_{I-\{i_j\}} - f_{\{i_j < \ldots < i_m < i_+\}} \circ f_{\{i_- < i_1 < \ldots < i_j\}}). \qquad (1.1)$$

If $\alpha : [m] \to [n]$ is a nondecreasing function, then the induced map $N_{dg}(\mathcal{C})_n \to N_{dg}(\mathcal{C})_m$ is given by

$$(\{X_i\}_{0 \le i \le n}, \{f_I\}) \mapsto (\{X_{\alpha(j)}\}_{0 \le j \le m}, \{g_J\}),$$

where $g_J = \begin{cases} f_{\alpha(J)} & \text{if } \alpha|J \text{ is injective} \\ \text{id}_{X_i} & \text{if } J = \{j, j'\} \text{ with } \alpha(j) = \alpha(j') = i \\ 0 & \text{otherwise.} \end{cases}$

Remark 1.3.1.7. The theory of differential graded categories can be regarded as a special case of the more general theory of A_∞-*categories* (see [131] for an exposition). Construction 1.3.1.6 (and many of the results proven below) can be generalized to the case of an A_∞-category $\mathcal{C}$: for this, one needs to replace equation 1.1 by a more elaborate version, involving the higher-order multiplications on $\mathcal{C}$.

Example 1.3.1.8. Let $\mathcal{C}$ be a differential graded category. Then:

- A 0-simplex of $N_{dg}(\mathcal{C})$ is simply an object of $\mathcal{C}$.

- A 1-simplex of $N_{dg}(\mathcal{C})$ is a morphism of $\mathcal{C}$: that is, a pair of objects $X, Y \in \mathcal{C}$ together with an element $f \in \text{Map}_{\mathcal{C}}(X, Y)_0$ satisfying $df = 0$.

- A 2-simplex of $N_{dg}(\mathcal{C})$ consists of a triple of objects $X, Y, Z \in \mathcal{C}$, a triple of morphisms

$$f \in \text{Map}_{\mathcal{C}}(X, Y)_0 \qquad g \in \text{Map}_{\mathcal{C}}(Y, Z)_0 \qquad h \in \text{Map}_{\mathcal{C}}(X, Z)_0$$

  satisfying $df = dg = dh = 0$, together with an element $z \in \text{Map}_{\mathcal{C}}(X, Z)_1$ with $dz = h - (g \circ f)$.

Remark 1.3.1.9. Let $\mathcal{C}$ be a differential graded category and let $\mathcal{C}_0$ denote its underlying ordinary category (see Remark 1.3.1.4). Then the simplicial set $N(\mathcal{C}_0)$ is isomorphic to the simplicial subset of the differential graded nerve $N_{dg}(\mathcal{C})$ whose n-simplices are given by pairs $(\{X_i\}_{0 \le i \le n}, \{f_I\})$, where $f_I = 0$ whenever the set I has more than two elements. In particular, the map $N(\mathcal{C}_0) \to N_{dg}(\mathcal{C})$ is bijective on n-simplices for $n \le 1$.

Proposition 1.3.1.10. *Let $\mathcal{C}$ be a differential graded category. Then the simplicial set $N_{dg}(\mathcal{C})$ is an ∞-category.*

Proof. Suppose we are given $0 < j < n$ and a map $\phi_0 : \Lambda^n_j \to N_{dg}(\mathcal{C})$; we wish to show that ϕ_0 can be extended to an n-simplex of $N_{dg}(\mathcal{C})$. Unwinding the definitions, we see that ϕ_0 can be identified with the data of a pair $(\{X_i\}_{0 \le i \le n}, \{f_I\})$, where $\{X_i\}_{0 \le i \le n}$ is a collection of objects of $\mathcal{C}$ and $f_I \in \text{Map}_{\mathcal{C}}(X_{i_0}, X_{i_k})_{k-1}$ is defined for every subset $I = \{i_- < i_m < \ldots < i_1 < i_+\} \subseteq [n]$ with $m \ge 0$ and $I \ne [n], [n] - \{j\}$, satisfying the equation 1.1 of Construction 1.3.1.6. This data extends uniquely to an n-simplex $(\{X_i\}, \{f_I\})$ of $N_{dg}(\mathcal{C})$ satisfying $f_{[n]} = 0$, if we set

$$f_{[n]-\{j\}} = \sum_{0 < p < j} (-1)^{p-j} f_{\{p, p+1, \ldots, n\}} \circ f_{\{0, \ldots, p\}} - \sum_{0 < p < k, p \ne j} (-1)^{p-j} f_{[n] - \{p\}}$$

$$f_{[n]} = 0.$$

$\square$

Remark 1.3.1.11. Let $\mathcal{C}$ be a differential graded category. Then the homotopy category $h\mathcal{C}$ of Remark 1.3.1.5 is canonically isomorphic to the homotopy category $hN_{dg}(\mathcal{C})$ of the ∞-category $N_{dg}(\mathcal{C})$. To see this, let $\mathcal{C}_0$ be the underlying category of $\mathcal{C}$. Remark 1.3.1.9 supplies a map

$$\mathcal{C}_0 \simeq hN(\mathcal{C}_0) \to hN_{dg}(\mathcal{C})$$

which is bijective on objects and surjective on morphisms. It therefore suffices to show that for every pair of objects $X, Y \in \mathcal{C}$, the induced equivalence relation on $\text{Hom}_{\mathcal{C}_0}(X, Y)$ agrees with the relation of homology (that is, two cycles $f, g \in \text{Map}_{\mathcal{C}}(X, Y)_0$ are homologous if and only if $f - g = dz$ for some $z \in \text{Map}_{\mathcal{C}}(X, Y)_1$). This follows immediately from the description of the 2-simplices in $N_{dg}(\mathcal{C})$ supplied by Example 1.3.1.8.

Remark 1.3.1.12. Let $\mathcal{D}$ be an ∞-category. Recall that for $X, Y \in \mathcal{D}$, the mapping space $\operatorname{Hom}_{\mathcal{C}}^{\mathrm{R}}(X, Y)$ is defined as the simplicial set whose n-simplices are maps $\Delta^{n+1} \to \mathcal{D}$ which carry the simplicial subset $\Delta^n \subseteq \Delta^{n+1}$ to the vertex X and the opposite vertex $\{n+1\} \subseteq \Delta^{n+1}$ to the vertex Y. Let $\mathcal{C}$ be a differential graded category containing objects X and Y. Unwinding the definitions, we see that an n-simplex of $\operatorname{Hom}_{\mathrm{N}_{\mathrm{dg}}(\mathcal{C})}^{\mathrm{R}}(X, Y)$ is determined by specifying, for every subset $I = \{i_m < i_{m-1} < \ldots < i_0 < n+1\} \subseteq [n+1]$ with $m \geq 0$, an element $f_I \in \operatorname{Map}_{\mathcal{C}}(X, Y)_m$, satisfying the equations

$$-df_I = \sum_{0 \leq j \leq m} (-1)^j f_{I-\{i_j\}}.$$

After a change of signs, we can identify n-simplices of $\operatorname{Hom}_{\mathrm{N}_{\mathrm{dg}}(\mathcal{C})}^{\mathrm{R}}(X, Y)$ with chain complex homomorphisms $N_*(\mathbf{Z}\Delta^n) \to \operatorname{Map}_{\mathcal{C}}(X, Y)_*$. This identification is functorial in Δ^n, and (using Lemma 1.2.3.12) yields an isomorphism of simplicial sets

$$\operatorname{Hom}_{\mathrm{N}_{\mathrm{dg}}(\mathcal{C})}^{\mathrm{R}}(X, Y) \simeq \mathrm{DK}_\bullet(\tau_{\geq 0} \operatorname{Map}_{\mathcal{C}}(X, Y)),$$

where $\mathrm{DK}_\bullet$ denotes the functor given by the Dold-Kan correspondence (see Construction 1.2.3.5).

Our next goal is to compare the formation of differential graded nerves of differential graded categories (Construction 1.3.1.6) with the formation of homotopy coherent nerves of simplicial categories (§T.1.1.5).

Construction 1.3.1.13. Let $\mathcal{A}b$ denote the category of abelian groups. Using the Alexander-Whitney construction, we see that the composite functor

$$\operatorname{Ch}(\mathcal{A}b) \overset{\tau_{\geq 0}}{\Rightarrow} \operatorname{Ch}(\mathcal{A}b)_{\geq 0} \xrightarrow{\mathrm{DK}} \operatorname{Fun}(\mathbf{\Delta}^{op}, \mathcal{A}b) \to \operatorname{Fun}(\mathbf{\Delta}^{op}, \mathcal{S}et) = \mathcal{S}et_\Delta$$

is right-lax monoidal (see Example 1.2.3.26). In particular, every differential graded category $\mathcal{C}$ determines a simplicial category $\mathcal{C}_\Delta$, which may be described more concretely as follows:

- The objects of $\mathcal{C}_\Delta$ are the objects of $\mathcal{C}$.

- Given objects $X, Y \in \mathcal{C}$, the mapping space $\operatorname{Map}_{\mathcal{C}_\Delta}(X, Y)$ is the underlying simplicial set of the simplicial abelian group $\mathrm{DK}_\bullet(\tau_{\geq 0} \operatorname{Map}_{\mathcal{C}}(X, Y)_*)$.

We will refer to $\mathcal{C}_\Delta$ as the *underlying simplicial category* of $\mathcal{C}$.

Remark 1.3.1.14. For every differential graded category $\mathcal{C}$, the underlying simplicial category $\mathcal{C}_\Delta$ is automatically fibrant (see Remark 1.2.3.14).

Notation 1.3.1.15. Let $\mathcal{C}$ be a simplicial category. Recall that the simplicial set $\mathrm{N}(\mathcal{C})$ is defined by the formula

$$\mathrm{N}(\mathcal{C})_n = \operatorname{Hom}_{\mathcal{C}at_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C}),$$

where $\mathfrak{C}[\Delta^n]$ denotes the category with set of objects $\{0, 1, \ldots, n\}$ and

$$\operatorname{Map}_{\mathfrak{C}[\Delta^n]}(i, j) = \begin{cases} \emptyset & \text{if } i > j \\ \mathrm{N}(P_{i,j}) & \text{if } i \leq j, \end{cases}$$

where $P_{i,j}$ is the partially ordered set consisting of those subsets $S \subseteq [n]$ having least element i and greatest element j. Composition in $\mathfrak{C}[\Delta^n]$ is induced by the maps $P_{i,j} \times P_{j,k} \to P_{i,k}$ given by $(S, T) \mapsto S \cup T$.

Let $I = \{i_- < i_m < \ldots < i_1 < i_+\}$ be a subset of $[n]$ and let $\sigma \in \Sigma_m$ be a permutation of the set $\{1, \ldots, m\}$. We let $\tau_{I,\sigma}$ denote the m-simplex of $\operatorname{Map}_{\mathfrak{C}[\Delta^n]}(i_-, i_+)$ given by the chain

$$\{i_-, i_+\} \subset \{i_-, i_{\sigma(1)}, i_+\} \subset \{i_-, i_{\sigma(1)}, i_{\sigma(2)}, i_+\} \subset \cdots \subset \{i_-, i_{\sigma(1)}, \ldots, i_{\sigma(m)}, i_+\}$$

in the partially ordered set P_{i_-, i_+}.

Construction 1.3.1.16. Let $\mathcal{C}$ be a differential graded category. For every n-simplex $\tau \in \mathrm{Map}_{\mathcal{C}_\Delta}(X, Y)_n$, we let $[\tau] \in \mathrm{Map}_{\mathcal{C}}(X, Y)_n$ denote the corresponding n-chain. Let $\alpha : \mathfrak{C}[\Delta^n] \to \mathcal{C}_\Delta$ be a functor of simplicial categories. For every subset $I = \{i_- < i_m < \ldots < i_1 < i_+\} \subseteq [n]$, we set

$$f_I = \sum_{\sigma \in \Sigma_m} (-1)^\sigma [\alpha(\tau_{I,\sigma})] \in \mathrm{Map}_{\mathcal{C}}(\alpha(i_-), \alpha(i_+))_m$$

where $\tau_{I,\sigma}$ is defined as in Notation 1.3.1.15 and $(-1)^\sigma$ denotes the sign of the permutation σ. A straightforward calculation shows that

$$df_I = \sum_{1 \leq j \leq m} (-1)^j (f_{\{i_j < \ldots < i_m < i_+\}} \circ f_{\{i_- < i_1 < \ldots < i_j\}} - f_{I - \{i_j\}}),$$

so that we can regard the pair $(\{\alpha(i)\}_{0 \leq i \leq n}, \{f_I\})$ as an n-simplex of $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})$. This construction determines a map of sets $\mathrm{N}(\mathcal{C}_\Delta)_n \to \mathrm{N}_{\mathrm{dg}}(\mathcal{C})_n$ which depends functorially on the linearly ordered set $[n]$, and therefore defines a functor of ∞-categories $\mathrm{N}(\mathcal{C}_\Delta) \to \mathrm{N}_{\mathrm{dg}}(\mathcal{C})$.

If $\mathcal{C}$ is a differential graded category, then the map $\mathrm{N}(\mathcal{C}_\Delta) \to \mathrm{N}_{\mathrm{dg}}(\mathcal{C})$ of Construction 1.3.1.16 is bijective when restricted to simplices of dimension ≤ 2. It is generally not an isomorphism. However, we do have the following result:

Proposition 1.3.1.17. *Let $\mathcal{C}$ be a differential graded category. Then the functor $\theta : \mathrm{N}(\mathcal{C}_\Delta) \to \mathrm{N}_{\mathrm{dg}}(\mathcal{C})$ of Construction 1.3.1.16 is an equivalence of ∞-categories.*

Proof. Since θ is bijective on vertices, it will suffice to show that θ is fully faithful. Choose objects $X, Y \in \mathcal{C}$; we claim that θ induces a homotopy equivalence of Kan complexes $\phi : \mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}(\mathcal{C}_\Delta)}(X, Y) \to \mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}_{\mathrm{dg}}(\mathcal{C})}(X, Y)$. Let $Q^\bullet$ be the cosimplicial object of $\mathcal{S}\mathrm{et}_\Delta$ defined in §T.2.2.2, so that we have a canonical isomorphism of simplicial sets

$$\mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}(\mathcal{C}_\Delta)}(X, Y) \simeq \mathrm{Sing}_{Q^\bullet} \mathrm{Map}_{\mathcal{C}_\Delta}(X, Y) \simeq \mathrm{Sing}_{Q^\bullet} \mathrm{DK}_\bullet(\tau_{\geq 0} \mathrm{Map}_{\mathcal{C}}(X, Y)_*)$$

(see Proposition T.2.2.2.13). Remark 1.3.1.12 yields an isomorphism of simplicial sets $\mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}_{\mathrm{dg}}(\mathcal{C})}(X, Y) \simeq \mathrm{DK}_\bullet(\tau_{\geq 0} \mathrm{Map}_{\mathcal{C}}(X, Y)_*)$. Let $\Delta^\bullet$ denote the cosimplicial object of $\mathcal{S}\mathrm{et}_\Delta$ given by $[n] \mapsto \Delta^n$, so that we have a map $Q^\bullet \to \Delta^\bullet$ of cosimplicial objects of $\mathcal{S}\mathrm{et}_\Delta$ (see Proposition T.2.2.2.7) which induces a map $\psi_Z : Z \to \mathrm{Sing}_{Q^\bullet} Z$ for every simplicial set Z. When $Z = \mathrm{DK}_\bullet(\tau_{\geq 0} \mathrm{Map}_{\mathcal{C}}(X, Y)_*)$, the composite map $\phi \circ \psi_Z$ is an isomorphism of simplicial sets. It will therefore suffice to show that ψ_Z is a homotopy equivalence. Since Z is a Kan complex, this follows from Propositions T.2.2.2.9 and T.2.2.2.7. $\square$

We now discuss the functoriality of the construction $\mathcal{C} \mapsto \mathrm{N}_{\mathrm{dg}}(\mathcal{C})$.

Definition 1.3.1.18. Let $\mathcal{C}$ and $\mathcal{D}$ be differential graded categories over a field k. A *differential graded functor* F from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data:

- For every object $X \in \mathcal{C}$, an object $F(X) \in \mathcal{D}$.

- For every pair of objects $X, Y \in \mathcal{C}$, a map of chain complexes

$$\alpha_{X,Y} : \mathrm{Map}_{\mathcal{C}}(X, Y)_* \to \mathrm{Map}_{\mathcal{D}}(F(X), F(Y))_*.$$

This data is required to be compatible with composition in the following sense:

- For every $X \in \mathcal{C}$, we have $\alpha_{X,X}(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

- For every triple of objects $X, Y, Z \in \mathcal{C}$ and every $f \in \mathrm{Map}_{\mathcal{C}}(X, Y)_p$, $g \in \mathrm{Map}_{\mathcal{C}}(Y, Z)_q$, we have $\alpha_{X,Z}(g \circ f) = \alpha_{Y,Z}(g) \circ \alpha_{X,Y}(f)$.

The collection of differential graded categories over k forms a category $\mathrm{Cat}_{\mathrm{dg}}\,k$, whose morphisms are given by differential graded functors.

Proposition 1.3.1.19. *Let k be a commutative ring. There is a combinatorial model structure on the category $\mathrm{Cat}_{\mathrm{dg}}\,k$ of differential graded categories over k, which is characterized by the following properties:*

(W) *A differential graded functor $F : \mathcal{C} \to \mathcal{D}$ is a weak equivalence if and only if F induces an equivalence of homotopy categories $\mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ and, for every pair of objects $X, Y \in \mathcal{C}$, the induced map $\mathrm{Map}_{\mathcal{C}}(X, Y)_* \to \mathrm{Map}_{\mathcal{D}}(F(X), F(Y))_*$ is a quasi-isomorphism of chain complexes of k-modules.*

(F) *A differential graded functor $F : \mathcal{C} \to \mathcal{D}$ is a fibration if and only if F satisfies the following pair of conditions:*

- *The underlying functor $\mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is a quasi-fibration. That is, given an object $X \in \mathcal{C}$ and an isomorphism $\beta : F(X) \to Y$ in $\mathrm{h}\mathcal{D}$, β can be lifted to an isomorphism $\overline{\beta} : X \to \overline{Y}$ in $\mathrm{h}\mathcal{C}$.*
- *For every pair of objects $X, Y \in \mathcal{C}$, the map of chain complexes*

$$\mathrm{Map}_{\mathcal{C}}(X, Y)_* \to \mathrm{Map}_{\mathcal{D}}(F(X), F(Y))_*$$

is degreewise surjective.

For a proof, we refer the reader to [141] (in the case where k is a field, Proposition 1.3.1.19 can be deduced from Proposition T.A.3.2.4 and Theorem T.A.3.2.24).

Proposition 1.3.1.20. *Let k be a commutative ring. Then the formation of differential graded nerves $\mathcal{C} \mapsto \mathrm{N}_{\mathrm{dg}}(\mathcal{C})$ determines a right Quillen functor from the category $\mathrm{Cat}_{\mathrm{dg}}\,k$ (endowed with the model structure of Proposition 1.3.1.19) to the category of simplicial sets (endowed with the Joyal model structure).*

Proof. The functor N_{dg} preserves small limits and filtered colimits, and therefore admits admits a left adjoint by virtue of the adjoint functor theorem (Corollary T.5.5.2.9). We next claim that the functor N_{dg} preserves weak equivalences. Let $F : \mathcal{C} \to \mathcal{D}$ be a weak equivalence of differential graded categories over k. Then F induces an equivalence of homotopy categories $\mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$. Using Remark 1.3.1.11, we deduce that $\mathrm{N}_{\mathrm{dg}}(F) : \mathrm{N}_{\mathrm{dg}}(\mathcal{C}) \to \mathrm{N}_{\mathrm{dg}}(\mathcal{D})$ induces an equivalence of homotopy categories. To prove that $\mathrm{N}_{\mathrm{dg}}(F)$ is an equivalence of ∞-categories, it suffices to show that it is fully faithful. In other words, we must show that for every pair of objects $X, Y \in \mathcal{C}$, the induced map $\mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}_{\mathrm{dg}}(\mathcal{C})}(X, Y) \to \mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}_{\mathrm{dg}}(\mathcal{D})}(F(X), F(Y))$ is a homotopy equivalence of Kan complexes. This follows immediately from Remark 1.3.1.12, since the map of chain complexes $\mathrm{Map}_{\mathcal{C}}(X, Y)_* \to \mathrm{Map}_{\mathcal{D}}(F(X), F(Y))_*$ is a quasi-isomorphism.

To complete the proof, it will suffice to show that the functor N_{dg} preserves fibrations. Let $F : \mathcal{C} \to \mathcal{D}$ be a fibration of differential graded categories over k. We wish to prove that F induces a categorical fibration of simplicial sets $\mathrm{N}_{\mathrm{dg}}(\mathcal{C}) \to \mathrm{N}_{\mathrm{dg}}(\mathcal{D})$. The functor F induces a quasi-fibration of ordinary categories $\mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$, and therefore also a quasi-fibration $\mathrm{h}\mathrm{N}_{\mathrm{dg}}(\mathcal{C}) \to \mathrm{h}\mathrm{N}_{\mathrm{dg}}(\mathcal{D})$ (Remark 1.3.1.11). It will therefore suffice to show that the map $\mathrm{N}_{\mathrm{dg}}(F)$ is an inner fibration of simplicial sets (Corollary T.2.4.6.5). That is, we must show that for $0 < j < n$, every lifting problem of the form

$$
\begin{array}{ccc}
\Lambda^n_j & \xrightarrow{\phi_0} & \mathrm{N}_{\mathrm{dg}}(\mathcal{C}) \\
\downarrow & {\phi\nearrow} & \downarrow \\
\Delta^n & \xrightarrow[\overline{\phi}]{} & \mathrm{N}_{\mathrm{dg}}(\mathcal{D})
\end{array}
$$

admits a solution. The map ϕ_0 determines objects $X_0, X_1, \ldots, X_n \in \mathcal{C}$ together with elements $f_I \in \mathrm{Map}_{\mathcal{C}}(X_{i_-}, X_{i_+})_m$ for every subset $I = \{i_- < i_m < \ldots < i_1 < i_+\} \subseteq [n]$ with at least two elements such that $[n] \neq I \neq [n] - \{j\}$, satisfying equation 1.1 of Construction 1.3.1.6. For every I, let $\overline{f}_I$ denote the image of f_I in $\mathrm{Map}_{\mathcal{D}}(F(X_{i_-}), F(X_{i_+}))_m$. The extension $\overline{\phi}$ determines a pair of elements

$$\overline{f}_{[n]} \in \mathrm{Map}_{\mathcal{D}}(F(X_0), F(X_n))_{n-1} \qquad \overline{f}_{[n]-\{j\}} \in \mathrm{Map}_{\mathcal{D}}(F(X_0), F(X_n))_{n-2}$$

$$d\overline{f}_{[n]} = \sum_{1 \le i \le n-1} (-1)^{n-i}(\overline{f}_{[n]-\{i\}} - \overline{f}_{\{i,...,n\}} \circ \overline{f}_{\{0,...,i\}}).$$

Since F is a fibration, we can lift $\overline{f}_{[n]}$ to an element $f_{[n]} \in \mathrm{Map}_{\mathcal{C}}(X_0, X_n)_{n-1}$. This choice of lift extends uniquely to an n-simplex of $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})$ lifting $\overline{\phi}$, by setting

$$f_{[n]-\{j\}} = (-1)^{n-j}df_{[n]} + \sum_{1 \le i \le n-1} (-1)^{i-j}f_{\{i,...,n\}} \circ f_{\{0,...,i\}} - \sum_{1 \le i \le n-1, i \ne j} (-1)^{i-j}f_{[n]-\{i\}}.$$

$\square$

1.3.2   Derived ∞-Categories

Let $\mathcal{A}$ be an abelian category. To every pair of objects $X, Y \in \mathcal{A}$ and every integer $n \ge 0$, one can define a (Yoneda) Ext-group $\mathrm{Ext}_{\mathcal{A}}^n(X, Y)$. If $\mathcal{A}$ has enough projective objects, then X admits a projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to X,$$

and the Ext-groups $\mathrm{Ext}_{\mathcal{A}}^n(X, Y)$ are given by the cohomologies of the cochain complex $\mathrm{Hom}_{\mathcal{A}}(P_\bullet, Y)$. The functors $\mathrm{Ext}_{\mathcal{A}}^n$ are examples of *derived functors*: that is, they are functors which can be computed by choosing a projective (or injective) resolution of one of their arguments. In working with derived functors, it is often convenient to replace the abelian category $\mathcal{A}$ by its *derived category*: that is, to work not with objects of $\mathcal{A}$ but with chain complexes of objects of $\mathcal{A}$. In this section, we will study a slightly more elaborate object: the *derived ∞-category* $\mathcal{D}^-(\mathcal{A})$ of an abelian category $\mathcal{A}$. Roughly speaking, $\mathcal{D}^-(\mathcal{A})$ is an ∞-category whose objects are (projective and right-bounded) chain complexes with values in $\mathcal{A}$, whose morphisms are given by maps of chain complexes, 2-morphisms are given by chain homotopies, and so forth. Our goal in this section is to define the stable ∞-category $\mathcal{D}^-(\mathcal{A})$ and establish some of its basic properties. In particular, we will show that $\mathcal{D}^-(\mathcal{A})$ is stable (Corollary 1.3.2.18) and admits a t-structure, whose heart can be identified with the abelian category $\mathcal{A}$ (Proposition 1.3.2.19).

Definition 1.3.2.1. chain complex!differential graded category of Let $\mathcal{A}$ be an additive category, and let M_* and N_* be chain complexes with values in $\mathcal{A}$ (see Definition 1.2.3.1). For each integer p, we let $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, N_*)_p$ denote the product $\prod_{n \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{A}}(M_n, N_{n+p})$. We will regard the collection

$$\{\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, N_*)_p\}_{p \in \mathbf{Z}}$$

as a chain complex of abelian groups, with differential given by the formula

$$(df)(x) = d(f(x)) - (-1)^p f(dx)$$

for $f \in \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, N_*)_p$.

Given a triple of chain complexes $M_*, N_*, P_* \in \mathrm{Ch}(\mathcal{A})$, composition gives a bilinear map

$$\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(N_*, P_*)_p \times \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, N_*)_q \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, P_*)_{p+q}$$

satisfying the Leibniz rule $d(f \circ g) = df \circ g + (-1)^p f \circ dg$. Using this notion of composition, we can regard $\mathrm{Ch}(\mathcal{A})$ as a differential graded category.

Remark 1.3.2.2. For every additive category $\mathcal{A}$, we can apply Construction 1.3.1.6 to the differential graded category $\mathrm{Ch}(\mathcal{A})$ to obtain an ∞-category $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$. Note that the objects of $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$ are chain complexes M_* with values in $\mathcal{A}$ and the morphisms in $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$ are morphisms of chain complexes. Two morphisms $f, g : M_* \to N_*$ are homotopic in $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$ if and only if there exists a *chain homotopy* from f to g: that is, if and only if there is a sequence of maps $h_n : M_n \to N_{n+1}$ satisfying

$$d \circ h_n + h_{n-1} \circ d = f - g.$$

Remark 1.3.2.3. Let $\mathcal{A}$ be an additive category, and let $X_*, Y_* \in \mathrm{Ch}(\mathcal{A})$. The homotopy group

$$\pi_n \, \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(X_*, Y_*)$$

can be identified with the group of chain-homotopy classes of maps from X_* to the shifted complex Y_{*+n}.

Definition 1.3.2.4. Let $\mathcal{A}$ be an abelian category. An object $P \in \mathcal{A}$ is said to be *projective* if, for every epimorphism $M \to N$ in $\mathcal{A}$, the induced map $\mathrm{Hom}_{\mathcal{A}}(P, M) \to \mathrm{Hom}_{\mathcal{A}}(P, N)$ is surjective. We say that $\mathcal{A}$ has *enough projective objects* if, for every object $M \in \mathcal{A}$, there exists an epimorphism $P \to M$, where P is a projective object of $\mathcal{A}$.

Remark 1.3.2.5. Let $\mathcal{A}$ be an abelian category. Then an object $P \in \mathcal{A}$ is projective in the sense of Definition 1.3.2.4 if and only if it is projective in the sense of Definition T.5.5.8.18, when regarded as an object of the ∞-category $\mathrm{N}(\mathcal{A})$ (see Example T.5.5.8.21.

Notation 1.3.2.6. Let $\mathcal{A}$ be an additive category. We let $\mathrm{Ch}^-(\mathcal{A})$ denote the full subcategory of $\mathrm{Ch}(\mathcal{A})$ spanned by those chain complexes M_* such that $M_n \simeq 0$ for $n \ll 0$, and $\mathrm{Ch}^+(\mathcal{A})$ the full subcategory of $\mathrm{Ch}(\mathcal{A})$ spanned by those chain complexes such that $M_n \simeq 0$ for $n \gg 0$.

Definition 1.3.2.7. Let $\mathcal{A}$ be an abelian category with enough projective objects. We let $\mathcal{D}^-(\mathcal{A})$ denote the ∞-category $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^-(\mathcal{A}_{\mathrm{proj}}))$. We will refer to $\mathcal{D}^-(\mathcal{A})$ as the *derived ∞-category of $\mathcal{A}$.*

Variant 1.3.2.8. If $\mathcal{A}$ is an abelian category with enough injective objects, we let $\mathcal{D}^+(\mathcal{A})$ denote the nerve $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^+(\mathcal{A}_{\mathrm{inj}}))$, where $\mathcal{A}_{\mathrm{inj}}$ denotes the full subcategory of $\mathcal{A}$ spanned by the injective objects. We have a canonical equivalence $\mathcal{D}^+(\mathcal{A})^{op} \simeq \mathcal{D}^-(\mathcal{A}^{op})$.

Remark 1.3.2.9. The homotopy category $\mathrm{h}\mathcal{D}^-(\mathcal{A})$ can be described as follows: objects are given by (right bounded) chain complexes of projective objects of $\mathcal{A}$, and morphisms are given by homotopy classes of chain maps. Consequently, $\mathrm{h}\mathcal{D}^-(\mathcal{A})$ can be identified with the derived category of $\mathcal{A}$ studied in classical homological algebra (with appropriate boundedness conditions imposed).

Our first goal in this section is to verify the stability of the derived category of an abelian category. We begin by studying a more general situation.

Proposition 1.3.2.10. *Let $\mathcal{A}$ be an additive category. Then the ∞-category $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$ is stable.*

The proof of Proposition 1.3.2.10 will require some preliminary calculations. We will need the following simple observation:

Proposition 1.3.2.11. *Let $f : A_\bullet \to B_\bullet$ be a map of simplicial abelian groups. Then f is a Kan fibration if and only if the associated map of chain complexes $N_*(A) \to N_*(B)$ induces a surjection $N_n(A) \to N_n(B)$ for $n > 0$.*

Proof. For every integer n, let $E(n)_*$ denote the acyclic chain complex given by

$$E(n)_k = \begin{cases} \mathbf{Z} & \text{if } k \in \{n, n-1\} \\ 0 & \text{otherwise,} \end{cases}$$

where the differential $E(n)_n \to E(n)_{n-1}$ is an isomorphism. We note that $E(n)_*$ enjoys the following universal property: if M_* is any chain complex in the category $\mathcal{A}b$ of abelian groups, then there is a canonical isomorphism $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A}b)}(E(n)_*, M_*) \simeq M_n$. In particular, there is a map $\theta : E(n)_* \to N_*(\Delta^n)$ corresponding to a generator of the group $N_n(\Delta^n) \simeq \mathbf{Z}$. Assume that $n > 0$. For any $0 \leq i \leq n$, the map θ determines an isomorphism $N_*(\Lambda_i^n) \oplus E(n)_* \to N_*(\Delta^n)$. Consequently, the map $N_*(A) \to N_*(B)$ has the right lifting property with respect to the inclusion $N_*(\Lambda_i^n) \hookrightarrow N_*(\Delta^n)$ if and only if it has the extension property with respect to $E(n)_*$. Invoking the universal property of $E(n)_*$, we see that this lifting property is equivalent to the requirement that the map $N_n(A) \to N_n(B)$ is surjective. $\square$

Corollary 1.3.2.12. *Let $A_\bullet$ be a simplicial abelian group. Then $A_\bullet$ is a Kan complex.*

Notation 1.3.2.13. Let $\mathcal{A}$ be an additive category. We let $\mathrm{Ch}(\mathcal{A})_\Delta$ underlying simplicial category of the differential graded category $\mathrm{Ch}(\mathcal{A})$ (see Construction 1.3.1.13).

Suppose we are given a map $f : M_* \to M'_*$ of chain complexes with values in an additive category $\mathcal{A}$, and let $Q_* \in \mathrm{Ch}(\mathcal{A})$ be another chain complex. Composition with f induces a map of chain complexes

$$\theta : \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M'_*, Q_*) \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, Q_*).$$

Suppose that, for every pair of integers $p < q$, f induces a surjection $\mathrm{Hom}_{\mathcal{A}}(M'_p, Q_q) \to \mathrm{Hom}_{\mathcal{A}}(M_p, Q_q)$. It follows from Proposition 1.3.2.11 that f induces a Kan fibration of simplicial sets $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})_\Delta}(M'_*, Q_*) \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})_\Delta}(M_*, Q_*)$. In particular, we have the following result:

Corollary 1.3.2.14. *Let $\mathcal{A}$ be an additive category. Suppose we are given a map of chain complexes $f : M_* \to M'_*$ in $\mathrm{Ch}(\mathcal{A})$ and another chain complex Q_* in $\mathrm{Ch}(\mathcal{A})$. Assume that, for every pair of integers $p < q$, composition with f induces a surjection $\mathrm{Hom}_{\mathcal{A}}(M'_p, Q_q) \to \mathrm{Hom}_{\mathcal{A}}(M_p, Q_q)$. Then f induces a Kan fibration of simplicial sets $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})_\Delta}(M'_*, Q_*) \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})_\Delta}(M_*, Q_*)$.*

Remark 1.3.2.15. The hypotheses of Corollary 1.3.2.14 are satisfied in either of the following situations:

(a) The map f is *degreewise split*: that is, each of the maps $M_p \to M'_p$ admits a left inverse.

(b) The category $\mathcal{A}$ is abelian, each of the maps $M_p \to M'_p$ is a monomorphism, and each of the objects $Q_q \in \mathcal{A}$ is injective.

Corollary 1.3.2.16. *Let $\mathcal{A}$ be an additive category, and suppose we are given a pushout diagram σ :*

$$
\begin{array}{ccc}
M_* & \xrightarrow{\ f\ } & M'_* \\
\downarrow & & \downarrow \\
N_* & \longrightarrow & N'_*
\end{array}
$$

in $\mathrm{Ch}(\mathcal{A})$. If f is degreewise split (as in Remark 1.3.2.15), then σ is a homotopy pushout diagram in the simplicial category $\mathrm{Ch}(\mathcal{A})_\Delta$.

Proof. We must show that for any $Q_* \in \mathrm{Ch}(\mathcal{A})$, the diagram of Kan complexes

$$
\begin{array}{ccc}
\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})_\Delta}(N'_*, Q_*) & \longrightarrow & \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})_\Delta}(N_*, Q_*) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})_\Delta}(M'_*, Q_*) & \xrightarrow{\ g\ } & \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})_\Delta}(M_*, Q_*)
\end{array}
$$

is a homotopy pullback square. This diagram is evidently a pullback square. Since the Kan model structure on Set_Δ is right proper, it suffices to observe that the map g is a Kan fibration (by Corollary 1.3.2.14). $\square$

Remark 1.3.2.17. Let $\mathcal{A}$ be an additive category, and suppose we are given any map $f : M_* \to M'_*$ in $\mathrm{Ch}(\mathcal{A})$. We have a pushout diagram of chain complexes

$$
\begin{array}{ccc}
M_* & \longrightarrow & E(1)_* \otimes M_* \\
\downarrow & & \downarrow \\
M'_* & \longrightarrow & C(f)_*,
\end{array}
$$

where $C(f)_*$ is the *mapping cone* of f (so that $C(f)_n \simeq M'_n \oplus M_{n-1}$). It follows from Corollary 1.3.2.16, Theorem T.4.2.4.1, and Proposition 1.3.1.17 that $C(f)_*$ can be identified with a cofiber of f in the ∞-category $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$.

Proof of Proposition 1.3.2.10. Let $\mathcal{A}$ be an additive category. We first claim that the ∞-category $\mathrm{N_{dg}}(\mathrm{Ch}(\mathcal{A}))$ admits pushouts. Using Proposition 1.3.1.17 and Theorem T.4.2.4.1, we are reduced to proving that the simplicial category $\mathrm{Ch}(\mathcal{A})_\Delta$ admits homotopy pushouts. This follows from Corollary 1.3.2.16, since any map of chain complexes $f : M_* \to M'_*$ is chain homotopy-equivalent to a map which is degreewise split (replace M'_* by the mapping cylinder of f).

It is obvious that $\mathrm{N_{dg}}(\mathrm{Ch}(\mathcal{A}))$ has a zero object (since $\mathrm{Ch}(\mathcal{A})$ has a zero object). We can describe the suspension functor explicitly as follows. Let $E(1)_*$ be the chain complex of abelian groups described in Proposition 1.3.2.11. There is a pushout diagram of differential graded functors from $\mathrm{Ch}(\mathcal{A})$ to itself, which carries each $M_* \in \mathrm{Ch}(\mathcal{A})$ to the diagram

$$
\begin{array}{ccc}
M_* & \longrightarrow & E(1)_* \otimes M_* \\
\downarrow & & \downarrow \\
0 & \longrightarrow & M_{*-1}.
\end{array}
$$

The above arguments show that this diagram determines a pushout diagram in the ∞-category of functors from $\mathrm{N_{dg}}(\mathrm{Ch}(\mathcal{A}))$ to itself. Note that $E(1)_* \otimes M_*$ is chain homotopy contractible and therefore a zero object of $\mathrm{N_{dg}}(\mathrm{Ch}(\mathcal{A}))$. It follows that the suspension functor $\Sigma : \mathrm{N_{dg}}(\mathrm{Ch}(\mathcal{A})) \to \mathrm{N_{dg}}(\mathrm{Ch}(\mathcal{A}))$ is induced by the shift functor $M_* \mapsto M_{*-1}$. Since the shift functor is an equivalence of differential graded categories, Σ is an equivalence of ∞-categories. Using Proposition 1.4.2.27, we deduce that $\mathrm{N_{dg}}(\mathrm{Ch}(\mathcal{A}))$ is stable, as desired. $\square$

In the situation of Proposition 1.3.2.10, suppose that $\mathcal{C}$ is a full subcategory of $\mathrm{Ch}(\mathcal{A})$, which is closed under the formation of shifts and under the formation of mapping cones. Then the differential graded nerve $\mathrm{N_{dg}}(\mathcal{C})$ is a stable subcategory of $\mathrm{N_{dg}}(\mathrm{Ch}(\mathcal{A}))$, and is therefore itself a stable ∞-category. In particular, we obtain the following result:

Corollary 1.3.2.18. *Let $\mathcal{A}$ be an abelian category with enough projective objects. Then the ∞-category $\mathcal{D}^-(\mathcal{A})$ is stable.*

We next construct a t-structure on the stable ∞-category $\mathcal{D}^-(\mathcal{A})$.

Proposition 1.3.2.19. *Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $\mathcal{D}_{\geq 0}^-(\mathcal{A})$ denote the full subcategory of $\mathcal{D}^-(\mathcal{A})$ spanned by those chain complexes A_* such that the homology objects $\mathrm{H}_n(A) \in \mathcal{A}$ vanish for $n < 0$, and define $\mathcal{D}_{\leq 0}^-(\mathcal{A})$ similarly. Then the pair $(\mathcal{D}_{\geq 0}^-(\mathcal{A}), \mathcal{D}_{\leq 0}^-(\mathcal{A}))$ determines a t-structure on $\mathcal{D}^-(\mathcal{A})$. Moreover, the heart of $\mathcal{D}^-(\mathcal{A})$ is canonically equivalent to (the nerve of) the abelian category $\mathcal{A}$.*

Lemma 1.3.2.20. *Let $\mathcal{A}$ be an abelian category, and let $P_* \in \mathrm{Ch}(\mathcal{A})$ be a complex of projective objects of $\mathcal{A}$ such that $P_n \simeq 0$ for $n \ll 0$. Let $Q_* \to Q'_*$ be a quasi-isomorphism in $\mathrm{Ch}(\mathcal{A})$. Then the induced map*

$$
\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(P_*, Q_*) \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(P_*, Q'_*)
$$

is a quasi-isomorphism.

Proof. We observe that P_* is a homotopy colimit of its naive truncations

$$
\ldots \to 0 \to P_n \to P_{n-1} \to \ldots.
$$

It therefore suffices to prove the result for each of these truncations, so we may assume that P_* is concentrated in finitely many degrees. Working by induction, we can reduce to the case where P_* is concentrated in a single degree. Shifting, we can reduce to the case where P_* consists of a single projective object P concentrated in degree zero. Since P is projective, we have isomorphisms

$$
\mathrm{H}_i \, \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(P_*, Q_*) \simeq \mathrm{Hom}_\mathcal{A}(P, \mathrm{H}_i(Q_*)) \simeq \mathrm{Hom}_\mathcal{A}(P, \mathrm{H}_i(Q'_*)) \simeq \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(P_*, Q'_*).
$$

$\square$

Lemma 1.3.2.21. *Let $\mathcal{A}$ be an abelian category. Suppose that $P_*, Q_* \in \mathrm{Ch}(\mathcal{A})$ have the following properties:*

(1) *Each P_n is projective, and $P_n \simeq 0$ for $n < 0$.*

(2) *The homologies $\mathrm{H}_n(Q_*)$ vanish for $n > 0$.*

Then the space $\mathrm{Map}_{\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))}(P_*, Q_*)$ is discrete, and we have a canonical isomorphism of abelian groups

$$\mathrm{Ext}^0(P_*, Q_*) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathrm{H}_0(P_*), \mathrm{H}_0(Q_*)).$$

Proof. Let Q'_* be the complex

$$\ldots \to 0 \to \mathrm{coker}(Q_1 \to Q_0) \to Q_{-1} \to \ldots.$$

Condition (2) implies that the canonical map $Q_* \to Q'_*$ is a quasi-isomorphism. In view of (1) and Lemma 1.3.2.20, it will suffice to prove the result after replacing Q_* by Q'_*. In this case, we have

$$\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(P_*, Q_*)_m \simeq \begin{cases} 0 & \text{if } m > 0 \\ \mathrm{Hom}_{\mathcal{A}}(P_0, Q_0) & \text{if } m = 0. \end{cases}$$

Unwinding the definitions, we see that $\mathrm{H}_0(\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(P_*, Q_*))$ is the subgroup of $\mathrm{Hom}_{\mathcal{A}}(P_0, Q_0)$ given by $\mathrm{Hom}_{\mathcal{A}}(\mathrm{coker}(P_1 \to P_0), \ker(Q_0 \to Q_{-1})) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathrm{H}_0(P), \mathrm{H}_0(Q))$. The desired result now follows from Remark 1.3.1.12.   □

Proof of Proposition 1.3.2.19. We begin with the following observation:

(∗) For any object $A_* \in \mathrm{Ch}(\mathcal{A})$, there exists a map $f : P_* \to A_*$ where each P_n is projective, $P_n \simeq 0$ for $n < 0$, and the induced map $\mathrm{H}_k(P_*) \to \mathrm{H}_k(A_*)$ is an isomorphism for $k \geq 0$.

To prove (∗), we construct projective objects $P_n \in \mathcal{A}$, differentials $d_n : P_n \to P_{n-1}$, and maps $f_n : P_n \to A_n$ which are compatible with the differentials, using induction on n. For $n < 0$ set $P_n = 0$ (so that f_n and d_n are uniquely determined). Suppose that $n \geq 0$ and that P_{n-1}, d_{n-1}, and f_{n-1} have already been defined. Since $\mathcal{A}$ has enough projective objects, we can choose a projective object P_n equipped with an epimorphism

$$g : P_n \to A_n \times_{\ker(A_{n-1} \to A_{n-2})} \ker(P_{n-1} \to P_{n-2}).$$

We now define $f_n : P_n \to A_n$ to be the composition of g with the projection onto the first factor, and $d_n : P_n \to P_{n-1}$ to be the composition of g with the projection onto the second factor (followed by the inclusion of $\ker(P_{n-1} \to P_{n-2})$ into P_{n-1}). It is easy to see that this construction yields a map of chain complexes $f : P_* \to A_*$ with the desired properties. Note also that if $A_* \in \mathcal{D}^-(\mathcal{A})$ and the homologies $\mathrm{H}_n(A_*)$ vanish for $n < 0$, then f is a quasi-isomorphism between projective complexes and therefore a chain homotopy equivalence.

It is clear that $\mathcal{D}^-_{\leq 0}(\mathcal{A})[-1] \subseteq \mathcal{D}^-_{\leq 0}(\mathcal{A})$ and $\mathcal{D}^-_{\geq 0}(\mathcal{A})[1] \subseteq \mathcal{D}^-_{\geq 0}(\mathcal{A})$. Suppose now that $A_* \in \mathcal{D}^-_{\geq 0}(\mathcal{A})$ and $B_* \in \mathcal{D}^-_{\leq -1}(\mathcal{A})$; we wish to show that $\mathrm{Ext}^0_{\mathcal{D}^-(\mathcal{A})}(A_*, B_*) \simeq 0$. Using (∗), we may reduce to the case where $A_n \simeq 0$ for $n < 0$. The desired result now follows immediately from Lemma 1.3.2.21. Finally, choose an arbitrary object $A_* \in \mathcal{D}^-(\mathcal{A})$, and let $f : P_* \to A_*$ be as in (∗). Using the construction of cofibers given in the proof of Proposition 1.3.2.10, we deduce that $\mathrm{cofib}(f)[1] \in \mathcal{D}^-_{\leq 0}(\mathcal{A})$. This completes the proof that $(\mathcal{D}^-_{\geq 0}(\mathcal{A}), \mathcal{D}^-_{\leq 0}(\mathcal{A}))$ is a t-structure on $\mathcal{D}^-(\mathcal{A})$.

It remains to describe the heart of the stable ∞-category $\mathcal{D}^-(\mathcal{A})$. Note that the construction $A_* \mapsto \mathrm{H}_0(A)$ determines a functor $\theta : \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A})) \to \mathrm{N}(\mathcal{A})$. Let $\mathcal{C} \subseteq \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$ be the full subcategory spanned by complexes P_* such that each P_n is projective, $P_n \simeq 0$ for $n < 0$, and $\mathrm{H}_n(P_*) \simeq 0$ for $n \neq 0$. Assertion (∗) implies that the inclusion $\mathcal{C} \subseteq \mathcal{D}^-(\mathcal{A})^\heartsuit$ is an equivalence of ∞-categories. Lemma 1.3.2.21 implies that $\theta | \mathcal{C}$ is fully faithful. Finally, we can apply (∗) in the case where A_* is concentrated in degree zero to deduce that $\theta | \mathcal{C}$ is essentially surjective. It follows that θ restricts to an equivalence $\mathcal{D}^-(\mathcal{A})^\heartsuit \to \mathrm{N}(\mathcal{A})$.   □

If $\mathcal{A}$ is an abelian category with enough projective objects, then the full subcategory $\mathcal{D}^-_{\geq 0}(\mathcal{A}) \subseteq \mathcal{D}^-(\mathcal{A})$ admits an alternative description by means of the Dold-Kan correspondence. Note first that for any additive category $\mathcal{B}$, the category of simplicial objects $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{B})$ is naturally tensored over the category of finite simplicial sets: the tensor product of a simplicial object $P_\bullet$ with a simplicial set K is given by $[n] \mapsto (\mathbf{Z}K_n) \otimes P_n$. We may therefore view $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{B})$ as a simplicial category, whose mapping spaces are characterized by the formula

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(K, \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{B})}(P_\bullet, P'_\bullet)) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{B})}(\mathbf{Z}K_\bullet \otimes P_\bullet, P'_\bullet)$$

for every finite simplicial set K. The following result will play an important role in §1.3.3:

Proposition 1.3.2.22. *Let $\mathcal{A}$ be an abelian category with enough projective objects and let $\mathcal{A}_{\mathrm{proj}}$ denote the full subcategory of $\mathcal{A}$ spanned by the projective objects. Then $\mathcal{D}^-_{\geq 0}(\mathcal{A})$ is equivalent to the underlying ∞-category of the simplicial category $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A}_{\mathrm{proj}})$.*

The proof of Proposition 1.3.2.22 will require some preliminary remarks. Suppose that $\mathcal{A}$ is an idempotent complete additive category, so that the Dold-Kan correspondence supplies an equivalence of categories $\mathrm{DK} : \mathrm{Ch}(\mathcal{A})_{\geq 0} \to \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A})$ (Theorem 1.2.3.7). For every pair of chain complexes $M_*, M'_* \in \mathrm{Ch}(\mathcal{A})_{\geq 0}$ and any finite simplicial set K, a morphism of simplicial sets $\phi : K \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, M'_*)$ determines a composite map

$$
\begin{aligned}
\mathbf{Z}K \otimes \mathrm{DK}_\bullet\, M &\simeq (\mathrm{DK}_\bullet \circ N_*)(\mathbf{Z}K \otimes \mathrm{DK}_\bullet(M)) \\
&\overset{\mathrm{AW}}{\to} \mathrm{DK}_\bullet(N_*(K) \otimes N_* \mathrm{DK}_\bullet(M)) \\
&\simeq \mathrm{DK}_\bullet(N_*K \otimes M_*) \\
&\overset{\phi}{\to} \mathrm{DK}_\bullet(M'_*);
\end{aligned}
$$

here AW denotes the Alexander-Whitney map of Construction 1.2.3.22. This construction is natural in K and therefore determines a map of simplicial sets

$$\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, M'_*) \to \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A})}(\mathrm{DK}_\bullet(M), \mathrm{DK}_\bullet(M')).$$

Using the associativity properties of the Alexander-Whitney construction (Remark 1.2.3.25), we see that these maps are compatible with composition and therefore endow $\mathrm{DK} : \mathrm{Ch}(\mathcal{A})_{\geq 0} \to \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A})$ with the structure of a simplicial functor (where we view $\mathrm{Ch}(\mathcal{A})_{\geq 0}$ as a simplicial category using Construction 1.3.1.13). Although DK is an equivalence of ordinary categories, it is not an equivalence of simplicial categories, because the Alexander-Whitney maps are not isomorphisms. Nevertheless, we have the following result:

Proposition 1.3.2.23. *Let $\mathcal{A}$ be an idempotent complete additive category. Then the functor*

$$\mathrm{DK} : \mathrm{Ch}(\mathcal{A})_{\geq 0} \to \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A})$$

is a weak equivalence of simplicial categories.

Proof. Since Theorem 1.2.3.7 implies that DK is essentially surjective, it will suffice to show that for every pair of chain complexes $P_*, Q_* \in \mathrm{Ch}(\mathcal{A})_{\geq 0}$, the map

$$\theta_P : \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})_\Delta}(P_*, Q_*) \to \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A})}(\mathrm{DK}_\bullet(P_*), \mathrm{DK}_\bullet(Q_*))$$

is a weak homotopy equivalence. In the argument which follows, we will regard Q_* as fixed. For each object $A \in \mathcal{A}$, let $A[n] \in \mathrm{Ch}(\mathcal{A})$ denote the chain complex consisting of the single object A, concentrated in degree n. Let us say that a chain complex M_* is *good* if the map θ_M is a weak homotopy equivalence. We now proceed as follows:

(1) Let A be an object of $\mathcal{A}$. For any simplicial set K, the Alexander-Whitney map AW : $N_*(K \otimes$ DK$(A[0])) \to N_*(K) \otimes A[0]$ is an isomorphism of chain complexes. It follows that $A[0]$ is good: in fact, the map $\theta_{A[0]}$ is an isomorphism of simplicial sets. In particular, the simplicial set $X = \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}^{op},\mathcal{A})}(\mathrm{DK}_\bullet(A[0]), \mathrm{DK}_\bullet(Q))$ is a Kan complex.

(2) Let A be an object of $\mathcal{A}$ and $n > 0$. Then the map

$$
\begin{aligned}
\mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}^{op},\mathcal{A})}(\mathbf{Z}\Delta^n \otimes \mathrm{DK}_\bullet(A[0]), \mathrm{DK}_\bullet(Q)) &\simeq \mathrm{Fun}(\Delta^n, X) \to \mathrm{Fun}(\Lambda_0^n, X) \\
&\simeq \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}^{op},\mathcal{A})}(\mathbf{Z}\Lambda_0^n \otimes \mathrm{DK}_\bullet(A[0]), \mathrm{DK}_\bullet(Q))
\end{aligned}
$$

is a trivial Kan fibration. Let $E(n)_*$ be defined as in the proof of Proposition 1.3.2.11, so that we have an isomorphism of simplicial abelian groups $N_*(\Delta^n) \simeq N_*(\Lambda_0^n) \oplus E(n)_*$. We conclude that the mapping space $\mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}^{op},\mathcal{A})}(\mathrm{DK}(E(n) \otimes A[0]), \mathrm{DK}(M'))$ is a contractible Kan complex. Since the zero map from $E(n)_* \otimes A[0]$ to itself is chain homotopic to the identity, we conclude also that $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(E(n)_* \otimes A[0], M'_*)$ is a contractible Kan complex. In particular, $E(n)_* \otimes A[0]$ is good.

(3) Let $n \geq 1$, and regard $\mathbf{Z}[n-1]$ as a subcomplex of $E(n)_*$. A choice of isomorphism $N_n(\Delta^n) \simeq \mathbf{Z} \simeq E(n)_n$ extends uniquely to a map of chain complexes $N_*(\Delta^n) \to E(n)_*$ which fits into a pushout diagram

$$
\begin{array}{ccc}
N_*(\partial\, \Delta^n) & \longrightarrow & N_*(\Delta^n) \\
\downarrow & & \downarrow \\
\mathbf{Z}[n-1] & \longrightarrow & E(n)_*.
\end{array}
$$

It follows that for any $A \in \mathcal{A}$, the induced map

$$
\mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}^{op},\mathcal{A})}(\mathrm{DK}_\bullet(E(n) \otimes A), \mathrm{DK}_\bullet(Q)) \to \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}^{op},\mathcal{A})}(\mathrm{DK}_\bullet(A[n-1]), \mathrm{DK}_\bullet(Q))
$$

is a pullback of the map $\mathrm{Fun}(\Delta^n, X) \to \mathrm{Fun}(\partial\, \Delta^n, X)$, and therefore a Kan fibration.

(4) Let $P_* \in \mathrm{Ch}(\mathcal{A})_{\geq 0}$, and suppose we are given an object $A \in \mathcal{A}$ and a map $A \to P_{n-1}$ for $n > 0$. Form a pushout diagram of chain complexes

$$
\begin{array}{ccc}
A[n-1] & \longrightarrow & E(n)_* \otimes A[0] \\
\downarrow & & \downarrow \\
P_* & \longrightarrow & P'_*.
\end{array}
$$

We obtain diagrams of mapping spaces

$$
\begin{array}{ccc}
\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})_\Delta}(P'_*, Q_*) & \longrightarrow & \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})_\Delta}(P_*, Q_*) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})_\Delta}(E(n)_* \otimes A[0], Q_*) & \longrightarrow & \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})_\Delta}(A[n-1], Q_*)
\end{array}
$$

$$
\begin{array}{ccc}
\mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}^{op},\mathcal{A})}(\mathrm{DK}(P'), \mathrm{DK}(Q)) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}^{op},\mathcal{A})}(\mathrm{DK}(P), \mathrm{DK}(Q)) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}^{op},\mathcal{A})}(\mathrm{DK}(E(n)_* \otimes A[0]), \mathrm{DK}(Q)) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}^{op},\mathcal{A})}(\mathrm{DK}(A[n-1]), \mathrm{DK}(Q).
\end{array}
$$

The first diagram is a homotopy pullback square by Corollary 1.3.2.16, and the second diagram is a homotopy pullback square by (3). Since $E(n)_* \otimes A[0]$ is good by (2), we conclude that if P_* and $A[n-1]$ are good, then P'_* is also good.

(5) Taking $P_* = 0$ in (4), we conclude that if $A[n-1]$ is good then $A[n]$ is good. Combining this with (1), we deduce that each $A[n]$ is good using induction on n.

(6) Let P_* be arbitrary. For each $n \geq 0$, let $P(n)_*$ be the chain complex

$$\ldots \to 0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to 0 \to \cdots$$

We have an evident pushout diagram

$$
\begin{array}{ccc}
P_n[n-1] & \longrightarrow & E(n)_* \otimes P_n \\
\downarrow & & \downarrow \\
P(n-1)_* & \longrightarrow & P(n)_*.
\end{array}
$$

Using (4) and (5), we conclude that each $P(n)_*$ is good using induction on n. Then θ_P is the homotopy limit of a tower of homotopy equivalences $\{\theta_{P^n}\}$. It follows that θ_P is itself a homotopy equivalence: that is, P_* is good.

$\square$

Remark 1.3.2.24. Let $\mathcal{A}$ be an arbitrary additive category. Then $\mathcal{A}$ admits a fully faithful embedding into an idempotent complete additive category. It follows from Proposition 1.3.2.23 that the functor DK : $\mathrm{Ch}(\mathcal{A})_{\geq 0} \to \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A})$ induces a fully faithful embedding of ∞-categories $\mathrm{N}(\mathrm{Ch}(\mathcal{A})_{\geq 0}) \to \mathrm{N}(\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A}))$.

Proof of Proposition 1.3.2.22. Let $\mathcal{A}$ be an abelian category with enough projective objects. If P_* is an object of $\mathcal{D}_{\geq 0}^-(\mathcal{A})$, then assertion $(*)$ appearing in the proof of Proposition 1.3.2.19 guarantees the existence of a quasi-isomorphism $P'_* \to P_*$, where $P'_* \in \mathrm{Ch}(\mathcal{A}_{\mathrm{proj}})_{\geq 0}$. Such a map is automatically a chain homotopy equivalence, and therefore an equivalence in the ∞-category $\mathcal{D}_{\geq 0}^-(\mathcal{A})$. It follows that the inclusion $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}_{\mathrm{proj}})_{\geq 0}) \hookrightarrow \mathcal{D}_{\geq 0}^-(\mathcal{A})$ is an equivalence of ∞-categories. Combining this observation with Propositions 1.3.2.23 and 1.3.1.17, we obtain equivalences

$$\mathcal{D}_{\geq 0}^-(\mathcal{A}) \hookleftarrow \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}_{\mathrm{proj}})_{\geq 0}) \leftarrow \mathrm{N}(\mathrm{Ch}(\mathcal{A}_{\mathrm{proj}})_{\geq 0}) \to \mathrm{N}(\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{A}_{\mathrm{proj}})).$$

$\square$

1.3.3 The Universal Property of $\mathcal{D}^-(\mathcal{A})$

Let $\mathcal{A}$ be an abelian category with enough projective objects. In §1.3.2, we introduced the derived ∞-category $\mathcal{D}^-(\mathcal{A})$, whose objects are right-bounded chain complexes of projective objects of $\mathcal{A}$. Our goal in this section is to characterize the ∞-category $\mathcal{D}^-(\mathcal{A})$ by a universal mapping property. To formulate this property, we need to introduce a bit of terminology.

Definition 1.3.3.1. Let $\mathcal{C}$ and $\mathcal{C}'$ be stable ∞-categories equipped with t-structures. We will say that a functor $f : \mathcal{C} \to \mathcal{C}'$ is *right t-exact* if it is exact and carries $\mathcal{C}_{\geq 0}$ into $\mathcal{C}'_{\geq 0}$. We say that f is *left t-exact* if it is exact and carries $\mathcal{C}_{\leq 0}$ into $\mathcal{C}'_{\leq 0}$.

Our main result can now be stated as follows:

Theorem 1.3.3.2. *Let $\mathcal{A}$ be an abelian category with enough projective objects, let $\mathcal{C}$ be a stable ∞-category equipped with a left complete t-structure, and let $\mathcal{E} \subseteq \mathrm{Fun}(\mathcal{D}^-(\mathcal{A}), \mathcal{C})$ be the full subcategory spanned by those right t-exact functors which carry projective objects of $\mathcal{A}$ into the heart of $\mathcal{C}$. The construction $F \mapsto \tau_{\leq 0} \circ (F | \mathcal{D}^-(\mathcal{A})^{\heartsuit})$ determines an equivalence from $\mathcal{E}$ to (the nerve of) the ordinary category of right exact functors from $\mathcal{A}$ to the heart $\mathcal{C}^{\heartsuit}$ of $\mathcal{C}$.*

Remark 1.3.3.3. Let $\mathcal{A}$ be an abelian category with enough projective objects. We will prove below that the ∞-category $\mathcal{D}^-(\mathcal{A})$ is left complete (with respect to the t-structure of Proposition 1.3.2.19); see Proposition 1.3.3.16. It follows that $\mathcal{D}^-(\mathcal{A})$ is determined (up to canonical equivalence) by the universal property of Theorem 1.3.3.2.

If $\mathcal{A}$ and $\mathcal{C}$ are as in Theorem 1.3.3.2, then any right exact functor from $\mathcal{A}$ to $\mathcal{C}^\heartsuit$ can be extended (in an essentially unique way) to a functor $\mathcal{D}^-(\mathcal{A}) \to \mathcal{C}$. In particular, if the abelian category $\mathcal{C}^\heartsuit$ has enough projective objects, then we obtain an induced map $\mathcal{D}^-(\mathcal{C}^\heartsuit) \to \mathcal{C}$.

Example 1.3.3.4. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories equipped with enough projective objects. Then any right-exact functor $f : \mathcal{A} \to \mathcal{B}$ extends to a right t-exact functor $F : \mathcal{D}^-(\mathcal{A}) \to \mathcal{D}^-(\mathcal{B})$. One typically refers to F as the *left derived functor* of f.

Example 1.3.3.5. Let Sp be the stable ∞-category of spectra (see §1.4.3), with its natural t-structure. Then the heart of Sp is equivalent to the category $\mathcal{A}$ of abelian groups. We therefore obtain a functor $\mathcal{D}^-(\mathcal{A}) \to \mathrm{Sp}$, which carries a complex of abelian groups to the corresponding *generalized Eilenberg-MacLane spectrum.*

Remark 1.3.3.6. Let $\mathcal{A}$ be an abelian category with enough projective objects, let $\mathcal{C}$ be a stable ∞-category equipped with a left-complete t-structure, and let $F : \mathcal{D}^-(\mathcal{A}) \to \mathcal{C}$ be a right t-exact functor which carries projective objects of $\mathcal{A}$ into the heart of $\mathcal{C}$. Then the following conditions are equivalent:

(i) The functor F is left t-exact: that is, F carries $\mathcal{D}^-(\mathcal{A})_{\leq 0}$ into $\mathcal{C}_{\leq 0}$.

(ii) The functor F carries $\mathrm{N}(\mathcal{A}) \subseteq \mathcal{D}^-(\mathcal{A})$ into the heart of $\mathcal{C}$.

(iii) The underlying right-exact functor $f : \mathcal{A} \to \mathcal{C}^\heartsuit$ is exact.

The implication $(i) \Rightarrow (ii)$ is obvious. To prove that $(ii) \Rightarrow (i)$, we prove by induction on n that for every object $A \in \mathcal{D}^-(\mathcal{A})_{\leq 0} \cap \mathcal{D}^-(\mathcal{A})_{\geq -n}$, we have $F(A) \in \mathcal{C}_{\leq 0}$. If $n = 0$, this follows from (ii). More generally, we have a fiber sequence

$$A' \to A \to A''$$

where $A' \in \mathcal{D}^-(\mathcal{A})_{\leq 0} \cap \mathcal{D}^-(\mathcal{A})_{\geq 1-n}$ and $A''[n] \in \mathcal{D}^-(\mathcal{A})^\heartsuit \simeq \mathrm{N}(\mathcal{A})$. Then $F(A') \in \mathcal{C}_{\leq 0}$ by the inductive hypothesis, and $F(A'') = F(A''[n])[-n] \in \mathcal{C}_{\leq -n} \subseteq \mathcal{C}_{\leq 0}$ by virtue of (i), so that $F(A) \in \mathcal{C}_{\leq 0}$.

To prove $(ii) \Rightarrow (iii)$, we note that every short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$ gives rise to a fiber sequence

$$F(A') \to F(A) \to F(A'')$$

in $\mathcal{C}_{\geq 0}$, hence a short exact sequence

$$\pi_1 F(A'') \to f(A') \xrightarrow{\phi} f(A) \to f(A'') \to 0$$

in the abelian category $\mathrm{h}\mathcal{C}^\heartsuit$. If (ii) is satisfied, then $\pi_1 F(A'') \simeq 0$, so that the map ϕ is injective.

It remains to prove that (iii) implies (ii). Assume that (iii) is satisfied. Using induction on $n > 0$, we prove that for each $A \in \mathcal{A}$, the object $\pi_n F(A) \in \mathcal{C}^\heartsuit$ vanishes. Since $\mathcal{C}$ is left-complete, this will guarantee that $\tau_{\geq 1} F(A)$ vanishes, so that $F(A)$ belongs to the heart of $\mathcal{C}$. To carry out the induction, we choose a short exact sequence

$$0 \to A' \xrightarrow{\psi} P \to A \to 0$$

in the abelian category $\mathcal{A}$, where P is projective, so that we have an exact sequence

$$\pi_n F(P) \to \pi_n F(A) \to \pi_{n-1} F(A') \to \pi_{n-1} F(P).$$

If $n > 1$, then $\pi_n F(P) \simeq \pi_{n-1} F(P) \simeq 0$, so that $\pi_n F(A) \simeq \pi_{n-1} A'$ vanishes by the inductive hypothesis. If $n = 1$, then $\pi_n F(P)$ vanishes so that $\pi_n F(A)$ can be identified with the kernel of the map $f(\psi) : \pi_0 F(A') \to \pi_0 F(P)$. If the functor f is exact, then $f(\psi)$ is a monomorphism so that $\pi_n F(A) \simeq 0$.

We have the following recognition criterion for derived categories:

Proposition 1.3.3.7. *Let $\mathcal{C}$ be a stable ∞-category equipped with a left complete t-structure. Suppose that the abelian category $\mathcal{A} = h\mathcal{C}^{\heartsuit}$ has enough projective objects. Then there exists an essentially unique t-exact functor $F : \mathcal{D}^{-}(\mathcal{A}) \to \mathcal{C}$ extending the inclusion $f : N(\mathcal{A}) \simeq \mathcal{C}^{\heartsuit} \subseteq \mathcal{C}$. Moreover, the following conditions are equivalent:*

(i) *The functor F is fully faithful.*

(ii) *For every pair of objects $X, Y \in \mathcal{A}$, if X is projective, then the abelian groups $\mathrm{Ext}_{\mathcal{C}}^{i}(X,Y)$ vanish for $i > 0$.*

(iii) *For every pair of objects $X, Y \in \mathcal{A}$, if X is projective, then there exists a monomorphism $Y \to Z$ such that the abelian groups $\mathrm{Ext}_{\mathcal{C}}^{i}(X,Z)$ vanish for $i > 0$.*

Moreover, if these conditions are satisfied, then the essential image of F is the full subcategory $\bigcup_n \mathcal{C}_{\geq -n}$ of right-bounded objects of $\mathcal{C}$.

Proof. The existence of F follows from Theorem 1.3.3.2 and Remark 1.3.3.6. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious. We prove that $(iii) \Rightarrow (ii)$ by induction on $i > 0$. Assume that $X, Y \in \mathcal{A}$ where X is projective, and choose a monomorphism $Y \hookrightarrow Z$ satisfying (iii). Then we have an exact sequence

$$\mathrm{Ext}_{\mathcal{C}}^{i-1}(X,Z) \to \mathrm{Ext}_{\mathcal{C}}^{i-1}(X,Z/Y) \to \mathrm{Ext}_{\mathcal{C}}^{i}(X,Y) \to \mathrm{Ext}_{\mathcal{C}}^{i}(X,Z).$$

If $i \geq 2$, then we have $\mathrm{Ext}_{\mathcal{C}}^{i-1}(X,Z) \simeq \mathrm{Ext}_{\mathcal{C}}^{i}(X,Z) \simeq 0$, so that $\mathrm{Ext}_{\mathcal{C}}^{i}(X,Y) \simeq \mathrm{Ext}_{\mathcal{C}}^{i-1}(X,Z/Y)$ vanishes by the inductive hypothesis. If $i = 1$, then $\mathrm{Ext}_{\mathcal{C}}^{i}(X,Y)$ can be identified with the cokernel of the map

$$\mathrm{Ext}_{\mathcal{C}}^{0}(X,Z) \simeq \mathrm{Hom}_{\mathcal{A}}(X,Z) \to \mathrm{Hom}_{\mathcal{A}}(X,Z/Y) \simeq \mathrm{Ext}_{\mathcal{C}}^{0}(X,Z/Y).$$

Since X is a projective object of $\mathcal{A}$, this map is a surjection, so that $\mathrm{Ext}_{\mathcal{C}}^{i}(X,Y)$ vanishes as required.

We now prove that (ii) implies (i). Choose any pair of objects $X, Y \in \mathcal{D}^{-}(\mathcal{A})$; we wish to prove that F induces a homotopy equivalence

$$\theta_{X,Y} : \mathrm{Map}_{\mathcal{D}^{-}(\mathcal{A})}(X,Y) \to \mathrm{Map}_{\mathcal{C}}(FX, FY).$$

Since $\mathcal{D}^{-}(\mathcal{A})$ and $\mathcal{C}$ are both left-complete (see Proposition 1.3.3.16), both sides can be identified with a homotopy limit of the tower of maps

$$\mathrm{Map}_{\mathcal{D}^{-}(\mathcal{A})}(\tau_{\leq n}X, \tau_{\leq n}Y) \to \mathrm{Map}_{\mathcal{C}}(F\tau_{\leq n}X, F\tau_{\leq n}Y).$$

It therefore suffices to show that each of these maps is a homotopy equivalence. We may therefore replace Y by $\tau_{\leq n}Y$ and thereby reduce to the case where Y belongs to the full subcategory $\mathcal{D}^{b}(\mathcal{A}) = \bigcup_n \mathcal{D}^{-}(\mathcal{A})_{\leq n}$ spanned by the (homologically) bounded objects of $\mathcal{D}^{-}(\mathcal{A})$.

Let $\mathcal{D}'$ denote the full subcategory of $\mathcal{D}^{b}(\mathcal{A})$ spanned by those objects Y for which the map $\theta_{X,Y}$ is a homotopy equivalence, for every object $X \in \mathcal{D}^{-}(\mathcal{A})$. We wish to prove that $\mathcal{D}' = \mathcal{D}^{b}(\mathcal{A})$. Since $\mathcal{D}'$ is stable under translations and extensions in $\mathcal{D}^{b}(\mathcal{A})$, it will suffice to show that $\mathcal{D}'$ contains the heart of $\mathcal{D}^{b}(\mathcal{A})$. We may therefore reduce to the case where Y is in the heart of $\mathcal{D}^{-}(\mathcal{A})$.

The object X can be represented by a chain complex of projective objects

$$\cdots \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$$

in the abelian category $\mathcal{A}$. Let X' denote the truncated chain complex

$$\cdots \to 0 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots,$$

so that we have a fiber sequence

$$X' \to X \to X''$$

in $\mathcal{D}^-(\mathcal{A})$ with $X'' \in \mathcal{C}_{\geq 2}$. We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Map}_{\mathcal{D}^-(\mathcal{A})}(X, Y) & \xrightarrow{\theta_{X,Y}} & \mathrm{Map}_{\mathcal{C}}(FX, FY) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathcal{D}^-(\mathcal{A})}(X', Y) & \xrightarrow{\theta_{X',Y}} & \mathrm{Map}_{\mathcal{C}}(FX', FY)
\end{array}
$$

where the vertical maps are homotopy equivalences. Consequently, to prove that $\theta_{X,Y}$ is a homotopy equivalence, it suffices to show that $\theta_{X',Y}$ is a homotopy equivalence. We may therefore replace X by X' and thereby reduce to the case where X belongs to the full subcategory $\mathcal{E} \subseteq \mathcal{D}^-(\mathcal{A})$ spanned by finite-length chain complexes of projective objects. Let $\mathcal{E}' \subseteq \mathcal{E}$ be the full subcategory spanned by those objects X for which the functor F induces an isomorphism of abelian groups $\mathrm{Ext}^n_{\mathcal{D}^-(\mathcal{A})}(X, Y) \to \mathrm{Ext}^n_{\mathcal{C}}(FX, FY)$ for every integer n. We wish to prove that $\mathcal{E}' = \mathcal{E}$. Since $\mathcal{E}'$ is stable under extensions and translations, it will suffice to show that $\mathcal{E}'$ contains every projective object of $\mathcal{A}$. In other words, we must show that if $X, Y \in \mathcal{A}$ and X is projective, then F induces an isomorphism

$$\mathrm{Ext}^n_{\mathcal{D}^-(\mathcal{A})}(X, Y) \to \mathrm{Ext}^n_{\mathcal{C}}(FX, FY).$$

If $n < 0$, then both sides vanish; if $n = 0$, then both sides can be identified with $\mathrm{Hom}_{\mathcal{A}}(X, Y)$. If $n > 0$, then the left side vanishes since X is projective, and the right side vanishes by virtue of assumption (ii).

We now complete the proof by describing the essential image of F. Replacing $\mathcal{C}$ by $\bigcup_n \mathcal{C}_{\geq -n}$, we may assume that $\mathcal{C}$ is right-bounded; we wish to prove that F is an equivalence. Since $\mathcal{D}^-(\mathcal{A})$ and $\mathcal{C}$ are both left-complete, we can identify F with a homotopy limit of a tower of fully faithful functors $\{F_n : \mathcal{D}^-(\mathcal{A})_{\leq n} \to \mathcal{C}_{\leq n}\}_{n \geq 0}$; it will therefore suffice to show that each F_n is an equivalence of ∞-categories. Each of the functors F_n is a restriction of F, and therefore fully faithful. It therefore suffices to show that each F_n is essentially surjective. Let C be an object of $\mathcal{C}_{\leq n}$. Since $\mathcal{C}$ is right-bounded, we may suppose that $C \in \mathcal{C}_{\geq m}$, for some integer m. We will prove that C belongs to the essential image of F_n using descending induction on m. If $m > n$, then $C \simeq 0$ and the result is obvious. Otherwise, we choose a distinguished triangle

$$C' \to C \to C''[m] \xrightarrow{\alpha} C'[1]$$

where $C' \in \mathcal{C}_{\geq m+1}$ and $C'' \in \mathcal{C}^{\heartsuit}$; note that C' and $C''[m]$ both belong to $\mathcal{C}_{\leq n}$. The inductive hypothesis guarantees that $C' \simeq F(X)$ for some object $X \in \mathcal{D}^-(\mathcal{A})_{\leq n}$, and we can identify C'' with an object A of the abelian category $\mathcal{A}$. Since F is fully faithful, the map α is represented by a morphism $\beta : A[m] \to X[1]$ in $\mathcal{D}^-(\mathcal{A})$. Let $Y = \mathrm{fib}(\beta)$, so that we have a fiber sequence

$$X \to Y \to A[m]$$

which proves that $Y \in \mathcal{D}^-(\mathcal{A})_{\leq n}$. Since F is exact, we deduce that $FY \simeq \mathrm{fib}(F(\beta)) \simeq C$, so that C belongs to the essential image of F_n as required. $\square$

We will deduce Theorem 1.3.3.2 from the following more precise characterization of the ∞-category $\mathcal{D}^-_{\geq 0}(\mathcal{A})$:

Theorem 1.3.3.8. *Let $\mathcal{A}$ be an abelian category with enough projective objects, $\mathcal{A}_{\mathrm{proj}} \subseteq \mathcal{A}$ the full subcategory spanned by the projective objects, and $\mathcal{C}$ an arbitrary ∞-category which admits geometric realizations of simplicial objects. Let $\mathrm{Fun}'(\mathcal{D}^-_{\geq 0}(\mathcal{A}), \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{D}^-_{\geq 0}(\mathcal{A}), \mathcal{C})$ spanned by those functors which preserve geometric realizations of simplicial objects. Then:*

(1) *The restriction functor $\mathrm{Fun}'(\mathcal{D}^-_{\geq 0}(\mathcal{A}), \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}(\mathcal{A}_{\mathrm{proj}}), \mathcal{C})$ is an equivalence of ∞-categories.*

(2) *A functor $F \in \mathrm{Fun}'(\mathcal{D}_{\geq 0}^{-}(\mathcal{A}), \mathcal{C})$ preserves preserves finite coproducts if and only if the restriction $F| \mathrm{N}(\mathcal{A}_{\mathrm{proj}})$ preserves finite coproducts.*

Let us assume Theorem 1.3.3.8 for the moment, and explain how to use it to deduce Theorem 1.3.3.2. The proof of Theorem 1.3.3.8 itself will be given at the end of this section.

Lemma 1.3.3.9. *Let $\mathcal{A}$ be an abelian category with enough projective objects, and let $\mathcal{B}$ be an arbitrary category which admits finite colimits. Let $\mathcal{C}$ be the category of right exact functors from $\mathcal{A}$ to $\mathcal{B}$, and let $\mathcal{C}'$ be the category of coproduct-preserving functors from $\mathcal{A}_{\mathrm{proj}}$ to $\mathcal{B}$ (here $\mathcal{A}_{\mathrm{proj}}$ denotes the full subcategory of $\mathcal{A}$ spanned by the projective objects). Then the restriction functor $\theta : \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories.*

Proof. We will give an explicit construction of an inverse functor. Let $f : \mathcal{A}_{\mathrm{proj}} \to \mathcal{B}$ be a functor which preserves finite coproducts. Let $A \in \mathcal{A}$ be an arbitrary object. Since $\mathcal{A}$ has enough projectives, there exists a projective resolution

$$\ldots \to P_1 \overset{u}{\to} P_0 \to A \to 0.$$

We now define $F(A)$ to be the coequalizer of the map

$$f(P_1) \overset{f(0)}{\underset{f(u)}{\rightrightarrows}} f(P_0) \ .$$

Of course, this definition appears to depend not only on A but on a choice of projective resolution. However, because any two projective resolutions of A are chain homotopy equivalent to one another, $F(A)$ is well-defined up to canonical isomorphism. It is easy to see that $F : \mathcal{A} \to \mathcal{B}$ is a right exact functor which extends f, and that F is uniquely determined (up to unique isomorphism) by these properties. $\square$

Lemma 1.3.3.10. (1) *Let $\mathcal{C}$ be an ∞-category which admits finite coproducts and geometric realizations of simplicial objects. Then $\mathcal{C}$ admits all finite colimits. Conversely, if $\mathcal{C}$ is an n-category which admits finite colimits, then $\mathcal{C}$ admits geometric realizations of simplicial objects.*

(2) *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories which admit finite coproducts and geometric realizations of simplicial objects. If F preserves finite coproducts and geometric realizations of simplicial objects, then F is right exact. The converse holds if $\mathcal{C}$ and $\mathcal{D}$ are n-categories.*

Proof. We will prove (1); the proof of (2) follows by the same argument. Now suppose that $\mathcal{C}$ admits finite coproducts and geometric realizations of simplicial objects. We wish to show that $\mathcal{C}$ admits all finite colimits. According to Proposition T.4.4.3.2, it will suffice to show that $\mathcal{C}$ admits coequalizers. Let $\boldsymbol{\Delta}_{s,\leq 1}$ be the subcategory of $\boldsymbol{\Delta}$ spanned by the objects $[0]$ and $[1]$ and *injective* maps between them, so that a coequalizer diagram in $\mathcal{C}$ can be identified with a functor $\mathrm{N}(\boldsymbol{\Delta}_{s,\leq 1})^{op} \to \mathcal{C}$. Let $j : \mathrm{N}(\boldsymbol{\Delta}_{s,\leq 1})^{op} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$ be the inclusion functor. We claim that every diagram $f : \mathrm{N}(\boldsymbol{\Delta}_{s,\leq 1})^{op} \to \mathcal{C}$ has a left Kan extension along j. To prove this, it suffices to show that for each $n \geq 0$, the associated diagram

$$\mathrm{N}(\boldsymbol{\Delta}_{s,\leq 1})^{op} \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} (\mathrm{N}(\boldsymbol{\Delta}_{[n]/})^{op}) \to \mathcal{C}$$

has a colimit. We now observe that this last colimit is equivalent to a coproduct: more precisely, we have $(j_! f)([n]) \simeq f([0]) \coprod f([1]) \coprod \ldots \coprod f([1])$, where there are precisely n summands equivalent to $f([1])$. Since $\mathcal{C}$ admits finite coproducts, the desired Kan extension $j_! f$ exists. We now observe that $\varinjlim(f)$ can be identified with $\varinjlim(j_! f)$, and the latter exists in virtue of our assumption that $\mathcal{C}$ admits geometric realizations for simplicial objects.

Now suppose that $\mathcal{C}$ is an n-category which admits finite colimits; we wish to show that $\mathcal{C}$ admits geometric realizations. Passing to a larger universe if necessary, we may suppose that $\mathcal{C}$ is small. Let $\mathcal{D} = \mathrm{Ind}(\mathcal{C})$, and let $\mathcal{C}' \subseteq \mathcal{D}$ denote the essential image of the Yoneda embedding $j : \mathcal{C} \to \mathcal{D}$. Then $\mathcal{D}$ admits small colimits (Theorem T.5.5.1.1) and j is fully faithful (Proposition T.5.1.3.1); it will therefore suffice to show that $\mathcal{C}'$ is stable under geometric realization of simplicial objects in $\mathcal{D}$.

Fix a simplicial object $U_\bullet : N(\mathbf{\Delta})^{op} \to \mathcal{C}' \subseteq \mathcal{D}$. Let $V_\bullet : N(\mathbf{\Delta})^{op} \to \mathcal{D}$ be a left Kan extension of $U_\bullet | N(\mathbf{\Delta}^{\leq n})^{op}$, and $\alpha_\bullet : V_\bullet \to U_\bullet$ the induced map. The geometric realization of $V_\bullet$ can be identified with the colimit of $U_\bullet | N(\mathbf{\Delta}^{\leq n})^{op}$, and therefore belongs to $\mathcal{C}'$ since $\mathcal{C}'$ is stable under finite colimits in $\mathcal{D}$ (Proposition T.5.3.5.14). Consequently, it will suffice to prove that $\alpha_\bullet$ induces an equivalence from the geometric realization of $V_\bullet$ to the geometric realization of $U_\bullet$.

Let $L : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ be a left adjoint to the inclusion. Let $|U_\bullet|$ and $|V_\bullet|$ be colimits of $U_\bullet$ and $V_\bullet$ in the ∞-category $\mathcal{P}(\mathcal{C})$, and let $|\alpha_\bullet| : |V_\bullet| \to |U_\bullet|$ be the induced map. We wish to show that $L|\alpha_\bullet|$ is an equivalence in $\mathcal{D}$. Since $\mathcal{C}$ is an n-category, we have inclusions $\mathrm{Ind}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{op}, \tau_{\leq n-1}\, \mathcal{S}) \subseteq \mathcal{P}(\mathcal{C})$. It follows that L factors through the truncation functor $\tau_{\leq n-1} : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$. Consequently, it will suffice to prove that $\tau_{\leq n-1}|\alpha_\bullet|$ is an equivalence in $\mathcal{P}(\mathcal{C})$. For this, it will suffice to show that the morphism $|\alpha_\bullet|$ is n-connective (in the sense of Definition T.6.5.1.10). This follows from Lemma T.6.5.3.10, since $\alpha_k : V_k \to U_k$ is an equivalence for $k \leq n$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 1.3.3.11. *Let $\mathcal{C}$ and $\mathcal{C}'$ be stable ∞-categories equipped with t-structures. Let $\theta : \mathrm{Fun}(\mathcal{C}, \mathcal{C}') \to \mathrm{Fun}(\mathcal{C}_{\geq 0}, \mathcal{C}')$ be the restriction map. Then:*

(1) *If $\mathcal{C}$ is right-bounded, then θ induces an equivalence from the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ spanned by the right t-exact functors to the full subcategory of $\mathrm{Fun}(\mathcal{C}_{\geq 0}, \mathcal{C}'_{\geq 0})$ spanned by the right exact functors.*

(2) *Let $\mathcal{C}$ and $\mathcal{C}'$ be left complete. Then the ∞-categories $\mathcal{C}_{\geq 0}$ and $\mathcal{C}'_{\geq 0}$ admit geometric realizations of simplicial objects. Furthermore, a functor $F : \mathcal{C}_{\geq 0} \to \mathcal{C}'_{\geq 0}$ is right exact if and only if it preserves finite coproducts and geometric realizations of simplicial objects.*

Proof. We first prove (1). If $\mathcal{C}$ is right bounded, then $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ is the (homotopy) inverse limit of the tower of ∞-categories

$$\ldots \to \mathrm{Fun}(\mathcal{C}_{\geq -1}, \mathcal{C}') \to \mathrm{Fun}(\mathcal{C}_{\geq 0}, \mathcal{C}'),$$

where the functors are given by restriction. The full subcategory of right t-exact functors is then given by the homotopy inverse limit

$$\ldots \to \mathrm{Fun}'(\mathcal{C}_{\geq -1}, \mathcal{C}'_{\geq -1}) \overset{\theta(0)}{\to} \mathrm{Fun}'(\mathcal{C}_{\geq 0}, \mathcal{C}'_{\geq 0})$$

where $\mathrm{Fun}'(\mathcal{C}, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the right exact functors. To complete the proof, it will suffice to show that this tower is essentially constant; in other words, that each $\theta(n)$ is an equivalence of ∞-categories. Without loss of generality, we may suppose that $n = 0$. Choose shift functors on the ∞-categories $\mathcal{C}$ and $\mathcal{C}'$, and define

$$\psi : \mathrm{Fun}'(\mathcal{C}_{\geq 0}, \mathcal{C}'_{\geq 0}) \to \mathrm{Fun}'(\mathcal{C}_{\geq -1}, \mathcal{C}'_{\geq -1})$$

by the formula $\psi(F) = \Sigma^{-1} \circ F \circ \Sigma$. We claim that ψ is a homotopy inverse to $\theta(0)$. To prove this, we observe that the right exactness of $F \in \mathrm{Fun}'(\mathcal{C}_{\geq 0}, \mathcal{C}'_{\geq 0})$, $G \in \mathrm{Fun}'(\mathcal{C}_{\geq -1}, \mathcal{C}'_{\geq -1})$ determines canonical equivalences

$$(\theta(0) \circ \psi)(F) \simeq F \qquad (\psi \circ \theta(0))(G) \simeq G.$$

We now prove (2). Since $\mathcal{C}$ is left complete, $\mathcal{C}_{\geq 0}$ is the (homotopy) inverse limit of the tower of ∞-categories $\{(\mathcal{C}_{\geq 0})_{\leq n}\}$ with transition maps given by right exact truncation functors. Lemma 1.3.3.10 implies that each $(\mathcal{C}_{\geq 0})_{\leq n}$ admits geometric realizations of simplicial objects, and that each of the truncation functors preserves geometric realizations of simplicial objects. It follows that $\mathcal{C}_{\geq 0}$ admits geometric realizations for simplicial objects. Similarly, $\mathcal{C}'_{\geq 0}$ admits geometric realizations for simplicial objects.

If F preserves finite coproducts and geometric realizations of simplicial objects, then F is right exact (Lemma 1.3.3.10). Conversely, suppose that F is right exact; we wish to prove that F preserves geometric realizations of simplicial objects. It will suffice to show that each composition

$$\mathcal{C}_{\geq 0} \overset{F}{\to} \mathcal{C}'_{\geq 0} \overset{\tau_{\leq n}}{\Rightarrow} (\mathcal{C}'_{\geq 0})_{\leq n}$$

preserves geometric realizations of simplicial objects. We observe that, in virtue of the right exactness of F, this functor is equivalent to the composition

$$\mathcal{C}_{\geq 0} \stackrel{\tau_{\leq n}}{\to} (\mathcal{C}_{\geq 0})_{\leq n} \stackrel{\tau_{\leq n} \circ F}{\to} (\mathcal{C}'_{\geq 0})_{\leq n}.$$

It will therefore suffice to prove that $\tau_{\leq n} \circ F$ preserves geometric realizations of simplicial objects, which follows from Lemma 1.3.3.10 since both the source and target are equivalent to n-categories. $\square$

Proposition 1.3.3.12. *Let $\mathcal{A}$ be an abelian category with enough projective objects, and let $\mathcal{C}$ be a stable ∞-category equipped with a left complete t-structure. Then the restriction functor*

$$\mathrm{Fun}(\mathcal{D}^-(\mathcal{A}), \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}(\mathcal{A}_{\mathrm{proj}}), \mathcal{C})$$

induces an equivalence from the full subcategory of $\mathrm{Fun}(\mathcal{D}^-(\mathcal{A}), \mathcal{C})$ spanned by the right t-exact functors to the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathcal{A}_{\mathrm{proj}}), \mathcal{C}_{\geq 0})$ spanned by functors which preserve finite coproducts (here $\mathcal{A}_{\mathrm{proj}}$ denotes the full subcategory of $\mathcal{A}$ spanned by the projective objects).

Proof. Let $\mathrm{Fun}'(\mathcal{D}^-(\mathcal{A}), \mathcal{C})$ be the full subcategory of $\mathrm{Fun}(\mathcal{D}^-(\mathcal{A}), \mathcal{C})$ spanned by the right t-exact functors. Lemma 1.3.3.11 implies that $\mathrm{Fun}'(\mathcal{D}^-(\mathcal{A}), \mathcal{C})$ is equivalent (via restriction) to the full subcategory

$$\mathrm{Fun}'(\mathcal{D}^-_{\geq 0}(\mathcal{A}), \mathcal{C}_{\geq 0}) \subseteq \mathrm{Fun}(\mathcal{D}^-_{\geq 0}(\mathcal{A}), \mathcal{C}_{\geq 0})$$

spanned by those functors which preserve finite coproducts and geometric realizations of simplicial objects. Theorem 1.3.3.8 allows us to identify $\mathrm{Fun}'(\mathcal{D}^-_{\geq 0}(\mathcal{A}), \mathcal{C}_{\geq 0})$ with the ∞-category of finite-coproduct preserving functors from $\mathrm{N}(\mathcal{A}_{\mathrm{proj}})$ into $\mathcal{C}_{\geq 0}$. $\square$

We now return to the proof of our main result.

Proof of Theorem 1.3.3.2. Proposition 1.3.3.12 implies that the restriction map $\mathcal{E} \to \mathrm{Fun}(\mathrm{N}(\mathcal{A}_{\mathrm{proj}}), \mathcal{C}^{\heartsuit})$ is fully faithful, and that the essential image of θ consists of the collection of coproduct-preserving functors from $\mathrm{N}(\mathcal{A}_{\mathrm{proj}})$ to $\mathcal{C}^{\heartsuit}$. Lemma 1.3.3.9 allows us to identify the latter ∞-category with the nerve of the category of right exact functors from $\mathcal{A}$ to the heart of $\mathcal{C}$. $\square$

We now turn to the proof of Theorem 1.3.3.8. Passing to a larger universe if necessary, we may assume without loss of generality that the abelian category $\mathcal{A}$ is small. We will analyze the ∞-category $\mathcal{D}^-(\mathcal{A})_{\geq 0}$ by embedding it into a larger ∞-category which admits sifted colimits. To this end, let $\mathcal{A}_{\mathrm{proj}} \subseteq \mathcal{A}$ denote the full subcategory of $\mathcal{A}$ spanned by the projective objects, and let $\mathbf{A}$ denote the category of product-preserving functors from $\mathcal{A}^{op}_{\mathrm{proj}}$ to the category of simplicial sets, as in §T.5.5.9. Let $\mathcal{A}^{\wedge}$ denote the category of product-preserving functors from $\mathcal{A}^{op}_{\mathrm{proj}}$ to sets, so that we can identify $\mathbf{A}$ with the category of simplicial objects of $\mathcal{A}^{\wedge}$. Our first goal is to understand the category $\mathcal{A}^{\wedge}$.

Proposition 1.3.3.13. *Let $\mathcal{A}$ be a small abelian category with enough projective objects. Then:*

(1) *The category $\mathcal{A}^{\wedge}$ can be identified with the category of Ind-objects of $\mathcal{A}$.*

(2) *The category $\mathcal{A}^{\wedge}$ is abelian.*

(3) *The abelian category $\mathcal{A}^{\wedge}$ has enough projective objects.*

Proof. Assertion (1) follows immediately from Lemma 1.3.3.9 (taking $\mathcal{B}$ to be the opposite of the category of sets). Assertion (2) follows formally from (1) and the assumption that $\mathcal{A}$ is an abelian category (see, for example, [5]). We may identify $\mathcal{A}$ with a full subcategory of $\mathcal{A}^{\wedge}$ via the Yoneda embedding. Moreover, if P is a projective object of $\mathcal{A}$, then P is also projective when viewed as an object of $\mathcal{A}^{\wedge}$. An arbitrary object of $\mathcal{A}^{\wedge}$ can be written as a filtered colimit $A = \varinjlim\{A_{\alpha}\}$, where each $A_{\alpha} \in \mathcal{A}$. Using the assumption that $\mathcal{A}$ has enough projective objects, we can choose epimorphisms $P_{\alpha} \to A_{\alpha}$, where each P_{α} is projective. We then have an epimorphism $\oplus P_{\alpha} \to A$. Since $\oplus P_{\alpha}$ is projective, we conclude that $\mathcal{A}^{\wedge}$ has enough projectives. $\square$

In what follows, we fix an abelian category $\mathcal{A}$ with enough projective objects, and let $\mathbf{A}$ denote the category of product-preserving functors $F : \mathcal{A}_{\mathrm{proj}}^{op} \to \mathrm{Set}_{\Delta}$. We will regard $\mathbf{A}$ as endowed with the simplicial model structure described in Proposition T.5.5.9.1. Note that $\mathbf{A}$ can be identified with the category of simplicial objects of the abelian category $\mathcal{A}^{\wedge}$. Corollary 1.3.2.12 implies that every object of $\mathbf{A}$ is fibrant. An object $M_{\bullet}$ of $\mathbf{A}$ is cofibrant if and only if the chain complex $N_{*}(M_{\bullet})$ consists of projective objects of $\mathcal{A}^{\wedge}$. Using Proposition 1.3.2.22, we obtain an equivalence of ∞-categories

$$\mathcal{D}_{\geq 0}^{-}(\mathcal{A}^{\wedge}) \simeq \mathrm{N}(\mathbf{A}^{\mathrm{o}}).$$

Composing with the equivalence of Corollary T.5.5.9.3, we obtain the following result:

Proposition 1.3.3.14. *Let $\mathcal{A}$ be an abelian category with enough projective objects. Then there exists an equivalence of ∞-categories*

$$\psi : \mathcal{D}_{\geq 0}^{-}(\mathcal{A}^{\wedge}) \to \mathcal{P}_{\Sigma}(\mathrm{N}(\mathcal{A}_{\mathrm{proj}}))$$

whose composition with the inclusion $\mathrm{N}(\mathcal{A}_{\mathrm{proj}}) \simeq \mathcal{D}^{-}(\mathcal{A}^{\wedge})^{\heartsuit} \subseteq \mathcal{D}_{\geq 0}^{-}(\mathcal{A}^{\wedge})$ is equivalent to the Yoneda embedding $\mathrm{N}(\mathcal{A}_{\mathrm{proj}}) \to \mathcal{P}_{\Sigma}(\mathrm{N}(\mathcal{A}_{\mathrm{proj}}))$.

Remark 1.3.3.15. We can identify $\mathcal{D}^{-}(\mathcal{A})$ with a full subcategory of $\mathcal{D}^{-}(\mathcal{A}^{\wedge})$. Moreover, an object $P_{\bullet} \in \mathcal{D}^{-}(\mathcal{A}^{\wedge})$ belongs to the essential image of $\mathcal{D}^{-}(\mathcal{A})$ if and only if the homologies $\mathrm{H}_{n}(P_{\bullet})$ belong to $\mathcal{A}$, for all $n \in \mathbf{Z}$.

Proposition 1.3.3.16. *Let $\mathcal{A}$ be an abelian category with enough projective objects. Then the t-structure on $\mathcal{D}^{-}(\mathcal{A})$ is right bounded and left complete.*

Proof. The right boundedness of $\mathcal{D}^{-}(\mathcal{A})$ is obvious. To prove the left completeness, we must show that $\mathcal{D}^{-}(\mathcal{A})$ is a homotopy inverse limit of the tower of ∞-categories

$$\ldots \to \mathcal{D}^{-}(\mathcal{A})_{\leq 1} \to \mathcal{D}^{-}(\mathcal{A})_{\leq 0} \to \ldots$$

Invoking the right boundedness of $\mathcal{D}^{-}(\mathcal{A})$, we may reduce to proving that for each $n \in \mathbf{Z}$, $\mathcal{D}^{-}(\mathcal{A})_{\geq n}$ is a homotopy inverse limit of the tower

$$\ldots \to \mathcal{D}^{-}(\mathcal{A})_{\leq 1, \geq n} \to \mathcal{D}^{-}(\mathcal{A})_{\leq 0, \geq n} \to \ldots$$

Shifting if necessary, we may suppose that $n = 0$. Using Remark 1.3.3.15, we can replace $\mathcal{A}$ by $\mathcal{A}^{\wedge}$. For each $k \geq 0$, we let $\mathcal{P}_{\Sigma}^{\leq k}(\mathrm{N}(\mathcal{A}_{\mathrm{proj}}))$ denote the ∞-category of product-preserving functors from $\mathrm{N}(\mathcal{A}_{\mathrm{proj}})^{op}$ to $\tau_{\leq k} \mathcal{S}$; equivalently, we can define $\mathcal{P}_{\Sigma}^{\leq k}(\mathrm{N}(\mathcal{A}_{\mathrm{proj}}))$ to be the ∞-category of k-truncated objects of $\mathcal{P}_{\Sigma}(\mathrm{N}(\mathcal{A}_{\mathrm{proj}}))$. We observe that the equivalence ψ of Proposition 1.3.3.14 restricts to an equivalence

$$\psi(k) : \mathcal{D}_{\geq 0}^{-}(\mathcal{A}^{\wedge}) \cap \mathcal{D}_{\leq k}^{-}(\mathcal{A}^{\wedge}) \simeq \mathcal{P}_{\Sigma}^{\leq k}(\mathrm{N}(\mathcal{A}_{\mathrm{proj}})).$$

Consequently, it will suffice to show that $\mathcal{P}_{\Sigma}(\mathrm{N}(\mathcal{A}_{\mathrm{proj}}))$ is a homotopy inverse limit for the tower

$$\ldots \to \mathcal{P}_{\Sigma}^{\leq 1}(\mathrm{N}(\mathcal{A}_{\mathrm{proj}})) \to \mathcal{P}_{\Sigma}^{\leq 0}(\mathrm{N}(\mathcal{A}_{\mathrm{proj}})).$$

Since the truncation functors on $\mathcal{S}$ commute with finite products (Lemma T.6.5.1.2), this follows from the observation that $\mathcal{S}$ is a homotopy inverse limit of the tower

$$\ldots \to \tau_{\leq 1} \mathcal{S} \to \tau_{\leq 0} \mathcal{S};$$

see Proposition T.7.2.1.10. $\square$

Lemma 1.3.3.17. *Let $\mathcal{A}$ be a small abelian category with enough projective objects, and let $\mathcal{C} \subseteq \mathcal{P}_{\Sigma}(\mathrm{N}(\mathcal{A}_{\mathrm{proj}}))$ be the essential image of $\mathcal{D}_{\geq 0}^{-}(\mathcal{A}) \subseteq \mathcal{D}_{\geq 0}^{-}(\mathcal{A}^{\wedge})$ under the equivalence $\psi : \mathcal{D}_{\geq 0}^{-}(\mathcal{A}^{\wedge}) \to \mathcal{P}_{\Sigma}(\mathrm{N}(\mathcal{A}_{\mathrm{proj}}))$ of Proposition 1.3.3.14. Then $\mathcal{C}$ is the smallest full subcategory of $\mathcal{P}(\mathrm{N}(\mathcal{A}_{\mathrm{proj}}))$ which is closed under geometric realizations and contains the essential image of the Yoneda embedding.*

Proof. It is clear that $\mathcal{C}$ contains the essential image of the Yoneda embedding. Lemma 1.3.3.11 implies that $\mathcal{D}_{\geq 0}^{-}(\mathcal{A})$ admits geometric realizations and that the inclusion $\mathcal{D}_{\geq 0}^{-}(\mathcal{A}) \subseteq \mathcal{D}_{\geq 0}^{-}(\mathcal{A}^{\wedge})$ preserves geometric realizations. It follows that $\mathcal{C}$ is closed under geometric realizations in $\mathcal{P}(\mathrm{N}(\mathcal{A}_{\mathrm{proj}}))$.

To complete the proof, we will show that every object of $X \in \mathcal{D}_{\geq 0}^{-}(\mathcal{A})$ can be obtained as the geometric realization, in $\mathcal{D}_{\geq 0}^{-}(\mathcal{A}^{\wedge})$, of a simplicial object $P_{\bullet}$ such that each $P_n \in \mathcal{D}_{\geq 0}^{-}(\mathcal{A}^{\wedge})$ consists of a projective object of $\mathcal{A}$, concentrated in degree zero. In fact, we can take $P_{\bullet}$ to be the simplicial object of $\mathcal{A}_{\mathrm{proj}}$ which corresponds to $X \in \mathrm{Ch}_{\geq 0}(\mathcal{A}_{\mathrm{proj}})$ under the Dold-Kan correspondence. It follows from Theorem T.4.2.4.1 and Proposition T.5.5.9.14 that X can be identified with the geometric realization of $P_{\bullet}$. $\square$

We are now ready to establish our characterization of $\mathcal{D}_{\geq 0}^{-}(\mathcal{A})$.

Proof of Theorem 1.3.3.8. Part (1) follows by combining Lemma 1.3.3.17, Remark T.5.3.5.9, and Proposition T.4.3.2.15. The "only if" direction of (2) is obvious. To prove the "if" direction, let us suppose that $F | \mathrm{N}(\mathcal{A}_0)$ preserves finite coproducts. We may assume without loss of generality that $\mathcal{C}$ admits filtered colimits (Lemma T.5.3.5.7), so that F extends to a functor $F' : \mathcal{D}_{\geq 0}^{-}(\mathcal{A}^{\wedge}) \to \mathcal{C}$ which preserves filtered colimits and geometric realizations (Propositions 1.3.3.14 and T.5.5.8.15). It follows from Proposition T.5.5.8.15 that F' preserves finite coproducts, so that $F = F' | \mathcal{D}_{\geq 0}^{-}(\mathcal{A})$ also preserves finite coproducts. $\square$

1.3.4 Inverting Quasi-Isomorphisms

Let $\mathcal{A}$ be an abelian category with enough projective objects, and let $\mathcal{D}$ denote its (right-bounded) derived category. The category $\mathcal{D}$ admits multiple descriptions:

(a) We can define $\mathcal{D}$ explicitly as the category whose objects are chain complexes P_* with each P_n a projective object of $\mathcal{A}$, and $P_n \simeq 0$ for $n \ll 0$; the morphisms in $\mathcal{D}$ are given by chain homotopy equivalence classes of morphisms of chain complexes.

(b) The category $\mathcal{D}$ can be obtained by starting with the category $\mathrm{Ch}^{-}(\mathcal{A})$ of *all* right bounded chain complexes with values in $\mathcal{A}$, and formally inverting quasi-isomorphisms.

In §1.3.2, we defined an ∞-category $\mathcal{D}^{-}(\mathcal{A})$, whose homotopy category $\mathrm{h}\mathcal{D}^{-}(\mathcal{A})$ is canonically equivalent to the derived category $\mathcal{D}$. Our definition $\mathcal{D}^{-}(\mathcal{A})$ can be regarded as an elaboration of (a): objects of $\mathcal{D}^{-}(\mathcal{A})$ are right-bounded chain complexes of projective objects of $\mathcal{A}$, morphisms are given by chain maps, 2-morphisms by chain homotopies, and so forth (see Definition 1.3.2.7). In this section, we will obtain an alternative description of the ∞-category $\mathcal{D}^{-}(\mathcal{A})$, which can be regarded as a generalization of (b). To formulate this description, we will need a bit of notation.

Definition 1.3.4.1. Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms in $\mathcal{C}$. We will say that a morphism $f : \mathcal{C} \to \mathcal{D}$ *exhibits $\mathcal{D}$ as the ∞-category obtained from $\mathcal{C}$ by inverting the set of morphisms W* if, for every ∞-category $\mathcal{E}$, composition with f induces a fully faithful embedding $\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$, whose essential image is the collection of functors $F : \mathcal{C} \to \mathcal{E}$ which carry each morphism in W to an equivalence in $\mathcal{E}$. In this case, the ∞-category $\mathcal{D}$ is determined uniquely up to equivalence by $\mathcal{C}$ and W, and will be denoted by $\mathcal{C}[W^{-1}]$.

If $\mathcal{C}$ is an ordinary category and W is a collection of morphisms in $\mathcal{C}$, we let $\mathcal{C}[W^{-1}]$ denote the ∞-category $\mathrm{N}(\mathcal{C})[W^{-1}]$.

Remark 1.3.4.2. For any ∞-category $\mathcal{C}$ and any collection W of morphisms of $\mathcal{C}$, the ∞-category $\mathcal{C}[W^{-1}]$ is defined: that is, there exists a functor $f : \mathcal{C} \to \mathcal{D}$ which exhibits $\mathcal{D}$ as the ∞-category obtained from $\mathcal{C}$ by inverting W. To prove this, we can assume without loss of generality that W contains all degenerate edges of $\mathcal{C}$, in which case $\mathcal{C}[W^{-1}]$ can be identified with a fibrant replacement for the pair $(\mathcal{C}, W)$ in the category $\mathrm{Set}_{\Delta}^{+}$ of marked simplicial sets (see §T.3.1).

Example 1.3.4.3. Let $\mathcal{C}$ be an ∞-category, and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a localization of $\mathcal{C}$: that is, $\mathcal{C}_0$ is a full subcategory of $\mathcal{C}$, and the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ admits a left adjoint L. Let W be the collection of those morphisms α in $\mathcal{C}$ such that $L(\alpha)$ is an equivalence. Then the composite functor $\mathcal{C}_0 \hookrightarrow \mathcal{C} \to \mathcal{C}[W^{-1}]$ is an equivalence of ∞-categories, so that we can identify $\mathcal{C}_0$ with the ∞-category $\mathcal{C}[W^{-1}]$ (see Proposition T.5.2.7.12).

We now can formulate our main result as follows:

Theorem 1.3.4.4. *Let $\mathcal{A}$ be an abelian category with enough projective objects, let $\mathbf{A} = \mathrm{Ch}^-(\mathcal{A})$ (regarded as an ordinary category), and let W be the collection of morphisms in $\mathbf{A}$ which are quasi-isomorphisms of chain complexes (that is, the collection of all morphisms which induce isomorphisms on homology). Then there is a canonical equivalence of ∞-categories $\mathbf{A}[W^{-1}] \simeq \mathcal{D}^-(\mathcal{A})$.*

The main ingredient in the proof of Theorem 1.3.4.4 is the following result, which we will prove later in this section:

Proposition 1.3.4.5. *Let $\mathcal{A}$ be an additive category, let $\mathrm{Ch}'(\mathcal{A})$ be a full subcategory of $\mathrm{Ch}(\mathcal{A})$, let $\mathbf{A}$ denote the underlying discrete category of $\mathrm{Ch}'(\mathcal{A})$, and let W be the collection of chain homotopy equivalences in $\mathbf{A}$. Assume that for every object $M_* \in \mathrm{Ch}'(\mathcal{A})$, the tensor product $N_*(\Delta^1) \otimes M_*$ (that is, the mapping cylinder of the identity map $\mathrm{id} : M_* \to M_*$) also belongs to $\mathrm{Ch}'(\mathcal{A})$. Then the inclusion $\theta : \mathrm{N}(\mathbf{A}) \hookrightarrow \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}'(\mathcal{A}))$ of Remark 1.3.1.9 induces an equivalence of ∞-categories $\theta' : \mathbf{A}[W^{-1}] \simeq \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}'(\mathcal{A}))$ (see Notation 4.1.3.3).*

We will also need the following observation:

Proposition 1.3.4.6. *Let $\mathcal{A}$ be an abelian category with enough projective objects. Then:*

(1) *The inclusion of ∞-categories $\mathcal{D}^-(\mathcal{A}) \hookrightarrow \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^-(\mathcal{A}))$ admits a right adjoint G.*

(2) *Let α be a morphism in $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^-(\mathcal{A}))$. Then $G(\alpha)$ is an equivalence in $\mathcal{D}^-(\mathcal{A})$ if and only if α is a quasi-isomorphism of chain complexes.*

(3) *Let W be the collection of all morphisms in $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^-(\mathcal{A}))$ which are quasi-isomorphisms of chain complexes. Then we have a canonical equivalence of ∞-categories*

$$\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^-(\mathcal{A}))[W^{-1}] \simeq \mathcal{D}^-(\mathcal{A}).$$

Proof. We first note that if $f : M_* \to N_*$ is a quasi-isomorphism of chain complexes with values in $\mathcal{A}$, then Lemma 1.3.2.20 implies that f induces a quasi-isomorphism

$$\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(P_*, M_*) \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(P_*, N_*)$$

for of chain complexes of abelian groups, for every object $P_* \in \mathcal{D}^-(\mathcal{A})$. Assertion $(*)$ in the proof of Proposition 1.3.2.19 shows that for every object $N_* \in \mathrm{Ch}^-(\mathcal{A})$, we can choose a quasi-isomorphism $f : M_* \to N_*$, where $M_* \in \mathcal{D}^-(\mathcal{A})$. Assertion (1) now follows from Proposition T.5.2.7.8. Moreover, we immediately deduce the "if" direction of (2). For the converse, suppose that $f : M_* \to N_*$ is a morphism in $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^-(\mathcal{A}))$ and that $G(f)$ is an equivalence; we wish to prove that f is a quasi-isomorphism. We have a commutative diagram

$$
\begin{array}{ccc}
G(M_*) & \xrightarrow{\ G(f)\ } & G(N_*) \\
\downarrow & & \downarrow \\
M_* & \xrightarrow{\ \ f\ \ } & N_*
\end{array}
$$

in the ∞-category $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^-(\mathcal{A}))$. The vertical maps are quasi-isomorphisms by construction, and the upper horizontal map is a chain homotopy equivalence. It follows that f is also a quasi-isomorphism. This completes the proof of (2); assertion (3) now follows from Example 1.3.4.3. $\square$

Proof of Theorem 1.3.4.4. Let $\mathbf{A}$ denote the ordinary category underlying $\mathrm{Ch}^{-}(\mathcal{A})$, let W be the collection of quasi-isomorphisms in $\mathbf{A}$, and let $W_0 \subseteq W$ be the collection of chain homotopy equivalences in $\mathbf{A}$. Propositions 1.3.4.5 and 1.3.4.6 supply equivalences of ∞-categories

$$\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^{-}(\mathcal{A})) \simeq \mathbf{A}[W_0^{-1}] \qquad \mathcal{D}^{-}(\mathcal{A}) \simeq \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^{-}(\mathcal{A}))[W^{-1}],$$

from which we immediately obtain an equivalence of ∞-categories $\mathcal{D}^{-}(\mathcal{A}) \simeq \mathbf{A}[W^{-1}]$. $\qquad\square$

We now turn to the proof of Proposition 1.3.4.5. We have already seen that the differential graded nerve $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}'(\mathcal{A}))$ is equivalent to the nerve of the simplicial category underlying $\mathrm{Ch}'(\mathcal{A})$ (Proposition 1.3.1.17). We therefore ask the following general question: given a simplicial category $\mathcal{C}$, under what circumstances can we recover the homotopy coherent nerve $\mathrm{N}(\mathcal{C})$ from the underlying ordinary category of $\mathcal{C}$, by inverting the class of homotopy equivalences? The following result provides a useful criterion:

Proposition 1.3.4.7. *Let $\mathcal{C}$ be a simplicial category and let W be a collection of morphisms in $\mathcal{C}$. Assume that the following conditions are satisfied:*

(1) *Every isomorphism in $\mathcal{C}$ belongs to W.*

(2) *Given a commutative diagram*

in $\mathcal{C}$, if any two of the morphisms f, g, and h belong to W, then so does the third.

(3) *For every object $X \in \mathcal{C}$, there exists an interval object $\Delta^1 \otimes X$ equipped with a map $h : \Delta^1 \to \mathrm{Map}_{\mathcal{C}}(X, \Delta^1 \otimes X)$ having the following universal property: for every object $Y \in \mathcal{C}$, composition with h determines an isomorphism of simplicial sets*

$$\mathrm{Map}_{\mathcal{C}}(\Delta^1 \otimes X, Y) \to \mathrm{Map}_{\mathrm{Set}_\Delta}(\Delta^1, \mathrm{Map}_{\mathcal{C}}(X, Y)).$$

(4) *For each $X \in \mathcal{C}$, the canonical map $\Delta^1 \otimes X \to X$ belongs to W.*

Let $\mathcal{C}_0$ denote the underlying ordinary category of $\mathcal{C}$ and let $\mathcal{C}'$ be a fibrant replacement for $\mathcal{C}$ (in the category Cat_Δ of simplicial categories). Then the canonical map $\theta : (\mathrm{N}(\mathcal{C}_0), W) \to (\mathrm{N}(\mathcal{C}'), W)$ is a weak equivalence of marked simplicial sets.

Example 1.3.4.8. If $\mathcal{C}$ is a *fibrant* simplicial category, then we can take $\mathcal{C}' = \mathcal{C}$. Assume that $\mathcal{C}$ admits interval objects (that is, $\mathcal{C}$ satisfies condition (3) of Proposition 1.3.4.7), and let W be the collection of homotopy equivalences in $\mathcal{C}$. Then Proposition 1.3.4.7 provides an equivalence of ∞-categories $\mathcal{C}_0[W^{-1}] \simeq \mathrm{N}(\mathcal{C})$.

Warning 1.3.4.9. As stated, Proposition 1.3.4.7 does quite apply to the underlying simplicial category of the differential graded category $\mathrm{Ch}'(\mathcal{A})$ appearing in Proposition 1.3.4.5. The problem is with hypothesis (3): the construction $(K, M_*) \mapsto N_*(K) \otimes M_*$ does not exhibit $\mathrm{Ch}(\mathcal{A})$ as tensored over the category of simplicial sets (see Warning 1.3.5.4). However, we obtain a proof of Proposition 1.3.4.5 by slightly modifying the proof of Proposition 1.3.4.7 given below.

The proof of Proposition 1.3.4.7 will require some preliminaries.

Lemma 1.3.4.10. *Let $\mathcal{J}$ be a category and suppose we are given a diagram $u : \mathcal{J} \to \mathrm{Cat}_\Delta$, which we will denote by $\alpha \mapsto \mathcal{C}_\alpha$, together with a map of simplicial categories $\theta : \varinjlim \mathcal{C}_\alpha \to \mathcal{C}$. Suppose furthermore that the following conditions are satisfied:*

(1) *For every index α, the simplicial functor $f_\alpha : \mathcal{C}_\alpha \to \mathcal{C}$ is bijective on objects.*

(2) *For every pair of objects $x, y \in \mathcal{C}$, the canonical map*

$$\varinjlim_{\alpha} \operatorname{Map}_{\mathcal{C}_\alpha}(f_\alpha^{-1} x, f_\alpha^{-1} y) \to \operatorname{Map}_{\mathcal{C}}(x, y)$$

exhibits $\operatorname{Map}_{\mathcal{C}}(x, y)$ as a homotopy colimit of the diagram of simplicial sets $\{\operatorname{Map}_{\mathcal{C}_\alpha}(f_\alpha^{-1} x, f_\alpha^{-1} y)\}_{\alpha \in \mathcal{J}}$.

(3) *The ∞-category $\mathrm{N}(\mathcal{J})$ is sifted.*

Then θ exhibits $\mathcal{C}$ as a homotopy colimit of the diagram $\{\mathcal{C}_\alpha\}$.

Proof. Let S be the set of objects of $\mathcal{C}$. Let $\mathbf{A} = \operatorname{Set}_\Delta^{S \times S}$ be the category whose objects are collections of simplicial sets $\{X_{s,t}\}_{s, t \in S}$. The standard model structure on $\operatorname{Set}_\Delta$ determines a model structure on the category $\mathbf{A}$, where fibrations, cofibrations, and weak equivalences are given pointwise. Note that $\mathbf{A}$ has the structure of a monoidal model category, where the tensor product is given by the formula

$$(X \otimes Y)_{s,t} = \coprod_{u \in S} X_{s,u} \times Y_{u,t}.$$

Since every object of $\mathbf{A}$ is cofibrant, Proposition 4.1.4.3 implies that $\operatorname{Alg}(\mathbf{A})$ inherits the structure of a combinatorial model category. The objects of $\operatorname{Alg}(\mathbf{A})$ can be identified simplicial categories whose underlying set of objects is S. Consequently, we have a forgetful functor $F : \operatorname{Alg}(\mathbf{A}) \to (\operatorname{Cat}_\Delta)_{S/}$ (where we regard S as a simplicial category via the formula $\operatorname{Map}_S(s, t) = \begin{cases} \Delta^0 & \text{if } s = t \\ \emptyset & \text{if } s \neq t. \end{cases}$). Note that $\mathcal{C} \simeq F(A)$ for some $A \in \operatorname{Alg}(\mathbf{A})$.

Condition (1) implies that u is isomorphic to a functor given by the composition $\mathcal{J} \xrightarrow{u'} \operatorname{Alg}(\mathbf{A}) \xrightarrow{F} \operatorname{Cat}_\Delta$, and that θ is obtained from a map $\theta' : \varinjlim(u') \to A$ in $\operatorname{Alg}(\mathbf{A})$. Since F is a left Quillen functor which preserves weak equivalences, it preserves homotopy colimits; consequently, it will suffice to show that θ' exhibits A as a homotopy colimit of u' in the model category $\operatorname{Alg}(\mathbf{A})$. Using condition (3) and Lemma 4.1.4.13, we are reduced to proving that θ' exhibits A as a homotopy colimit of u' in the model category $\mathbf{A}$, which is equivalent to condition (2). $\qquad\square$

Notation 1.3.4.11. Let $\mathcal{C}$ be a simplicial category. We can regard $\mathcal{C}$ as a simplicial object in the category Cat: that is, for each $[n] \in \boldsymbol{\Delta}$, let $\mathcal{C}_n$ denote the category whose objects are the objects of $\mathcal{C}$, with $\operatorname{Hom}_{\mathcal{C}_n}(x, y) = \operatorname{Hom}_{\operatorname{Set}_\Delta}(\Delta^n, \operatorname{Map}_{\mathcal{C}}(x, y))$.

For $[n] \in \boldsymbol{\Delta}$, let $t_{[n]} : \operatorname{Set}_\Delta \to \operatorname{Set}_\Delta$ be the translation functor described by the formula

$$\operatorname{Hom}_{\operatorname{Set}_\Delta}(X, t_{[n]} Y) \simeq \operatorname{Hom}_{\operatorname{Set}_\Delta}(\Delta^n \star X, Y).$$

Then $t_{[n]}$ preserves products and therefore induces a functor $T_{[n]} : \operatorname{Cat}_\Delta \to \operatorname{Cat}_\Delta$. For any simplicial category $\mathcal{C}$, there is an evident pair of maps

$$\mathcal{C}_n \leftarrow T_{[n]} \mathcal{C} \to \mathcal{C}.$$

Allowing $[n]$ to vary, we obtain a simplicial object $T_\bullet \mathcal{C}$ of $\operatorname{Cat}_\Delta$, equipped with a map of simplicial objects $T_\bullet \mathcal{C} \to \mathcal{C}_\bullet$ and a map of simplicial categories $\varinjlim T_\bullet \mathcal{C} \to \mathcal{C}$.

Lemma 1.3.4.12. *Let X be a simplicial set. Then for each $m \geq 0$, the canonical map*

$$t_{[m]} X \to \operatorname{Hom}_{\operatorname{Set}_\Delta}(\Delta^m, X)$$

is a homotopy equivalence.

Proof. Unwinding the definitions, we have an isomorphism $t_{[m]} X \simeq \coprod_{\sigma : \Delta^m \to X} X_{\sigma/}$, where each summand is weakly contractible. $\qquad\square$

Lemma 1.3.4.13. *Let X be a simplicial set. Then the canonical map $\varinjlim t_{[n]}(X) \to X$ exhibits X as a homotopy colimit of the diagram $[n] \mapsto t_{[n]}(X)$.*

Proof. Since the collection of homotopy colimit diagrams is stable under filtered colimits, we may assume without loss of generality that X is finite. We work by induction on the dimension n of X and the number of nondegenerate n-simplices of X. If X is empty, the result is obvious; otherwise, we have a homotopy pushout diagram

$$\begin{array}{ccc} \partial\,\Delta^n & \longrightarrow & \Delta^n \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X. \end{array}$$

Using the inductive hypothesis, we deduce that X' and $\partial\,\Delta^n$ can be identified with the homotopy colimit of the diagrams $t_\bullet X'$ and $t_\bullet(\partial\,\Delta^n)$. Consequently, to prove that X is the homotopy colimit of $t_\bullet X$, it suffices to show that Δ^n is a homotopy colimit of $t_\bullet\Delta^n$: in other words, we wish to show that the homotopy colimit of $t_\bullet\Delta^n$ is contractible. Using Lemma 1.3.4.12, we are reduced to showing that the diagram $[m] \mapsto \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, \Delta^n)$ has a contractible homotopy colimit, which follows from Corollary T.A.2.9.30. $\square$

Proposition 1.3.4.14. *Let $\mathcal{C}$ be a simplicial category. Then $\mathcal{C}$ can be identified with the geometric realization of the simplicial object $\mathcal{C}_\bullet$ in $\mathrm{Cat} \subseteq \mathrm{Cat}_\Delta$. More precisely, the map $T_\bullet\,\mathcal{C} \to \mathcal{C}_\bullet$ is a weak equivalence of simplicial objects of Cat_Δ, and the map $\varinjlim T_\bullet\,\mathcal{C} \to \mathcal{C}$ exhibits $\mathcal{C}$ as a homotopy colimit of the simplicial object $T_\bullet\,\mathcal{C}$.*

Proof. We first claim that for $n \geq 0$, the canonical map $T_{[n]}\,\mathcal{C} \to \mathcal{C}_n$ is a weak equivalence of simplicial categories. Since this map is bijective on objects, it suffices to show that for every pair of objects $x, y \in \mathcal{C}$, the map $\mathrm{Map}_{T_{[n]}\,\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}_n}(x, y)$ is a weak homotopy equivalence, which follows from Lemma 1.3.4.12.

We next show that $\mathcal{C}$ can be identified with the homotopy colimit of the diagram $T_\bullet\,\mathcal{C}$. Using Lemma 1.3.4.10, we are reduced to proving that for every pair of objects $x, y \in \mathcal{C}$, the $\mathrm{Map}_\mathcal{C}(x, y)$ is a homotopy colimit of the diagram $[n] \mapsto t_{[n]}\,\mathrm{Map}_\mathcal{C}(x, y)$; this follows from Lemma 1.3.4.13. $\square$

Proof of Proposition 1.3.4.7. We let $|\mathcal{C}|$ denote the topological category obtained from the simplicial category $\mathcal{C}$ by geometric realization (that is, $|\mathcal{C}|$ has the same objects as $\mathcal{C}$, with morphism spaces given by $\mathrm{Map}_{|\mathcal{C}|}(X, Y) = |\mathrm{Map}_\mathcal{C}(X, Y)|$). We may assume without loss of generality that $\mathcal{C}'$ is the underlying simplicial category of $|\mathcal{C}|$, so that the homotopy coherent nerve $\mathrm{N}(\mathcal{C}')$ is isomorphic to the nerve of the topological category $|\mathcal{C}|$.

Fix an object $X \in \mathcal{C}$, and let $h : \Delta^1 \to \mathrm{Map}_\mathcal{C}(X, \Delta^1 \otimes X)$ be as in (3). Restricting to vertices, we obtain a pair of morphisms $h_0, h_1 : X \to \Delta^1 \otimes X$. These maps are both right inverse to the projection $\Delta^1 \otimes X \to X$. Using (1), (2), and (4), we deduce that h_0 and h_1 belong to W. Suppose that $f : X \to Y$ is a morphism in $\mathcal{C}$ which belongs to W, and that $f' : X \to Y$ is another morphism which is connected to f by an edge of $\mathrm{Map}_\mathcal{C}(X, Y)$. This edge determines a map $F : \Delta^1 \otimes X \to Y$ such that $f = F \circ h_0$ and $f' = F \circ h_1$. Using (2) we deduce that $F \in W$ and therefore $f' \in W$. It follows that if $g : X \to Y$ is any morphism which belongs to the same connected component of $\mathrm{Map}_\mathcal{C}(X, Y)$ as f, then $g \in W$.

For each $n \geq 0$ let W_n be the collection of morphisms in $\mathcal{C}_n$ which correspond to maps $\Delta^n \to \mathrm{Map}_\mathcal{C}(X, Y)$ which carry each vertex to an element of W, and let W'_n be the collection of morphisms in the simplicial category $T_{[n]}\,\mathcal{C}$ whose image in $\mathcal{C}_n$ belongs to W_n. The map θ factors as a composition

$$(\mathrm{N}(\mathcal{C}_0), W) \xrightarrow{\theta'} \mathrm{hocolim}(\mathrm{N}(|T_\bullet\,\mathcal{C}|), W'_\bullet) \xrightarrow{\theta''} (\mathrm{N}(|\mathcal{C}|), W).$$

It will therefore suffice to prove that θ' and θ'' are weak equivalences. For the map θ'', this is an immediate consequence of Proposition 1.3.4.14.

We now show that θ' is a weak equivalence. Consider the composite map

$$(\mathrm{N}(\mathcal{C}_0), W) \xrightarrow{\theta'} \mathrm{hocolim}(\mathrm{N}(|T_\bullet\,\mathcal{C}|), W'_\bullet) \xrightarrow{\phi} \mathrm{hocolim}(\mathrm{N}(\mathcal{C}_\bullet), W_\bullet).$$

Each of the maps $T_n \, \mathcal{C} \to \mathcal{C}_n$ is a weak equivalence of simplicial categories, so that ϕ is a weak equivalence of marked simplicial sets. It therefore suffices to show that the composite map $\phi \circ \theta'$ is a weak equivalence. Since $N(\boldsymbol{\Delta})^{op}$ is weakly contractible, it will suffice to show that $[n] \mapsto (N(\mathcal{C}_n), W_n)$ determines a constant diagram in the homotopy category $h\mathcal{S}et_\Delta^+$. Equivalently, we must show that for each $n \geq 0$, the map $[n] \to [0]$ induces a weak equivalence of marked simplicial sets $F : (N(\mathcal{C}_0), W_0) \to (N(\mathcal{C}_n), W_n)$. Let $G : (N(\mathcal{C}_n), W_n) \to (N(\mathcal{C}_0), W_0)$ be the map induced by the inclusion $[0] \hookrightarrow [n]$. Then $G \circ F = \mathrm{id}$. To complete the proof, it will suffice to show that $F \circ G$ is homotopic to the identity map. To this end, we define a functor $U : \mathcal{C}_n \to \mathcal{C}_n$ as follows:

- On objects, U is given by $U(X) = \Delta^1 \otimes X$.

- Let $\alpha : X \to Y$ be a morphism in $\mathcal{C}_n$ corresponding to a map $\Delta^n \to \mathrm{Map}_{\mathcal{C}}(X, Y)$. Then $U(\alpha)$ corresponds to the map $\Delta^n \to \mathrm{Map}_{\mathcal{C}}(\Delta^1 \otimes X, \Delta^1 \otimes Y)$ given by

$$\Delta^n \times \Delta^1 \xrightarrow{r} \Delta^n \times \Delta^1 \xrightarrow{\alpha} \mathrm{Map}_{\mathcal{C}}(X, Y) \times \Delta^1 \to \mathrm{Map}_{\mathcal{C}}(X, \Delta^1 \otimes Y),$$

  where r is given on vertices by the formula

$$r(i, j) = \begin{cases} (0, 0) & \text{if } j = 0 \\ (i, 1) & \text{if } j = 1. \end{cases}$$

The inclusions $\{0\} \hookrightarrow \Delta^1 \hookleftarrow \{1\}$ determine natural transformations $F \circ G \to U \leftarrow \mathrm{id}$. Conditions (1), (2), and (4) guarantee that these transformations are given by morphisms in W_n, and therefore determine a homotopy from $F \circ G$ to the identity. $\qquad\square$

We now prove Proposition 1.3.4.5 using a similar strategy.

Proof of Proposition 1.3.4.5. For each $[n] \in \boldsymbol{\Delta}$, let $\mathcal{C}_n$ denote the category whose objects are the objects of $\mathrm{Ch}'(\mathcal{A})$, with morphisms given by

$$\mathrm{Hom}_{\mathcal{C}_n}(M_*, M_*') \simeq \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(N_*(\Delta^n) \otimes M_*, M_*').$$

Then $\mathcal{C}_\bullet$ is a simplicial object of $\mathcal{C}at$ arising from a simplicial category $\mathcal{C}$, and Proposition 1.3.1.17 supplies an equivalence of ∞-categories $N(\mathcal{C}) \to N_{\mathrm{dg}}(\mathrm{Ch}'(\mathcal{A}))$. It will therefore suffice to show that the inclusion $N(\mathcal{C}_0) \hookrightarrow N(\mathcal{C})$ induces an equivalence $\mathcal{C}_0[W^{-1}] \to N(\mathcal{C})$. For each $n \geq 0$, let W_n be the collection of morphisms in the category $\mathcal{C}_n$ defined as in the proof of Proposition 1.3.4.7.

Let $T_\bullet \, \mathcal{C}$ be the simplicial object of $\mathcal{C}at_\Delta$ introduced in Notation 1.3.4.11, and let $\overline{W}_n$ be the collection of morphisms in $T_n \, \mathcal{C}$ whose images in $\mathcal{C}_n$ belong to W_n. Proposition 1.3.4.14 yields weak equivalences of simplicial categories

$$\mathrm{hocolim} \, \mathcal{C}_\bullet \leftarrow \mathrm{hocolim} \, T_\bullet \, \mathrm{Ch}'(\mathcal{A}) \to \mathrm{Ch}'(\mathcal{A}).$$

Moreover, the map on the right carries each morphism in $\overline{W}_n$ to a homotopy equivalence in $\mathrm{Ch}'(\mathcal{A})$, so we obtain categorical equivalences

$$\mathrm{hocolim} \, N(\mathcal{C}_\bullet)[W_\bullet^{-1}]] \leftarrow \mathrm{hocolim} \, N(T_\bullet \, \mathrm{Ch}'(\mathcal{A}))[\overline{W}_\bullet^{-1}] \to N(\mathrm{Ch}'(\mathcal{A})).$$

The map θ factors naturally through $N(T_0 \, \mathrm{Ch}'(\mathcal{A}))$ so that the composite map $N(\mathbf{A}) \to N(T_0 \, \mathrm{Ch}'(\mathcal{A})) \to N(\mathcal{C}_0)$ is an isomorphism. Consequently, to prove that θ' is an equivalence of ∞-categories, it will suffice to show that the natural map $N(\mathcal{C}_0)[W_0^{-1}] \to \mathrm{hocolim} \, N(\mathcal{C}_\bullet)[W_\bullet^{-1}]$ is a categorical equivalence. We will prove this by showing that the diagram $N(\mathcal{C}_\bullet)[W_\bullet^{-1}]$ is essentially constant: that is, for each $n \geq 0$, the map $[n] \to [0]$ in $\boldsymbol{\Delta}$ induces a weak equivalence of marked simplicial sets

$$F : (N(\mathcal{C}_0), W_0^{-1}) \to (N(\mathcal{C}_n), W_n^{-1}).$$

Let $G : (\mathrm{N}(\mathcal{C}_n), W_n) \to (\mathrm{N}(\mathcal{C}_0), W_0)$ be the map induced by the inclusion $[0] \hookrightarrow [n]$. Then $G \circ F = \mathrm{id}$. To complete the proof, it will suffice to show that $F \circ G$ is homotopic to the identity map in the model category $\mathcal{S}et_\Delta^+$. We will prove this by an explicit construction.

Given a sequence $i_0 < i_1 < \cdots < i_k$ corresponding to a nondegenerate k-simplex of Δ^m, we let $[i_0, \dots, i_k]$ denote the corresponding generator for the free abelian group $N_*(\Delta^k)$. We define a map of chain complexes $\chi : N_*(\Delta^n) \otimes N_*(\Delta^1) \to N_*(\Delta^1) \otimes N_*(\Delta^n)$ by the formula

$$\chi([i_0, \dots, i_k] \otimes [0]) = \begin{cases} [0] \otimes [0] & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\chi([i_0, \dots, i_k] \otimes [1]) = [1] \otimes [i_0, \dots, i_k]$$

$$\chi([i_0, \dots, i_k] \otimes [0,1]) = \begin{cases} (-1)^k [0,1] \otimes [i_0, \dots, i_k] & \text{if } i_0 = 0 \\ (-1)^k [0,1] \otimes [i_0, \dots, i_k] + (-1)^k [0] \otimes [0, i_0, \dots, i_k] & \text{if } i_0 > 0. \end{cases}$$

For objects $M_*, M_*' \in \mathrm{Ch}'(\mathcal{A})$, every morphism $\phi \in \mathrm{Hom}_{\mathcal{C}_n}(M_*, M_*')$ determines a map

$$\phi' : N_*(\Delta^n) \otimes N_*(\Delta^1) \otimes M_* \xrightarrow{\chi} N_*(\Delta^1) \otimes N_*(\Delta^n) \otimes M_* \xrightarrow{\phi} N_*(\Delta^1) \otimes M_*'.$$

The construction $\phi \mapsto \phi'$ determines a functor U from $\mathcal{C}_n$ to itself, given on objects by $M_* \mapsto N_*(\Delta^1) \otimes M_*$. Moreover, the inclusions $\{0\} \hookrightarrow \Delta^1 \hookleftarrow \{1\}$ determine natural transformations

$$F \circ G \to U \leftarrow \mathrm{id}.$$

Since the underlying maps $M_* \to N_*(\Delta^1) \otimes M_*$ are chain homotopy equivalences, these natural transformations show that id and $F \circ G$ are both homotopic to U, and are therefore homotopic to each other. $\qquad\square$

We conclude this section by describing some other applications of Proposition 1.3.4.7. Recall that we can associate an ∞-category to each simplicial model category $\mathbf{A}$, via the construction $\mathbf{A} \mapsto \mathrm{N}(\mathbf{A}^\circ)$ (here $\mathbf{A}^\circ$ denotes the full subcategory of $\mathbf{A}$ spanned by the fibrant-cofibrant objects). However, many model categories which naturally arise which are not simplicial (we will study some examples in §1.3.5). In these cases, we cannot produce an ∞-category directly using the homotopy coherent nerve. However, we can still associate an underlying ∞-category via the following procedure:

Definition 1.3.4.15. Let $\mathbf{A}$ be a model category. We let $\mathbf{A}^c$ denote the full subcategory of $\mathbf{A}$ spanned by the cofibrant objects. Let $\mathcal{C}$ be an ∞-category. We will say that a functor $f : \mathrm{N}(\mathbf{A}^c) \to \mathcal{C}$ *exhibits* $\mathcal{C}$ *as the underlying ∞-category of* $\mathbf{A}$ if f induces an equivalence $\mathrm{N}(\mathbf{A}^c)[W^{-1}] \simeq \mathcal{C}$, where W is the collection of weak equivalences in $\mathbf{A}^c$.

Remark 1.3.4.16. In Definition 1.3.4.15, we restrict our attention to cofibrant objects of $\mathbf{A}$ in order to facilitate applications to the study of monoidal model categories: if $\mathbf{A}$ is a monoidal model category, then the tensor product on $\mathbf{A}^c$ preserves weak equivalences. Other variations on Definition 1.3.4.15 are possible. For example, we could define the underlying ∞-category of $\mathbf{A}$ to be the ∞-category obtained from $\mathbf{A}$, from the fibrant objects of $\mathbf{A}$, or from the fibrant-cofibrant objects of $\mathbf{A}$, by formally inverting all weak equivalences. If we assume that the morphisms $f : X \to Y$ in $\mathbf{A}$ admit factorizations

$$X \xrightarrow{\alpha} U(f) \xrightarrow{\beta} Y \qquad X \xrightarrow{\alpha'} V(f) \xrightarrow{\beta'} Y$$

(where α is a cofibration, α' a trivial cofibration, β a trivial fibration, and β' a fibration) which can be chosen functorially in f, then all of these approaches are equivalent to Definition 1.3.4.15. The functorial factorization condition is always satisfied in practice (and is sometimes taken as part of the definition of a model category); it is automatic, for example, if $\mathbf{A}$ is a combinatorial model category.

Our goal now is to show that if **A** is a simplicial model category, then the underlying ∞-category of **A** is given by the homotopy coherent nerve $N(\mathbf{A}^o)$ (Theorem 1.3.4.20). We begin by constructing a functor from the ordinary category $\mathbf{A}^c$ to the ∞-category $N(\mathbf{A}^o)$.

Notation 1.3.4.17. In what follows, we will always regard $\mathbf{A}^c$ as a *discrete* category, even in cases where **A** is equipped with the structure of a simplicial model category.

Construction 1.3.4.18. Let **A** be a simplicial model category. We define a simplicial category $\mathcal{M}$ as follows:

(1) An object of $\mathcal{M}$ is a pair (i, A), where $i \in \{0, 1\}$ and A is a cofibrant object of **A**, which is fibrant when $i = 1$.

(2) Given a pair of objects (i, A) and (j, B) in $\mathcal{M}$, we have

$$\mathrm{Map}_{\mathcal{M}}((i, A), (j, B)) = \begin{cases} \mathrm{Map}_{\mathbf{A}}(A, B) & \text{if } j = 1 \\ \mathrm{Hom}_{\mathbf{A}}(A, B) & \text{if } i = j = 0 \\ \emptyset & \text{if } j = 0 < 1 = i. \end{cases}$$

We note that the mapping spaces in $\mathcal{M}$ are fibrant, so that $N(\mathcal{M})$ is an ∞-category. There is an evident forgetful functor $p : N(\mathcal{M}) \to \Delta^1$, which exhibits $N(\mathcal{M})$ as a correspondence from $N(\mathcal{M})_0 \simeq N(\mathbf{A}^c)$ to $N(\mathcal{M})_1 \simeq N(\mathbf{A}^o)$.

Lemma 1.3.4.19. *Let **A** be a simplicial model category, and let $\mathcal{M}$ be defined as in Construction 1.3.4.18. Then the projection map $p : N(\mathcal{M}) \to \Delta^1$ is a coCartesian fibration. Moreover, if $f : (i, A) \to (j, B)$ is a morphism in $N(\mathcal{M})$ with $i = 0 < 1 = j$, then f is p-coCartesian if and only if the induced map $A \to B$ is a weak equivalence in **A**.*

Proof. Choose an object $(0, A) \in N(\mathcal{M})_0$, and choose a trivial cofibration $A \to B$ where B is a fibrant object of **A**. We will show that the induced map $(0, A) \to (1, B)$ in $N(\mathcal{M})$ is p-coCartesian. This will prove that p is a coCartesian fibration and the "if" direction of the final assertion; the "only if" will then follow from the uniqueness of coCartesian morphisms up to equivalence. Using Proposition T.2.4.4.3, we are reduced to proving the following assertion: for every object $C \in \mathbf{A}^o$, composition with f induces a homotopy equivalence $\mathrm{Map}_{\mathbf{A}}(B, C) \to \mathrm{Map}_{\mathbf{A}}(A, C)$. This follows from the fact that C is fibrant and f is a weak equivalence between cofibrant objects. $\square$

It follows from Lemma 1.3.4.19 that the correspondence $N(\mathcal{M}) \to \Delta^1$ determines a functor $\theta : N(\mathbf{A}^c) \to N(\mathbf{A}^o)$, which is well-defined up to equivalence. We now have the following result:

Theorem 1.3.4.20. *[Dwyer-Kan] Let **A** be a simplicial model category, let $\theta : N(\mathbf{A}^c) \to N(\mathbf{A}^o)$ be the functor constructed above, and let W be the collection of weak equivalences in $\mathbf{A}^c$. Then θ induces an equivalence $\mathbf{A}^c[W^{-1}] \to N(\mathbf{A}^o)$ (that is, θ exhibits $N(\mathbf{A}^o)$ as the underlying ∞-category of **A**, in the sense of Definition 1.3.4.15).*

Before giving the proof of Theorem 1.3.4.20, let us collect a few consequences concerning the structure of the underlying ∞-category of an arbitrary combinatorial model category.

Lemma 1.3.4.21. *Let $F : \mathbf{A} \to \mathbf{B}$ be a left Quillen equivalence between combinatorial model categories. Let $\mathbf{A}^c$ and $\mathbf{B}^c$ denote the full subcategories of **A** and **B** spanned by the cofibrant objects, and let $W_{\mathbf{A}}$ and $W_{\mathbf{B}}$ be the collection of weak equivalences in $\mathbf{A}^c$ and $\mathbf{B}^c$, respectively. Then F induces a weak equivalence of marked simplicial sets*

$$f : (N(\mathbf{A}^c), W_{\mathbf{A}}) \to (N(\mathbf{B}^c), W_{\mathbf{B}}).$$

In other words, a left Quillen equivalence between combinatorial model categories induces an equivalence between their underlying ∞-categories.

Proof. Since $\mathbf{A}$ is combinatorial, there exists a cofibrant replacement functor $P : \mathbf{A} \to \mathbf{A}$. That is, P is a functor equipped with a natural transformation $u : P \to \mathrm{id}$ such that, for every object $X \in \mathbf{A}$, the induced map $u_X : P(X) \to X$ is a weak equivalence and $P(X)$ is cofibrant. Similarly, we can choose a fibrant replacement functor $Q : \mathbf{B} \to \mathbf{B}$. Let G be a right adjoint to F, and let $G' : \mathbf{B}^c \to \mathbf{A}^c$ be the functor given by the composition

$$\mathbf{B}^c \overset{Q}{\to} \mathbf{B} \overset{G}{\to} \mathbf{A} \overset{P}{\to} \mathbf{A}^c.$$

Since P and Q preserve weak equivalences and G preserves weak equivalences between fibrant objects, we conclude that G' carries $W_{\mathbf{B}}$ into $W_{\mathbf{A}}$, and therefore induces a map of marked simplicial sets

$$g' : (\mathrm{N}(\mathbf{B}^c), W_{\mathbf{B}}) \to (\mathrm{N}(\mathbf{A}^c), W_{\mathbf{A}}).$$

We claim that this map is homotopy inverse to f. We will argue that $g' \circ f$ is homotopic to the identity; the proof for $f \circ g'$ is similar. We have a diagram of natural transformations

$$\mathrm{id}_{\mathbf{A}^c} \leftarrow P \to P \circ (G \circ F) \to (P \circ G \circ Q) \circ F \simeq G' \circ F.$$

It will therefore suffice to show that for every cofibrant object $X \in \mathbf{A}^c$, the resulting maps

$$X \overset{u_X}{\leftarrow} P(X) \overset{v_X}{\to} (G' \circ F)(X)$$

are weak equivalences. For the map u_X, this is clear. The map v_X fits into a commutative diagram

$$
\begin{array}{ccc}
P(X) & \overset{v_X}{\longrightarrow} & (G' \circ F)(X) \\
\downarrow & & \downarrow \\
X & \overset{v'_X}{\longrightarrow} & (G \circ Q \circ F)(X)
\end{array}
$$

where the vertical maps are weak equivalences. It will therefore suffice to show that v'_X is a weak equivalence. Since X is cofibrant and $(Q \circ F)(X)$ is a fibrant object of $\mathbf{B}$, our assumption that F is a Quillen equivalence shows that v'_X is a weak equivalence if and only if the adjoint map $F(X) \to (Q \circ F)(X)$ is a weak equivalence, which is clear. □

It follows from the main result of [36] that every combinatorial model category is Quillen equivalent to a combinatorial simplicial model category. Combining this result, Lemma 1.3.4.21, Theorem 1.3.4.20, and Proposition T.A.3.7.6, we obtain the following:

Proposition 1.3.4.22. *Let $\mathbf{A}$ be a combinatorial model category. Then the underlying ∞-category of $\mathbf{A}$ is presentable.*

Similarly, Theorem T.4.2.4.1 implies the following:

Proposition 1.3.4.23. *Let $\mathbf{A}$ be a combinatorial model category, let $\mathcal{I}$ be a small category, let $F : \mathcal{I} \to \mathbf{A}^c$ be a functor, and let $\alpha : X \to \varprojlim_{I \in \mathcal{I}} F(I)$ be a morphism in $\mathbf{A}^c$. The following conditions are equivalent:*

(1) *The map α exhibits X as a homotopy limit of the diagram F (in the model category $\mathbf{A}$).*

(2) *The induced map*

$$\mathrm{N}(\mathcal{I})^{\triangleleft} \to \mathrm{N}(\mathbf{A}^c) \to \mathrm{N}(\mathbf{A}^c)[W^{-1}]$$

is a limit diagram in the underlying ∞-category $\mathrm{N}(\mathbf{A}^c)[W^{-1}]$ of $\mathbf{A}$.

Proposition 1.3.4.24. *Let $\mathbf{A}$ be a combinatorial model category, let $\mathcal{I}$ be a small category, let $F : \mathcal{I} \to \mathbf{A}^c$ be a functor, and let $\alpha : \varinjlim_{I \in \mathcal{I}} F(I) \to X$ be a morphism in $\mathbf{A}^c$. The following conditions are equivalent:*

(1) *The map α exhibits X as a homotopy colimit of the diagram F (in the model category $\mathbf{A}$).*

(2) *The induced map*

$$\mathrm{N}(\mathcal{I})^{\triangleright} \to \mathrm{N}(\mathbf{A}^c) \to \mathrm{N}(\mathbf{A}^c)[W^{-1}]$$

is a colimit diagram in the underlying ∞-category $\mathrm{N}(\mathbf{A}^c)[W^{-1}]$ of $\mathbf{A}$.

Finally, by reducing to the simplicial case and invoking Proposition T.4.2.4.4, we obtain:

Proposition 1.3.4.25. *Let $\mathbf{A}$ be a combinatorial model category and let $\mathcal{I}$ be a small category. Let $\mathbf{A}^{\mathcal{I}}$ be the category of functors from $\mathcal{I}$ to $\mathbf{A}$ (endowed with either the injective or projective model structure), let $(\mathbf{A}^{\mathcal{I}})^c$ be the full subcategory of $\mathbf{A}^{\mathcal{I}}$ spanned by the cofibrant objects, and let W' be the collection of weak equivalences in $(\mathbf{A}^{\mathcal{I}})^c$. Then the evident map*

$$\mathrm{N}(\mathcal{I}) \times \mathrm{N}((\mathbf{A}^{\mathcal{I}})^c) \to \mathrm{N}(\mathbf{A}^c)$$

induces an equivalence of ∞-categories

$$\mathrm{N}((\mathbf{A}^{\mathcal{I}})^c)[W'^{-1}] \to \mathrm{Fun}(\mathrm{N}(\mathcal{I}), \mathrm{N}(\mathbf{A})[W^{-1}]).$$

Corollary 1.3.4.26. *Let $F : \mathbf{A} \to \mathbf{B}$ be a left Quillen functor between combinatorial model categories. Let $\mathbf{A}^c$ and $\mathbf{B}^c$ denote the full subcategories of $\mathbf{A}$ and $\mathbf{B}$ spanned by the cofibrant objects, and let $W_{\mathbf{A}}$ and $W_{\mathbf{B}}$ be the collection of weak equivalences in $\mathbf{A}^c$ and $\mathbf{B}^c$, respectively. Then the induced functor $f : \mathrm{N}(\mathbf{A}^c)[W_{\mathbf{A}}^{-1}] \to \mathrm{N}(\mathbf{B}^c)[W_{\mathbf{B}}^{-1}]$ preserves small colimits.*

Proof. In view of Proposition T.4.4.2.7, it will suffice to show that f preserves small colimits indexed by $\mathrm{N}(\mathcal{I})$, where $\mathcal{I}$ is a small category. By virtue of Proposition 1.3.4.25, it suffices consider colimits of diagrams which arise from functors $\mathcal{I} \to \mathbf{A}^c$. This follows from Proposition 1.3.4.24, since F preserves homotopy colimits. $\qquad\square$

Remark 1.3.4.27. Let $F : \mathbf{A} \to \mathbf{B}$ be as in Corollary 1.3.4.26. Using Proposition 1.3.4.22 and Corollary T.5.5.2.9, we conclude that the induced map of ∞-categories $f : \mathrm{N}(\mathbf{A}^c)[W_{\mathbf{A}}^{-1}] \to \mathrm{N}(\mathbf{B}^c)[W_{\mathbf{B}}^{-1}]$ admits a right adjoint g. It is not hard to construct g explicitly, by composing a right adjoint to F with a fibrant replacement functor in $\mathbf{B}$ and a cofibrant replacement functor in $\mathbf{A}$ (as in the proof of Lemma 1.3.4.21).

We conclude this section by giving the proof of Theorem 1.3.4.20.

Proof of Theorem 1.3.4.20. Let $\mathbf{A}$ be a simplicial model category and let $\theta : \mathrm{N}(\mathbf{A}^c) \to \mathrm{N}(\mathbf{A}^{\mathrm{o}})$ be the functor determined by Lemma 1.3.4.19. Let W be the collection of weak equivalences in $\mathbf{A}^c$ and W^{o} the collection of weak equivalences in $\mathbf{A}^{\mathrm{o}}$. We wish to show that θ induces a weak equivalence of marked simplicial sets $(\mathrm{N}(\mathbf{A}^c), W) \to (\mathrm{N}(\mathbf{A}^{\mathrm{o}}), W^{\mathrm{o}})$. Let $\mathcal{C} = \mathbf{A}^c$, regarded as a simplicial category, and let ϕ denote the composite map

$$\mathrm{N}(\mathbf{A}^c) \xrightarrow{\theta} \mathrm{N}(\mathbf{A}^{\mathrm{o}}) \to \mathrm{N}(|\,\mathcal{C}\,|)$$

(here $|\,\mathcal{C}\,|$ denotes the topological category associated to $\mathcal{C}$, as in the proof of Proposition 1.3.4.7). There is an evident natural transformation $\phi' \to \phi$, where ϕ' is the inclusion $\mathrm{N}(\mathbf{A}^c) \subseteq \mathrm{N}(|\,\mathcal{C}\,|)$. It follows from Lemma 1.3.4.19 that this natural transformation is given by morphisms belonging to W, so that ϕ and ϕ' induce the same morphism $(\mathrm{N}(\mathbf{A}^c), W) \to (\mathrm{N}(|\,\mathcal{C}\,|), W)$ in the homotopy category of marked simplicial sets. Proposition 1.3.4.7 implies that ϕ' determines a weak equivalence of marked simplicial sets, so that ϕ is also a weak equivalence of marked simplicial sets. It will therefore suffice to show that the composite map

$$(\mathrm{N}(\mathbf{A}^{\mathrm{o}}), W^{\mathrm{o}}) \xrightarrow{\psi'} \mathrm{N}(|\mathbf{A}^{\mathrm{o}}|, W^{\mathrm{o}}) \xrightarrow{\psi} (\mathrm{N}(|\,\mathcal{C}\,|), W)$$

is a weak equivalence of marked simplicial sets. It is clear that ψ' is a weak equivalence (since $\mathbf{A}^{\mathrm{o}}$ is a fibrant simplicial category). The argument above shows that $\psi \circ \psi'$ (and therefore also ψ) admits a right homotopy inverse. We will complete the proof by constructing a left homotopy inverse to ψ.

We claim that the inclusion $\mathrm{N}(|\mathbf{A}^\circ|) \subseteq \mathrm{N}(|\mathcal{C}|)$ admits a left adjoint. To prove this, it suffices to show that for every cofibrant object $X \in \mathbf{A}$, there exists a morphism $f : X \to Y$ where $Y \in \mathbf{A}^\circ$ which induces a homotopy equivalence $\mathrm{Map}_{\mathbf{A}}(Y, Z) \to \mathrm{Map}_{\mathbf{A}}(X, Z)$ for each $Z \in \mathbf{A}^\circ$. For this, it suffices to take f to be any weak equivalence from X to a fibrant-cofibrant object of Y. Let $L : \mathrm{N}(|\mathcal{C}|) \to \mathrm{N}(|\mathbf{A}^\circ|)$ be a left adjoint to the inclusion. The above argument shows that the unit transformation $u : \mathrm{id} \to L$ carries each cofibrant object $X \in \mathbf{A}$ to a weak equivalence $f : X \to Y$ in $\mathbf{A}$. It follows that L carries W into W°, and therefore induces a map of marked simplicial sets $\zeta : (\mathrm{N}(|\mathcal{C}|), W) \to (\mathrm{N}(\mathbf{A}^\circ), W^\circ)$ which is the desired left homotopy inverse to ψ. $\qquad\square$

1.3.5 Grothendieck Abelian Categories

In §1.3.3, we studied the right-bounded derived ∞-category $\mathcal{D}^-(\mathcal{A})$ of an abelian category $\mathcal{A}$ with enough projective objects. If $\mathcal{A}$ instead has enough injective objects, then we can consider instead its left-bounded derived ∞-category $\mathcal{D}^+(\mathcal{A}) \simeq \mathcal{D}^-(\mathcal{A}^{op})^{op}$ (see Variant 1.3.2.8). For many applications, it is convenient to work with chain complexes which are unbounded in both directions. In this section, we will study an unbounded version of the derived ∞-category introduced in §1.3.3, following ideas introduced in [136]. We will discuss a variant of this construction in §1.3.6.

We begin by singling out a convenient class of abelian categories to work with.

Definition 1.3.5.1. Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ is *Grothendieck* if it is presentable and the collection of monomorphisms in $\mathcal{A}$ is closed under small filtered colimits.

Remark 1.3.5.2. The notion of a Grothendieck abelian category was introduced by Grothendieck in the paper [64].

Proposition 1.3.5.3. *Let $\mathcal{A}$ be a Grothendieck abelian category. Then $\mathrm{Ch}(\mathcal{A})$ admits a left proper combinatorial model structure, which can be described as follows:*

(C) *A map of chain complexes $f : M_* \to N_*$ is a cofibration if, for every integer k, the induced map $M_k \to N_k$ is a monomorphism in $\mathcal{A}$.*

(W) *A map of chain complexes $f : M_* \to N_*$ is a weak equivalence if it is a quasi-isomorphism: that is, if it induces an isomorphism on homology.*

(F) *A map of chain complexes $f : M_* \to N_*$ is a fibration if it has the right lifting property with respect to every map which is both a cofibration and a weak equivalence.*

Proof. Since $\mathcal{A}$ is presentable, there exists a small collection of objects $X_i \in \mathcal{A}$ which generate $\mathcal{A}$ under small colimits. Let $X = \bigoplus X_i$. Then each X_i is a retract of X, so that the object X generates $\mathcal{A}$ under small colimits. In particular, for every object $Y \in \mathcal{A}$ the canonical map $\coprod_{\alpha : X \to Y} X \to Y$ is an epimorphism. Every subobject of Y is the image of a coproduct $\coprod_{\alpha \in S} X$ for some subset $S \subseteq \mathrm{Hom}_{\mathcal{A}}(X, Y)$; it follows that the category of subobjects of Y is essentially small.

For every monomorphism $u : X_0 \hookrightarrow X$ and every integer n, let $E(u, n)_*$ denote the chain complex given by

$$E(u, n)_k = \begin{cases} X_0 & \text{if } k = n \\ X & \text{if } k = n - 1 \\ 0 & \text{otherwise,} \end{cases}$$

where the differential is given by the map u. Let C_0 be the collection of all monomorphisms of the form $E(u, n)_* \to E(\mathrm{id}_X, n)_*$. We first claim that the collection of cofibrations in $\mathrm{Ch}(\mathcal{A})$ is the smallest collection of morphisms containing C_0 which is weakly saturated, in the sense of Definition T.A.1.2.2. It is clear that the collection of cofibrations contains C_0 and is weakly saturated. Conversely, suppose we are given a cofibration of chain complexes $f : M_* \to N_*$. We define a compatible sequence of monomorphisms $f_\alpha : M(\alpha)_* \to N_*$ by transfinite induction. Set $f_0 = f$, and for α a nonzero limit ordinal let f_α be the

induced map $\varinjlim_{\beta<\alpha} M(\beta)_* \to N_*$ (our assumption that $\mathcal{A}$ is Grothendieck guarantees that this map is again a monomorphism).

Assume that f_α has been defined. We can choose a map $\lambda : X \to N_n$ which does not factor through $M(\alpha)_n$. Replacing λ by the composite map $X \xrightarrow{\lambda} N_n \xrightarrow{d} N_{n-1}$ if necessary, we may assume in addition that the composite map $d \circ \lambda : X \to N_{n-1}$ does factor through $M(\alpha)_{n-1}$. Let $X_0 = M(\alpha)_n \times_{N_n} X$ and let $u : X_0 \to X$ be the projection onto the second factor. Then we have a commutative diagram

$$
\begin{array}{ccc}
E(u,n)_* & \longrightarrow & E(\mathrm{id}_X, n)_* \\
\downarrow & & \downarrow \\
M(\alpha)_* & \xrightarrow{\ f_\alpha\ } & N_*.
\end{array}
$$

Taking $M(\alpha+1)_*$ to be the pushout $M(\alpha)_* \coprod_{E(\alpha,n)_*} E(\mathrm{id}_X, n)_*$, we obtain a monomorphism $f_{\alpha+1} : M(\alpha+1)_* \to N_*$ compatible with α. Moreover, the map λ factors through $f_{\alpha+1}$, so that $M(\alpha)_*$ and $M(\alpha+1)_*$ are not isomorphic as subobjects of N_*.

Since the collection of subobjects of N_* is bounded, this process must eventually terminate. In other words, for α sufficiently large, the map f_α is an equivalence. This implies that the map f can be obtained as a transfinite composition of monomorphisms $M(\beta)_* \to M(\beta+1)_*$, each of which is a pushout of a morphism belonging to C_0. It follows that f belongs to the smallest weakly saturated class of morphisms containing C_0, as desired.

To complete the proof, it will suffice to show that the class of cofibrations and weak equivalences satisfy the hypotheses of Proposition T.A.2.6.13:

(1) The class of weak equivalences is perfect. To prove this, we note that a morphism $f : M_* \to N_*$ is a weak equivalence if and only if its image under the homology functor $\mathrm{H}_* : \mathrm{Ch}(\mathcal{A}) \to \prod_{n\in\mathbf{Z}} \mathcal{A}$ is an isomorphism. Since $\mathcal{A}$ is a Grothendieck abelian category, the functor H_* commutes with filtered colimits and the result follows from Corollary T.A.2.6.12.

(2) Suppose we are given a pushout diagram of chain complexes σ :

$$
\begin{array}{ccc}
M'_* & \xrightarrow{\ f\ } & M_* \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\
N'_* & \xrightarrow{\ f'\ } & N_*
\end{array}
$$

where f is a cofibration and g is a weak equivalence; we must show that g' is a weak equivalence. For every integer n, we have a map of exact sequences

$$
\begin{array}{ccccccccc}
\mathrm{H}_{n+1}(M/M') & \longrightarrow & \mathrm{H}_n(M') & \longrightarrow & \mathrm{H}_n(M) & \longrightarrow & \mathrm{H}_n(M/M') & \longrightarrow & \mathrm{H}_{n-1}(M') \\
\downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \theta_3} & & \downarrow{\scriptstyle \theta_4} \\
\mathrm{H}_{n+1}(N/N') & \longrightarrow & \mathrm{H}_n(N') & \longrightarrow & \mathrm{H}_n(N) & \longrightarrow & \mathrm{H}_n(N/N') & \longrightarrow & \mathrm{H}_{n-1}(N').
\end{array}
$$

Since g is a quasi-isomorphism, the maps θ_1 and θ_4 are isomorphisms. The maps θ_0 and θ_3 are induced by the map of chain complexes $(M/M')_* \to (N/N')_*$, which is an isomorphism because σ is assumed to be a pushout diagram. It follows that θ_2 is also an isomorphism, as desired.

(3) Let $f : M_* \to N_*$ be a map of chain complexes which has the right lifting property with respect to every cofibration. We must show that f is a quasi-isomorphism. Fix an integer n and let Z denote the kernel of the differential $d : N_n \to N_{n-1}$. Let $Z[n]$ denote the chain complex consisting of the object $Z \in \mathcal{A}$, concentrated in degree n. Then the inclusion $0 \hookrightarrow Z[n]$ is a cofibration, so that f has the

right lifting property with respect to f. It follows that the monomorphism $Z[n] \hookrightarrow N_n$ lifts to a map $Z \to Z'$, where Z' denotes the kernel of the differential $M_n \to M_{n-1}$. This guarantees that the map on homology $\theta : \mathrm{H}_n(M) \to \mathrm{H}_n(N)$ is an epimorphism. We now show that θ is a monomorphism. To prove this, let Z'' denote the fiber product $Z' \times_{N_n} N_{n+1}$, and let $u : 0 \to Z''$ be the zero map. Then f has the right lifting property with respect to the cofibration $E(u, n+1)_* \to E(\mathrm{id}_{Z''}, n+1)_*$. It follows that there exists a map $\mu : Z'' \to M_{n+1}$ such that $Z'' \overset{\mu}{\to} M_{n+1} \overset{f}{\to} N_{n+1}$ is given by the projection onto the second factor, and the map $Z'' \overset{\mu}{\to} M_{n+1} \overset{d}{\to} M_n$ is given by the projection onto the first factor Z' (which we view as a subobject of M_n). We note that Z'' is given by the fiber product $Z \times_{\mathrm{H}_n(M)} \ker(\theta)$, so the map $Z'' \to \ker(\theta)$ is an epimorphism. However, the existence of the map μ shows that the map $Z'' \to \ker(\theta)$ is zero, so that θ is a monomorphism as desired.

$\square$

Warning 1.3.5.4. Let $\mathcal{A}$ be a Grothendieck abelian category. The category $\mathrm{Ch}(\mathcal{A})$ is endowed with both a model structure and a simplicial enrichment (via Construction 1.3.1.13). However, it is not a simplicial model category in the sense of Definition T.A.3.1.5, because it is not tensored over the category of simplicial sets. For every simplicial set K and any pair of objects $M_*, M'_* \in \mathrm{Ch}(\mathcal{A})$, we have a canonical bijection

$$\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(N_*(K) \otimes M_*, M'_*) \simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(K, \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})_\Delta}(M_*, M'_*)).$$

This bijection extends to a map of simplicial sets

$$\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})_\Delta}(N_*(K) \otimes M_*, M'_*) \to \mathrm{Map}_{\mathrm{Set}_\Delta}(K, \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, M'_*)),$$

which is generally not an isomorphism (because the Alexander-Whitney map $N_*(K \times K') \to N_*(K) \otimes N_*(K')$ generally fails to be an isomorphism).

Remark 1.3.5.5. In the situation of Proposition 1.3.5.3, the collection of weak equivalences in $\mathrm{Ch}(\mathcal{A})$ is closed under filtered colimits (since the formation of homology commutes with filtered colimits).

If $\mathcal{A}$ is a Grothendieck abelian category, then every object of $\mathrm{Ch}(\mathcal{A})$ is cofibrant. Our next goal is to say something about the fibrant objects.

Proposition 1.3.5.6. *Let $\mathcal{A}$ be a Grothendieck abelian category and let $M_* \in \mathrm{Ch}(\mathcal{A})$ be a chain complex. If M_* is fibrant (with respect to the model structure described in Proposition 1.3.5.3), then each M_n is an injective object of $\mathcal{A}$. Conversely, if each M_n is injective and $M_n \simeq 0$ for $n \gg 0$, then M_* is a fibrant object of $\mathrm{Ch}(\mathcal{A})$.*

Corollary 1.3.5.7 (Grothendieck). *Let $\mathcal{A}$ be a Grothendieck abelian category. Then $\mathcal{A}$ has enough injective objects: that is, for every object $M \in \mathcal{A}$, there exists a monomorphism $M \to Q$, where $Q \in \mathcal{A}$ is injective.*

Proof. Let $M[0]$ denote the chain complex consisting of object $M \in \mathcal{A}$, concentrated in degree zero. Choose a trivial cofibration $M[0] \to Q_*$, where $Q_* \in \mathrm{Ch}(\mathcal{A})$ is fibrant. Then the induced map $M \to Q_0$ is a monomorphism, and $Q_0 \in \mathcal{A}$ is injective by Proposition 1.3.5.6. $\square$

Proof of Proposition 1.3.5.6. For every object $X \in \mathcal{A}$, we let $E(X, n)_*$ denote the chain complex given by $E(X, n)_k = \begin{cases} X & \text{if } k \in \{n, n-1\} \\ 0 & \text{otherwise,} \end{cases}$ where the differential is given by the identity map $\mathrm{id}_X : X \to X$. If $X \to Y$ is a monomorphism in $\mathcal{A}$, then the induced map $i : E(X, n)_* \to E(Y, n)_*$ is a trivial cofibration in $\mathrm{Ch}(\mathcal{A})$. If M_* is fibrant, then M_* has the extension property with respect to i, so that every map $X \to M_n$ extends to a map $Y \to M_n$; this proves that M_n is injective.

Conversely, suppose that each M_n is injective and that $M_n \simeq 0$ for $n \gg 0$. We wish to show that M_* is fibrant. Choose a trivial cofibration $u : A_* \to A'_*$ and suppose we are given a chain map $f : A_* \to M_*$; we must show that f can be extended to a chain map $f' : A'_* \to M_*$. For each $n \in \mathbf{Z}$, we let $Z_n(A)$ denote the

kernel of the differential $A_n \to A_{n-1}$ and $B_n(A)$ the image of the differential $A_{n+1} \to A_n$, let $A(n)_*$ denote the chain complex

$$\cdots \to A_{n+2} \to A_{n+1} \to B_n(A) \to 0 \to 0 \to \cdots ,$$

and let $f_n = f|A(n)_*$. We define $B_n(A')$, $Z_n(A')$, and $A'(n)_*$ similarly. For $n \gg 0$, the map f_n is zero and therefore extends to a map $f'_n : A'(n)_* \to M_*$. We will construct a compatible sequence of maps $\{f'_i : A'(i)_* \to M_*\}_{i \leq n}$ extending the map f_n; passing to the limit, we will obtain the desired chain map $f' : A'_* \to M_*$ extending f. Assume therefore that $i \leq n$ and that the map f'_i has already been constructed. We show that it is possible to construct a chain map $f'_{i-1} : A'(i-1)_* \to M_*$ which extends both f'_i and f_{i-1}. Our assumption that u is a quasi-isomorphism guarantees that the diagram

$$
\begin{array}{ccc}
B_i(A) & \longrightarrow & Z_i(A) \\
\downarrow & & \downarrow \\
B_i(A') & \longrightarrow & Z_i(A')
\end{array}
$$

is a pushout square in $\mathcal{A}$, so that the maps $f|Z_i(A)$ and $f'_i|B_n(A')$ determine a unique map $g : Z_i(A') \to M_i$. The map g carries $Z_i(A)$ into the kernel of the differential $d : M_i \to M_{i-1}$ (since f is a chain map) and $B_i(A')$ into the image of the differential $M_{i+1} \to M_i$ (since f'_i is a chain map). Since $B_{i-1}(A') \simeq A'_n/Z_n A'$, we conclude that giving a chain map $f'_{i-1} : A'(i)_* \to M_*$ compatible with both f'_i and g is equivalent to choosing a map $\overline{g} : A'_i \to M_i$ extending g. To complete the construction, we must show that it is possible to choose $\overline{g}$ so that the composite map

$$A_i \xrightarrow{u} A'_i \xrightarrow{\overline{g}} M_i$$

is given by f. We note that f and g determine a map

$$\overline{g}_0 : A_i \coprod_{Z_i(A)} Z_i(A') \to M_i.$$

Since M_i is an injective object of $\mathcal{A}$, to prove the existence of $\overline{g}$ it will suffice to show that the map

$$\theta : A_i \coprod_{Z_i(A)} Z_i(A') \to A'_i$$

is a monomorphism. This map fits into a diagram of short exact sequences

$$
\begin{array}{ccccccccc}
0 \to & Z_i(A') & \longrightarrow & A_i \coprod_{Z_i(A)} Z_i(A') & \longrightarrow & B_{i-1}(A) & \longrightarrow & 0 \\
& \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta''} & & \\
0 \to & Z_i(A') & \longrightarrow & A'_i & \longrightarrow & B_{i-1}(A') & \longrightarrow & 0
\end{array}
$$

The map θ' is an isomorphism and the map θ'' is a monomorphism (since u is a cofibration), so that θ is a monomorphism by the snake lemma. $\qquad\square$

Definition 1.3.5.8. Let $\mathcal{A}$ be a Grothendieck abelian category. We let $\mathrm{Ch}(\mathcal{A})^{\circ}$ denote the full subcategory of $\mathrm{Ch}(\mathcal{A})$ spanned by the fibrant objects (which are automatically cofibrant). We let $\mathcal{D}(\mathcal{A})$ denote the differential graded nerve $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A})^{\circ})$. We will refer to $\mathcal{D}(\mathcal{A})$ as the *derived ∞-category of $\mathcal{A}$*.

Proposition 1.3.5.9. *Let $\mathcal{A}$ be a Grothendieck abelian category. Then the ∞-category $\mathcal{D}(\mathcal{A})$ is stable.*

Proof. The ∞-category $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$ is stable, by Proposition 1.3.2.10. It will therefore suffice to show that $\mathcal{D}(\mathcal{A})$ is a stable subcategory of $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$. Since $\mathcal{D}(\mathcal{A})$ is evidently invariant under translation, it will suffice to show that it is closed under taking cofibers. For this, it suffices to show that if $f : M_* \to N_*$

is a map between fibrant objects of $\mathrm{Ch}(\mathcal{A})$, then the mapping cone $C_*(f)$ of f is also a fibrant object of $\mathrm{Ch}(\mathcal{A})$. Since $M_*[1]$ is fibrant, it suffices to show that the epimorphism of chain complexes $C_*(f) \to M_*[1]$ is a fibration. For this, we must show that every lifting problem of the form

$$
\begin{array}{ccc}
A_* & \longrightarrow & C_*(f) \\
{\scriptstyle i}\downarrow & \nearrow & \downarrow \\
B_* & \longrightarrow & M_*[1]
\end{array}
$$

has a solution, provided that i is both a monomorphism and a quasi-isomorphism. To prove this, it suffices to show that the chain complex $N_*[1]$ has the right lifting property with respect to the monomorphism $j : C_*(i) \hookrightarrow C_*(\mathrm{id}_B)$. This follows from our assumption that N_* is fibrant, since j is a trivial cofibration in $\mathrm{Ch}(\mathcal{A})$. $\qquad\square$

Remark 1.3.5.10. Let $\mathcal{A}$ be a Grothendieck abelian category. Then $\mathcal{A}$ has enough injectives (Corollary 1.3.5.7), so that the ∞-category $\mathcal{D}^+(\mathcal{A})$ is defined as in Variant 1.3.2.8. The characterization of the fibrant objects of $\mathrm{Ch}(\mathcal{A})$ supplied by Proposition 1.3.5.6 implies that $\mathcal{D}^+(\mathcal{A})$ is a full subcategory of $\mathcal{D}(\mathcal{A})$.

Lemma 1.3.5.11. *Let $\mathcal{A}$ be a Grothendieck abelian category. If $M_* \in \mathrm{Ch}(\mathcal{A})$ is acyclic and $Q_* \in \mathrm{Ch}(\mathcal{A})$ is fibrant, then the chain complex $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, Q_*)$ is acyclic.*

Proof. Let η be an n-cycle in $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, Q_*)$, given by a map of chain complexes $f_0 : M_*[-n] \to Q_*$. Let C_* denote the cone on $M_*[-n]$. To prove that η is homologous to zero, it will suffice to show that f_0 extends to a map $f : C_* \to Q_*$. Since Q_* is fibrant, we are reduced to proving that the inclusion $j : M_*[-n] \hookrightarrow C_*$ is a trivial cofibration. The map j is evidently a monomorphism; it is a quasi-isomorphism by virtue of our assumption that M_* is acyclic (which implies that the complexes $M_*[-n]$ and C_* are also acyclic). $\qquad\square$

Lemma 1.3.5.12. *Let $\mathcal{A}$ be a Grothendieck abelian category and let $f : M'_* \to M_*$ be a trivial cofibration in $\mathrm{Ch}(\mathcal{A})$. For every fibrant object $Q_* \in \mathrm{Ch}(\mathcal{A})$, the induced map*

$$\theta : \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, Q_*) \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M'_*, Q_*)$$

is a quasi-isomorphism of chain complexes.

Proof. Proposition 1.3.5.6 implies that each Q_n is injective, so that each of the maps $\mathrm{Hom}_{\mathcal{A}}(M_p, Q_n) \to \mathrm{Hom}_{\mathcal{A}}(M'_p, Q_n)$ is surjective. It follows that θ is a surjection of chain complexes. It will therefore suffice to show that the chain complex $\ker(\theta) \simeq \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}((M/M')_*, Q_*)$ is acyclic. This follows from Lemma 1.3.5.11 (the complex $(M/M')_*$ is acyclic by virtue of our assumption that f is a quasi-isomorphism). $\qquad\square$

Proposition 1.3.5.13. *Let $\mathcal{A}$ be a Grothendieck abelian category. Then $\mathcal{D}(\mathcal{A})$ is a localization of the ∞-category $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$.*

Proof. For every object $M_* \in \mathrm{Ch}(\mathcal{A})$, we can choose a trivial cofibration $f : M_* \to Q_*$, where Q_* is fibrant. Lemma 1.3.5.12 implies that f exhibits Q_* as a $\mathcal{D}(\mathcal{A})$-localization of M_*. $\qquad\square$

Proposition 1.3.5.14. *Let $\mathcal{A}$ be a Grothendieck abelian category and let $f : M_* \to M'_*$ be a map of chain complexes. If f is a quasi-isomorphism and $Q_* \in \mathrm{Ch}(\mathcal{A})$ is fibrant, then f induces a quasi-isomorphism $\theta : \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M'_*, Q_*) \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, Q_*)$. In particular, if f is a quasi-isomorphism between fibrant chain complexes, then f is a chain homotopy equivalence (and therefore induces an equivalence in $\mathcal{D}(\mathcal{A})$).*

Proof. The map f admits a factorization

$$M_* \xrightarrow{f'} M''_* \xrightarrow{f''} M'_*$$

where f' is a trivial cofibration in $\mathrm{Ch}(\mathcal{A})$ and f'' is a trivial fibration in $\mathrm{Ch}(\mathcal{A})$. We may therefore assume either that f is a trivial cofibration or a trivial fibration. If f is a trivial cofibration, the result follows from Lemma 1.3.5.12. If f is a trivial fibration, then it admits a section s (since M'_* is cofibrant). The map s is then a trivial cofibration, and therefore induces a quasi-isomorphism $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, Q_*) \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M'_*, Q_*)$ which is left inverse to θ, so that θ is also a quasi-isomorphism. $\square$

Our next goal is to prove that the derived ∞-category $\mathcal{D}(\mathcal{A})$ can be identified with the underlying ∞-category of the model category $\mathrm{Ch}(\mathcal{A})$, in the sense of Definition 1.3.4.15: that is, we can regard $\mathcal{D}(\mathcal{A})$ as the ∞-category obtained from the ordinary category of chain complexes over $\mathcal{A}$ by inverting quasi-isomorphisms. This does not follow formally from Theorem 1.3.4.20, because $\mathrm{Ch}(\mathcal{A})$ is not a simplicial model category (Warning 1.3.5.4). Nevertheless, we have the following result:

Proposition 1.3.5.15. *Let $\mathcal{A}$ be a Grothendieck abelian category, and let $\mathbf{A}$ denote the category of chain complexes $\mathrm{Ch}(\mathcal{A})$, regarded as a discrete category. Then the composite map $\mathrm{N}(\mathbf{A}) \to \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A})) \xrightarrow{L} \mathcal{D}(\mathcal{A})$ exhibits $\mathcal{D}(\mathcal{A})$ as the underlying ∞-category of the model category $\mathbf{A}$; here $L : \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A})) \to \mathcal{D}(\mathcal{A})$ denotes a left adjoint to the inclusion $\mathcal{D}(\mathcal{A}) \hookrightarrow \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$.*

Proof. Let $\mathbf{A}^{\circ}$ be the full subcategory of $\mathbf{A}$ spanned by the fibrant chain complexes, and let W be the collection of weak equivalences in $\mathbf{A}^{\circ}$. It is easy to see that $\mathbf{A}^{\circ}$ is closed under tensor product by the finite chain complex $N_*(\Delta^1)$ (note that tensor product with $N_*(\Delta^1)$ is a right Quillen functor, since it is right adjoint to the left Quillen functor given by tensor product with the dual chain complex $N_*(\Delta^1)^{\vee}$). Using Propositions 1.3.5.14 and 1.3.4.5, we deduce that the map $\mathrm{N}(\mathbf{A}^{\circ})[W^{-1}] \to \mathcal{D}(\mathcal{A})$ is an equivalence of ∞-categories. This is equivalent to the asserted result, by virtue of Remark 1.3.4.16. $\square$

Definition 1.3.5.16. Let $\mathcal{A}$ be a Grothendieck abelian category. For each integer n, we let $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))_{\geq n}$ denote the full subcategory of $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$ spanned by those chain complexes M_* such that $\mathrm{H}_k(M) \simeq 0$ for $k < n$, and $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))_{\leq n}$ the full subcategory of $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$ spanned by those chain complexes M_* such that $\mathrm{H}_k(M) \simeq 0$ for $k > n$. Set

$$\mathcal{D}(\mathcal{A})_{\geq n} = \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))_{\geq n} \cap \mathcal{D}(\mathcal{A}) \qquad \mathcal{D}(\mathcal{A})_{\leq n} = \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))_{\leq n} \cap \mathcal{D}(\mathcal{A}).$$

Remark 1.3.5.17. In the situation of Definition 1.3.5.16, suppose we are given an object $M_* \in \mathcal{D}(\mathcal{A})_{\leq n}$. Then there exists an equivalence $M_* \simeq M'_*$ in $\mathcal{D}(\mathcal{A})_{\leq n}$, where $M'_k \simeq 0$ for $k > n$. To prove this, we note that our assumption that $\mathrm{H}_k(M) \simeq 0$ for $k > n$ implies that the canonical map $M_* \to \tau_{\leq n} M_*$ is a quasi-isomorphism, where $\tau_{\leq n} M_*$ denotes the truncated chain complex

$$\cdots \to 0 \to M_n / d(M_{n+1}) \to M_{n-1} \to M_{n-2} \to \cdots$$

This complex may not be fibrant, but the fact that $\mathcal{A}$ has enough injectives (Corollary 1.3.5.7) guarantees that we can construct a quasi-isomorphism $\tau_{\leq n} M_* \to M'_*$, where $M'_k \simeq 0$ for $k > n$ and each M'_k is injective (see the proof of Proposition 1.3.2.19). Then M'_* is fibrant by Proposition 1.3.5.6, and the composite map $M_* \to \tau_{\leq n} M_* \to M'_*$ is a quasi-isomorphism.

Proposition 1.3.5.18. *Let $\mathcal{A}$ be a Grothendieck abelian category. Then the full subcategories*

$$\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))_{\geq 0}, \mathcal{D}(\mathcal{A})_{\leq 0}, \subseteq \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$$

determine a t-structure on the stable ∞-category $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$.

Proof. Fix any object $M_* \in \mathrm{Ch}(\mathcal{A})$. Let X_* be the truncated chain complex

$$\cdots \to 0 \to M_{-1} / d(M_0) \to M_{-2} \to M_{-3} \to \cdots$$

and choose a quasi-isomorphism $X_* \to M''_*$, where M''_* is a chain complex of injective objects of $\mathcal{A}$ and $M''_k \simeq 0$ for $k \geq 0$. Let $f : M_* \to M''_*$ be the composite map and let M'_* denote the shifted mapping cone $C(f)[-1]$. We have a termwise split exact sequence of chain complexes

$$M''_*[-1] \to M'_* \to M_*$$

which gives a fiber sequence

$$M_*' \to M_* \to M_*''$$

in $N_{dg}(\mathrm{Ch}(\mathcal{A}))$, where $M_*' \in N_{dg}(\mathrm{Ch}(\mathcal{A}))_{\geq 0}$ and $M_*'' \in \mathcal{D}(\mathcal{A})_{\leq -1}$. To complete the proof, it will suffice to show that if $M_* \in N_{dg}(\mathrm{Ch}(\mathcal{A}))_{\geq 0}$ and $Q_* \in \mathcal{D}(\mathcal{A})_{\leq -1}$, then the mapping space $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, Q_*)$ is contractible. In view of Remark 1.3.5.17, we may assume without loss of generality that $Q_k \simeq 0$ for $k \geq 0$. Let Y_* denote the chain complex

$$\cdots \to M_2 \to M_1 \to \ker(d : M_0 \to M_{-1}) \to 0 \to \cdots$$

Since $M_* \in N_{dg}(\mathrm{Ch}(\mathcal{A}))_{\geq 0}$, the monomorphism $Y_* \hookrightarrow M_*$ is a quasi-isomorphism. It follows from Lemma 1.3.5.12 that the map

$$\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, Q_*) \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(Y_*, Q_*)$$

is a quasi-isomorphism, so that $\mathrm{Map}_{N_{dg}(\mathrm{Ch}(\mathcal{A}))}(M_*, Q_*) \simeq \mathrm{Map}_{N_{dg}(\mathrm{Ch}(\mathcal{A}))}(Y_*, Q_*)$ is a contractible Kan complex. $\square$

Remark 1.3.5.19. Let $\mathcal{A}$ be a Grothendieck abelian category and suppose that $M_* \in N_{dg}(\mathrm{Ch}(\mathcal{A}))_{\geq 0}$ and $Q_* \in \mathcal{D}(\mathcal{A})_{\leq 0}$. Then the canonical map

$$\mathrm{Map}_{N_{dg}(\mathrm{Ch}(\mathcal{A}))}(M_*, Q_*) \to \mathrm{Hom}_{\mathcal{A}}(H_0(M), H_0(Q))$$

is a homotopy equivalence. To see this, we use Remark 1.3.5.17 to reduce to the case where $Q_* \in \mathrm{Ch}(\mathcal{A})_{\leq 0}$, and Lemma 1.3.5.12 to reduce to the case where $M_* \in \mathrm{Ch}(\mathcal{A})_{\geq 0}$, in which case the result is obvious. It follows that the functor $M \mapsto H_0(M)$ induces an equivalence of abelian categories $N_{dg}(\mathrm{Ch}(\mathcal{A}))^{\heartsuit} \to \mathcal{A}$ (the homotopy inverse functor can be described as assigning to each object $M \in \mathcal{A}$ an injective resolution of M).

Definition 1.3.5.20. Let $\mathcal{C}$ be a stable ∞-category which admits small filtered colimits. We will say that a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ on $\mathcal{C}$ is *compatible with filtered colimits* if $\mathcal{C}_{\leq 0}$ is closed under small filtered colimits in $\mathcal{C}$.

Proposition 1.3.5.21. *Let $\mathcal{A}$ be a Grothendieck abelian category. Then:*

(1) *The ∞-category $\mathcal{D}(\mathcal{A})$ is presentable.*

(2) *The pair of subcategories $(\mathcal{D}(\mathcal{A})_{\geq 0}, \mathcal{D}(\mathcal{A})_{\leq 0})$ determines a t-structure on $\mathcal{D}(\mathcal{A})$.*

(3) *The t-structure of (2) is accessible, right complete, and compatible with filtered colimits.*

Proof. Assertion (1) follows from Propositions 1.3.5.15 and 1.3.4.22, and assertion (2) is an immediate consequence of Proposition 1.3.5.18. To prove (3), we note an object of $\mathcal{D}(\mathcal{A})$ belongs to $\mathcal{D}(\mathcal{A})_{\leq 0}$ if and only if its image under the homology functor $\prod_{n>0} H_n : \mathcal{D}(\mathcal{A}) \to (\prod_{n>0} N(\mathcal{A}))$ vanishes. Using Propositions 1.3.4.24, 1.3.4.25, and our assumption that filtered colimits in $\mathcal{A}$ are left exact, we conclude that each of the functors H_n commutes with filtered colimits. It is now obvious that $\mathcal{D}(\mathcal{A})_{\leq 0}$ is closed under filtered colimits, and Proposition T.5.4.6.6 guarantees that $\mathcal{D}(\mathcal{A})_{\leq 0}$ is accessible. The right completeness of $\mathcal{D}(\mathcal{A})$ follows from the dual of Proposition 1.2.1.19. $\square$

Warning 1.3.5.22. The stable ∞-category $\mathcal{D}(\mathcal{A})$ is generally not left complete.

Remark 1.3.5.23. Let $\mathcal{C}$ be a presentable stable ∞-category equipped with an accessible t-structure (see Definition 1.4.4.12) which is compatible with filtered colimits. Then the heart $\mathcal{C}^{\heartsuit}$ is a presentable abelian category, and the canonical map $N(\mathcal{C}^{\heartsuit}) \to \mathcal{C}$ preserves filtered colimits. If $\{f_\alpha : A_\alpha \to B_\alpha\}$ is a filtered diagram of monomorphisms in $\mathcal{C}^{\heartsuit}$, then we have a filtered diagram of fiber sequences

$$A_\alpha \to B_\alpha \to B_\alpha/A_\alpha$$

in $\mathcal{C}$. Passing to filtered colimits, we obtain an fiber sequence

$$A \to B \to B/A$$

where A, B, and B/A belong to the heart of $\mathcal{C}$, so that $f = \varinjlim f_\alpha$ is again a monomorphism. It follows that $\mathcal{C}^\heartsuit$ is a Grothendieck abelian category.

Assume that $\bigcap_{n\geq 0} \mathcal{C}_{\leq -n}$ contains only zero objects of $\mathcal{C}$. Using Proposition 1.2.1.19, we conclude that $\mathcal{C}$ is right complete. It follows from Theorem 1.3.3.2 and Remark 1.3.3.6 that the inclusion $\mathrm{N}(\mathcal{C}^\heartsuit) \subseteq \mathcal{C}$ extends in an essentially unique way to a t-exact functor $\mathcal{D}^+(\mathcal{C}^\heartsuit) \to \mathcal{C}$.

Suppose that $\mathcal{A}$ is a Grothendieck abelian category with enough projective objects (for example, the category of R-modules for some ring R). Then, in addition to the derived ∞-category $\mathcal{D}(\mathcal{A})$, we can consider the derived ∞-category $\mathcal{D}^-(\mathcal{A})$ introduced in Definition 1.3.2.7. These two ∞-categories are a priori quite different from one another: one is defined using complexes of injective objects of $\mathcal{A}$, the other using complexes of projective objects of $\mathcal{A}$. Nevertheless we have the following result:

Proposition 1.3.5.24. *Let $\mathcal{A}$ be a Grothendieck abelian category with enough projective objects, and let $L : \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A})) \to \mathcal{D}(\mathcal{A})$ be a left adjoint to the inclusion. Then the composite functor*

$$F : \mathcal{D}^-(\mathcal{A}) \hookrightarrow \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A})) \xrightarrow{L} \mathcal{D}(\mathcal{A})$$

is a fully faithful embedding, whose essential image is the subcategory $\bigcup_{n\geq 0} \mathcal{D}(\mathcal{A})_{\geq -n} \subseteq \mathcal{D}(\mathcal{A})$.

Proof. Let $M_*, M'_* \in \mathcal{D}^-(\mathcal{A})$. We will show that the composite map

$$\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, M'_*) \xrightarrow{\theta} \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, LM'_*) \xrightarrow{\theta'} \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(LM_*, LM'_*)$$

is a quasi-isomorphism of chain complexes of abelian groups. The map θ is a quasi-isomorphism by Lemma 1.3.2.20, and the definition of L guarantees that θ' is a quasi-isomorphism. This proves that F is fully faithful. It is obvious that the essential image of F is contained in $\bigcup_{n\geq 0} \mathcal{D}(\mathcal{A})_{\geq -n}$. Conversely, suppose that $M_* \in \mathcal{D}(\mathcal{A})_{\geq -n}$. Since $\mathcal{A}$ has enough projective objects, we can choose an object $P_* \in \mathcal{D}^-(\mathcal{A})$ and a quasi-isomorphism from P_* to the subcomplex

$$\cdots \to M_{1-n} \to \ker(d : M_{-n} \to M_{-1-n}) \to 0 \to \cdots$$

of M_*. Using Lemma 1.3.2.20, we conclude that $M_* \simeq LP_*$ belongs to the essential image of L. $\square$

1.3.6 Complexes of Injective Objects

Let $\mathcal{A}$ be an abelian category with enough injective objects. In §1.3.2 we introduced the derived ∞-category $\mathcal{D}^+(\mathcal{A})$, whose objects are left-bounded chain complexes of injective objects of $\mathcal{A}$. The restriction to left-bounded complexes is sometimes overly restrictive: for example, the category $\mathcal{D}^+(\mathcal{A})$ generally does not admit infinite limits and colimits, even if the abelian category $\mathcal{A}$ is well-behaved. We can attempt to correct this defect by enlarging the ∞-category $\mathcal{D}^+(\mathcal{A})$. For example, if $\mathcal{A}$ is a Grothendieck abelian category, we can replace $\mathcal{D}^+(\mathcal{A})$ by the larger ∞-category $\mathcal{D}(\mathcal{A})$ introduced in §1.3.5. The ∞-category $\mathcal{D}(\mathcal{A})$ is presentable (Proposition 1.3.5.21), and therefore admits small limits and colimits. However, for some purposes it is more convenient to work with a different enlargement of $\mathcal{D}(\mathcal{A})$. For example, if $\mathcal{A}$ is a locally Noetherian abelian category (see Definition 1.3.6.5), then $\mathcal{D}^+(\mathcal{A})$ is equipped with a good supply of "finite" objects: namely, those chain complexes Q_* whose homologies $\mathrm{H}_n(Q_*)$ are compact objects of $\mathcal{A}$, which vanish for almost every $n \in \mathbf{Z}$. The collection of chain complexes satisfying this condition span a full subcategory $\mathcal{D}_0 \subseteq \mathcal{D}^+(\mathcal{A})$. We will prove below that $\mathcal{D}^+(\mathcal{A})$ can be identified with a full subcategory of $\mathrm{Ind}(\mathcal{D}_0)$. In particular, we can regard $\mathrm{Ind}(\mathcal{D}_0)$ as an enlargement of $\mathcal{D}^+(\mathcal{A})$. The ∞-category $\mathrm{Ind}(\mathcal{D}_0)$ is not only presentable: it is compactly generated, which is often very useful in applications. Our goal in this section is to study the

∞-category $\mathrm{Ind}(\mathcal{D}_0)$. Following work of Krause ([90]), we show that this ∞-category admits an explicit description in terms of homological algebra: namely, it can be identified with the differential graded nerve of $\mathrm{Ch}(\mathcal{A}_{\mathrm{inj}})$, where $\mathcal{A}_{\mathrm{inj}}$ denotes the full subcategory $\mathcal{A}$ spanned by the injective objects (Theorem 1.3.6.7).

Definition 1.3.6.1. Let $\mathcal{A}$ be a Grothendieck abelian category, and let $\mathcal{A}_{\mathrm{inj}}$ denote the full subcategory of $\mathcal{A}$ spanned by the injective objects. We let $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ denote the ∞-category $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}_{\mathrm{inj}}))$.

Proposition 1.3.6.2. *Let $\mathcal{A}$ be a Grothendieck abelian category. Then:*

(1) *The ∞-category $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ is stable.*

(2) *The ∞-category $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ contains $\mathcal{D}(\mathcal{A})$ as a full subcategory.*

(3) *Let $\mathcal{D}^{\mathrm{inj}}_{\geq 0}(\mathcal{A})$ denote the intersection $\mathcal{D}^{\mathrm{inj}}(\mathcal{A}) \cap \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))_{\geq 0}$, where $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))_{\geq 0}$ is as in Definition 1.3.5.16. Then the pair $(\mathcal{D}^{\mathrm{inj}}_{\geq 0}(\mathcal{A}), \mathcal{D}^{+}_{\leq 0}(\mathcal{A}))$ determines a t-structure on $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$.*

Proof. Assertion (1) is a special case of Proposition 1.3.2.10, assertion (2) follows from Proposition 1.3.5.6, and assertion (3) follows from Proposition 1.3.5.18. □

Remark 1.3.6.3. Let $\mathcal{A}$ be a Grothendieck abelian category, and regard $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ as endowed with the t-structures described in Propositions 1.3.5.21 and 1.3.6.2, respectively. Then the inclusion $\mathcal{D}(\mathcal{A}) \hookrightarrow \mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ is t-exact, and induces an isomorphism $\mathcal{D}(\mathcal{A})_{\leq 0} \simeq \mathcal{D}^{\mathrm{inj}}(\mathcal{A})_{\leq 0} = \mathcal{D}^{-}_{\leq 0}(\mathcal{A})$. In particular, the heart of $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ is equivalent to (the nerve of) the abelian category $\mathcal{A}$ (see Remark 1.3.5.23).

Remark 1.3.6.4. Let $\mathcal{A}$ be a Grothendieck abelian category. Proposition 1.3.5.13 implies that the inclusion functor $\mathcal{D}(\mathcal{A}) \hookrightarrow \mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ admits a left adjoint L. If $f : Q_* \to Q'_*$ is a morphism in $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$, then Lf is an equivalence if and only if f is a quasi-isomorphism of chain complexes. In particular, we have $LQ_* \simeq 0$ in the ∞-category $\mathcal{D}(\mathcal{A})$ if and only if the chain complex Q_* is acyclic. The collection of chain complexes Q_* satisfying this condition span the full subcategory $\bigcap_n \mathcal{D}^{\mathrm{inj}}_{\geq n}(\mathcal{A})$. We can summarize the situation as follows: the stable ∞-category $\mathcal{D}(\mathcal{A})$ is obtained from the larger stable ∞-category $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ by dividing out by the stable subcategory $\bigcap_n \mathcal{D}^{\mathrm{inj}}_{\geq n}(\mathcal{A}) \subseteq \mathcal{D}^{\mathrm{inj}}(\mathcal{A})$.

We now review some finiteness conditions on an abelian category $\mathcal{A}$ which guarantee the good behavior of the stable ∞-category $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$.

Definition 1.3.6.5. Let $\mathcal{A}$ be an abelian category containing an object M. We say that M is *Noetherian* if every nonempty collection of subobjects of M contains a maximal element. We say that an abelian category $\mathcal{A}$ is *locally Noetherian* if it is a Grothendieck abelian category and every object $M \in \mathcal{A}$ can be written as a filtered colimit of Noetherian subobjects of M.

Remark 1.3.6.6. Let $\mathcal{A}$ be an abelian category. Then an object $M \in \mathcal{A}$ is Noetherian if and only if it satisfies the following condition: for every ascending chain of subobjects

$$M_0 \hookrightarrow M_1 \hookrightarrow \cdots \hookrightarrow M,$$

we have $M_n \simeq M_{n+1}$ for $n \gg 0$.

We can now state the main result of this section:

Theorem 1.3.6.7 (Krause). *Let $\mathcal{A}$ be a locally Noetherian abelian category. Then the ∞-category $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ is compactly generated. Moreover, an object $Q_* \in \mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ is compact if and only if it satisfies the following conditions:*

(a) *The object Q_* lies in the essential image of the inclusion $\mathcal{D}^{+}(\mathcal{A}) \hookrightarrow \mathcal{D}^{\mathrm{inj}}(\mathcal{A})$; in particular, $\mathrm{H}_n(Q_*) \simeq 0$ for $n \gg 0$.*

(b) *The homology objects* $\mathrm{H}_n(Q_*)$ *vanish for* $n \ll 0$.

(c) *For every integer* n, $\mathrm{H}_n(Q_*)$ *is a Noetherian object of the abelian category* $\mathcal{A}$.

Corollary 1.3.6.8. *Let* $\mathcal{A}$ *be a locally Noetherian abelian category. Then the* ∞-*category* $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ *is presentable.*

The proof of Theorem 1.3.6.7 will be given at the end of this section. Let us begin by reviewing the theory of locally Noetherian abelian categories.

Proposition 1.3.6.9. *Let* $\mathcal{A}$ *be an abelian category, and let* $\mathcal{A}_0 \subseteq \mathcal{A}$ *be the full subcategory spanned by the Noetherian objects. Then:*

(1) *The subcategory* $\mathcal{A}_0 \subseteq \mathcal{A}$ *is closed under the formation of subobjects, quotient objects, and extensions. In particular,* $\mathcal{A}_0$ *is an abelian category, and the inclusion* $\mathcal{A}_0 \hookrightarrow \mathcal{A}$ *is injective.*

(2) *If the abelian category* $\mathcal{A}$ *is locally Noetherian, then* $\mathcal{A}_0$ *is essentially small, and the inclusion* $\mathcal{A}_0 \hookrightarrow \mathcal{A}$ *induces an equivalence of categories* $\mathrm{Ind}(\mathcal{A}_0) \simeq \mathcal{A}$.

Proof. We first note that if $N \in \mathcal{A}$ is a subobject or quotient of an object $M \in \mathcal{A}$, then the partially ordered set of isomorphism classes of subobjects of N can be regarded as a subset of the partially ordered set of isomorphism classes of subobjects of M. It follows immediately that if M is Noetherian, then N is Noetherian. To complete the proof of (1), suppose that we are given an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

where M' and M'' are Noetherian; we wish to prove that M is Noetherian. Suppose we are given a nonempty collection $\{M_\alpha\}_{\alpha \in A}$ of subobjects of M. Since M'' is Noetherian, the collection of subobjects $\{\mathrm{im}(M_\alpha \to M'')\}_{\alpha \in A}$ contains a maximal element N. Let $B = \{\alpha \in A : \mathrm{im}(M_\alpha \to M'') \simeq N\}$. Since M' is Noetherian, the collection $\{M_\beta \times_M M'\}_{\beta \in B}$ of subobjects of M' contains a maximal element $M_\beta \times_M M'$. It is now easy to see that M_β is a maximal element of the collection of subobjects $\{M_\alpha\}_{\alpha \in A}$.

We now prove (2). Assume that $\mathcal{A}$ is locally Noetherian. Fix a small collection of objects $\{M_i\}_{i \in I}$ which generates $\mathcal{A}$ under colimits. Then, for every object $M \in \mathcal{A}$, there exists an epimorphism $\coprod N_\alpha \to M$, where each N_α belongs to the set $\{M_i\}_{i \in I}$. It follows that the canonical map

$$\coprod_{i \in I, f : M_i \to M} M_i \to M$$

is an epimorphism. In particular, every subobject M' of M can be identified with the image of the map $\coprod_{i \in I : f : M_i \to M'} M_i \to M$. It follows that M' is determined (up to canonical isomorphism) by the images of the injective group homomorphisms $\{\mathrm{Hom}_{\mathcal{A}}(M_i, M') \to \mathrm{Hom}_{\mathcal{A}}(M_i, M)\}_{i \in I}$. In particular, there exists a set of representatives for the collection of all isomorphism classes of subobjects of M.

We now prove that $\mathcal{A}_0$ is locally small. Let $M \in \mathcal{A}$ be Noetherian, and choose an epimorphism $\coprod_{\alpha \in J} N_\alpha \to M$ where each N_α belongs to $\{M_i\}_{i \in I}$. For every finite subset $J' \subseteq J$, let $M_{J'}$ denote the image of the composite map

$$\coprod_{\alpha \in J'} N_\alpha \to \coprod_{\alpha \in J} N_\alpha \to M.$$

Since $\mathcal{A}$ is a Grothendieck abelian category, the formation of images in $\mathcal{A}$ commutes with filtered colimits, so that M is given by the filtered colimit $\varinjlim_{J' \subseteq J} M_{J'}$. Since M is Noetherian, the collection of subobjects $\{M_{J'}\}$ contains a maximal element. It follows that there exists a finite subset $J' \subseteq J$ such that $M_{J'} \simeq M$. Replacing J by J', we may assume that J is finite. Then $M \simeq \mathrm{coker}(K \to \coprod_{\alpha \in J} N_\alpha)$, for some subobject K of $\coprod_{\alpha \in J} N_\alpha$. Since there are only a bounded number of choices for the isomorphism type of the coproduct $\coprod_{\alpha \in J} N_\alpha$ and a bounded number of choices for the subobject J, we conclude that there are only a bounded number of possibilities for the isomorphism class of M: that is, the category $\mathcal{A}_0$ is essentially small.

We next claim that if $M \in \mathcal{A}$ is Noetherian, then M is a compact object of $\mathcal{A}$. To prove this, suppose we are given a diagram $\{N_\alpha\}_{\alpha \in A}$ indexed by a filtered partially ordered set A, having colimit $N \in \mathcal{A}$. We wish to prove that the canonical map

$$\theta : \varinjlim_\alpha \operatorname{Hom}_{\mathcal{A}}(M, N_\alpha) \to \operatorname{Hom}_{\mathcal{A}}(M, N)$$

is bijective. We first show that θ is injective. Suppose we are given a map $\phi : M \to N_\alpha$ such that the composite map $\phi' : M \xrightarrow{\phi} N_\alpha \to N$ vanishes. For each $\beta \geq \alpha$, let ϕ_β denote the composite map $M \xrightarrow{\phi} N_\alpha \to N_\beta$. Since $\mathcal{A}$ is a Grothendieck abelian category, we have $M \simeq \ker(\phi') \simeq \varinjlim \ker(\phi_\beta)$. Since M is Noetherian, we conclude that $M \simeq M_\beta$ for some $\beta \geq \alpha$: that is, the map ϕ_β is zero. This completes the proof that θ is injective.

We now verify the surjectivity of θ. Fix a morphism $\phi : M \to N$. For each index $\alpha \in A$, let $\widetilde{M}_\alpha = M \times_N N_\alpha$. Then M can be identified with the filtered colimit $\varinjlim \operatorname{im}(\widetilde{M}_\alpha \to M)$. Since M is Noetherian, there exists an index $\alpha \in A$ such that the map $\widetilde{M}_\alpha \to M$ is an epimorphism. Write $\widetilde{M}_\alpha$ as a filtered colimit of Noetherian subobjects $\widetilde{M}_{\alpha,\gamma}$. Then $M \simeq \varinjlim_\beta \operatorname{im}(\widetilde{M}_{\alpha,\gamma} \to M)$. Since M is Noetherian, we can choose γ such that $\widetilde{M}_{\alpha,\gamma} \to M$ is an epimorphism. Set $\overline{M} = \widetilde{M}_{\alpha,\gamma}$, and let K denote the kernel of the epimorphism $\overline{M} \to M$. Then the composite map

$$\overline{M} \hookrightarrow \widetilde{M}_\alpha \to N_\alpha \to N$$

annihilates K. For each $\beta \geq \alpha$, let K_β denote the kernel of the composite map

$$K \hookrightarrow \widetilde{M}_\alpha \to N_\alpha \to N.$$

Then $K \simeq \varinjlim_\beta K_\beta$. Since K is a subobject of $\overline{M}$, it is Noetherian. We therefore have $K \simeq K_\beta$ for some $\beta \geq \alpha$. It follows that the map $\overline{M} \to N_\alpha \to N_\beta$ factors through $\overline{M}/K \simeq M$, so that ϕ lies in the image of $\operatorname{Hom}_{\mathcal{A}}(M, N_\beta)$.

Applying Proposition T.5.3.5.11, we deduce that the inclusion $\mathcal{A}_0 \hookrightarrow \mathcal{A}$ extend to a fully faithful embedding $F : \operatorname{Ind}(\mathcal{A}_0) \to \mathcal{A}$. Since $\mathcal{A}$ is locally Noetherian, it is generated by $\mathcal{A}_0$ under filtered colimits; it follows immediately that F is an equivalence of categories. $\square$

Remark 1.3.6.10. Proposition 1.3.6.9 has a converse. Suppose that $\mathcal{A}_0$ is a small abelian category in which every object is Noetherian. Then $\operatorname{Ind}(\mathcal{A}_0)$ is a locally Noetherian abelian category, and an object of $\operatorname{Ind}(\mathcal{A}_0)$ is Noetherian if and only if it belongs to the essential image of the fully faithful embedding $j : \mathcal{A}_0 \hookrightarrow \operatorname{Ind}(\mathcal{A}_0)$. To prove this, we first note that $\operatorname{Ind}(\mathcal{A}_0)$ compactly generated. It follows that filtered colimits in $\operatorname{Ind}(\mathcal{A}_0)$ are left exact and therefore $\operatorname{Ind}(\mathcal{A}_0)$ is a Grothendieck abelian category. We next claim that if $M \in \operatorname{Ind}(\mathcal{A}_0)$ belongs to the essential image of j, then every subobject of M belongs to the essential image of j. To prove this, write $M = j(N)$ for some $N \in \mathcal{A}_0$. Let M' be a subobject of M, and write $M' \simeq \varinjlim_{\alpha \in A} j(N_\alpha)$ where the colimit is taken over some filtered partially ordered set A. Then $M' \simeq \varinjlim_{\alpha \in A} j(\operatorname{im}(N_\alpha \to N))$. Since N is Noetherian, we deduce that there exists $\alpha \in A$ such that $M' \simeq \operatorname{im}(N_\alpha \to N)$, so that M' belongs to the image of j.

The above argument shows that for each object $N \in \mathcal{A}_0$, the partially ordered set of isomorphism classes of subobjects of N is isomorphic to the partially ordered set of isomorphism classes of subobjects of $j(N)$. Since $N \in \mathcal{A}_0$ is Noetherian, we conclude that $j(N) \in \operatorname{Ind}(\mathcal{A}_0)$ is Noetherian. Now suppose that $M \simeq \varinjlim_{\alpha \in A} j(N_\alpha)$ is an arbitrary object of $\operatorname{Ind}(\mathcal{A}_0)$. Then M is equivalent to the filtered colimit of subobjects $\varinjlim_{\alpha \in A} \operatorname{im}(j(N_\alpha) \to M)$. Each of these subobjects is a quotient of $j(N_\alpha)$, and therefore Noetherian. This proves that the abelian category $\operatorname{Ind}(\mathcal{A}_0)$ is locally Noetherian. If the object M above is Noetherian, then there exists an index $\alpha \in A$ such that $M \simeq \operatorname{im}(j(N_\alpha) \to M)$. Then M is a quotient of $j(N_\alpha)$. Since $\ker(j(N_\alpha) \to M)$ is a subobject of $j(N_\alpha)$, it belongs to the essential image of j. It follows that every Noetherian object $M \in \operatorname{Ind}(\mathcal{A}_0)$ belongs to the essential image of j.

Proposition 1.3.6.9 implies that a locally Noetherian abelian category $\mathcal{A}$ is "controlled" by the full sub-category $\mathcal{A}_0 \subseteq \mathcal{A}$ spanned by the Noetherian objects. In particular, we have the following characterization of injective objects of $\mathcal{A}$:

Proposition 1.3.6.11. *Let $\mathcal{A}$ be a locally Noetherian abelian category, and let $Q \in \mathcal{A}$ be an object. The following conditions are equivalent:*

(1) *The object $Q \in \mathcal{A}$ is injective.*

(2) *For every Noetherian object $M \in \mathcal{A}$ and every subobject M' of M, the restriction map $\operatorname{Hom}_{\mathcal{A}}(M, Q) \to \operatorname{Hom}_{\mathcal{A}}(M', Q)$ is surjective.*

Proof. The implication $(1) \Rightarrow (2)$ is obvious. Conversely, suppose that (2) is satisfied; we wish to show that Q is injective. Let $i_0 : M_0 \to M$ be a monomorphism in $\mathcal{A}$ and suppose we are given a map $f_0 : M_0 \to Q$; we wish to show that f_0 factors through i_0. Let P be set of all isomorphism classes of pairs $(i_1 : M_1 \to M, f_1)$, where i_1 is a monomorphism, the map i_0 factors (automatically uniquely) as a composition $M_0 \xrightarrow{j} M_1 \xrightarrow{i_1} M$, and $f_1 : M_1 \to Q$ is a map satisfying $f_1 \circ j = f_0$. Using Zorn's Lemma, we deduce that P has a maximal element $(i_1 : M_1 \to M, f_1)$. We will complete the proof by showing that i_1 is an isomorphism. Since $\mathcal{A}$ is locally Noetherian, M can be written as a filtered colimit of Noetherian subobjects. It will therefore suffice to show that M_1 contains every Noetherian subobject N of M. Assume otherwise, and let $N' = M_1 \times_M N$. Then $M_2 = M_1 \coprod_{N'} N$ is a subobject of M properly containing M_1. By maximality, the map $f_1 : M_1 \to Q$ cannot be extended to a map $f_2 : M_2 \to Q$. It follows that the restriction of f_1 to N' cannot be extended to a map from N into Q, contradicting assumption (2). $\square$

Corollary 1.3.6.12. *Let $\mathcal{A}$ be a locally Noetherian abelian category. Then the collection of injective objects of $\mathcal{A}$ is closed under filtered colimits.*

Corollary 1.3.6.13. *Let $\mathcal{A}$ be a locally Noetherian abelian category. Then the collection of injective objects of $\mathcal{A}$ is closed under (possibly infinite) sums.*

Proof. It is clear that the collection of injective objects is closed under finite sums, and an arbitrary sum can be written as a filtered colimit of finite sums. $\square$

Corollary 1.3.6.14. *Let $\mathcal{A}$ be a locally Noetherian abelian category, and suppose we are given a collection of objects $\{Q(\alpha)_* \in \operatorname{Ch}(\mathcal{A}_{\mathrm{inj}})\}_{\alpha \in A}$. Then:*

(1) *The coproduct $Q_* \simeq \coprod_{\alpha \in A} Q(\alpha)_*$ is a chain complex of injective objects of $\mathcal{A}$.*

(2) *The canonical maps $Q(\alpha)_* \to Q_*$ exhibit Q_* as a coproduct of the objects $Q(\alpha)_*$ in the ∞-category $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$.*

Proof. Assertion (1) follows from Corollary 1.3.6.13, and assertion (2) from the description of mapping spaces in $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ supplied by Remark 1.3.1.12. $\square$

Corollary 1.3.6.15. *Let $\mathcal{A}$ be a locally Noetherian abelian category. Then the ∞-category $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ admits small colimits.*

Proof. Combine Corollary 1.3.6.14 with Proposition 1.4.4.1. $\square$

We now turn to the problem of computing mapping spaces in $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$.

Proposition 1.3.6.16. *Let $\mathcal{A}$ be a Grothendieck abelian category, let $i : M'_* \to M_*$ be a quasi-isomorphism in $\operatorname{Ch}_{\leq 0}(\mathcal{A})$, and let Q_* be a complex of injective objects of $\mathcal{A}$. Then composition with i induces a quasi-isomorphism of chain complexes*

$$\operatorname{Map}_{\operatorname{Ch}(\mathcal{A})}(M_*, Q_*) \to \operatorname{Map}_{\operatorname{Ch}(\mathcal{A})}(M'_*, Q_*).$$

Proof. Since Q_* is degreewise injective, we have a short exact sequence of chain complexes

$$0 \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*/M'_*, Q_*) \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*, Q_*) \to \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M'_*, Q_*) \to 0.$$

It will therefore suffice to show that the chain complex of abelian groups $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_*/M'_*, Q_*)$ is acyclic. In other words, we must show that for every integer n, every chain map $M_*/M'_* \to Q_*[n]$ is nullhomotopic. To prove this, we are free to replace Q_* by the chain complex

$$\cdots \to 0 \to Q_{1-n} \to Q_{-n} \to \cdots$$

In this case, Q_* is fibrant with respect to the model structure of Proposition 1.3.5.3 (see Proposition 1.3.5.6), so the desired result follows from Proposition 1.3.5.11 (since the quotient M_*/M'_* is acyclic). $\qquad\square$

Example 1.3.6.17. Let $\mathcal{A}$ be a locally Noetherian abelian category. The identification of $\mathcal{A}$ with the heart of $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ determines a functor $f : \mathrm{N}(\mathcal{A}) \to \mathcal{D}^{\mathrm{inj}}(\mathcal{A})$. Unwinding the definitions, we see that this functor carries an object $M \in \mathcal{A}$ to a chain complex

$$\cdots \to 0 \to Q_0 \to Q_{-1} \to \cdots$$

which is an injective resolution of M. Using Proposition 1.3.6.16, we conclude that for every complex $Q'_* \in \mathcal{D}^{\mathrm{inj}}(\mathcal{A})$, we can identify the mapping space $\mathrm{Map}_{\mathcal{D}^{\mathrm{inj}}(\mathcal{A})}(f(M), Q'_*)$ with the simplicial set obtained by applying the Dold-Kan correspondence to the truncation of the chain complex

$$\cdots \to \mathrm{Hom}_{\mathcal{A}}(M, Q'_1) \to \mathrm{Hom}_{\mathcal{A}}(M, Q'_0) \to \mathrm{Hom}_{\mathcal{A}}(M, Q'_{-1}) \to \cdots$$

In particular, we have canonical isomorphisms

$$\mathrm{Ext}^n_{\mathcal{D}^{\mathrm{inj}}(\mathcal{A})}(f(M), Q'_*) \simeq \mathrm{H}_{-n}(\mathrm{Hom}_{\mathcal{A}}(M, Q'_*)).$$

Corollary 1.3.6.18. *Let $\mathcal{A}$ be a locally Noetherian abelian category. Then every Noetherian object of the heart $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})^{\heartsuit} \simeq \mathrm{N}(\mathcal{A})$ is compact when viewed as an object of $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$.*

Proof. Let $f : \mathrm{N}(\mathcal{A}) \to \mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ be as in Example 1.3.6.17, and let M be a Noetherian object of $\mathcal{A}$. We wish to show that $f(M) \in \mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ is compact. Using Proposition 1.4.4.1, we are reduced to proving that for every collection of objects $Q(\alpha)_* \in \mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ having coproduct Q_*, the canonical map

$$\bigoplus_{\alpha} \mathrm{Ext}^0_{\mathcal{D}^{\mathrm{inj}}(\mathcal{A})}(f(M), Q(\alpha)_*) \to \mathrm{Ext}^0_{\mathcal{D}^{\mathrm{inj}}(\mathcal{A})}(f(M), Q_*)$$

is an isomorphism. Using Corollary 1.3.6.13, we can take Q_* to be the direct sum of the chain complexes $Q(\alpha)_*$ in the ordinary category $\mathrm{Ch}(\mathcal{A})$. In this case, the desired result follows immediately from the description of the relevant Ext-groups supplied by Example 1.3.6.17. $\qquad\square$

Proposition 1.3.6.19. *Let $\mathcal{A}$ be a locally Noetherian abelian category, and let $f : \mathrm{N}(\mathcal{A}) \to \mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ be the functor determined by the identification of $\mathcal{A}$ with the heart of $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$. Then f preserves filtered colimits.*

Proof. Let $\mathcal{A}_0$ denote the full subcategory of $\mathcal{A}$ spanned by the Noetherian objects, and let f_0 be the restriction of f to $\mathrm{N}(\mathcal{A}_0)$. Let $f' : \mathrm{N}(\mathcal{A}) \to \mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ be a left Kan extension of f_0, so that f' commutes with filtered colimits (Lemma T.5.3.5.8). The identity map from $f' | \mathrm{N}(\mathcal{A}_0)$ to $f | \mathrm{N}(\mathcal{A}_0)$ extends to a natural transformation of functors $\phi : f' \to f$. We will complete the proof by showing that γ is an equivalence.

Let M be an object of $\mathcal{A}$; we wish to show that ϕ induces an equivalence $f'(M) \to f(M)$ in the ∞-category $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$. We define a transfinite sequence $\{M_\alpha\}$ of subobjects of M as follows:

- Let $M_0 = 0$.

- If λ is a limit ordinal, set $M_\lambda = \varinjlim_{\alpha < \lambda} M_\alpha$.

- Suppose that M_α is defined and not isomorphic to M. Since $\mathcal{A}$ is locally Noetherian, there exists a nonzero Noetherian subobject N of the quotient M/M_α. Let $M_{\alpha+1}$ be the inverse image of N in M.

Since the collection of all isomorphism classes of subobjects of M is bounded in size (see the proof of Proposition 1.3.6.9), this construction must eventually stop: that is, there exists an ordinal α such that $M_\alpha \simeq M$. For each $\beta \leq \alpha$, let ϕ_β be the morphism from $f'(M_\beta)$ to $f(M_\beta)$ determined by ϕ. We will prove that each of the maps ϕ_β is an equivalence. When $\beta = 0$, this is clear (since $M_0 \simeq 0$ is a Noetherian object of $\mathcal{A}$). We now proceed by induction on β. There are two cases to consider:

- Suppose that $\beta < \alpha$ and that ϕ_β is an equivalence. We will show that $\phi_{\beta+1}$ is an equivalence. By construction, $M_{\beta+1}$ is the inverse image of a Noetherian subobject $N \subseteq M/M_\beta$. Since $\mathcal{A}$ is locally Noetherian, there exists a Noetherian subobject $\overline{N}$ of $M_{\beta+1}$ such that the composite map

$$\overline{N} \hookrightarrow M_{\beta+1} \to N$$

is an epimorphism. Let $\overline{N}_0$ denote the fiber product $\overline{N} \times_{M_{\beta+1}} M_\beta$. We then have a pushout diagram σ :

$$
\begin{array}{ccc}
\overline{N}_0 & \longrightarrow & N \\
\downarrow & & \downarrow \\
M_\beta & \longrightarrow & M_{\beta+1}
\end{array}
$$

in the abelian category $\mathcal{A}$, where the horizontal maps are monomorphisms. It follows that $f(\sigma)$ is a pushout diagram in $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$. Since $\overline{N}_0$ and $\overline{N}$ are Noetherian, the natural transformation ϕ induces equivalences

$$f'(\overline{N}) \to f(\overline{N}) \qquad f'(\overline{N}_0) \to f(\overline{N}_0).$$

Consequently, to prove that $\phi_{\beta+1}$ is an equivalence, it will suffice to show that the diagram $f'(\sigma)$ is a pushout diagram in $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$. Since $\mathcal{A}$ is locally Noetherian, we can write M_β as a filtered colimit of Noetherian subobjects $M_{\beta,i}$ which contain $\overline{N}_0$. For every such index i, form a pushout diagram

$$
\begin{array}{ccc}
\overline{N}_0 & \longrightarrow & N \\
\downarrow & & \downarrow \\
M_{\beta,i} & \longrightarrow & P_i
\end{array}
$$

in the abelian category $\mathcal{A}$. Then $\sigma \simeq \varinjlim_i \sigma_i$. Since the functor f' preserves filtered colimits, we have $f'(\sigma) \simeq \varinjlim_i f'(\sigma_i)$. It will therefore suffice to show that each of the diagrams $f'(\sigma_i)$ is a pushout square in $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$. This is clear, since σ_i is a diagram of Noetherian objects of $\mathcal{A}$ so that ϕ induces an equivalence $f'(\sigma_i) \to f(\sigma_i)$.

- Suppose that $\lambda \leq \alpha$ is a limit ordinal, and that ϕ_β is an equivalence for all ordinals $\beta < \lambda$. We wish to show that ϕ_λ is an equivalence. For this, it suffices to show that $\phi_\lambda \simeq \varinjlim_{\beta < \lambda} \phi_\beta$. Because the functor f' preserves filtered colimits, we are reduced to proving that $f(M_\lambda)$ is a filtered colimit of the diagram $\{f(M_\beta)\}_{\beta < \lambda}$. For each object $N \in \mathcal{A}$, let $Q_*(N) \in \mathrm{Ch}(\mathcal{A}_{\mathrm{inj}})$ denote a (functorial) choice of injective resolution of N. Using Theorem T.4.2.4.1, we are reduced to proving that $Q_*(M_\lambda)$ is a homotopy colimit of the diagram $\{Q_*(M_\beta)\}_{\beta < \lambda}$ in the underlying simplicial category of $\mathrm{Ch}(\mathcal{A}_{\mathrm{inj}})$. Let $Q'_* \in \mathrm{Ch}(\mathcal{A}_{\mathrm{inj}})$. Using Example 1.3.6.17, we are reduced to proving that the canonical map

$$\theta : \mathrm{DK}(\tau_{\geq 0} \mathrm{Hom}_\mathcal{A}(M_\lambda, Q'_*)) \to \varprojlim \mathrm{DK}(\tau_{\geq 0} \mathrm{Hom}_\mathcal{A}(M_\beta, Q'_*))$$

exhibits the simplicial set $\mathrm{DK}(\tau_{\geq 0} \mathrm{Hom}_\mathcal{A}(M_\lambda, Q'_*))$ as a homotopy limit of the tower of simplicial sets $\{\mathrm{DK}(\tau_{\geq 0} \mathrm{Hom}_\mathcal{A}(M_\beta, Q'_*))\}_{\beta < \lambda}$. Since $M_\lambda \simeq \varinjlim M_\beta$, the map θ is an isomorphism. It will therefore

suffice to show that the diagram of simplicial sets $\{\mathrm{DK}(\tau_{\geq 0}\operatorname{Hom}_{\mathcal{A}}(M_{\beta}, Q'_*))\}_{\beta < \lambda}$ is fibrant. Using the criterion given in Corollary T.A.2.9.24, we are reduced to proving that for $\beta \leq \lambda$, the map

$$\theta' : \mathrm{DK}(\tau_{\geq 0}\operatorname{Hom}_{\mathcal{A}}(M_{\beta}, Q'_*)) \to \varprojlim_{\gamma < \beta} \mathrm{DK}(\tau_{\geq 0}\operatorname{Hom}_{\mathcal{A}}(M_{\gamma}, Q'_*))$$

is a Kan fibration. This is obvious if β is a limit ordinal (in this case, θ' is an isomorphism). If $\beta = \gamma + 1$ is a successor ordinal, the desired result follows from Proposition 1.3.2.11.

$\square$

We are now ready to prove our main result:

Proof of Theorem 1.3.6.7. Let $\mathcal{C}$ denote the full subcategory of $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ spanned by those objects which satisfy conditions (1), (2), and (3). It follows from Corollary 1.3.6.18 that $\mathcal{C}$ consists of compact objects of $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$. Using Proposition T.5.3.5.11, we see that the inclusion $f : \mathcal{C} \hookrightarrow \mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ extends to a fully faithful embedding $F : \operatorname{Ind}(\mathcal{C}) \to \mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ which preserves filtered colimits. Since f is exact, the functor F preserves small colimits. We will complete the proof by showing that F is essentially surjective (since $\mathcal{C}$ is idempotent complete, this will imply that F induces an equivalence from $\mathcal{C}$ to the full subcategory of $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ spanned by the compact objects).

Let $\overline{\mathcal{C}}$ denote the essential image of the functor F (equivalently, $\overline{\mathcal{C}}$ can be described as the full subcategory of $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ generated by $\mathcal{C}$ under small colimits). We wish to show that every chain complex Q_* of injective objects of $\mathcal{A}$ belongs to $\overline{\mathcal{C}}$. Let $\operatorname{Ch}(\mathcal{A}_{\mathrm{inj}})_{\Delta}$ denote the underlying simplicial category of the differential graded category $\operatorname{Ch}(\mathcal{A}_{\mathrm{inj}})$. Note that Q_* can be identified with the homotopy colimit (in the simplicial category $\operatorname{Ch}(\mathcal{A}_{\mathrm{inj}})_{\Delta}$) of chain complexes of the form

$$\cdots \to 0 \to 0 \to Q_n \to Q_{n-1} \to Q_{n-2} \to \cdots$$

It will therefore suffice to show that every such chain complex belongs to $\overline{\mathcal{C}}$: that is, we may suppose that $Q_* \in \mathcal{D}^+(\mathcal{A})$.

Replacing Q_* by a shift, we may suppose that $Q_* \in \mathcal{D}_{\leq 0}^+(\mathcal{A})$. Since the t-structure on $\mathcal{D}^+(\mathcal{A})$ is right complete, Q_* is given by the colimit of the sequence $\tau_{\geq -m} Q_*$. It will therefore suffice to show that each truncation $\tau_{\geq -m} Q_*$ belongs to $\overline{\mathcal{C}}$. We proceed by induction on m, the case $m < 0$ being trivial. We have a fiber sequence

$$\tau_{\geq 1-m} Q_* \to \tau_{\geq -m} Q_* \to M[-m],$$

where M is an object belonging to the heart of $\mathcal{D}^-(\mathcal{A})$ (which we can identify with an object of $\mathcal{A}$). Using the inductive hypothesis, we are reduced to proving that $M \in \overline{\mathcal{C}}$. For this, it suffices to show that $\overline{\mathcal{C}}$ contains the heart of $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$, which follows immediately from Proposition 1.3.6.19. $\square$

We conclude this section with an additional remark about the t-structure on $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$.

Proposition 1.3.6.20. *Let $\mathcal{A}$ be a locally Noetherian abelian category. Then the t-structure on $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ is accessible (see Definition 1.4.4.12) and compatible with filtered colimits (that is, the full subcategory $\mathcal{D}_{\leq 0}^{\mathrm{inj}}(\mathcal{A}) \subseteq \mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ is closed under filtered colimits).*

Proof. To prove that the t-structure on $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ is accessible, it will suffice to show that the full subcategory $\mathcal{D}_{\leq 0}^{\mathrm{inj}}(\mathcal{A})$ is presentable. This follows from Proposition 1.3.5.21, since the inclusion $\mathcal{D}(\mathcal{A}) \hookrightarrow \mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ induces an equivalence $\mathcal{D}(\mathcal{A})_{\leq 0} \simeq \mathcal{D}_{\leq 0}^{\mathrm{inj}}(\mathcal{A})$.

We now show that the full subcategory $\mathcal{D}_{\leq 0}^{\mathrm{inj}}(\mathcal{A}) \subseteq \mathcal{D}^{\mathrm{inj}}(\mathcal{A})$ is closed under filtered colimits. Applying Corollary 1.3.6.13, we deduce that $\mathcal{D}_{\leq 0}^{\mathrm{inj}}(\mathcal{A})$ is closed under (possibly infinite) coproducts in $\mathcal{D}^{\mathrm{inj}}(\mathcal{A})$. Suppose next that we are given a sequence

$$Q(0)_* \to Q(1)_* \to Q(2)_* \to \cdots$$

of objects of $\mathcal{D}^{\text{inj}}_{\leq 0}(\mathcal{A})$, having a colimit Q_* in $\mathcal{D}^{\text{inj}}(\mathcal{A})$. Then Q_* can be identified with the cofiber of a map $\coprod Q(n)_* \to \coprod Q(n)_*$, and therefore belongs to $\mathcal{D}^{\text{inj}}_{\leq 1}(\mathcal{A})$. In particular, Q_* is a colimit of the sequence $\{Q(n)_*\}$ in the full subcategory $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}^{\text{inj}}(\mathcal{A})$, so that $Q_* \in \mathcal{D}^{\text{inj}}_{\leq 0}(\mathcal{A})$ by virtue of Proposition 1.3.5.21.

Now let Q_* be an arbitrary object of $\mathcal{D}^{\text{inj}}_{\leq 0}(\mathcal{A})$. The above argument shows that $\varinjlim \tau_{\geq -n} Q_*$ belongs to $\mathcal{D}^{\text{inj}}_{\leq 0}(\mathcal{A})$. Since the t-structure on $\mathcal{D}^+(\mathcal{A})$ is right complete, we deduce that Q_* is a colimit of the sequence $\{\tau_{\geq -n} Q_*\}$ in the ∞-category $\mathcal{D}^+(\mathcal{A})$, and therefore also in the larger ∞-category $\mathcal{D}^{\text{inj}}(\mathcal{A})$.

Now suppose we are given a filtered diagram $\{Q(\alpha)_*\}$ of objects of $\mathcal{D}^{\text{inj}}_{\leq 0}(\mathcal{A})$. We wish to show that $\varinjlim_\alpha Q(\alpha)_*$ belongs to $\mathcal{D}^{\text{inj}}_{\leq 0}(\mathcal{A})$. We have

$$\varinjlim_\alpha Q(\alpha)_* \simeq \varinjlim_\alpha \varinjlim_n \tau_{\geq -n} Q(\alpha)_* \simeq \varinjlim_n \varinjlim_\alpha \tau_{\geq -n} Q(\alpha)_*.$$

It will therefore suffice to show that each of the colimits $\varinjlim_\alpha \tau_{\geq -n} Q(\alpha)_*$ belongs to $\mathcal{D}^{\text{inj}}_{\leq 0}$. We proceed by induction on n, the case $n < 0$ being trivial. To carry out the inductive step, we observe that there is a cofiber sequence

$$\varinjlim_\alpha \tau_{\geq 1-n} Q(\alpha)_* \to \varinjlim_\alpha \tau_{\geq -n} Q(\alpha)_* \to (\varinjlim_\alpha \pi_{-n} Q(\alpha)_*)[-n].$$

Using the inductive hypothesis, we are reduced to proving that $\varinjlim_\alpha \pi_{-n} Q(\alpha)_*$ belongs to $\mathcal{D}^{\text{inj}}_{\leq n} Q(\alpha)_*$. In fact, Proposition 1.3.6.19 gives the stronger statement $\varinjlim_\alpha \pi_{-n} Q(\alpha)_* \in \mathcal{D}^{\text{inj}}_{\leq 0}(\mathcal{A})$. □

1.4　Spectra and Stabilization

One very broad goal of homotopy theory is to classify continuous maps between topological spaces up to homotopy. To formulate the problem more precisely, let X and Y be topological spaces equipped with base points, let $\text{Map}_*(X, Y)$ denote the space of continuous pointed maps from X to Y, and let $[X, Y] = \pi_0 \text{Map}_*(X, Y)$ be the set of homotopy classes of pointed maps from X to Y; one would like to describe the set $[X, Y]$. This is difficult in part because the problem is essentially nonlinear: in general, the set $[X, Y]$ does not have any algebraic structure. However, the situation is better in some special cases. For example, if X is the suspension of another pointed space X', then $[X, Y] \simeq \pi_1 \text{Map}_*(X', Y)$ admits a group structure. If X' is itself the suspension of another space X'', then the group $[X, Y] \simeq \pi_2 \text{Map}_*(X'', Y)$ is abelian. One can attempt to use these observations to study the mapping sets $[X, Y]$ in general: the construction $X \mapsto \Sigma(X)$ is functorial in X, so we have natural maps

$$[X, Y] \to [\Sigma(X), \Sigma(Y)] \to [\Sigma^2(X), \Sigma^2(Y)] \to \cdots$$

In particular, we can view each $[\Sigma^n(X), \Sigma^n(Y)]$ as an approximation to $[X, Y]$; these approximations are often easier to study, since they admit group structures for $n > 0$ (and are abelian for $n > 1$). If X and Y are finite pointed CW complexes, the direct limit $\varinjlim_n [\Sigma^n(X), \Sigma^n(Y)]$ is an abelian group, called the *group of homotopy classes of stable maps from X to Y*; we will denote this group by $[X, Y]_s$.

The abelian groups $[X, Y]_s$ can be regarded as simplified (or linearized) versions of the homotopy sets $[X, Y]$. To study them systematically, it is useful to linearize the homotopy category $\mathcal{H}_*$ of (pointed) spaces itself: that is, to work with a version of the homotopy category where the morphisms are given by homotopy classes of stable maps, rather than homotopy classes of maps. The relevant category is often called the *stable homotopy category*, or the *homotopy category of spectra*. It can be described in several different ways:

(A) There is an obvious candidate for a category $\mathcal{C}_0$ which satisfies the requirement given above: namely, we take the objects of $\mathcal{C}_0$ to be finite pointed CW complexes, and the morphisms to be given by the formula $\text{Hom}_{\mathcal{C}_0}(X, Y) = [X, Y]_s$. By construction, we have canonical bijections $[X, Y]_s \simeq [\Sigma(X), \Sigma(Y)]_s$: in other words, the suspension functor $X \mapsto \Sigma(X)$ determines a fully faithful embedding from $\mathcal{C}_0$ to itself. For many purposes, it is convenient to work in a slightly larger category $\mathcal{C}$, on which the suspension

functor $X \mapsto \Sigma(X)$ is an equivalence of categories. One can achieve this end by formally introducing objects of the form $\Sigma^n(X)$ for all integers n. More precisely, we let $\mathcal{C}$ be the category whose objects are pairs (X, n), where X is a pointed finite CW complex and $n \in \mathbf{Z}$ an integer, with morphisms given by the formula

$$\mathrm{Hom}_{\mathcal{C}}((X, m), (Y, n)) = \varinjlim[\Sigma^{m+k}(X), \Sigma^{n+k}(Y)].$$

The construction $X \mapsto (X, 0)$ determines a fully faithful embedding $\mathcal{C}_0 \hookrightarrow \mathcal{C}$, and the suspension functor $X \mapsto \Sigma(X)$ on $\mathcal{C}_0$ extends (up to isomorphism) to an equivalence of $\mathcal{C}$ with itself, given by the formula $(X, n) \mapsto (X, n+1)$.

We will refer to $\mathcal{C}$ as the *homotopy category of finite spectra*. Unlike the homotopy category of spaces (or pointed spaces), it possesses a rich algebraic structure: for example, it is a triangulated category. To prove this, it suffices (by Theorem 1.1.2.14) to realize $\mathcal{C}$ as the homotopy category of a stable ∞-category. This ∞-category can be obtained by the same formal procedure used to define $\mathcal{C}$. Namely, we begin with the ∞-category $\mathcal{S}^{\mathrm{fin}}_*$ of finite pointed spaces (Notation 1.4.2.5), and formally invert the suspension functor by passing to the colimit of the sequence

$$\mathcal{S}^{\mathrm{fin}}_* \xrightarrow{\Sigma} \mathcal{S}^{\mathrm{fin}}_* \xrightarrow{\Sigma} \cdots$$

We will denote this colimit by $\mathrm{Sp}^{\mathrm{fin}}$, and refer to it as the ∞-*category of finite spectra*. We denote the the Ind-completion of $\mathrm{Sp}^{\mathrm{fin}}$ by Sp, and refer to it as the ∞-*category of spectra*. As we will see, Sp is a stable ∞-category, whose homotopy category can be identified with the classical stable homotopy category.

(B) The passage from the ∞-category $\mathrm{Sp}^{\mathrm{fin}}$ to its Ind-completion Sp is important if we wish to work with an ∞-category which admits arbitrary limits and colimits. This is clear, since the ∞-category $\mathrm{Sp}^{\mathrm{fin}}$ has strong finiteness conditions built into its definition. We can attempt to remove these conditions by beginning not with the ∞-category $\mathcal{S}^{\mathrm{fin}}_*$ of finite pointed spaces, but its Ind-completion $\mathrm{Ind}(\mathcal{S}^{\mathrm{fin}}_*) \simeq \mathcal{S}_*$. A formal argument shows that the Ind-completion of the direct limit

$$\mathcal{S}^{\mathrm{fin}}_* \xrightarrow{\Sigma} \mathcal{S}^{\mathrm{fin}}_* \xrightarrow{\Sigma} \mathcal{S}^{\mathrm{fin}}_* \xrightarrow{\Sigma} \cdots$$

is equivalent to the homotopy inverse limit of the tower

$$\mathrm{Ind}(\mathcal{S}^{\mathrm{fin}}_*) \xleftarrow{\Omega} \mathrm{Ind}(\mathcal{S}^{\mathrm{fin}}_*) \xleftarrow{\Omega} \cdots,$$

where Ω denotes the loop space functor (the right adjoint of the suspension Σ). We can therefore describe Sp as an ∞-category of *infinite loop spaces*: that is, infinite sequences of pointed spaces $\{E(n)\}$ equipped with homotopy equivalences $E(n) \simeq \Omega E(n+1)$.

(C) Another approach to the subject of stable homotopy theory is to study invariants of (pointed) topological spaces which are invariant under suspension. For example, singular cohomology has this property: for every pointed topological space X, there are canonical isomorphisms $\widetilde{\mathrm{H}}^n(X) \simeq \widetilde{\mathrm{H}}^{n+1}(\Sigma(X))$, where $\widetilde{\mathrm{H}}$ denotes the functor of reduced (integral) cohomology. More generally, one can consider *generalized cohomology theories*: that is, sequences of functors $\{h^n\}_{n \in \mathbf{Z}}$ from the homotopy category of pointed spaces to the category of abelian groups, together with natural isomorphisms $\gamma_n : h^n X \simeq h^{n+1} \Sigma(X)$, satisfying a suitable collection of axioms (see Definition 5.3.3.8). The celebrated *Brown representability theorem* (Theorem 1.4.1.2) guarantees that each of the functors h^n is representable by a pointed space $E(n)$, and the natural isomorphisms γ_n give homotopy equivalences $E(n) \simeq \Omega E(n+1)$. In other words, any cohomology theory $\{h^n\}_{n \in \mathbf{Z}}$ can be *represented* by a spectrum $\{E(n)\}_{n \in \mathbf{Z}}$: we can therefore regard Sp as an ∞-category whose objects are cohomology theories. (This perspective merits a word of caution: every morphism $f : E \to E'$ in Sp induces a natural transformation between the corresponding cohomology theories, but this latter map can be zero even if f is not nullhomotopic. In other words, passage from a spectrum E to the underlying cohomology theory is not faithful in general.)

(D) Let $\{E(n)\}_{n \in \mathbf{Z}}$ be a spectrum. Then $E(0) \simeq \Omega E(1)$ is a loop space: in particular, it admits a multiplication $E(0) \times E(0) \to E(0)$ given by concatenation of loops, which is associative up to coherent homotopy. However, much more is true: the identifications $E(0) \simeq \Omega^n E(n)$ show that $E(0)$ has the structure of an n-fold loop space for each $n \geq 0$. This structure allows us to view $E(0)$ as a *commutative monoid* object of the ∞-category $\mathcal{S}$ of spaces. In fact, there is a converse to this observation: the construction $\{E(n)\} \mapsto E(0)$ determines an equivalence between the full subcategory $\mathrm{Sp}^{\mathrm{cn}} \subseteq \mathrm{Sp}$ of *connective spectra* and the ∞-category $\mathrm{Mon}^{\mathrm{gp}}_{\mathrm{CRing}}(\mathcal{S})$ of grouplike commutative monoids in $\mathcal{S}$ (see §5.1.3 for further discussion). This provides an algebraic way of thinking about the ∞-category of spectra: roughly speaking, the ∞-category of spectra bears the same relationship to the ∞-category of spaces as the ordinary category of abelian groups bears to the ordinary category of sets. Later in this book, we will elaborate on this analogy by describing homotopy-theoretic analogues of the theory of commutative and associative rings.

Our goal in this section is to provide a quick introduction to stable homotopy theory by elaborating on perspectives (A) through (C) (we will return to (D) briefly later in the book, once we have the technology to discuss algebraic structures in an ∞-categorical context; see Remark 5.1.3.17). We will begin in §1.4.1 with a review of Brown's representability theorem. More precisely, we will show that if $\mathcal{C}$ is a pointed ∞-category satisfying some mild hypotheses, then it is possible to give necessary and sufficient conditions for a functor $F : h\mathcal{C} \to \mathcal{S}\mathrm{et}$ to be representable by an object of $\mathcal{C}$. We can apply this to give a classification of *cohomology theories* on $\mathcal{C}$ in terms of *infinite loop objects* of $\mathcal{C}$: that is, sequences of objects $\{E(n) \in \mathcal{C}\}_{n \in \mathbf{Z}}$ equipped with equivalences $E(n) \simeq \Omega E(n+1)$. The collection of such infinite loop objects can be organized into an ∞-category $\mathrm{Sp}(\mathcal{C})$; we will refer to $\mathrm{Sp}(\mathcal{C})$ as the ∞-category of *spectrum objects* of $\mathcal{C}$. We will define this ∞-category in §1.4.2, and show that it is a stable ∞-category. Of greatest interest to us is the case where $\mathcal{C}$ is the ∞-category $\mathcal{S}$ of spaces. In this case, we will denote the ∞-category $\mathrm{Sp}(\mathcal{C})$ by Sp, and refer to it as the ∞-category of spectra. We will study this ∞-category in §1.4.3, and show that it can be identified with the ∞-category $\mathrm{Ind}(\mathrm{Sp}^{\mathrm{fin}})$ described in (A).

It should be emphasized that there are many definitions of the stable homotopy category $h\mathrm{Sp}$ in the literature, some of which look quite different from the definition given in this book. To facilitate the comparison of our approach with others, it is convenient to have not only a construction of the ∞-category Sp, but also an abstract characterization of it. We will provide such a characterization by showing that $\mathrm{Sp}(\mathcal{C})$ is in some sense *universal* among stable ∞-categories equipped with a forgetful functor $\mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ (Corollary 1.4.2.23).

There is another characterization of the ∞-category Sp which is worthy of mention: among stable ∞-categories, it is freely generated by one object (the *sphere spectrum*) under small colimits. We will prove this result in §1.4.4 (see Corollary 1.4.4.6), after embarking on a general study of the behavior of colimits in stable ∞-categories.

1.4.1   The Brown Representability Theorem

Let $\mathcal{D}$ be a category. A functor $F : \mathcal{D}^{op} \to \mathcal{S}\mathrm{et}$ is said to be *representable* if there exists an object $X \in \mathcal{D}$ and a point $\eta \in F(X)$ which induces bijections $\mathrm{Hom}_{\mathcal{D}}(Y, X) \to F(Y)$ for every object $Y \in \mathcal{D}$. If we assume that the category $\mathcal{D}$ is presentable, then the functor F is representable if and only if it carries colimits in $\mathcal{D}$ to limits in $\mathcal{S}\mathrm{et}$ (see Proposition T.5.5.2.2). Our goal in this section is to study representability in a slightly different situation: namely, we will suppose that $\mathcal{D}$ is given as the homotopy category of a presentable ∞-category $\mathcal{C}$. In this case, the category $\mathcal{D}$ need not admit colimits. Nevertheless, one can often characterize the representable functors on $\mathcal{D}$ in terms of the behavior with respect to colimits in the underlying ∞-category $\mathcal{C}$.

We begin by recalling a bit of terminology. Let $\mathcal{D}$ be a category which admits finite coproducts. A *cogroup object* of $\mathcal{D}$ is an object $X \in \mathcal{D}$ equipped with a comultiplication $X \to X \amalg X$ with the following property: for every object $Y \in \mathcal{D}$, the induced multiplication

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) \times \mathrm{Hom}_{\mathcal{D}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{D}}(X \amalg X, Y) \to \mathrm{Hom}_{\mathcal{D}}(X, Y)$$

determines a group structure on the set $\mathrm{Hom}_{\mathcal{D}}(X, Y)$.

Example 1.4.1.1. Let $\mathcal{C}$ be an ∞-category which admits finite colimits, let $\emptyset$ denote the initial object of $\mathcal{C}$, and suppose we are given a map $\epsilon : X \to \emptyset$. Then the pushout $\Sigma(X) = \emptyset \amalg_X \emptyset$ is a cogroup object of the homotopy category $h\mathcal{C}$. Namely, there is a "fold" map

$$\Sigma(X) \amalg \Sigma(X) \simeq \emptyset \amalg_X \emptyset \amalg_X \emptyset \to \emptyset \amalg_X X \amalg_X \emptyset \simeq \emptyset \amalg_X \emptyset$$

which, for every object $Y \in \mathcal{C}$, induces the canonical group structure on the set $\mathrm{Hom}_{h\mathcal{C}}(\Sigma(X), Y) \simeq \pi_1(\mathrm{Map}_{\mathcal{C}}(X, Y), f)$. Here $f \in \mathrm{Map}_{\mathcal{C}}(X, Y)$ is the point given by the composition $X \xrightarrow{\epsilon} \emptyset \to Y$.

The main result of this section is the following:

Theorem 1.4.1.2 (Brown Representability). *Let $\mathcal{C}$ be a presentable ∞-category containing a set of objects $\{S_\alpha\}_{\alpha \in A}$ with the following properties:*

 (i) *Each object S_α is a cogroup object of the homotopy category $h\mathcal{C}$.*

 (ii) *Each object $S_\alpha \in \mathcal{C}$ is compact.*

 (iii) *The ∞-category $\mathcal{C}$ is generated by the objects S_α under small colimits.*

Then a functor $F : h\mathcal{C}^{op} \to \mathrm{Set}$ is representable if and only if it satisfies the following conditions:

 (a) *For every collection of objects C_β in $\mathcal{C}$, the map $F(\coprod_\beta C_\beta) \to \prod_\beta F(C_\beta)$ is a bijection.*

 (b) *For every pushout square*

$$
\begin{array}{ccc}
C & \longrightarrow & C' \\
\downarrow & & \downarrow \\
D & \longrightarrow & D'
\end{array}
$$

 in $\mathcal{C}$, the induced map $F(D') \to F(C') \times_{F(C)} F(D)$ is surjective.

We will give the proof of Theorem 1.4.1.2 at the end of this section.

Example 1.4.1.3. Let $\mathcal{C}$ be a presentable stable ∞-category. Then the homotopy category of $\mathcal{C}$ is additive (Lemma 1.1.2.9), so every object of $\mathcal{C}$ is a cogroup object of $h\mathcal{C}$. If $\mathcal{C}$ is compactly generated, then it satisfies the hypotheses of Theorem 1.4.1.2.

Example 1.4.1.4. Let $\mathcal{S}_*$ denote the ∞-category of pointed spaces, and let $\mathcal{S}_*^{\geq 1}$ denote the full subcategory spanned by the connected spaces. We claim that $\mathcal{S}_*^{\geq 1}$ satisfies the hypotheses of Theorem 1.4.1.2: that is, $\mathcal{S}_*^{\geq 1}$ is generated under colimits by connective cogroup objects. In fact, $\mathcal{S}_*^{\geq 1}$ is generated under colimits by the 1-sphere S^1 (which corepresents the group-valued functor $X \mapsto \pi_1(X)$, and is therefore a cogroup object of the homotopy category $h\mathcal{S}_*^{\geq 1}$). This is equivalent to the assertion that a map of connected pointed spaces $f : X \to Y$ is a homotopy equivalence if and only if the induced map $\mathrm{Map}_{\mathcal{S}_*}(S^1, X) \to \mathrm{Map}_{\mathcal{S}_*}(S^1, Y)$ is a homotopy equivalence. This is clear, since we have isomorphisms

$$\pi_n X \simeq \pi_{n-1} \mathrm{Map}_{\mathcal{S}_*}(S^1, X) \qquad \pi_n Y \simeq \pi_{n-1} \mathrm{Map}_{\mathcal{S}_*}(S^1, Y)$$

for $n > 0$.

Remark 1.4.1.5. In the special case $\mathcal{C} = \mathcal{S}_*$, the conclusion of Theorem 1.4.1.2 reproduces the classical Brown representability theorem (see [27]).

We now discuss some of consequences of Theorem 1.4.1.2 for the classification of *cohomology theories*.

Definition 1.4.1.6. Let $\mathcal{C}$ be a pointed ∞-category which admits small colimits and let $\Sigma : \mathcal{C} \to \mathcal{C}$ be the suspension functor. A *cohomology theory* on $\mathcal{C}$ is a sequence of functors $\{H^n : h\mathcal{C}^{op} \to \mathcal{S}et\}_{n \in \mathbf{Z}}$ together with isomorphisms $\delta^n : H^n \simeq H^{n+1} \circ \Sigma$, satisfying the following pair of conditions:

(1) For every collection of objects $\{C_\alpha\}$ in $\mathcal{C}$, the canonical map $H^n(\coprod C_\alpha) \to \prod H^n(C_\alpha)$ is a bijection. In particular, if $*$ denotes a zero object of $\mathcal{C}$, then $H^n(*)$ consists of a single point. For any object $C \in \mathcal{C}$, the canonical map $C \to *$ induces a map $H^n(*) \to H^n(C)$ which we can identify with an element $0 \in H^n(C)$.

(2) Suppose we are given a cofiber sequence

$$C' \to C \to C''$$

in the ∞-category $\mathcal{C}$. If $\eta \in H^n(C)$ has image $0 \in H^n(C')$, then η lies in the image of the map $H^n(C'') \to H^n(C)$.

Remark 1.4.1.7. Let $\mathcal{C}$ be a pointed ∞-category which admits small colimits, and let C be an object of $\mathcal{C}$. The two-fold suspension $\Sigma^2(C)$ is a commutative cogroup object of the homotopy category $h\mathcal{C}$ (we have canonical isomorphisms $\mathrm{Hom}_{h\mathcal{C}}(\Sigma^2(C), D) \simeq \pi_2 \mathrm{Map}_{\mathcal{C}}(C, D)$). Let $\{H^n, \delta^n\}$ be a cohomology theory on $\mathcal{C}$. Since the functor H^{n+2} carries coproducts in $h\mathcal{C}$ to products of sets, it carries commutative cogroup objects of $h\mathcal{C}$ to abelian groups. In particular, for every object $C \in \mathcal{C}$, the set $H^n(C) \simeq H^{n+2}(\Sigma^2(C))$ has the structure of an abelian group, depending functorially on the object C: that is, we can regard each H^n as a functor from the homotopy category $h\mathcal{C}$ to the category of abelian groups. In particular, for every object $C \in \mathcal{C}$, the map $H^n(*) \to H^n(C)$ carries the unique element of $H^n(*)$ to the identity element $0 \in H^n(C)$.

Remark 1.4.1.8. Let $\mathcal{C}$ be a pointed ∞-category which admits small colimits, and suppose we are given a cofiber sequence $C' \xrightarrow{f} C \xrightarrow{g} C''$ in $\mathcal{C}$. Such a triangle induces a map $C'' \to \Sigma(C')$, well-defined up to homotopy. If we are given a cohomology theory $\{H^n, \delta^n\}$ on $\mathcal{C}$, we obtain a boundary map

$$\partial : H^n(C') \xrightarrow{\delta^n} H^{n+1}(\Sigma(C')) \to H^{n+1}(C'').$$

These boundary maps can be used to splice together a sequence of abelian groups

$$\cdots \to H^{n-1}(C') \xrightarrow{\partial} H^n(C'') \xrightarrow{g^*} H^n(C) \xrightarrow{f^*} H^n(C') \xrightarrow{\partial} H^{n+1}(C'') \to \cdots$$

We claim that this sequence is exact. Exactness at $H^n(C)$ follows immediately from condition (2) of Definition 1.4.1.6. Exactness at $H^n(C'')$ follows by applying the same argument to the cofiber sequence $C \to C'' \to \Sigma(C')$ (which gives rise to the same abelian groups and the same group homomorphisms up to sign, by virtue of Lemma 1.1.2.13), and exactness at $H^n(C')$ follows by applying the same argument to the cofiber sequence $C'' \to \Sigma(C') \to \Sigma(C)$.

Remark 1.4.1.9. Let $\mathcal{C}$ be a pointed ∞-category which admits small colimits, and let $\{H^n, \delta^n\}$ be a cohomology theory on $\mathcal{C}$. Then each of the functors $H^n : h\mathcal{C}^{op} \to \mathcal{S}et$ satisfies conditions (a) and (b) of Theorem 1.4.1.2. Condition (a) is obvious. To prove (b), suppose we are given a pushout square

$$
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\downarrow & & \downarrow \\
D & \xrightarrow{g} & D'
\end{array}
$$

Let $E = \mathrm{cofib}(f) \simeq \mathrm{cofib}(g)$. Using Remark 1.4.1.8, we get a map of exact sequences

$$
\begin{array}{ccccccc}
H^n(E) & \longrightarrow & H^n(D') & \longrightarrow & H^n(D) & \longrightarrow & H^{n+1}(E) \\
\downarrow{\scriptstyle\phi} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\psi} \\
H^n(E) & \longrightarrow & H^n(C') & \longrightarrow & H^n(C) & \longrightarrow & H^{n+1}(E).
\end{array}
$$

Using the injectivity of ψ and the surjectivity of ϕ, we deduce that the map $H^n(D') \to H^n(C') \times_{H^n(C)} H^n(D)$ is surjective.

Combining Remark 1.4.1.9 with Theorem 1.4.1.2, we obtain the following result:

Corollary 1.4.1.10. *Let $\mathcal{C}$ be a presentable pointed ∞-category. Assume that $\mathcal{C}$ is generated under colimits by compact objects which are cogroup objects of the homotopy category $h\mathcal{C}$, and let $\{H^n, \delta^n\}$ be a cohomology theory on $\mathcal{C}$. Then for every integer n, the functor $H^n : h\mathcal{C}^{op} \to \mathrm{Set}$ is representable by an object $E(n) \in \mathcal{C}$.*

Remark 1.4.1.11. In the situation of Corollary 1.4.1.10, the isomorphisms $\delta^n : H^n \simeq H^{n+1} \circ \Sigma$ determine canonical isomorphisms $E(n) \simeq \Omega E(n+1)$ in the homotopy category $h\mathcal{C}$. Choosing equivalences in $\mathcal{C}$ which represent these isomorphisms, we can promote the sequence $\{E(n)\}$ to an object E in the homotopy limit $\mathrm{Sp}(\mathcal{C})$ of the tower

$$\cdots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \cdots$$

The object E is well-defined up to (non-unique) isomorphism in the homotopy category $h\mathrm{Sp}(\mathcal{C})$. We will return to the study of the ∞-category $\mathrm{Sp}(\mathcal{C})$ in §1.4.2.

Proof of Theorem 1.4.1.2. The necessity of conditions (a) and (b) is obvious. We will prove that these conditions are sufficient. Let $\emptyset$ denote an initial object of $\mathcal{C}$. If S is an object of $\mathcal{C}$ equipped with a map $\epsilon : S \to \emptyset$, we define the suspension $\Sigma(S)$ to be the pushout $\emptyset \amalg_S \emptyset$, so that $\Sigma(S)$ has the structure of a cogroup in $h\mathcal{C}$ (Example 1.4.1.1). Each of the objects S_α is equipped with a counit map $S_\alpha \to \emptyset$ (by virtue of (i)), so that the suspension $\Sigma(S_\alpha)$ is well-defined. Enlarging the collection $\{S_\alpha\}$ if necessary, we may assume that this collection is stable under the formation of suspensions.

We first prove the following:

($*$) Let $f : C \to C'$ be a morphism in $\mathcal{C}$ such that the induced map $\mathrm{Hom}_{h\mathcal{C}}(S_\alpha, C) \to \mathrm{Hom}_{h\mathcal{C}}(S_\alpha, C')$ is an isomorphism, for every index α. Then f is an equivalence in $\mathcal{C}$.

To prove ($*$), it will suffice to show that for every object $X \in \mathcal{C}$, the map f induces a homotopy equivalence $\phi_X : \mathrm{Map}_{\mathcal{C}}(X, C) \to \mathrm{Map}_{\mathcal{C}}(X, C')$. Let $\mathcal{C}'$ denote the full subcategory of $\mathcal{C}$ spanned by those objects X for which ϕ_X is an equivalence. The full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is stable under colimits; we wish to prove that $\mathcal{C}' = \mathcal{C}$. By virtue of assumption (iii), it suffices to show that each of the objects S_α belongs to $\mathcal{C}'$. Since S_α is a cogroup object of $h\mathcal{C}$, ϕ_{S_α} is a map between group objects of the homotopy category $\mathcal{H}$ of spaces. It follows that ϕ_{S_α} is a homotopy equivalence if and only if it induces an isomorphism of groups $\pi_n \mathrm{Map}_{\mathcal{C}}(X, C) \to \pi_n \mathrm{Map}_{\mathcal{C}}(X, C')$ for each $n \geq 0$ (here the homotopy groups are taken with respect to the base points given by the group structures). Replacing S_α by $\Sigma^n(S_\alpha)$, we can reduce to the case $n = 0$: that is, to the bijectivity of the maps $\mathrm{Hom}_{h\mathcal{C}}(S_\alpha, C) \to \mathrm{Hom}_{h\mathcal{C}}(S_\alpha, C')$.

Now suppose that F is a functor satisfying conditions (a) and (b). We will prove the following:

($*'$) Let $X \in \mathcal{C}$ and let $\eta \in F(X)$. Then there exists a map $f : X \to X'$ in $\mathcal{C}$ and an object $\eta' \in F(X')$ lifting η with the following property: for every index $\alpha \in A$, η' induces a bijection $\mathrm{Hom}_{h\mathcal{C}}(S_\alpha, X') \to F(S_\alpha)$.

To prove ($*'$), we begin by defining X_0 to be the coproduct of X with $\coprod_{\alpha \in A, \gamma \in F(S_\alpha)} S_\alpha$. Using (a), we deduce the existence of an element $\eta_0 \in F(X_0)$ lifting η. By construction, η_0 induces a *surjection* $\mathrm{Hom}_{h\mathcal{C}}(S_\alpha, X_0) \to F(S_\alpha)$ for each index α.

We now define a sequence of morphisms $X_0 \to X_1 \to X_2 \to \cdots$ and a compatible family of elements $\eta_n \in F(X_n)$ using induction on n. Suppose that X_n and η_n have already been constructed. For each index $\alpha \in A$, let K_α be the kernel of the group homomorphism $\mathrm{Hom}_{h\mathcal{C}}(S_\alpha, X_n) \to F(S_\alpha)$, and define X_{n+1} to fit into a pushout diagram

$$\begin{array}{ccc}
\coprod_{\alpha \in A, \gamma \in K_\alpha} S_\alpha & \longrightarrow & \emptyset \\
\downarrow & & \downarrow \\
X_n & \longrightarrow & X_{n+1}
\end{array}$$

where the upper horizontal map is given by the counit on each S_α. The existence of a point $\eta_{n+1} \in F(X_{n+1})$ lifting η_n follows from assumption (a).

Let $X' = \varinjlim_n X_n$. We have a pushout diagram

$$\begin{array}{ccc} \coprod_n X_n & \longrightarrow & \coprod_n X_{2n} \\ \downarrow & & \downarrow \\ \coprod_n X_{2n+1} & \longrightarrow & X'. \end{array}$$

Using (a) and (b), we deduce the existence of a point $\eta' \in F(X')$ lifting the sequence $\{\eta_n \in F(X_n)\}$. We claim that η' satisfies the condition described in $(*)$. Fix an index α; we wish to prove that the map $\psi : \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(S_\alpha, X') \to F(S_\alpha)$ is bijective. It is clear that ψ is surjective (since the composite map $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(S_\alpha, X_0) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(S_\alpha, X') \to F(S_\alpha)$ is surjective by construction). To prove that ψ is injective, it will suffice to show that the kernel of ψ is trivial (since ψ is a group homomorphism, using the cogroup structure on S_α given by (i)). Fix an element $\gamma \in \ker(\psi)$, represented by a map $f : S_\alpha \to X'$. Assumption (ii) guarantees that S_α is compact, so that f factors through some map $\overline{f} : S_\alpha \to X_n$, which determines an element of the kernel K of the map $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(S_\alpha, X_n) \to F(S_\alpha)$. It follows from our construction that the composite map $S_\alpha \to X_n \to X_{n+1}$ factors through the counit of S_α, so that f is the unit element of $\ker(\psi)$. This completes the proof of $(*')$.

Assertion (b) guarantees that $F(\emptyset)$ consists of a single element. Applying $(*')$ in the case $X = \emptyset$, we obtain an element $\eta' \in F(X')$ which induces isomorphisms $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(S_\alpha, X') \to F(S_\alpha)$ for each index α. We will complete the proof by showing that η' exhibits F as the functor on $\mathrm{h}\mathcal{C}$ represented by the object X'. In other words, we claim that for every object $Y \in \mathcal{C}$, the element η' induces a bijection $\theta : \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, X') \to F(Y)$.

We begin by showing that θ is surjective. Fix an element $\eta'' \in F(Y)$. Assumption (b) guarantees that (η', η'') determines an element of $F(Y \coprod X')$. Applying assertion $(*')$ to this element, we deduce the existence of a map $X' \coprod Y \to Z$ and an element $\overline{\eta} \in F(Z)$ lifting the pair (η', η'') which induces isomorphisms $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(S_\alpha, Z) \to F(S_\alpha)$ for each index α. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{h}\mathcal{C}}(S_\alpha, X') & \longrightarrow & \mathrm{Map}_{\mathrm{h}\mathcal{C}}(S_\alpha, Z) \\ & \searrow \quad \swarrow & \\ & F(S_\alpha) & \end{array}$$

for each index α, in which the vertical maps are bijective. It follows that the horizontal map is also bijective. Invoking $(*)$, we deduce that $X' \to Z$ is an equivalence. The composite map $Y \to Z \simeq X'$ is then a preimage of η'' in the set $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, X')$.

We now complete the proof by showing that θ is injective. Fix a pair of maps $f, g : Y \to X'$ which determine the same element of $F(Y)$. Form a pushout diagram

$$\begin{array}{ccc} Y \coprod Y & \xrightarrow{(f,g)} & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z. \end{array}$$

Using assumption (b), we deduce that $\eta' \in F(X')$ can be lifted to an element $\overline{\eta} \in F(Z)$. Applying $(*')$, we deduce the existence of a map $Z \to Z'$ and an element $\overline{\eta}' \in F(Z')$ lifting $\overline{\eta}$ and inducing bijections $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(S_\alpha, Z') \to F(S_\alpha)$ for each index α. We have commutative diagrams

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{h}\mathcal{C}}(S_\alpha, X') & \longrightarrow & \mathrm{Map}_{\mathrm{h}\mathcal{C}}(S_\alpha, Z') \\ & \searrow \quad \swarrow & \\ & F(S_\alpha) & \end{array}$$

in which the vertical maps are bijective. It follows that the horizontal maps are also bijective, so that $(*)$ guarantees that the map $h : X' \to Z'$ is an equivalence in $\mathcal{C}$. Since the compositions $h \circ f$ and $h \circ g$ are homotopic, we deduce that f and g are homotopic and therefore represent the same element of $\mathrm{Hom}_{h\mathcal{C}}(Y, X')$, as desired. $\qquad\square$

1.4.2 Spectrum Objects

In this section, we will describe a method for constructing stable ∞-categories: for any ∞-category $\mathcal{C}$ which admits finite limits, one can consider an ∞-category $\mathrm{Sp}(\mathcal{C})$ of *spectrum objects* of $\mathcal{C}$. In the special case where $\mathcal{C}$ is the ∞-category of spaces, this construction will recover classical stable homotopy theory; we will discuss this example in more detail in §1.4.3.

If the ∞-category $\mathcal{C}$ is pointed, then the ∞-category $\mathrm{Sp}(\mathcal{C})$ of spectrum objects of $\mathcal{C}$ can be described as the homotopy inverse limit of the tower of ∞-categories

$$\cdots \to \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C}.$$

As we saw in §1.4.1, the objects of this homotopy inverse limit are closely related to *cohomology theories* defined on the ∞-category $\mathcal{C}$. In this section, it will be more convenient to adopt a dual perspective: we will identify spectrum objects of $\mathcal{C}$ with *homology theories* defined on pointed spaces, taking values in $\mathcal{C}$.

Before giving any formal definitions, let us consider the most classical example of a homology theory: the singular homology of topological spaces. This theory associates to every topological space X the singular homology groups $\mathrm{H}_n(X; \mathbf{Z})$. These groups are covariantly functorial in X, and have the following additional property: for every pair of open sets $U, V \subseteq X$ which cover X, we have a long exact Mayer-Vietoris sequence

$$\cdots \mathrm{H}_1(U; \mathbf{Z}) \oplus \mathrm{H}_1(V; \mathbf{Z}) \to \mathrm{H}_1(X; \mathbf{Z}) \to \mathrm{H}_0(U \cap V; \mathbf{Z}) \to \mathrm{H}_0(U; \mathbf{Z}) \oplus \mathrm{H}_0(V; \mathbf{Z}) \to \mathrm{H}_0(X; \mathbf{Z}) \to 0.$$

Note that the singular homology $\mathrm{H}_n(X; \mathbf{Z})$ can be defined as the homology of the (normalized or unnormalized) chain complex associated to the simplicial abelian group $\mathbf{Z}\,\mathrm{Sing}(X)_\bullet$ freely generated by the simplicial set $\mathrm{Sing}(X)_\bullet$. As such, they can be viewed as the homotopy groups of $\mathbf{Z}\,\mathrm{Sing}(X)_\bullet$, regarded as a Kan complex. The above long exact sequence results from the observation that that diagram

$$
\begin{array}{ccc}
\mathbf{Z}\,\mathrm{Sing}(U \cap V)_\bullet & \longrightarrow & \mathbf{Z}\,\mathrm{Sing}(U)_\bullet \\
\downarrow & & \downarrow \\
\mathbf{Z}\,\mathrm{Sing}(V)_\bullet & \longrightarrow & \mathbf{Z}\,\mathrm{Sing}(X)_\bullet
\end{array}
$$

is a homotopy pullback square of Kan complexes. This is a consequence of the following more general fact: the construction $X \mapsto \mathbf{Z}\,\mathrm{Sing}(X)_\bullet$ carries homotopy pushout diagrams (of topological spaces) to homotopy pullback diagrams (of Kan complexes). We now proceed to axiomatize this phenomenon:

Definition 1.4.2.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories.

(*i*) If $\mathcal{C}$ admits pushouts, then we will say that F is *excisive* if F carries pushout squares in $\mathcal{C}$ to pullback squares in $\mathcal{D}$.

(*ii*) If $\mathcal{C}$ admits a final object $*$, we will say that F is *reduced* if $F(*)$ is a final object of $\mathcal{D}$.

If $\mathcal{C}$ admits pushouts, we let $\mathrm{Exc}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the excisive functors. If $\mathcal{C}$ admits a final object, we let $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the reduced functors. If $\mathcal{C}$ admits pushouts and a final object, we let $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$ denote the intersection $\mathrm{Exc}(\mathcal{C}, \mathcal{D}) \cap \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$.

Remark 1.4.2.2. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories, and suppose that $\mathcal{C}$ is a pointed ∞-category which admits finite colimits. If $\mathcal{C}$ is stable, then F is reduced and excisive if and only if it is left exact (Proposition 1.1.3.4). If instead $\mathcal{D}$ is stable, then F is reduced and excisive if and only if it is right exact. In particular, if both $\mathcal{C}$ and $\mathcal{D}$ are stable, then F is reduced and excisive if and only if it is exact (Proposition 1.1.4.1).

Remark 1.4.2.3. Let K be a simplicial set, let $\mathcal{C}$ be an ∞-category which admits pushouts, and let $\mathcal{D}$ be an ∞-category which admits K-indexed limits. Then $\text{Fun}(\mathcal{C}, \mathcal{D})$ admits K-indexed limits. Moreover, the collection of excisive functors from $\mathcal{C}$ to $\mathcal{D}$ is closed under K-indexed limits. Similarly, if $\mathcal{C}$ has a final object $*$, the the full subcategory $\text{Fun}_*(\mathcal{C}, \mathcal{D}) \subseteq \text{Fun}(\mathcal{C}, \mathcal{D})$ is closed under K-indexed limits.

Remark 1.4.2.4. Suppose that $\mathcal{C}$ is a small pointed ∞-category which admits finite colimits, and let $\mathcal{D}$ be a presentable ∞-category. Then $\text{Fun}_*(\mathcal{C}, \mathcal{D})$, $\text{Exc}(\mathcal{C}, \mathcal{D})$, and $\text{Exc}_*(\mathcal{C}, \mathcal{D})$ are accessible localizations of the ∞-category $\text{Fun}(\mathcal{C}, \mathcal{D})$ (Lemmas T.5.5.4.18 and T.5.5.4.19). In particular, each is a presentable ∞-category.

Notation 1.4.2.5. Let $\mathcal{S}_*$ denote the ∞-category of pointed objects of $\mathcal{S}$. That is, $\mathcal{S}_*$ denotes the full subcategory of $\text{Fun}(\Delta^1, \mathcal{S})$ spanned by those morphisms $f : X \to Y$ for which X is a final object of $\mathcal{S}$ (Definition T.7.2.2.1). Let $\mathcal{S}^{\text{fin}}$ denote the smallest full subcategory of $\mathcal{S}$ which contains the final object $*$ and is stable under finite colimits. We will refer to $\mathcal{S}^{\text{fin}}$ as the ∞-*category of finite spaces*. We let $\mathcal{S}_*^{\text{fin}} \subseteq \mathcal{S}_*$ denote the ∞-category of pointed objects of $\mathcal{S}^{\text{fin}}$. We observe that the suspension functor $\Sigma : \mathcal{S}_* \to \mathcal{S}_*$ carries $\mathcal{S}_*^{\text{fin}}$ to itself. For each $n \geq 0$, we let $S^n \in \mathcal{S}_*$ denote a representative for the (pointed) n-sphere.

Remark 1.4.2.6. It follows from Remark T.5.3.5.9 and Proposition T.4.3.2.15 that $\mathcal{S}^{\text{fin}}$ is characterized by the following universal property: for every ∞-category $\mathcal{D}$ which admits finite colimits, evaluation at $*$ induces an equivalence of ∞-categories $\text{Fun}^{\text{Rex}}(\mathcal{S}^{\text{fin}}, \mathcal{D}) \to \mathcal{D}$. Here $\text{Fun}^{\text{Rex}}(\mathcal{S}^{\text{fin}}, \mathcal{D})$ denotes the full subcategory of $\text{Fun}(\mathcal{S}^{\text{fin}}, \mathcal{D})$ spanned by the right exact functors. More informally: the ∞-category $\mathcal{S}^{\text{fin}}$ is freely generated by a single object (the space $*$) under finite colimits.

Warning 1.4.2.7. The ∞-category $\mathcal{S}^{\text{fin}}$ does not coincide with the ∞-category of compact objects $\mathcal{S}^c \subseteq \mathcal{S}$. Instead, there is an inclusion $\mathcal{S}^{\text{fin}} \subseteq \mathcal{S}^c$, which realizes $\mathcal{S}^c$ as an idempotent completion of $\mathcal{S}^{\text{fin}}$. An object of $X \in \mathcal{S}^c$ belongs to $\mathcal{S}^{\text{fin}}$ if and only if its *Wall finiteness obstruction* vanishes. We refer the reader to [158] for further details.

Definition 1.4.2.8. Let $\mathcal{C}$ be an ∞-category which admits finite limits. A *spectrum object* of $\mathcal{C}$ is a reduced, excisive functor $X : \mathcal{S}_*^{\text{fin}} \to \mathcal{C}$. Let $\text{Sp}(\mathcal{C}) = \text{Exc}_*(\mathcal{S}_*^{\text{fin}}, \mathcal{C})$ denote the full subcategory of $\text{Fun}(\mathcal{S}_*^{\text{fin}}, \mathcal{C})$ spanned by the spectrum objects of $\mathcal{C}$.

Remark 1.4.2.9. Let $\mathcal{C}$ be an ∞-category which admits finite limits, and K an arbitrary simplicial set. Then we have a canonical isomorphism $\text{Sp}(\text{Fun}(K, \mathcal{C})) \simeq \text{Fun}(K, \text{Sp}(\mathcal{C}))$.

We next show that if $\mathcal{C}$ is an ∞-category which admits finite limits, then the ∞-category $\text{Sp}(\mathcal{C})$ is stable. We begin with the following observation:

Lemma 1.4.2.10. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits, and let $\mathcal{D}$ be an ∞-category which admits finite limits. Then the ∞-category $\text{Exc}_*(\mathcal{C}, \mathcal{D})$ is pointed and admits finite limits.*

Proof. The existence of finite limits in $\text{Exc}_*(\mathcal{C}, \mathcal{D})$ follows from Remark 1.4.2.3. Let $*$ denote a final object of $\mathcal{C}$ and $*'$ a final object of $\mathcal{D}$. Let $X : \mathcal{C} \to \mathcal{D}$ be the constant functor taking the value $*'$. Then X is a final object of $\text{Fun}(\mathcal{C}, \mathcal{D})$, and in particular a final object of $\text{Exc}_*(\mathcal{C}, \mathcal{D})$. We claim that X is also an initial object of $\text{Fun}_*(\mathcal{C}, \mathcal{D})$ (and in particular an initial object of $\text{Exc}_*(\mathcal{C}, \mathcal{D})$). To prove this, choose any other object $Y \in \text{Fun}_*(\mathcal{C}, \mathcal{D})$; we wish to show that the mapping space $\text{Map}_{\text{Fun}(\mathcal{C}, \mathcal{D})}(X, Y)$ is contractible. Since the functor Y is reduced, the mapping space $\text{Map}_{\text{Fun}(\mathcal{C}, \mathcal{D})}(X(*), Y(*))$ is contractible. It will therefore suffice to show that the restriction map

$$\text{Map}_{\text{Fun}(\mathcal{C}, \mathcal{D})}(X, Y) \to \text{Map}_{\text{Fun}(\mathcal{C}, \mathcal{D})}(X(*), Y(*))$$

is a homotopy equivalence. This follows from the observation that X is a left Kan extension of its restriction along the inclusion $\{*\} \hookrightarrow \mathcal{C}$. $\square$

We will deduce the stability of $\mathrm{Sp}(\mathcal{C})$ using the following general criterion:

Proposition 1.4.2.11. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite limits and colimits. Then:*

(1) *If the suspension functor $\Sigma_{\mathcal{C}}$ is fully faithful, then every pushout square in $\mathcal{C}$ is a pullback square.*

(2) *If the loop functor $\Omega_{\mathcal{C}}$ is fully faithful, then every pullback square in $\mathcal{C}$ is a pushout square.*

(3) *If the loop functor $\Omega_{\mathcal{C}}$ is an equivalence of ∞-categories, then $\mathcal{C}$ is stable.*

We will deduce Proposition 1.4.2.11 from a more general assertion regarding functors between pointed ∞-categories. The formulation of this result will require a bit of terminology.

Notation 1.4.2.12. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories, and assume that $\mathcal{D}$ admits finite limits. For every commutative square τ:

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

in $\mathcal{C}$, we obtain a commutative square $F(\tau)$:

$$\begin{array}{ccc} F(W) & \longrightarrow & F(X) \\ \downarrow & & \downarrow \\ F(Y) & \longrightarrow & F(Z) \end{array}$$

in $\mathcal{D}$. This diagram determines a map $\eta_{\tau} : F(W) \to F(X) \times_{F(Z)} F(Y)$ in the ∞-category $\mathcal{D}$, which is well-defined up to homotopy. If we suppose further that X and Y are zero objects of $\mathcal{C}$, that $F(X)$ and $F(Y)$ are zero objects of $\mathcal{D}$, and that τ is a pushout diagram, then we obtain a map $F(W) \to \Omega_{\mathcal{D}} F(\Sigma_{\mathcal{C}} W)$, which we will denote simply by η_W.

Proposition 1.4.2.13. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits, $\mathcal{D}$ a pointed ∞-category which admits finite limits, and let $F : \mathcal{C} \to \mathcal{D}$ a reduced functor. The following conditions are equivalent:*

(1) *The functor F is excisive (Definition 1.4.2.1): that is, F carries pushout squares in $\mathcal{C}$ to pullback squares in $\mathcal{D}$.*

(2) *For every object $X \in \mathcal{C}$, the canonical map $\eta_X : F(X) \to \Omega_{\mathcal{D}} F(\Sigma_{\mathcal{C}} X)$ is an equivalence in $\mathcal{D}$ (see Notation 1.4.2.12).*

Assuming Proposition 1.4.2.13 for the moment, it is easy to verify Proposition 1.4.2.11:

Proof of Proposition 1.4.2.11. Assertion (1) follows by applying Proposition 1.4.2.13 to the identity functor $\mathrm{id}_{\mathcal{C}}$, and assertion (2) follows from (1) by passing to the opposite ∞-category. Assertion (3) is an immediate consequence of (1) and (2) (note that if $\Omega_{\mathcal{C}}$ is an equivalence of ∞-categories, then its left adjoint $\Sigma_{\mathcal{C}}$ is also an equivalence of ∞-categories). $\square$

Restricting our attention to stable ∞-categories, Proposition 1.4.2.13 yields the following:

Corollary 1.4.2.14. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between stable ∞-categories. Then F is exact if and only if the following conditions are satisfied:*

(1) *The functor F carries zero objects of $\mathcal{C}$ to zero objects of $\mathcal{D}$.*

(2) *For every object $X \in \mathcal{C}$, the canonical map $\Sigma_{\mathcal{D}} F(X) \to F(\Sigma_{\mathcal{C}} X)$ is an equivalence in $\mathcal{D}$.*

The proof of Proposition 1.4.2.13 makes use of the following lemma:

Lemma 1.4.2.15. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits, $\mathcal{D}$ a pointed ∞-category which admits finite limits, and $F : \mathcal{C} \to \mathcal{D}$ a reduced functor. Suppose given a pushout diagram τ:*

$$
\begin{array}{ccc}
W & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Z
\end{array}
$$

in $\mathcal{C}$. Then there exists a map $\theta_\tau : F(X) \times_{F(Z)} F(Y) \to \Omega_{\mathcal{D}} F(\Sigma_{\mathcal{C}} W)$ with the following properties:

(1) *The composition $\theta_\tau \circ \eta_\tau$ is homotopic to η_W. Here η_τ and η_W are defined as in Notation 1.4.2.12.*

(2) *Let $\Sigma_{\mathcal{C}}(\tau)$ denote the induced diagram*

$$
\begin{array}{ccc}
\Sigma_{\mathcal{C}} W & \longrightarrow & \Sigma_{\mathcal{C}} X \\
\downarrow & & \downarrow \\
\Sigma_{\mathcal{C}} Y & \longrightarrow & \Sigma_{\mathcal{C}} Z.
\end{array}
$$

Then there is a pullback square

$$
\begin{array}{ccc}
\eta_{\Sigma_{\mathcal{C}}(\tau)} \circ \theta_\tau & \longrightarrow & \eta_X \\
\downarrow & & \downarrow \\
\eta_Y & \longrightarrow & \eta_Z
\end{array}
$$

in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{D})$ of morphisms in $\mathcal{D}$.

Proof. In the ∞-category $\mathcal{C}$, we have the following commutative diagram (in which every square is a pushout):

$$
\begin{array}{ccccccc}
W & \longrightarrow & X & \longrightarrow & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
Y & \longrightarrow & X \amalg_W Y & \longrightarrow & 0 \amalg_W Y & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & X \amalg_W 0 & \longrightarrow & \Sigma_{\mathcal{C}} W & \longrightarrow & \Sigma_{\mathcal{C}} Y \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & \longrightarrow & \Sigma_{\mathcal{C}} X & \longrightarrow & \Sigma_{\mathcal{C}}(X \amalg_W Y).
\end{array}
$$

Applying the functor F, and replacing the upper left square by a pullback, we obtain a new diagram

$$
\begin{array}{ccccccc}
F(X) \times_{F(Z)} F(Y) & \longrightarrow & F(X) & \longrightarrow & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
F(Y) & \longrightarrow & F(Z) & \longrightarrow & F(0 \amalg_W Y) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F(X \amalg_W 0) & \longrightarrow & F(\Sigma_{\mathcal{C}} W) & \longrightarrow & F(\Sigma_{\mathcal{C}} Y) \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & \longrightarrow & F(\Sigma_{\mathcal{C}} X) & \longrightarrow & F(\Sigma_{\mathcal{C}} Z).
\end{array}
$$

Restricting attention to the large square in the upper left, we obtain the desired map $\theta_\tau : F(X) \times_{F(Z)} F(Y) \to \Omega_{\mathcal{D}} F(\Sigma_{\mathcal{C}} W)$. It is easy to verify that θ_τ has the desired properties. $\qquad\square$

Proof of Proposition 1.4.2.13. The implication $(1) \Rightarrow (2)$ is obvious. Conversely, suppose that (2) is satisfied. We must show that for every pushout square τ :

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
Z & \longrightarrow & Y \amalg_X Z
\end{array}
$$

in the ∞-category $\mathcal{C}$, the induced map η_τ is an equivalence in $\mathcal{D}$. Let θ_τ be as in the statement of Lemma 1.4.2.15. Then $\theta_\tau \circ \eta_\tau$ is homotopic to η_X, and is therefore an equivalence (by virtue of assumption (2)). It will therefore suffice to show that θ_τ is an equivalence. The preceding argument shows that θ_τ has a right homotopy inverse. To show that θ_τ admits a left homotopy inverse, it will suffice to show that $\eta_{\Sigma_{\mathcal{C}}\tau} \circ \theta_\tau$ is an equivalence. This follows from the second assertion of Lemma 1.4.2.15, since the maps η_Y, η_Z, and $\eta_{Y \amalg_X Z}$ are equivalences (by assumption (2)). $\qquad\square$

Proposition 1.4.2.16. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits and $\mathcal{D}$ an ∞-category which admits finite limits. Then the ∞-category $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$ is stable.*

Proof. We may assume without loss of generality that $\mathcal{C}$ is small. Suppose first that $\mathcal{D}$ is presentable. Let $S : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be given by $F \mapsto F \circ \Sigma_{\mathcal{C}}$, where $\Sigma : \mathcal{C} \to \mathcal{C}$ denotes the suspension functor. Then S carries $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$ to itself. Using the definition of excisive functors, we conclude that S is a homotopy inverse to the functor $\Omega_{\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})}$. Since $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$ is pointed (Lemma 1.4.2.10) and admits finite limits and colimits (Remark 1.4.2.4), we conclude from Proposition 1.4.2.11 that it is stable.

To handle the general case, we may assume without loss of generality that $\mathcal{D}$ is small. Let $\mathcal{D}' = \mathcal{P}(\mathcal{D})$ be the ∞-category of presheaves on $\mathcal{D}$ and let $j : \mathcal{D} \to \mathcal{D}'$ be the Yoneda embedding. Since j is left exact, it induces a fully faithful embedding $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D}) \to \mathrm{Exc}_*(\mathcal{C}, \mathcal{D}')$. Then $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$ is equivalent to a full subcategory of the stable ∞-category $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D}')$, which is closed under finite limits and suspensions. It follows from Lemma 1.1.3.3 that $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$ is stable. $\qquad\square$

Corollary 1.4.2.17. *Let $\mathcal{C}$ be an ∞-category which admits finite limits. Then the ∞-category $\mathrm{Sp}(\mathcal{C})$ of spectrum objects of $\mathcal{C}$ is stable.*

Remark 1.4.2.18. Let $\mathcal{C}$ be an ∞-category which admits finite limits, and let $\mathcal{C}_*$ denote the ∞-category of pointed objects of $\mathcal{C}$. Then the forgetful functor $\mathcal{C}_* \to \mathcal{C}$ induces an equivalence of ∞-categories $\mathrm{Sp}(\mathcal{C}_*) \to \mathrm{Sp}(\mathcal{C})$. To see this, we observe that there is a canonical isomorphism of simplicial sets $\mathrm{Sp}(\mathcal{C}_*) \simeq \mathrm{Sp}(\mathcal{C})_*$. We are therefore reduced to proving that the forgetful functor $\mathrm{Sp}(\mathcal{C})_* \to \mathrm{Sp}(\mathcal{C})$ is an equivalence of ∞-categories, which follows from the fact that $\mathrm{Sp}(\mathcal{C})$ is pointed (Corollary 1.4.2.17).

Our next goal is to characterize the ∞-category $\mathrm{Sp}(\mathcal{C})$ by means of a universal property.

Lemma 1.4.2.19. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and a final object, let $f : \mathcal{C} \to \mathcal{C}_*$ be a left adjoint to the forgetful functor, and let $\mathcal{D}$ be a stable ∞-category. Let $\mathrm{Exc}'(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Exc}(\mathcal{C}, \mathcal{D})$ spanned by those functors which carry the initial object of $\mathcal{C}$ to a final object of $\mathcal{D}$. Then composition with f induces an equivalence of ∞-categories $\phi : \mathrm{Exc}_*(\mathcal{C}_*, \mathcal{D}) \to \mathrm{Exc}'(\mathcal{C}, \mathcal{D})$.*

Proof. Consider the composite functor

$$\theta : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \times \mathcal{C}_* \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \times \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{D}) \xrightarrow{\mathrm{cofib}} \mathcal{D} .$$

We can identify θ with a map $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_*, \mathcal{D})$. Since the collection of pullback squares in $\mathcal{D}$ is a stable subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{D})$, we conclude θ restricts to a map $\psi : \mathrm{Exc}'(\mathcal{C}, \mathcal{D}) \to \mathrm{Exc}_*(\mathcal{C}_*, \mathcal{D})$. It is not difficult to verify that ψ is a homotopy inverse to ϕ. $\qquad\square$

Notation 1.4.2.20. Let S^0 denote the 0-sphere, regarded as an object of the ∞-category $\mathbb{S}^{\mathrm{fin}}_*$ of pointed finite spaces. If $\mathcal{C}$ is an ∞-category which admits finite limits, we let $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ denote the functor given by evaluation at $S^0 \in \mathbb{S}^{\mathrm{fin}}_*$. More generally, in $n \in \mathbf{Z}$ is an integer, we let $\Omega^{\infty-n} : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ denote the functor given by composing $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ with the shift functor $X \mapsto X[n]$ on $\mathrm{Sp}(\mathcal{C})$ (if $n \geq 0$, then the functor $\Omega^{\infty-n} : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ is given by evaluation on the n-sphere S^n).

Proposition 1.4.2.21. *Let $\mathcal{D}$ be an ∞-category which admits finite limits. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{D}$ is stable.*

(2) *The functor $\Omega^\infty : \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$ is an equivalence of ∞-categories.*

Proof. The implication $(2) \Rightarrow (1)$ follows from Corollary 1.4.2.17. Conversely, suppose that (1) is satisfied, and let $f : \mathbb{S}^{\mathrm{fin}} \to \mathbb{S}^{\mathrm{fin}}_*$ be a left adjoint to the forgetful functor (obtained by adding a disjoint base point). Using Lemma 1.4.2.19, we are reduced to proving that evaluation at the object $* \in \mathbb{S}^{\mathrm{fin}}$ induces an equivalence of ∞-categories $\mathrm{Exc}'(\mathbb{S}^{\mathrm{fin}}, \mathcal{D})$. Note that a functor $X : \mathbb{S}^{\mathrm{fin}} \to \mathcal{D}$ belongs to $\mathrm{Exc}'(\mathbb{S}^{\mathrm{fin}}, \mathcal{D})$ if and only if it is right exact. The desired result now follows from Remark 1.4.2.6. $\square$

Proposition 1.4.2.22. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits and $\mathcal{D}$ an ∞-category which admits finite limits. Then composition with the functor $\Omega^\infty : \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$ induces an equivalence of ∞-categories*

$$\theta : \mathrm{Exc}_*(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \to \mathrm{Exc}_*(\mathcal{C}, \mathcal{D}).$$

Proof. Under the canonical isomorphism $\mathrm{Exc}_*(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \simeq \mathrm{Sp}(\mathrm{Exc}_*(\mathcal{C}, \mathcal{D}))$, the functor θ corresponds to evaluation map $\Omega^\infty : \mathrm{Sp}(\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})) \to \mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$. Since $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$ is stable by Proposition 1.4.2.16, Proposition 1.4.2.21 implies that θ is an equivalence of ∞-categories. $\square$

Corollary 1.4.2.23. *Let $\mathcal{C}$ be a stable ∞-category, let $\mathcal{D}$ an ∞-category which admits finite limits, and let*

$$\mathrm{Fun}'(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \qquad \mathrm{Fun}'(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \subseteq \mathrm{Fun}(\mathcal{C}, \mathrm{Sp}(\mathcal{D}))$$

denote the full subcategories spanned by the left-exact functors. Then composition with the functor $\Omega^\infty : \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$ induces an equivalence of ∞-categories

$$\mathrm{Fun}'(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \to \mathrm{Fun}'(\mathcal{C}, \mathcal{D}).$$

Proposition 1.4.2.24. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite limits. Then the functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ can be lifted to an equivalence of $\mathrm{Sp}(\mathcal{C})$ with the homotopy limit of the tower of ∞-categories*

$$\cdots \to \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C}.$$

Remark 1.4.2.25. Let $\mathcal{C}$ be an ∞-category which admits finite limits. Combining Remark 1.4.2.18 with Proposition 1.4.2.24, we can identify the ∞-category $\mathrm{Sp}(\mathcal{C})$ of spectrum objects of $\mathcal{C}$ with the homotopy limit of the tower

$$\cdots \to \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*.$$

Lemma 1.4.2.26. *Let $\mathcal{C}$ be a small pointed ∞-category, and let $\mathcal{P}_*(\mathcal{C})$ denote the full subcategory of $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathbb{S})$ spanned by those functors which carry zero objects of $\mathcal{C}$ to final objects of $\mathbb{S}$. Then:*

(1) *Let S denote the set consisting of a single morphism from an initial object of $\mathcal{P}(\mathcal{C})$ to a final object of $\mathcal{P}(\mathcal{C})$. Then $\mathcal{P}_*(\mathcal{C}) = S^{-1}\mathcal{P}(\mathcal{C})$.*

(2) *The ∞-category $\mathcal{P}_*(\mathcal{C})$ is an accessible localization of $\mathcal{P}(\mathcal{C})$. In particular, $\mathcal{P}_*(\mathcal{C})$ is presentable.*

(3) *The Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ factors through $\mathcal{P}_*(\mathcal{C})$, and the induced embedding $j : \mathcal{C} \to \mathcal{P}_*(\mathcal{C})$ preserves zero objects.*

(4) *Let $\mathcal{D}$ be an ∞-category which admits small colimits, and let $\operatorname{Fun}^L(\mathcal{P}_*(\mathcal{C}), \mathcal{D})$ denote the full subcategory of $\operatorname{Fun}(\mathcal{P}_*(\mathcal{C}), \mathcal{D})$ spanned by those functors which preserve small colimits. Then composition with j induces an equivalence of ∞-categories $\operatorname{Fun}^L(\mathcal{P}_*(\mathcal{C}), \mathcal{D}) \to \operatorname{Fun}_0(\mathcal{C}, \mathcal{D})$, where $\operatorname{Fun}_0(\mathcal{C}, \mathcal{D})$ denotes the full subcategory of $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which carry zero objects of $\mathcal{C}$ to initial objects of $\mathcal{D}$.*

(5) *The ∞-category $\mathcal{P}_*(\mathcal{C})$ is pointed.*

(6) *The full subcategory $\mathcal{P}_*(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$ is closed under small limits and under small colimits parametrized by weakly contractible simplicial sets. In particular, $\mathcal{P}_*(\mathcal{C})$ is stable under small filtered colimits in $\mathcal{P}(\mathcal{C})$.*

(7) *The functor $j : \mathcal{C} \to \mathcal{P}_*(\mathcal{C})$ preserves all small limits which exist in $\mathcal{C}$.*

(8) *The ∞-category $\mathcal{P}_*(\mathcal{C})$ is compactly generated.*

Proof. For every object $X \in \mathcal{S}$, let $F_X \in \mathcal{P}(\mathcal{C})$ denote the constant functor taking the value X. Then F_X is a left Kan extension of $F_X|\{0\}$, where 0 denotes a zero object of $\mathcal{C}$. It follows that for any object $G \in \mathcal{P}(\mathcal{C})$, evaluation at 0 induces a homotopy equivalence

$$\operatorname{Map}_{\mathcal{P}(\mathcal{C})}(F_X, G) \to \operatorname{Map}_{\mathcal{S}}(F_X(0), G(0)) = \operatorname{Map}_{\mathcal{S}}(X, G(0)).$$

We observe that the inclusion $\emptyset \subseteq \Delta^0$ induces a map $F_\emptyset \to F_{\Delta^0}$ from an initial object of $\mathcal{P}(\mathcal{C})$ to a final object of $\mathcal{P}(\mathcal{C})$. It follows that an object G of $\mathcal{P}(\mathcal{C})$ is S-local if and only if the induced map

$$G(0) \simeq \operatorname{Map}_{\mathcal{S}}(\Delta^0, G(0)) \to \operatorname{Map}_{\mathcal{S}}(\emptyset, G(0)) \simeq \Delta^0$$

is a homotopy equivalence: that is, if and only if $G \in \mathcal{P}_*(\mathcal{C})$. This proves (1).

Assertion (2) follows immediately from (1), and assertion (3) is obvious. Assertion (4) follows from (1), Theorem T.5.1.5.6, and Proposition T.5.5.4.20. To prove (5), we observe that F_{Δ^0} is a final object of $\mathcal{P}(\mathcal{C})$, and therefore a final object of $\mathcal{P}_*(\mathcal{C})$. It therefore suffices to show that F_{Δ^0} is an initial object of $\mathcal{P}_*(\mathcal{C})$. This follows from the observation that for every $G \in \mathcal{P}_*(\mathcal{C})$, we have homotopy equivalences $\operatorname{Map}_{\mathcal{P}(\mathcal{C})}(F_{\Delta^0}, G) \simeq \operatorname{Map}_{\mathcal{S}}(\Delta^0, G(0)) \simeq G(0)$ so that the mapping space $\operatorname{Map}_{\mathcal{P}(\mathcal{C})}(F_{\Delta^0}, G)$ is contractible if $G \in \mathcal{P}_*(\mathcal{C})$. Assertion (6) is obvious, and (7) follows from (6) together with Proposition T.5.1.3.2. We deduce (8) from (6) together with Corollary T.5.5.7.3. $\square$

Proof of Proposition 1.4.2.24. Let $\overline{\mathcal{C}}$ denote a homotopy limit of the tower

$$\cdots \to \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C}.$$

We begin by showing that $\overline{\mathcal{C}}$ is a stable ∞-category.

Assume first that $\mathcal{C}$ is presentable. Applying Theorem T.5.5.3.18, we deduce that $\overline{\mathcal{C}}$ is presentable. In particular, $\overline{\mathcal{C}}$ admits small limits and colimits. By construction, $\overline{\mathcal{C}}$ is pointed and the loop functor $\Omega_{\overline{\mathcal{C}}}$ is an equivalence of ∞-categories. Applying Proposition 1.4.2.11, we deduce that $\overline{\mathcal{C}}$ is stable.

We now prove that $\overline{\mathcal{C}}$ is stable in general. Without loss of generality, we may assume that $\mathcal{C}$ is small. Let $j : \mathcal{C} \to \mathcal{P}_*(\mathcal{C})$ be as in Lemma 1.4.2.26, and let $\overline{\mathcal{P}_*(\mathcal{C})}$ denote a homotopy limit of the tower

$$\cdots \to \mathcal{P}_*(\mathcal{C}) \xrightarrow{\Omega} \mathcal{P}_*(\mathcal{C}) \xrightarrow{\Omega} \mathcal{P}_*(\mathcal{C}).$$

The functor j is fully faithful and left exact, and therefore induces a fully faithful left exact embedding $\overline{\mathcal{C}} \to \overline{\mathcal{P}_*(\mathcal{C})}$. Then $\overline{\mathcal{C}}$ is closed under finite limits and shifts in the stable ∞-category $\overline{\mathcal{P}_*(\mathcal{C})}$, and is therefore stable by Lemma 1.1.3.3.

Let $G : \overline{\mathcal{C}} \to \mathcal{C}$ be the the canonical map. Then G is left exact. Applying Corollary 1.4.2.23, we deduce that G factors as a composition

$$\overline{\mathcal{C}} \xrightarrow{G'} \operatorname{Sp}(\mathcal{C}) \xrightarrow{\Omega^\infty} \mathcal{C}.$$

We will complete the proof by showing that G' is an equivalence of ∞-categories. To prove this, it will suffice to show that for every stable ∞-category $\mathcal{D}$, composition with G' induces an equivalence

$$\mathrm{Fun}'(\mathcal{D}, \overline{\mathcal{C}}) \to \mathrm{Fun}'(\mathcal{D}, \mathrm{Sp}(\mathcal{C})),$$

where $\mathrm{Fun}'(\mathcal{D}, \mathcal{X})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{D}, \mathcal{X})$ spanned by the left exact functors. Using Corollary 1.4.2.23, we are reduced to proving that composition with G induces an equivalence $\mathrm{Fun}'(\mathcal{D}, \overline{\mathcal{C}}) \to \mathrm{Fun}'(\mathcal{D}, \mathcal{C})$. For this, it suffices to show that the loop functor $\Omega_{\mathcal{C}}$ induces an equivalence $\mathrm{Fun}'(\mathcal{D}, \mathcal{C}) \to \mathrm{Fun}'(\mathcal{D}, \mathcal{C})$. A homotopy inverse to this functor is given by precomposition with the suspension functor $\Sigma_{\mathcal{D}}$ (as in the proof of Proposition 1.4.2.16). $\qquad\square$

Using Proposition 1.4.2.24, we can obtain a slight improvement on Proposition 1.4.2.11:

Corollary 1.4.2.27. *Let $\mathcal{C}$ be a pointed ∞-category. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is stable.*

(2) *The ∞-category $\mathcal{C}$ admits finite colimits and the suspension functor $\Sigma_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is an equivalence.*

(3) *The ∞-category $\mathcal{C}$ admits finite limits and the loop functor $\Omega_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is an equivalence.*

Proof. We will show that $(1) \Leftrightarrow (3)$; the dual argument will prove that $(1) \Leftrightarrow (2)$. The implication $(1) \Rightarrow (3)$ is clear. Conversely, suppose that $\mathcal{C}$ admits finite limits and that $\Omega_{\mathcal{C}}$ is an equivalence. Proposition 1.4.2.24 implies that $\mathrm{Sp}(\mathcal{C})$ can be identified with the homotopy inverse limit of the tower

$$\ldots \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C}\,.$$

If (3) is satisfied, then the loop functor $\Omega_{\mathcal{C}}$ is an equivalence, so this tower is essentially constant. It follows that $\Omega^{\infty} : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ is an equivalence of ∞-categories. Since $\mathrm{Sp}(\mathcal{C})$ is stable (Corollary 1.4.2.17), so is $\mathcal{C}$. $\qquad\square$

1.4.3   The ∞-Category of Spectra

In this section, we will discuss what is perhaps the most important example of a stable ∞-category: the ∞-category of spectra. In classical homotopy theory, one defines a spectrum to be a sequence of pointed spaces $\{X_n\}_{n \geq 0}$, equipped with homotopy equivalences (or homeomorphisms, depending on the author) $X_n \to \Omega(X_{n+1})$ for all $n \geq 0$. By virtue of Remark 1.4.2.25, this admits the following ∞-categorical translation:

Definition 1.4.3.1. A *spectrum* is a spectrum object of the ∞-category $\mathcal{S}$ of spaces. We let $\mathrm{Sp} = \mathrm{Sp}(\mathcal{S}_*)$ denote the ∞-category of spectra.

Remark 1.4.3.2. The homotopy category hSp of spectra can be identified with the classical *stable homotopy category*. There are many different constructions of the stable homotopy category in the literature. For a discussion of some other modern approaches, we refer the reader to [51] and [73].

Remark 1.4.3.3. According to Definition 1.4.3.1, a spectrum E is a reduced, excisive functor from the ∞-category $\mathcal{S}_*^{\mathrm{fin}}$ of pointed finite spaces to the ∞-category $\mathcal{S}$ of spaces. As suggested in §1.4.2, we can think of such a functor as defining a *homology theory* A. More precisely, given a pair of finite spaces $X_0 \subseteq X$, we can define the relative homology group $A_n(X, X_0)$ to be $\pi_n E(X/X_0)$, where X/X_0 denotes the pointed space obtained from X by collapsing X_0 to a point (here the homotopy group is taken with base point provided by the map $* \simeq E(*) \to E(X/X_0)$). The assumption that E is excisive guarantees the existence of excision and Mayer-Vietoris sequences for A.

It follows from Corollary 1.4.2.17 that the ∞-category Sp of spectra is stable. To analyze this ∞-category further, we observe that there is a t-structure on Sp. This is a special case of the following general observation:

Proposition 1.4.3.4. *Let* $\mathcal{C}$ *be a presentable* ∞*-category, and let* $\mathrm{Sp}(\mathcal{C})_{\leq -1}$ *be the full subcategory of* $\mathrm{Sp}(\mathcal{C})$ *spanned by those objects* X *such that* $\Omega^{\infty}(X)$ *is a final object of* $\mathcal{C}$*. Then* $\mathrm{Sp}(\mathcal{C})_{\leq -1}$ *determines an accessible t-structure on* $\mathrm{Sp}(\mathcal{C})$ *(see Definition 1.4.4.12).*

Proof. Note that the forgetful functor $\Omega^{\infty} : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ is accessible and preserves small limits, and therefore admits a left adjoint Σ^{∞}_{+} (Corollary T.5.5.2.9). Choose a small collection of objects $\{C_{\alpha}\}$ which generates $\mathcal{C}$ under colimits. We observe that an object $X \in \mathrm{Sp}(\mathcal{C})$ belongs to $\mathrm{Sp}(\mathcal{C})_{\leq -1}$ if and only if each of the spaces

$$\mathrm{Map}_{\mathcal{C}}(C_{\alpha}, \Omega^{\infty}(X)) \simeq \mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(\Sigma^{\infty}(C_{\alpha}), X)$$

is contractible. Let $\mathrm{Sp}(\mathcal{C})_{\geq 0}$ be the smallest full subcategory of $\mathrm{Sp}(\mathcal{C})$ which is stable under colimits and extensions, and contains each $\Sigma^{\infty}(C_{\alpha})$. Proposition 1.4.4.11 implies that $\mathrm{Sp}(\mathcal{C})_{\geq 0}$ is the collection of non-negative objects of the desired t-structure on $\mathrm{Sp}(\mathcal{C})$. $\square$

Remark 1.4.3.5. The proof of Proposition 1.4.3.4 gives another characterization of the t-structure on $\mathrm{Sp}(\mathcal{C})$: the full subcategory $\mathrm{Sp}(\mathcal{C})_{\geq 0}$ is generated, under extensions and colimits, by the essential image of the functor $\Sigma^{\infty}_{+} : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$.

We now apply Proposition 1.4.3.4 to the study of the ∞-category Sp:

Proposition 1.4.3.6. (1) *The* ∞*-category* Sp *is stable.*

(2) *Let* $(\mathrm{Sp})_{\leq -1}$ *denote the full subcategory of* Sp *spanned by those objects* X *such that the space* $\Omega^{\infty}(X) \in \mathcal{S}$ *is contractible. Then* $(\mathrm{Sp})_{\leq -1}$ *determines an accessible t-structure on* Sp *(see Definition 1.4.4.12).*

(3) *The t-structure on* Sp *is both left complete and right complete, and the heart* $\mathrm{Sp}^{\heartsuit}$ *is canonically equivalent to the (nerve of the) category of abelian groups.*

Proof. Assertion (1) follows immediately from Corollary 1.4.2.17 and assertion (2) from Proposition 1.4.3.4. We will prove (3). Note that a spectrum X can be identified with a sequence of pointed spaces $\{X(n)\}$, equipped with equivalences $X(n) \simeq \Omega X(n+1)$ for all $n \geq 0$. We observe that $X \in (\mathrm{Sp})_{\leq m}$ if and only if each $X(n)$ is $(n+m)$-truncated. In general, the sequence $\{\tau_{\leq n+m}X(n)\}$ itself determines a spectrum, which we can identify with the truncation $\tau_{\leq m}X$. It follows that $X \in (\mathrm{Sp})_{\geq m+1}$ if and only if each $X(n)$ is $(n+m+1)$-connective. In particular, X lies in the heart of Sp if and only if each $X(n)$ is an Eilenberg-MacLane object of $\mathcal{S}$ of degree n (see Definition T.7.2.2.1). It follows that the heart of Sp can be identified with the homotopy inverse limit of the tower of ∞-categories

$$\ldots \overset{\Omega}{\to} \mathcal{EM}_1(\mathcal{S}) \overset{\Omega}{\to} \mathcal{EM}_0(\mathcal{S}),$$

where $\mathcal{EM}_n(\mathcal{S})$ denotes the full subcategory of $\mathcal{S}_{*}$ spanned by the Eilenberg-MacLane objects of degree n. Proposition T.7.2.2.12 asserts that after the second term, this tower is equivalent to the constant diagram taking the value $\mathrm{N}(\mathcal{Ab})$, where $\mathcal{Ab}$ is category of abelian groups.

It remains to prove that Sp is both right and left complete. We begin by observing that if $X \in \mathrm{Sp}$ is such that $\pi_n X \simeq 0$ for all $n \in \mathbf{Z}$, then X is a zero object of Sp (since each $X(n) \in \mathcal{S}$ has vanishing homotopy groups, and is therefore contractible by Whitehead's theorem). Consequently, both $\bigcap (\mathrm{Sp})_{\leq -n}$ and $\bigcap (\mathrm{Sp})_{\geq n}$ coincide with the collection of zero objects of Sp. It follows that

$$(\mathrm{Sp})_{\geq 0} = \{X \in \mathrm{Sp} : (\forall n < 0)[\pi_n X \simeq 0]\}$$

$$(\mathrm{Sp})_{\leq 0} = \{X \in \mathrm{Sp} : (\forall n > 0)[\pi_n X \simeq 0]\}.$$

According to Proposition 1.2.1.19, to prove that Sp is left and right complete it will suffice to show that the subcategories $(\mathrm{Sp})_{\geq 0}$ and $(\mathrm{Sp})_{\leq 0}$ are stable under products and coproducts. In view of the above formulas, it will suffice to show that the homotopy group functors $\pi_n : \mathrm{Sp} \to \mathrm{N}(\mathcal{Ab})$ preserve products and coproducts. Since π_n obviously commutes with finite coproducts, it will suffice to show that π_n commutes with products

and filtered colimits. Shifting if necessary, we may reduce to the case $n = 0$. Since products and filtered colimits in the category of abelian groups can be computed at the level of the underlying sets, we are reduced to proving that the composition

$$\mathrm{Sp} \xrightarrow{\Omega^\infty} \mathcal{S} \xrightarrow{\pi_0} \mathrm{N}(\mathcal{S}et)$$

preserves products and filtered colimits. This is clear, since each of the factors individually preserves products and filtered colimits. $\square$

Our next goal is to show that the ∞-category Sp is compactly generated. This is a consequence of the following more general result:

Proposition 1.4.3.7. *Let $\mathcal{C}$ be a compactly generated ∞-category. Then the ∞-category $\mathrm{Sp}(\mathcal{C})$ is compactly generated.*

Proof. Let $\mathcal{D}$ be the full subcategory of $\mathrm{Sp}(\mathcal{C})$ spanned by the compact objects. Since $\mathrm{Sp}(\mathcal{C})$ is presentable, the ∞-category $\mathcal{D}$ is essentially small. It follows that the inclusion $\mathcal{D} \hookrightarrow \mathrm{Sp}(\mathcal{C})$ extends to a fully faithful embedding $\theta : \mathrm{Ind}(\mathcal{D}) \to \mathrm{Sp}(\mathcal{C})$ (Proposition T.5.3.5.10). We wish to show that θ is an equivalence of ∞-categories. Since θ preserves small colimits (Proposition T.5.5.1.9), it admits a right adjoint G; it will therefore suffice to show that the functor G is conservative. Let $\alpha : X \to Y$ be a morphism in $\mathrm{Sp}(\mathcal{C})$ such that $G(\alpha)$ is an equivalence. We wish to show that α is an equivalence. For this, it will suffice to show that for every integer n, the induced map $\Omega^{\infty-n} X \to \Omega^{\infty-n} Y$ is an equivalence in $\mathcal{C}$. Since $\mathcal{C}$ is compactly generated, it will suffice to show that α induces a homotopy equivalence

$$\theta : \mathrm{Map}_{\mathcal{C}}(C, \Omega^\infty X) \to \mathrm{Map}_{\mathcal{C}}(C, \Omega^\infty Y)$$

for every compact object $C \in \mathcal{C}$. To prove this, we note that filtered colimits in $\mathcal{C}$ are left exact, so that the full subcategory $\mathrm{Sp}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$ is closed under filtered colimits. It follows that the functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\Sigma_+^\infty : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$ which carries compact objects of $\mathcal{C}$ to compact objects of $\mathrm{Sp}(\mathcal{C})$. We can identify θ with the map $\mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(\Sigma_+^\infty(C), X) \to \mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(\Sigma_+^\infty(C), Y)$ given by composition with α. Since $\Sigma_+^\infty(C)$ is compact, our assumption that $G(\alpha)$ is an equivalence guarantees that θ is a homotopy equivalence as desired. $\square$

Remark 1.4.3.8. Let $\mathcal{A}b$ denote the category of abelian groups. For each $n \in \mathbf{Z}$, the construction $X \mapsto \pi_n X$ determines a functor $\mathrm{Sp} \to \mathrm{N}(\mathcal{A}b)$. Note that if $n \geq 2$, then π_n can be identified with the composition

$$\mathrm{Sp} \xrightarrow{\Omega_*^\infty} \mathcal{S}_* \xrightarrow{\pi_n} \mathrm{N}(\mathcal{A}b)$$

where the second map is the usual homotopy group functor. Since Sp is both left and right complete, we conclude that a map $f : X \to Y$ of spectra is an equivalence if and only if it induces isomorphisms $\pi_n X \to \pi_n Y$ for all $n \in \mathbf{Z}$.

We close this section with the following useful result, which relates colimits in the ∞-categories Sp and $\mathcal{S}$:

Proposition 1.4.3.9. *The functor $\Omega^\infty : \mathrm{Sp}_{\geq 0} \to \mathcal{S}$ preserves sifted colimits.*

Proof. Since every sifted simplicial set is weakly contractible, the forgetful functor $\mathcal{S}_* \to \mathcal{S}$ preserves sifted colimits (Proposition T.4.4.2.9). It will therefore suffice to prove that the functor $\Omega_*^\infty| \mathrm{Sp}_{\geq 0} \to \mathcal{S}_*$ preserves sifted colimits.

For each $n \geq 0$, let $\mathcal{S}^{\geq n}$ denote the full subcategory of $\mathcal{S}$ spanned by the n-connective objects, and let $\mathcal{S}_*^{\geq n}$ be the ∞-category of pointed objects of $\mathcal{S}^{\geq n}$. We observe that $\mathrm{Sp}_{\geq 0}$ can be identified with the homotopy inverse limit of the tower

$$\ldots \xrightarrow{\Omega} \mathcal{S}_*^{\geq 1} \xrightarrow{\Omega} \mathcal{S}_*^{\geq 0} .$$

It will therefore suffice to prove that for every $n \geq 0$, the loop functor $\Omega : \mathcal{S}_*^{\geq n+1} \to \mathcal{S}_*^{\geq n}$ preserves sifted colimits.

The ∞-category $\mathcal{S}^{\geq n}$ is the preimage (under $\tau_{\leq n-1}$) of the full subcategory of $\tau_{\leq n-1}\mathcal{S}$ spanned by the final objects. Since this full subcategory is stable under sifted colimits and since $\tau_{\leq n-1}$ commutes with all colimits, we conclude that $\mathcal{S}^{\geq n} \subseteq \mathcal{S}$ is stable under sifted colimits.

According to Lemmas T.7.2.2.11 and T.7.2.2.10, there is an equivalence of $\mathcal{S}_*^{\geq 1}$ with the ∞-category of group objects $\mathcal{G}\mathrm{rp}(\mathcal{S}_*)$. This restricts to an equivalence of $\mathcal{S}_*^{\geq n+1}$ with $\mathcal{G}\mathrm{rp}(\mathcal{S}_*^{\geq n})$ for all $n \geq 0$. Moreover, under this equivalence, the loop functor Ω can be identified with the composition

$$\mathcal{G}\mathrm{rp}(\mathcal{S}_*^{\geq n}) \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{S}_*^{\geq n}) \to \mathcal{S}_*^{\geq n},$$

where the second map is given by evaluation at the object $[1] \in \mathbf{\Delta}$. This evaluation map commutes with sifted colimits (Proposition T.5.1.2.2). Consequently, it will suffice to show that $\mathcal{G}\mathrm{rp}(\mathcal{S}_*^{\geq n}) \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{S}_*^{\geq n})$ is stable under sifted colimits.

Without loss of generality, we may suppose $n = 0$; now we are reduced to showing that $\mathcal{G}\mathrm{rp}(\mathcal{S}_*) \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{S}_*)$ is stable under sifted colimits. In view of Lemma T.7.2.2.10, it will suffice to show that $\mathcal{G}\mathrm{rp}(\mathcal{S}) \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{S})$ is stable under sifted colimits. Invoking Proposition T.7.2.2.4, we are reduced to proving that the formation of sifted colimits $\mathcal{S}$ commutes with finite products, which follows from Lemma T.5.5.8.11. $\qquad\square$

1.4.4 Presentable Stable ∞-Categories

In this section, we will study the class of *presentable* stable ∞-categories: that is, stable ∞-categories which admit small colimits and are generated (under colimits) by a set of small objects. In the stable setting, the condition of presentability can be formulated in a particularly simple way.

Proposition 1.4.4.1. (1) *A stable ∞-category $\mathcal{C}$ admits small colimits if and only if $\mathcal{C}$ admits small coproducts.*

(2) *Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor between stable ∞-categories which admit small colimits. Then F preserves small colimits if and only if F preserves small coproducts.*

(3) *Let $\mathcal{C}$ be a stable ∞-category which admits small colimits, and let X be an object of $\mathcal{C}$. Then X is compact if and only if the following condition is satisfied:*

($*$) *For every map $f : X \to \coprod_{\alpha \in A} Y_\alpha$ in $\mathcal{C}$, there exists a finite subset $A_0 \subseteq A$ such that f factors (up to homotopy) through $\coprod_{\alpha \in A_0} Y_\alpha$.*

Proof. The "only if" direction of (1) is obvious, and the converse follows from Proposition T.4.4.3.2. Assertion (2) can be proven in the same way.

The "only if" direction of (3) follows from the fact that an arbitrary coproduct $\coprod_{\alpha \in A} Y_\alpha$ can be obtained as a filtered colimit of finite coproducts $\coprod_{\alpha \in A_0} Y_\alpha$ (see §T.4.2.3). Conversely, suppose that an object $X \in \mathcal{C}$ satisfies ($*$); we wish to show that X is compact. Let $f : \mathcal{C} \to \widehat{\mathcal{S}}$ be the functor corepresented by X (recall that $\widehat{\mathcal{S}}$ denotes the ∞-category of spaces which are not necessarily small). Proposition T.5.1.3.2 implies that f is left exact. According to Proposition 1.4.2.22, we can assume that $f = \Omega^\infty \circ F$, where $F : \mathcal{C} \to \widehat{\mathrm{Sp}}$ is an exact functor; here $\widehat{\mathrm{Sp}}$ denotes the ∞-category of spectra which are not necessarily small. We wish to prove that f preserves filtered colimits. Since Ω^∞ preserves filtered colimits, it will suffice to show that F preserves *all* colimits. In view of (2), it will suffice to show that F preserves coproducts. In virtue of Remark 1.4.3.8, we are reduced to showing that each of the induced functors

$$\mathcal{C} \xrightarrow{F} \widehat{\mathrm{Sp}} \xrightarrow{\pi_n} \mathrm{N}(\mathcal{A}b)$$

preserves coproducts, where $\mathcal{A}b$ denotes the category of (not necessarily small) abelian groups. Shifting if necessary, we may suppose $n = 0$. In other words, we must show that for any collection of objects $\{Y_\alpha\}_{\alpha \in A}$, the natural map

$$\theta : \bigoplus \mathrm{Ext}^0_{\mathcal{C}}(X, Y_\alpha) \to \mathrm{Ext}^0_{\mathcal{C}}(X, \coprod Y_\alpha)$$

is an isomorphism of abelian groups. The surjectivity of θ amounts to the assumption $(*)$, while the injectivity follows from the observations that each Y_α is a retract of the coproduct $\coprod Y_\alpha$ and that the natural map $\bigoplus \mathrm{Ext}_\mathcal{C}^0(X, Y_\alpha) \to \prod \mathrm{Ext}_\mathcal{C}^0(X, Y_\alpha)$ is injective. $\square$

If $\mathcal{C}$ is a stable ∞-category, then we will say that an object $X \in \mathcal{C}$ *generates* $\mathcal{C}$ if the condition $\pi_0 \mathrm{Map}_\mathcal{C}(X, Y) \simeq *$ implies that Y is a zero object of $\mathcal{C}$.

Corollary 1.4.4.2. *Let $\mathcal{C}$ be a stable ∞-category. Then $\mathcal{C}$ is presentable if and only if the following conditions are satisfied:*

(1) *The ∞-category $\mathcal{C}$ admits small coproducts.*

(2) *The homotopy category $\mathrm{h}\mathcal{C}$ is locally small.*

(3) *There exists a regular cardinal κ and a κ-compact generator $X \in \mathcal{C}$.*

Proof. Suppose first that $\mathcal{C}$ is presentable. Conditions (1) and (2) are obvious. To establish (3), we may assume without loss of generality that $\mathcal{C}$ is an accessible localization of $\mathcal{P}(\mathcal{D})$, for some small ∞-category $\mathcal{D}$. Let $F : \mathcal{P}(\mathcal{D}) \to \mathcal{C}$ be the localization functor and G its right adjoint. Let $j : \mathcal{D} \to \mathcal{P}(\mathcal{D})$ be the Yoneda embedding, and let X be a coproduct of all suspensions (see §1.1.2) of objects of the form $F(j(D))$, where $D \in \mathcal{D}$. Since $\mathcal{C}$ is presentable, X is κ-compact provided that κ is sufficiently large. We claim that X generates $\mathcal{C}$. To prove this, we consider an arbitrary $Y \in \mathcal{C}$ such that $\pi_0 \mathrm{Map}_\mathcal{C}(X, Y) \simeq *$. It follows that the space

$$\mathrm{Map}_\mathcal{C}(F(j(D)), Y) \simeq \mathrm{Map}_{\mathcal{P}(\mathcal{D})}(j(D), G(Y)) \simeq G(Y)(D)$$

is contractible for all $D \in \mathcal{D}$, so that $G(Y)$ is a final object of $\mathcal{P}(\mathcal{D})$. Since G is fully faithful, we conclude that Y is a final object of $\mathcal{C}$, as desired.

Conversely, suppose that (1), (2), and (3) are satisfied. We first claim that $\mathcal{C}$ is itself locally small. It will suffice to show that for every morphism $f : X \to Y$ in $\mathcal{C}$ and every $n \geq 0$, the homotopy group $\pi_n(\mathrm{Hom}_\mathcal{C}^{\mathrm{R}}(X, Y), f)$ is small. We note that $\mathrm{Hom}_\mathcal{C}^{\mathrm{R}}(X, Y)$ is equivalent to the loop space of $\mathrm{Hom}_\mathcal{C}^{\mathrm{R}}(X, Y[1])$; the question is therefore independent of base point, so we may assume that f is the zero map. We conclude that the relevant homotopy group can be identified with $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X[n], Y)$, which is small by virtue of assumption (2).

Fix a regular cardinal κ and a κ-compact object X which generates $\mathcal{C}$. We now define a transfinite sequence of full subcategories

$$\mathcal{C}(0) \subseteq \mathcal{C}(1) \subseteq \ldots$$

as follows. Let $\mathcal{C}(0)$ be the full subcategory of $\mathcal{C}$ spanned by the objects $\{X[n]\}_{n \in \mathbf{Z}}$. If λ is a limit ordinal, let $\mathcal{C}(\lambda) = \bigcup_{\beta < \lambda} \mathcal{C}(\beta)$. Finally, let $\mathcal{C}(\alpha + 1)$ be the full subcategory of $\mathcal{C}$ spanned by all objects which can be obtained as the colimit of κ-small diagrams in $\mathcal{C}(\alpha)$. Since $\mathcal{C}$ is locally small, it follows that each $\mathcal{C}(\alpha)$ is essentially small. It follows by induction that each $\mathcal{C}(\alpha)$ consists of κ-compact objects of $\mathcal{C}$ and is stable under translation. Finally, we observe that $\mathcal{C}(\kappa)$ is stable under κ-small colimits. It follows from Lemma 1.1.3.3 that $\mathcal{C}(\kappa)$ is a stable subcategory of $\mathcal{C}$. Choose a small ∞-category $\mathcal{D}$ and an equivalence $f : \mathcal{D} \to \mathcal{C}(\kappa)$. According to Proposition T.5.3.5.11, we may suppose that f factors as a composition

$$\mathcal{D} \xrightarrow{j} \mathrm{Ind}_\kappa(\mathcal{D}) \xrightarrow{F} \mathcal{C}$$

where j is the Yoneda embedding and F is a κ-continuous, fully faithful functor. We will complete the proof by showing that F is an equivalence.

Proposition T.5.5.1.9 implies that F preserves small colimits. It follows that F admits a right adjoint $G : \mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{D})$ (Remark T.5.5.2.10). We wish to show that the counit map $u : F \circ G \to \mathrm{id}_\mathcal{C}$ is an equivalence of functors. Choose an object $Z \in \mathcal{C}$, and let Y be a cofiber for the induced map $u_Z : (F \circ G)(Z) \to Z$. Since F is fully faithful, $G(u_Z)$ is an equivalence. Because G is an exact functor, we deduce that $G(Y) = 0$. It follows that $\mathrm{Map}_\mathcal{C}(F(D), Y) \simeq \mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{D})}(D, G(Y)) \simeq *$ for all $D \in \mathrm{Ind}_\kappa(\mathcal{D})$. In particular, we conclude that $\pi_0 \mathrm{Map}_\mathcal{C}(X, Y) \simeq *$. Since X generates $\mathcal{C}$, we deduce that $Y \simeq 0$. Thus u_Z is an equivalence as desired. $\square$

Remark 1.4.4.3. In view of Proposition 1.4.4.1 and Corollary 1.4.4.2, the hypothesis that a stable ∞-category $\mathcal{C}$ be compactly generated can be formulated entirely in terms of the homotopy category $h\mathcal{C}$. Consequently, one can study this condition entirely in the setting of triangulated categories, without making reference to (or assuming the existence of) an underlying stable ∞-category. We refer the reader to [114] for further discussion.

Our next result provides a large class of examples of presentable stable ∞-categories.

Proposition 1.4.4.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories, and suppose that $\mathcal{D}$ is stable.*

(1) *The ∞-category $\mathrm{Sp}(\mathcal{C})$ is presentable.*

(2) *The functor $\Omega^{\infty} : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\Sigma_{+}^{\infty} : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$.*

(3) *An exact functor $G : \mathcal{D} \to \mathrm{Sp}(\mathcal{C})$ admits a left adjoint if and only if $\Omega^{\infty} \circ G : \mathcal{D} \to \mathcal{C}$ admits a left adjoint.*

Proof. We first prove (1). Assume that $\mathcal{C}$ is presentable, and let 1 be a final object of $\mathcal{C}$. Then $\mathcal{C}_*$ is equivalent to $\mathcal{C}_{1/}$, and therefore presentable (Proposition T.5.5.3.11). The loop functor $\Omega : \mathcal{C}_* \to \mathcal{C}_*$ admits a left adjoint $\Sigma : \mathcal{C}_* \to \mathcal{C}_*$. Consequently, we may view the tower

$$\ldots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*$$

as a diagram in the ∞-category Pr^{R}. Invoking Theorem T.5.5.3.18 and Remark 1.4.2.25, we deduce (1) and the following modified versions of (2) and (3):

(2′) The functor $\Omega_{*}^{\infty} : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}_*$ admits a left adjoint $\Sigma^{\infty} : \mathcal{C}_* \to \mathrm{Sp}(\mathcal{C})$.

(3′) An exact functor $G : \mathcal{D} \to \mathrm{Sp}(\mathcal{C})$ admits a left adjoint if and only if $\Omega_{*}^{\infty} \circ G : \mathcal{D} \to \mathcal{C}_*$ admits a left adjoint.

To complete the proof, it will suffice to verify the following:

(2″) The forgetful functor $\mathcal{C}_* \to \mathcal{C}$ admits a left adjoint $\mathcal{C} \to \mathcal{C}_*$.

(3″) A functor $G : \mathcal{D} \to \mathcal{C}_*$ admits a left adjoint if and only if the composition $\mathcal{D} \xrightarrow{G} \mathcal{C}_* \to \mathcal{C}$ admits a left adjoint.

To prove (2″) and (3″), we recall that a functor G between presentable ∞-categories admits a left adjoint if and only if G preserves small limits and small, κ-filtered colimits, for some regular cardinal κ (Corollary T.5.5.2.9). The desired results now follow from Propositions T.4.4.2.9 and T.1.2.13.8. $\square$

In what follows, if $\mathcal{C}$ and $\mathcal{D}$ are presentable ∞-categories, we let $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which admit right adjoints, and $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D})$ the full subcategory spanned by those functors which admit left adjoints.

Corollary 1.4.4.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories, and suppose that $\mathcal{D}$ is stable. Then composition with $\Sigma_{+}^{\infty} : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$ induces an equivalence*

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}).$$

Proof. This is equivalent to the assertion that composition with Ω^{∞} induces an equivalence

$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \mathrm{Sp}(\mathcal{C})) \to \mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \mathcal{C}),$$

which follows from Propositions 1.4.2.22 and 1.4.4.4. $\square$

Using Corollary 1.4.4.5, we obtain another characterization of the ∞-category of spectra. Let $S \in \mathrm{Sp}$ denote the image under $\Sigma^\infty : \mathcal{S} \to \mathrm{Sp}$ of the final object $* \in \mathcal{S}$. We will refer to S as the *sphere spectrum*.

Corollary 1.4.4.6. *Let $\mathcal{D}$ be a presentable stable ∞-category. Then evaluation on the sphere spectrum induces an equivalence of ∞-categories*

$$\theta : \mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}, \mathcal{D}) \to \mathcal{D} \, .$$

In other words, we may regard the ∞-category Sp as the stable ∞-category which is freely generated, under colimits, by a single object.

Proof. We can factor the evaluation map θ as a composition

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}(\mathcal{S}), \mathcal{D}) \xrightarrow{\theta'} \mathrm{Fun}^{\mathrm{L}}(\mathcal{S}, \mathcal{D}) \xrightarrow{\theta''} \mathcal{D}$$

where θ' is given by composition with Σ_+^∞ and θ'' by evaluation at the final object of $\mathcal{S}$. We now observe that θ' and θ'' are both equivalences of ∞-categories (Corollary 1.4.4.5 and Theorem T.5.1.5.6). $\square$

We conclude this section with yet another characterization of the class of presentable stable ∞-categories.

Lemma 1.4.4.7. *Let $\mathcal{C}$ be a stable ∞-category, and let $\mathcal{C}' \subseteq \mathcal{C}$ be a localization of $\mathcal{C}$. Let $L : \mathcal{C} \to \mathcal{C}'$ be a left adjoint to the inclusion. Then L is left exact if and only if $\mathcal{C}'$ is stable.*

Proof. The "if" direction follows from Proposition 1.1.4.1, since L is right exact. Conversely, suppose that L is left exact. Since $\mathcal{C}'$ is a localization of $\mathcal{C}$, it is closed under finite limits. In particular, it is closed under the formation of fibers and contains a zero object of $\mathcal{C}$. To complete the proof, it will suffice to show that $\mathcal{C}'$ is stable under the formation of pushouts in $\mathcal{C}$. Choose a pushout diagram $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

in $\mathcal{C}$, where $X, X', Y \in \mathcal{C}'$. Proposition 1.1.3.4 implies that σ is also a pullback square. Let $L : \mathcal{C} \to \mathcal{C}'$ be a left adjoint to the inclusion. Since L is left exact, we obtain a pullback square $L(\sigma)$:

$$\begin{array}{ccc} LX & \longrightarrow & LX' \\ \downarrow & & \downarrow \\ LY & \longrightarrow & LY'. \end{array}$$

Applying Proposition 1.1.3.4 again, we deduce that $L(\sigma)$ is a pushout square in $\mathcal{C}$. The natural transformation $\sigma \to L(\sigma)$ is an equivalence when restricted to Λ_0^2, and therefore induces an equivalence $Y' \to LY'$. It follows that Y' belongs to the essential image of $\mathcal{C}'$, as desired. $\square$

Lemma 1.4.4.8. *Let $\mathcal{C}$ be a stable ∞-category, $\mathcal{D}$ an ∞-category which admits finite limits, and $G : \mathcal{C} \to \mathrm{Sp}(\mathcal{D})$ an exact functor. Suppose that $g = \Omega^\infty \circ G : \mathcal{C} \to \mathcal{D}$ is fully faithful. Then G is fully faithful.*

Proof. It will suffice to show that each of the composite maps

$$g_n : \mathcal{C} \to \mathrm{Sp}(\mathcal{D}) \xrightarrow{\Omega_*^{\infty - n}} \mathcal{D}_*$$

is fully faithful. Since g_n can be identified with $g_{n+1} \circ \Omega$, where $\Omega : \mathcal{C} \to \mathcal{C}$ denotes the loop functor, we can reduce to the case $n = 0$. Fix objects $C, C' \in \mathcal{C}$; we will show that the map $\mathrm{Map}_{\mathcal{C}}(C, C') \to \mathrm{Map}_{\mathcal{D}_*}(g_0(C), g_0(C'))$ is a homotopy equivalence. We have a homotopy fiber sequence

$$\mathrm{Map}_{\mathcal{D}_*}(g_0(C), g_0(C')) \xrightarrow{\theta} \mathrm{Map}_{\mathcal{D}}(g(C), g(C')) \to \mathrm{Map}_{\mathcal{D}}(*, g(C')).$$

Here $*$ denotes a final object of $\mathcal{D}$. Since g is fully faithful, it will suffice to prove that θ is a homotopy equivalence. For this, it suffices to show that $\mathrm{Map}_{\mathcal{D}}(*, g(C'))$ is contractible. Since g is left exact, this space can be identified with $\mathrm{Map}_{\mathcal{D}}(g(*), g(C'))$, where $*$ is the final object of $\mathcal{C}$. Invoking once again our assumption that g is fully faithful, we are reduced to proving that $\mathrm{Map}_{\mathcal{C}}(*, C')$ is contractible. This follows from the assumption that $\mathcal{C}$ is pointed (since $*$ is also an initial object of $\mathcal{C}$). $\square$

Proposition 1.4.4.9. *Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is presentable and stable.*

(2) *There exists a presentable, stable ∞-category $\mathcal{D}$ and an accessible left-exact localization $L : \mathcal{D} \to \mathcal{C}$.*

(3) *There exists a small ∞-category $\mathcal{E}$ such that $\mathcal{C}$ is equivalent to an accessible left-exact localization of $\mathrm{Fun}(\mathcal{E}, \mathrm{Sp})$.*

Proof. The ∞-category Sp is stable and presentable. It follows that for every small ∞-category $\mathcal{E}$, the functor ∞-category $\mathrm{Fun}(\mathcal{E}, \mathrm{Sp})$ is also stable (Proposition 1.1.3.1) and presentable (Proposition T.5.5.3.6). This proves $(3) \Rightarrow (2)$. The implication $(2) \Rightarrow (1)$ follows from Lemma 1.4.4.7. We will complete the proof by showing that $(1) \Rightarrow (3)$.

Since $\mathcal{C}$ is presentable, there exists a small ∞-category $\mathcal{E}$ and a fully faithful embedding $g : \mathcal{C} \to \mathcal{P}(\mathcal{E})$, which admits a left adjoint (Theorem T.5.5.1.1). Propositions 1.4.2.22 and 1.4.4.4 implies that g is equivalent to a composition

$$\mathcal{C} \overset{G}{\to} \mathrm{Sp}(\mathcal{P}(\mathcal{E})) \overset{\Omega^{\infty}}{\to} \mathcal{P}(\mathcal{E}),$$

where the functor G admits a left adjoint. Lemma 1.4.4.8 implies that G is fully faithful. It follows that $\mathcal{C}$ is an (accessible) left exact localization of $\mathrm{Sp}(\mathcal{P}(\mathcal{E}))$. We now invoke Remark 1.4.2.9 to identify $\mathrm{Sp}(\mathcal{P}(\mathcal{E}))$ with $\mathrm{Fun}(\mathcal{E}^{op}, \mathrm{Sp})$. $\square$

Remark 1.4.4.10. Proposition 1.4.4.9 can be regarded as an analogue of Giraud's characterization of topoi as left exact localizations of presheaf categories ([56]). Other variations on this theme include the ∞-categorical version of Giraud's theorem (Theorem T.6.1.0.6) and the Gabriel-Popesco theorem for abelian categories (see [111]).

If $\mathcal{C}$ is a presentable stable ∞-category, then it is reasonably easy to construct t-structures on $\mathcal{C}$: for any small collection of objects $\{X_{\alpha}\}$ of $\mathcal{C}$, there exists a t-structure *generated* by the objects X_{α}. More precisely, we have the following result:

Proposition 1.4.4.11. *Let $\mathcal{C}$ be a presentable stable ∞-category.*

(1) *If $\mathcal{C}' \subseteq \mathcal{C}$ is a full subcategory which is presentable, closed under small colimits, and closed under extensions, then there exists a t-structure on $\mathcal{C}$ such that $\mathcal{C}' = \mathcal{C}_{\geq 0}$.*

(2) *Let $\{X_{\alpha}\}$ be a small collection of objects of $\mathcal{C}$, and let $\mathcal{C}'$ be the smallest full subcategory of $\mathcal{C}$ which contains each X_{α} and is closed under extensions and small colimits. Then $\mathcal{C}'$ is presentable.*

Proof. We will give the proof of (1) and defer the (somewhat technical) proof of (2) until the end of this section. Fix $X \in \mathcal{C}$, and let $\mathcal{C}'_{/X}$ denote the fiber product $\mathcal{C}_{/X} \times_{\mathcal{C}} \mathcal{C}'$. Using Proposition T.5.5.3.12, we deduce that $\mathcal{C}'_{/X}$ is presentable, so that it admits a final object $f : Y \to X$. It follows that composition with f induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}}(Z, Y) \to \mathrm{Map}_{\mathcal{C}}(Z, X)$$

for each $Z \in \mathcal{C}'$. Proposition T.5.2.7.8 implies that $\mathcal{C}'$ is a colocalization of $\mathcal{C}$. Since $\mathcal{C}'$ is stable under extensions, Proposition 1.2.1.16 implies the existence of a (uniquely determined) t-structure such that $\mathcal{C}' = \mathcal{C}_{\geq 0}$. $\square$

Definition 1.4.4.12. Let $\mathcal{C}$ be a presentable stable ∞-category. We will say that a t-structure on $\mathcal{C}$ is *accessible* if the subcategory $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ is presentable.

Proposition 1.4.4.11 can be summarized as follows: any small collection of objects $\{X_\alpha\}$ of a presentable stable ∞-category $\mathcal{C}$ determines an accessible t-structure on $\mathcal{C}$, which is minimal among t-structures such that each X_α belongs to $\mathcal{C}_{\geq 0}$.

Definition 1.4.4.12 has a number of reformulations:

Proposition 1.4.4.13. *Let $\mathcal{C}$ be a presentable stable ∞-category equipped with a t-structure. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}_{\geq 0}$ is presentable (equivalently: the t-structure on $\mathcal{C}$ is accessible).*

(2) *The ∞-category $\mathcal{C}_{\geq 0}$ is accessible.*

(3) *The ∞-category $\mathcal{C}_{\leq 0}$ is presentable.*

(4) *The ∞-category $\mathcal{C}_{\leq 0}$ is accessible.*

(5) *The truncation functor $\tau_{\leq 0} : \mathcal{C} \to \mathcal{C}$ is accessible.*

(6) *The truncation functor $\tau_{\geq 0} : \mathcal{C} \to \mathcal{C}$ is accessible.*

Proof. We observe that $\mathcal{C}_{\geq 0}$ is stable under all colimits which exist in $\mathcal{C}$, and that $\mathcal{C}_{\leq 0}$ is a localization of $\mathcal{C}$. It follows that $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ admit small colimits, so that $(1) \Leftrightarrow (2)$ and $(3) \Leftrightarrow (4)$. We have a fiber sequence of functors

$$\tau_{\geq 0} \xrightarrow{\alpha} \mathrm{id}_{\mathcal{C}} \xrightarrow{\beta} \tau_{\leq -1}.$$

The collection of accessible functors from $\mathcal{C}$ to itself is stable under shifts and under small colimits. Since $\tau_{\leq 0} \simeq \mathrm{cofib}(\alpha)[1]$ and $\tau_{\geq 0} \simeq \mathrm{cofib}(\beta)[-1]$, we conclude that $(5) \Leftrightarrow (6)$. The equivalence $(1) \Leftrightarrow (5)$ follows from Proposition T.5.5.1.2. We will complete the proof by showing that $(1) \Leftrightarrow (3)$.

Suppose first that (1) is satisfied. Then $\mathcal{C}_{\geq 1} = \mathcal{C}_{\geq 0}[1]$ is generated under colimits by a set of objects $\{X_\alpha\}$. Let S be the collection of all morphisms f in $\mathcal{C}$ such that $\tau_{\leq 0}(f)$ is an equivalence. Using Proposition 1.2.1.16, we conclude that S is generated by $\{0 \to X_\alpha\}$ as a quasisaturated class of morphisms, and therefore also as a strongly saturated class of morphisms (Definition T.5.5.4.5). We now apply Proposition T.5.5.4.15 to conclude that $\mathcal{C}_{\leq 0} = S^{-1}\,\mathcal{C}$ is presentable; this proves (3).

We now complete the proof by showing that $(3) \Rightarrow (1)$. If $\mathcal{C}_{\leq -1} = \mathcal{C}_{\leq 0}[-1]$ is presentable, then Proposition T.5.5.4.16 implies that S is of small generation (as a strongly saturated class of morphisms). Proposition 1.2.1.16 implies that S is generated (as a strongly saturated class) by the morphisms $\{0 \to X_\alpha\}_{\alpha \in A}$, where X_α ranges over the collection of all objects of $\mathcal{C}_{\geq 0}$. It follows that there is a small subcollection $A_0 \subseteq A$ such that S is generated by the morphisms $\{0 \to X_\alpha\}_{\alpha \in A_0}$. Let $\mathcal{D}$ be the smallest full subcategory of $\mathcal{C}$ which contains the objects $\{X_\alpha\}_{\alpha \in A_0}$ and is closed under colimits and extensions. Since $\mathcal{C}_{\geq 0}$ is closed under colimits and extensions, we have $\mathcal{D} \subseteq \mathcal{C}_{\geq 0}$. Consequently, $\mathcal{C}_{\leq -1}$ can be characterized as full subcategory of $\mathcal{C}$ spanned by those objects $Y \in \mathcal{C}$ such that $\mathrm{Ext}^k_{\mathcal{C}}(X, Y)$ for all $k \leq 0$ and $X \in \mathcal{D}$. Propositions 1.4.4.11 implies that $\mathcal{D}$ is the collection of nonnegative objects for some accessible t-structure on $\mathcal{C}$. Since the negative objects of this new t-structure coincide with the negative objects of the original t-structure, we conclude that $\mathcal{D} = \mathcal{C}_{\geq 0}$, which proves (1). $\qquad\square$

We conclude this section by completing the proof of Proposition 1.4.4.11.

Proof of part (2) *of Proposition 1.4.4.11.* Choose a regular cardinal κ such that every object of X_α is κ-compact, and let $\mathcal{C}^\kappa$ denote the full subcategory of $\mathcal{C}$ spanned by the κ-compact objects. Let $\mathcal{C}'^\kappa = \mathcal{C}' \cap \mathcal{C}^\kappa$, and let $\mathcal{C}''$ be the smallest full subcategory of $\mathcal{C}'$ which contains $\mathcal{C}'^\kappa$ and is closed under small colimits. The ∞-category $\mathcal{C}''$ is κ-accessible, and therefore presentable. To complete the proof, we will show that $\mathcal{C}' \subseteq \mathcal{C}''$. For this, it will suffice to show that $\mathcal{C}''$ is stable under extensions.

Let $\mathcal{D}$ be the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by those morphisms $f : X \to Y$ where $Y \in \mathcal{C}''$, $X \in \mathcal{C}''[-1]$. We wish to prove that the cofiber functor $\mathrm{cofib} : \mathcal{D} \to \mathcal{C}$ factors through $\mathcal{C}''$. Let $\mathcal{D}^\kappa$ be the full subcategory of $\mathcal{D}$ spanned by those morphisms $f : X \to Y$ where both X and Y are κ-compact objects of $\mathcal{C}$. By construction, $\mathrm{cofib} \,|\, \mathcal{D}^\kappa$ factors through $\mathcal{C}''$. Since $\mathrm{cofib} : \mathcal{D} \to \mathcal{C}$ preserves small colimits, it will suffice to show that $\mathcal{D}$ is generated (under small colimits) by $\mathcal{D}^\kappa$.

Fix an object $f : X \to Y$ in $\mathcal{D}$. To complete the proof, it will suffice to show that the canonical map $(\mathcal{D}^\kappa_{/f})^\triangleright \to \mathcal{D}$ is a colimit diagram. Since $\mathcal{D}$ is stable under colimits in $\mathrm{Fun}(\Delta^1, \mathcal{C})$ and colimits in $\mathrm{Fun}(\Delta^1, \mathcal{C})$ are computed pointwise (Proposition T.5.1.2.2), it will suffice to show that composition with the evaluation maps give colimit diagrams $(\mathcal{D}^\kappa_{/f})^\triangleright \to \mathcal{C}$. Lemma T.5.3.5.8 implies that the maps $(\mathcal{C}'^\kappa[-1])^\triangleright_{/X} \to \mathcal{C}$, $(\mathcal{C}'^\kappa)^\triangleright_{/Y} \to \mathcal{C}$ are colimit diagrams. It will therefore suffice to show that the evaluation maps

$$(\mathcal{C}'^\kappa[-1])_{/X} \xleftarrow{\theta} (\mathcal{D}^\kappa_{/f}) \xrightarrow{\theta'} (\mathcal{C}'^\kappa)_{/Y}$$

are left cofinal.

We first show that θ is left cofinal. According to Theorem T.4.1.3.1, it will suffice to show that for every morphism $\alpha : X' \to X$ in $\mathcal{C}'[-1]$, where X' is κ-compact, the ∞-category

$$\mathcal{E}_\theta : \mathcal{D}^\kappa_{/f} \times_{\mathcal{C}'^\kappa[-1]_{/X}} (\mathcal{C}'^\kappa[-1]_{/X})_{X'/}$$

is weakly contractible. For this, it is sufficient to show that $\mathcal{E}_\theta$ is filtered (Lemma T.5.3.1.18).

We will show that $\mathcal{E}_\theta$ is κ-filtered. Let K be a κ-small simplicial set, and $p : K \to \mathcal{E}_\theta$ a diagram; we will extend p to a diagram $\overline{p} : K^\triangleright \to \mathcal{E}_\theta$. We can identify p with two pieces of data:

(*i*) A map $p' : K^\triangleleft \to \mathcal{C}'^\kappa[-1]_{/X}$.

(*ii*) A map $p'' : (K \star \{\infty\}) \times \Delta^1 \to \mathcal{C}$, with the properties that $p''|(K \star \{\infty\}) \times \{0\}$ can be identified with p', $p''|\{\infty\} \times \Delta^1$ can be identified with f, and $p''|K \times \{1\}$ factors through $\mathcal{C}'^\kappa$.

Let $\overline{p}' : (K^\triangleleft)^\triangleright \to \mathcal{C}'^\kappa[-1]_{/X}$ be a colimit of p'. To complete the proof that $\mathcal{E}_\theta$ is κ-filtered, it will suffice to show that we can find a compatible extension $\overline{p}'' : (K^\triangleright \star \{\infty\}) \times \Delta^1 \to \mathcal{C}$ with the appropriate properties. Let L denote the full simplicial subset of $(K^\triangleright \star \{\infty\}) \times \Delta^1)$ spanned by every vertex except $(v, 1)$, where v denotes the cone point of $K^\triangleright$. We first choose a map $q : L \to \mathcal{C}$ compatible with p'' and $\overline{p}'$. This is equivalent to solving the lifting problem

$$\begin{array}{ccc} & & \mathcal{C}_{/f} \\ & \nearrow & \downarrow \\ K^\triangleright & \longrightarrow & \mathcal{C}_{/X}, \end{array}$$

which is possible since the vertical arrow is a trivial fibration. Let $L' = L \cap (K^\triangleright \times \Delta^1)$. Then q determines a map $q_0 : L' \to \mathcal{C}_{/Y}$. Finding the desired extension $\overline{p}''$ is equivalent to finding a map $\overline{q}_0 : L'^\triangleright \to \mathcal{C}_{/Y}$, which carries the cone point into $\mathcal{C}'^\kappa$.

Let $g : Z \to Y$ be a colimit of q_0 (in the ∞-category $\mathcal{C}_{/Y}$). We observe that Z is a κ-small colimit of κ-compact objects of $\mathcal{C}$, and therefore κ-compact. Since $Y \in \mathcal{C}''$, Y can be written as the colimit of a κ-filtered diagram $\{Y_\alpha\}$, taking values in $\mathcal{C}'^\kappa$. Since Z is κ-compact, the map g factors through some Y_α; it follows that there exists an extension $\overline{q}_0$ as above, which carries the cone point to Y_α. This completes the proof that $\mathcal{E}_\theta$ is κ-filtered, and also the proof that θ is left cofinal.

The proof that θ' is left cofinal is similar but slightly easier: it suffices to show that for every map $Y' \to Y$ in $\mathcal{C}'$, where Y' is κ-compact, the fiber product

$$\mathcal{E}_{\theta'} = \mathcal{D}^\kappa_{/f} \times_{\mathcal{C}'^\kappa_{/Y}} (\mathcal{C}'^\kappa_{/Y})_{Y'/}$$

is filtered. For this, we can either argue as above, or simply observe that $\mathcal{E}_{\theta'}$ admits κ-small colimits. $\qquad\square$

Chapter 2

∞-Operads

Recall that a *commutative monoid* is a set M equipped with a multiplication $M \times M \to M$ and a unit object $1 \in M$ satisfying the identities

$$1x = x \quad xy = yx \quad x(yz) = (xy)z$$

for all $x, y, z \in M$. Roughly speaking, a *symmetric monoidal category* is a category $\mathcal{C}$ equipped with the same type of structure: a unit object $1 \in \mathcal{C}$, and a bifunctor $\otimes : \mathcal{C} \times \mathcal{C}$. However, in the categorical setting, it is unnatural to require the identities displayed above to hold as equalities. For example, we do not expect $X \otimes (Y \otimes Z)$ to be *equal* to $(X \otimes Y) \otimes Z$. Instead, these identities should be formulated by requiring the existence of isomorphisms

$$\alpha_X : 1 \otimes X \simeq X \qquad \beta_{X,Y} : X \otimes Y \simeq Y \otimes X$$

$$\gamma_{X,Y,Z} : (X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z).$$

Moreover, these isomorphisms should be regarded as additional data, and are required to satisfy further conditions (such as naturality in the objects X, Y, and Z). If we try to generalize this definition to higher categories, then the equations satisfied by the isomorphisms α_X, $\beta_{X,Y}$, and $\gamma_{X,Y,Z}$ should themselves hold only up to isomorphism. It quickly becomes prohibitively complicated to explicitly specify all of the coherence conditions that these isomorphisms must satisfy. Consequently, to develop an ∞-categorical analogue of the theory of symmetric monoidal categories, it will be more convenient for us to proceed in another way.

We begin by considering an example of a symmetric monoidal category. Let $\mathcal{C}$ be the category of complex vector spaces, with monoidal structure given by tensor products of vector spaces. Given a pair of vector spaces U and V, the tensor product $U \otimes V$ is characterized by the requirement that $\operatorname{Hom}_{\mathcal{C}}(U \otimes V, W)$ can be identified with the set of *bilinear* maps $U \times V \to W$. In fact, this property really only determines $U \otimes V$ up to canonical isomorphism: in order to build an actual tensor product functor, we need to choose some particular construction of $U \otimes V$. Because this requires making certain decisions in an ad-hoc manner, it is unrealistic to expect an equality of vector spaces $U \otimes (V \otimes W) = (U \otimes V) \otimes W$. However, the existence of a canonical *isomorphism* between $U \otimes (V \otimes W)$ and $(U \otimes V) \otimes W$ is easily explained: linear maps from either into a fourth vector space X can be identified with trilinear maps $U \times V \times W \to X$.

The above example suggests that we might reformulate the definition of a symmetric monoidal category as follows. Rather than give a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, we instead specify, for each n-tuple $(C_1, \ldots, C_n)$ of objects of $\mathcal{C}$ and each $D \in \mathcal{C}$, the collection of morphisms $C_1 \otimes \ldots \otimes C_n \to D$. Of course, we also need to specify how such morphisms are to be composed. The relevant data can be encoded in a new category, which we will denote by $\mathcal{C}^{\otimes}$.

Construction 2.0.0.1. Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. We define a new category $\mathcal{C}^{\otimes}$ as follows:

    (i) An object of $\mathcal{C}^{\otimes}$ is a finite (possibly empty) sequence of objects of $\mathcal{C}$, which we will denote by $[C_1, \ldots, C_n]$.

(ii) A morphism from $[C_1, \ldots, C_n]$ to $[C'_1, \ldots, C'_m]$ in $\mathcal{C}^\otimes$ consists of a subset $S \subseteq \{1, \ldots, n\}$, a map of finite sets $\alpha : S \to \{1, \ldots, m\}$, and a collection of morphisms $\{f_j : \bigotimes_{\alpha(i)=j} C_i \to C'_j\}_{1 \leq j \leq m}$ in the category $\mathcal{C}$. (Here the tensor product $\bigotimes_{\alpha(i)=j} C_i$ is well-defined up to canonical isomorphism, since $\mathcal{C}$ is a symmetric monoidal category.)

(iii) Suppose we are given morphisms $f : [C_1, \ldots, C_n] \to [C'_1, \ldots, C'_m]$ and $g : [C'_1, \ldots, C'_m] \to [C''_1, \ldots, C''_l]$ in $\mathcal{C}^\otimes$, determining subsets $S \subseteq \{1, \ldots, n\}$ and $T \subseteq \{1, \ldots, m\}$ together with maps $\alpha : S \to \{1, \ldots, m\}$ and $\beta : T \to \{1, \ldots, l\}$. The composition $g \circ f$ is given by the subset $U = \alpha^{-1}T \subseteq \{1, \ldots, n\}$, the composite map $\beta \circ \alpha : U \to \{1, \ldots, l\}$, and the maps

$$\bigotimes_{(\beta \circ \alpha)(i)=k} C_i \simeq \bigotimes_{\beta(j)=k} \otimes_{\alpha(i)=j} C_i \to \bigotimes_{\beta(j)=k} C'_j \to C''_k$$

for $1 \leq k \leq l$.

To analyze this construction, we begin by recalling the definition of Segal's category $\mathcal{F}\mathrm{in}_*$ of pointed finite sets:

Notation 2.0.0.2. For any finite set I, let I_* denote the set $I \amalg \{*\}$ obtained from I by adjoining a new element $*$. For each $n \geq 0$, we let $\langle n \rangle^\circ$ denote the set $\{1, 2, \ldots, n-1, n\}$ and $\langle n \rangle = \langle n \rangle^\circ_* = \{*, 1, \ldots, n\}$ the pointed set obtained by adjoining a disjoint base point $*$ to $\langle n \rangle^\circ$. We define a category $\mathcal{F}\mathrm{in}_*$ as follows:

(1) The objects of $\mathcal{F}\mathrm{in}_*$ are the sets $\langle n \rangle$, where $n \geq 0$.

(2) Given a pair of objects $\langle m \rangle, \langle n \rangle \in \mathcal{F}\mathrm{in}_*$, a morphism from $\langle m \rangle$ to $\langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$ is a map $\alpha : \langle m \rangle \to \langle n \rangle$ such that $\alpha(*) = *$.

For every pair of integers $1 \leq i \leq n$, we let $\rho^i : \langle n \rangle \to \langle 1 \rangle$ denote the morphism given by the formula

$$\rho^i(j) = \begin{cases} 1 & \text{if } i = j \\ * & \text{otherwise.} \end{cases}$$

Remark 2.0.0.3. The category $\mathcal{F}\mathrm{in}_*$ is equivalent to the category of *all* finite sets equipped with a distinguished point $*$. We will often invoke this equivalence implicitly, using the following device. Let I be a finite linearly ordered set. Then there is a canonical bijection $\alpha : I_* \simeq \langle n \rangle$, where n is the cardinality of I; the bijection α is determined uniquely by the requirement that the restriction of α determines an order-preserving bijection of I with $\langle n \rangle^\circ$. We will generally identify I_* with the object $\langle n \rangle \in \mathcal{F}\mathrm{in}_*$ via this isomorphism.

Remark 2.0.0.4. If $\alpha : \langle n \rangle \to \langle m \rangle$ is a morphism in $\mathcal{F}\mathrm{in}_*$, it is convenient to think of α as a *partially defined* map from $\langle n \rangle^\circ$ to $\langle m \rangle^\circ$: namely, α is given by specifying a subset $S = \alpha^{-1}\langle m \rangle^\circ \subseteq \langle n \rangle^\circ$ together with a map $S \to \langle m \rangle^\circ$.

For every symmetric monoidal category $\mathcal{C}$, the category $\mathcal{C}^\otimes$ of Construction 2.0.0.1 comes equipped with a forgetful functor $\mathcal{C}^\otimes \to \mathcal{F}\mathrm{in}_*$, which carries an object $[C_1, \ldots, C_n]$ to the pointed finite set $\langle n \rangle$.

Remark 2.0.0.5. Let $\mathcal{C}_0$ be the category $[0]$, containing a unique object and a unique morphism. Then $\mathcal{C}_0$ admits a unique symmetric monoidal structure. Moreover, the forgetful functor $\mathcal{C}_0^\otimes \to \mathcal{F}\mathrm{in}_*$ described above is an isomorphism of categories. In other words, we may view $\mathcal{F}\mathrm{in}_*$ as obtained by applying Construction 2.0.0.1 in the simplest possible example. Moreover, for any symmetric monoidal category $\mathcal{C}$, the forgetful functor $\mathcal{C}^\otimes \to \mathcal{F}\mathrm{in}_*$ can be viewed as obtained from the (unique) symmetric monoidal functor $\mathcal{C} \to \mathcal{C}_0$ by means of the functoriality implicit in Construction 2.0.0.1.

For any symmetric monoidal category $\mathcal{C}$, the forgetful functor $\mathcal{C}^\otimes \to \mathcal{F}\mathrm{in}_*$ enjoys two special features:

(M1) The functor p is an *op-fibration of categories*. In other words, for every object $[C_1, \ldots, C_n] \in \mathcal{C}^{\otimes}$ and every morphism $f : \langle n \rangle \to \langle m \rangle$ in $\mathcal{F}\text{in}_*$, there exists a morphism $\overline{f} : [C_1, \ldots, C_n] \to [C'_1, \ldots, C'_m]$ which covers f, and is universal in the sense that composition with $\overline{f}$ induces a bijection

$$\text{Hom}_{\mathcal{C}^{\otimes}}([C'_1, \ldots, C'_m], [C''_1, \ldots, C''_l])$$

$$\downarrow$$

$$\text{Hom}_{\mathcal{C}^{\otimes}}([C_1, \ldots, C_n], [C''_1, \ldots, C''_l]) \times_{\text{Hom}_{\mathcal{F}\text{in}_*}(\langle n \rangle, \langle l \rangle)} \text{Hom}_{\mathcal{F}\text{in}_*}(\langle m \rangle, \langle l \rangle)$$

for every object $[C''_1, \ldots, C''_l] \in \mathcal{C}^{\otimes}$. To achieve this, it suffices to choose a morphism $\overline{f}$ which determines isomorphisms $C'_j \simeq \bigotimes_{f(i)=j} C_i$ for $1 \leq j \leq m$.

Remark 2.0.0.6. It follows from Condition $(M1)$ that every morphism $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\text{in}_*$ induces a functor between the fibers $\mathcal{C}^{\otimes}_{\langle m \rangle} \to \mathcal{C}^{\otimes}_{\langle n \rangle}$, which is well-defined up to canonical isomorphism.

(M2) Let $\mathcal{C}^{\otimes}_{\langle n \rangle}$ denote the fiber of p over the object $\langle n \rangle \in \mathcal{F}\text{in}_*$. Then $\mathcal{C}^{\otimes}_{\langle 1 \rangle}$ is equivalent to $\mathcal{C}$. More generally, $\mathcal{C}^{\otimes}_{\langle n \rangle}$ is equivalent to an n-fold product of copies of $\mathcal{C}$. This equivalence is induced by the functors associated to the maps $\{\rho^i : \langle n \rangle \to \langle 1 \rangle\}_{1 \leq i \leq n}$ in $\mathcal{F}\text{in}_*$.

We now observe that the symmetric monoidal structure on a category $\mathcal{C}$ is determined, up to symmetric monoidal equivalence, by the category $\mathcal{C}^{\otimes}$ together with its forgetful functor $\mathcal{C}^{\otimes} \to \mathcal{F}\text{in}_*$. More precisely, suppose we are given a functor $p : \mathcal{D} \to \mathcal{F}\text{in}_*$ satisfying conditions $(M1)$ and $(M2)$, and let $\mathcal{C}$ denote the fiber $\mathcal{D}_{\langle 1 \rangle}$. Then:

(a) Condition $(M2)$ implies that $\mathcal{D}_{\langle 0 \rangle}$ has a single object, up to equivalence. The unique morphism $\langle 0 \rangle \to \langle 1 \rangle$ in $\mathcal{F}\text{in}_*$ determines a functor $\mathcal{D}_{\langle 0 \rangle} \to \mathcal{D}_{\langle 1 \rangle} = \mathcal{C}$, which we can identify with an object $\mathbf{1} \in \mathcal{C}$.

(b) Let $\alpha : \langle 2 \rangle \to \langle 1 \rangle$ be the morphism given by

$$\alpha(1) = \alpha(2) = 1 \qquad \alpha(*) = *.$$

By virtue of $(M1)$, the functors α, ρ^1, and ρ^2 determine functors

$$\mathcal{C} \times \mathcal{C} = \mathcal{D}_{\langle 1 \rangle} \times \mathcal{D}_{\langle 1 \rangle} \xleftarrow{\rho^0 \times \rho^1} \mathcal{D}_{\langle 2 \rangle} \xrightarrow{\alpha} \mathcal{D}_{\langle 1 \rangle} = \mathcal{C}.$$

Condition $(M2)$ guarantees that the map on the left is an equivalence of categories, so that we obtain a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (well-defined up to canonical isomorphism), which we will denote by $\otimes$.

(c) Let $\sigma : \langle 2 \rangle \to \langle 2 \rangle$ denote the automorphism which exchanges the elements $1, 2 \in \langle 2 \rangle$. Then $\alpha \circ \sigma = \alpha$, while $\rho^i \circ \sigma = \rho^{\sigma(i)}$. It follows that there is a canonical isomorphism between the functors $(X, Y) \mapsto X \otimes Y$ and $(X, Y) \mapsto Y \otimes X$.

(d) For $1 \leq i < n$, let $\tau_i^n : \langle n \rangle \to \langle n-1 \rangle$ denote the map given by the formula

$$\tau_i^n(j) = \begin{cases} j & \text{if } 1 \leq j \leq i \\ j-1 & \text{if } i < j \leq n \\ * & \text{if } j = *. \end{cases}$$

The commutative diagram

$$\begin{array}{ccc} \langle 3 \rangle & \xrightarrow{\tau_1^3} & \langle 2 \rangle \\ \downarrow{\scriptstyle \tau_2^3} & & \downarrow{\scriptstyle \tau_1^2} \\ \langle 2 \rangle & \xrightarrow{\tau_1^2} & \langle 1 \rangle \end{array}$$

in $\mathcal{F}\mathrm{in}_*$ determines a diagram of categories and functors (which commutes up to canonical isomorphism):

$$
\begin{array}{ccc}
\mathcal{D}_{\langle 3 \rangle} & \longrightarrow & \mathcal{D}_{\langle 2 \rangle} \\
\downarrow & & \downarrow \\
\mathcal{D}_{\langle 2 \rangle} & \longrightarrow & \mathcal{D}_{\langle 1 \rangle} \, .
\end{array}
$$

Combining this with the equivalences $\mathcal{D}_{\langle n \rangle} \simeq \mathcal{C}^n$, we obtain a functorial isomorphism

$$\gamma_{A,B,C} : (A \otimes B) \otimes C \simeq A \otimes (B \otimes C).$$

A similar argument can be used to construct canonical isomorphisms

$$\mathbf{1} \otimes X \simeq X \simeq X \otimes \mathbf{1}.$$

It is not difficult to see that (a), (b), (c), and (d) endow $\mathcal{C}$ with the structure of a monoidal category. For example, MacLane's pentagon axiom asserts that the diagram

is commutative. This follows from the fact that all five expressions can be canonically identified with the image of (A, B, C, D) under the composite functor $\mathcal{C}^4 \simeq \mathcal{D}_{\langle 4 \rangle} \to \mathcal{D}_{\langle 1 \rangle} \simeq \mathcal{C}$, where the middle map is induced by a map $\alpha : \langle 4 \rangle \to \langle 1 \rangle$ in $\mathcal{F}\mathrm{in}_*$, characterized by the requirement that $\alpha^{-1}\{1\} = \langle 4 \rangle^\circ$. In the case where $\mathcal{D} = \mathcal{C}^\otimes$ is given by Construction 2.0.0.1, it is easy to see that the data provided by (a) through (d) recovers the original symmetric monoidal structure on $\mathcal{C}$ (up to canonical isomorphism). Conversely, an arbitrary functor $\mathcal{D} \to \mathcal{F}\mathrm{in}_*$ satisfying $(M1)$ and $(M2)$ determines a symmetric monoidal structure on $\mathcal{C} = \mathcal{D}_{\langle 1 \rangle}$ and an equivalence $\mathcal{D} \simeq \mathcal{C}^\otimes$. In other words, giving a symmetric monoidal category is *equivalent* to giving a functor $\mathcal{C}^\otimes \to \mathcal{F}\mathrm{in}_*$ satisfying $(M1)$ and $(M2)$. However, the second definition enjoys several advantages:

- As we saw above in the case of vector spaces, it is sometimes easier to specify the category $\mathcal{C}^\otimes$ than to specify the bifunctor $\otimes$, in the sense that it requires fewer arbitrary choices.

- Axioms $(M1)$ and $(M2)$ concerning the functor $\mathcal{C}^\otimes \to \mathcal{F}\mathrm{in}_*$ are a bit simpler than the usual definition of a (symmetric) monoidal category. Complicated assertions, such as the commutativity of MacLane's pentagon, are consequences of $(M1)$ and $(M2)$.

The significance of the latter point becomes more apparent in the ∞-categorical setting, where we expect the MacLane pentagon to be only the first step in a hierarchy of coherence conditions of ever-increasing complexity. Fortunately, the above discussion suggests an approach which does not require us to formulate these conditions explicitly:

Definition 2.0.0.7. A *symmetric monoidal ∞-category* is a coCartesian fibration of simplicial sets $p : \mathcal{C}^\otimes \to$ $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ with the following property:

$(*)$ For each $n \geq 0$, the maps $\{\rho^i : \langle n \rangle \to \langle 1 \rangle\}_{1 \leq i \leq n}$ induce functors $\rho_!^i : \mathcal{C}^\otimes_{\langle n \rangle} \to \mathcal{C}^\otimes_{\langle 1 \rangle}$ which determine an equivalence $\mathcal{C}^\otimes_{\langle n \rangle} \simeq (\mathcal{C}^\otimes_{\langle 1 \rangle})^n$.

One of our main goals in this book is to show that Definition 2.0.0.7 is reasonable: that is, it provides a robust generalization of the classical theory of symmetric monoidal categories. Moreover, by relaxing the assumption that p is a coCartesian fibration in Definition 2.0.0.7, one obtains a more general theory of ∞-*operads* (see Definition 2.1.1.10), which generalizes the classical theory of colored operads (or *multicategories*, as some authors refer to them).

Our objective in this chapter is to lay down the foundations for the study of ∞-operads. We begin in §2.1 by introducing the basic definitions. Roughly speaking, we can think of an ∞-operad $\mathcal{O}^{\otimes}$ as consisting of an underlying ∞-category $\mathcal{O}$, together with mapping spaces $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y)$ (for any finite set of objects $\{X_i \in \mathcal{O}\}_{i \in I}$ and any object $Y \in \mathcal{O}$) equipped with a coherently associative composition law (see Remark 2.1.1.17), which reduce to the usual mapping spaces in $\mathcal{O}$ when I has a single element. An ∞-operad can be regarded as a kind of generalized ∞-category: we recover the usual theory of ∞-categories if we require that $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y)$ be empty when I is of cardinality $\neq 1$ (see Proposition 2.1.4.11). Many ∞-categorical constructions can be generalized to the setting of ∞-operads: we will study a number of examples in §2.2.

In practice, it is quite common to encounter a family of symmetric monoidal ∞-categories which depend on some parameter. For example, to every commutative ring A we can associate the symmetric monoidal category of A-modules, which depends functorially on A. In §2.3 we will axiomatize this phenomenon by introducing the definition of a *generalized ∞-operad*. This notion will be useful when we discuss ∞-categories of modules in §3.3 and when we consider variants on the little cubes operads of Boardman-Vogt in §5.2.

One can obtain a rich source of examples of symmetric monoidal ∞-categories by purely categorical considerations: if $\mathcal{C}$ is an ∞-category which admits finite products (coproducts), then $\mathcal{C}$ can be promoted to a symmetric monoidal ∞-category $\mathcal{C}^{\times}$ ($\mathcal{C}^{\mathrm{II}}$) in which the tensor product operation is given by the Cartesian product (coproduct). We will close this chapter with a detailed discussion of these examples (§2.4).

Remark 2.0.0.8. An alternate approach to the theory of ∞-operads has been proposed by Cisinski and Moerdijk, based on the formalism of *dendroidal sets* (see [31] and [32]). It seems overwhelmingly likely that their theory is equivalent to the one presented in this chapter. More precisely, there should be a Quillen equivalence between the category of dendroidal sets and the category $\mathcal{P}\mathrm{Op}_{\infty}$ of ∞-preoperads which we describe in §2.4.

Nevertheless, the two perspectives differ somewhat at the level of technical detail. Our approach has the advantage of being phrased entirely in the language of simplicial sets, which allows us to draw heavily upon the preexisting theory of ∞-categories (as described in [97]) and to avoid direct contact with the combinatorics of trees (which play an essential role in the definition of a dendroidal set). However, the advantage comes at a cost: though our ∞-operads can be described using the relatively pedestrian language of simplicial sets, the actual simplicial sets which we need are somewhat unwieldy and tend to encode information in a inefficient way. Consequently, some of the proofs presented here are perhaps more difficult than they need to be: for example, we suspect that Theorems 3.1.2.3 and 3.4.4.3 admit much shorter proofs in the dendroidal setting.

2.1 Foundations

Our goal in this section is to introduce an ∞-categorical version of the classical theory of operads. We begin in §2.1.1 by reviewing the classical definition of a colored operad. The structure of an arbitrary colored operad $\mathcal{O}$ is completely encoded by a category which we will denote by $\mathcal{O}^{\otimes}$ (defined by a generalization of Construction 2.0.0.1), together with a forgetful functor from $\mathcal{O}^{\otimes}$ to Segal's category $\mathcal{F}\mathrm{in}_*$. Motivated by this observation, we will define an ∞-*operad* to be an ∞-category $\mathcal{O}^{\otimes}$ equipped with a map $\mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, satisfying an appropriate set of axioms (Definition 2.1.1.10).

To any ∞-operad $\mathcal{O}^{\otimes}$, one can associate a theory of $\mathcal{O}$-*monoidal ∞-categories*: namely, one considers coCartesian fibrations $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ satisfying an appropriate analogue of condition $(*)$ appearing in Definition 2.0.0.7. The relevant definitions will be given in §2.1.2. An important special case is obtained when we take $\mathcal{O}^{\otimes}$ to be the *commutative* ∞-operad: in this case, we recover the notion of symmetric monoidal ∞-category given in Definition 2.0.0.7. If $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category, then there is an associated theory

of *commutative algebra objects* of $\mathcal{C}^{\otimes}$: these can be defined as sections of the fibration $\mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ which satisfy a mild additional condition. More generally, to any fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$, we can associate an ∞-category $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ of $\mathcal{O}$-algebras in $\mathcal{C}$. We will give some basic definitions in §2.1.3, and undertake a much more comprehensive study in §3.

The collection of all ∞-operads can be organized into an ∞-category, where we take morphisms from an ∞-operad $\mathcal{O}^{\otimes}$ to another ∞-operad $\mathcal{O}'^{\otimes}$ to be given by $\mathcal{O}$-algebra objects of $\mathcal{O}'$. We will denote this ∞-category by Op_{∞} and refer to it as the *∞-category of ∞-operads*. In §2.1.4 we will illuminate the structure of Op_{∞} by showing that it can be realized as the underlying ∞-category of a combinatorial simplicial model category. The construction of this model category uses a general existence result which we discuss in §B.4.

2.1.1   From Colored Operads to ∞-Operads

Our goal in this section is to introduce the definition of an *∞-operad* (Definition 2.1.1.10), which will play a fundamental role throughout this book. We begin with a review of the classical theory of colored operads.

Definition 2.1.1.1. A *colored operad* $\mathcal{O}$ consists of the following data:

(1) A collection $\{X, Y, Z, \ldots\}$ which we will refer to as the collection of *objects* or *colors* of $\mathcal{O}$. We will indicate that X is an object of $\mathcal{O}$ by writing $X \in \mathcal{O}$.

(2) For every finite set I, every I-indexed collection of objects $\{X_i\}_{i \in I}$ in $\mathcal{O}$, and every object $Y \in \mathcal{O}$, a set $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y)$, which we call the set of *morphisms* from $\{X_i\}_{i \in I}$ to Y.

(3) For every map of finite sets $I \to J$ having fibers $\{I_j\}_{j \in J}$, every finite collection of objects $\{X_i\}_{i \in I}$, every finite collection of objects $\{Y_j\}_{j \in J}$, and every object $Z \in \mathcal{O}$, a composition map

$$\prod_{j \in J} \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I_j}, Y_j) \times \mathrm{Mul}_{\mathcal{O}}(\{Y_j\}_{j \in J}, Z) \to \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Z).$$

(4) A collection of morphisms $\{\mathrm{id}_X \in \mathrm{Mul}_{\mathcal{O}}(\{X\}, X)\}_{X \in \mathcal{O}}$ which are both left and right units for the composition on $\mathcal{O}$ in the following sense: for every finite collection of objects $\{X_i\}_{i \in I}$ and every objects $Y \in \mathcal{O}$, the compositions

$$
\begin{aligned}
\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y) &\simeq \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y) \times \{\mathrm{id}_Y\} \\
&\subseteq \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y) \times \mathrm{Mul}_{\mathcal{O}}(\{Y\}, Y) \\
&\to \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y) &\simeq (\prod_{i \in I}\{\mathrm{id}_{X_i}\}) \times \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y) \\
&\subseteq (\prod_{i \in I} \mathrm{Mul}_{\mathcal{O}}(\{X_i\}, X_i)) \times \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y) \\
&\to \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y)
\end{aligned}
$$

both coincide with the identity map from $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y)$ to itself.

(5) Composition is required to be associative in the following sense: for every sequence of maps $I \to J \to K$ of finite sets together with collections of objects $\{W_i\}_{i \in I}$, $\{X_j\}_{j \in J}$, $\{Y_k\}_{k \in K}$, and every object $Z \in \mathcal{O}$,

the diagram

$$\prod_{j\in J} \mathrm{Mul}_{\mathcal{O}}(\{W_i\}_{i\in I_j}, X_j)) \times \prod_{k\in K} \mathrm{Mul}_{\mathcal{O}}(\{X_j\}_{j\in J_k}, Y_k) \times \mathrm{Mul}_{\mathcal{O}}(\{Y_k\}_{k\in K}, Z)$$

$$\prod_{k\in K} \mathrm{Mul}_{\mathcal{O}}(\{W_i\}_{i\in I_k}, Y_k) \times \mathrm{Mul}_{\mathcal{O}}(\{Y_k\}_{k\in K}, Z) \qquad \prod_{j\in J} \mathrm{Mul}_{\mathcal{O}}(\{W_i\}_{i\in I_j}, X_j) \times \mathrm{Mul}_{\mathcal{O}}(\{X_j\}_{j\in J}, Z)$$

$$\mathrm{Mul}_{\mathcal{O}}(\{W_i\}_{i\in I}, Z)$$

is commutative.

Remark 2.1.1.2. Every colored operad $\mathcal{O}$ has an underlying category (which we will also denote by $\mathcal{O}$) whose objects are the colors of $\mathcal{O}$ and whose morphisms are defined by the formula $\mathrm{Hom}_{\mathcal{O}}(X, Y) = \mathrm{Mul}_{\mathcal{O}}(\{X\}, Y)$. Consequently, we can view a colored operad as a category with some additional structure: namely, the collections of "multilinear" maps $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y)$. Some authors choose to emphasize this analogy by using the term *multicategory* for what we refer to here as a colored operad.

Variant 2.1.1.3. We obtain a notion of a *simplicial colored operad* by replacing the morphism sets

$$\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y)$$

by simplicial sets in Definition 2.1.1.1.

Remark 2.1.1.4. Remark 2.1.1.2 describes a forgetful functor from the category $\mathcal{COp}$ of (small) colored operads to the category $\mathcal{Cat}$ of (small) categories. This functor has a left adjoint: every category $\mathcal{C}$ can be regarded as a colored operad by defining

$$\mathrm{Mul}_{\mathcal{C}}(\{X_i\}_{1\leq i\leq n}, Y) = \begin{cases} \mathrm{Hom}(X_1, Y) & \text{if } n = 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

This construction exhibits $\mathcal{Cat}$ as a full subcategory of $\mathcal{COp}$: namely, the full subcategory spanned by those colored operads $\mathcal{O}$ for which the sets $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y)$ are empty unless I has exactly one element.

Example 2.1.1.5. Let $\mathcal{C}$ be a symmetric monoidal category: that is, a category equipped with a tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is commutative and associative up to coherent isomorphism (see [99] for a careful definition). Then we can regard $\mathcal{C}$ as a colored operad by setting

$$\mathrm{Mul}_{\mathcal{C}}(\{X_i\}_{i\in I}, Y) = \mathrm{Hom}_{\mathcal{C}}(\otimes_{i\in I} X_i, Y).$$

We can recover the symmetric monoidal structure on $\mathcal{C}$ (up to canonical isomorphism) via Yoneda's lemma. For example, for every pair of objects $X, Y \in \mathcal{C}$, the tensor product $X \otimes Y$ is characterized up to canonical isomorphism by the fact that it corepresents the functor $Z \mapsto \mathrm{Mul}_{\mathcal{C}}(\{X, Y\}, Z)$. It follows from this analysis that we can regard the notion of a symmetric monoidal category as a special case of the notion of a colored operad.

Example 2.1.1.6. An *operad* is a colored operad $\mathcal{O}$ having only a single color $\mathbf{1}$. For every nonnegative integer n we let $\mathcal{O}_n = \mathrm{Mul}(\{\mathbf{1}\}_{1\leq i\leq n}, \mathbf{1})$. We sometimes refer to $\mathcal{O}_n$ as the *set of n-ary operations of* $\mathcal{O}$. The structure of $\mathcal{O}$ as a colored operad is determined by the sets $\{\mathcal{O}_n\}_{n\geq 0}$ together with the actions of the symmetric groups Σ_n on $\mathcal{O}_n$ and the "substitution maps"

$$\mathcal{O}_m \times (\prod_{1\leq i\leq m} \mathcal{O}_{n_i}) \to \mathcal{O}_{n_1+\cdots+n_m}.$$

Definition 2.1.1.1 is phrased in a somewhat complicated way because the notion of morphism in a colored operad $\mathcal{O}$ is asymetrical: the domain of a morphism consists of a finite collection of object, while the codomain consists of only a single object. We can correct this asymetry by packaging the data of a colored operad in a different way.

Construction 2.1.1.7. Let $\mathcal{O}$ be a colored operad. We define a category $\mathcal{O}^{\otimes}$ as follows:

(1) The objects of $\mathcal{O}^{\otimes}$ are finite sequences of colors $X_1, \ldots, X_n \in \mathcal{O}$.

(2) Given two sequences of objects

$$X_1, \ldots, X_m \in \mathcal{O} \qquad Y_1, \ldots, Y_n \in \mathcal{O},$$

a morphism from $\{X_i\}_{1 \leq i \leq m}$ to $\{Y_j\}_{1 \leq j \leq n}$ is given by a map $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\text{in}_*$ together with a collection of morphisms

$$\{\phi_j \in \text{Mul}_{\mathcal{O}}(\{X_i\}_{i \in \alpha^{-1}\{j\}}, Y_j)\}_{1 \leq j \leq n}$$

in $\mathcal{O}$.

(3) Composition of morphisms in $\mathcal{O}^{\otimes}$ is determined by the composition laws on $\mathcal{F}\text{in}_*$ and on the colored operad $\mathcal{O}$.

Let $\mathcal{O}$ be a colored operad. By construction, the category $\mathcal{O}^{\otimes}$ comes equipped with a forgetful functor $\pi : \mathcal{O}^{\otimes} \to \mathcal{F}\text{in}_*$. Using the functor π, we can reconstruct the colored operad $\mathcal{O}$ up to canonical equivalence. For example, the underlying category of $\mathcal{O}$ can be identified with the fiber $\mathcal{O}^{\otimes}_{\langle 1 \rangle} = \pi^{-1}\{\langle 1 \rangle\}$. More generally, suppose that $n \geq 0$. For $1 \leq i \leq n$, let $\rho^i : \langle n \rangle \to \langle 1 \rangle$ be as in Notation 2.0.0.2, so that ρ^i induces a functor $\rho^i_! : \mathcal{O}^{\otimes}_{\langle n \rangle} \to \mathcal{O}^{\otimes}_{\langle 1 \rangle} \simeq \mathcal{O}$ and these functors together determine an equivalence of categories $\mathcal{O}^{\otimes}_{\langle n \rangle} \simeq \mathcal{O}^n$. Given a finite sequence of colors $X_1, \ldots, X_n \in \mathcal{O}$, let $\vec{X}$ denote the corresponding object of $\mathcal{O}^{\otimes}_{\langle n \rangle}$ (which is well-defined up to equivalence). For every color $Y \in \mathcal{O}$, the set $\text{Mul}_{\mathcal{O}}(\{X_i\}_{1 \leq i \leq n}, Y)$ can be identified with the set of morphisms $f : \vec{X} \to Y$ in $\mathcal{O}^{\otimes}$ such that $\pi(f) : \langle n \rangle \to \langle 1 \rangle$ satisfies $\pi(f)^{-1}\{*\} = \{*\}$. This shows that $\pi : \mathcal{O}^{\otimes} \to \mathcal{F}\text{in}_*$ determines the morphism sets $\text{Mul}_{\mathcal{O}}(\{X_i\}_{1 \leq i \leq n}, Y)$ in the colored operad $\mathcal{O}$; a more elaborate argument shows that the composition law for morphisms in the colored operad $\mathcal{O}$ can be recovered from the composition law for morphisms in the category $\mathcal{O}^{\otimes}$.

The above construction suggests that it is possible to give an alternate version of Definition 2.1.1.1: rather than thinking of a colored operad as a category-like structure $\mathcal{O}$ equipped with an elaborate notion of morphism, we can think of a colored operad as an ordinary category $\mathcal{O}^{\otimes}$ equipped with a forgetful functor $\pi : \mathcal{O}^{\otimes} \to \mathcal{F}\text{in}_*$. Of course, we do not want to consider an *arbitrary* functor π: we only want to consider those functors which induce equivalences $\mathcal{O}^{\otimes}_{\langle n \rangle} \simeq (\mathcal{O}^{\otimes}_{\langle 1 \rangle})^n$, so that the category $\mathcal{O} = \mathcal{O}^{\otimes}_{\langle 1 \rangle}$ inherits the structure described in Definition 2.1.1.1. This is one drawback of the second approach: it requires us to formulate a somewhat complicated-looking assumption on the functor π. The virtue of the second approach is that it can be phrased entirely in the language of category theory. This allows us to generalize the theory of colored operads to the ∞-categorical setting. First, we need to introduce a bit of terminology.

Definition 2.1.1.8. We will say that a morphism $f : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\text{in}_*$ is *inert* if, for each element $i \in \langle n \rangle^{\circ}$, the inverse image $f^{-1}\{i\}$ has exactly one element.

Remark 2.1.1.9. Every inert morphism $f : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\text{in}_*$ induces an injective map of sets $\alpha : \langle n \rangle^{\circ} \to \langle m \rangle^{\circ}$, characterized by the formula $f^{-1}\{i\} = \{\alpha(i)\}$.

Definition 2.1.1.10. An ∞-*operad* is a functor $p : \mathcal{O}^{\otimes} \to \text{N}(\mathcal{F}\text{in}_*)$ between ∞-categories which satisfies the following conditions:

(1) For every inert morphism $f : \langle m \rangle \to \langle n \rangle$ in $\text{N}(\mathcal{F}\text{in}_*)$ and every object $C \in \mathcal{O}^{\otimes}_{\langle m \rangle}$, there exists a p-coCartesian morphism $\overline{f} : C \to C'$ in $\mathcal{O}^{\otimes}$ lifting f. In particular, f induces a functor $f_! : \mathcal{O}^{\otimes}_{\langle m \rangle} \to \mathcal{O}^{\otimes}_{\langle n \rangle}$.

(2) Let $C \in \mathcal{O}^{\otimes}_{\langle m \rangle}$ and $C' \in \mathcal{O}^{\otimes}_{\langle n \rangle}$ be objects, let $f : \langle m \rangle \to \langle n \rangle$ be a morphism in $\mathcal{F}\mathrm{in}_*$, and let $\mathrm{Map}^f_{\mathcal{O}^{\otimes}}(C, C')$ be the union of those connected components of $\mathrm{Map}_{\mathcal{O}^{\otimes}}(C, C')$ which lie over $f \in \mathrm{Hom}_{\mathcal{F}\mathrm{in}_*}(\langle m \rangle, \langle n \rangle)$. Choose p-coCartesian morphisms $C' \to C'_i$ lying over the inert morphisms $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \le i \le n$. Then the induced map
$$\mathrm{Map}^f_{\mathcal{O}^{\otimes}}(C, C') \to \prod_{1 \le i \le n} \mathrm{Map}^{\rho^i \circ f}_{\mathcal{O}^{\otimes}}(C, C'_i)$$
is a homotopy equivalence.

(3) For every finite collection of objects $C_1, \ldots, C_n \in \mathcal{O}^{\otimes}_{\langle 1 \rangle}$, there exists an object $C \in \mathcal{O}^{\otimes}_{\langle n \rangle}$ and a collection of p-coCartesian morphisms $C \to C_i$ covering $\rho^i : \langle n \rangle \to \langle 1 \rangle$.

Remark 2.1.1.11. Definition 2.1.1.10 is really an ∞-categorical generalization of the notion of a colored operad, rather than that of an operad. Our choice of terminology is motivated by a desire to avoid awkward language. To obtain an ∞-categorical analogue of the notion of an operad, we should consider instead ∞-operads $\mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ equipped with an essentially surjective functor $\Delta^0 \to \mathcal{O}^{\otimes}_{\langle 1 \rangle}$.

Remark 2.1.1.12. Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad. We will often abuse terminology by referring to $\mathcal{O}^{\otimes}$ as an ∞-operad (in this case, it is implicitly assumed that we are supplied with a map p satisfying the conditions listed in Definition 2.1.1.10). We will usually denote the fiber $\mathcal{O}^{\otimes}_{\langle 1 \rangle} \simeq p^{-1}\{\langle 1 \rangle\}$ by $\mathcal{O}$. We will sometimes refer to $\mathcal{O}$ as the *underlying ∞-category* of $\mathcal{O}^{\otimes}$.

Remark 2.1.1.13. Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad. Then p is a categorical fibration. To prove this, we first observe that p is an inner fibration (this follows from Proposition T.2.3.1.5 since $\mathcal{O}^{\otimes}$ is an ∞-category and $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is the nerve of an ordinary category). In view of Corollary T.2.4.6.5, it will suffice to show that if $C \in \mathcal{O}^{\otimes}_{\langle m \rangle}$ and $\alpha : \langle m \rangle \to \langle n \rangle$ is an isomorphism in $\mathcal{F}\mathrm{in}_*$, then we can lift α to an equivalence $\overline{\alpha} : C \to C'$ in $\mathcal{O}^{\otimes}$. Since α is inert, we can choose $\overline{\alpha}$ to be p-coCartesian; it then follows from Proposition T.2.4.1.5 that $\overline{\alpha}$ is an equivalence.

Remark 2.1.1.14. Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a functor between ∞-categories which satisfies conditions (1) and (2) of Definition 2.1.1.10. Then (3) is equivalent to the following apparently stronger condition:

(3′) For each $n \ge 0$, the functors $\{\rho^i_! : \mathcal{O}^{\otimes}_{\langle n \rangle} \to \mathcal{O}\}_{1 \le i \le n}$ determine an equivalence of ∞-categories $\phi : \mathcal{O}^{\otimes}_{\langle n \rangle} \to \mathcal{O}^n$.

It follows easily from (2) that ϕ is fully faithful, and condition (3) guarantees that ϕ is essentially surjective.

Remark 2.1.1.15. Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad and $\mathcal{O} = \mathcal{O}^{\otimes}_{\langle 1 \rangle}$ the underlying ∞-category. It follows from Remark 2.1.1.14 that we have a canonical equivalence $\mathcal{O}^{\otimes}_{\langle n \rangle} \simeq \mathcal{O}^n$. Using this equivalence, we can identify objects of $\mathcal{O}^{\otimes}_{\langle n \rangle}$ with finite sequences $(X_1, X_2, \ldots, X_n)$ of objects of $\mathcal{O}$. We will sometimes denote the corresponding object of $\mathcal{O}^{\otimes}$ by $X_1 \oplus X_2 \oplus \cdots \oplus X_n$ (this object is well-defined up to equivalence). More generally, given objects $X \in \mathcal{O}^{\otimes}_{\langle m \rangle}$ and $Y \in \mathcal{O}^{\otimes}_{\langle n \rangle}$ corresponding to sequences $(X_1, \ldots, X_m)$ and $(Y_1, \ldots, Y_n)$, we let $X \oplus Y$ denote an object of $\mathcal{O}^{\otimes}_{\langle m+n \rangle}$ corresponding to the sequence $(X_1, \ldots, X_m, Y_1, \ldots, Y_n)$. We will discuss the operation $\oplus$ more systematically in §2.2.4.

Notation 2.1.1.16. Let $\mathcal{O}^{\otimes}$ be an ∞-operad, and suppose we are given a finite sequence of objects $\{X_i\}_{1 \le i \le n}$ of $\mathcal{O}$ and another object $Y \in \mathcal{O}$. We let $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{1 \le i \le n}, Y)$ denote the union of those components of $\mathrm{Map}_{\mathcal{O}^{\otimes}}(X_1 \oplus \ldots \oplus X_n, Y)$ which lie over the unique morphism $\beta : \langle n \rangle \to \langle 1 \rangle$ such that $\beta^{-1}\{*\} = \{*\}$. We regard $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{1 \le i \le n}, Y)$ as an object in the homotopy category $\mathcal{H}$ of spaces, which is well-defined up to canonical isomorphism.

Remark 2.1.1.17. Let $\mathcal{O}^{\otimes}$ be an ∞-operad. Then we should imagine that $\mathcal{O}^{\otimes}$ consists of an ordinary ∞-category $\mathcal{O}$ which comes equipped with a more elaborate notion of morphism supplied by the spaces

$\mathrm{Mul}_{\mathcal{O}}(\bullet, \bullet)$ of Notation 2.1.1.16. These morphism spaces are equipped with a composition law satisfying the axiomatics of Definition 2.1.1.1 up to coherent homotopy. A precise formulation of this idea is cumbersome, and Definition 2.1.1.10 provides an efficient substitute.

Example 2.1.1.18. The identity map exhibits $\mathrm{Comm}^{\otimes} = \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ as an ∞-operad, whose underlying ∞-category Comm is isomorphic to Δ^0. We will refer to this ∞-operad as the *commutative ∞-operad*. When we wish to emphasize its role as an ∞-operad, we will sometimes denote it by $\mathrm{Comm}^{\otimes}$.

Example 2.1.1.19. Let $\mathcal{F}\mathrm{in}_*^{\mathrm{inj}}$ denote the subcategory of $\mathcal{F}\mathrm{in}_*$ spanned by all objects together with those morphisms $f : \langle m \rangle \to \langle n \rangle$ such that $f^{-1}\{i\}$ has at most one element for $1 \leq i \leq n$. The nerve $\mathrm{N}(\mathcal{F}\mathrm{in}_*^{\mathrm{inj}})$ is an ∞-operad, which we will denote by $\mathbb{E}_0^{\otimes}$.

Example 2.1.1.20. Let **Triv** be the subcategory of $\mathcal{F}\mathrm{in}_*$ whose objects are the objects of $\mathcal{F}\mathrm{in}_*$, and whose morphisms are the inert morphisms in $\mathcal{F}\mathrm{in}_*$. Let $\mathrm{Triv}^{\otimes} = \mathrm{N}(\mathbf{Triv})$. Then the inclusion $\mathbf{Triv} \subseteq \mathcal{F}\mathrm{in}_*$ induces a functor $\mathrm{Triv}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ which exhibits $\mathrm{Triv}^{\otimes}$ as an ∞-operad; we will refer to $\mathrm{Triv}^{\otimes}$ as the *trivial ∞-operad*.

Example 2.1.1.21. Let $\mathcal{O}$ be a colored operad in the sense of Definition 2.1.1.1, and let $\mathcal{O}^{\otimes}$ be the ordinary category given by Construction 2.1.1.7. Then the forgetful functor $\mathrm{N}(\mathcal{O}^{\otimes}) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is an ∞-operad. Examples 2.1.1.18, 2.1.1.19, and 2.1.1.20 all arise as special cases of this construction.

Notation 2.1.1.22. Example 2.1.1.21 admits a generalization to *simplicial* colored operads. If $\mathcal{O}$ is a simplicial colored operad (see Variation 2.1.1.3 on Definition 2.1.1.1), we let $\mathcal{O}^{\otimes}$ denote the simplicial category given by Construction 2.1.1.7:

(i) The objects of $\mathcal{O}^{\otimes}$ are pairs $(\langle n \rangle, (C_1, \ldots, C_n))$, where $\langle n \rangle \in \mathcal{F}\mathrm{in}_*$ and $C_1, \ldots, C_n$ are colors of $\mathcal{O}$.

(ii) Given a pair of objects $C = (\langle m \rangle, (C_1, \ldots, C_m))$ to $C' = (\langle n \rangle, (C_1', \ldots, C_n'))$ in $\mathcal{O}^{\otimes}$, the simplicial set $\mathrm{Map}_{\mathcal{O}^{\otimes}}(C, C')$ is defined to be

$$\coprod_{\alpha : \langle m \rangle \to \langle n \rangle} \prod_{1 \leq j \leq n} \mathrm{Mul}_{\mathcal{O}}(\{C_i\}_{\alpha(i)=j}, C_j').$$

(iii) Composition in $\mathcal{O}^{\otimes}$ is defined in the obvious way.

Definition 2.1.1.23. Let $\mathcal{O}$ be a simplicial colored operad. We will denote the simplicial nerve of the category $\mathcal{O}^{\otimes}$ by $\mathrm{N}^{\otimes}(\mathcal{O})$; we will refer to $\mathrm{N}^{\otimes}(\mathcal{O})$ as the *operadic nerve of $\mathcal{O}$*.

Remark 2.1.1.24. Let $\mathcal{O}$ be a simplicial colored operad. Then there is an evident forgetful functor from $\mathcal{O}^{\otimes}$ to the ordinary category $\mathcal{F}\mathrm{in}_*$ (regarded as a simplicial category). This forgetful functor induces a canonical map $\mathrm{N}^{\otimes}(\mathcal{O}) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. We may therefore regard $\mathrm{N}^{\otimes}$ as a functor from the category $\mathcal{M}\mathrm{Cat}_{\Delta}$ of simplicial colored operads to the category $(\mathcal{S}\mathrm{et}_{\Delta})_{/\,\mathrm{N}(\mathcal{F}\mathrm{in}_*)}$.

Remark 2.1.1.25. Let $\mathcal{O}$ be a simplicial colored operad. The fiber product $\mathrm{N}^{\otimes}(\mathcal{O}) \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \{\langle 1 \rangle\}$ is canonically isomorphic to the nerve of the simplicial category underlying $\mathcal{O}$; we will denote this simplicial set by $\mathrm{N}(\mathcal{O})$.

Definition 2.1.1.26. We will say that a simplicial colored operad $\mathcal{O}$ is *fibrant* if each of the simplicial sets $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y)$ is fibrant.

Proposition 2.1.1.27. *Let $\mathcal{O}$ be a fibrant simplicial colored operad. Then the operadic nerve $\mathrm{N}^{\otimes}(\mathcal{O})$ is an ∞-operad.*

Proof. If $\mathcal{O}$ is a fibrant simplicial colored operad, then $\mathcal{O}^{\otimes}$ is a fibrant simplicial category so that $\mathrm{N}^{\otimes}(\mathcal{O})$ is an ∞-category. Let $C = (\langle m \rangle, (C_1, \ldots, C_m))$ be an object of $\mathrm{N}^{\otimes}(\mathcal{C})$ and let $\alpha : \langle m \rangle \to \langle n \rangle$ be an inert morphism in $\mathcal{F}\mathrm{in}_*$. Then we have a canonical map $C \to C' = (\langle n \rangle, (C_{\alpha^{-1}(1)}, \ldots, C_{\alpha^{-1}(n)}))$ in $\mathcal{O}^{\otimes}$, which we can identify with an edge $\overline{\alpha}$ of $\mathrm{N}^{\otimes}(\mathcal{O})$ lying over α. Using Proposition T.2.4.1.10, we deduce that $\overline{\alpha}$ is p-coCartesian, where $p : \mathrm{N}^{\otimes}(\mathcal{O}) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

As a special case, we observe that there are p-coCartesian morphisms $\overline{\alpha}^{i, \langle m \rangle^{\circ}} : C \to (\langle 1 \rangle, C_i)$ covering $\rho^i : C \to C_i$ for $1 \leq i \leq m$. To prove that $\mathrm{N}^{\otimes}(\mathcal{O})$ is an ∞-operad, we must show that these maps determine a p-limit diagram $\langle m \rangle^{\circ \triangleleft} \to \mathrm{N}^{\otimes}(\mathcal{O})$. Unwinding the definitions, we must show that for every object $D = (\langle n \rangle, (D_1, \ldots, D_n))$ and every morphism $\beta : \langle n \rangle \to \langle m \rangle$, the canonical map

$$\mathrm{Map}_{\mathcal{O}^{\otimes}}^{\beta}(D, C) \to \prod_{1 \leq i \leq m} \mathrm{Map}_{\mathcal{O}^{\otimes}}^{\rho^i \circ \beta}(D, (\langle 1 \rangle, C_i))$$

is a homotopy equivalence; here $\mathrm{Map}_{\mathcal{O}^{\otimes}}^{\beta}(D, C)$ denotes the inverse image of $\{\beta\}$ in $\mathrm{Map}_{\mathcal{O}^{\otimes}}(D, C)$ and $\mathrm{Map}_{\mathcal{O}^{\otimes}}^{\rho^i \circ \beta}(D, (\langle 1 \rangle, C_i))$ the inverse image of $\{\rho^i \circ \beta\}$ in $\mathrm{Map}_{\mathcal{O}^{\otimes}}(D, (\langle 1 \rangle, C_i))$. We now observe that this map is an isomorphism of simplicial sets.

To complete the proof that $\mathrm{N}^{\otimes}(\mathcal{O})$ is an ∞-operad, it suffices to show that for each $m \geq 0$ the functors $\rho_!^i$ associated to p induces an essentially surjective map

$$\mathrm{N}^{\otimes}(\mathcal{O}) \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \{\langle m \rangle\} \to \prod_{1 \leq i \leq m} \mathrm{N}(\mathcal{O}).$$

In fact, $\mathrm{N}^{\otimes}(\mathcal{O}) \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \{\langle m \rangle\}$ is canonically isomorphic with $\mathrm{N}(\mathcal{O})^m$ and this isomorphism identifies the above map with the identity. $\qquad\square$

2.1.2 Maps of ∞-Operads

In order to make effective use of the theory of ∞-operads, we must understand them not only in isolation but also in relation to one another. To this end, we will introduce the notion of ∞-*operad fibration* (Definition 2.1.2.10). We begin with the more general notion of an ∞-*operad map*, which we will study in more detail in §2.1.3.

We begin with a brief digression on the structure of morphisms in an ∞-operad. According to Definition 2.1.1.8, a morphism $\gamma : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$ is *inert* if exhibits $\langle n \rangle$ as the quotient of $\langle m \rangle$ obtained by identifying a subset of $\langle m \rangle^{\circ}$ with the base point $*$. In this case, γ^{-1} determines an injective map from $\langle n \rangle^{\circ}$ to $\langle m \rangle^{\circ}$. By design, for any ∞-operad $\mathcal{O}^{\otimes}$, the morphism γ induces a functor $\gamma_! : \mathcal{O}^{\otimes}_{\langle m \rangle} \to \mathcal{O}^{\otimes}_{\langle n \rangle}$. This functor can be identified with the projection map $\mathcal{O}^m \to \mathcal{O}^n$ determined by γ^{-1}. Our choice of terminology is intended to emphasize the role of $\gamma_!$ as a forgetful functor, which is not really encoding the essential structure of $\mathcal{O}^{\otimes}$. We now introduce a class of morphisms which lies at the other extreme:

Definition 2.1.2.1. A morphism $f : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$ is *active* if $f^{-1}\{*\} = \{*\}$.

Remark 2.1.2.2. Every morphism f in $\mathcal{F}\mathrm{in}_*$ admits a factorization $f = f' \circ f''$, where f'' is inert and f' is active; moreover, this factorization is unique up to (unique) isomorphism. In other words, the collections of inert and active morphisms determine a factorization system on $\mathcal{F}\mathrm{in}_*$.

The classes of active and inert morphisms determine a factorization system on the category $\mathcal{F}\mathrm{in}_*$ which induces an analogous factorization system on any ∞-operad $\mathcal{O}^{\otimes}$.

Definition 2.1.2.3. Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad. We will say that a morphism f in $\mathcal{O}^{\otimes}$ is *inert* if $p(f)$ is inert and f is p-coCartesian. We will say that a morphism f in $\mathcal{O}^{\otimes}$ is *active* if $p(f)$ is active.

Proposition 2.1.2.4. *Let $\mathcal{O}^{\otimes}$ be an ∞-operad. Then the collections of active and inert morphisms determine a factorization system on $\mathcal{O}^{\otimes}$.*

Proposition 2.1.2.4 is an immediate consequence of Remark 2.1.2.2 together with the following general assertion:

Proposition 2.1.2.5. *Let* $p : \mathcal{C} \to \mathcal{D}$ *be an inner fibration of* ∞-*categories. Suppose that* $\mathcal{D}$ *admits a factorization system* (S_L, S_R) *satisfying the following condition:*

($*$) *For every object* $C \in \mathcal{C}$ *and every morphism* $\alpha : p(C) \to D$ *in* $\mathcal{D}$ *which belongs to* S_L, *there exists a* p-*coCartesian morphism* $\overline{\alpha} : C \to \overline{D}$ *lifting* α.

Let $\overline{S}_L$ *denote the collection of all* p-*coCartesian morphisms* $\overline{\alpha}$ *in* $\mathcal{C}$ *such that* $p(\overline{\alpha}) \in S_L$, *and let* $\overline{S}_R = p^{-1} S_R$. *Then* $(\overline{S}_L, \overline{S}_R)$ *is a factorization system on* $\mathcal{C}$.

Proof. We will prove that $(\overline{S}_L, \overline{S}_R)$ satisfies conditions (1) through (3) of Definition T.5.2.8.8:

(1) The collections $\overline{S}_L$ and $\overline{S}_R$ are stable under retracts. This follows from the stability of S_L and S_R under retracts, together with the observation that the collection of p-coCartesian morphisms is stable under retracts.

(2) Every morphism in $\overline{S}_L$ is left orthogonal to every morphism in $\overline{S}_R$. To prove this, let $\overline{\alpha} : \overline{A} \to \overline{B}$ belong to $\overline{S}_L$ and $\overline{\beta} : \overline{X} \to \overline{Y}$ belong to $\overline{S}_R$. Let $\alpha : A \to B$ and $\beta : X \to Y$ denote the images of $\overline{\alpha}$ and $\overline{\beta}$ under the functor p. We wish to prove that the space $\mathrm{Map}_{\mathcal{C}_{\overline{A}//\overline{Y}}}(\overline{B}, \overline{X})$ is contractible. Using the fact that $\overline{\alpha}$ is p-coCartesian, we deduce that the map $\mathrm{Map}_{\mathcal{C}_{\overline{A}//\overline{Y}}}(\overline{B}, \overline{X}) \to \mathrm{Map}_{\mathcal{D}_{A//Y}}(B, X)$ is a trivial Kan fibration. The desired result now follows from the fact that $\alpha \in S_L$ is left orthogonal to $\beta \in S_R$.

(3) Every morphism $\overline{\alpha} : \overline{X} \to \overline{Z}$ admits a factorization $\overline{\alpha} = \overline{\beta} \circ \overline{\gamma}$, where $\overline{\gamma} \in \overline{S}_L$ and $\overline{\beta} \in \overline{S}_R$. To prove this, let $\alpha : X \to Z$ denote the image of α under p. Using the fact that (S_L, S_R) is a factorization system on $\mathcal{D}$, we deduce that α fits into a commutative diagram

$$
\begin{array}{ccc}
 & Y & \\
\gamma \nearrow & & \searrow \beta \\
X & \xrightarrow{\quad \alpha \quad} & Z
\end{array}
$$

where $\gamma \in S_L$ and $\beta \in S_R$. Using assumption ($*$), we can lift γ to a p-coCartesian morphism $\overline{\gamma} \in \overline{S}_R$. Using the fact that $\overline{\gamma}$ is p-coCartesian, we can lift the above diagram to a commutative triangle

$$
\begin{array}{ccc}
 & \overline{Y} & \\
\overline{\gamma} \nearrow & & \searrow \overline{\beta} \\
\overline{X} & \xrightarrow{\quad \overline{\alpha} \quad} & \overline{Z}
\end{array}
$$

in $\mathcal{C}$ having the desired properties.

$\square$

Remark 2.1.2.6. Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad, and suppose we are given a collection of inert morphisms $\{f_i : X \to X_i\}_{1 \le i \le m}$ in $\mathcal{O}^{\otimes}$ covering maps $\langle n \rangle \to \langle n_i \rangle$ in $\mathcal{F}\mathrm{in}_*$ which induce a bijection $\coprod_{1 \le i \le m} \langle n_i \rangle^{\circ} \to \langle n \rangle^{\circ}$. These morphisms determine a p-limit diagram $q : (\langle m \rangle^{\circ})^{\triangleleft} \to \mathcal{O}^{\otimes}$. To prove this, choose inert morphisms $g_{i,j} : X_i \to X_{i,j}$ covering the maps $\rho^j : \langle n_i \rangle \to \langle 1 \rangle$ for $1 \le j \le n_i$. Using the fact that $\mathcal{O}^{\otimes}$ is an ∞-category, we obtain a diagram $\overline{q} : (\coprod_{1 \le i \le m} (\langle n_i \rangle^{\circ})^{\triangleleft})^{\triangleleft} \to \mathcal{O}^{\otimes}$. Since the inclusion $\langle m \rangle^{\circ} \subseteq \coprod_{1 \le i \le m} \langle n_i \rangle^{\circ \triangleright}$ is left cofinal, it will suffice to show that $\overline{q}$ is a p-limit diagram. Using the assumption that $\mathcal{O}^{\otimes}$ is an ∞-operad, we deduce that $\overline{q}|(\coprod_{1 \le i \le m} (\langle n_i \rangle^{\circ})^{\triangleleft})$ is a p-right Kan extension of $\overline{q}|\langle n \rangle^{\circ}$. According to Lemma T.4.3.2.7, it will suffice to show that $\overline{q}|\langle n \rangle^{\circ \triangleleft}$ is a p-limit diagram, which again follows from the assumption that $\mathcal{O}^{\otimes}$ is an ∞-operad.

We are now ready to discuss a notion of map between ∞-operads.

Definition 2.1.2.7. Let $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ be ∞-operads. An ∞-*operad map* from $\mathcal{O}^{\otimes}$ to $\mathcal{O}'^{\otimes}$ is a map of simplicial sets $f : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ satisfying the following conditions:

(1) The diagram

$$\mathcal{O}^{\otimes} \xrightarrow{\quad f \quad} \mathcal{O}'^{\otimes}$$
$$\mathrm{N}(\mathcal{F}\mathrm{in}_*)$$

commutes.

(2) The functor f carries inert morphisms in $\mathcal{O}^{\otimes}$ to inert morphisms in $\mathcal{O}'^{\otimes}$.

We let $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$ denote the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes})$ spanned by the ∞-operad maps.

Warning 2.1.2.8. The notation $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$ is somewhat abusive, since it depends on the ∞-operads $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ (and their maps to $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$) and not only on the underlying ∞-categories $\mathcal{O}$ and $\mathcal{O}'$.

Remark 2.1.2.9. Let $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ be ∞-operads, and let $F : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ be a functor satisfying condition (1) of Definition 2.1.2.7. Then F preserves all inert morphisms if and only if it preserves inert morphisms in $\mathcal{O}^{\otimes}$ lying over the maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$. For suppose that $X \to Y$ is an inert morphism in $\mathcal{O}^{\otimes}$ lying over $\beta : \langle m \rangle \to \langle n \rangle$; we wish to prove that the induced map $\beta_! F(X) \to F(Y)$ is an equivalence. Since $\mathcal{O}'^{\otimes}$ is an ∞-operad, it suffices to show that the induced map $\rho^i_! \beta_! F(X) \to \rho^i_! F(Y)$ is an equivalence in $\mathcal{O}'$ for $1 \leq i \leq n$. Our hypothesis allows us to identify this with the morphism $F(\rho^i_! \beta_! X) \to F(\rho^i_! Y)$, which is an equivalence since $Y \simeq \beta_! X$.

Definition 2.1.2.10. We will say that a map of ∞-operads $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a *fibration of ∞-operads* if q is a categorical fibration.

Remark 2.1.2.11. Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, and suppose we are given a collection of inert morphisms $\{f_i : X \to X_i\}_{1 \leq i \leq m}$ in $\mathcal{C}^{\otimes}$ covering maps $\langle n \rangle \to \langle n_i \rangle$ in $\mathcal{F}\mathrm{in}_*$ which induce a bijection $\coprod_{1 \leq i \leq m} \langle n_i \rangle^{\circ} \to \langle n \rangle^{\circ}$. Then these morphisms determine a p-limit diagram $q : \langle m \rangle^{\circ \triangleleft} \to \mathcal{C}^{\otimes}$. This follows from Remark 2.1.2.6 and Proposition T.4.3.1.5.

The following result describes an important special class of ∞-operad fibrations:

Proposition 2.1.2.12. *Let $\mathcal{O}^{\otimes}$ be an ∞-operad, and let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration. The following conditions are equivalent:*

(a) *The composite map $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ exhibits $\mathcal{C}^{\otimes}$ as an ∞-operad.*

(b) *For every object $T \simeq T_1 \oplus \cdots \oplus T_n \in \mathcal{O}^{\otimes}_{\langle n \rangle}$, the inert morphisms $T \to T_i$ induce an equivalence of ∞-categories $\mathcal{C}^{\otimes}_T \to \prod_{1 \leq i \leq n} \mathcal{C}^{\otimes}_{T_i}$.*

Proof. Suppose that (a) is satisfied. We first claim that p preserves inert morphisms. To prove this, choose an inert morphism $f : C \to C'$ in $\mathcal{C}^{\otimes}$. Let $g : p(C) \to X$ be an inert morphism in $\mathcal{O}^{\otimes}$ lifting $q(f)$, and let $\bar{g} : C \to \overline{X}$ be a p-coCartesian lift of g. It follows from Proposition T.2.4.1.3 that $\bar{g}$ is a q-coCartesian lift of $q(f)$ and therefore equivalent to f. We conclude that $p(f)$ is equivalent to $p(\bar{g}) = g$ and is therefore inert.

The above argument guarantees that for each $n \geq 0$, the maps $\{\rho^i\}_{1 \leq i \leq n}$ determine a homotopy commutative diagram

$$\begin{array}{ccc} \mathcal{C}^{\otimes}_{\langle n \rangle} & \longrightarrow & \mathcal{O}^{\otimes}_{\langle n \rangle} \\ \downarrow & & \downarrow \\ \mathcal{C}^n & \longrightarrow & \mathcal{O}^n \end{array}$$

and the assumption that $\mathcal{C}^{\otimes}$ and $\mathcal{O}^{\otimes}$ are ∞-operads guarantees that the vertical maps are categorical equivalences. Let T be an object of $\mathcal{O}^{\otimes}_{\langle n \rangle}$. Passing to the homotopy fibers over the vertices T and $(T_i)_{1 \leq i \leq n}$ (which are equivalent to the actual fibers by virtue of Corollary T.3.3.1.4), we deduce that the canonical map $\mathcal{C}^{\otimes}_T \to \prod_{1 \leq i \leq n} \mathcal{C}^{\otimes}_{T_i}$ is an equivalence, which proves (b).

Now suppose that (b) is satisfied. We will prove that the functor $q : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ satisfies the conditions of Definition 2.1.1.10. To prove (1), consider an object $C \in \mathcal{C}^{\otimes}$ and an inert morphism $\alpha : q(C) \to \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$. Since $\mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is an ∞-operad, there exists an inert morphism $\widetilde{\alpha} : p(C) \to X$ in $\mathcal{O}^{\otimes}$ lying over α. Since p is a coCartesian fibration, we can lift $\widetilde{\alpha}$ to a p-coCartesian morphism $\overline{\alpha} : C \to \overline{X}$ in $\mathcal{C}^{\otimes}$. Proposition T.2.4.1.3 implies that $\overline{\alpha}$ is p-coCartesian, which proves (1).

Now let $C \in \mathcal{C}^{\otimes}_{\langle m \rangle}$, $C' \in \mathcal{C}^{\otimes}_{\langle n \rangle}$, and $f : \langle m \rangle \to \langle n \rangle$ be as in condition (2) of Definition 2.1.1.10. Set $T = p(C)$, set $T' = p(C')$, choose inert morphisms $g_i : T' \to T'_i$ lying over ρ^i for $1 \leq i \leq n$, and choose p-coCartesian morphisms $\overline{g}_i : C' \to C'_i$ lying over g_i for $1 \leq i \leq n$. Let $f_i = \rho^i \circ f$ for $1 \leq i \leq n$. We have a homotopy coherent diagram

$$
\begin{array}{ccc}
\mathrm{Map}^f_{\mathcal{C}^{\otimes}}(C, C') & \longrightarrow & \prod_{1 \leq i \leq n} \mathrm{Map}^{f_i}_{\mathcal{C}^{\otimes}}(C, C'_i) \\
\downarrow & & \downarrow \\
\mathrm{Map}^f_{\mathcal{O}^{\otimes}}(T, T') & \longrightarrow & \prod_{1 \leq i \leq n} \mathrm{Map}^{f_i}_{\mathcal{O}^{\otimes}}(T, T'_i).
\end{array}
$$

Since $\mathcal{O}^{\otimes}$ is an ∞-operad, the bottom horizontal map is a homotopy equivalence. Consequently, to prove that the top map is a homotopy equivalence, it will suffice to show that it induces a homotopy equivalence after passing to the homotopy fiber over any $h : T \to T'$ lying over f. Let $D = h_! C$ and let $D_i = (g_i)_! D$ for $1 \leq i \leq n$. Using the assumption that p is a coCartesian fibration and Proposition T.2.4.4.2, we see that the map of homotopy fibers can be identified with

$$
\mathrm{Map}_{\mathcal{C}^{\otimes}_{T'}}(D, C) \to \prod_{1 \leq i \leq n} \mathrm{Map}_{\mathcal{C}^{\otimes}_{T_i}}(D_i, C_i),
$$

which is a homotopy equivalence by virtue of assumption (b).

We now prove (3). Fix a sequence of objects $\{C_i \in \mathcal{C}^{\otimes}_{\langle 1 \rangle}\}_{1 \leq i \leq n}$, and set $T_i = p(C_i)$. Since $\mathcal{O}^{\otimes}$ is an ∞-operad, we can choose an object $T \in \mathcal{O}^{\otimes}$ equipped with inert morphisms $f_i : T \to T_i$ lifting ρ^i for $1 \leq i \leq n$. Invoking (b), we conclude that there exists an object C together with p-coCartesian morphisms $\overline{f}_i : C \to C_i$ lifting f_i. It follows from Proposition T.2.4.1.3 that each $\overline{f}_i$ is q-coCartesian, so that condition (3) is satisfied and the proof is complete. $\qquad\square$

Definition 2.1.2.13. Let $\mathcal{O}^{\otimes}$ be an ∞-operad. We will say that a map $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a *coCartesian fibration of ∞-operads* if it satisfies the hypotheses of Proposition 2.1.2.12. In this case, we also say that p *exhibits* $\mathcal{C}^{\otimes}$ *as a $\mathcal{O}$-monoidal ∞-category*.

Remark 2.1.2.14. Let $\mathcal{O}^{\otimes}$ be an ∞-operad and let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads. Then p is a map of ∞-operads: this follows from the first step of the proof of Proposition 2.1.2.12. Combining this observation with Proposition T.3.3.1.7, we conclude that p is a fibration of ∞-operads in the sense of Definition 2.1.2.10.

Remark 2.1.2.15. In the situation of Definition 2.1.2.13, we will generally denote the fiber product $\mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{O}$ by $\mathcal{C}$, and abuse terminology by saying that $\mathcal{C}$ *is an $\mathcal{O}$-monoidal ∞-category*.

Remark 2.1.2.16. Let $\mathcal{O}^{\otimes}$ be an ∞-operad and let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads. For every object $X \in \mathcal{O}^{\otimes}$, we let $\mathcal{C}^{\otimes}_X$ denote the inverse image of T under p. If $X \in \mathcal{O}$, we will also denote this ∞-category by $\mathcal{C}_X$. Note that if $X \in \mathcal{O}^{\otimes}_{\langle n \rangle}$ corresponds to a sequence of objects $\{X_i\}_{1 \leq i \leq n}$ in $\mathcal{O}$, then we have a canonical equivalence $\mathcal{C}^{\otimes}_X \simeq \prod_{1 \leq i \leq n} \mathcal{C}_{X_i}$.

Given a morphism $f \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{1 \leq i \leq n}, Y)$ in $\mathcal{O}$, the coCartesian fibration p determines a functor

$$\prod_{1 \leq i \leq n} \mathcal{C}_{X_i} \simeq \mathcal{C}_X^{\otimes} \to \mathcal{C}_Y$$

which is well-defined up to equivalence; we will sometimes denote this functor by $\otimes_f$.

Remark 2.1.2.17. Let $\mathcal{O}^{\otimes}$ be an ∞-operad, and let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a $\mathcal{O}$-monoidal ∞-category. Then the underlying map $\mathcal{C} \to \mathcal{O}$ is a coCartesian fibration of ∞-categories, which is classified by a functor $\chi : \mathcal{O} \to \mathrm{Cat}_{\infty}$. In other words, we can think of a $\mathcal{O}$-monoidal ∞-category as assigning to each color $X \in \mathcal{O}$ an ∞-category $\chi(X)$ of X-*colored objects*. We will later see that χ can be extended to a map of ∞-operads, and that this map of ∞-operads determines $\mathcal{C}^{\otimes}$ up to equivalence (Example 2.4.2.4 and Proposition 2.4.2.5).

Example 2.1.2.18. A symmetric monoidal ∞-category is an ∞-category $\mathcal{C}^{\otimes}$ equipped with a coCartesian fibration of ∞-operads $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ (Definition 2.0.0.7).

Remark 2.1.2.19. In other words, a symmetric monoidal ∞-category is a coCartesian fibration $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ which induces equivalences of ∞-categories $\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq \mathcal{C}^n$ for each $n \geq 0$, where $\mathcal{C}$ denotes the ∞-category $\mathcal{C}^{\otimes}_{\langle 1 \rangle}$.

Remark 2.1.2.20. Let $\mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a symmetric monoidal ∞-category. We will refer to the fiber $\mathcal{C}^{\otimes}_{\langle 1 \rangle}$ as the *underlying ∞-category* of $\mathcal{C}^{\otimes}$; it will often be denoted by $\mathcal{C}$. We will sometimes abuse terminology by referring to $\mathcal{C}$ as a symmetric monoidal ∞-category, or say that $\mathcal{C}^{\otimes}$ determines a symmetric monoidal structure on $\mathcal{C}$.

Using the constructions of Remark 2.1.2.16, we see that the active morphisms $\alpha : \langle 0 \rangle \to \langle 1 \rangle$ and $\beta : \langle 2 \rangle \to \langle 1 \rangle$ determine functors

$$\Delta^0 \to \mathcal{C} \qquad \mathcal{C} \times \mathcal{C} \to \mathcal{C},$$

which are well-defined up to a contractible space of choice. The first of these functors determines an object of $\mathcal{C}$ which we will denote by $\mathbf{1}$ (or sometimes $\mathbf{1}_{\mathcal{C}}$ if we wish to emphasize the dependence on $\mathcal{C}$) and refer to as the *unit object* of $\mathcal{C}$.

It is not difficult to verify that the unit object $\mathbf{1} \in \mathcal{C}$ and the tensor product $\otimes$ on $\mathcal{C}$ satisfy all of the usual axioms for a symmetric monoidal category up to homotopy. In particular, these operations endow the homotopy category $h\mathcal{C}$ with a symmetric monoidal structure.

Example 2.1.2.21. Let $\mathcal{C}$ be a symmetric monoidal category (see [99]), so that we can regard $\mathcal{C}$ as a colored operad as in Example 2.1.1.5. The ∞-operad $\mathrm{N}(\mathcal{C}^{\otimes})$ of Example 2.1.1.21 is a symmetric monoidal ∞-category (whose underlying ∞-category is the nerve $\mathrm{N}(\mathcal{C})$).

We conclude this section with a useful criterion for detecting ∞-operad fibrations:

Proposition 2.1.2.22. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a map of ∞-operads which is an inner fibration. The following conditions are equivalent:*

(1) *The map q is a fibration of ∞-operads.*

(2) *For every object $C \in \mathcal{C}^{\otimes}$ and every inert morphism $f : q(C) \to X$ in $\mathcal{O}^{\otimes}$, there exists an inert morphism $\overline{f} : C \to \overline{X}$ in $\mathcal{C}^{\otimes}$ such that $f = q(\overline{f})$.*

Moreover, if these conditions are satisfied, then the inert morphisms of $\mathcal{C}^{\otimes}$ are precisely the q-coCartesian morphisms in $\mathcal{C}^{\otimes}$ whose image in $\mathcal{O}^{\otimes}$ is inert.

Proof. According to Corollary T.2.4.6.5, condition (1) is satisfied if and only if condition (2) is satisfied whenever f is an equivalence; this proves that (2) $\Rightarrow$ (1). For the reverse implication, suppose that q is a categorical fibration and let $f : q(C) \to X$ be as in (2). Let $f_0 : \langle m \rangle \to \langle n \rangle$ denote the image of f in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$,

and let $\overline{f}' : C \to \overline{X}'$ be an inert morphism in $\mathcal{C}^{\otimes}$ lifting f_0. Since f and $q(\overline{f}')$ are both inert lifts of the morphism f_0, we can find a 2-simplex

$$
\begin{array}{ccc}
 & X' & \\
 q(\overline{f}') \nearrow & & \searrow g \\
 q(C) \xrightarrow{\quad f \quad} & & X
\end{array}
$$

in $\mathcal{O}^{\otimes}$ where g is an equivalence. Since q is a categorical fibration, Corollary T.2.4.6.5 guarantees that we can choose an equivalence $\overline{g} : \overline{X}' \to \overline{X}$ in $\mathcal{C}^{\otimes}$ lifting g. Using the fact that q is an inner fibration, we can lift the above diagram to a 2-simplex

$$
\begin{array}{ccc}
 & \overline{X}' & \\
 \overline{f}' \nearrow & & \searrow \overline{g} \\
 C \xrightarrow{\quad \overline{f} \quad} & & \overline{X}
\end{array}
$$

in $\mathcal{C}^{\otimes}$. Since $\overline{f}$ is the composition of an inert morphism with an equivalence, it is an inert morphism which lifts f, as desired. This completes the proof of the implication $(1) \Rightarrow (2)$; the final assertion follows immediately from Proposition T.2.4.1.3 (and the assumption that q preserves inert morphisms). $\qquad\square$

2.1.3 Algebra Objects

In this section, we will undertake a more systematic study of maps between ∞-operads. We begin by introducing a variation on Definition 2.1.2.7.

Definition 2.1.3.1. Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, and suppose we are given ∞-operads $\alpha : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$. We let $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$ spanned by the maps of ∞-operads. Equivalently, $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ can be described as the fiber over the vertex α of the categorical fibration $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}'}(\mathcal{O})$ given by composition with p. In the special case where $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes}$ and α is the identity map, we will denote the ∞-category $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ by $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$.

In the special case where $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes} = \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, we will denote the ∞-category $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) = \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) = \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ by $\mathrm{CAlg}(\mathcal{C})$. We will refer to $\mathrm{CAlg}(\mathcal{C})$ as the ∞-*category of commutative algebra objects of* $\mathcal{C}$.

Remark 2.1.3.2. In the situation of Definition 2.1.3.1, we will sometimes abuse terminology by referring to $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ as the ∞-category of $\mathcal{O}'$-algebra objects of $\mathcal{C}$. Note that this ∞-category is generally not equivalent to the ∞-category $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$ of Definition 2.1.2.7 (except in the special case $\mathcal{O}^{\otimes} = \mathrm{N}(\mathcal{F}\mathrm{in}_*)$).

Example 2.1.3.3. Let $\mathcal{C}$ be a symmetric monoidal category, and let us regard $\mathrm{N}(\mathcal{C})$ as a symmetric monoidal ∞-category via the construction described in Example 2.1.2.21. Then $\mathrm{CAlg}(\mathrm{N}(\mathcal{C}))$ can be identified with the nerve of the category of *commutative algebra objects* of $\mathcal{C}$: that is, objects $A \in \mathcal{C}$ equipped with a unit map $1_{\mathcal{C}} \to A$ and a multiplication $A \otimes A \to A$ which is commutative, unital, and associative.

Remark 2.1.3.4. Let $\mathcal{O}^{\otimes}$ be an ∞-operad, let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads, and let K be an arbitrary simplicial set. Then the induced map

$$
\mathrm{Fun}(K, \mathcal{C}^{\otimes}) \times_{\mathrm{Fun}(K, \mathcal{O}^{\otimes})} \mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes}
$$

is again a coCartesian fibration of ∞-operads. For every object $X \in \mathcal{O}$, we have a canonical isomorphism $\mathcal{D}_X \simeq \mathrm{Fun}(K, \mathcal{C}_X)$, and the operations $\otimes_f$ of Remark 2.1.2.16 are computed pointwise. We have a canonical isomorphism

$$
\mathrm{Alg}_{/\mathcal{O}}(\mathcal{D}) \simeq \mathrm{Fun}(K, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})).
$$

Example 2.1.3.5. Let $q : \mathcal{O}^{\otimes} \to N(\mathcal{F}in_*)$ be an ∞-operad such that the image of q is contained in $\mathcal{T}riv^{\otimes} \subseteq N(\mathcal{F}in_*)$, and let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a $\mathcal{O}$-monoidal ∞-category. We observe that a section $A : \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ of p is a $\mathcal{O}$-algebra if and only if A is a p-right Kan extension of its restriction to $\mathcal{O} \subseteq \mathcal{O}^{\otimes}$. Using Proposition T.4.3.2.15, we deduce that the restriction map $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ is a trivial Kan fibration. In particular, taking $\mathcal{O}^{\otimes}$ to be the trivial ∞-operad $\mathcal{T}riv^{\otimes}$, we obtain a trivial Kan fibration $\mathrm{Alg}_{\mathcal{T}riv}(\mathcal{C}) \to \mathcal{C}$.

Remark 2.1.3.6. Let $\mathcal{O}^{\otimes}$ be an ∞-operad. Since $\mathcal{T}riv^{\otimes}$ can be identified with a simplicial subset of $N(\mathcal{F}in_*)$, an ∞-operad map $p : \mathcal{O}^{\otimes} \to \mathcal{T}riv^{\otimes}$ is unique if it exists. If this condition is satisfied, then p is automatically a coCartesian fibration.

Now let $\mathcal{O}^{\otimes}$ be a general ∞-operad. The ∞-category $\mathrm{Alg}_{\mathcal{T}riv}(\mathcal{O})$ can be identified with the collection of $\mathcal{T}riv$-algebras in ∞-category $\mathcal{O}^{\otimes} \times_{N(\mathcal{F}in_*)} \mathcal{T}riv^{\otimes}$. By virtue of Example 2.1.3.5, we deduce that evaluation at $\langle 1 \rangle$ induces a trivial Kan fibration $\mathrm{Alg}_{\mathcal{T}riv}(\mathcal{O}) \to \mathcal{O}$.

Suppose that $p : \mathcal{C}^{\otimes} \to N(\mathcal{F}in_*)$ and $q : \mathcal{D}^{\otimes} \to N(\mathcal{F}in_*)$ are symmetric monoidal ∞-categories. An ∞-operad map $F \in \mathrm{Alg}_{\mathcal{C}}(\mathcal{D})$ can be thought of as a functor $F : \mathcal{C} \to \mathcal{D}$ which is compatible with the symmetric monoidal structures in the lax sense that we are given maps

$$F(C) \otimes F(C') \to F(C \otimes C') \qquad \mathbf{1} \to F(\mathbf{1})$$

which are compatible with the commutativity and associativity properties of the tensor products on $\mathcal{C}$ and $\mathcal{D}$. Our next definition singles out a special class of ∞-category maps for which the morphisms above are required to be equivalences.

Definition 2.1.3.7. Let $\mathcal{O}^{\otimes}$ be an ∞-operad, and let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $q : \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ be coCartesian fibrations of ∞-operads. We will say that an ∞-operad map $f \in \mathrm{Alg}_{\mathcal{C}}(\mathcal{D})$ is a $\mathcal{O}$-*monoidal functor* if it carries p-coCartesian morphisms to q-coCartesian morphisms. We let $\mathrm{Fun}_{\mathcal{O}}^{\otimes}(\mathcal{C}, \mathcal{D})$ denote the full subcategories $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ spanned by the $\mathcal{O}$-monoidal functors.

In the special case where $\mathcal{O}^{\otimes} = N(\mathcal{F}in_*)$, we will denote $\mathrm{Fun}_{\mathcal{O}}^{\otimes}(\mathcal{C}, \mathcal{D})$ by $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$; we will refer to objects of $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ as *symmetric monoidal functors* from $\mathcal{C}^{\otimes}$ to $\mathcal{D}^{\otimes}$.

Remark 2.1.3.8. Let $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a $\mathcal{O}$-monoidal functor between $\mathcal{O}$-monoidal ∞-categories, where $\mathcal{O}^{\otimes}$ is an ∞-operad. Using Corollary T.2.4.4.4, we deduce that the following conditions are equivalent:

(1) The functor F is an equivalence.

(2) The underlying map of ∞-categories $\mathcal{C} \to \mathcal{D}$ is an equivalence.

(3) For every object $X \in \mathcal{O}$, the induced map of fibers $\mathcal{C}_X \to \mathcal{D}_X$ is an equivalence.

The analogous statement fails if we assume only that $F \in \mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})$.

We conclude this section by considering a variant on Example 2.1.3.5, which can be used to describe algebras over the ∞-operad $\mathbb{E}_0^{\otimes}$ of Example 2.1.1.19.

Proposition 2.1.3.9. *Let* $p : \mathcal{O}^{\otimes} \to \mathbb{E}_0^{\otimes}$ *be a fibration of ∞-operads, and consider the map* $\Delta^1 \to \mathbb{E}_0^{\otimes}$ *corresponding to the unique morphism* $\alpha : \langle 0 \rangle \to \langle 1 \rangle$ *of* $\mathcal{F}in_*^{\mathrm{inj}} \subseteq \mathcal{F}in_*$. *Then the restriction functor*

$$\theta : \mathrm{Alg}_{/\mathbb{E}_0}(\mathcal{O}) \to \mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\Delta^1, \mathcal{O}^{\otimes})$$

is a trivial Kan fibration.

Remark 2.1.3.10. Suppose that $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category with unit object $\mathbf{1}$. Proposition 2.1.3.9 implies that we can identify $\mathrm{Alg}_{/\mathbb{E}_0}(\mathcal{C})$ with the ∞-category $\mathcal{C}_{\mathbf{1}/}$ consisting of maps $\mathbf{1} \to A$ in $\mathcal{C}$. In other words, an $\mathbb{E}_0$-algebra object of $\mathcal{C}$ is an object $A \in \mathcal{C}$ equipped with a unit map $\mathbf{1} \to A$, but no other additional structures.

Proof. Since p is a categorical fibration, the map θ is also a categorical fibration. The morphism α determines a functor $s : [1] \to \mathcal{F}\mathrm{in}_*^{\mathrm{inj}}$. Let $\mathcal{J}$ denote the categorical mapping cylinder of s. More precisely, we define the category $\mathcal{J}$ as follows:

- An object of $\mathcal{J}$ is either an object $\langle n \rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{inj}}$ or an object $i \in [1]$.

- Morphisms in $\mathcal{J}$ are described by the formulas

$$\mathrm{Hom}_{\mathcal{J}}(\langle m \rangle, \langle n \rangle) = \mathrm{Hom}_{\mathcal{F}\mathrm{in}_*^{\mathrm{inj}}}(\langle m \rangle, \langle n \rangle) \qquad \mathrm{Hom}_{\mathcal{J}}(\langle m \rangle, i) = \mathrm{Hom}_{\mathcal{F}\mathrm{in}_*^{\mathrm{inj}}}(\langle m \rangle, s(i))$$

$$\mathrm{Hom}_{\mathcal{J}}(i, j) = \mathrm{Hom}_{[1]}(i, j) \qquad\qquad \mathrm{Hom}_{\mathcal{J}}(i, \langle m \rangle) = \emptyset.$$

Note that there is a canonical retraction r of $\mathrm{N}(\mathcal{J})$ onto the simplicial subset $\mathbb{E}_0^{\otimes} = \mathrm{N}(\mathcal{F}\mathrm{in}_*^{\mathrm{inj}})$. We let $\mathcal{C}$ denote the full subcategory of $\mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\mathrm{N}(\mathcal{J}), \mathcal{O}^{\otimes})$ spanned by those functors F satisfying the following conditions:

(a) For $i \in [1]$, the morphism $F(s(i)) \to F(i)$ is an equivalence in $\mathcal{O}^{\otimes}$.

(b) The restriction $F | \mathbb{E}_0$ belongs to $\mathrm{Alg}_{/\mathbb{E}_0}(\mathcal{C})$.

We observe that a functor F satisfies condition $(*)$ if and only if it is a left Kan extension of $F | \mathbb{E}_0$. Using Proposition T.4.3.2.15, we deduce that the restriction map $\phi : \mathcal{C} \to \mathrm{Alg}_{\mathbb{E}_0}(\mathcal{C})$ is a trivial Kan fibration. Let $\theta' : \mathcal{C} \to \mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\Delta^1, \mathcal{O}^{\otimes})$ be the map given by restriction to $\Delta^1 \simeq \mathrm{N}([1])$. The map θ can be written as a composition

$$\mathrm{Alg}_{/\mathbb{E}_0}(\mathcal{O}) \to \mathcal{C} \xrightarrow{\theta'} \mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\Delta^1, \mathcal{O}^{\otimes}),$$

where the first map is the section of ϕ given by composition with r (and therefore a categorical equivalence). By a two-out-of-three argument, we are reduced to proving that θ' is a categorical equivalence.

We will show that θ' is a trivial Kan fibration. By virtue of Proposition T.4.3.2.15, it will suffice to prove the following:

(i) A functor $\mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\mathrm{N}(\mathcal{J}), \mathcal{O}^{\otimes})$ belongs to $\mathcal{C}$ if and only if F is a p-right Kan extension of $F | \Delta^1$.

(ii) Every functor $F_0 \in \mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\Delta^1, \mathcal{O}^{\otimes})$ admits a p-right Kan extension $F \in \mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\mathrm{N}(\mathcal{J}), \mathcal{O}^{\otimes})$.

To see this, fix an object $F \in \mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\mathrm{N}(\mathcal{J}), \mathcal{O}^{\otimes})$ and consider an object $\langle n \rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{inj}}$. Let $\mathcal{I} = \mathcal{J}_{\langle n \rangle /} \times_{\mathcal{J}} [1]$, let $\mathcal{I}_0$ denote the full subcategory of $\mathcal{I}$ obtained by omitting the map $\langle n \rangle \to 1$ corresponding to the nonsurjective map $\langle n \rangle \to \langle 1 \rangle$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{inj}}$, and let $\mathcal{I}_1$ denote the full subcategory of $\mathcal{I}_0$ obtained by omitting the unique map $\langle n \rangle \to 0$. We observe that the inclusion $\mathrm{N}(\mathcal{I}_0) \subseteq \mathrm{N}(\mathcal{I})$ is right cofinal and that the restriction $F | \mathrm{N}(\mathcal{I}_0)$ is a p-right Kan extension of $F | \mathrm{N}(\mathcal{I}_1)$. Using Lemma T.4.3.2.7, we deduce that F is a p-right Kan extension of $F | \Delta^1$ at $\langle n \rangle$ if and only if the induced map

$$\mathrm{N}(\mathcal{I}_1)^{\triangleleft} \to \mathrm{N}(\mathcal{J}) \xrightarrow{F} \mathcal{O}^{\otimes}$$

is a p-limit diagram. Combining this with the observation that $\mathcal{I}_1$ is a discrete category (whose objects can be identified with the elements of $\langle n \rangle^{\circ}$), we obtain the following version of (i):

(i') A functor $F \in \mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\mathrm{N}(\mathcal{J}), \mathcal{O}^{\otimes})$ is a p-right Kan extension of $F | \Delta^1$ if and only if, for every nonnegative integer n, the maps $F(\rho^i) : F(\langle n \rangle) \to F(1)$ exhibit $F(\langle n \rangle)$ as a p-product of the objects $\{F(1)\}_{1 \le i \le n}$.

Since $\mathcal{O}^{\otimes}$ is an ∞-operad, condition (i') is equivalent to the requirement that each of the maps $F(\rho^i) : F(\langle n \rangle) \to F(1)$ is inert.

We now prove (i). Suppose first that F is a p-right Kan extension of $F | \Delta^1$. We will show that $F \in \mathcal{C}$. We first show that the map $F(s(i)) \to F(i)$ is an equivalence in $\mathcal{O}^{\otimes}$ for $i \in [1]$. If $i = 0$ this is clear (the ∞-category $\mathcal{O}_{\langle 0 \rangle}^{\otimes}$ is a contractible Kan complex, so every morphism in $\mathcal{O}_{\langle 0 \rangle}^{\otimes}$ is an equivalence). If $i = 1$, we

apply condition (i') in the case $n = 1$. Using this observation, we see that the condition of (i') is equivalent to the requirement that the maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$ induce inert morphisms $F(\langle n \rangle) \to F(\langle 1 \rangle)$ for $1 \leq i \leq n$, so that $F|\mathbb{E}_0^{\otimes} \in \mathrm{Alg}_{\mathbb{E}_0}(\mathcal{O})$. This proves that $F \in \mathcal{C}$ as desired.

Conversely, suppose that $F \in \mathcal{C}$. If $1 \leq i \leq n$, then the map $F(\rho^i) : F(\langle n \rangle) \to F(1)$ factors as a composition

$$F(\langle n \rangle) \to F(\langle 1 \rangle) \to F(1).$$

The first map is inert by virtue of our assumption that $F|\mathbb{E}_0^{\otimes} \in \mathrm{Alg}_{/\mathbb{E}_0}(\mathcal{O})$, and the second map is an equivalence since F satisfies condition (a). It follows that F satisfies the hypothesis of (i') and is therefore a p-right Kan extension of $F|\Delta^1$. This completes the proof of (i).

The proof of (ii) is similar: using Lemma T.4.3.2.7, we can reduce to showing that for each $n \geq 0$ the composite map

$$\mathrm{N}(\mathcal{I}_1) \to \{1\} \subseteq \Delta^1 \xrightarrow{F_0} \mathcal{O}^{\otimes}$$

can be extended to a p-limit diagram (lying over the canonical map $\mathrm{N}(\mathcal{I}_1)^{\triangleleft} \to \mathrm{N}(\mathcal{J}) \to \mathbb{E}_0^{\otimes}$). The existence of such a diagram follows immediately from Proposition 2.1.2.12. $\qquad\square$

2.1.4 ∞-Preoperads

The collection of all ∞-operads can be organized into a simplicial category $\mathrm{Op}_{\infty}^{\Delta}$, which we define as follows:

- The objects of $\mathrm{Op}_{\infty}^{\Delta}$ are (small) ∞-operads $\mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

- For every pair of ∞-operads $\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes} \in \mathrm{Op}_{\infty}^{\Delta}$, we define $\mathrm{Map}_{\mathrm{Op}_{\infty}^{\Delta}}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes})$ to be the Kan complex $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')^{\simeq}$ of ∞-operad maps from $\mathcal{O}^{\otimes}$ to $\mathcal{O}'^{\otimes}$.

Definition 2.1.4.1. We define Op_{∞} to be the nerve of the simplicial category $\mathrm{Op}_{\infty}^{\Delta}$. We will refer to Op_{∞} as the ∞-*category of ∞-operads*.

Because the mapping spaces in $\mathrm{Op}_{\infty}^{\Delta}$ are Kan complexes, the simplicial set Op_{∞} is an ∞-category (Proposition T.1.1.5.10). Our goal in this section is to study the ∞-category Op_{∞}. We will show that Op_{∞} is a presentable ∞-category: in particular, it admits small limits and colimits. To prove this, we will exhibit Op_{∞} as the underlying ∞-category of a combinatorial simplicial model category $\mathcal{P}\mathrm{Op}_{\infty}$, which we call the *category of ∞-preoperads*.

Recall that a *marked simplicial set* is a pair (X, M), where X is a simplicial set and M is a collection of edges of X, which contains all degenerate edges (see §T.3.1). If X is a simplicial set, we let $X^{\sharp}$ denote the marked simplicial set (X, M) where M consists of all edges of X, and $X^{\flat}$ the marked simplicial set (X, M) where M consists of all the degenerate edges of X. The category of marked simplicial sets will be denoted by $\mathcal{S}\mathrm{et}_{\Delta}^{+}$.

Definition 2.1.4.2. An ∞-*preoperad* is a marked simplicial set (X, M) equipped with a map of simplicial sets $f : X \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ with the following property: for each edge e of X which belongs to M, the image $f(e)$ is an inert morphism in $\mathcal{F}\mathrm{in}_*$.

A morphism from an ∞-preoperad (X, M) to an ∞-preoperad (Y, N) is a map of marked simplicial sets $(X, M) \to (Y, N)$ such that the diagram

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
& \searrow \quad \swarrow & \\
& \mathrm{N}(\mathcal{F}\mathrm{in}_*) &
\end{array}
$$

commutes. The collection of ∞-preoperads (and ∞-preoperad morphisms) forms a category, which we will denote by $\mathcal{P}\mathrm{Op}_{\infty}$.

Remark 2.1.4.3. The category $\mathcal{POp}_\infty$ can be identified with the overcategory $\mathcal{S}\mathrm{et}^+_{/(\mathrm{N}(\mathcal{F}\mathrm{in}_*),M)}$, where M is the collection of all inert morphisms in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

Remark 2.1.4.4. The category $\mathcal{POp}_\infty$ of ∞-preoperads is naturally tensored over simplicial sets: if $\overline{X}$ is an ∞-preoperad and K is a simplicial set, we let $\overline{X} \otimes K = \overline{X} \times K^\sharp$. This construction endows $\mathcal{POp}_\infty$ with the structure of a simplicial category.

Notation 2.1.4.5. Let $\mathcal{O}^\otimes$ be an ∞-operad. We let $\mathcal{O}^{\otimes,\natural}$ denote the ∞-preoperad $(\mathcal{O}^\otimes, M)$, where M is the collection of all inert morphisms in $\mathcal{O}^\otimes$.

Proposition 2.1.4.6. *There exists a left proper combinatorial simplicial model structure on $\mathcal{POp}_\infty$ which may be characterized as follows:*

- (C) *A morphism $f : \overline{X} \to \overline{Y}$ in $\mathcal{POp}_\infty$ is a cofibration if and only if it induces a monomorphism between the underlying simplicial sets of $\overline{X}$ and $\overline{Y}$.*

- (W) *A morphism $f : \overline{X} \to \overline{Y}$ in $\mathcal{POp}_\infty$ is a weak equivalence if and only if, for every ∞-operad $\mathcal{O}^\otimes$, the induced map*
$$\mathrm{Map}_{\mathcal{POp}_\infty}(\overline{Y}, \mathcal{O}^{\otimes,\natural}) \to \mathrm{Map}_{\mathcal{POp}_\infty}(\overline{X}, \mathcal{O}^{\otimes,\natural})$$
is a weak homotopy equivalence of simplicial sets.

This model structure is compatible with the simplicial structure of Remark 2.1.4.4. Moreover, if $\mathcal{O}^\otimes$ is an ∞-operad, then a map $\alpha : \overline{X} \to \mathcal{O}^{\otimes,\natural}$ in $\mathcal{POp}_\infty$ is a fibration if and only if there exists a fibration of ∞-operads $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ (see Definition 2.1.2.10) such that α is induced by an equivalence $\overline{X} \simeq \mathcal{C}^{\otimes,\natural}$. In particular (taking $\mathcal{O}^\otimes = \mathrm{N}(\mathcal{F}\mathrm{in}_*)$), we deduce that an object $\overline{X} \in \mathcal{POp}_\infty$ is fibrant if and only if it has the form $\mathcal{C}^{\otimes,\natural}$, for some ∞-operad $\mathcal{C}^\otimes$.

Proposition 2.1.4.6 can be deduced from much more general results which we will prove in §B.4. In what follows, we will freely use the terminology developed in the appendix.

Proof of Proposition 2.1.4.6. Let $\mathfrak{P} = (M, T, \{p_\alpha : \Lambda_0^2 \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)\}_{\alpha \in A})$ be the categorical pattern on $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ where M consists of all inert morphisms, T the collection of all 2-simplices, and A ranges over all diagrams
$$\langle p \rangle \leftarrow \langle n \rangle \to \langle q \rangle$$
where the maps are inert and determine a bijection $\langle p \rangle^\circ \coprod \langle q \rangle^\circ \to \langle n \rangle^\circ$. The main result now follows from Theorem B.0.19, and the characterization of fibrations between fibrant objects follows from Proposition B.2.7. $\qquad\square$

Remark 2.1.4.7. We will refer to the model structure of Proposition 2.1.4.6 as the ∞-*operadic model structure* on $\mathcal{POp}_\infty$.

Example 2.1.4.8. The inclusion $\{\langle 1 \rangle\}^\flat \subseteq \mathcal{T}\mathrm{riv}^{\otimes,\natural}$ is a weak equivalence of ∞-preoperads. This is an immediate consequence of Example 2.1.3.5. In other words, we can view $\mathcal{T}\mathrm{riv}^{\otimes,\natural}$ as a fibrant replacement for the object $\{\langle 1 \rangle\}^\flat \in \mathcal{POp}_\infty$.

Example 2.1.4.9. Using Proposition 2.1.3.9, we deduce that the morphism $\langle 0 \rangle \to \langle 1 \rangle$ in $\mathcal{F}\mathrm{in}_*$ induces a weak equivalence of ∞-preoperads $(\Delta^1)^\flat \subseteq \mathbb{E}_0^{\otimes,\natural}$, so that $\mathbb{E}_0^{\otimes,\natural}$ can be viewed as a fibrant replacement for the ∞-preoperad $(\Delta^1)^\flat$.

Remark 2.1.4.10. The ∞-operadic model structure on $\mathcal{POp}_\infty$ induces a model structure on the category $(\mathcal{POp}_\infty)_{/\{\langle 1 \rangle\}^\flat} \simeq \mathcal{S}\mathrm{et}_\Delta^+$. Unwinding the definitions, we see that a morphism of marked simplicial sets $\alpha : (X, M) \to (X', M')$ is a cofibration (with respect to this model structure) if and only if the underlying map of simplicial sets $X \to X'$ is a monomorphism, and is a weak equivalence if and only if it induces a homotopy equivalence $\mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta^+}((X', M'), \overline{Y}) \to \mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta^+}((X, M), \overline{Y})$ whenever $\overline{Y}$ is a marked simplicial set of the form

$\mathcal{O}^{\otimes,\natural} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)^{\natural}} \{\langle 1 \rangle\}^{\flat}$, for some ∞-operad $\mathcal{O}^{\otimes}$. This is true if and only if $\overline{Y}$ has the form $(\mathcal{C}, M)$, where $\mathcal{C}$ is an ∞-category and M is the collection of all equivalences in $\mathcal{C}$ (the "only if" assertion is obvious, and the converse implication follows by taking $\mathcal{O}^{\otimes}$ to be the ∞-operad $\mathcal{C}^{\times}$ of Proposition 2.4.1.5). It follows that the induced model structure on $\mathcal{S}\mathrm{et}_{\Delta}^{+}$ coincides with the marked model structure of §T.3.1, so that we can identify the underlying ∞-category $\mathrm{N}((\mathcal{S}\mathrm{et}_{\Delta}^{+})^{\circ})$ with $\mathcal{C}\mathrm{at}_{\infty}$.

Composing with the inclusion $\{\langle 1 \rangle\}^{\flat} \hookrightarrow \mathrm{N}(\mathcal{F}\mathrm{in}_*)^{\natural}$, we obtain a left Quillen functor

$$F : \mathcal{S}\mathrm{et}_{\Delta}^{+} \simeq (\mathcal{P}\mathrm{Op}_{\infty})_{/\{\langle 1 \rangle\}^{\flat}} \to \mathcal{P}\mathrm{Op}_{\infty}.$$

This functor has a right adjoint $G : \mathcal{P}\mathrm{Op}_{\infty} \to \mathcal{S}\mathrm{et}_{\Delta}^{+}$, which assigns to an ∞-preoperad $\overline{X}$ the fiber product $\overline{X} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)^{\natural}} \{\langle 1 \rangle\}^{\flat}$. Passing to the underlying ∞-categories, we see that G induces the functor $g : \mathrm{Op}_{\infty} \to \mathcal{C}\mathrm{at}_{\infty}$ which assigns to each ∞-operad $\mathcal{O}^{\otimes}$ its underlying ∞-category $\mathcal{O}$. The left Quillen functor F induces a left adjoint $f : \mathcal{C}\mathrm{at}_{\infty} \to \mathrm{Op}_{\infty}$.

Proposition 2.1.4.11. *The functor $f : \mathcal{C}\mathrm{at}_{\infty} \to \mathrm{Op}_{\infty}$ is fully faithful. An ∞-operad $q : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ belongs to the essential image of f if and only if the functor q factors through the subcategory $\mathcal{T}\mathrm{riv}^{\otimes} \subset \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.*

Proof. Let $\mathrm{Op}_{\infty}' \subseteq \mathrm{Op}_{\infty}$ be the full subcategory spanned by those ∞-operads $q : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ such that q factors through $\mathcal{T}\mathrm{riv}^{\otimes}$. It is easy to see that the inclusion $f' : \mathrm{Op}_{\infty}' \subseteq \mathrm{Op}_{\infty}$ admits a right adjoint g', given by $\mathcal{O}^{\otimes} \mapsto \mathcal{O}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathcal{T}\mathrm{riv}^{\otimes}$. The evaluation functor $g : \mathrm{Op}_{\infty} \to \mathcal{C}\mathrm{at}_{\infty}$ factors as a composition

$$\mathrm{Op}_{\infty} \xrightarrow{g'} \mathrm{Op}_{\infty}' \xrightarrow{g''} \mathcal{C}\mathrm{at}_{\infty},$$

where $g'' = g | \mathrm{Op}_{\infty}'$. It follows that $f \simeq f' \circ f''$, where f'' is a left adjoint to g''. Note that if $q : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ belongs to Op_{∞}', then q is a coCartesian fibration; moreover, every morphism in Op_{∞}' corresponds to a commutative diagram

where T carries q-coCartesian morphisms to q'-coCartesian morphisms. Using Theorem T.3.2.0.1, we deduce that Op_{∞}' is equivalent to a full subcategory $\mathrm{Fun}'(\mathcal{T}\mathrm{riv}^{\otimes}, \mathcal{C}\mathrm{at}_{\infty}) \to \mathrm{Fun}(\mathcal{T}\mathrm{riv}^{\otimes}, \mathcal{C}\mathrm{at}_{\infty})$: namely, the full subcategory consisting of those functors $\chi : \mathcal{T}\mathrm{riv}^{\otimes} \to \mathcal{C}\mathrm{at}_{\infty}$ which are right Kan extensions of $\chi | \{\langle 1 \rangle\}$. Using Proposition T.4.3.2.15, we deduce that evaluation at $\{\langle 1 \rangle\}$ induces an equivalence of ∞-categories $\mathrm{Fun}'(\mathcal{T}\mathrm{riv}^{\otimes}, \mathcal{C}\mathrm{at}_{\infty}) \to \mathcal{C}\mathrm{at}_{\infty}$. It follows that g'' is an equivalence of ∞-categories, so that $f'' \simeq (g'')^{-1}$ is an equivalence and $f \simeq f' \circ f''$ is fully faithful. $\square$

Remark 2.1.4.12. Proposition 2.1.4.11 is equivalent to the assertion that the weak equivalence $\{\langle 1 \rangle\}^{\flat} \subseteq \mathcal{T}\mathrm{riv}^{\otimes,\natural}$ of Example 2.1.4.8 induces a left Quillen equivalence of model categories

$$(\mathcal{P}\mathrm{Op}_{\infty})_{/\{\langle 1 \rangle\}^{\flat}} \to (\mathcal{P}\mathrm{Op}_{\infty})_{/\mathcal{T}\mathrm{riv}^{\otimes,\natural}}.$$

This statement is not completely formal: the model category $\mathcal{P}\mathrm{Op}_{\infty}$ is not right proper, and the object $\{\langle 1 \rangle\}^{\flat} \in \mathcal{P}\mathrm{Op}_{\infty}$ is not fibrant.

Variant 2.1.4.13. We define a subcategory $\mathcal{C}\mathrm{at}_{\infty}^{\otimes} \subset \mathrm{Op}_{\infty}$ as follows:

- An object $\mathcal{C}^{\otimes}$ of Op_{∞} belongs to $\mathcal{C}\mathrm{at}_{\infty}^{\otimes}$ if and only if it is a symmetric monoidal ∞-category.

- A morphism $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ in Op_{∞} belongs to $\mathcal{C}\mathrm{at}_{\infty}^{\otimes}$ if and only if it is a symmetric monoidal functor.

We will refer to $\mathcal{C}\mathrm{at}_{\infty}^{\otimes}$ as the ∞-*category of symmetric monoidal ∞-categories*. Using Theorem B.0.19, we deduce that $\mathcal{C}\mathrm{at}_{\infty}^{\otimes}$ can be realized as the underlying ∞-category of the combinatorial simplicial model category $(\mathcal{S}\mathrm{et}_{\Delta}^{+})_{/\mathfrak{P}}$, where $\mathfrak{P}$ denotes the categorical pattern $(M', T, \{p_{\alpha} : \Lambda_0^2 \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)\}_{\alpha \in A})$ on the simplicial set $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, where T and $\{p_{\alpha} : \Lambda_0^2 \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)\}_{\alpha \in A}$ are defined as in the proof of Proposition 2.1.4.6, while M' consists of *all* edges of $\mathfrak{P}$.

2.2   Constructions of ∞-Operads

In this section, we will describe how various categorical constructions can be generalized to the setting of symmetric monoidal ∞-categories (or the more general setting of ∞-operads). Suppose that $\mathcal{C}$ is an ∞-category equipped with a symmetric monoidal structure, and that $\mathcal{D}$ is another ∞-category which is obtained from $\mathcal{C}$ via some natural procedure; under what conditions does $\mathcal{D}$ inherit a symmetric monoidal structure?

We will begin in §2.2.1 by considering the construction of full subcategories. If $\mathcal{C}$ is a symmetric monoidal ∞-category and S is a collection of objects of $\mathcal{C}$, then we can consider the full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ spanned by the objects of S (see §T.1.2.11). If the collection of objects S is stable under tensor products, then $\mathcal{C}_0$ inherits a symmetric monoidal structure (Proposition 2.2.1.1). However, there are other situations where the same phenomenon occurs: for example, if $\mathcal{C}_0$ is a localization of $\mathcal{C}$, then a symmetric monoidal structure on $\mathcal{C}$ sometimes yields a symmetric monoidal structure on $\mathcal{C}_0$ by passage to the quotient (Proposition 2.2.1.9).

If $\mathcal{C}$ is a symmetric monoidal ∞-category and $X \in \mathcal{C}$ is an object, then the ∞-categories $\mathcal{C}_{X/}$ and $\mathcal{C}_{/X}$ generally do not inherit symmetric monoidal structures. However, if X is a commutative algebra object of $\mathcal{C}$, then the situation is somewhat better. In §2.2.2, we will show that the overcategory $\mathcal{C}_{/X}$ inherits a symmetric monoidal structure when X is a commutative algebra object of $\mathcal{C}$. In this situation, the undercategory $\mathcal{C}_{X/}$ generally does not inherit a symmetric monoidal structure, but it can nevertheless be regarded as an ∞-operad (see Theorem 2.2.2.4). Some other categorical constructions are better behaved at the level of ∞-operads. For example, suppose we are given a pair of ∞-operad $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$, having underlying ∞-categories $\mathcal{O}$ and $\mathcal{O}'$. In §2.2.3, we will show that the disjoint union $\mathcal{O} \amalg \mathcal{O}'$ is equivalent to the underlying ∞-category of an ∞-operad which we will denote by $\mathcal{O}^{\otimes} \boxplus \mathcal{O}'^{\otimes}$. Moreover, the ∞-operad $\mathcal{O}^{\otimes} \boxplus \mathcal{O}'^{\otimes}$ plays the role of the coproduct in the setting of ∞-operads (Theorem 2.2.3.6).

In §2.2.4, we will study the interplay between the theories of symmetric monoidal ∞-categories and ∞-operads. By definition, every symmetric monoidal ∞-category $\mathcal{C}^{\otimes} \to N(\mathcal{F}in_*)$ can be regarded as an ∞-operad. The resulting forgetful functor from symmetric monoidal ∞-categories to ∞-operads has a left adjoint: that is, to every ∞-operad $\mathcal{O}^{\otimes}$, one can associate a symmetric monoidal ∞-category $\mathrm{Env}(\mathcal{O})^{\otimes}$, so that $\mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{O}, \mathcal{C})) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ for any symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ (see Proposition 2.2.4.9).

Some categorical constructions admit more than one generalization to the ∞-operadic setting. For example, the formation of Cartesian products of ∞-categories admits two natural generalizations: in addition to the Cartesian product, the ∞-category Op_{∞} admits a symmetric monoidal structure $\otimes$ which can be described informally as follows: for every pair of ∞-operads $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$, the tensor product $\mathcal{O}^{\otimes} \otimes \mathcal{O}'^{\otimes} = \mathcal{O}''^{\otimes}$ is characterized by the requirement that for every symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, we have an equivalence of ∞-categories

$$\mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}));$$

here we view $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$ as a symmetric monoidal ∞-category via pointwise tensor product. We will give a careful definition of this tensor product (and prove that it endows Op_{∞} with a symmetric monoidal structure) in §2.2.5, essentially by formalizing the universal property stated above.

Remark 2.2.0.1. Our tensor product of ∞-operads $\mathcal{O}^{\otimes} \otimes \mathcal{O}'^{\otimes}$ is a version of the Boardman-Vogt tensor product. For more details, we refer the reader to [19], [20], and [41].

2.2.1   Subcategories of $\mathcal{O}$-Monoidal ∞-Categories

Let $\mathcal{C}^{\otimes}$ be an ∞-operad, and let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory of the underlying ∞-category of $\mathcal{C}^{\otimes}$ which is stable under equivalence. In this case, we let $\mathcal{D}^{\otimes}$ denote the full subcategory of $\mathcal{C}^{\otimes}$ spanned by those objects $D \in \mathcal{C}^{\otimes}$ having the form $D_1 \oplus \cdots \oplus D_n$, where each object D_i belongs to $\mathcal{D}$. It follows immediately from the definitions that $\mathcal{D}^{\otimes}$ is again an ∞-operad, and that the inclusion $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ is a map of ∞-operads. In this section, we will consider some refinements of the preceding statement: namely, we will suppose that there exists a coCartesian fibration of ∞-operads $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$, and obtain criteria which guarantee that $p| \mathcal{D}^{\otimes}$ is again a coCartesian fibration of ∞-operads. The most obvious case to consider is that in which $\mathcal{D}$ is stable under the relevant tensor product operations on $\mathcal{C}$.

Proposition 2.2.1.1. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads, let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory which is stable under equivalence, and let $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ be defined as above. Suppose that, for every operation $f \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}, Y)$, the functor $\otimes_f : \prod_{1 \leq i \leq n} \mathcal{C}_{X_i} \to \mathcal{C}_Y$ defined in Remark 2.1.2.16 carries $\prod_{1 \leq i \leq n} \mathcal{D}_{X_i}$ into $\mathcal{D}_Y$. Then:*

(1) *The restricted map $\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ is a coCartesian fibration of ∞-operads.*

(2) *The inclusion $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ is a $\mathcal{O}$-monoidal functor.*

(3) *Suppose that, for every object $X \in \mathcal{O}$, the inclusion $\mathcal{D}_X \subseteq \mathcal{C}_X$ admits a right adjoint L_X (so that $\mathcal{D}_X$ is a colocalization of $\mathcal{C}_X$). Then there exists a commutative diagram*

$$
\begin{array}{ccc}
\mathcal{C}^{\otimes} & \xrightarrow{\;\;L^{\otimes}\;\;} & \mathcal{D}^{\otimes} \\
& {\scriptstyle p} \searrow \quad \swarrow & \\
& \mathcal{O}^{\otimes} &
\end{array}
$$

and a natural transformation of functors $\alpha : L^{\otimes} \to \mathrm{id}_{\mathcal{C}^{\otimes}}$ which exhibits $L^{\otimes}$ as a colocalization functor (see Proposition T.5.2.7.4) and such that, for every object $C \in \mathcal{C}^{\otimes}$, the image $p(\alpha(C))$ is a degenerate edge of $\mathcal{O}^{\otimes}$.

(4) *Under the hypothesis of (3), the functor $L^{\otimes}$ is a map of ∞-operads.*

Remark 2.2.1.2. Assume that $\mathcal{O}^{\otimes}$ is the commutative ∞-operad, and let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a symmetric monoidal ∞-category. A full subcategory $\mathcal{D} \subseteq \mathcal{C}$ (stable under equivalence) satisfies the hypotheses of Proposition 2.2.1.1 if and only if $\mathcal{D}$ contains the unit object of $\mathcal{C}$ and is closed under the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

Example 2.2.1.3. Let $\mathcal{O}^{\otimes}$ be an ∞-operad, and suppose we are given a coCartesian fibration of ∞-operads $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$. Assume that for each $X \in \mathcal{O}$, the ∞-category $\mathcal{C}_X$ is stable, and that for every operation $\phi \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y)$ the associated functor $\otimes_{\phi} : \prod_{i \in I} \mathcal{C}_{X_i} \to \mathcal{C}_Y$ is exact in each variable. We will say that a family of t-structures $((\mathcal{C}_X)_{\geq 0}, (\mathcal{C}_X)_{\leq 0})$ is *compatible* with p if, for each $\phi \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y)$, the functor $\otimes_{\phi}$ carries $\prod_{i \in I}(\mathcal{C}_{X_i})_{\geq 0}$ into $(\mathcal{C}_Y)_{\geq 0}$. Let $\mathcal{C}^{\otimes}_{\geq 0} \subseteq \mathcal{C}^{\otimes}$ be the full subcategory spanned by the objects $C \in \mathcal{C}^{\otimes}$ such that, for every inert morphism $C \to C'$, where $C' \in \mathcal{C}_X$ for some $X \in \mathcal{O}$, the object C' belongs to $(\mathcal{C}_X)_{\geq 0}$. Proposition 2.2.1.1 implies that the induced map $\mathcal{C}^{\otimes}_{\geq 0} \to \mathcal{O}^{\otimes}$ is again a coCartesian fibration of ∞-operads.

Remark 2.2.1.4. Let $\mathcal{C}^{\otimes} \to \mathcal{A}ss^{\otimes}$ be a coCartesian fibration of ∞-operads satisfying the hypotheses of Example 2.2.1.3. Then, for every pair of integers $m, n \in \mathbf{Z}$, the tensor product functor $\otimes$ carries $\mathcal{C}_{\geq m} \times \mathcal{C}_{\geq n}$ into $\mathcal{C}_{\geq n+m}$. This follows immediately from the exactness of the tensor product in each variable, and the assumption that the desired result holds in the case $m = n = 0$.

Proof of Proposition 2.2.1.1. Assertions (1) and (2) follow immediately from the definitions. Now suppose that the hypotheses of (3) are satisfied. Let us say that a morphism $f : D \to C$ in $\mathcal{C}^{\otimes}$ is *colocalizing* if $D \in \mathcal{D}^{\otimes}$, and the projection map

$$
\psi : \mathcal{D}^{\otimes} \times_{\mathcal{C}^{\otimes}} \mathcal{C}^{\otimes}_{/f} \to \mathcal{D}^{\otimes} \times_{\mathcal{C}^{\otimes}} \mathcal{C}^{\otimes}_{/C}
$$

is a trivial Kan fibration. Since ψ is automatically a right fibration (Proposition T.2.1.2.1), the map f is colocalizing if and only if the fibers of ψ are contractible (Lemma T.2.1.3.4). For this, it suffices to show that for each $D' \in \mathcal{D}^{\otimes}$, the induced map of fibers

$$
\psi_{D'} : \{D'\} \times_{\mathcal{C}^{\otimes}} \mathcal{C}^{\otimes}_{/f} \to \{D'\} \times_{\mathcal{C}^{\otimes}} \mathcal{C}^{\otimes}_{/C}
$$

has contractible fibers. Since $\psi_{D'}$ is a right fibration between Kan complexes, it is a Kan fibration (Lemma T.2.1.3.3). Consequently, $\psi_{D'}$ has contractible fibers if and only if it is a homotopy equivalence. In other

words, f is colocalizing if and only if composition with f induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}^{\otimes}}(D', D) \to \mathrm{Map}_{\mathcal{C}^{\otimes}}(D', C)$ for all $D' \in \mathcal{D}^{\otimes}$.

Suppose now that $f : D \to C$ is a morphism belonging to a particular fiber $\mathcal{C}_X^{\otimes}$ of p. Then, for every object $D' \in \mathcal{C}_Y^{\otimes}$, we have canonical maps

$$\mathrm{Map}_{\mathcal{C}^{\otimes}}(D', D) \to \mathrm{Map}_{\mathcal{O}^{\otimes}}(Y, X) \leftarrow \mathrm{Map}_{\mathcal{C}^{\otimes}}(D', C)$$

whose homotopy fibers over a morphisms $g : Y \to X$ in $\mathcal{O}^{\otimes}$ can be identified with $\mathrm{Map}_{\mathcal{C}_X^{\otimes}}(g_! D', D)$ and $\mathrm{Map}_{\mathcal{C}_X^{\otimes}}(g_! D', C)$ (Proposition T.2.4.4.2). Our hypothesis that $\mathcal{D}$ is stable under the tensor operations on $\mathcal{C}$ implies that $g_! D' \in \mathcal{D}_X^{\otimes}$. It follows that f is colocalizing in $\mathcal{C}^{\otimes}$ if and only if it is colocalizing in $\mathcal{C}_X^{\otimes}$.

To prove (3), we need to construct a commutative diagram

$$
\begin{array}{ccc}
\mathcal{C}^{\otimes} \times \Delta^1 & \xrightarrow{\;\;\alpha\;\;} & \mathcal{C}^{\otimes} \\
 & \searrow \qquad \swarrow_{p} & \\
 & \mathcal{O}^{\otimes} &
\end{array}
$$

with $\alpha|\,\mathcal{C}^{\otimes} \times \{1\}$ equal to the identity, and having the property that, for every object $C \in \mathcal{C}^{\otimes}$, the restriction $\alpha|\{C\} \times \Delta^1$ is colocalizing. For this, we construct α one simplex at a time. To define α on a vertex $C \in \mathcal{C}_X^{\otimes}$ where $X \in \mathcal{O}_{\langle n \rangle}^{\otimes}$ corresponds to a sequence of objects $\{X_i\}_{1 \le i \le n}$ in $\mathcal{O}$, we use the equivalence $\mathcal{C}_X^{\otimes} \simeq \prod_{1 \le i \le X} \mathcal{C}_{X_i}$ and the assumption that each $\mathcal{D}_{X_i} \subseteq \mathcal{C}_{X_i}$ is a colocalization. For simplices of higher dimension, we need to solve extensions problems of the form

$$
\begin{array}{ccc}
(\partial \Delta^n \times \Delta^1) \coprod_{\partial \Delta^n \times \{1\}} (\Delta^n \times \{1\}) & \xrightarrow{\;\alpha_0\;} & \mathcal{C}^{\otimes} \\
\downarrow & \nearrow & \downarrow{p} \\
\Delta^n \times \Delta^1 & \longrightarrow & \mathcal{O}^{\otimes},
\end{array}
$$

where $n > 0$ and α_0 carries each of the edges $\{i\} \times \Delta^1$ to a colocalizing morphism in $\mathcal{C}^{\otimes}$. We now observe that $\Delta^n \times \Delta^1$ admits a filtration

$$X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq X_{n+1} = \Delta^n \times \Delta^1,$$

where $X_0 = (\partial \Delta^n \times \Delta^1) \coprod_{\partial \Delta^n \times \{1\}} (\Delta^n \times \{1\})$ and there exist pushout diagrams

$$
\begin{array}{ccc}
\Lambda_{i+1}^{n+1} & \hookrightarrow & \Delta^{n+1} \\
\downarrow & & \downarrow \\
X_i & \longrightarrow & X_{i+1}.
\end{array}
$$

We now argue, by induction on i, that the map α_0 admits an extension to X_i (compatible with the projection p). For $i \le n$, this follows from the fact that p is an inner fibration. For $i = n+1$, it follows from the definition of a colocalizing morphism. This completes the proof of (3). Assertion (4) follows from the construction. $\square$

Remark 2.2.1.5. Let $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ be as in the statement of Proposition 2.2.1.1, and suppose that each inclusion $\mathcal{D}_X \subseteq \mathcal{C}_X$ admits a right adjoint L_X. The inclusion $i : \mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ is a $\mathcal{O}$-monoidal functor, and therefore induces a fully faithful embedding $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{D}) \subseteq \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$. Let $L^{\otimes}$ be the functor constructed in Proposition 2.2.1.1. Since $L^{\otimes}$ is a map of ∞-operads, it also induces a functor $f : \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{D})$. It is not difficult to see that f is right adjoint to the inclusion $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{D}) \subseteq \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$. Moreover, if $\theta : \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathcal{C}_X$ denotes the evaluation functor associated to an object $X \in \mathcal{O}$, then for each $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ the map $\theta(f(A)) \to \theta(A)$ induced by the colocalization map $f(A) \to A$ determines an equivalence $\theta(f(A)) \simeq L_X \theta(A)$.

Proposition 2.2.1.1 has an obvious converse: if $i : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ is a fully faithful $\mathcal{O}$-monoidal functor between $\mathcal{O}$-monoidal ∞-categories, then then the essential image of $\mathcal{D}$ in $\mathcal{C}$ is stable under the tensor operations of Remark 2.1.2.16. However, it is possible for this converse to fail if we assume only that i is lax monoidal. We now discuss a general class of examples where $\mathcal{D}$ is *not* stable under tensor products, yet $\mathcal{D}$ nonetheless inherits a $\mathcal{O}$-monoidal structure from that of $\mathcal{C}$.

Definition 2.2.1.6. Let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads and suppose we are given a family of localization functors $L_X : \mathcal{C}_X \to \mathcal{C}_X$ for $X \in \mathcal{O}$. We will say that the family $\{L_X\}_{X \in \mathcal{O}}$ is *compatible with the $\mathcal{O}$-monoidal structure on $\mathcal{C}$* if the following condition is satisfied:

($\ast$) Let $f \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{1 \leq i \leq n}, Y)$ be an operation in $\mathcal{O}$, and suppose we are given morphisms g_i in $\mathcal{C}_{X_i}$ for $1 \leq i \leq n$. If each g_i is a L_{X_i}-equivalence, then the morphism $\otimes_f \{g_i\}_{1 \leq i \leq n}$ in $\mathcal{C}_Y$ is an L_Y-equivalence.

Example 2.2.1.7. In the situation where $\mathcal{O}^{\otimes}$ is the commutative ∞-operad, the condition of Definition 2.2.1.6 can be simplified: a localization functor $L : \mathcal{C} \to \mathcal{C}$ is compatible with a symmetric monoidal structure on $\mathcal{C}$ if and only if, for every L-equivalence $X \to Y$ in $\mathcal{C}$ and every object $Z \in \mathcal{C}$, the induced map $X \otimes Z \to Y \otimes Z$ is again an L-equivalence.

Proposition 2.2.1.8. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads. Assume that for each $X \in \mathcal{O}$, the ∞-category $\mathcal{C}_X$ is stable, and that for every operation $\phi \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y)$ the associated functor $\otimes_{\phi} : \prod_{i \in I} \mathcal{C}_{X_i} \to \mathcal{C}_Y$ is exact in each variable. Suppose we are given a collection of t-structures on the ∞-categories $\mathcal{C}_X$ which are compatible with p (in the sense of Example 2.2.1.3), so that we have an induced coCartesian fibration $\mathcal{C}_{\geq 0}^{\otimes} \to \mathcal{O}^{\otimes}$. Then the collection of localization functors $\tau_{\leq n} : (\mathcal{C}_X)_{\geq 0} \to (\mathcal{C}_X)_{\geq 0}$ is compatible with the $\mathcal{O}$-monoidal structure on $\mathcal{C}_{\geq 0}$ (in the sense of Definition 2.2.1.6).*

Proof. Suppose we are given an operation $\phi \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y)$, so that ϕ induces a tensor product functor $\otimes_{\phi} : \prod_{i \in I} \mathcal{C}_{X_i} \to \mathcal{C}_Y$. We must show that if we are given a collection of morphisms $\alpha_i : C_i \to C_i'$ in $(\mathcal{C}_{X_i})_{\geq 0}$ which induce equivalences $\tau_{\leq k} C_i \to \tau_{\leq k} C_i'$, then the induced map

$$\tau_{\leq k} \otimes_{\phi} (\{C_i\}) \to \tau_{\leq k} \otimes_{\phi} (\{C_i'\})$$

is an equivalence in $\mathcal{C}_Y$. Without loss of generality, we may assume that there exists an element $j \in I$ such that α_i is an equivalence for $i \neq j$, and that $C_j' = \tau_{\leq k} C_j$. We have a fiber sequence

$$C_j''[k + 1] \to C_j \to C_j'$$

for some $C_j'' \in (\mathcal{C}_{X_j})_{\geq 0}$. Let $C_i'' = C_i$ for $i \neq j$. Since $\otimes_{\phi}$ is exact in each variable, we conclude that there is a fiber sequence

$$\otimes_{\phi}(\{C_i''\})[k + 1] \to \otimes_{\phi}(\{C_i\}) \xrightarrow{\beta} \otimes_{\phi}(\{C_i'\}).$$

Since our t-structures are assumed to be compatible with p, the object $\otimes_{\phi}(\{C_i''\})$ belongs to $(\mathcal{C}_Y)_{\geq 0}$, from which it follows that $\tau_{\leq k}(\beta)$ is an equivalence. $\square$

Proposition 2.2.1.9. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a be a coCartesian fibration of ∞-operads and suppose we are given a family of localization functors $\{L_X : \mathcal{C}_X \to \mathcal{C}_X\}_{X \in \mathcal{O}}$ which are compatible with the $\mathcal{O}$-monoidal structure on $\mathcal{C}$. Let $\mathcal{D}$ denote the collection of all objects of $\mathcal{C}$ which lie in the image of some L_X, and let $\mathcal{D}^{\otimes}$ be defined as above. Then:*

(1) *There exists a commutative diagram*

$$
\begin{array}{ccc}
\mathcal{C}^{\otimes} & \xrightarrow{\ L^{\otimes}\ } & \mathcal{D}^{\otimes} \\
 & \searrow{\scriptstyle p} \quad \swarrow & \\
 & \mathcal{O}^{\otimes} &
\end{array}
$$

and a natural transformation $\alpha : \mathrm{id}_{\mathcal{C}^{\otimes}} \to L^{\otimes}$ which exhibits $L^{\otimes}$ as a left adjoint to the inclusion $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ and such that $p(\alpha)$ is the identity natural transformation from p to itself.

(2) *The restriction $p|\,\mathcal{D}^{\otimes} : \mathcal{D}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a coCartesian fibration of ∞-operads.*

(3) *The inclusion functor $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ is a map of ∞-operads and $L^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a $\mathcal{O}$-monoidal functor.*

Example 2.2.1.10. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure and a compatible symmetric monoidal structure, in the sense of Example 2.2.1.3. Applying Proposition 2.2.1.8, we deduce that the truncation functors $\tau_{\leq n} : \mathcal{C}_{\geq 0} \to (\mathcal{C}_{\geq 0})_{\leq n}$ are compatible with the symmetric monoidal structure on $\mathcal{C}_{\geq 0}$. Consequently, Proposition 2.2.1.9 implies that $(\mathcal{C}_{\geq 0})_{\leq n}$ inherits the structure of a symmetric monoidal ∞-category. Taking $n = 0$, we deduce that the heart of $\mathcal{C}$ inherits the structure of a symmetric monoidal category (in the sense of classical category theory).

The proof of Proposition 2.2.1.9 relies on the following observation:

Lemma 2.2.1.11. *Let $p : \mathcal{C} \to \mathcal{D}$ be a coCartesian fibration of ∞-categories. Let $L : \mathcal{C} \to \mathcal{C}$ and $L' : \mathcal{D} \to \mathcal{D}$ be localization functors, with essential images $L\,\mathcal{C}$ and $L'\,\mathcal{D}$. Suppose that L and L' are compatible in the following sense:*

(i) *The functor p restricts to a functor $p' : L\,\mathcal{C} \to L'\,\mathcal{D}$.*

(ii) *If f is a morphism in $\mathcal{C}$ such that Lf is an equivalence, then $L'p(f)$ is an equivalence in $\mathcal{D}$.*

Then:

(1) *The functor L carries p-coCartesian morphisms of $\mathcal{C}$ to p'-coCartesian morphisms of $L\,\mathcal{C}$.*

(2) *The functor p' is a coCartesian fibration.*

Proof. Let $f : X \to Y$ be a p-coCartesian morphism of $\mathcal{C}$. We wish to prove that Lf is p'-coCartesian. According to Proposition T.2.4.4.3, it will suffice to show that for every $Z' \in L\,\mathcal{C}$, the diagram of Kan complexes

$$\begin{array}{ccc} \mathcal{C}_{Lf/} \times_{\mathcal{C}} \{Z'\} & \longrightarrow & \mathcal{C}_{LX/} \times_{\mathcal{C}} \{Z'\} \\ \downarrow & & \downarrow \\ \mathcal{D}_{p(Lf)/} \times_{\mathcal{D}} \{p(Z')\} & \longrightarrow & \mathcal{D}_{p(LX)/} \times_{\mathcal{D}} \{p(Z')\} \end{array}$$

is homotopy Cartesian. Let $Z' = LZ$. Since $L^2 \simeq L$, we can assume without loss of generality that $Z \in L\,\mathcal{C}$.

Since f is p-coCartesian, Proposition T.2.4.4.3 implies that the diagram

$$\begin{array}{ccc} \mathcal{C}_{f/} \times_{\mathcal{C}} \{Z\} & \longrightarrow & \mathcal{C}_{X/} \times_{\mathcal{C}} \{Z\} \\ \downarrow & & \downarrow \\ \mathcal{D}_{p(f)/} \times_{\mathcal{D}} \{p(Z)\} & \longrightarrow & \mathcal{D}_{p(X)/} \times_{\mathcal{D}} \{p(Z)\} \end{array}$$

is homotopy Cartesian. Choose a natural transformation $\alpha : \mathrm{id}_{\mathcal{C}} \to L$ which exhibits L as a localization functor. Then α induces a natural transformation between the above diagrams. It will therefore suffice to show that each of the induced maps

$$\mathcal{C}_{f/} \times_{\mathcal{C}} \{Z\} \to \mathcal{C}_{Lf/} \times_{\mathcal{C}} \{LZ\} \quad \mathcal{C}_{X/} \times_{\mathcal{C}} \{Z\} \to \mathcal{C}_{LX/} \times_{\mathcal{C}} \{LZ\}$$

$$\mathcal{D}_{p(f)/} \times_{\mathcal{D}} \{p(Z)\} \to \mathcal{D}_{p(Lf)/} \times_{\mathcal{D}} \{p(LZ)\} \quad \mathcal{D}_{p(LX)/} \times_{\mathcal{D}} \{p(Z)\} \to \mathcal{D}_{p(LX)/} \times_{\mathcal{D}} \{p(LZ)\}$$

is a homotopy equivalence. For the first pair of maps, this follows from the fact that $Z \in L\,\mathcal{C}$. For the second pair, we observe that (i) and (ii) imply that for every $C \in \mathcal{C}$, the map $p(\alpha(C)) : p(C) \to p(LC)$ is equivalent to the L'-localization $p(C) \to L'p(C)$, and $p(Z) \in L'\,\mathcal{D}$. This completes the proof of (1).

To prove (2), choose any object $C \in L\,\mathcal{C}$ and a morphism $f : p(C) \to D$ in $L'\,\mathcal{D}$. Choose a p-coCartesian morphism $\overline{f} : C \to \overline{D}$ in $\mathcal{C}$. According to (1), the morphism $L(\overline{f}) : LC \to L\overline{D}$ is p'-coCartesian. We now use the fact that p' is a categorical fibration to lift the equivalence $p(\alpha(\overline{f}))$ to an equivalence $L(\overline{f}) \simeq \overline{f}'$, where $\overline{f}' : C \to \overline{D}'$ is a p'-coCartesian morphism lifting f. $\square$

Proof of Proposition 2.2.1.9. We first prove (1). Let us say that a map $f : C \to D$ in $\mathcal{C}^{\otimes}$ is *localizing* if it induces a trivial Kan fibration

$$\mathcal{D}^{\otimes} \times_{\mathcal{C}^{\otimes}} \mathcal{C}^{\otimes}_{f/} \to \mathcal{D}^{\otimes} \times_{\mathcal{C}^{\otimes}} \mathcal{C}^{\otimes}_{D/} \, .$$

Arguing as in the proof of Proposition 2.2.1.1, we see that f is localizing if and only if, for every object $D' \in \mathcal{D}^{\otimes}$, composition with f induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}^{\otimes}}(D, D') \to \mathrm{Map}_{\mathcal{C}^{\otimes}}(C, D')$. Suppose that f lies in a fiber $\mathcal{C}^{\otimes}_X$ of p, and let $D' \in \mathcal{C}^{\otimes}_Y$ of p. We have canonical maps

$$\mathrm{Map}_{\mathcal{C}^{\otimes}}(D, D') \to \mathrm{Map}_{\mathcal{O}^{\otimes}}(X, Y) \leftarrow \mathrm{Map}_{\mathcal{C}^{\otimes}}(C, D')$$

and Proposition T.2.4.4.2 allows us to identify the homotopy fibers of these maps over a morphism $g : X \to Y$ in $\mathcal{O}^{\otimes}$ with $\mathrm{Map}_{\mathcal{C}^{\otimes}_Y}(g_! D, D')$ and $\mathrm{Map}_{\mathcal{C}^{\otimes}_Y}(g_! C, D')$. It follows that f is localizing in $\mathcal{C}^{\otimes}$ if and only if $g_!(f)$ is localizing in $\mathcal{C}^{\otimes}_Y$ for every map $g : X \to Y$ in $\mathcal{O}^{\otimes}$. Let $X \in \mathcal{O}^{\otimes}_{\langle n \rangle}$ and $Y \in \mathcal{O}^{\otimes}_{\langle m \rangle}$ correspond to sequences of objects $\{X_i\}_{1 \le i \le n}$ and $\{Y_j\}_{1 \le j \le m}$ in $\mathcal{O}$. Using the equivalence $\mathcal{C}^{\otimes}_Y \simeq \prod_{1 \le j \le m} \mathcal{C}_{Y_i}$ we see that it suffices to check this in the case $m = 1$. Invoking the assumption that the localization functors Invoking the assumption that the localization functors $\{L_Z\}_{Z \in \mathcal{O}}$ are compatible with the $\mathcal{O}$-monoidal structure on $\mathcal{C}^{\otimes}$, we see that f is localizing if and only if it induces a L_{X_i}-localizing morphism in $\mathcal{C}_{X_i}$ for $1 \le i \le n$.

We now argue as in the proof of Proposition 2.2.1.1. To prove (1), we need to construct a commutative diagram

$$\begin{array}{ccc} \mathcal{C}^{\otimes} \times \Delta^1 & \xrightarrow{\quad \alpha \quad} & \mathcal{C}^{\otimes} \\ & \searrow \quad\quad \swarrow{\scriptstyle p} & \\ & \mathrm{N}(\mathcal{F}\mathrm{in}_*) & \end{array}$$

with $\alpha | \, \mathcal{C}^{\otimes} \times \{0\}$ equal to the identity, and having the property that, for every object $C \in \mathcal{C}^{\otimes}$, the restriction $\alpha | \{C\} \times \Delta^1$ is localizing. For this, we construct α one simplex at a time. To define α on a vertex $C \in \mathcal{C}^{\otimes}_X$ for $X \in \mathcal{O}^{\otimes}_{\langle n \rangle}$, we choose a morphism $f : C \to C'$ corresponding to the localization morphisms $C_i \to L_{X_i} C_i$ under the equivalence $\mathcal{C}^{\otimes}_X \simeq \prod_{1 \le i \le n} \mathcal{C}_{X_i}$ and apply the argument above. For simplices of higher dimension, we need to solve extensions problems of the form

$$\begin{array}{ccc} (\partial \, \Delta^n \times \Delta^1) \coprod_{\partial \, \Delta^n \times \{0\}} (\Delta^n \times \{0\}) & \xrightarrow{\alpha_0} & \mathcal{C}^{\otimes} \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \Delta^n \times \Delta^1 & \longrightarrow & \mathcal{O}^{\otimes}, \end{array}$$

where $n > 0$ and α_0 carries each of the edges $\{i\} \times \Delta^1$ to a localizing morphism in $\mathcal{C}^{\otimes}$. We now observe that $\Delta^n \times \Delta^1$ admits a filtration

$$X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq X_{n+1} = \Delta^n \times \Delta^1,$$

where $X_0 = (\partial \, \Delta^n \times \Delta^1) \coprod_{\partial \, \Delta^n \times \{0\}} (\Delta^n \times \{0\})$ and there exist pushout diagrams

$$\begin{array}{ccc} \Lambda^{n+1}_{n-i} & \longhookrightarrow & \Delta^{n+1} \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & X_{i+1}. \end{array}$$

We now argue, by induction on i, that the map α_0 admits an extension to X_i (compatible with the projection p). For $i \le n$, this follows from the fact that p is an inner fibration. For $i = n + 1$, it follows from the definition of a localizing morphism. This completes the proof of (1).

Lemma 2.2.1.11 implies that $p' = p | \mathcal{D}^{\otimes}$ is a coCartesian fibration. It follows immediately from the definition that for every object $X \in \mathcal{O}^{\otimes}_{\langle m \rangle}$ corresponding to $\{X_i \in \mathcal{O}\}_{1 \le i \le m}$, the equivalence $\mathcal{C}^{\otimes}_X \simeq \prod_{1 \le i \le m} \mathcal{C}_{X_i}$

restricts to an equivalence $\mathcal{D}_X^\otimes \simeq \prod_{1 \le i \le m} \mathcal{D}_{X_i}$. This proves that $p' : \mathcal{D}^\otimes \to \mathcal{O}^\otimes$ is a coCartesian fibration of ∞-operads and that the inclusion $\mathcal{D}^\otimes \subseteq \mathcal{C}^\otimes$ is a map of ∞-operads. Lemma 2.2.1.11 implies that $L^\otimes$ carries p-coCartesian edges to p'-coCartesian edges and is therefore a $\mathcal{O}$-monoidal functor.   □

2.2.2   Slicing ∞-Operads

Let $\mathcal{C}$ be a symmetric monoidal category and let A be a commutative algebra object of $\mathcal{C}$. Then the overcategory $\mathcal{C}_{/A}$ inherits the structure of a symmetric monoidal category: the tensor product of a map $X \to A$ with a map $Y \to A$ is given by the composition

$$X \otimes Y \to A \otimes A \xrightarrow{m} A,$$

where m denotes the multiplication on A. Our goal in this section is to establish an ∞-categorical analogue of this observation (and a weaker result concerning undercategories). Before we can state our result, we need to introduce a bit of notation.

Definition 2.2.2.1. Let $q : X \to S$ be a map of simplicial sets, and suppose we are given a commutative diagram

We define a simplicial set $X_{p_S/}$ equipped with a map $q' : X_{p_S/} \to S$ so that the following universal property is satisfied: for every map of simplicial sets $Y \to S$, there is a canonical bijection of $\mathrm{Fun}_S(Y, X_{p_S/})$ with the collection of commutative diagrams

Similarly, we define a map of simplicial sets $X_{/p_S} \to S$ so that $\mathrm{Fun}_S(Y, X_{/p_S})$ is in bijection with the set of diagrams

Remark 2.2.2.2. If S consists of a single point, then $X_{p_S/}$ and $X_{/p_S}$ coincide with the usual overcategory and undercategory constructions $X_{p/}$ and $X_{/p}$. In general, the fiber of the morphism $X_{p_S/} \to S$ over a vertex $s \in S$ can be identified with $(X_s)_{p_s/}$, where $X_s = X \times_S \{s\}$ and $p_s : K \to X_s$ is the induced map; similarly, we can identify $X_{/p_S} \times_S \{s\}$ with $(X_s)_{/p_s}$.

Notation 2.2.2.3. Let $q : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a fibration of ∞-operads, and let $p : K \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ be a diagram. We let $\mathcal{C}_{p_\mathcal{O}/}^\otimes$ and $\mathcal{C}_{/p_\mathcal{O}}^\otimes$ denote the simplicial sets $(\mathcal{C}^\otimes)_{p_{\mathcal{O}^\otimes}/}$ and $(\mathcal{C}^\otimes)_{/p_{\mathcal{O}^\otimes}}$ described in Definition 2.2.2.1.

   In the special case where $K = \Delta^0$, the diagram p is simply given by a $\mathcal{O}$-algebra object $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$; in this case, we will denote $\mathcal{C}_{p_\mathcal{O}/}^\otimes$ and $\mathcal{C}_{/p_\mathcal{O}}^\otimes$ by $\mathcal{C}_{A_\mathcal{O}/}^\otimes$ and $\mathcal{C}_{/A_\mathcal{O}}^\otimes$, respectively.

   We can now state the main result of this section.

Theorem 2.2.2.4. *Let $q : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a fibration of ∞-operads, and let $p : K \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ be a diagram. Then:*

(1) *The maps* $\mathcal{C}_{p_{\mathcal{O}}/}^{\otimes} \xrightarrow{q'} \mathcal{O}^{\otimes} \xleftarrow{q''} \mathcal{C}_{/p_{\mathcal{O}}}^{\otimes}$ *are fibrations of ∞-operads.*

(2) *A morphism in* $\mathcal{C}_{p_{\mathcal{O}}/}^{\otimes}$ *is inert if and only if its image in* $\mathcal{C}^{\otimes}$ *is inert; similarly, a morphism in* $\mathcal{C}_{/p_{\mathcal{O}}}^{\otimes}$ *is inert if and only if its image in* $\mathcal{C}^{\otimes}$ *is inert.*

(3) *If q is a coCartesian fibration of ∞-operads, then q'' is a coCartesian fibration of ∞-operads. If, in addition, $p(k) : \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ is a $\mathcal{O}$-monoidal functor for each vertex $k \in K$, then q' is also a coCartesian fibration of ∞-operads.*

Remark 2.2.2.5. In the special case where $\mathcal{O}^{\otimes}$ is the commutative ∞-operad and $K = \Delta^0$, we can state Theorem 2.2.2.4 more informally as follows: let $\mathcal{C}^{\otimes}$ be an ∞-operad equipped with a commutative algebra object $A \in \mathrm{CAlg}(\mathcal{C})$. Then the ∞-categories $\mathcal{C}_{A/}$ and $\mathcal{C}_{/A}$ can be regarded as the underlying ∞-categories of ∞-operads $\mathcal{C}_{A_{\mathcal{O}}/}^{\otimes}$ and $\mathcal{C}_{/A_{\mathcal{O}}}^{\otimes}$. Moreover, if $\mathcal{C}$ is a symmetric monoidal ∞-cateogry, then $\mathcal{C}_{/A}$ is a symmetric monoidal ∞-category; the same result holds for $\mathcal{C}_{A/}$ if we assume that A is a trivial algebra in the sense of Definition 3.2.1.7.

The remainder of this section is devoted to the proof of Theorem 2.2.2.4. We will need a few lemmas.

Lemma 2.2.2.6. *Suppose we are given a diagram of simplicial sets*

$$
\begin{array}{ccc}
 & & X \\
 & \nearrow^{p} & \downarrow^{q} \\
S \times K & \longrightarrow & S
\end{array}
$$

where q is an inner fibration, and let $q' : X_{p_S/} \to S$ be the induced map. Then q' is a inner fibration. Similarly, if q is a categorical fibration, then q' is a categorical fibration.

Proof. We will prove the assertion regarding inner fibrations; the case of categorical fibrations is handled similarly. We wish to show that every lifting problem of the form

$$
\begin{array}{ccc}
A & \longrightarrow & X_{p_S/} \\
\downarrow^{j} & \nearrow & \downarrow^{q'} \\
B & \longrightarrow & S
\end{array}
$$

admits a solution, provided that j is inner anodyne. Unwinding the definitions, we arrive at an equivalent lifting problem

$$
\begin{array}{ccc}
(A \times K^{\triangleright}) \coprod_{A \times K} (B \times K) & \longrightarrow & X \\
\downarrow^{j'} & & \downarrow^{q} \\
B \times K^{\triangleright} & \longrightarrow & S,
\end{array}
$$

which admits a solution by virtue of the fact that q is an inner fibration and j' is inner anodyne (Corollary T.2.3.2.4). □

Lemma 2.2.2.7. *Let $q : X \to S$ be an innert fibration of simplicial sets and let K and Y be simplicial sets. Suppose that $\overline{h} : K \times Y^{\triangleright} \to X$ is a map such that, for each $k \in K$, the induced map $\{k\} \times Y^{\triangleright} \to X$ is a q-colimit diagram. Let $h = \overline{h}|K \times Y$. Then the map*

$$
X_{\overline{h}/} \to X_{h/} \times_{S_{qh/}} S_{q\overline{h}/}
$$

is a trivial Kan fibration.

Proof. We will prove more generally that if $K_0 \subseteq K$ is a simplicial subset and $\overline{h}_0 = \overline{h}|(K \times Y) \coprod_{K_0 \times Y} (K \times Y^{\triangleright})$, then the induced map $\theta : X_{\overline{h}/} \to X_{\overline{h}_0/} \times_{S_{q\overline{h}_0/}} S_{q\overline{h}/}$ is a trivial Kan fibration. Working simplex-by-simplex, we can reduce to the case where $K = \partial \Delta^n$ and $K' = \partial \Delta^n$. Let us identify $Y \star \Delta^n$ with the full simplicial subset of $Y^{\triangleright} \times \Delta^n$ spanned by Δ^n and $Y^{\triangleright} \times \{0\}$. Let $\overline{g} = \overline{h}|Y \star \Delta^n$, and let $g = \overline{g}|Y \star \partial \Delta^n$. Then θ is a pullback of the map

$$\theta' : X_{\overline{g}/} \to X_{g/} \times_{S_{qg/}} S_{q\overline{g}/}.$$

It will now suffice to show that θ' has the right lifting property with respect to every inclusion $\partial \Delta^m \subseteq \Delta^m$. Unwinding the definition, this is equivalent to solving a lifting problem of the form

$$
\begin{array}{ccc}
Y \star \partial \Delta^{n+m+1} & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow q \\
Y \star \Delta^{n+m+1} & \longrightarrow & S.
\end{array}
$$

This lifting problem admits a solution by virtue of our assumption that $\overline{h}|\{0\} \times Y^{\triangleright}$ is a q-colimit diagram. $\square$

Lemma 2.2.2.8. *Let*

$$
\begin{array}{ccc}
 & & X \\
 & \nearrow^{p} & \downarrow q \\
S \times K & \longrightarrow & S
\end{array}
$$

be a diagram of simplicial sets, where q is an inner fibration, let $q' : X^{pS/} \to S$ be the induced map, and suppose we are given a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{f} & X^{pS/} \\
\downarrow & \nearrow^{\overline{f}} & \downarrow q' \\
Y^{\triangleright} & \xrightarrow{g} & S
\end{array}
$$

satisfying the following conditions:

(i) *For each vertex $k \in K$, the diagram*

$$Y^{\triangleright} \xrightarrow{g} S \simeq S \times \{k\} \hookrightarrow S \times K \xrightarrow{p} X$$

   is a q-colimit diagram.

(ii) *The composite map $Y \xrightarrow{f} X^{pS/} \to X$ can be extended to a q-colimit diagram $Y^{\triangleright} \to X$ lying over g.*

Then:

(1) *Let $\overline{f} : Y^{\triangleright} \to X^{pS/}$ be a map rendering the diagram commutative. Then $\overline{f}$ is a q'-colimit diagram if and only if the composite map $Y^{\triangleright} \xrightarrow{\overline{f}} X^{pS/} \to X$ is a q-colimit diagram.*

(2) *There exists a map $\overline{f}$ satisfying the equivalent conditions of (1).*

Proof. Let Z be the full simplicial subset of $K^{\triangleright} \times Y^{\triangleright}$ obtained by removing the final object, so we have a canonical isomorphism $Z^{\triangleright} \simeq K^{\triangleright} \times Y^{\triangleright}$. The maps f and g determine a diagram $h : Z \to X$. We claim that h can be extended to a q-colimit diagram $\overline{h} : Z^{\triangleright} \to X$ lying over the map $Z^{\triangleright} \to Y^{\triangleright} \xrightarrow{g} S$. To prove this, let $h_0 = h|K^{\triangleright} \times Y$, $h_1 = h|K \times Y^{\triangleright}$, and $h_2 = h|K \times Y$. Using (i) we deduce that the map $\theta : X_{h_1/} \to X_{h_2/} \times_{S_{qh_2/}} S_{qh_1/}$ is a trivial Kan fibration (Lemma 2.2.2.7). The map $X_{h/} \to X_{h_0/} \times_{S_{qh_0/}} S_{qh/}$ is a pullback of θ, and therefore also a trivial Kan fibration. Consequently, to show that h admits a q-colimit

diagram compatible with g, it suffices to show that h_0 admits a q-colimit diagram compatible with g. Since the inclusion $Y \hookrightarrow K^\triangleright \times Y$ is left cofinal, this follows immediately from (ii). This proves the existence of $\overline{h}$: moreover, it shows that an arbitrary extension $\overline{h}$ of h (compatible with g) is a p-colimit diagram if and only if it restricts to a p-colimit diagram $Y^\triangleright \to Z$.

The map $\overline{h}$ determines an extension $\overline{f} : Y^\triangleright \to X_{pS/}$ of f. We will show that $\overline{f}$ is a q'-colimit diagram. This will prove the "if" direction of (1) and (2); the "only if" direction of (1) will then follow from the uniqueness properties of q'-colimit diagrams.

We wish to show that every lifting problem of the form

$$
\begin{array}{ccc}
Y \star \partial \Delta^n & \xrightarrow{F} & X_{pS/} \\
\downarrow & \nearrow & \downarrow{\scriptstyle q'} \\
Y \star \Delta^n & \longrightarrow & S
\end{array}
$$

admits a solution, provided that $n > 0$ and $F|Y \star \{0\}$ coincides with $\overline{f}$. This is equivalent to a lifting problem of the form

$$
\begin{array}{ccc}
((Y \star \partial \Delta^n) \times K^\triangleright) \coprod_{(Y \star \partial \Delta^n) \times K} ((Y \star \Delta^n) \times K) & \longrightarrow & X \\
\downarrow{\scriptstyle j} & \nearrow & \downarrow{\scriptstyle q} \\
(Y \star \Delta^n) \times K^\triangleright & \longrightarrow & S
\end{array}
$$

It now suffices to observe that the map j is a pushout of the inclusion $Z \star \partial \Delta^n \hookrightarrow Z \star \Delta^n$, so the desired lifting problem can be solved by virtue of our assumption that $\overline{h}$ is a q-colimit diagram. $\qquad \square$

The following result is formally similar to Lemma 2.2.2.8 but requires a slightly different proof:

Lemma 2.2.2.9. *Let*

$$
\begin{array}{ccc}
 & & X \\
 & \nearrow{\scriptstyle p} & \downarrow{\scriptstyle q} \\
S \times K & \longrightarrow & S
\end{array}
$$

be a diagram of simplicial sets, where q is an inner fibration, let $q' : X^{pS/} \to S$ be the induced map, and suppose we are given a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{f} & X^{pS/} \\
\downarrow & \nearrow{\scriptstyle \overline{f}} & \downarrow{\scriptstyle q'} \\
Y^\triangleleft & \xrightarrow{g} & S
\end{array}
$$

satisfying the following condition:

$(*)$ *The composite map $Y \xrightarrow{f} X^{pS/} \to X$ can be extended to a q-limit diagram $g' : Y^\triangleleft \to X$ lying over g.*

Then:

(1) *Let $\overline{f} : Y^\triangleleft \to X^{pS/}$ be a map rendering the diagram commutative. Then $\overline{f}$ is a q'-limit diagram if and only if the composite map $Y^\triangleleft \xrightarrow{\overline{f}} X^{pS/} \to X$ is a q-limit diagram.*

(2) *There exists a map $\overline{f}$ satisfying the equivalent conditions of (1).*

Proof. Let v be the cone point of $K^\rhd$ and v' the cone point of $Y^\lhd$. Let Z be the full subcategory of $K^\rhd \times Y^\lhd$ obtained by removing the vertex (v, v'). The maps f and g determine a map $h : Z \to X$. Choose any map g' as in (i), and let $g'_0 = g'|Y$. We claim that there exists an extension $\overline{h} : K^\rhd \times Y^\lhd \to X$ of h which is compatible with g, such that $\overline{h}|\{v\} \times Y^\lhd = g'$. Unwinding the definitions, we see that providing such a map $\overline{h}$ is equivalent to solving a lifting problem of the form

$$
\begin{array}{ccc}
\emptyset & \longrightarrow & X_{/g'} \\
\downarrow & & \downarrow \\
K & \longrightarrow & X_{g'_0} \times_{S_{/qg'_0}} S_{/g},
\end{array}
$$

which is possible since the left vertical map is a trivial Kan fibration (since g' is a q-limit diagram).

The map $\overline{h}$ determines a diagram $\overline{f} : Y^\lhd \to X_{p_S/}$. We will prove that $\overline{f}$ is a q'-limit diagram. This will prove the "if" direction of (1) and (2); the "only if" direction of (1) will then follow from the uniqueness properties of q-limit diagrams.

To show that $\overline{f}$ is a q-limit diagram, we must show that every lifting problem of the form

$$
\begin{array}{ccc}
\partial\,\Delta^n \star Y & \xrightarrow{\ F\ } & X_{p_S/} \\
\downarrow & \nearrow & \downarrow{\scriptstyle q'} \\
\Delta^n \star Y & \longrightarrow & S
\end{array}
$$

admits a solution, provided that $n > 0$ and $F|\{n\}\star Y = \overline{f}$. Unwinding the definitions, we obtain an equivalent lifting problem

$$
\begin{array}{ccc}
(\partial\,\Delta^n \star Y) \times K^\rhd) \coprod_{(\partial\,\Delta^n\star Y)\times K}((\Delta^n \star Y) \times K) & \longrightarrow & X \\
\downarrow{\scriptstyle j} & \nearrow & \downarrow{\scriptstyle q} \\
(\Delta^n \star Y) \times K^\rhd & \longrightarrow & S.
\end{array}
$$

It now suffices to observe that j is a pushout of the inclusion $K \star \partial\,\Delta^n \star Y \hookrightarrow K \star \Delta^n \star Y$, so that the desired extension exists because $\overline{h}|\{v\} \times Y^\lhd = g'$ is a q-limit diagram. $\square$

Proof of Theorem 2.2.2.4. We will prove (1), (2), and (3) for the simplicial set $\mathcal{C}^\otimes_{p_\mathcal{O}/}$; the analogous assertions for $\mathcal{C}^\otimes_{/p_\mathcal{O}}$ will follow by the same reasoning. We first observe that q' is a categorical fibration (Lemma 2.2.2.6). Let $\overline{X} \in \mathcal{C}^\otimes_{p_\mathcal{O}/}$, and suppose we are given an inert morphism $\alpha : q(\overline{X}) \to Y$ in $\mathcal{O}^\otimes$; we wish to show that there exists a q'-coCartesian morphism $\overline{X} \to \overline{Y}$ in $\mathcal{C}^\otimes_{p_\mathcal{O}/}$ lifting α. This follows immediately from Lemma 2.2.2.8.

Suppose next that we are given an object $X \in \mathcal{O}^\otimes$ lying over $\langle n\rangle \in \mathcal{F}\mathrm{in}_*$, and a collection of inert morphisms $\alpha^i : X \to X_i$ lying over $\rho^i : \langle n\rangle \to \langle 1\rangle$ for $1 \le i \le n$. We wish to prove that the maps α^i induce an equivalence

$$
\theta : (\mathcal{C}^\otimes_{p_\mathcal{O}/})_X \simeq \prod_{1 \le i \le n} (\mathcal{C}^\otimes_{p_\mathcal{O}/})_{X_i}.
$$

Let $p_X : K \to \mathcal{C}^\otimes_X$ be the map induced by p, and define maps $p_{X_i} : K \to \mathcal{C}_{X_i}$ similarly. We observe that p_{X_i} can be identified with the composition of p_X with $\alpha^i_! : \mathcal{C}^\otimes_X \to \mathcal{C}_{X_i}$. Since q is a fibration of ∞-operads, we have an equivalence of ∞-categories

$$
\mathcal{C}^\otimes_X \to \prod_{1 \le i \le n} \mathcal{C}_{X_i}.
$$

Passing to the ∞-categories of objects under p, we deduce that θ is also an equivalence.

Now suppose that X is as above, that $\overline{X} \in \mathcal{C}^\otimes_{p_\mathcal{O}/}$ is a preimage of X, and that we are given q'-coCartesian morphisms $\overline{X} \to \overline{X}_i$ lying over the maps α^i. We wish to show that the induced map $\delta : \langle n\rangle^{\circ\lhd} \mathcal{C}^\otimes_{p_\mathcal{O}/}$ is a

q'-limit diagram. This follows from Lemma 2.2.2.9, since the image of δ in $\mathcal{C}^{\otimes}$ is a q-limit diagram. This completes the proof of (1). Moreover, our characterization of q'-coCartesian morphisms immediately implies (2). Assertion (3) follows immediately from Lemma 2.2.2.8. □

2.2.3 Coproducts of ∞-Operads

Let Op_{∞} denote the ∞-category of ∞-operads (Definition 2.1.4.1). Because Op_{∞} can be realized as underlying ∞-category of the combinatorial simplicial model category $\mathcal{P}\mathrm{Op}_{\infty}$ of ∞-preoperads, it admits all small limits and colimits (Corollary T.4.2.4.8). The limit of a diagram σ in Op_{∞} can usually be described fairly explicitly: namely, choose an injectively fibrant diagram $\overline{\sigma}$ in $\mathcal{P}\mathrm{Op}_{\infty}$ representing σ, and then take the limit of $\overline{\sigma}$ in the ordinary category of ∞-preoperads. The case of colimits is more difficult: we can apply the same procedure to construct an ∞-preoperad which represents $\varinjlim(\sigma)$, but this representative will generally not be fibrant and the process of "fibrant replacement" is fairly inexplicit. Our goal in this section is to give a more direct construction of colimits in a special case: namely, the case of coproducts. We can summarize our main results as follows: for every pair of ∞-operads $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$, we can explicitly construct a new ∞-operad $\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}$. This ∞-category comes equipped with fully faithful embeddings

$$\mathcal{C}^{\otimes} \hookrightarrow \mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes} \hookleftarrow \mathcal{D}^{\otimes}$$

(well-defined up to homotopy) which exhibit $\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}$ as a coproduct of $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ in the ∞-category Op_{∞} (Theorem 2.2.3.6), and exhibit the underlying ∞-category of $\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}$ as a coproduct of $\mathcal{C}$ and $\mathcal{D}$ in Cat_{∞}.

Before describing the construction of $\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}$, we need to establish some notation.

Notation 2.2.3.1. Given an object $\langle n \rangle \in \mathcal{F}\mathrm{in}_*$ and a subset $S \subseteq \langle n \rangle$ which contains the base point, there is a unique integer k and bijection $\langle k \rangle \simeq S$ whose restriction to $\langle k \rangle^{\circ}$ is order-preserving; we will denote the corresponding object of $\mathcal{F}\mathrm{in}_*$ by $[S]$.

Definition 2.2.3.2. We define a category Sub as follows:

(1) The objects of Sub are triples $(\langle n \rangle, S, T)$ where $\langle n \rangle \in \mathcal{F}\mathrm{in}_*$, S and T are subsets of $\langle n \rangle$ such that $S \cup T = \langle n \rangle$ and $S \cap T = \{*\}$.

(2) A morphism from $(\langle n \rangle, S, T)$ to $(\langle n' \rangle, S', T')$ in Sub is a morphism $f : \langle n \rangle \to \langle n' \rangle$ in $\mathcal{F}\mathrm{in}_*$ such that $f(S) \subseteq S'$ and $f(T) \subseteq T'$.

There is an evident triple of functors $\pi, \pi_-, \pi_+ : \mathrm{Sub} \to \mathcal{F}\mathrm{in}_*$, given by the formulas

$$\pi_-(\langle n \rangle, S, T) = [S] \qquad \pi(\langle n \rangle, S, T) = \langle n \rangle \qquad \pi_+(\langle n \rangle, S, T) = [T].$$

Construction 2.2.3.3. For any pair of simplicial sets $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ equipped with maps $\mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \leftarrow \mathcal{D}^{\otimes}$, we define a new simplicial set $\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}$ so that we have a pullback diagram

$$\begin{array}{ccc} \mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes} & \longrightarrow & \mathcal{C}^{\otimes} \times \mathcal{D}^{\otimes} \\ \downarrow & & \downarrow \\ \mathrm{N}(\mathrm{Sub}) & \xrightarrow{\pi_- \times \pi_+} & \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \mathrm{N}(\mathcal{F}\mathrm{in}_*). \end{array}$$

We regard $\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}$ as equipped with a map to $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, given by the composition

$$\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes} \to \mathrm{N}(\mathrm{Sub}) \xrightarrow{\pi} \mathrm{N}(\mathcal{F}\mathrm{in}_*).$$

Remark 2.2.3.4. The product functor $(\pi_- \times \pi_+) : \mathrm{Sub} \to \mathcal{F}\mathrm{in}_* \times \mathcal{F}\mathrm{in}_*$ is an equivalence of categories. Consequently, $\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}$ is equivalent (as an ∞-category) to the product $\mathcal{C}^{\otimes} \times \mathcal{D}^{\otimes}$. However, it is slightly better behaved in the following sense: the composite map

$$\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes} \to \mathrm{N}(\mathrm{Sub}) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$$

is a categorical fibration, since it is the composition of a pullback of the categorical fibration $\mathcal{C}^{\otimes} \times \mathcal{D}^{\otimes} \to$ $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ with the categorical fibration $\mathrm{N}(\mathrm{Sub}) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

The main results of this section can now be stated as follows:

Proposition 2.2.3.5. *Let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be ∞-operads. Then $\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}$ is an ∞-operad.*

Theorem 2.2.3.6. *Let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be ∞-operads. We let $(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_-$ denote the full subcategory of $\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}$ spanned by those objects whose image in Sub has the form $(\langle n \rangle, \langle n \rangle, \{*\})$, and let $(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_+$ denote the full subcategory spanned by those objects whose image in Sub has the form $(\langle n \rangle, \{*\}, \langle n \rangle)$. Then:*

(1) *The projection maps $(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_- \to \mathcal{C}^{\otimes}$ and $(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_+ \to \mathcal{D}^{\otimes}$ are trivial Kan fibrations.*

(2) *The map $\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ exhibits both $(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_-$ and $(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_+$ as ∞-operads.*

(3) *For any ∞-operad $\mathcal{E}^{\otimes}$, the inclusions*

$$(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_- \overset{i}{\hookrightarrow} \mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes} \overset{j}{\hookleftarrow} (\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_+$$

induce an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{lax}}(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}, \mathcal{E}^{\otimes}) \to \mathrm{Fun}^{\mathrm{lax}}((\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_-, \mathcal{E}^{\otimes}) \times \mathrm{Fun}^{\mathrm{lax}}((\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_+, \mathcal{E}^{\otimes});$$

here we let $\mathrm{Fun}^{\mathrm{lax}}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes})$ denote the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$ of ∞-operad maps from $\mathcal{O}^{\otimes}$ to $\mathcal{O}'^{\otimes}$, for any pair of ∞-operads $\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}$. In particular, i and j exhibit $\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}$ as a coproduct of $(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_- \simeq \mathcal{C}^{\otimes}$ and $(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_+ \simeq \mathcal{D}^{\otimes}$ in the ∞-category Op_{∞}.

Remark 2.2.3.7. The operation $\boxplus$ of Construction 2.2.3.3 is commutative and associative up to coherent isomorphism, and determines a symmetric monoidal structure on the category $(\mathcal{S}\mathrm{et}_{\Delta})_{/\mathrm{N}(\mathcal{F}\mathrm{in}_*)}$ of simplicial sets X endowed with a map $X \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. This restricts to a symmetric monoidal structure on the (ordinary) category of ∞-operads and maps of ∞-operads.

Remark 2.2.3.8. We can informally summarize Theorem 2.2.3.6 as follows: for every triple of ∞-operads $\mathcal{O}_-^{\otimes}$, $\mathcal{O}_+^{\otimes}$, and $\mathcal{C}^{\otimes}$, we have a canonical equivalence of ∞-categories

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}_-}(\mathcal{C}) \times \mathrm{Alg}_{\mathcal{O}_+}(\mathcal{C}),$$

where $\mathcal{O}^{\otimes} = \mathcal{O}_-^{\otimes} \boxplus \mathcal{O}_+^{\otimes}$.

Proof of Proposition 2.2.3.5. Let X be an object of $(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_{\langle n \rangle}$, given by a quintuple $(\langle n \rangle, S, T, C, D)$ where $(\langle n \rangle, S, T) \in \mathrm{Sub}$, $C \in \mathcal{C}_{[S]}^{\otimes}$, and $D \in \mathcal{D}_{[T]}^{\otimes}$. Suppose we are given an inert map $\alpha : \langle n \rangle \to \langle n' \rangle$ in $\mathcal{F}\mathrm{in}_*$. Let $S' = \alpha(S)$ and $T' = \alpha(T)$. Then α induces inert morphisms $\alpha_- : [S] \to [S']$ and $\alpha_+ : [T] \to [T']$. Choose an inert morphism $f_- : C \to C'$ in $\mathcal{C}^{\otimes}$ lifting α_- and an inert morphism $f_+ : D \to D'$ in $\mathcal{D}^{\otimes}$ lifting α_+. These maps together determine a morphism $f : (\langle n \rangle, S, T, C, D) \to (\langle n' \rangle, S', T', C', D')$. Since (f_-, f_+) is coCartesian with respect to the projection $\mathcal{C}^{\otimes} \times \mathcal{D}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, the map f is p-coCartesian, where p denotes the map $\mathcal{C}^{\otimes} \boxtimes \mathcal{D}^{\otimes} \to \mathrm{N}(\mathrm{Sub})$. Let $\pi : \mathrm{N}(\mathrm{Sub}) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be as in Definition 2.2.3.2. It is easy to see that $p(f)$ is π-coCartesian, so that f is $(\pi \circ p)$-coCartesian by virtue of Proposition T.2.4.1.3.

Choose $(\pi \circ p)$-coCartesian morphisms $X \to X_i$ covering the inert morphisms $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \le i \le n$. We claim that these maps exhibit X as a $(\pi \circ p)$-product of the objects X_i. Using our assumption that $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are ∞-operads, we deduce that these maps exhibit X as a p-product of the objects $\{X_i\}$. It therefore suffices to show that they exhibit $p(X)$ as a π-product of the objects $p(X_i)$ in the ∞-category $\mathrm{N}(\mathrm{Sub})$, which follows immediately from the definitions.

It remains only to show that for each $n \ge 0$, the canonical functor $\phi : (\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_{\langle n \rangle} \to (\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_{\langle 1 \rangle}^n$ is essentially surjective. We can identify the latter with $(\mathcal{C} \coprod \mathcal{D})^n$. Given a collection of objects $X_1, \dots, X_n$ of $\mathcal{C} \coprod \mathcal{D}$, we let $S = \{*\} \cup \{i : X_i \in \mathcal{C}\}$ and $T = \{*\} \cup \{i : X_i \in \mathcal{D}\}$. Let $C = \bigoplus_{X_i \in \mathcal{C}} X_i \in \mathcal{C}_{[S]}^{\otimes}$ and let $D = \bigoplus_{X_i \in \mathcal{D}} X_i \in \mathcal{D}_{[T]}^{\otimes}$. Then $X = (\langle n \rangle, S, T, C, D)$ is an object of $(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_{\langle n \rangle}$ such that $\phi(X) \simeq (X_1, \dots, X_n)$. $\square$

Proof of Theorem 2.2.3.6. Assertion (1) follows from the evident isomorphisms

$$(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_{-} \simeq \mathcal{C}^{\otimes} \times \mathcal{D}^{\otimes}_{\langle 0 \rangle} \qquad (\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_{+} \simeq \mathcal{C}^{\otimes}_{\langle 0 \rangle} \times \mathcal{D}^{\otimes},$$

together with the observation that $\mathcal{C}^{\otimes}_{\langle 0 \rangle}$ and $\mathcal{D}^{\otimes}_{\langle 0 \rangle}$ are contractible Kan complexes. Assertion (2) follows immediately from (1). To prove (3), let $\mathcal{X} = (\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_{-} \cap (\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_{+} \simeq \mathcal{C}^{\otimes}_{\langle 0 \rangle} \times \mathcal{D}^{\otimes}_{\langle 0 \rangle}$ and let $\mathcal{Y} = (\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_{-} \cup (\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_{+}$. Let $\mathcal{A}$ denote the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{Y}, \mathcal{E}^{\otimes})$ spanned by those functors whose restriction to both $(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_{-}$ and $(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_{+}$ are ∞-operad maps. We have homotopy pullback diagram

$$
\begin{array}{ccc}
\mathcal{A} & \longrightarrow & \mathrm{Fun}^{\mathrm{lax}}((\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_{-}, \mathcal{E}^{\otimes}) \\
\downarrow & & \downarrow \\
\mathrm{Fun}^{\mathrm{lax}}((\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_{+}, \mathcal{E}^{\otimes}) & \longrightarrow & \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{X}, \mathcal{E}^{\otimes}).
\end{array}
$$

Since $\mathcal{E}^{\otimes}_{\langle 0 \rangle}$ is a contractible Kan complex, the simplicial set $\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{X}, \mathcal{E}^{\otimes})$ is also a contractible Kan complex, so the map

$$\mathcal{A} \to \mathrm{Fun}^{\mathrm{lax}}((\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_{-}, \mathcal{E}^{\otimes}) \times \mathrm{Fun}^{\mathrm{lax}}((\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_{+}, \mathcal{E}^{\otimes})$$

is a categorical equivalence. We will complete the proof by showing that the map $\mathrm{Fun}^{\mathrm{lax}}(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}, \mathcal{E}^{\otimes}) \to \mathcal{A}$ is a trivial Kan fibration.

Let $q : \mathcal{E}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ denote the projection map. In view of Proposition T.4.3.2.15, it will suffice to show the following:

(a) An arbitrary map $A \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}, \mathcal{E}^{\otimes})$ is an ∞-operad map if and only if it satisfies the following conditions:

  (i) The restriction $A_0 = A | \mathcal{Y}$ belongs to $\mathcal{A}$.

  (ii) The map A is a q-right Kan extension of A_0.

(b) For every object $A_0 \in \mathcal{A}$, there exists an extension $A \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}, \mathcal{E}^{\otimes})$ of A_0 which satisfies the equivalent conditions of (a).

To prove (a), consider an object $A \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}, \mathcal{E}^{\otimes})$ and an object $X = (\langle n \rangle, S, T, C, D) \in \mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}$. Choose morphisms $\alpha : C \to C_0$ and $\beta : D \to D_0$, where $C_0 \in \mathcal{C}^{\otimes}_{\langle 0 \rangle}$ and $D_0 \in \mathcal{D}^{\otimes}_{\langle 0 \rangle}$. Set

$$X_{-} = ([S], [S], \{*\}, C, D_0) \qquad X_0 = (\langle 0 \rangle, \{*\}, \{*\}, C_0, D_0) \qquad X_{+} = ([T], \{*\}, [T], C_0, D).$$

Then α and β determine a commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X_{-} \\
\downarrow & & \downarrow \\
X_{+} & \longrightarrow & X_0.
\end{array}
$$

Using Theorem T.4.1.3.1, we deduce that this diagram determines a map

$$\phi : \Lambda^2_2 \to \mathcal{Y} \times_{\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}} (\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_{X/}$$

such that ϕ is right cofinal. It follows that A is a q-right Kan extension of A_0 at X if and only if the diagram

$$
\begin{array}{ccc}
A(X) & \longrightarrow & A(X_{-}) \\
\downarrow & & \downarrow \\
A(X_{+}) & \longrightarrow & A(X_0)
\end{array}
$$

is a q-pullback diagram. Since $q : \mathcal{E}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is an ∞-operad, this is equivalent to the requirement that the maps $A(X_-) \leftarrow A(X) \to A(X_+)$ are inert. In other words, we obtain the following version of (a):

(a') A map $A \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}, \mathcal{E}^{\otimes})$ is a q-right Kan extension of $A_0 = A|\mathcal{Y}$ if and only if, for every object $X \in \mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}$ as above, the maps $A(X_-) \leftarrow A(X) \to A(X_+)$ are inert.

We now prove (a) Suppose first that A is an ∞-operad map; we wish to prove that A satisfies conditions (i) and (ii). Condition (i) follows immediately from the description of the inert morphisms in $\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes}$ provided by the proof of Proposition 2.2.3.5, and condition (ii) follows from (a'). Conversely, suppose that (i) and (ii) are satisfied. We wish to prove that A preserves inert morphisms. In view of Remark 2.1.2.9, it will suffice to show that $A(X) \to A(Y)$ is inert whenever $X \to Y$ is an inert morphism such that $Y \in (\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_{\langle 1 \rangle}$. It follows that $Y \in \mathcal{Y}$; we may therefore assume without loss of generality that $Y \in (\mathcal{C}^{\otimes} \boxplus \mathcal{D}^{\otimes})_-$. The map $X \to Y$ then factors as a composition of inert morphisms $X \to X_- \to Y$, where X_- is defined as above. Then $A(X) \to A(X_-)$ is inert by virtue of (ii) and (a'), while $A(X_-) \to A(Y)$ is inert by virtue of assumption (i).

To prove (b), it will suffice (by virtue of Lemma T.4.3.2.13) to show that for each $X \in \mathcal{C}^{\otimes} \boxtimes \mathcal{D}^{\otimes}$, the induced diagram

$$\mathcal{Y}_{X/} \to \mathcal{Y} \xrightarrow{A_0} \mathcal{E}^{\otimes}$$

admits a q-limit. Since ϕ is right cofinal, it suffices to show that there exists a q-limit of the diagram

$$A_0(X_-) \to A_0(X_0) \leftarrow A_0(X_+).$$

The existence of this q-limit follows immediately from the assumption that $\mathcal{E}^{\otimes}$ is an ∞-operad. $\square$

2.2.4   Monoidal Envelopes

Every symmetric monoidal ∞-category $\mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ can be regarded as an ∞-operad, and every symmetric monoidal functor determines a map between the underlying ∞-operads. This observation provides a forgetful functor from the ∞-category $\mathcal{C}\mathrm{at}_{\infty}^{\otimes} \simeq \mathrm{CAlg}(\mathcal{C}\mathrm{at}_{\infty})$ of symmetric monoidal ∞-categories to the ∞-category Op_{∞} of ∞-operads. Our goal in this section is to construct a left adjoint to this forgetful functor. More generally, we will give a construction which converts an arbitrary fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ into a coCartesian fibration of ∞-operads $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$.

Construction 2.2.4.1. Let $\mathcal{O}^{\otimes}$ be an ∞-operad. We let $\mathrm{Act}(\mathcal{O}^{\otimes})$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$ spanned by the active morphisms. Suppose that $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a fibration of ∞-operads. We let $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ denote the fiber product

$$\mathcal{C}^{\otimes} \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} \mathrm{Act}(\mathcal{O}^{\otimes}).$$

We will refer to $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ as the $\mathcal{O}$-*monoidal envelope* of $\mathcal{C}^{\otimes}$. In the special case where $\mathcal{O}^{\otimes}$ is the commutative ∞-operad, we will denote $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ simply by $\mathrm{Env}(\mathcal{C})^{\otimes}$.

Remark 2.2.4.2. More informally, we can identify $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ with the ∞-category of pairs (C, α), where $C \in \mathcal{C}^{\otimes}$ and $\alpha : p(C) \to X$ is an active morphism in $\mathcal{O}^{\otimes}$.

Remark 2.2.4.3. Let $\mathcal{C}^{\otimes}$ be an ∞-operad. Evaluation at $\{1\} \subseteq \Delta^1$ induces a map $\mathrm{Env}(\mathcal{C})^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. We let $\mathrm{Env}(\mathcal{C})$ denote the fiber $\mathrm{Env}(\mathcal{C})_{\langle 1 \rangle}^{\otimes}$. Unwinding the definitions, we deduce that $\mathrm{Env}(\mathcal{C})$ can be identified with the subcategory $\mathcal{C}_{\mathrm{act}}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ spanned by all objects and active morphisms between them.

We will defer the proof of the following basic result until the end of this section:

Proposition 2.2.4.4. *Let* $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ *be a fibration of ∞-operads. Then evaluation at the vertex $\{1\} \subseteq \Delta^1$ induces a coCartesian fibration of ∞-operads* $q : \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$.

Taking $\mathcal{O}^{\otimes}$ to be the commutative ∞-operad, we deduce the following:

Corollary 2.2.4.5. *Let $\mathcal{C}^{\otimes}$ be an ∞-operad. Then there is a canonical symmetric monoidal structure on the ∞-category $\mathcal{C}^{\otimes}_{\mathrm{act}}$ of active morphisms.*

Remark 2.2.4.6. If $\mathcal{C}^{\otimes}$ is an ∞-operad, we let $\oplus : \mathcal{C}^{\otimes}_{\mathrm{act}} \times \mathcal{C}^{\otimes}_{\mathrm{act}} \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ denote the functor induced by the symmetric monoidal structure described in Corollary 2.2.4.5. This operation can be described informally as follows: if $X \in \mathcal{C}^{\otimes}_{\langle m \rangle}$ classifies a sequence of objects $\{X_i \in \mathcal{C}\}_{1 \le i \le m}$ and $Y \in \mathcal{C}^{\otimes}_{\langle n \rangle}$ classifies a sequence $\{Y_j \in \mathcal{C}\}_{1 \le j \le n}$, then $X \oplus Y \in \mathcal{C}^{\otimes}_{\langle m+n \rangle}$ corresponds to the sequence of objects $\{X_i \in \mathcal{C}\}_{1 \le i \le m} \cup \{Y_j \in \mathcal{C}\}_{1 \le j \le n}$ obtained by concatenation.

Remark 2.2.4.7. The symmetric monoidal structure on $\mathcal{C}^{\otimes}_{\mathrm{act}}$ described in Corollary 2.2.4.5 can actually be extended to a symmetric monoidal structure on $\mathcal{C}^{\otimes}$ itself, but we will not need this.

Remark 2.2.4.8. Let $\mathcal{O}^{\otimes}$ be an ∞-operad, and let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a $\mathcal{O}$-monoidal ∞-category. Then the collection of active q-coCartesian morphisms in $\mathcal{C}^{\otimes}$ is stable under the operation $\oplus : \mathcal{C}^{\otimes}_{\mathrm{act}} \times \mathcal{C}^{\otimes}_{\mathrm{act}} \to \mathcal{C}^{\otimes}_{\mathrm{act}}$. To see this, let $\alpha : C \to C'$ and $\beta : D \to D'$ be active q-coCartesian morphisms in $\mathcal{C}^{\otimes}$. Let $\gamma : C \oplus D \to E$ be a q-coCartesian morphism lifting $q(\alpha \oplus \beta)$. We have a commutative diagram in $\mathcal{O}^{\otimes}$

$$
\begin{array}{ccccc}
p(C) & \longleftarrow & p(C \oplus D) & \longrightarrow & p(D) \\
\downarrow & & \downarrow & & \downarrow \\
p(C') & \longleftarrow & p(C' \oplus D') & \longrightarrow & p(D')
\end{array}
$$

We can lift this to a diagram of q-coCartesian morphisms

$$
\begin{array}{ccccc}
C & \longleftarrow & C \oplus D & \longrightarrow & D \\
\downarrow & & \downarrow & & \downarrow \\
C' & \longleftarrow & E & \longrightarrow & D'.
\end{array}
$$

Let $\delta : E \to C' \oplus D'$ be the canonical map in $\mathcal{C}^{\otimes}_{p(C' \oplus D')}$; the above diagram shows that the image of δ is an equivalence in both $\mathcal{C}^{\otimes}_{p(C')}$ and $\mathcal{C}^{\otimes}_{p(D')}$. Since $\mathcal{C}^{\otimes}_{p(C' \oplus D')} \simeq \mathcal{C}^{\otimes}_{p(C')} \times \mathcal{C}^{\otimes}_{p(D')}$, it follows that δ is an equivalence, so that $\alpha \oplus \beta$ is q-coCartesian as desired.

For any ∞-operad $\mathcal{O}^{\otimes}$, the diagonal embedding $\mathcal{O}^{\otimes} \to \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$ factors through $\mathrm{Act}(\mathcal{O}^{\otimes})$. Pullback along this embedding induces an inclusion $\mathcal{C}^{\otimes} \subseteq \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ for any fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$. It follows from Proposition 2.2.4.4 and Lemma 2.2.4.16 (below) that this inclusion is a map of ∞-operads. The terminology "$\mathcal{O}$-monoidal envelope" is justified by the following result:

Proposition 2.2.4.9. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads and $p' : \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ the induced coCartesian fibration of ∞-operads, and let $q : \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ be another coCartesian fibration of ∞-operads. The inclusion $i : \mathcal{C}^{\otimes} \subseteq \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ induces an equivalence of ∞-categories*

$$
\mathrm{Fun}^{\otimes}_{\mathcal{O}}(\mathrm{Env}_{\mathcal{O}}(\mathcal{C}), \mathcal{D}) \to \mathrm{Alg}_{\mathcal{C}}(\mathcal{D}).
$$

Here $\mathrm{Fun}^{\otimes}_{\mathcal{O}}(\mathcal{U}_{\mathcal{O}}(\mathcal{C}), \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}, \mathcal{D}^{\otimes})$ spanned by those functors which carry p'-coCartesian morphisms to q-coCartesian morphisms, and $\mathrm{Alg}_{\mathcal{C}}(\mathcal{D})$ the full subcategory of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ spanned by the maps of ∞-operads.

Remark 2.2.4.10. Since every diagonal embedding $\mathcal{O}^{\otimes} \to \mathrm{Act}(\mathcal{O}^{\otimes})$ is fully faithful, we conclude that for every fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ the inclusion $\mathcal{C}^{\otimes} \hookrightarrow \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ is fully faithful. In particular, we deduce that for every ∞-operad $\mathcal{C}^{\otimes}$ there exists a fully faithful ∞-operad map $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$, where $\mathcal{D}^{\otimes}$ is a symmetric monoidal ∞-category.

We now turn to the proofs of Propositions 2.2.4.4 and 2.2.4.9. We will need several preliminary results.

Lemma 2.2.4.11. *Let $p : \mathcal{C} \to \mathcal{D}$ be a coCartesian fibration of ∞-categories. Let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}$ satisfying the following conditions:*

(i) For each $D \in \mathcal{D}$, the inclusion $\mathcal{C}'_D \subseteq \mathcal{C}_D$ admits a left adjoint L_D.

(ii) For every morphism $f : D \to D'$ in $\mathcal{D}$, the associated functor $f_! : \mathcal{C}_D \to \mathcal{C}_{D'}$ carries L_D-equivalences to $L_{D'}$-equivalences.

Then:

(1) The restriction $p' = p| \, \mathcal{C}'$ is a coCartesian fibration.

(2) Let $f : C \to C'$ be a morphism in $\mathcal{C}'$ lying over $g : D \to D'$ in $\mathcal{D}$, and let $g_! : \mathcal{C}_D \to \mathcal{C}_{D'}$ be the functor induced by the coCartesian fibration p. Then f is p'-coCartesian if and only if the induced map $\alpha : g_! C \to C'$ is an $L_{D'}$-equivalence.

Proof. We first prove the "if" direction of (2). According to Proposition T.2.4.4.3, it will suffice to show that for every object $C'' \in \mathcal{C}'$ lying over $D'' \in \mathcal{D}$, the outer square in the homotopy coherent diagram

$$
\begin{array}{ccccc}
\mathrm{Map}_{\mathcal{C}}(C', C'') & \xrightarrow{\;\theta\;} & \mathrm{Map}_{\mathcal{C}}(g_! C, C'') & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(C, C'') \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Map}_{\mathcal{D}}(D', D'') & \longrightarrow & \mathrm{Map}_{\mathcal{D}}(D', D'') & \longrightarrow & \mathrm{Map}_{\mathcal{D}}(D, D'')
\end{array}
$$

is a homotopy pullback square. Since the right square is a homotopy pullback (Proposition T.2.4.4.3), it will suffice to show that θ induces a homotopy equivalence after passing the homotopy fiber over any map $h : D' \to D''$. Using Proposition T.2.4.4.2, we see that this is equivalent to the assertion that the map $h_!(\alpha)$ is an $L_{D''}$-equivalence. This follows from (ii), since α is an $L_{D'}$-equivalence.

To prove (1), it will suffice to show that for every $C \in \mathcal{C}'$ and every $g : p(C) \to D'$ in $\mathcal{D}$, there exists a morphism $f : C \to C'$ lying over g satisfying the criterion of (2). We can construct f as a composition $C \xrightarrow{f'} C'' \xrightarrow{f''} C'$, where f' is a p-coCartesian lift of g in $\mathcal{C}$, and $f'' : C'' \to C'$ exhibits C' as an $\mathcal{C}'_{D'}$-localization of C''.

We conclude by proving the "only if" direction of (2). Let $f : C \to C'$ be a p'-coCartesian morphism in $\mathcal{C}'$ lying over $g : D \to D'$. Choose a factorization of f as a composition $C \xrightarrow{f'} C'' \xrightarrow{f''} C'$, where f' is p-coCartesian and f'' is a morphism in $\mathcal{C}_{D'}$. The map f'' admits a factorization as a composition $C'' \xrightarrow{h} C''' \xrightarrow{h'} C'$, where h exhibits C''' as a $\mathcal{C}'_{D'}$-localization of C''. The first part of the proof shows that the composition $h \circ f'$ is a p'-coCartesian lift of g. Since f is also a p'-coCartesian lift of g, we deduce that h' is an equivalence. It follows that $f'' = h' \circ h$ exhibits C' as a $\mathcal{C}'_{D'}$-localization of $C'' \simeq g_! C$, so that f satisfies the criterion of (2). $\qquad\square$

Remark 2.2.4.12. Let $\mathcal{C}' \subseteq \mathcal{C}$ and $p : \mathcal{C} \to \mathcal{D}$ be as in Lemma 2.2.4.11. Hypotheses (i) and (ii) are equivalent to the following:

(i′) The inclusion $\mathcal{C}' \subseteq \mathcal{C}$ admits a left adjoint L.

(ii′) The functor p carries each L-equivalence in $\mathcal{C}$ to an equivalence in $\mathcal{D}$.

Suppose first that (i) and (ii) are satisfied. To prove (i') and (ii'), it will suffice (by virtue of Proposition T.5.2.7.8) to show that for each object $C \in \mathcal{C}_D$, if $f : C \to C'$ exhibits C' as a $\mathcal{C}'_D$-localization of C, then f exhibits C' as a $\mathcal{C}'$-localization of C. In other words, we must show that for each $C'' \in \mathcal{C}'$ lying over $D'' \in \mathcal{D}$, composition with f induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(C', C'') \to \mathrm{Map}_{\mathcal{C}}(C, C'')$. Using Proposition T.2.4.4.2, we can reduce to showing that for every morphism $g : D \to D''$ in $\mathcal{D}$, the induced map

$\mathrm{Map}_{\mathcal{C}_{D''}}(g_!C', C'') \to \mathrm{Map}_{\mathcal{C}_{D''}}(g_!C, C'')$ is a homotopy equivalence. For this, it suffices to show that $g_!(f)$ is an $L_{D''}$-equivalence, which follows immediately from (ii).

Conversely, suppose that (i') and (ii') are satisfied. We first prove (i). Fix $D \in \mathcal{D}$. To prove that the inclusion $\mathcal{C}'_D \subseteq \mathcal{C}_D$ admits a left adjoint, it will suffice to show that for each object $C \in \mathcal{C}_D$ there exists a $\mathcal{C}'_D$-localization of C (Proposition T.5.2.7.8). Fix a map $f : C \to C'$ in $\mathcal{C}$ which exhibits C' as a $\mathcal{C}'$-localization of C. Assumption (ii') guarantees that $p(f)$ is an equivalence in $\mathcal{D}$. Replacing f by an equivalent morphism if necessary, we may suppose that $p(f) = \mathrm{id}_D$ so that f is a morphism in $\mathcal{C}_D$. We claim that f exhibits C' as a $\mathcal{C}'_D$-localization of C. To prove this, it suffices to show that for each $C'' \in \mathcal{C}'_D$, composition with f induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}_D}(C', C'') \to \mathrm{Map}_{\mathcal{C}_D}(C, C'')$. Using Proposition T.2.4.4.2, we can reduce to showing that f induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(C', C'') \to \mathrm{Map}_{\mathcal{C}}(C, C'')$, which follows from the assumption that f exhibits C' as a $\mathcal{C}'$-localization of C.

We now prove (ii). Let $f : C \to C'$ be an L_D-equivalence in $\mathcal{C}_D$, and let $g : D \to D''$ be a morphism in $\mathcal{D}$. We wish to show that $g_!(f)$ is an $L_{D''}$-equivalence in $\mathcal{C}_{D''}$. In other words, we wish to show that for each object $C'' \in \mathcal{C}'_{D''}$, the map $\mathrm{Map}_{\mathcal{C}_{D''}}(g_!C', C'') \to \mathrm{Map}_{\mathcal{C}_{D''}}(g_!C, C'')$ is a homotopy equivalence. This follows from Proposition T.2.4.4.2 and the fact that $\mathrm{Map}_{\mathcal{C}}(C', C'') \to \mathrm{Map}_{\mathcal{C}}(C, C'')$ is a homotopy equivalence.

Lemma 2.2.4.13. *Let $p : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories. Let $\mathcal{D}' \subseteq \mathcal{D}$ be a full subcategory and set $\mathcal{C}' = \mathcal{D}' \times_{\mathcal{D}} \mathcal{C}$. Assume that:*

(i) *The inclusion $\mathcal{D}' \subseteq \mathcal{D}$ admits a left adjoint.*

(ii) *Let $C \in \mathcal{C}$ be an object and let $g : p(C) \to D$ be a morphism which exhibits D as a $\mathcal{D}'$-localization of $p(C)$. Then g can be lifted to a p-coCartesian morphism $C \to C'$.*

Then:

(1) *A morphism $f : C \to C'$ exhibits C' as a $\mathcal{C}'$-localization of C if and only if $p(f)$ exhibits $p(C')$ as a $\mathcal{D}'$-localization of $p(C)$ and f is p-coCartesian.*

(2) *The inclusion $\mathcal{C}' \subseteq \mathcal{C}$ admits a left adjoint.*

Proof. We first prove the "if" direction of (1). Fix an object $C'' \in \mathcal{C}'$; we wish to prove that f induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(C', C'') \to \mathrm{Map}_{\mathcal{C}}(C, C'')$. Using Proposition T.2.4.4.3, we deduce that the homotopy coherent diagram

$$
\begin{array}{ccc}
\mathrm{Map}_{\mathcal{C}}(C', C'') & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(C, C'') \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathcal{D}}(p(C'), p(C'')) & \longrightarrow & \mathrm{Map}_{\mathcal{D}}(p(C), p(C''))
\end{array}
$$

is a homotopy pullback square. It therefore suffices to show that the bottom horizontal map is a homotopy equivalence, which follows from the assumptions that $p(C'') \in \mathcal{D}'$ and $p(f)$ exhibits $p(C')$ as a $\mathcal{D}'$-localization of $p(C)$.

Assertion (2) now follows from (ii) together with Proposition T.5.2.7.8. To complete the proof, we verify the "only if" direction of (1). Let $f : C \to C'$ be a map which exhibits C' as a $\mathcal{C}'$-localization of C, and let $g : D \to D'$ be the image of f in $\mathcal{D}$. Then f factors as a composition

$$C \xrightarrow{f'} g_!C \xrightarrow{f''} C';$$

we wish to prove that f'' is an equivalence. This follows from the first part of the proof, which shows that f' exhibits $g_!C$ as a $\mathcal{C}'$-localization of C. $\square$

Lemma 2.2.4.14. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, and let $\mathcal{D} = \mathcal{C}^{\otimes} \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$. Then the inclusion $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes} \subseteq \mathcal{D}$ admits a left adjoint. Moreover, a morphism $\alpha : D \to D'$ in $\mathcal{D}$ exhibits D' as an $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$-localization of D if and only if D' is active, the image of α in $\mathcal{C}^{\otimes}$ is inert, and the image of α in $\mathcal{O}^{\otimes}$ is an equivalence.*

Proof. According to Proposition 2.1.2.4, the active and inert morphisms determine a factorization system on $\mathcal{O}^{\otimes}$. It follows from Lemma T.5.2.8.19 that the inclusion $\mathrm{Act}(\mathcal{O}^{\otimes}) \subseteq \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$ admits a left adjoint, and that a morphism $\alpha : g \to g'$ in $\mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$ corresponding to a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\;f\;} & X' \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\
Y & \xrightarrow{\;f'\;} & Y'
\end{array}
$$

in $\mathcal{O}^{\otimes}$ exhibits g' as an $\mathrm{Act}(\mathcal{O}^{\otimes})$-localization of g if and only if g' is active, f is inert, and f' is an equivalence. The desired result now follows from Lemma 2.2.4.13. $\square$

Lemma 2.2.4.15. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, and let $\mathcal{D} = \mathcal{C}^{\otimes} \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$. Then:*

(1) *Evaluation at $\{1\}$ induces a coCartesian fibration $q' : \mathcal{D} \to \mathcal{O}^{\otimes}$.*

(2) *A morphism in $\mathcal{D}$ is q'-coCartesian if and only if its image in $\mathcal{C}^{\otimes}$ is an equivalence.*

(3) *The map q' restricts to a coCartesian fibration $q : \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$.*

(4) *A morphism f in $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ is q-coCartesian if and only if its image in $\mathcal{C}^{\otimes}$ is inert.*

Proof. Assertions (1) and (2) follow from Corollary T.2.4.7.12. Assertions (3) and (4) follow by combining Lemma 2.2.4.14, Remark 2.2.4.12, and Lemma 2.2.4.11. $\square$

Lemma 2.2.4.16. *Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{F}\mathrm{in}_*))$ spanned by the active morphisms, and let $p : \mathcal{C} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be given by evaluation on the vertex 1. Let $X \in \mathcal{C}$ be an object with $p(X) = \langle n \rangle$, and choose p-coCartesian morphisms $f_i : X \to X_i$ covering the maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \le i \le n$. These morphisms determine a p-limit diagram $\langle n \rangle^{\circ \triangleleft} \to \mathcal{C}$.*

Proof. Let X be given by an active morphism $\beta : \langle m \rangle \to \langle n \rangle$. Each of the maps f_i can be identified with a commutative diagram

$$
\begin{array}{ccc}
\langle m \rangle & \xrightarrow{\;\gamma_i\;} & \langle m_i \rangle \\
\downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta_i} \\
\langle n \rangle & \xrightarrow{\;\rho^i\;} & \langle 1 \rangle
\end{array}
$$

where β_i is active and γ_i is inert. Unwinding the definitions, we must show the following:

(∗) Given an active morphism $\beta' : \langle m' \rangle \to \langle n' \rangle$ in $\mathcal{F}\mathrm{in}_*$, a map $\delta : \langle n' \rangle \to \langle n \rangle$, and a collection of commutative diagrams

$$
\begin{array}{ccc}
\langle m' \rangle & \xrightarrow{\;\epsilon_i\;} & \langle m_i \rangle \\
\downarrow & & \downarrow{\scriptstyle \beta_i} \\
\langle n' \rangle & \xrightarrow{\;\rho^i \circ \delta\;} & \langle 1 \rangle,
\end{array}
$$

there is a unique morphism $\epsilon : \langle m' \rangle \to \langle m \rangle$ such that $\epsilon_i = \gamma_i \circ \epsilon$.

For each $j \in \langle m' \rangle$, let $j' = (\delta \circ \beta')(j) \in \langle n \rangle$. Then ϵ is given by the formula

$$
\epsilon(j) = \begin{cases} * & \text{if } j' = * \\ \gamma_j'^{-1}(\epsilon_i(j)) & \text{if } j' \ne *. \end{cases}
$$

$\square$

Proof of Proposition 2.2.4.4. Lemma 2.2.4.15 implies that q is a coCartesian fibration. It will therefore suffice to show that $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ is an ∞-operad. Let r denote the composition

$$\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes} \xrightarrow{q} \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*).$$

Let $X \in \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}_{\langle n \rangle}$, and choose r-coCartesian morphisms $X \to X_i$ covering $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \le i \le n$. We wish to prove that these morphisms determine an r-limit diagram $\alpha : \langle n \rangle^{\circ \triangleleft} \to \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$.

Let $\mathcal{D} = \mathcal{C}^{\otimes} \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$. Then r extends to a map $r' : \mathcal{D} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. To show that α is an r-limit diagram, it will suffice to show that α is an r'-limit diagram. Write r' as a composition

$$\mathcal{D} \xrightarrow{r'_0} \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes}) \xrightarrow{r'_1} \mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{F}\mathrm{in}_*)) \xrightarrow{r'_2} \mathrm{N}(\mathcal{F}\mathrm{in}_*).$$

In view of Proposition T.4.3.1.5, it will suffice to show that α is an r'_0-limit diagram, that $r'_0 \circ \alpha$ is an r'_1-limit diagram, and that $r'_1 \circ r'_0 \circ \alpha$ is an r'_2-limit diagram. The second of these assertions follows from Remark 2.1.2.6 and Lemma 3.2.2.9, and the last from Lemma 2.2.4.16. To prove that α is an r'_0-limit diagram, we consider the pullback diagram

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes}) \\ \downarrow & & \downarrow \\ \mathcal{C}^{\otimes} & \longrightarrow & \mathcal{O}^{\otimes} \, . \end{array}$$

Using Proposition T.4.3.1.5, we are reduced to the problem of showing that the induced map $\langle n \rangle^{\circ \triangleleft} \to \mathcal{C}^{\otimes}$ is a p-limit diagram; this follows from Remark 2.1.2.11.

To complete the proof, it will suffice to show for any finite collection of objects $X_i \in \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}_{\langle 1 \rangle}$ (parametrized by $1 \le i \le n$), there exists an object $X \in \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}_{\langle n \rangle}$ and a collection of r-coCartesian morphisms $X \to X_i$ covering the maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$. Each X_i can be identified with an object $C_i \in \mathcal{C}^{\otimes}$ and an active morphism $\beta_i : p(C_i) \to Y_i$ in $\mathcal{O}^{\otimes}$, where $Y_i \in \mathcal{O}$. The objects Y_i determine a diagram $g : \langle n \rangle^{\circ} \to \mathcal{O}^{\otimes}$. Using the assumption that $\mathcal{O}^{\otimes}$ is an ∞-operad, we deduce the existence of an object $Y \in \mathcal{O}^{\otimes}_{\langle n \rangle}$ and a collection of inert morphisms $Y \to Y_i$ covering the maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$. We regard these morphisms as providing an object $\overline{Y} \in \mathcal{O}^{\otimes}_{/g}$ lifting Y. Let $C_i \in \mathcal{C}^{\otimes}$ lie over $\langle m_i \rangle$ in $\mathcal{F}\mathrm{in}_*$. Since $\mathcal{C}^{\otimes}$ is an ∞-operad, there exists an object $C \in \mathcal{C}^{\otimes}_{\langle m \rangle}$ and a collection of inert morphisms $C \to C_i$, where $m = m_1 + \cdots + m_n$. Composing these maps with the β_i, we can lift $p(C)$ to an object $Z \in \mathcal{O}^{\otimes}_{/g}$. To construct the object X and the maps $X \to X_i$, it suffices to select a morphism $Z \to \overline{Y}$ in $\mathcal{O}^{\otimes}_{/g}$. The existence of such a morphism follows from the observation that $\overline{Y}$ is a final object of $\mathcal{O}^{\otimes}_{/g}$. $\square$

Proof of Proposition 2.2.4.9. Let $\mathcal{E}^{\otimes}$ denote the essential image of $i : \mathcal{C}^{\otimes} \to \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$. We can identify $\mathcal{E}^{\otimes}$ with the full subcategory of $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ spanned by those objects $(X, \alpha : p(X) \to Y)$ for which $X \in \mathcal{C}^{\otimes}$ and α is an equivalence in $\mathcal{O}^{\otimes}$. We observe that i induces an equivalence of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{E}^{\otimes}$. It will therefore suffice to prove that the restriction functor

$$\mathrm{Fun}^{\otimes}_{\mathcal{O}}(\mathrm{Env}_{\mathcal{O}}(\mathcal{C}), \mathcal{D}) \to \mathrm{Alg}_{\mathcal{E}/\mathcal{O}}(\mathcal{D})$$

is an equivalence of ∞-categories. In view of Proposition T.4.3.2.15, it will suffice to show the following:

(a) Every ∞-operad map $\theta_0 : \mathcal{E}^{\otimes} \to \mathcal{D}^{\otimes}$ admits a q-left Kan extension $\theta : \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{D}^{\otimes}$.

(b) An arbitrary map $\theta : \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{D}^{\otimes}$ in $(\mathrm{Set}_{\Delta})_{/\mathcal{O}^{\otimes}}$ is a $\mathcal{O}$-monoidal functor if and only if it is a q-left Kan extension of $\theta_0 = \theta | \mathcal{E}^{\otimes}$ and θ_0 is an ∞-operad map.

To prove (a), we will use criterion of Lemma T.4.3.2.13: it suffices to show that for every object $\overline{X} = (X, \alpha : p(X) \to Y)$ in $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$, the induced diagram $\mathcal{E}^{\otimes} \times_{\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}} \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}_{/\overline{X}} \to \mathcal{D}^{\otimes}$ admits a q-colimit

covering the natural map $(\mathcal{E}^{\otimes} \times_{\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}} \mathrm{End}_{\mathcal{O}}(\mathcal{C})_{/X}^{\otimes})^{\triangleright} \to \mathcal{D}^{\otimes}$. To see this, we observe that the ∞-category $\mathcal{E}^{\otimes} \times_{\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}} \mathrm{Env}_{\mathcal{O}}(\mathcal{C})_{/X}^{\otimes}$ has a final object, given by the pair $(X, \mathrm{id}_{p(X)})$. It therefore suffices to show that there exists a q-coCartesian morphism $\theta_0(X, \mathrm{id}_{p(X)}) \to C$ lifting $\alpha : p(X) \to Y$, which follows from the assumption that q is a coCartesian fibration. This completes the proof of (a) and yields the following version of (b):

(b') Let $\theta : \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{D}^{\otimes}$ be a morphism in $(\mathcal{S}\mathrm{et}_{\Delta})_{/\mathcal{O}^{\otimes}}$ such that the restriction $\theta_0 = \theta|\,\mathcal{E}^{\otimes}$ is an ∞-operad map. Then θ is a q-left Kan extension of θ_0 if and only if, for every object $(X, \alpha : p(X) \to Y) \in \mathrm{Env}(\mathcal{O})^{\otimes}$, the canonical map $\theta(X, \mathrm{id}_{p(X)}) \to \theta(X, \alpha)$ is q-coCartesian.

We now prove (b). Let $\theta : \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{D}^{\otimes}$ be such that the restriction $\theta_0 = \theta|\,\mathcal{E}^{\otimes}$ is an ∞-operad map. In view of (b'), it will suffice to show that θ is a $\mathcal{O}$-monoidal functor if and only if $\theta(X, \mathrm{id}_{p(X)}) \to \theta(X, \alpha)$ is q-coCartesian, for each $(X, \alpha) \in \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$. The "only if" direction is clear, since Lemma 2.2.4.16 implies that the morphism $(X, \mathrm{id}_{p(X)}) \to (X, \alpha)$ in $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ is p'-coCartesian, where $p' : \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ denotes the projection. For the converse, suppose that we are given a p'-coCartesian morphism $f : (X, \alpha) \to (Y, \alpha')$ in $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$. Let $\beta : p(X) \to p(Y)$ be the induced map in $\mathcal{O}^{\otimes}$, and choose a factorization $\beta \simeq \beta'' \circ \beta'$ where β' is inert and β'' is active. Choose a p-coCartesian morphism $\overline{\beta}' : X \to X''$ lifting β'. We then have a commutative diagram

$$
\begin{array}{ccc}
(X, \mathrm{id}_{p(X)}) & \xrightarrow{\ f'\ } & (X'', \mathrm{id}_{p(X'')}) \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\
(X, \alpha) & \xrightarrow{\ f\ } & (X', \alpha').
\end{array}
$$

The description of p'-coCartesian morphisms supplied by Lemma 2.2.4.16 shows that the map $X'' \to X'$ is an equivalence in $\mathcal{O}^{\otimes}$. If θ satisfies the hypotheses of (b'), then $\theta(g)$ and $\theta(g')$ are q-coCartesian. The assumption that θ_0 is an ∞-operad map guarantees that $\theta(f')$ is q-coCartesian. It follows from Proposition T.2.4.1.7 that $\theta(f)$ is q-coCartesian. By allowing f to range over all morphisms in $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ we deduce that θ is a $\mathcal{O}$-monoidal functor, as desired. $\qquad\square$

2.2.5 Tensor Products of ∞-Operads

Let $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ be a pair of ∞-operads. Our goal in this section is to introduce a new ∞-operad $\mathcal{O}''^{\otimes}$, which we call the *tensor product* of $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$. This tensor product is characterized (up to equivalence) by the existence of a map $\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ with a certain universal property: see Definition 2.2.5.3. This universal property guarantees the existence of an equivalence of ∞-categories $\mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}))$, for every symmetric monoidal ∞-category $\mathcal{C}$: here $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$ is endowed with a symmetric monoidal structure determined by the tensor product in $\mathcal{C}$.

Notation 2.2.5.1. We define a functor $\wedge : \mathcal{F}\mathrm{in}_* \times \mathcal{F}\mathrm{in}_* \to \mathcal{F}\mathrm{in}_*$ as follows:

(i) On objects, $\wedge$ is given by the formula $\langle m \rangle \wedge \langle n \rangle = \langle mn \rangle$.

(ii) If $f : \langle m \rangle \to \langle m' \rangle$ and $g : \langle n \rangle \to \langle n' \rangle$ are morphisms in $\mathcal{F}\mathrm{in}_*$, then $f \wedge g$ is given by the formula

$$
(f \wedge g)(an + b - n) = \begin{cases} * & \text{if } f(a) = * \text{ or } g(b) = * \\ f(a)n' + g(b) - n' & \text{otherwise.} \end{cases}
$$

In other words, $\wedge$ is given by the formula $\langle m \rangle \wedge \langle n \rangle = (\langle m \rangle^{\circ} \times \langle n \rangle^{\circ})_*$, where we identify $\langle m \rangle^{\circ} \times \langle n \rangle^{\circ}$ with $\langle mn \rangle^{\circ}$ via the lexicographical ordering.

Remark 2.2.5.2. The operation $\wedge$ of Notation 2.2.5.1 is associative, and endows $\mathcal{F}\mathrm{in}_*$ with the structure of a *strict* monoidal category (that is, we have equalities $(\langle l \rangle \wedge \langle m \rangle) \wedge \langle n \rangle = \langle l \rangle \wedge (\langle m \rangle \wedge \langle n \rangle)$, rather than just isomorphisms). In particular, the nerve $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ has the structure of a simplicial monoid.

Definition 2.2.5.3. Let $\mathcal{O}^{\otimes}$, $\mathcal{O}'^{\otimes}$, and $\mathcal{O}''^{\otimes}$ be ∞-operads. A *bifunctor of ∞-operads* is a map $f : \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ with the following properties:

(*i*) The diagram

$$
\begin{array}{ccc}
\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} & \longrightarrow & \mathcal{O}''^{\otimes} \\
\downarrow & & \downarrow \\
\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \overset{\wedge}{\longrightarrow} & & \mathrm{N}(\mathcal{F}\mathrm{in}_*)
\end{array}
$$

commutes.

(*ii*) For every inert morphism α in $\mathcal{O}^{\otimes}$ and every inert morphism β in $\mathcal{O}'^{\otimes}$, the image $f(\alpha, \beta)$ is an inert morphism in $\mathcal{O}''^{\otimes}$.

We let $\mathrm{BiFunc}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}; \mathcal{O}''^{\otimes})$ denote the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes})$ spanned by those maps which satisfy (*ii*); we will refer to $\mathrm{BiFunc}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}; \mathcal{O}''^{\otimes})$ as the *∞-category of ∞-operad bifunctors from $\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes}$ into $\mathcal{O}''^{\otimes}$.*

Given an ∞-operad bifunctor $f : \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ and another ∞-operad $\mathcal{C}^{\otimes}$, composition with f determines a functor $\theta : \mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}) \to \mathrm{BiFunc}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}; \mathcal{C}^{\otimes})$. We will say that f *exhibits $\mathcal{O}''^{\otimes}$ as a tensor product of $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$* if the functor θ is an equivalence for every ∞-operad $\mathcal{C}^{\otimes}$.

Remark 2.2.5.4. For every triple of ∞-operads $\mathcal{O}^{\otimes}$, $\mathcal{O}'^{\otimes}$, and $\mathcal{O}''^{\otimes}$, we let $\mathrm{Mul}^{\wedge}_{\mathrm{Op}^{\Delta}_{\infty}}(\{\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}\}, \mathcal{O}''^{\otimes})$ denote the largest Kan complex contained in the ∞-category $\mathrm{BiFunc}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}; \mathcal{O}''^{\otimes})$. We claim that a bifunctor of ∞-operads $f : \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ exhibits $\mathcal{O}''^{\otimes}$ as an tensor product of $\mathcal{O}^{\otimes}$ with $\mathcal{O}'^{\otimes}$ if and only if, for every ∞-operad $\mathcal{C}^{\otimes}$, composition with f induces a homotopy equivalence $\theta_{\mathcal{C}^{\otimes}} : \mathrm{Alg}_{\mathcal{O}''}(\mathcal{C})^{\simeq} \to \mathrm{Mul}^{\wedge}_{\mathrm{Op}^{\Delta}_{\infty}}(\{\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}\}, \mathcal{O}''^{\otimes})$. The "only if" direction is clear. For the converse, we observe that to show that the functor $\mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}) \to \mathrm{BiFunc}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}; \mathcal{C}^{\otimes})$ is an equivalence of ∞-categories, it suffices to show that for every simplicial set K the induced functor $\mathrm{Fun}(K, \mathrm{Alg}_{\mathcal{O}''}(\mathcal{C})) \to \mathrm{Fun}(K, \mathrm{BiFunc}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}; \mathcal{C}^{\otimes}))$ induces a homotopy equivalence from $\mathrm{Fun}(K, \mathrm{Alg}_{\mathcal{O}''}(\mathcal{C}))^{\simeq}$ to $\mathrm{Fun}(K, \mathrm{BiFunc}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}; \mathcal{C}^{\otimes}))^{\simeq}$. Unwinding the definitions, we see that this map is given by $\theta_{\mathcal{D}^{\otimes}}$, where $\mathcal{D}^{\otimes}$ denotes the ∞-operad $\mathrm{Fun}(K, \mathcal{C}^{\otimes}) \times_{\mathrm{Fun}(K, \mathrm{N}(\mathcal{F}\mathrm{in}_*))} \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

It is immediate from the definition that if a pair of ∞-operads $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ admits a tensor product $\mathcal{O}''^{\otimes}$, then $\mathcal{O}''^{\otimes}$ is determined uniquely up to equivalence. To prove the existence of the tensor product it is convenient to work in the more general setting of ∞-preoperads:

Notation 2.2.5.5. Let $\overline{X} = (X, M)$ and $\overline{Y} = (Y, M')$ be ∞-preoperads: that is, X and Y are simplicial sets equipped with maps $X \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \leftarrow Y$, and M and M' are collections of edges of X and Y. We let $\overline{X} \odot \overline{Y}$ denote the ∞-preoperad $(X \times Y, M \times M')$, where we regard $X \times Y$ as an object of $(\mathcal{S}\mathrm{et}_{\Delta})_{/\mathrm{N}(\mathcal{F}\mathrm{in}_*)}$ via the map $X \times Y \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \overset{\wedge}{\to} \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

Unwinding the definitions (and using Remark 2.2.5.4), we see that a bifunctor of ∞-operads

$$ f : \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes} $$

exhibits $\mathcal{O}''^{\otimes}$ as a tensor product of $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ if and only if the induced map $\mathcal{O}^{\otimes,\natural} \odot \mathcal{O}'^{\otimes,\natural} \to \mathcal{O}''^{\otimes,\natural}$ is a weak equivalence of ∞-preoperads. In other words, a tensor product of a pair of ∞-operads $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ can be identified with a fibrant replacement for the object $\mathcal{O}^{\otimes,\natural} \odot \mathcal{O}'^{\otimes,\natural}$ in the category $\mathcal{P}\mathrm{Op}_{\infty}$. This proves:

Proposition 2.2.5.6. *Let $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ be ∞-operads. Then there exists a bifunctor of ∞-operads $\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ which exhibits $\mathcal{O}''^{\otimes}$ as a tensor product of $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$.*

We now show that the operation $\odot$ is compatible with the ∞-operadic model structure on $\mathcal{P}\mathrm{Op}_\infty$.

Proposition 2.2.5.7. *The functor $\odot$ endows $\mathcal{P}\mathrm{Op}_\infty$ with the structure of a monoidal model category.*

Remark 2.2.5.8. In the approach to ∞-operads based on dendroidal sets, one can do better: the tensor product of ∞-operads is modelled by an operation which is commutative up to isomorphism. We refer to [31] for a discussion.

Proof of Proposition 2.2.5.7. Since every object of $\mathcal{P}\mathrm{Op}_\infty$ is cofibrant, it will suffice to show that the functor

$$\odot : \mathcal{P}\mathrm{Op}_\infty \times \mathcal{P}\mathrm{Op}_\infty \to \mathcal{P}\mathrm{Op}_\infty$$

is a left Quillen bifunctor. Let $\mathfrak{P}$ be as in the proof of Proposition 2.1.4.6. Using Remark B.2.5, we deduce that the Cartesian product functor $\mathcal{P}\mathrm{Op}_\infty \times \mathcal{P}\mathrm{Op}_\infty \to (\mathcal{S}et_\Delta^+)_{/\mathfrak{P} \times \mathfrak{P}}$ is a left Quillen bifunctor. The desired result now follows by applying Proposition B.2.9 to the product functor $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.   □

Our next goal is to discuss the symmetry properties of the tensor product construction on ∞-operads. Here we encounter a subtlety: the functor $\wedge : \mathcal{F}\mathrm{in}_* \times \mathcal{F}\mathrm{in}_* \to \mathcal{F}\mathrm{in}_*$ endows $\mathcal{F}\mathrm{in}_*$ with the structure of a symmetric monoidal category. However, this symmetric monoidal structure is not strict: the natural symmetry isomorphism $\langle mn \rangle \simeq \langle m \rangle \wedge \langle n \rangle \simeq \langle n \rangle \wedge \langle m \rangle \simeq \langle mn \rangle$ is not the identity, because the two different lexicographical orderings of $\langle m \rangle^\circ \times \langle n \rangle^\circ$ do not coincide. We can address this problem by replacing the specific functor $\wedge : \mathcal{F}\mathrm{in}_* \times \mathcal{F}\mathrm{in}_* \to \mathcal{F}\mathrm{in}_*$ of Notation 2.2.5.1 by the collection of *all* functors which are isomorphic to $\wedge$.

Notation 2.2.5.9. Let $F : \mathcal{F}\mathrm{in}_* \times \cdots \times \mathcal{F}\mathrm{in}_* \to \mathcal{F}\mathrm{in}_*$ be a functor. We will say that F is a *smash product functor* if it has the following properties:

 (1) There exists an isomorphism (automatically unique) $F(\langle 1 \rangle, \ldots, \langle 1 \rangle) \simeq \langle 1 \rangle$.

 (2) The functor F preserves coproducts separately in each variable.

Example 2.2.5.10. The functor $F : \mathcal{F}\mathrm{in}_* \times \cdots \times \mathcal{F}\mathrm{in}_* \to \mathcal{F}\mathrm{in}_*$ given by $F(\langle n_1 \rangle, \ldots, \langle n_k \rangle) = \langle n_1 \rangle \wedge \cdots \wedge \langle n_k \rangle = \langle n_1 \cdots n_k \rangle$ is a smash product functor.

Notation 2.2.5.11. For every finite set I, the collection of smash product functors $F : \mathcal{F}\mathrm{in}_*{}^I \to \mathcal{F}\mathrm{in}_*$ forms a category, which we will denote by $\mathfrak{S}(I)$. For every pair of objects $F, G \in \mathfrak{S}(I)$, there is a unique morphism from F to G (which is an isomorphism). Consequently, $\mathfrak{S}(I)$ is equivalent to the discrete category $[0]$, consisting of the smash product functor described in Example 2.2.5.10.

Construction 2.2.5.12. Given a collection of ∞-operads $\{\mathcal{O}_i^\otimes\}_{i \in I}$, another ∞-operad $\mathcal{O}'^\otimes$, and a smash product functor F, we let $\mathrm{Fun}'_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\prod_i \mathcal{O}_i^\otimes, \mathcal{O}'^\otimes)$ denote the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\prod_i \mathcal{O}_i^\otimes, \mathcal{O}'^\otimes)$ spanned by those diagrams

$$
\begin{array}{ccc}
\prod_i \mathcal{O}_i^\otimes & \xrightarrow{\ f\ } & \mathcal{O}'^\otimes \\
\downarrow & & \downarrow \\
\mathrm{N}(\mathcal{F}\mathrm{in}_*)^I & \xrightarrow{\ F\ } & \mathrm{N}(\mathcal{F}\mathrm{in}_*)
\end{array}
$$

with the following property: given any collection of morphisms $\{\alpha_i \in \mathrm{Fun}(\Delta^1, \mathcal{O}_i^\otimes)\}_{i \in I}$, if each α_i is an inert morphism in $\mathcal{O}_i^\otimes$, then $f(\{\alpha_i\})$ is an inert morphism in $\mathcal{O}'^\otimes$.

Let $\mathrm{Mul}_{\mathrm{Op}_\infty^\Delta}^F(\{\mathcal{O}_i^\otimes\}, \mathcal{O}'^\otimes)$ denote the Kan complex $\mathrm{Fun}'_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\prod_i \mathcal{O}_i^\otimes, \mathcal{O}'^\otimes)^\simeq$. The construction $F \mapsto \mathrm{Mul}_{\mathrm{Op}_\infty^\Delta}^F(\{\mathcal{O}_i^\otimes\}, \{\mathcal{O}'\})$ defines a functor $\lambda : \mathfrak{S}(I) \to \mathcal{S}et_\Delta$. Let $\mathrm{Mul}_{\mathrm{Op}_\infty^\Delta}(\{\mathcal{O}_i^\otimes\}, \mathcal{O}'^\otimes)$ denote the relative nerve $\mathrm{N}_\lambda(\mathfrak{S}(I))$ (see Definition T.3.2.5.2), so that we have a Kan fibration

$$\mathrm{Mul}_{\mathrm{Op}_\infty^\Delta}(\{\mathcal{O}_i^\otimes\}, \mathcal{O}'^\otimes) \to \mathrm{N}(\mathfrak{S}(I))$$

whose fiber over an object $F \in \mathfrak{S}(I)$ is given by $\operatorname{Mul}^F_{\operatorname{Op}^\Delta_\infty}(\{\mathcal{O}^\otimes_i\}, \mathcal{O}'^\otimes)$. With the evident composition maps, these mapping spaces determine a simplicial colored operad $\operatorname{Op}^\Delta_\infty$, whose objects are small ∞-operads.

Let $\operatorname{Op}(\infty)^\otimes$ denote the operadic nerve $N^\otimes(\operatorname{Op}^\Delta_\infty)$ (see Definition 2.1.1.23). Since the simplicial colored operad $\operatorname{Op}^\Delta_\infty$ is fibrant (Definition 2.1.1.26), we conclude that $\operatorname{Op}(\infty)^\otimes$ is an ∞-operad (Proposition 2.1.1.27).

Proposition 2.2.5.13. (*a*) *The ∞-operad $\operatorname{Op}(\infty)^\otimes \to N(\mathfrak{Fin}_*)$ of Construction 2.2.5.12 is a symmetric monoidal ∞-category.*

(*b*) *There is a canonical equivalence of ∞-categories $\operatorname{Op}_\infty \to \operatorname{Op}(\infty)$.*

(*c*) *The bifunctor*

$$\operatorname{Op}_\infty \times \operatorname{Op}_\infty \simeq \operatorname{Op}(\infty) \times \operatorname{Op}(\infty) \overset{\otimes}{\to} \operatorname{Op}(\infty) \simeq \operatorname{Op}_\infty$$

is given by the tensor product construction of Definition 2.2.5.3.

(*d*) *The unit object of $\operatorname{Op}(\infty)$ can be identified with a fibrant replacement for the ∞-preoperad $\{\langle 1 \rangle\}^\flat$*

Proof. For the proof, we will borrow some results from Chapter 4. By virtue of Proposition 4.1.1.20, assertion (*a*) is equivalent to the statement that $\operatorname{Op}(\infty)^\otimes \times_{N(\mathfrak{Fin}_*)} \mathcal{A}ss^\otimes$ is a monoidal ∞-category. This follows immediately from Example 4.1.3.18 and Variant 4.1.3.17. Assertion (*b*) is immediate from Example 4.1.3.18. Assertion (*c*) and (*d*) follow from Example 4.1.3.18 and the proof of Proposition 4.1.3.10. $\square$

Remark 2.2.5.14. It follows from Example 2.1.4.8 that the $\mathcal{Triv}^{\otimes,\natural}$ can be identified with a fibrant replacement for the ∞-preoperad $\{\langle 1 \rangle\}^\flat$. It follows that the trivial ∞-operad $\mathcal{Triv}^\otimes$ is a unit object of $\operatorname{Op}(\infty)$.

We now describe the relationship between tensor products of ∞-operads and ordinary products of ∞-categories. Let $\mathcal{S}et^+_\Delta$ denote the category of marked simplicial sets, equipped with the marked model structure described in §T.3.1.3. The subcategory $(\mathcal{S}et^+_\Delta)^\circ$ of fibrant-cofibrant objects is endowed with a symmetric monoidal structure, given by the Cartesian product. We will denote the associated symmetric monoidal ∞-category $N^\otimes((\mathcal{S}et^+_\Delta)^\circ)$ by $\mathcal{C}at^\times_\infty$: its underlying ∞-category is the ∞-category $\mathcal{C}at_\infty$ of small ∞-categories, and the symmetric monoidal structure is given by the formation of Cartesian products of ∞-categories. The construction $\mathcal{O}^\otimes \mapsto \mathcal{O}$ determines a map of simplicial colored operads $\operatorname{Op}^\Delta_\infty \to (\mathcal{S}et^+_\Delta)^\circ$. Passing to the operadic nerve, we get a map of ∞-operads $\theta : \operatorname{Op}(\infty)^\otimes \to \mathcal{C}at^\times_\infty$.

Proposition 2.2.5.15. *Let $\operatorname{Op}(\infty)'$ denote the full subcategory of $\operatorname{Op}(\infty) \simeq \operatorname{Op}_\infty$ spanned by those ∞-operads $p : \mathcal{O}^\otimes \to N(\mathfrak{Fin}_*)$ for which p factors through $\mathcal{Triv}^\otimes \subseteq N(\mathfrak{Fin}_*)$ (see Proposition 2.1.4.11). Then:*

(*a*) *The subcategory $\operatorname{Op}(\infty)'$ contains the unit object of $\operatorname{Op}(\infty)$ and is stable under tensor products. Consequently, $\operatorname{Op}(\infty)'$ inherits a symmetric monoidal structure $\operatorname{Op}(\infty)'^\otimes$ (Proposition 2.2.1.1).*

(*b*) *The functor θ restricts to a symmetric monoidal equivalence $\operatorname{Op}(\infty)'^\otimes \to \mathcal{C}at^\times_\infty$.*

In other words, when restricted to those ∞-operads which belong to the image of the fully faithful embedding $\mathcal{C}at_\infty \to \operatorname{Op}_\infty$ of Proposition 2.1.4.11, the operation of tensor product recovers the usual Cartesian product of ∞-categories.

Proof. As in the proof of Proposition 2.2.5.13, it will be convenient to borrow some ideas from Chapter 4. According to Remark 2.2.5.14, the unit object of $\operatorname{Op}(\infty)$ can be identified with $\mathcal{Triv}^\otimes$, which obviously belongs to $\operatorname{Op}(\infty)'$. If $\mathcal{O}^\otimes$ and $\mathcal{O}'^\otimes$ are ∞-operads belonging to $\operatorname{Op}(\infty)'$ having tensor product $\mathcal{O}''^\otimes$, then $\mathcal{O}''^\otimes$ admits a map to the tensor product of $\mathcal{Triv}^\otimes$ with itself. Since $\mathcal{Triv}^\otimes$ is a unit object of $\operatorname{Op}(\infty)'$, this tensor product can be identified with $\mathcal{Triv}^\otimes$, so that $\mathcal{O}''^\otimes$ belongs to $\operatorname{Op}(\infty)'$. This proves (*a*).

To prove (*b*), we note that θ induces an equivalence between the underlying ∞-categories $\operatorname{Op}(\infty)' \to \mathcal{C}at_\infty$ by Proposition 2.1.4.11. In view of Remark 2.1.3.8, to prove that $\theta|\operatorname{Op}(\infty)'^\otimes$ is an equivalence of ∞-categories, it will suffice to show that θ is a symmetric monoidal functor. Let $q : \operatorname{Op}(\infty)'^\otimes \to N(\mathfrak{Fin}_*)$ and

$r : \mathcal{C}at_\infty^\times \to \mathrm{N}(\mathcal{F}in_*)$ be the projection maps; we wish to show that θ carries q-coCartesian morphisms to r-coCartesian morphisms. Let $q' : \mathrm{N}^\otimes_{\mathcal{A}ss}(\mathcal{P}\mathrm{Op}_\infty{}^\circ) \to \mathcal{A}ss^\otimes$ be the ∞-category of Example 4.1.3.18. Every q-coCartesian morphism in $\mathrm{Op}(\infty)'^\otimes$ is the image of a q'-coCartesian morphism in $\mathrm{N}^\otimes_{\mathcal{A}ss}(\mathcal{P}\mathrm{Op}_\infty{}^\circ)$ (see Example 4.1.3.18). It will therefore suffice to show that the induced map

$$\mathrm{N}^\otimes_{\mathcal{A}ss}(\mathcal{P}\mathrm{Op}_\infty{}^\circ) \times_{\mathrm{Op}(\infty)^\otimes} \mathrm{Op}(\infty)'^\otimes \subseteq \mathrm{N}^\otimes_{\mathcal{A}ss}(\mathcal{P}\mathrm{Op}_\infty{}^\circ) \to \mathcal{C}at_\infty^\times$$

carries q'-coCartesian morphisms to r-coCartesian morphisms. Using the proof of Proposition 4.1.3.10, we are reduced to the following assertion:

($*$) Let $\mathcal{O}_1^\otimes, \mathcal{O}_2^\otimes, \ldots, \mathcal{O}_n^\otimes$ be a sequence of ∞-operads belonging to $\mathrm{Op}(\infty)'$, and let $f : \mathcal{O}_1^\otimes \times \cdots \times \mathcal{O}_n^\otimes \to \mathcal{O}'^\otimes$ be a map which exhibits $\mathcal{O}'^\otimes$ as a tensor product of the ∞-operads $\{\mathcal{O}_i^\otimes\}_{1 \le i \le n}$. Then f induces an equivalence of ∞-categories $\mathcal{O}_1 \times \cdots \times \mathcal{O}_n \to \mathcal{O}'$.

For each index i, we let $\mathcal{O}_i^\natural$ denote the marked simplicial set $(\mathcal{O}_i, M_i)$, where M_i is the collection of equivalences in $\mathcal{O}_i$. We claim that the inclusion $\phi : \mathcal{O}_i^\natural \hookrightarrow \mathcal{O}_i^{\otimes,\natural}$ is a weak equivalence of ∞-preoperads. To prove this, it suffices to show that for any ∞-operad $\mathcal{C}^\otimes$, composition with ϕ induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{P}\mathrm{Op}_\infty}(\mathcal{O}_i^{\otimes,\natural}, \mathcal{C}^\otimes) \to \mathrm{Map}_{\mathcal{P}\mathrm{Op}_\infty}(\mathcal{O}_i^\natural, \mathcal{C}^\otimes).$$

Without loss of generality, we may replace $\mathcal{C}^\otimes$ by $\mathcal{C}^\otimes \times_{\mathrm{N}(\mathcal{F}in_*)} \mathcal{T}riv^\otimes$ (this does not change either of the relevant mapping spaces), and thereby reduce to Proposition 2.1.4.11.

The hypothesis that f exhibits $\mathcal{O}'^\otimes$ as a tensor product of the ∞-operads $\mathcal{O}_i^\otimes$ is equivalent to the requirement that f induces a weak equivalence of ∞-preoperads

$$\mathcal{O}_1^{\otimes,\natural} \odot \cdots \odot \mathcal{O}_n^{\otimes,\natural} \to \mathcal{O}'^{\otimes,\natural}.$$

It follows that the induced map $\alpha : \prod_i \mathcal{O}_i^\natural \to \mathcal{O}'^{\otimes,\natural}$ is a weak equivalence of ∞-preoperads. According to Proposition 2.1.4.11, there exists a map of ∞-operads $\mathcal{C}^\otimes \to \mathcal{T}riv^\otimes$ and an equivalence of ∞-categories $\prod_i \mathcal{O}_i \to \mathcal{C}$. The above argument shows that α induces a weak equivalence of ∞-preoperads $\beta : \prod_i \mathcal{O}_i^\natural \to \mathcal{C}^{\otimes,\natural}$. In particular, the map α can be factored as a composition

$$\prod_i \mathcal{O}_i^\natural \xrightarrow{\beta} \mathcal{C}^{\otimes,\natural} \xrightarrow{\gamma} \mathcal{O}'^{\otimes,\natural}.$$

Since α and β are weak equivalences, we conclude that γ is a weak equivalence between fibrant objects of $\mathcal{P}\mathrm{Op}_\infty$: that is, γ induces an equivalence of ∞-operads $\mathcal{C}^\otimes \to \mathcal{O}'^\otimes$. In particular, the underlying map of ∞-categories $\mathcal{C} \to \mathcal{O}'$ is a categorical equivalence. It follows that the composite functor $\prod_i \mathcal{O}_i \to \mathcal{C} \to \mathcal{O}'$ is also an equivalence, as desired. $\square$

2.3   Disintegration and Assembly

Let A be an associative ring. Recall that an *involution* on A is a map $\sigma : A \to A$ satisfying the conditions

$$(a + b)^\sigma = a^\sigma + b^\sigma \qquad (ab)^\sigma = b^\sigma a^\sigma \qquad (a^\sigma)^\sigma = a;$$

here a^σ denotes the image of a under the map σ. Let Ring denote the category of associative rings, and let Ring$^\sigma$ denote the category of associative rings equipped with an involution (whose morphisms are ring homomorphisms that commute with the specified involutions). To understand the relationship between these two categories, we observe that the construction $A \mapsto A^{op}$ defines an action of the symmetric group Σ_2 on the category Ring. The category Ring$^\sigma$ can be described as the category of (homotopy) fixed points for the action of Σ_2 on Ring. In particular, we can reconstruct the category Ring$^\sigma$ by understanding the category Ring together with its action of Σ_2.

The category Ring can be described as the category of algebras over the associative operad $\mathcal{O}$ in the (symmetric monoidal) category of abelian groups. Similarly, we can describe Ring^σ as the category of $\mathcal{O}'$-algebra objects in the category of abelian groups, where $\mathcal{O}'$ is a suitably defined enlargment of the associative operad. The relationship between Ring and Ring^σ reflects a more basic relationship between the operads $\mathcal{O}$ and $\mathcal{O}'$: namely, the operad $\mathcal{O}$ carries an action of the group Σ_2, and the operad $\mathcal{O}'$ can be recovered as a kind of semidirect product $\mathcal{O} \rtimes \Sigma_2$. This assertion is useful because $\mathcal{O}$ is, in many respects, simpler than $\mathcal{O}'$. For example, the operad $\mathcal{O}$ has only a single unary operation (the identity) while $\mathcal{O}'$ has a pair of unary operations (the identity and the involution).

In this section, we will describe a generalization of this phenomenon. We begin in §2.3.1 by introducing the notion of a *unital* ∞-operad. Roughly speaking, an ∞-operad $\mathcal{O}^\otimes$ is unital if it has a unique nullary operation (more precisely, if it has a unique nullary operation for each object of the ∞-category $\mathcal{O}$; see Definition 2.3.1.1). Many of ∞-operads which arise naturally are unital, and nonunital ∞-operads can be replaced by unital ∞-operads via the process of *unitalization* (Definition 2.3.1.10). We say that a unital ∞-operad $\mathcal{O}^\otimes$ is *reduced* if the ∞-category $\mathcal{O}$ is a contractible Kan complex. In §2.3.4, we will show that if $\mathcal{O}^\otimes$ is any unital ∞-operad whose underlying ∞-category $\mathcal{O}$ is a Kan complex, then $\mathcal{O}^\otimes$ can be "assembled" from a family of reduced ∞-operads parametrized by $\mathcal{O}$ (Theorem 2.3.4.4). A precise formulation of this assertion requires the notion of a *generalized ∞-operad*, which we discuss in §2.3.2. The proof will require a somewhat technical criterion for detecting weak equivalences of ∞-preoperads, which we discuss in §2.3.3.

Remark 2.3.0.1. The assembly process described in §2.3.4 can be regarded as a generalization of the semidirect product construction mentioned above, and will play an important role in §5.2.2.

2.3.1 Unital ∞-Operads

Let $\mathcal{O}^\otimes$ be an ∞-operad. Then for every n-tuple of objects $\{X_i\}_{1 \le i \le n}$ in $\mathcal{O}$ and every object $Y \in \mathcal{O}$, we can consider the mapping space $\mathrm{Mul}_\mathcal{O}(\{X_i\}, Y)$ defined in Notation 2.1.1.16. We can think of this as the space of n-ary operations (taking inputs of type $\{X_i\}_{1 \le i \le n}$ and producing an output of type Y) in $\mathcal{O}^\otimes$. In general, these operation spaces are related to one another via very complicated composition laws. When $n = 1$ the situation is dramatically simpler: the 1-ary operation spaces $\mathrm{Mul}_\mathcal{O}(\{X\}, Y)$ are simply the mapping spaces $\mathrm{Map}_\mathcal{O}(X, Y)$ in the underlying ∞-category $\mathcal{O}$. In this section, we will consider what is in some sense an even more basic invariant of $\mathcal{O}^\otimes$: namely, the structure of the nullary operation spaces $\mathrm{Mul}_\mathcal{O}(\emptyset, Y)$. More precisely, we will be interested in the situation where this invariant is trivial:

Definition 2.3.1.1. We will say that an ∞-operad $\mathcal{O}^\otimes$ is *unital* if, for every object $X \in \mathcal{O}$, the space $\mathrm{Mul}_\mathcal{O}(\emptyset, X)$ is contractible.

Warning 2.3.1.2. In the literature, the term *unital operad* is used with two very different meanings:

(i) To describe an operad $\{\mathcal{O}_n\}_{n \ge 0}$ which has a distinguished unary operation $\mathrm{id} \in \mathcal{O}_1$, which is a left and right unit with respect to composition.

(ii) To describe an operad $\{\mathcal{O}_n\}_{n \ge 0}$ which has a unique nullary operation $e \in \mathcal{O}_0$.

Definition 2.3.1.1 should be regarded as an ∞-categorical generalization of (*ii*); the analogue of condition (*i*) is built-in to our definition of an ∞-operad.

Example 2.3.1.3. The ∞-operads $\mathrm{Comm}^\otimes$ and $\mathbb{E}_0^\otimes$ of Examples 2.1.1.18 and 2.1.1.19 are unital. The trivial ∞-operad of Example 2.1.1.20 is not unital.

Here is a purely categorical description of the class of unital ∞-operads:

Proposition 2.3.1.4. *Let* $\mathcal{O}^\otimes$ *be an ∞-operad. The following conditions are equivalent:*

(1) *The ∞-category* $\mathcal{O}^\otimes$ *is pointed (that is, there exists an object of* $\mathcal{O}^\otimes$ *which is both initial and final).*

(2) *The ∞-operad* $\mathcal{O}^\otimes$ *is unital.*

The proof depends on the following observation:

Lemma 2.3.1.5. *Let* $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *be an* ∞-*operad. Then an object* X *of the* ∞-*category* $\mathcal{O}^{\otimes}$ *is final if and only if* $p(X) = \langle 0 \rangle$. *Moreover, there exists an object of* $\mathcal{O}^{\otimes}$ *satisfying this condition.*

Proof. Since $\mathcal{O}^{\otimes}$ is an ∞-operad, we have an equivalence $\mathcal{O}^{\otimes}_{\langle 0 \rangle} \simeq \mathcal{O}^0 \simeq \Delta^0$; this proves the existence of an object $X \in \mathcal{O}^{\otimes}$ such that $p(X) = \langle 0 \rangle$. Since $\mathcal{O}^{\otimes}$ is an ∞-operad, the object $X \in \mathcal{O}^{\otimes}$ is p-final. Since $p(X) = \langle 0 \rangle$ is a final object of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, it follows that X is a final object of $\mathcal{O}^{\otimes}$.

To prove the converse, suppose that X' is any final object of $\mathcal{O}^{\otimes}$. Then $X' \simeq X$ so that $p(X') \simeq \langle 0 \rangle$. It follows that $p(X') = \langle 0 \rangle$, as desired. $\square$

Proof of Proposition 2.3.1.4. According to Lemma 2.3.1.5, the ∞-category $\mathcal{O}^{\otimes}$ admits a final object Y. Assertion (1) is equivalent to the requirement that Y is also initial: that is, that the space $\mathrm{Map}_{\mathcal{O}^{\otimes}}(Y, X)$ is contractible for every $X \in \mathcal{O}^{\otimes}$. Let $\langle n \rangle$ denote the image of X in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, and choose inert morphisms $X \to X_i$ covering $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \leq i \leq n$. Since X is a p-limit of the diagram $\{X_i\}_{1 \leq i \leq n} \to \mathcal{O}^{\otimes}$, we conclude that $\mathrm{Map}_{\mathcal{O}^{\otimes}}(Y, X) \simeq \prod_{1 \leq i \leq n} \mathrm{Map}_{\mathcal{O}^{\otimes}}(Y, X_i)$. Assertion (1) is therefore equivalent to the requirement that $\mathrm{Map}_{\mathcal{O}^{\otimes}}(Y, X)$ is contractible for $X \in \mathcal{O}$, which is a rewording of condition (2). $\square$

The class of unital ∞-operads also has a natural characterization in terms of the tensor product of ∞-operads (see §2.2.5). Let $\wedge : \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be the functor described in Notation 2.2.5.1. If $\alpha : \langle m \rangle \to \langle m' \rangle$ and $\beta : \langle n \rangle \to \langle n' \rangle$ are morphisms in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and $\gamma = \alpha \wedge \beta : \langle mn \rangle \to \langle m'n' \rangle$ is the induced map, then $\gamma^{-1}(an + b - n) \simeq \alpha^{-1}\{a\} \times \beta^{-1}\{b\}$. In particular, if each of the fibers $\alpha^{-1}\{a\}$ and $\beta^{-1}\{b\}$ has cardinality ≤ 1, then each fiber of γ has cardinality ≤ 1. It follows that $\wedge$ induces a map $f : \mathbb{E}_0^{\otimes} \times \mathbb{E}_0^{\otimes} \to \mathbb{E}_0^{\otimes}$, where $\mathbb{E}_0^{\otimes} \subseteq \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is the ∞-operad defined in Example 2.1.1.19.

Proposition 2.3.1.6. *The map* $f : \mathbb{E}_0^{\otimes} \times \mathbb{E}_0^{\otimes} \to \mathbb{E}_0^{\otimes}$ *is a bifunctor of* ∞-*operads, which exhibits* $\mathbb{E}_0^{\otimes}$ *as a tensor product of* $\mathbb{E}_0^{\otimes}$ *with itself.*

Proof. Consider the map $g : \Delta^1 \to \mathbb{E}_0$ determined by the morphism $\langle 0 \rangle \to \langle 1 \rangle$ in $\mathcal{F}\mathrm{in}_*$. Example 2.1.4.9 asserts that g induces a weak equivalence of ∞-preoperads $(\Delta^1)^\flat \to \mathbb{E}_0^{\otimes,\natural}$. We can factor the weak equivalence g as a composition

$$(\Delta^1)^\flat \xrightarrow{\delta} (\Delta^1)^\flat \odot (\Delta^1)^\flat \xrightarrow{g \odot g} \mathbb{E}_0^{\otimes,\natural} \odot \mathbb{E}_0^{\otimes,\natural} \xrightarrow{f} \mathbb{E}_0^{\otimes,\natural},$$

where δ is the diagonal map. The map $g \odot g$ is a weak equivalence of ∞-operads. By a two-out-of-three argument, we are reduced to proving that the diagonal $\delta : (\Delta^1)^\flat \to (\Delta^1)^\flat \odot (\Delta^1)^\flat$ is a weak equivalence of ∞-preoperads.

Unwinding the definitions, it suffices to show the following: for every ∞-operad $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, composition with δ induces a trivial Kan fibration

$$\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\Delta^1 \times \Delta^1, \mathcal{O}^{\otimes}) \to \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\Delta^1, \mathcal{O}^{\otimes}).$$

This follows from Proposition T.4.3.2.15, since every functor $F \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\Delta^1 \times \Delta^1, \mathcal{O}^{\otimes})$ is a p-left Kan extension of $F \circ \delta$ (because every morphism in $\mathcal{O}^{\otimes}_{\langle 0 \rangle}$ is an equivalence). $\square$

Corollary 2.3.1.7. *Let* $i : \{\langle 0 \rangle\}^\flat \to \mathbb{E}_0^{\otimes,\natural}$ *denote the inclusion. Then composition with* i *induces weak equivalences of* ∞-*preoperads*

$$\mathbb{E}_0^{\otimes,\natural} \simeq \mathbb{E}_0^{\otimes,\natural} \odot \{\langle 0 \rangle\}^\flat \to \mathbb{E}_0^{\otimes,\natural} \odot \mathbb{E}_0^{\otimes,\natural}$$

$$\mathbb{E}_0^{\natural} \simeq \{\langle 0 \rangle\}^\flat \odot \mathbb{E}_0^{\natural} \to \mathbb{E}_0^{\otimes,\natural} \odot \mathbb{E}_0^{\otimes,\natural}.$$

Corollary 2.3.1.8. *Let* Op_∞ *denote the* ∞-*category of (small)* ∞-*operads, which we identify with the underlying* ∞-*category* $\mathrm{N}(\mathcal{P}\mathrm{Op}_\infty{}^o)$ *of the simplicial monoidal model category* $\mathcal{P}\mathrm{Op}_\infty$. *Let* $U : \mathrm{Op}_\infty \to \mathrm{Op}_\infty$ *be induced by the left Quillen functor* $\overline{X} \mapsto \overline{X} \odot \mathbb{E}_0^{\natural}$. *Then* U *is a localization functor from* Op_∞ *to itself.*

Proof. Combine Corollary 2.3.1.7 with Proposition T.5.2.7.4. □

Proposition 2.3.1.9. *Let $\mathcal{O}^{\otimes}$ be an ∞-operad. The following conditions are equivalent:*

(1) *The ∞-operad $\mathcal{O}^{\otimes}$ is unital.*

(2) *The ∞-operad $\mathcal{O}^{\otimes}$ belongs to the essential image of the localization functor $U : \mathrm{Op}_{\infty} \to \mathrm{Op}_{\infty}$ of Corollary 2.3.1.8.*

Proof. We first show that (1) implies (2). Let $\alpha : \mathcal{O}^{\otimes} \to U\,\mathcal{O}^{\otimes}$ be a morphism in Op_{∞} which exhibits $U\,\mathcal{O}^{\otimes}$ as a U-localization of $\mathcal{O}^{\otimes}$ (so that $U\,\mathcal{O}^{\otimes}$ is a tensor product of $\mathcal{O}^{\otimes}$ with $\mathbb{E}_0^{\otimes}$). We will prove that there exists a morphism $\beta : U\,\mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes}$ such that $\beta \circ \alpha$ is equivalent to $\mathrm{id}_{\mathcal{O}^{\otimes}}$. We claim that β is a homotopy inverse to α: to prove this, it suffices to show that $\alpha \circ \beta$ is homotopic to the identity $\mathrm{id}_{U\,\mathcal{O}^{\otimes}}$. Since $U\,\mathcal{O}^{\otimes}$ is U-local and α is a U-equivalence, it suffices to show that $\alpha \circ \beta \circ \alpha$ is homotopic to α, which is clear.

To construct the map β, we observe that $U\,\mathcal{O}^{\otimes}$ can be identified with a fibrant replacement for the object $\mathcal{O}^{\otimes,\natural} \odot (\Delta^1)^{\flat} \in \mathcal{P}\mathrm{Op}_{\infty}$ (here we regard $(\Delta^1)^{\flat}$ as an ∞-preoperad as in the proof of Proposition 2.3.1.6). It will therefore suffice to construct a map $h : \mathcal{O}^{\otimes} \times \Delta^1 \to \mathcal{O}^{\otimes}$ such that $h|(\mathcal{O}^{\otimes} \times \{1\}) = \mathrm{id}_{\mathcal{O}^{\otimes}}$ and $h|(\mathcal{O}^{\otimes} \times \{0\})$ factors through $\mathcal{O}^{\otimes}_{\langle 0 \rangle}$. The existence of h follows immediately the fact that $\mathcal{O}^{\otimes}$ is a pointed ∞-category (Proposition 2.3.1.4).

To show that $(2) \Rightarrow (1)$, we reverse the above reasoning: if $\mathcal{O}^{\otimes}$ is U-local, then there exists a morphism $\beta : U\,\mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes}$ which is right inverse to α, which is equivalent to the existence of a map $h : \mathcal{O}^{\otimes} \times \Delta^1 \to \mathcal{O}^{\otimes}$ as above. We may assume without loss of generality that $h|(\mathcal{O}^{\otimes} \times \{0\})$ is the constant map taking some value $X \in \mathcal{O}^{\otimes}_{\langle 0 \rangle}$. Then h can be regarded as a section of the left fibration $(\mathcal{O}^{\otimes})^{X/} \to \mathcal{O}^{\otimes}$. This proves that X is an initial object of $\mathcal{O}^{\otimes}$. Since X is also a final object of $\mathcal{O}^{\otimes}$, we deduce that $\mathcal{O}^{\otimes}$ is pointed as an ∞-category and therefore unital as an ∞-operad (Proposition 2.3.1.4). □

It follows from Proposition 2.3.1.9 that the full subcategory of Op_{∞} spanned by the unital ∞-operads is a localization of Op_{∞}. Our next goal is to show that this subcategory is also a colocalization of Op_{∞}.

Definition 2.3.1.10. *Let $f : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a map of ∞-operads. We will say that f exhibits $\mathcal{O}'^{\otimes}$ as a unitalization of $\mathcal{O}^{\otimes}$ if the following conditions are satisfied:*

(1) *The ∞-operad $\mathcal{O}'^{\otimes}$ is unital.*

(2) *For every unital ∞-operad $\mathcal{C}^{\otimes}$, composition with f induces an equivalence of ∞-categories $\mathrm{Alg}_{\mathcal{C}}(\mathcal{O}') \to \mathrm{Alg}_{\mathcal{C}}(\mathcal{O})$.*

It is clear that if an ∞-operad $\mathcal{O}^{\otimes}$ admits a unitalization $\mathcal{O}'^{\otimes}$, then $\mathcal{O}'^{\otimes}$ is determined uniquely up to equivalence. We now prove the existence of $\mathcal{O}'^{\otimes}$ by means of a simple explicit construction.

Proposition 2.3.1.11. *Let $\mathcal{O}^{\otimes}$ be an ∞-operad, and let $\mathcal{O}^{\otimes}_{*}$ be the ∞-category of pointed objects of $\mathcal{O}^{\otimes}$. Then:*

(1) *The forgetful map $p : \mathcal{O}^{\otimes}_{*} \to \mathcal{O}^{\otimes}$ is a fibration of ∞-operads (in particular, $\mathcal{O}^{\otimes}_{*}$ is an ∞-operad).*

(2) *The ∞-operad $\mathcal{O}^{\otimes}_{*}$ is unital.*

(3) *For every unital ∞-operad $\mathcal{C}^{\otimes}$, composition with p induces a trivial Kan fibration $\theta : \mathrm{Alg}_{\mathcal{C}}(\mathcal{O}_{*}) \to \mathrm{Alg}_{\mathcal{C}}(\mathcal{O})$ (here $\mathrm{Alg}_{\mathcal{C}}(\mathcal{O}_{*})$ denotes the ∞-category of $\mathcal{C}$-algebra objects in the ∞-operad $\mathcal{O}^{\otimes}_{*}$).*

(4) *The map p exhibits $\mathcal{O}^{\otimes}_{*}$ as a unitalization of the ∞-operad $\mathcal{O}^{\otimes}$.*

Lemma 2.3.1.12. *Let $\mathcal{C}$ be a pointed ∞-category, and let $\mathcal{D}$ be an ∞-category with a final object. Let $\mathrm{Fun}'(\mathcal{C}, \mathcal{D})$ be the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which preserve final objects, and let $\mathrm{Fun}'(\mathcal{C}, \mathcal{D}_{*})$ be defined similarly. Then the forgetful functor*

$$\mathrm{Fun}'(\mathcal{C}, \mathcal{D}_{*}) \to \mathrm{Fun}'(\mathcal{C}, \mathcal{D})$$

is a trivial Kan fibration.

Proof. Let $\mathcal{E} \subseteq \mathcal{C} \times \Delta^1$ be the full subcategory spanned by objects (C, i), where either C is a zero object of $\mathcal{C}$ or $i = 1$. Let $\mathrm{Fun}'(\mathcal{E}, \mathcal{D})$ be the full subcategory of $\mathrm{Fun}(\mathcal{E}, \mathcal{D})$ spanned by those functors F such that $F(C, i)$ is a final object of $\mathcal{D}$, whenever $C \in \mathcal{C}$ is a zero object. We observe that a functor $F \in \mathrm{Fun}(\mathcal{E}, \mathcal{D})$ belongs to $\mathrm{Fun}'(\mathcal{E}, \mathcal{D})$ if and only if $F_0 = F | \mathcal{C} \times \{1\}$ belongs to $\mathrm{Fun}'(\mathcal{C}, \mathcal{D})$, and F is a right Kan extension of F_0. We can identify $\mathrm{Fun}'(\mathcal{C}, \mathcal{D}_*)$ with the full subcategory of $\mathrm{Fun}(\mathcal{C} \times \Delta^1, \mathcal{D})$ spanned by those functors G such that $G_0 = G | \mathcal{E} \in \mathrm{Fun}'(\mathcal{E}, \mathcal{D})$ and G is a left Kan extension of G_0. It follows from Proposition T.4.3.2.15 that the restriction maps

$$\mathrm{Fun}'(\mathcal{C}, \mathcal{D}_*) \to \mathrm{Fun}'(\mathcal{E}, \mathcal{D}) \to \mathrm{Fun}'(\mathcal{C}, \mathcal{D})$$

are trivial Kan fibrations, so that their composition is a trivial Kan fibration as desired. □

Proof of Proposition 2.3.1.11. We first prove (1). Fix an object $X_* \in \mathcal{O}_*^{\otimes}$ lying over $X \in \mathcal{O}^{\otimes}$, and let $\alpha : X \to Y$ be an inert morphism in $\mathcal{O}^{\otimes}$. Since the map $q : \mathcal{O}_*^{\otimes} \to \mathcal{O}^{\otimes}$ is a left fibration, we can lift α to a morphism $X_* \to Y_*$, which is automatically q-coCartesian. Let $\langle n \rangle$ denote the image of X in $\mathcal{F}\mathrm{in}_*$, and choose inert morphisms $\alpha^i : X \to X^i$ covering the maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \leq i \leq n$. We claim that the induced functors $\alpha_!^i$ induce an equivalence $(\mathcal{O}_*^{\otimes})_X \to \prod_{1 \leq i \leq n} (\mathcal{O}_*^{\otimes})_{X_i}$. Fix a final object 1 in $\mathcal{O}^{\otimes}$, so that $\mathcal{O}_*^{\otimes}$ is equivalent to $\mathcal{O}_{1/}^{\otimes}$. The desired assertion is not equivalent to the assertion that the maps α^i induce a homotopy equivalence

$$\mathrm{Map}_{\mathcal{O}^{\otimes}}(1, X) \to \prod_{1 \leq i \leq n} \mathrm{Map}_{\mathcal{O}^{\otimes}}(1, X_i),$$

which follows immediately from our assumptions that $\mathcal{O}^{\otimes}$ is an ∞-operad and that each α^i is inert.

To complete the proof that p is an ∞-operad fibration, let X_* be as above, let $\langle n \rangle$ be its image in $\mathcal{F}\mathrm{in}_*$, and suppose we have chosen morphisms $X_* \to X_*^i$ in $\mathcal{O}_*^{\otimes}$ whose images in $\mathcal{O}^{\otimes}$ are inert and which cover the inert morphisms $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \leq i \leq n$; we wish to show that the induced diagram $\overline{\delta} : \langle n \rangle^{\circ \triangleleft} \to \mathcal{O}_*^{\otimes}$ is a p-limit diagram. Let $\delta = \overline{\delta} | \langle n \rangle^{\circ}$; we wish to prove that the map

$$(\mathcal{O}_*^{\otimes})_{/\overline{\delta}} \to (\mathcal{O}_*^{\otimes})_{/\delta} \times_{\mathcal{O}_{/p\delta}^{\otimes}} \mathcal{O}_{/p\overline{\delta}}^{\otimes}$$

is a trivial Kan fibration. Since $\mathcal{O}_*^{\otimes}$ is equivalent to $\mathcal{O}_{1/}^{\otimes}$, this is equivalent to the requirement that every extension problem of the form

$$
\begin{array}{ccc}
\partial \Delta^m \star \langle n \rangle^{\circ} & \xrightarrow{f} & \mathcal{O}^{\otimes} \\
\downarrow & \nearrow & \\
\Delta^m \star \langle n \rangle^{\circ} & &
\end{array}
$$

admits a solution, provided that $m \geq 2$, f carries the initial vertex of Δ^m to $1 \in \mathcal{O}^{\otimes}$, and $f | \{m\} \star \langle n \rangle^{\circ} = p \circ \overline{\delta}$. Let $\pi : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. The map $\pi \circ f$ admits a unique extension to $\Delta^m \star \langle n \rangle^{\circ}$: this is obvious if $m > 2$, and for $m = 2$ it follows from the observation that $\pi(1) = \langle 0 \rangle$ is an initial object of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. The solubility of the relevant lifting problem now follows from the observation that $p \circ \overline{\delta}$ is a π-limit diagram.

Assertion (2) is clear (since $\mathcal{O}_*^{\otimes}$ has a zero object), assertion (3) follows from the observation that θ is a pullback of the morphism $\mathrm{Fun}'(\mathcal{C}^{\otimes}, \mathcal{O}_*^{\otimes}) \to \mathrm{Fun}'(\mathcal{C}^{\otimes}, \mathcal{O}^{\otimes})$ described in Lemma 2.3.1.12, and assertion (4) follows immediately from (2) and (3). □

We conclude this section with two results concerning the behavior of unitalization in families.

Proposition 2.3.1.13. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads, where $\mathcal{O}^{\otimes}$ is unital. The following conditions are equivalent:*

(1) *The ∞-operad $\mathcal{C}^{\otimes}$ is unital.*

(2) *For every object $X \in \mathcal{C}$, the unit object of $\mathcal{C}_X$ (see §3.2.1) is initial in $\mathcal{C}_X$.*

Proof. Choose an object $1 \in \mathcal{C}^{\otimes}_{\langle 0 \rangle}$. Assertion (1) is equivalent to the requirement that 1 be an initial object of $\mathcal{C}^{\otimes}$. Since $p(1)$ is an initial object of $\mathcal{O}^{\otimes}$, this is equivalent to the requirement that $\emptyset$ is p-initial (Proposition T.4.3.1.5). Since p is a coCartesian fibration, (1) is equivalent to the requirement that for every morphism $\beta : p(1) \to X$ in $\mathcal{O}$, the object $\beta_!(1)$ is an initial object of $\mathcal{C}^{\otimes}_X$ (Proposition T.4.3.1.10). Write $X = \bigoplus X_i$, where each $X_i \in \mathcal{O}$. Using the equivalence $\mathcal{C}^{\otimes}_X \simeq \prod_i \mathcal{C}_{X_i}$, we see that it suffices to check this criterion when $X \in \mathcal{O}$, in which case we are reduced to assertion (2). $\qquad\square$

Proposition 2.3.1.14. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads, where $\mathcal{O}^{\otimes}$ is unital. Then:*

(1) *Let $q : \mathcal{C}'^{\otimes} \to \mathcal{C}^{\otimes}$ be a categorical fibration which exhibits $\mathcal{C}'^{\otimes}$ as a unitalization of $\mathcal{C}^{\otimes}$. Then the map $p \circ q : \mathcal{C}'^{\otimes} \to \mathcal{O}^{\otimes}$ is a coCartesian fibration of ∞-operads.*

(2) *For every map of unital ∞-operads $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$, the map q induces an equivalence of ∞-categories $\theta : \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}') \to \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$.*

Proof. By virtue of Proposition 2.3.1.11, we may assume without loss of generality that $\mathcal{C}'^{\otimes} = \mathcal{C}^{\otimes}_*$. In this case, the map $p \circ q$ factors as a composition $\mathcal{C}^{\otimes}_* \to \mathcal{O}^{\otimes}_* \to \mathcal{O}^{\otimes}$. The functor $\mathcal{C}^{\otimes}_* \to \mathcal{O}^{\otimes}_*$ is equivalent to $\mathcal{C}^{\otimes}_{1/} \to \mathcal{O}^{\otimes}_{p(1)/}$, where $1 \in \mathcal{C}^{\otimes}_{\langle 0 \rangle}$ is a final object of $\mathcal{C}^{\otimes}$, and therefore a coCartesian fibration (Proposition T.2.4.3.1), and the map $\mathcal{O}^{\otimes}_* \to \mathcal{O}^{\otimes}$ is a trivial Kan fibration by virtue of our assumption that $\mathcal{O}^{\otimes}$ is unital. This proves (1). To prove (2), it suffices to observe that θ is a pullback of the map $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}_*) \to \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$, which is a trivial Kan fibration by Proposition 2.3.1.11 (here $\mathcal{C}_*$ denotes the underlying ∞-category of the ∞-operad $\mathcal{C}^{\otimes}_*$, which is generally *not* the ∞-category of pointed objects of $\mathcal{C}$). $\qquad\square$

2.3.2 Generalized ∞-Operads

Let $\mathcal{O}^{\otimes}$ be an ∞-operad. Then, for each $n \geq 0$, we have a canonical equivalence of ∞-categories $\mathcal{O}^{\otimes}_{\langle n \rangle} \simeq \mathcal{O}^n$. In particular, the ∞-category $\mathcal{O}^{\otimes}_{\langle 0 \rangle}$ is a contractible Kan complex. In this section, we will introduce the notion of a *generalized ∞-operad* (Definition 2.3.2.1), where we relax the assumption that $\mathcal{O}^{\otimes}_{\langle 0 \rangle}$ is contractible, and replace the absolute nth power $\mathcal{O}^n$ with the nth fiber power over the ∞-category $\mathcal{O}^{\otimes}_{\langle 0 \rangle}$. We will also introduce the closely related notion of a $\mathcal{C}$-*family of ∞-operads*, where $\mathcal{C}$ is an ∞-category (Definition 2.3.2.10). We will see that giving a generalized ∞-operad $\mathcal{O}^{\otimes}$ is equivalent to giving an ∞-category $\mathcal{C}$ (which can be identified with $\mathcal{O}^{\otimes}_{\langle 0 \rangle}$) and a $\mathcal{C}$-family of ∞-operads.

Definition 2.3.2.1. A *generalized ∞-operad* is an ∞-category $\mathcal{O}^{\otimes}$ equipped with a map $q : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ satisfying the following conditions:

(1) For every object $X \in \mathcal{O}^{\otimes}$ and every inert morphism $\alpha : p(X) \to \langle n \rangle$, there exists a q-coCartesian morphism $\overline{\alpha} : X \to Y$ with $q(\overline{\alpha}) = \alpha$.

(2) Suppose we are given a commutative diagram σ:

$$
\begin{array}{ccc}
\langle m \rangle & \longrightarrow & \langle n \rangle \\
\downarrow & & \downarrow \\
\langle m' \rangle & \longrightarrow & \langle n' \rangle
\end{array}
$$

in $\mathcal{F}\mathrm{in}_*$ which consists of inert morphisms and induces a bijection of finite sets $\langle m' \rangle^{\circ} \coprod_{\langle n' \rangle^{\circ}} \langle n \rangle^{\circ} \to \langle m \rangle^{\circ}$. Then the induced diagram

$$
\begin{array}{ccc}
\mathcal{O}^{\otimes}_{\langle m \rangle} & \longrightarrow & \mathcal{O}^{\otimes}_{\langle n \rangle} \\
\downarrow & & \downarrow \\
\mathcal{O}^{\otimes}_{\langle m' \rangle} & \longrightarrow & \mathcal{O}^{\otimes}_{\langle n' \rangle}
\end{array}
$$

is a pullback square of ∞-categories.

(3) Let σ be as in (2), and suppose that σ can be lifted to a diagram $\overline{\sigma}$

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X' & \longrightarrow & Y'
\end{array}
$$

consisting of q-coCartesian morphisms in $\mathcal{O}^{\otimes}$. The $\overline{\sigma}$ is a q-limit diagram.

Definition 2.3.2.2. Let $q : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a generalized ∞-operad. We will say that a morphism α in $\mathcal{O}^{\otimes}$ is *inert* if $q(\alpha)$ is an inert morphism in $\mathcal{F}\mathrm{in}_*$ and α is q-coCartesian. We let $\mathcal{O}^{\otimes,\natural}$ denote the marked simplicial set $(\mathcal{O}^{\otimes}, M)$, where M is the collection of all inert morphisms in $\mathcal{O}^{\otimes}$.

If $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ are generalized ∞-operads, then we will say that a morphism of simplicial sets $f : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ is a *map of generalized ∞-operads* if the following conditions are satisfied:

(a) The diagram

$$
\begin{array}{ccc}
\mathcal{O}^{\otimes} & \longrightarrow & \mathcal{O}'^{\otimes} \\
 & \searrow \quad \swarrow & \\
 & \mathrm{N}(\mathcal{F}\mathrm{in}_*) &
\end{array}
$$

commutes.

(b) The map f carries inert morphisms in $\mathcal{O}^{\otimes}$ to inert morphisms in $\mathcal{O}'^{\otimes}$.

We let $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$ denote the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes})$ spanned by the maps of generalized ∞-operads.

Variant 2.3.2.3. Given a categorical fibration of generalized ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and a map $\alpha : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$, we let $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ denote the fiber of the induced map $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}'}(\mathcal{O})$ over the vertex α. If $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes}$ and α is the identity map, we let $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ denote the ∞-category $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$.

Remark 2.3.2.4. Let $\mathcal{POp}_{\infty}$ be the category of ∞-preoperads (Definition 2.1.4.2). There exists a left proper combinatorial simplicial model structure on the category $\mathcal{POp}_{\infty}$ with the following properties:

(1) A morphism $\alpha : \overline{X} \to \overline{Y}$ in $\mathcal{POp}_{\infty}$ is a cofibration if and only if the underlying map of simplicial sets $X \to Y$ is a monomorphism.

(2) An object $\overline{X}$ in $\mathcal{POp}_{\infty}$ is fibrant if and only if it has the form $\mathcal{O}^{\otimes,\natural}$, for some generalized ∞-operad $\mathcal{O}^{\otimes}$.

We will refer to this model structure on $\mathcal{POp}_{\infty}$ as the *generalized ∞-operadic model structure*.

To verify the existence (and uniqueness) of this model structure, we apply Theorem B.0.19 to the categorical pattern $\mathfrak{P} = (M, T, \{\sigma_{\alpha} : \Delta^1 \times \Delta^1 \to \mathcal{O}^{\otimes}\}_{\alpha \in A})$ on $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, where M is the collection of inert morphisms in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, T is the collection of all 2-simplices of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, and A is the collection of all diagrams

$$
\begin{array}{ccc}
\langle m \rangle & \longrightarrow & \langle n \rangle \\
\downarrow & & \downarrow \\
\langle m' \rangle & \longrightarrow & \langle n' \rangle
\end{array}
$$

consisting of inert morphisms which induce a bijection $\langle m' \rangle^{\circ} \coprod_{\langle n' \rangle^{\circ}} \langle n \rangle^{\circ} \to \langle m \rangle^{\circ}$. Moreover, Proposition B.2.7 implies the following additional property:

(3) Let $\mathcal{O}^{\otimes}$ be a generalized ∞-operad. Then a map of ∞-preoperads $\overline{X} \to \mathcal{O}^{\otimes,\natural}$ is a fibration (with respect to the generalized ∞-operadic model structure) if and only if $\overline{X}$ has the form $\mathcal{O}'^{\otimes,\natural}$, where $\mathcal{O}'^{\otimes}$ is a generalized ∞-operad and the underlying map $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ is a categorical fibration which carries inert morphisms in $\mathcal{O}'^{\otimes}$ to an inert morphism in $\mathcal{O}^{\otimes}$.

The following result shows that the theory of generalized ∞-operads really is a generalization of the theory of ∞-operads:

Proposition 2.3.2.5. *Let* $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *be a map of simplicial sets. The following conditions are equivalent:*

(1) *The map* p *exhibits* $\mathcal{O}^{\otimes}$ *as a generalized* ∞*-operad, and the fiber* $\mathcal{O}^{\otimes}_{\langle 0 \rangle}$ *is a contractible Kan complex.*

(2) *The map* p *exhibits* $\mathcal{O}^{\otimes}$ *as an* ∞*-operad.*

Proposition 2.3.2.5 is a consequence of a more general result (Proposition 2.3.2.11) which we will prove at the end of this section.

Corollary 2.3.2.6. *Let* $\mathbf{A} = \mathbf{B} = \mathcal{P}\mathrm{Op}_{\infty}$*, where we regard* $\mathbf{A}$ *as endowed with with the generalized* ∞*-operadic model structure of Remark 2.3.2.4 and* $\mathbf{B}$ *as endowed with the* ∞*-operadic model structure of Proposition 2.1.4.6. Then the identify functor* $F : \mathbf{A} \to \mathbf{B}$ *is a left Quillen functor: that is, we can regard* ∞*-operadic model structure as a localization of the generalized* ∞*-operadic model structure (see* §*T.A.3.7).*

Proof. Since $\mathbf{A}$ and $\mathbf{B}$ have the same class of cofibrations, it will suffice to show that the functor F preserves weak equivalences. Let $\alpha : \overline{X} \to \overline{Y}$ be a map of ∞-preoperads. Then α is a weak equivalence in $\mathbf{A}$ if and only if, for every generalized ∞-operad $\mathcal{O}^{\otimes}$, composition with α induces a homotopy equivalence $\mathrm{Map}_{\mathcal{P}\mathrm{Op}_{\infty}}(\overline{Y}, \mathcal{O}^{\otimes,\natural}) \to \mathrm{Map}_{\mathcal{P}\mathrm{Op}_{\infty}}(\overline{X}, \mathcal{O}^{\otimes,\natural})$. Since every ∞-operad is a generalized ∞-operad (Proposition 2.3.2.5), this condition implies that α is a weak equivalence in $\mathbf{B}$ as well. $\square$

Notation 2.3.2.7. We let $\mathrm{Op}^{\mathrm{gn}}_{\infty}$ denote the underlying ∞-category $\mathrm{N}(\mathcal{P}\mathrm{Op}_{\infty}{}^{\mathrm{o}})$ of the simplicial model category $\mathcal{P}\mathrm{Op}_{\infty}$, with respect to the generalized ∞-operadic model structure of Remark 2.3.2.4. We will refer to $\mathrm{Op}^{\mathrm{gn}}_{\infty}$ as the ∞*-category of generalized* ∞*-operads*. It contains the ∞-category Op_{∞} of ∞-operads as a full subcategory.

Remark 2.3.2.8. The terminology introduced in Definition 2.3.2.2 for discussing generalized ∞-operads is compatible with the corresponding terminology for ∞-operads. For example, if $\mathcal{O}^{\otimes}$ is an ∞-operad, then a morphism in $\mathcal{O}^{\otimes}$ is inert in the sense of Definition 2.3.2.2 if and only if it is inert in the sense of Definition 2.1.2.3. If $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ are ∞-operads, then a functor $f : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ is a map of ∞-operads if and only if it is a map of generalized ∞-operads, and the notation $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$ is unambiguous. Similarly, the notation of Variant 2.3.2.3 is compatible with the notation for ∞-operads introduced in Definition 2.1.3.1.

According to Proposition 2.3.2.5, the discrepancy between Op_{∞} and $\mathrm{Op}^{\mathrm{gn}}_{\infty}$ is controlled by the forgetful functor $F : \mathrm{Op}^{\mathrm{gn}}_{\infty} \to \mathcal{C}\mathrm{at}_{\infty}$, given by the formula $F(\mathcal{O}^{\otimes}) = \mathcal{O}^{\otimes}_{\langle 0 \rangle}$.

Proposition 2.3.2.9. (1) *For every* ∞*-category* $\mathcal{C}$*, the product* $\mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *is a generalized* ∞*-operad.*

(2) *The construction* $\mathcal{C} \mapsto \mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *determines a functor* $G : \mathcal{C}\mathrm{at}_{\infty} \to \mathrm{Op}^{\mathrm{gn}}_{\infty}$*.*

(3) *The functor* G *is a fully faithful right adjoint to the forgetful functor* $F : \mathrm{Op}^{\mathrm{gn}}_{\infty} \to \mathcal{C}\mathrm{at}_{\infty}$ *described above.*

Proof. Assertions (1) and (2) are obvious. We have a canonical equivalence $v : F \circ G \to \mathrm{id}$ of functors from $\mathcal{C}\mathrm{at}_{\infty}$ to itself. To complete the proof of (3), it will suffice to show that v is the counit of an adjunction between F and G. In other words, we must show that for every generalized ∞-operad $\mathcal{O}^{\otimes}$ and every ∞-category $\mathcal{C}$, the restriction functor $\theta : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{O}^{\otimes}_{\langle 0 \rangle}, \mathcal{C})$ induces a homotopy equivalence from $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\simeq}$ to the Kan complex $\mathrm{Fun}(\mathcal{O}^{\otimes}_{\langle 0 \rangle}, \mathcal{C})^{\simeq}$ (here we identify $\mathcal{C}$ with the underlying ∞-category of the generalized ∞-operad $\mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$). In fact, we will show that θ is a trivial Kan fibration.

We observe that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ can be identified with the full subcategory of $\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C})$ spanned by those functors which carry each inert morphism of $\mathcal{O}^{\otimes}$ to an equivalence in $\mathcal{C}$. In view of Proposition T.4.3.2.15, it will suffice to prove the following:

(a) A functor $F : \mathcal{O}^{\otimes} \to \mathcal{C}$ is a right Kan extension of $F | \mathcal{O}^{\otimes}_{\langle 0 \rangle}$ if and only if F carries each inert morphism in $\mathcal{O}^{\otimes}$ to an equivalence in $\mathcal{C}$.

(b) Every functor $F_0 : \mathcal{O}^{\otimes}_{\langle 0 \rangle} \to \mathcal{C}$ can be extended to a functor $F : \mathcal{O}^{\otimes} \to \mathcal{C}$ satisfying the equivalent conditions of (a).

To prove (a), we note that for each object $X \in \mathcal{O}^{\otimes}$, the ∞-category $\mathcal{O}^{\otimes}_{X/} \times_{\mathcal{O}^{\otimes}} \mathcal{O}^{\otimes}_{\langle 0 \rangle}$ contains an initial object: namely, any inert morphism $X \to X_0$ where $X_0 \in \mathcal{O}^{\otimes}_{\langle 0 \rangle}$. Consequently, a functor $F : \mathcal{O}^{\otimes} \to \mathcal{C}$ is a right Kan extension of $F | \mathcal{O}^{\otimes}_{\langle 0 \rangle}$ if and only if $F(\alpha)$ is an equivalence for every morphism $\alpha : X \to X_0$ such that $X_0 \in \mathcal{O}^{\otimes}_{\langle 0 \rangle}$. This proves the "if" direction of (a). The "only if" direction follows from the two-out-of-three property, since every inert morphism $X \to Y$ in $\mathcal{O}^{\otimes}$ fits into a commutative diagram of inert morphisms

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
 & \searrow \quad \swarrow & \\
 & Z &
\end{array}
$$

where $Z \in \mathcal{O}^{\otimes}_{\langle 0 \rangle}$. Assertion (b) follows immediately from Lemma T.4.3.2.13. $\square$

Our next goal is to describe the fiber of the forgetful functor $F : \mathrm{Op}^{\mathrm{gn}}_{\infty} \to \mathcal{C}\mathrm{at}_{\infty}$ over a general ∞-category $\mathcal{C}$.

Definition 2.3.2.10. Let $\mathcal{C}$ be an ∞-category. A $\mathcal{C}$-*family of ∞-operads* is a categorical fibration $p : \mathcal{O}^{\otimes} \to \mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ with the following properties:

(a) Let $C \in \mathcal{C}$ be an object, let $X \in \mathcal{O}^{\otimes}_C$ have image $\langle m \rangle \in \mathcal{F}\mathrm{in}_*$, and let $\alpha : \langle m \rangle \to \langle n \rangle$ be an inert morphism. Then there exists a p-coCartesian morphism $\overline{\alpha} : X \to Y$ in $\mathcal{O}^{\otimes}_C$.

We will say that a morphism $\overline{\alpha}$ of $\mathcal{O}^{\otimes}$ is *inert* if $\overline{\alpha}$ is p-coCartesian, the image of $\overline{\alpha}$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is inert, and the image of $\overline{\alpha}$ in $\mathcal{C}$ is an equivalence.

(b) Let $X \in \mathcal{O}^{\otimes}$ have images $C \in \mathcal{C}$ and $\langle n \rangle \in \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. For $1 \leq i \leq n$, let $f_i : X \to X_i$ be an inert morphism in $\mathcal{O}^{\otimes}_C$ which covers $\rho^i : \langle n \rangle \to \langle 1 \rangle$. Then the collection of morphisms $\{f_i\}_{1 \leq i \leq n}$ determines a p-limit diagram $\langle n \rangle^{\circ \triangleleft} \to \mathcal{O}^{\otimes}$.

(c) For each object $C \in \mathcal{C}$, the induced map $\mathcal{O}^{\otimes}_C \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is an ∞-operad.

The main result of this section is the following:

Proposition 2.3.2.11. *Let $\mathcal{C}$ be an ∞-category and let $p : \mathcal{O}^{\otimes} \to \mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a categorical fibration of simplicial sets. The following conditions are equivalent:*

(1) *The map p is a fibration of generalized ∞-operads, and the underlying map $\mathcal{O}^{\otimes}_{\langle 0 \rangle} \to \mathcal{C}$ is a trivial Kan fibration.*

(2) *The map p exhibits $\mathcal{O}^{\otimes}$ as a $\mathcal{C}$-family of ∞-operads.*

Remark 2.3.2.12. In the special case $\mathcal{C} = \Delta^0$, the definition of a $\mathcal{C}$-operad family reduces to the usual definition of an ∞-operad. Consequently, Proposition 2.3.2.11 implies Proposition 2.3.2.5 (by taking $\mathcal{C} = \Delta^0$).

Corollary 2.3.2.13. *Let $\mathcal{C}$ be an ∞-category. Then the fiber product $\mathrm{Op}^{\mathrm{gn}}_{\infty} \times_{\mathcal{C}\mathrm{at}_{\infty}} \{\mathcal{C}\}$ can be identified with the full subcategory of $(\mathrm{Op}^{\mathrm{gn}}_{\infty})_{/\mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)}$ spanned by those maps $\mathcal{O}^{\otimes} \to \mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ which exhibit $\mathcal{O}^{\otimes}$ as a $\mathcal{C}$-family of ∞-operads.*

In other words, we can think of a generalized ∞-operad as consisting of a pair $(\mathcal{C}, \mathcal{O}^\otimes)$, where $\mathcal{C}$ is an ∞-category and $\mathcal{O}^\otimes$ is a $\mathcal{C}$-family of ∞-operads.

Proof of Corollary 2.3.2.13. We can identify the fiber product $\mathrm{Op}^{\mathrm{gn}}_\infty \times_{\mathcal{C}\mathrm{at}_\infty} \{\mathcal{C}\}$ with the full subcategory of $\mathrm{Op}^{\mathrm{gn}}_\infty \times_{\mathcal{C}\mathrm{at}_\infty} (\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{C}}$ spanned by those pairs $(\mathcal{O}^\otimes, \alpha : \mathcal{O}^\otimes_{\langle 0 \rangle} \to \mathcal{C})$ where $\mathcal{O}^\otimes$ is a generalized ∞-operad and α is an equivalence of ∞-categories. Using Proposition 2.3.2.9, we can identify this fiber product with the full subcategory of $(\mathrm{Op}^{\mathrm{gn}}_\infty)_{/\mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)}$ spanned by those maps of generalized ∞-operads $\mathcal{O}^\otimes \to \mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ which induce a categorical equivalence $\mathcal{O}^\otimes_{\langle 0 \rangle} \to \mathcal{C}$. This subcategory evidently contains all $\mathcal{C}$-families of ∞-operads. Conversely, if $f : \mathcal{O}^\otimes \to \mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is an arbitrary map of generalized ∞-operads which induces a categorical equivalence $\mathcal{O}^\otimes_{\langle 0 \rangle} \to \mathcal{C}$, then we can factor f as a composition

$$\mathcal{O}^\otimes \xrightarrow{f'} \mathcal{O}'^\otimes \xrightarrow{f''} \mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*),$$

where f' is an equivalence of generalized ∞-operads and f'' is a categorical fibration. It follows that f'' induces a trivial Kan fibration $\mathcal{O}'^\otimes_{\langle 0 \rangle} \to \mathcal{C}$ and therefore exhibits $\mathcal{O}'^\otimes$ as a $\mathcal{C}$-family of ∞-operads. $\square$

We conclude this section by proving Proposition 2.3.2.11.

Proof of Proposition 2.3.2.11. We first prove that $(1) \Rightarrow (2)$. Assume that $p : \mathcal{O}^\otimes \to \mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a fibration of generalized ∞-operads. Then $\mathcal{O}^\otimes$ is a generalized ∞-operad and p is a categorical fibration which carries inert morphisms in $\mathcal{O}^\otimes$ to equivalences in $\mathcal{C}$. We will show that p satisfies conditions (a), (b), and (c) of Definition 2.3.2.10. To prove (a), suppose that $X \in \mathcal{O}^\otimes$ lies over $(C, \langle m \rangle) \in \mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and that we are given an inert morphism $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Since $\mathcal{O}^\otimes$ is a generalized ∞-operad, we can lift α to an inert $\overline{\alpha} : X \to X'$ in $\mathcal{O}^\otimes$, lying over a map $(C, \langle m \rangle) \to (C', \langle n \rangle)$. Because p preserves inert morphisms, the underlying map $C \to C'$ is an equivalence. Choosing a homotopy inverse, we get an equivalence $(C', \langle n \rangle) \to (C, \langle n \rangle)$ in $\mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ which (since p is a categorical fibratino) can be lifted to an equivalence $\beta : X' \to X''$ in $\mathcal{O}^\otimes$. Since p has the right lifting property with respect to the horn inclusion $\Lambda^2_1 \subseteq \Delta^2$, we can choose a composition $\gamma \simeq \beta \circ \overline{\alpha}$ lying over the morphism $(\mathrm{id}_C, \alpha) : (C, \langle m \rangle) \to (C, \langle n \rangle)$. We claim that γ is p-coCartesian. Since γ is equivalent to $\overline{\alpha}$, it suffices to show that α is p-coCartesian. This follows from Proposition T.2.4.1.3, since α is inert and $p(\alpha)$ is coCartesian with respect to the projection $\pi : \mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

We now prove (b). Suppose we are given an object $X \in \mathcal{O}^\otimes_{(C, \langle n \rangle)}$ together with inert morphisms $X \to X_i$ in $\mathcal{O}^\otimes_C$ covering the maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$. We wish to show that the induced map $q : \langle n \rangle^{\triangleleft} \to \mathcal{O}^\otimes$ is a p-limit diagram. The proof proceeds by induction on n.

If $n = 0$, then we must show that every object $X \in \mathcal{O}^\otimes_{C, \langle 0 \rangle}$ is p-final. In other words, we must show that for every object $Y \in \mathcal{O}^\otimes_{D, \langle m \rangle}$, the homotopy fiber of the map

$$\mathrm{Map}_{\mathcal{O}^\otimes}(Y, X) \to \mathrm{Map}_{\mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)}((D, \langle m \rangle), (C, \langle 0 \rangle))$$

is contractible: that is, $\mathrm{Map}_{\mathcal{O}^\otimes}(Y, X) \to \mathrm{Map}_{\mathcal{C}}(D, C)$ is a homotopy equivalence. To prove this, we choose an inert morphism $\alpha : Y \to Y'$ covering the unique map $\langle m \rangle \to \langle 0 \rangle$ in $\mathcal{F}\mathrm{in}_*$. Since the image of α in $\mathcal{C}$ is an equivalence, we are free to replace Y by Y' and to thereby assume that $m = 0$. In this case, the desired assertion follows from the assumption that $\mathcal{O}^\otimes_{\langle 0 \rangle} \to \mathcal{C}$ is a trivial Kan fibration.

If $n = 1$, there is nothing to prove. Assume that $n > 1$. Let $\beta : \langle n \rangle \to \langle n - 1 \rangle$ be defined by the formula

$$\beta(i) = \begin{cases} i & \text{if } 1 \leq i \leq n - 1 \\ * & \text{otherwise,} \end{cases}$$

and choose a p-coCartesian morphism $g : X \to X'$ lying over (id_C, β). Using the assumption that g is p-coCartesian, we obtain factorizations of f_i as a composition

$$X \xrightarrow{g} X' \xrightarrow{f'_i} X_i$$

for $1 \leq i \leq n$. These factorizations determine a diagram

$$q' : (\langle n-1 \rangle^{\circ \lhd} \coprod \{n\})^{\lhd} \to \mathcal{O}_C^{\otimes}$$

extending q. Fix an object $X_0 \in \mathcal{O}_{C,\langle 0 \rangle}^{\otimes}$. We have seen that X_0 is a p-final object of $\mathcal{C}^{\otimes}$. Since $\langle 0 \rangle$ is also a final object of $N(\mathcal{F}in_*)$, we deduce that X_0 is a final object of $\mathcal{O}_C^{\otimes}$ We may therefore extend q' to a diagram

$$q'' : \{x\} \star (\langle n-1 \rangle^{\circ \lhd} \coprod \{n\}) \star \{x_0\} \to \mathcal{O}_C^{\otimes}$$

carrying x to X and x_0 to X_0.

In view of Lemma T.4.3.2.7, to prove that q is a p-limit diagram it will suffice to show the following:

(i) The restriction $q''|(\langle n-1 \rangle^{\circ \lhd} \coprod \{n\}) \star \{x_0\}$ is a p-right Kan extension of $q''|\langle n \rangle^{\circ}$.

(ii) The diagram q'' is a p-limit.

Using Proposition T.4.3.2.8, we can break the proof of (a) into two parts:

(i') The restriction $q''|(\langle n-1 \rangle^{\circ \lhd} \coprod \{n\}) \star \{x_0\}$ is a p-right Kan extension of $q''|(\langle n-1 \rangle^{\circ \lhd} \coprod \{n\})$.

(i'') The restriction $q''|(\langle n-1 \rangle^{\circ \lhd} \coprod \{n\})$ is a p-right Kan extension of $q''|\langle n \rangle^{\circ}$.

Assertion (i') follows from the observation that X_0 is a p-final object of $\mathcal{C}^{\otimes}$, and (i'') follows from the inductive hypothesis.

To prove (ii), we observe that the inclusion $(\emptyset^{\lhd} \coprod \{n\}) \star \{x_0\} \subseteq (\langle n-1 \rangle^{\circ \lhd} \coprod \{n\}) \star \{x_0\}$ is left cofinal (for example, using Theorem T.4.1.3.1). Consequently, it suffices to show that the restriction of $q''' = q|(\{x\} \star (\emptyset^{\lhd} \coprod \{n\}) \star \{x_0\})$ is a p-limit diagram. Since $p \circ q'''$ is π-coCartesian (the projection of $p \circ q'''$ to $\mathcal{C}$ is constant, and therefore a pullback square in $\mathcal{C}$), it will suffice to show that q''' is a $\pi \circ p$-limit diagram (Proposition T.2.4.1.3). This follows from our assumption that $\mathcal{O}^{\otimes}$ is a generalized ∞-operad, since q''' is a p-coCartesian lift of the inert diagram

$$
\begin{array}{ccc}
\langle n \rangle & \xrightarrow{\ \rho^i\ } & \langle 1 \rangle \\
\downarrow{\scriptstyle \beta} & & \downarrow \\
\langle n-1 \rangle & \longrightarrow & \langle 0 \rangle.
\end{array}
$$

We next verify (c): that is, for $n \geq 0$ and every object $C \in \mathcal{C}$, the maps

$$\rho_!^i : \mathcal{O}_{(C,\langle n \rangle)}^{\otimes} \to \mathcal{O}_{(C,\langle 1 \rangle)}^{\otimes}$$

induce an equivalence of ∞-categories $\theta_n : \mathcal{O}_{(C,\langle n \rangle)}^{\otimes} \to (\mathcal{O}_{(C,\langle 1 \rangle)}^{\otimes})^n$. The proof again proceeds by induction on n. When $n = 0$, this follows from our assumption that $\mathcal{O}_{\langle 0 \rangle}^{\otimes} \to \mathcal{C}$ is a trivial Kan fibration (and therefore has contractible fibers). When $n = 1$ there is nothing to prove. Assume therefore that $n \geq 2$ and observe that θ_n is equivalent to the composition

$$\mathcal{O}_{(C,\langle n \rangle)}^{\otimes} \xrightarrow{\ \beta_! \times \alpha_!\ } \mathcal{O}_{(C,\langle n-1 \rangle)}^{\otimes} \times \mathcal{O}_{(C,\langle 1 \rangle)}^{\otimes} \xrightarrow{\ \theta_{n-1} \times \mathrm{id}\ } (\mathcal{O}_{(C,\langle 1 \rangle)}^{\otimes})^n,$$

where $\beta : \langle n \rangle \to \langle n-1 \rangle$ is defined as above and $\alpha = \rho^n$. By virtue of the inductive hypothesis, it suffices to show that the map $\beta_! \times \alpha_!$ is an equivalence of ∞-categories. We have a homotopy coherent diagram of ∞-categories

$$
\begin{array}{ccc}
\mathcal{O}_{(C,\langle n \rangle)}^{\otimes} & \xrightarrow{\ \beta_!\ } & \mathcal{O}_{(C,\langle n-1 \rangle)}^{\otimes} \\
\downarrow{\scriptstyle \alpha_!} & & \downarrow \\
\mathcal{O}_{(C,\langle 1 \rangle)}^{\otimes} & \longrightarrow & \mathcal{O}_{(C,\langle 0 \rangle)}^{\otimes}.
\end{array}
$$

Because $\mathcal{O}_C^\otimes \simeq \mathcal{O}^\otimes \times_{\mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)} (\{C\} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*))$ is a generalized ∞-operad, this square is a homotopy pullback. Since $\mathcal{O}_{(C,\langle 0 \rangle)}^\otimes$ is a contractible Kan complex, we conclude that $\beta_! \times \alpha_!$ is a categorical equivalence as desired. This completes the proof that $(1) \Rightarrow (2)$.

We now prove that $(2) \Rightarrow (1)$. Assume that p exhibits $\mathcal{O}^\otimes$ as a $\mathcal{C}$-family of ∞-operads. We wish to show that $\mathcal{O}^\otimes$ is a generalized ∞-operad and that p carries inert morphisms in $\mathcal{O}^\otimes$ to equivalences in $\mathcal{C}$. Suppose first that we are given an object $X \in \mathcal{O}_C^\otimes$ lying over $\langle m \rangle \in \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and an inert morphism $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$. We wish to prove that α can be lifted to a $(\pi \circ p)$-coCartesian morphism $\overline{\alpha}$ in $\mathcal{O}^\otimes$ such that $p(\overline{\alpha})$ induces an equivalence in $\mathcal{C}$. In view of Proposition T.2.4.1.3, it will suffice to show that the morphism (id_C, α) in $\mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ can be lifted to a p-coCartesian morphism in $\mathcal{O}^\otimes$, which follows from assumption (a) of Definition 2.3.2.10.

To complete the proof that $\mathcal{O}^\otimes$ is a generalized ∞-operad, we fix a diagram $\sigma : \Delta^1 \times \Delta^1 \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ of inert morphisms

$$
\begin{array}{ccc}
\langle n \rangle & \xrightarrow{\ \alpha\ } & \langle m \rangle \\
\downarrow{\scriptstyle \beta} & {\scriptstyle \gamma}\searrow & \downarrow \\
\langle m' \rangle & \longrightarrow & \langle k \rangle
\end{array}
$$

which induces a bijection $\langle m \rangle^\circ \coprod_{\langle k \rangle^\circ} \langle m' \rangle^\circ$. We wish to prove the following:

(iii) Every map $\overline{\sigma} : \Delta^1 \times \Delta^1 \to \mathcal{O}^\otimes$ lifting σ which carries every morphism in $\Delta^1 \times \Delta^1$ to an inert morphism in $\mathcal{O}^\otimes$ is a $(\pi \circ p)$-limit diagram (since p carries inert morphisms to equivalences in $\mathcal{C}$ and the simplicial set $\Delta^1 \times \Delta^1$ is weakly contractible, we know automatically that $p(\widetilde{\sigma})$ is a π-limit diagram; by virtue of Proposition T.2.4.1.3, it suffices to show that $\overline{\sigma}$ is a p-limit diagram).

(iv) Let σ_0 denote the restriction of σ to the full subcategory K of $\Delta^1 \times \Delta^1$ obtained by omitting the initial object. If $\overline{\sigma}_0 : K \to \mathcal{O}^\otimes$ is a map lifting σ_0 which carries every edge of K to an inert morphism in $\mathcal{O}^\otimes$, then $\overline{\sigma}_0$ can be extended to a map $\overline{\sigma} : \Delta^1 \times \Delta^1 \to \mathcal{O}^\otimes$ satisfying the hypothesis of (i).

To prove these claims, consider the ∞-category $\widetilde{A} = (\Delta^1 \times \Delta^1) \star \langle n \rangle^\circ$, and let A denote the subcategory obtained by removing those morphisms of the form $(1,1) \to i$ where $i \in \gamma^{-1}\{*\}$, $(0,1) \to i$ where $i \in \beta^{-1}\{*\}$, and $(1,0) \to i$ where $i \in \alpha^{-1}\{*\}$. We observe that σ can be extended uniquely to a diagram $\tau : A \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ such that $\tau(i) = \langle 1 \rangle$ for $i \in \langle n \rangle^\circ$, and τ carries the morphism $(0,0) \to i$ to the map $\rho^i : \langle n \rangle \to \langle 1 \rangle$. The assumption that $\langle n \rangle^\circ \simeq \langle m \rangle^\circ \coprod_{\langle k \rangle^\circ} \langle m' \rangle^\circ$. guarantees that for each $i \in \langle n \rangle^\circ$, the ∞-category $(\Delta^1 \times \Delta^1) \times_A A_{/i}$ contains a final object corresponding to a morphism $(j, j') \to i$ in A, where $(j, j') \neq (0,0)$. Note that the image of this morphism in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is inert.

Let $\overline{\sigma} : \Delta^1 \times \Delta^1$ be as (iii). We may assume without loss of generality that the composition $\Delta^1 \times \Delta^1 \to \mathcal{O}^\otimes \xrightarrow{p} \mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathcal{C}$ is the constant functor taking some value $C \in \mathcal{C}$. Using Lemma T.4.3.2.13, we can choose a p-left Kan extension $\overline{\tau} : A \to \mathcal{O}_C^\otimes$ of $\overline{\sigma}$ such that $p \circ \overline{\tau} = \tau$. Let A^0 denote the full subcategory of A obtained by removing the object $(0,0)$. We observe that the inclusion $K \subseteq A^0$ is right cofinal (Theorem T.4.1.3.1). Consequently, to prove that $\overline{\sigma}$ is a p-limit diagram, it suffices to show that $\overline{\tau}$ is a p-limit diagram. Since $\mathcal{O}^\otimes$ is a $\mathcal{C}$-family of ∞-operads, the restriction of $\overline{\tau}$ to $\{(0,0)\} \star \langle n \rangle^\circ$ is a p-limit diagram. To complete the proof, it will suffice (by virtue of Lemma T.4.3.2.7) to show that $\overline{\tau}|A^0$ is a p-right Kan extension of $\overline{\tau}|\langle n \rangle^\circ$. This again follows immediately from our assumption that $\mathcal{O}^\otimes$ is a $\mathcal{C}$-family of ∞-operads.

We now prove (iv). Let $\overline{\sigma}_0 : K \to \mathcal{O}^\otimes$ be as in (iv); we may again assume without loss of generality that this diagram factors through $\mathcal{O}_C^\otimes$ for some $C \in \mathcal{C}$. Using Lemma T.4.3.2.13, we can choose a p-left Kan extension $\overline{\tau}_0 : A^0 \to \mathcal{O}_C^\otimes$ of $\overline{\sigma}_0$ covering the map $\tau_0 = \tau|A^0$. Using the assumption that $\mathcal{O}_C^\otimes$ is an ∞-operads, we deduce that $\overline{\tau}_0$ is a p-right Kan extension of $\overline{\tau}_0|\langle n \rangle^\circ$, and that $\overline{\tau}_0|\langle n \rangle^\circ$ can be extended to a p-limit diagram $\widetilde{\tau}_0 : \{(0,0)\} \star \langle n \rangle^\circ \to \mathcal{O}_C^\otimes$ lifting $\tau|(\{(0,0)\} \star \langle n \rangle^\circ)$; moreover, any such diagram carries each edge of $\{(0,0)\} \star \langle n \rangle^\circ$ to an inert morphism in $\mathcal{O}_C^\otimes$. Invoking Lemma T.4.3.2.7, we can amalgamate $\widetilde{\tau}_0$ and $\overline{\tau}_0$ to obtain a diagram $\overline{\tau} : A \to \mathcal{O}_C^\otimes$ covering τ. We claim that $\overline{\sigma} = \overline{\tau}|\Delta^1 \times \Delta^1$ is the desired extension of $\overline{\sigma}_0$. To

prove this, it suffices to show that $\overline{\tau}$ carries each morphism of $\Delta^1 \times \Delta^1$ to an inert morphism of $\mathcal{O}^{\otimes}$. Since the composition of inert morphisms in $\mathcal{O}^{\otimes}$ is inert, it will suffice to show that the maps

$$\overline{\tau}(0,1) \xleftarrow{\overline{\beta}} \overline{\tau}(0,0) \xrightarrow{\overline{\alpha}} \overline{\tau}(1,0)$$

are inert, where $\overline{\alpha}$ and $\overline{\beta}$ are the morphisms lying over α and β determined by τ. We will prove that $\overline{\alpha}$ is inert; the case of $\overline{\beta}$ follows by the same argument. We can factor $\overline{\alpha}$ as a composition

$$\overline{\tau}(0,0) \xrightarrow{\overline{\alpha}'} \alpha_! \overline{\tau}(0,0) \xrightarrow{\overline{\alpha}''} \overline{\tau}(1,0).$$

We wish to prove that $\overline{\alpha}''$ is an equivalence in the ∞-category $\mathcal{O}^{\otimes}_{(C,\langle m \rangle)}$. Since $\mathcal{O}^{\otimes}_C$ is an ∞-operad, it will suffice to show that for $1 \leq j \leq m$, the functor $\rho^j_! : \mathcal{O}^{\otimes}_{(C,\langle m \rangle)} \to \mathcal{O}_C$ carries $\overline{\alpha}''$ to an equivalence in $\mathcal{O}_C$. Unwinding the definitions, this is equivalent to the requirement that the map $\overline{\tau}(0,0) \to \overline{\tau}(i)$ is inert, where $i = \alpha^{-1}(j) \in \langle n \rangle^{\circ}$, which follows immediately from our construction. $\qquad\square$

2.3.3   Approximations to ∞-Operads

In §2.3.2, we introduced the notion of a generalized ∞-operad. We can regard the ∞-category Op_{∞} of ∞-operads as a full subcategory of the ∞-category $\mathrm{Op}^{\mathrm{gn}}_{\infty}$ of generalized ∞-operads. We now observe that the inclusion $\mathrm{Op}_{\infty} \hookrightarrow \mathrm{Op}^{\mathrm{gn}}_{\infty}$ admits a left adjoint.

Definition 2.3.3.1. Let $\mathcal{O}^{\otimes}$ be a generalized ∞-operad and $\mathcal{O}'^{\otimes}$ an ∞-operad. We will say that a map $\gamma : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ of generalized ∞-operads *assembles* $\mathcal{O}^{\otimes}$ *to* $\mathcal{O}'^{\otimes}$ if, for every ∞-operad $\mathcal{O}''^{\otimes}$, composition with γ induces an equivalence of ∞-categories $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{O}'') \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{O}'')$. In this case we will also say that $\mathcal{O}'^{\otimes}$ is an *assembly* of $\mathcal{O}^{\otimes}$, or that γ *exhibits* $\mathcal{O}'^{\otimes}$ *as an assembly of* $\mathcal{O}^{\otimes}$.

Remark 2.3.3.2. A map of generalized ∞-operads $\gamma : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ assembles $\mathcal{O}^{\otimes}$ to $\mathcal{O}'^{\otimes}$ if and only if it exhibits $\mathcal{O}'^{\otimes}$ as an Op_{∞}-localization of $\mathcal{O}^{\otimes} \in \mathrm{Op}^{\mathrm{gn}}_{\infty}$. In other words, γ assembles $\mathcal{O}^{\otimes}$ to $\mathcal{O}'^{\otimes}$ if and only if $\mathcal{O}'^{\otimes}$ is an ∞-operad, and for every ∞-operad $\mathcal{O}''^{\otimes}$, composition with γ induces a homotopy equivalence $\theta(\mathcal{O}''^{\otimes}) : \mathrm{Map}_{\mathrm{Op}^{\mathrm{gn}}_{\infty}}(\mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes}) \to \mathrm{Map}_{\mathrm{Op}^{\mathrm{gn}}_{\infty}}(\mathcal{O}^{\otimes}, \mathcal{O}''^{\otimes})$. The "only if" direction is clear, since the mapping spaces $\mathrm{Map}_{\mathrm{Op}^{\mathrm{gn}}_{\infty}}(\mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes})$ and $\mathrm{Map}_{\mathrm{Op}^{\mathrm{gn}}_{\infty}}(\mathcal{O}^{\otimes}, \mathcal{O}''^{\otimes})$ can be identified with the Kan complexes $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{O}'')^{\simeq}$ and $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}'')^{\simeq}$. Conversely, suppose that $\theta(\mathcal{O}''^{\otimes})$ is a homotopy equivalence for every ∞-operad $\mathcal{O}''^{\otimes}$; we wish to show that each of the maps $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{O}'') \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{O}'')$ is a categorical equivalence. It suffices to show that for every simplicial set K, the map $\mathrm{Fun}(K, \mathrm{Alg}_{\mathcal{O}'}(\mathcal{O}'')) \to \mathrm{Fun}(K, \mathrm{Alg}_{\mathcal{O}}(\mathcal{O}''))$ induces a homotopy equivalence on the underlying Kan complexes; this map can be identified with $\theta(\mathcal{C}^{\otimes})$, where $\mathcal{C}^{\otimes}$ is the ∞-operad $\mathrm{Fun}(K, \mathcal{O}''^{\otimes}) \times_{\mathrm{Fun}(K, \mathrm{N}(\mathcal{F}\mathrm{in}_*))} \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

Remark 2.3.3.3. Since the ∞-category Op_{∞} is a localization of the ∞-category $\mathrm{Op}^{\mathrm{gn}}_{\infty}$ (Corollary 2.3.2.6), we conclude that for every generalized ∞-operad $\mathcal{O}^{\otimes}$ there exists an assembly map $\gamma : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$, which is uniquely determined up to equivalence. The process of assembly determines a functor $\mathrm{Assem} : \mathrm{Op}^{\mathrm{gn}}_{\infty} \to \mathrm{Op}_{\infty}$, which is left adjoint to the inclusion $\mathrm{Op}_{\infty} \subseteq \mathrm{Op}^{\mathrm{gn}}_{\infty}$. This functor can be described concretely as follows: for every every generalized ∞-operad $\mathcal{O}^{\otimes}$, we can identify $\mathrm{Assem}(\mathcal{O}^{\otimes})$ with an ∞-operad $\mathcal{O}'^{\otimes}$, where $\mathcal{O}'^{\otimes,\natural}$ as a fibrant replacement for the ∞-preoperad $\mathcal{O}^{\otimes,\natural}$ with respect to the ∞-operadic model structure on $\mathcal{POp}_{\infty}$.

Remark 2.3.3.4. In the situation of Definition 2.3.3.1, suppose that $\mathcal{O}^{\otimes} \to \mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a $\mathcal{C}$-family of ∞-operads. We can think of an object of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}'')$ as a family of ∞-operad maps $\mathcal{O}^{\otimes}_C \to \mathcal{O}'^{\otimes}$ parametrized by the objects $C \in \mathcal{C}$. The map γ assembles $\mathcal{O}^{\otimes}$ if this is equivalent to the data of a single ∞-operad map $\mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$. In this case, we can view $\mathcal{O}'^{\otimes}$ as a sort of colimit of the family of ∞-operads $\{\mathcal{O}^{\otimes}_C\}_{C \in \mathcal{C}}$. This description is literally correct in the case where $\mathcal{C}$ is a Kan complex.

Our goal in §2.3.4 is to analyze the relationship between a generalized ∞-operad $\mathcal{O}^{\otimes}$ and its assembly $\mathrm{Assem}(\mathcal{O}^{\otimes})$ (under some mild hypotheses in $\mathcal{O}$). To carry out this analysis, we need a criterion for detecting

weak equivalences in the model category $\mathcal{POp}_\infty$ of ∞-preoperads. Our goal in this section is to establish such a criterion, whose formulation involves the notion of a (weak) *approximation* to an ∞-operad, which will have many other applications in later chapters of this book.

Warning 2.3.3.5. The remaining material in this section is of a rather technical nature. We recommend that the casual reader skip it for the time being, referring back to it as necessary.

Definition 2.3.3.6. Let $p : \mathcal{O}^\otimes \to \mathrm{N}(\mathcal{F}in_*)$ be an ∞-operad. We will say that a categorical fibration $f : \mathcal{C} \to \mathcal{O}^\otimes$ is an *approximation to* $\mathcal{O}^\otimes$ if it satisfies the following conditions:

(1) Let $p' = p \circ f$, let $C \in \mathcal{C}$ be an object, and let $\langle n \rangle = p'(C)$. For $1 \leq i \leq n$, there exists a locally p'-coCartesian morphism $\alpha_i : C \to C_i$ in $\mathcal{C}$ covering the map $\rho^i : \langle n \rangle \to \langle 1 \rangle$. Moreover, $f(\alpha_i)$ is an inert morphism in $\mathcal{O}^\otimes$.

(2) Let $C \in \mathcal{C}$ and let $\alpha : X \to f(C)$ be an active morphism in $\mathcal{O}^\otimes$. Then there exists an f-Cartesian morphism $\overline{\alpha} : \overline{X} \to C$ lifting α.

We will say that a categorical fibration $f : \mathcal{C} \to \mathcal{O}^\otimes$ is a *weak approximation* to $\mathcal{O}^\otimes$ if it satisfies condition (1) together with the following:

(2′) Let $C \in \mathcal{C}$ and let $\alpha : X \to f(C)$ be an arbitrary morphism in $\mathcal{O}^\otimes$. Let $\mathcal{E} \subseteq \mathcal{C}_{/C} \times_{\mathcal{O}^\otimes_{/f(C)}} \mathcal{O}^\otimes_{X//f(C)}$ be the full subcategory spanned by those objects corresponding to pairs $(\beta : C' \to C, \gamma : X \to f(C'))$ such that γ is inert. Then the ∞-category $\mathcal{E}$ is weakly contractible.

If $f : \mathcal{C} \to \mathcal{O}^\otimes$ is an arbitrary map of ∞-categories, we will say that f is a (weak) approximation if it factors as a composition $\mathcal{C} \xrightarrow{f'} \mathcal{C}' \xrightarrow{f''} \mathcal{O}^\otimes$, where f' is a categorical equivalence and f'' is a categorical fibration which is a (weak) approximation to $\mathcal{O}^\otimes$.

Remark 2.3.3.7. In the situation of Definition 2.3.3.6, the condition that $f : \mathcal{C} \to \mathcal{O}^\otimes$ is a (weak) approximation does not depend on the choice of factorization $\mathcal{C} \xrightarrow{f'} \mathcal{C}' \xrightarrow{f''} \mathcal{O}^\otimes$, provided that f' is a categorical equivalence and f'' is a categorical fibration.

Remark 2.3.3.8. Let $\mathcal{O}^\otimes$ be an ∞-operad and let $f : \mathcal{C} \to \mathcal{O}^\otimes$ be a categorical fibration which is an approximation to $\mathcal{O}^\otimes$. We will say that a morphism α in $\mathcal{C}$ is f-*active* if α is f-Cartesian and $f(\alpha)$ is an active morphism in $\mathcal{O}^\otimes$. We will say that α is f-*inert* if $f(\alpha)$ is an inert morphism in $\mathcal{O}^\otimes$. It follows from Proposition 2.1.2.5 that $\mathcal{C}$ admits a factorization system (S_L, S_R), where S_L is the collection of f-inert morphisms in $\mathcal{C}$ and S_R is the collection of f-active morphisms in $\mathcal{C}$.

Remark 2.3.3.9. Let $\mathcal{O}^\otimes$ be an ∞-operad and let $f : \mathcal{C} \to \mathcal{O}^\otimes$ be an approximation to $\mathcal{O}^\otimes$. If $u : \mathcal{O}'^\otimes \to \mathcal{O}^\otimes$ is a fibration of ∞-operads, then the induced map $\mathcal{C} \times_{\mathcal{O}^\otimes} \mathcal{O}'^\otimes \to \mathcal{O}'^\otimes$ is an approximation to $\mathcal{O}'^\otimes$. Indeed, the assumption that u is a fibration of ∞-operads guarantees that the fiber product $\mathcal{C} \times_{\mathcal{O}^\otimes} \mathcal{O}'^\otimes$ is also a homotopy fiber product. We may therefore replace $\mathcal{C}$ by an equivalent ∞-category and thereby reduce to the case where f is a categorical fibration, in which case the result follows readily from Definition 2.3.3.6.

Lemma 2.3.3.10. *Let $\mathcal{O}^\otimes$ be an ∞-operad and let $f : \mathcal{C} \to \mathcal{O}^\otimes$ be an approximation to $\mathcal{O}^\otimes$. Then f is a weak approximation to $\mathcal{O}^\otimes$.*

Proof. Fix an object $C \in \mathcal{C}$ and a morphism $\alpha : X \to f(C)$, and let $\mathcal{C}' \subseteq \mathcal{C}_{/C} \times_{\mathcal{O}^\otimes_{/f(C)}} \mathcal{O}^\otimes_{X//f(C)}$ be as in Definition 2.3.3.6. The map α fits into a commutative diagram σ :

$$
\begin{array}{ccc}
& X' & \\
\alpha' \nearrow & & \searrow \alpha'' \\
X \xrightarrow{\quad \alpha \quad} & & f(C),
\end{array}
$$

where α' is inert and α'' is active. Since f is an approximation to $\mathcal{O}^\otimes$, we can lift α'' to an f-Cartesian morphism $\overline{\alpha}'' : C' \to C$. The pair $(\overline{\alpha}'', \sigma)$ is a final object of $\mathcal{C}'$, so that $\mathcal{C}'$ is weakly contractible. $\qquad\square$

We will be primarily interested in the case where $f : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a map of generalized ∞-operads. In this case, the condition that f be a weak approximation is much easier to formulate:

Proposition 2.3.3.11. *Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad, and let $f : \mathcal{C} \to \mathcal{O}^{\otimes}$ be a categorical fibration. Assume that $\mathcal{C}$ satisfies the following condition:*

($*$) *For every object $C \in \mathcal{C}$ and every inert morphism $\beta : (p \circ f)(C) \to \langle n \rangle$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, there exists a $(p \circ f)$-coCartesian morphism $\overline{\beta} : C \to C'$ in $\mathcal{C}$ lifting β, and the image $f(\overline{\beta})$ is an inert morphism in $\mathcal{O}^{\otimes}$.*

Then f is a weak approximation if and only if the following condition is satisfied:

($*'$) *For every object $C \in \mathcal{C}$ and every active morphism $\alpha : X \to f(C)$ in $\mathcal{O}^{\otimes}$, the ∞-category*

$$\mathcal{C}_{/C} \times_{\mathcal{O}^{\otimes}_{/f(C)}} \{X\}$$

is weakly contractible.

Remark 2.3.3.12. Condition ($*$) of Proposition 2.3.3.11 is automatically satisfied if f is a fibration of generalized ∞-operads.

Proof. It is obvious that condition (1) of Definition 2.3.3.6 satisfies ($*$). It will therefore suffice to show that if ($*$) is satisfied, then condition ($2'$) of Definition 2.3.3.6 is equivalent to ($*'$). We first show that ($2'$) $\Rightarrow$ ($*'$). Consider an arbitrary morphism $\alpha : X \to f(C)$ in $\mathcal{O}^{\otimes}$ and let $\mathcal{E}$ be defined as in ($2'$). Let $\mathcal{E}_0$ be the full subcategory of $\mathcal{E}$ spanned by those objects which correspond to factorizations

$$X \overset{\beta}{\to} f(D) \overset{f(\gamma)}{\to} f(C)$$

of α, where β is an inert morphism in $\mathcal{O}^{\otimes}$ and $f(\gamma)$ is an active morphism in $\mathcal{O}^{\otimes}$. Using ($*$), we conclude that the inclusion $\mathcal{E}_0 \subseteq \mathcal{E}$ admits a left adjoint and is therefore a weak homotopy equivalence. Let $\mathcal{X}$ be the full subcategory of $\mathcal{O}^{\otimes}_{X//f(C)}$ spanned by those diagrams

$$X \overset{\alpha'}{\to} X' \overset{\alpha''}{\to} f(C)$$

such that α" is inert and α'' is active, so that $\mathcal{X}$ is a contractible Kan complex. Then $\mathcal{E}_0$ can be identified with the fiber product $\mathcal{C}^{\otimes}_{/C} \times_{\mathcal{O}^{\otimes}_{/f(C)}} \mathcal{X}$, and is therefore categorically equivalent to the fiber $\mathcal{C}^{\otimes}_{/C} \times_{\mathcal{O}^{\otimes}_{/f(C)}} \{X'\}$ for any object $X' \in \mathcal{E}$. If ($*'$) is satisfied, then this fiber product is weakly contractible and ($2'$) follows. Conversely, assume that ($2'$) is satisfied. If $\alpha : X \to f(C)$ is active, then we take $X' = X$ to conclude that $\mathcal{C}^{\otimes}_{/C} \times_{\mathcal{O}^{\otimes}_{/f(C)}} \{X\} \simeq \mathcal{E}_0$ is weakly contractible, which proves ($*'$). $\square$

To state our next result, we need a bit of notation. For each integer $n \geq 0$, we let Tup_n denote the subcategory of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle n \rangle}$ whose objects are active morphisms $\langle m \rangle \to \langle n \rangle$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, and whose morphisms are commutative diagrams

$$
\begin{array}{ccc}
\langle m \rangle & \overset{\alpha}{\longrightarrow} & \langle m \rangle \\
 & \searrow \quad \swarrow & \\
 & \langle n \rangle &
\end{array}
$$

where α is a bijection of pointed finite sets. The ∞-category Tup_n is equivalent to the nerve of the groupoid of n-tuples of finite sets.

Lemma 2.3.3.13. *Let $\mathfrak{X}$ be a Kan complex and let $\theta : \mathfrak{Y} \to \mathfrak{X}$ be a categorical fibration. Then, for every vertex $X \in \mathfrak{X}$, the pullback diagram*

$$
\begin{array}{ccc}
\mathfrak{Y}_X & \longrightarrow & \mathfrak{Y} \\
\downarrow & & \downarrow{\scriptstyle \theta} \\
\{X\} & \longrightarrow & \mathfrak{X}
\end{array}
$$

is a homotopy pullback diagram (with respect to the usual model structure on Set_Δ).

Proof. Choose a factorization of the inclusion $i : \{X\} \hookrightarrow \mathfrak{X}$ as a composition

$$
\{X\} \xrightarrow{i'} \overline{\mathfrak{X}} \xrightarrow{i''} \mathfrak{X},
$$

where i'' is a Kan fibration and $\overline{\mathfrak{X}}$ is a contractible Kan complex, so we have a commutative diagram

$$
\begin{array}{ccccc}
\mathfrak{Y}_X & \xrightarrow{\ j'\ } & \mathfrak{Y} \times_{\mathfrak{X}} \overline{\mathfrak{X}} & \longrightarrow & \mathfrak{Y} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \theta} \\
\{X\} & \xrightarrow{\ i'\ } & \overline{\mathfrak{X}} & \xrightarrow{\ i''\ } & \mathfrak{X}.
\end{array}
$$

Since i'' is a Kan fibration, the right square is a homotopy pullback diagram with respect to the usual model structure (since the usual model structure is right-proper). To prove that the outer square is a homotopy pullback diagram, it will suffice to show that i' and j' are weak homotopy equivalences. We will complete the proof by showing that j' is a categorical equivalence. Since θ is a categorical fibration and the simplicial sets $\{X\}$, $\overline{\mathfrak{X}}$, and $\mathfrak{X}$ are ∞-categories, the left square is a homotopy pullback diagram with respect to the Joyal model structure. It will therefore suffice to show that i' is a categorical equivalence, which is obvious. $\square$

Proposition 2.3.3.14. *Let $p : \mathcal{O}^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad and let $f : \mathcal{C} \to \mathcal{O}^\otimes$ be a functor. Assume that f satisfies condition $(*)$ of Proposition 2.3.3.11 and that the ∞-category $\mathcal{O}$ is a Kan complex. Then f is a weak approximation to $\mathcal{O}^\otimes$ if and only if the following condition is satisfied:*

$(*'')$ Let $C \in \mathcal{C}$ and let $\langle n \rangle = (p \circ f)(C) \in \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Then f induces a weak homotopy equivalence

$$
\theta : \mathcal{C}_{/C} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle n \rangle}} \mathrm{Tup}_n \to \mathcal{O}^\otimes_{/f(C)} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle n \rangle}} \mathrm{Tup}_n \, .
$$

Remark 2.3.3.15. In the situation of Corollary 2.3.3.14, the assumption that $\mathcal{O}$ is a Kan complex guarantees that the ∞-category $\mathcal{O}^\otimes_{/f(C)} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle n \rangle}} \mathrm{Tup}_n$ is a Kan complex. However, in many applications, the ∞-category $\mathcal{C}_{/C} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle n \rangle}} \mathrm{Tup}_n$ will not be a Kan complex.

Proof. We may assume without loss of generality that f is a categorical fibration, so that θ is also a categorical fibration. The map θ is a homotopy equivalence if and only if each of its homotopy fibers is weakly contractible. Since $\mathcal{O}^\otimes_{/f(C)} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle n \rangle}} \mathrm{Tup}_n$ is a Kan complex (Remark 2.3.3.15), we see that θ is a homotopy equivalence if and only if each fiber of θ is weakly contractible (Lemma 2.3.3.13). According to Proposition 2.3.3.11, this is equivalent to the requirement that f be a weak approximation to $\mathcal{O}^\otimes$. $\square$

Corollary 2.3.3.16. *Let $\mathcal{O}^\otimes$ be an ∞-operad and let $f : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a map of generalized ∞-operads. Assume that the ∞-category $\mathcal{O}$ is a Kan complex. Then f is a weak approximation if and only if, for every object $C \in \mathcal{C}^\otimes_{\langle n \rangle}$ and every active morphism $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$, the induced map*

$$
\theta_{C,\alpha} : \mathcal{C}^\otimes_{/C} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle n \rangle}} \{\langle m \rangle\} \to \mathcal{O}^\otimes_{/f(C)} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle n \rangle}} \{\langle m \rangle\}
$$

is a weak homotopy equivalence of simplicial sets.

Proof. For each $C \in \mathcal{C}^{\otimes}_{\langle n \rangle}$, we have a commutative diagram of simplicial sets

$$\mathcal{C}^{\otimes}_{/C} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle n \rangle}} \mathrm{Tup}_n \xrightarrow{\theta_C} \mathcal{O}^{\otimes}_{/f(C)} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle n \rangle}} \mathrm{Tup}_n$$

$$\mathrm{Tup}_n \, .$$

According to Proposition 2.3.3.14, the map f is an approximation to $\mathcal{O}^{\otimes}$ if and only if each of the maps θ_C is a weak homotopy equivalence. This is equivalent to the requirement that θ_C induces a weak homotopy equivalence after taking the homotopy fibers over any vertex of Tup_n, corresponding to an active morphism $\alpha : \langle m \rangle \to \langle n \rangle$. Using Lemma 2.3.3.13, we can identify the relevant map of homotopy fibers with $\theta_{C,\alpha}$. $\square$

For our next statement, we use the following notational convention: if $\mathcal{O}^{\otimes}$ is an ∞-operad, we let $\mathcal{O}^{\otimes}_{\mathrm{act}}$ denote the subcategory of $\mathcal{O}^{\otimes}$ spanned by the active morphisms.

Corollary 2.3.3.17. *Let* $f : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ *be a map of ∞-operads. Assume that $\mathcal{C}$ and $\mathcal{O}$ are Kan complexes. The following conditions are equivalent:*

(a) *The map f is an approximation.*

(b) *The map f is a weak approximation.*

(c) *The map $f_{\mathrm{act}} : \mathcal{C}^{\otimes}_{\mathrm{act}} \to \mathcal{O}^{\otimes}_{\mathrm{act}}$ is the composition of a categorical equivalence with a right fibration.*

Proof. We may assume without loss of generality that f is a fibration of ∞-operads. The implication $(a) \Rightarrow (b)$ follows from Lemma 2.3.3.10. We now show that $(b) \Rightarrow (c)$. Assume that f is a weak approximation and choose an active morphism $\alpha : Y \to Z$ in $\mathcal{C}^{\otimes}$; we wish to show that α is f_{act}-Cartesian. Unwinding the definitions, it will suffice to show that for every active morphism $\beta : X \to Z$ in $\mathcal{C}^{\otimes}$, the induced map

$$\theta : \mathrm{Map}_{\mathcal{C}^{\otimes}_{/Z}}(X, Y) \to \mathrm{Map}_{\mathcal{O}^{\otimes}_{/fZ}}(fX, fY)$$

is a homotopy equivalence. Let $\alpha_0 : \langle m \rangle \to \langle n \rangle$ and $\beta_0 : \langle k \rangle \to \langle n \rangle$ be the images of α and β in the ∞-category $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Then θ is given by a disjoint union of maps

$$\theta^\gamma : \mathrm{Map}^\gamma_{\mathcal{C}^{\otimes}_{/Z}}(X, Y) \to \mathrm{Map}^\gamma_{\mathcal{O}^{\otimes}_{/fZ}}(fX, fY),$$

where γ ranges over those maps $\langle k \rangle \to \langle m \rangle$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ such that $\beta = \alpha \circ \gamma$ and the superscripts indicate the relevant summand of the mapping spaces. The map θ^γ is given by taking vertical homotopy fibers of the diagram

$$\mathcal{C}^{\otimes}_{/Y} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle m \rangle}} \{\langle k \rangle\} \longrightarrow \mathcal{O}^{\otimes}_{/Y} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle m \rangle}} \{\langle k \rangle\}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\mathcal{C}^{\otimes}_{/Z} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle n \rangle}} \{\langle k \rangle\} \longrightarrow \mathcal{C}^{\otimes}_{/Z} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle m \rangle}} \{\langle k \rangle\}$$

(the hypothesis that $\mathcal{O}$ and $\mathcal{C}$ are Kan complexes guarantee that the entries in this diagram are Kan complexes). It therefore suffices to show that the horizontal maps in the above diagram are homotopy equivalences, which follows from our assumption that f is a weak approximation (Corollary 2.3.3.16).

We now complete the proof by showing that $(c) \Rightarrow (a)$. For any object $Z \in \mathcal{C}$ and any active morphism $\alpha_0 : Y_0 \to Z$, there is an essentially unique morphism $\alpha : Y \to Z$ in $\mathcal{C}^{\otimes}$ lifting α_0. We wish to show that α is f-Cartesian. Unwinding the definitions, we must show that for any morphism $\beta : X \to Z$ in $\mathcal{C}^{\otimes}$, the map f induces a homotopy $\mathrm{Map}_{\mathcal{C}^{\otimes}_{/Z}}(X, Y) \to \mathrm{Map}_{\mathcal{O}^{\otimes}_{/fZ}}(fX, fY)$. The map β factors as a composition

$$X \xrightarrow{\beta'} X' \xrightarrow{\beta''} Z$$

where β' is inert and β'' is active. Since f is a map of ∞-operads, $f(\beta')$ is inert and $f(\beta'')$ is active; we may therefore replace β by β'' and thereby reduce to the case where β is active. The desired result now follows immediately from assumption (c). $\qquad\square$

Corollary 2.3.3.18. *Suppose we are given maps of ∞-operads $\mathcal{O}^{\otimes} \xrightarrow{f} \mathcal{O}'^{\otimes} \xrightarrow{g} \mathcal{O}''^{\otimes}$, where $\mathcal{O}$, $\mathcal{O}'$, and $\mathcal{O}''$ are Kan complexes and g is an approximation. Then f is an approximation if and only if $g \circ f$ is an approximation.*

Remark 2.3.3.19. Let $\mathcal{O}^{\otimes}$ be an ∞-operad and let $f : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a map of generalized ∞-operads. Assume that $\mathcal{C}^{\otimes}_{\langle 0 \rangle}$ is a Kan complex. Then f is an approximation if and only if, for each object $C \in \mathcal{C}^{\otimes}_{\langle 0 \rangle}$, the induced map of ∞-operads $\mathcal{C}^{\otimes}_{/C} \to \mathcal{O}^{\otimes}$ is an approximation.

Definition 2.3.3.20. Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and $q : \mathcal{O}'^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be ∞-operads, and let $f : \mathcal{C} \to \mathcal{O}^{\otimes}$ be a weak approximation to $\mathcal{O}^{\otimes}$. Let $p' = p \circ f$. We will say that a functor $A : \mathcal{C} \to \mathcal{O}'^{\otimes}$ is a $\mathcal{C}$-*algebra object of* $\mathcal{O}'^{\otimes}$ if it satisfies the following conditions:

 (a) The diagram of simplicial sets

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\;\;A\;\;} & \mathcal{O}'^{\otimes} \\
\big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle q} \\
\mathcal{O}^{\otimes} & \xrightarrow{\;\;p\;\;} & \mathrm{N}(\mathcal{F}\mathrm{in}_*)
\end{array}
$$

 is commutative.

 (b) Let $C \in \mathcal{C}$ be such that $p'(C) = \langle n \rangle$, and for $1 \leq i \leq n$ choose a locally p'-coCartesian morphism $\alpha_i : C \to C_i$ in $\mathcal{C}$ covering the map $\rho^i : \langle n \rangle \to \langle 1 \rangle$. Then $A(\alpha_i)$ is an inert morphism in $\mathcal{O}'^{\otimes}$.

We will say that a $\mathcal{C}$-algebra object A of $\mathcal{O}'^{\otimes}$ is *locally constant* if it satisfies the following further condition:

 (c) For every morphism α in $\mathcal{C}$ such that $p'(\alpha) = \mathrm{id}_{\langle 1 \rangle}$, the image $A(\alpha)$ is an equivalence in $\mathcal{O}'$.

We let $\mathrm{Alg}_{\mathcal{C}}(\mathcal{O}')$ denote the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{C}, \mathcal{O}'^{\otimes})$ spanned by the $\mathcal{C}$-algebra objects of $\mathcal{O}'^{\otimes}$, and $\mathrm{Alg}^{\mathrm{loc}}_{\mathcal{C}}(\mathcal{O}')$ the full subcategory of $\mathrm{Alg}_{\mathcal{C}}(\mathcal{O}')$ spanned by the locally constant $\mathcal{C}$-algebra objects of $\mathcal{O}'^{\otimes}$.

 If $f : \mathcal{C} \to \mathcal{O}^{\otimes}$ is an essential weak approximation, so that f factors as a composition $\mathcal{C} \xrightarrow{f'} \mathcal{C}' \xrightarrow{f''} \mathcal{O}'^{\otimes}$ where f' is a categorical equivalence and f'' is a weak approximation, then we let $\mathrm{Alg}_{\mathcal{C}}(\mathcal{O}')$ denote the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathbf{\Delta})^{op}}(\mathcal{C}, \mathcal{O}'^{\otimes})$ given by the essential image of $\mathrm{Alg}_{\mathcal{C}'}(\mathcal{O}')$ under the equivalence of ∞-categories $\mathrm{Fun}_{\mathrm{N}(\mathbf{\Delta})^{op}}(\mathcal{C}', \mathcal{O}'^{\otimes}) \to \mathrm{Fun}_{\mathrm{N}(\mathbf{\Delta})^{op}}(\mathcal{C}, \mathcal{O}'^{\otimes})$ given by f', and $\mathrm{Alg}^{\mathrm{loc}}_{\mathcal{C}}(\mathcal{O}')$ the essential image of $\mathrm{Alg}'^{\mathrm{loc}}_{\mathcal{C}}(\mathcal{O}')$.

Example 2.3.3.21. If $\mathcal{O}^{\otimes}$ is an ∞-operad and $f : \mathcal{C} \to \mathcal{O}^{\otimes}$ is an essential weak approximation to $\mathcal{O}^{\otimes}$, then f is a $\mathcal{C}$-algebra object of $\mathcal{O}^{\otimes}$.

Example 2.3.3.22. Let $\mathcal{O}^{\otimes}$ be an ∞-operad, and suppose we are given a map of generalized ∞-operads $f : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$. Assume that f is an essential weak approximation to $\mathcal{O}^{\otimes}$. For any other ∞-operad $\mathcal{O}'^{\otimes}$, a map $A : \mathcal{C}^{\otimes} \to \mathcal{O}'^{\otimes}$ is a $\mathcal{C}^{\otimes}$-algebra object of $\mathcal{O}'^{\otimes}$ (in the sense of Definition 2.3.3.20) if and only if it is a map of generalized ∞-operads (Definition 2.3.2.2); this follows from the argument of Remark 2.1.2.9.

 Our main result in this section is the following:

Theorem 2.3.3.23. *Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and $q : \mathcal{O}'^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be ∞-operads, and let $f : \mathcal{C} \to \mathcal{O}^{\otimes}$ be a weak approximation to $\mathcal{O}^{\otimes}$. Let $\theta : \mathrm{Alg}_{\mathcal{O}}(\mathcal{O}') \to \mathrm{Alg}_{\mathcal{C}}(\mathcal{O}')$ be the map given by composition with f, and let $\mathcal{C}_{\langle 1 \rangle}$ denote the fiber $\mathcal{C} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \{\langle 1 \rangle\}$.*
 Then:

(1) *If f induces an equivalence of ∞-categories $\mathcal{C}_{\langle 1 \rangle} \to \mathcal{O}$, then θ is an equivalence of ∞-categories.*

(2) *If $\mathcal{O}$ is a Kan complex and f induces a weak homotopy equivalence $\mathcal{C}_{\langle 1 \rangle} \to \mathcal{O}$, then θ induces an equivalence of ∞-categories $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}') \to \mathrm{Alg}_{\mathcal{C}}^{\mathrm{loc}}(\mathcal{O}')$.*

Proof. Replacing $\mathcal{C}$ by an equivalent ∞-category if necessary, we may assume that f is a categorical approximation. Choose a Cartesian fibration $u : \mathcal{M} \to \Delta^1$ associated to the functor f, so that we have isomorphisms $\mathcal{O}^{\otimes} \simeq \mathcal{M} \times_{\Delta^1} \{0\}$, $\mathcal{C} \simeq \mathcal{M} \times_{\Delta^1} \{1\}$, and choose a retraction r from $\mathcal{M}$ onto $\mathcal{O}^{\otimes}$ such that $r | \mathcal{C} = f$. Let $\mathfrak{X}$ denote the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{M}, \mathcal{O}'^{\otimes})$ spanned by those functors $F : \mathcal{M} \to \mathcal{O}'^{\otimes}$ satisfying the following conditions:

(*i*) The restriction $F | \mathcal{O}^{\otimes}$ belongs to $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$.

(*ii*) For every u-Cartesian morphism α in $\mathcal{M}$, the image $F(\alpha)$ is an equivalence in $\mathcal{O}'^{\otimes}$.

Condition (*ii*) is equivalent to the requirement that F be a q-left Kan extension of $F | \mathcal{O}^{\otimes}$. Using Proposition T.4.3.2.15, we conclude that the restriction functor $\mathfrak{X} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$ is a trivial Kan fibration. Composition with r determines a section s of this trivial Kan fibration. Let $\psi : \mathfrak{X} \to \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{C}, \mathcal{O}'^{\otimes})$ be the other restriction functor. Then θ is given by the composition $\psi \circ s$. It will therefore suffice to show that ψ determines an equivalence from $\mathfrak{X}$ onto $\mathrm{Alg}_{\mathcal{C}}(\mathcal{O}')$ (in case (1)) or $\mathrm{Alg}_{\mathcal{C}}^{\mathrm{loc}}(\mathcal{O}')$ (in case (2)). In view of Proposition T.4.3.2.15, it will suffice to verify the following:

(*a*) Let $F_0 \in \mathrm{Alg}_{\mathcal{C}}(\mathcal{O}')$, and assume that F_0 is locally constant if we are in case (2). Then there exists a functor $F \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{M}, \mathcal{O}'^{\otimes})$ which is a q-right Kan extension of F_0.

(*b*) A functor $F \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{M}, \mathcal{O}'^{\otimes})$ belongs to $\mathfrak{X}$ if and only if F is a q-right Kan extension of $F_0 = F | \mathcal{C}$, and $F_0 \in \mathrm{Alg}_{\mathcal{C}}(\mathcal{O}')$ (in case (1)) or $F_0 \in \mathrm{Alg}_{\mathcal{C}}^{\mathrm{loc}}(\mathcal{O}')$ (in case (2)).

We begin by proving (*a*). Fix an object $X \in \mathcal{O}^{\otimes}$, let $\mathcal{C}_{X/}$ denote the fiber product $\mathcal{M}_{X/} \times_{\mathcal{M}} \mathcal{C}$, and let $F_X = F_0 | \mathcal{C}_{X/}$. According to Lemma T.4.3.2.13, it will suffice to show that the functor F_X can be extended to a q-limit diagram $\mathcal{C}_{X/}^{\triangleleft} \to \mathcal{O}'^{\otimes}$ (covering the map $\mathcal{C}_{X/}^{\triangleleft} \to \mathcal{M}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$). Let $\mathcal{C}'_{X/}$ denote the full subcategory of $\mathcal{C}_{X/}$ spanned by those morphisms $X \to C$ in $\mathcal{M}$ which correspond to inert morphisms $X \to f(C)$ in $\mathcal{O}^{\otimes}$. Since f is a weak approximation to $\mathcal{O}^{\otimes}$, Theorem T.4.1.3.1 implies that the inclusion $\mathcal{C}'_{X/} \hookrightarrow \mathcal{C}_{X/}$ is right cofinal. It will therefore suffice to show that the restriction $F'_X = F_X | \mathcal{C}'_{X/}$ can be extended to a q-limit diagram $\mathcal{C}'^{\triangleleft}_{X/} \to \mathcal{O}'^{\otimes}$.

Let $\langle n \rangle = p(X)$, and let $\mathcal{C}''_{X/}$ denote the full subcategory of $\mathcal{C}'_{X/}$ corresponding to inert morphisms $X \to f(C)$ for which $(p \circ f)(C) = \langle 1 \rangle$. We claim that F'_X is a q-right Kan extension of $F''_X = F | \mathcal{C}''_{X/}$. To prove this, let us choose an arbitrary object of $\mathcal{C}'_{X/}$, given by a map $\alpha : X \to C$ in $\mathcal{M}$. The fiber product $\mathcal{C}''_{X/} \times_{\mathcal{C}'_{X/}} (\mathcal{C}'_{X/})_{\alpha/}$ can be identified with the full subcategory of $\mathcal{M}_{\alpha/}$ spanned by those diagrams $X \xrightarrow{\alpha} C \xrightarrow{\beta} C'$ such that $(p \circ f)(\beta)$ has the form $\rho^i : \langle n \rangle \to \langle 1 \rangle$, for some $1 \le i \le n$. In particular, this ∞-category is a disjoint union of full subcategories $\{\mathcal{D}(i)\}_{1 \le i \le n}$, where each $\mathcal{D}(i)$ is equivalent to the full subcategory of $\mathcal{C}_{C/}$ spanned by morphisms $C \to C'$ covering the map ρ^i. Our assumption that f is a weak approximation to $\mathcal{O}^{\otimes}$ guarantees that each of these ∞-categories has a final object, given by a locally $(p \circ f)$-coCartesian morphism $C \to C_i$ in $\mathcal{C}$. It will therefore suffice to show that $F_0(C)$ is a q-product of the objects $\{F_0(C_i)\}_{1 \le i \le n}$. Since $\mathcal{O}'^{\otimes}$ is an ∞-operad, we are reduced to proving that each of the maps $F_0(C) \to F_0(C_i)$ is inert, which follows from our assumption that $F_0 \in \mathrm{Alg}_{\mathcal{C}}(\mathcal{O}')$.

Using Lemma T.4.3.2.7, we are reduced to proving that the diagram F''_X can be extended to a q-limit diagram $\mathcal{C}''^{\triangleleft}_{X/} \to \mathcal{O}'^{\otimes}$ (covering the natural map $\mathcal{C}''^{\triangleleft}_{X/} \to \mathcal{M} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$). For $1 \le i \le n$, let $\mathcal{C}(i)''_{X/}$ denote the full subcategory of $\mathcal{C}''_{X/}$ spanned by those objects for which the underlying morphism $X \to C$ covers $\rho^i : p(X) \simeq \langle n \rangle \to \langle 1 \rangle$. Then $\mathcal{C}''_{X/}$ is the disjoint union of the full subcategories $\{\mathcal{C}(i)''_{X/}\}$. Let $\mathcal{O}(i)$ denote the full subcategory of $\mathcal{O}^{\otimes}_{X/} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{\langle n \rangle/}} \{\rho^i\}$, so that we have a left fibration of simplicial sets $\mathcal{O}(i) \to \mathcal{O}$ and

a categorical equivalence $\mathcal{C}(i)''_{X/} \simeq \mathcal{O}(i) \times_{\mathcal{O}} \mathcal{C}_{\langle 1 \rangle}$. Choose inert morphisms $X \to X_i$ in $\mathcal{O}^{\otimes}$ for $1 \leq i \leq n$, so that each X_i determines an initial object of $\mathcal{O}(i)$. If f induces a categorical equivalence $\mathcal{C}_{\langle 1 \rangle} \to \mathcal{O}$, then we can write $X_i \simeq f(C_i)$ for some $C_i \in \mathcal{C}_{\langle 1 \rangle}$, and that the induced map $X \to C_i$ can be identified with a final object of $\mathcal{C}(i)''_{X/}$. Consequently, we are reduced to proving the existence of a q-product for the set of objects $\{F_0(C_i)\}_{1 \leq i \leq n}$, which follows from our assumption that $\mathcal{O}'^{\otimes}$ is an ∞-operad. This completes the proof of (a) in case (1).

In case (2), we must work a bit harder. Assume that $\mathcal{O}$ is a Kan complex and that $\mathcal{C}_{\langle 1 \rangle} \to \mathcal{O}$ is a weak homotopy equivalence. We again have $X_i \simeq f(C_i)$ for some $C_i \in \mathcal{C}_{\langle 1 \rangle}$. The map $\mathcal{O}(i) \to \mathcal{O}$ is a left fibration and therefore a Kan fibration. Using the right-properness of the usual model structure on $\mathcal{S}\mathrm{et}_\Delta$, we conclude that the diagram

$$
\begin{array}{ccc}
\mathcal{C}(i)''_{X/} & \longrightarrow & \mathcal{C}_{\langle 1 \rangle} \\
\downarrow & & \downarrow \\
\mathcal{O}(i) & \longrightarrow & \mathcal{O}
\end{array}
$$

is a homotopy pullback diagram, so that $\mathcal{C}(i)''_{X/} \to \mathcal{O}(i)$ is a weak homotopy equivalence and therefore $\mathcal{C}(i)''_{X/}$ is weakly contractible. Since F_0 is locally constant, Corollary T.4.4.4.10 and Proposition T.4.3.1.5 imply that $F''_X | \mathcal{C}(i)''_{X/}$ admits a q-limit, given by the object $F_0(C_i)$. We are therefore again reduced to proving the existence of a q-product for the set of objects $\{F_0(C_i)\}_{1 \leq i \leq n}$, which follows from our assumption that $\mathcal{O}'^{\otimes}$ is an ∞-operad. This completes the proof of (a) in case (2).

The arguments above (in either case) yield the following version of (b):

(b') Let $F \in \mathrm{Fun}_{N(\mathcal{F}\mathrm{in}_*)}(\mathcal{M}, \mathcal{O}'^{\otimes})$ be such that $F_0 = F | \mathcal{C} \in \mathrm{Alg}_{\mathcal{C}}(\mathcal{O}')$ (in case (1)) or $F_0 = F | \mathcal{C} \in \mathrm{Alg}_{\mathcal{C}}^{\mathrm{loc}}(\mathcal{O}')$ (in case (2)). Then F is a q-right Kan extension of F_0 if and only if, for every object $X \in \mathcal{O}^{\otimes}_{\langle n \rangle}$, if we choose $C_i \in \mathcal{C}_{\langle 1 \rangle}$ and maps $\alpha_i : X \to C_i$ in $\mathcal{M}$ having image $\rho^i : \langle n \rangle \to \langle 1 \rangle$ in $\mathcal{F}\mathrm{in}_*$ for $1 \leq i \leq n$, then $F(\alpha_i)$ is an inert morphism in $\mathcal{O}'^{\otimes}$ for $1 \leq i \leq n$.

We now prove (b). Assume first that $F \in \mathcal{X}$. Then $F_0 = F | \mathcal{C}$ is equivalent to the functor $(F | \mathcal{O}^{\otimes}) \circ f$. It follows immediately that $F_0 \in \mathrm{Alg}_{\mathcal{C}}(\mathcal{O}')$. In case (2), the assumption that $\mathcal{O}$ is a Kan complex immediately implies that F_0 is locally constant. Criterion (b') immediately implies that F is a q-right Kan extension of F_0. This proves the "only if" direction.

For the converse, assume that $F_0 \in \mathrm{Alg}_{\mathcal{C}}(\mathcal{O}')$, that F_0 is locally constant if we are in case (2), and that F is a q-right Kan extension of F_0. We wish to prove that $F \in \mathcal{X}$. We first verify that F satisfies (ii). Pick an object $C \in \mathcal{C}$ and choose locally $(p \circ f)$-coCartesian morphisms $\alpha_i : C \to C_i$ for $1 \leq i \leq n$. Let $X = f(C)$; we wish to show that the induced map $F(X) \to F(C)$ is an equivalence in $\mathcal{O}'^{\otimes}$. Since $\mathcal{O}'^{\otimes}$ is an ∞-operad, and the maps $F_0(C) \to C_0(C_i)$ are inert for $1 \leq i \leq n$ (by virtue of our assumption that $F \in \mathrm{Alg}_{\mathcal{C}}(\mathcal{O}')$), it will suffice to show that each of the maps $F(X) \to F_0(C_i)$ is inert, which follows from (b').

To complete the proof, we must show that $F | \mathcal{O}^{\otimes}$ is a map of ∞-operads. In view of Remark 2.1.2.9, it will suffice to show that if $X \in \mathcal{O}^{\otimes}_{\langle n \rangle}$ and $\alpha : X \to X_i$ is an inert morphism of $\mathcal{O}^{\otimes}$ covering the map $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \leq i \leq n$, then the induced map $F(X) \to F(X_i)$ is an inert morphism in $\mathcal{O}'^{\otimes}$. Arguing as above, we can assume that $X_i = f(C_i)$. Condition (ii) implies that $F(X_i) \to F(C_i)$ is an equivalence in $\mathcal{O}'$; it will therefore suffice to show that the composite map $F(X) \to F(X_i) \to F(C_i)$ is inert, which follows from criterion (b'). $\square$

Corollary 2.3.3.24. *Let* $f : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ *be a map of* ∞-*operads. Assume that* f *is a weak approximation to* $\mathcal{O}^{\otimes}$ *and that* f *induces an equivalence of* ∞-*categories* $\mathcal{C} \to \mathcal{O}$*. Then* f *is an equivalence of* ∞-*operads.*

Proof. Theorem 2.3.3.23 implies that for every ∞-operad $\mathcal{O}'^{\otimes}$, composition with f induces an equivalence of ∞-categories $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}') \to \mathrm{Alg}_{\mathcal{C}}(\mathcal{O}')$. $\square$

2.3.4 Disintegration of ∞-Operads

In §2.3.1, we introduced the definition of a unital ∞-operad. Roughly speaking, an ∞-operad $\mathcal{O}^{\otimes}$ is unital if there is no information contained in its spaces of nullary operations (more precisely, if these spaces are contractible). We now introduce a stronger condition, which guarantees also that the unary operation spaces of $\mathcal{O}^{\otimes}$ are trivial:

Definition 2.3.4.1. Let $\mathcal{O}^{\otimes}$ be an ∞-operad. We will say that $\mathcal{O}^{\otimes}$ is *reduced* if $\mathcal{O}^{\otimes}$ is unital and the underlying ∞-category $\mathcal{O}$ is a contractible Kan complex.

Our main goal in this section is to show that if $\mathcal{O}^{\otimes}$ is an arbitrary unital ∞-operad whose underlying ∞-category $\mathcal{O}$ is a Kan complex, then $\mathcal{O}^{\otimes}$ can be obtained by assembling a $\mathcal{O}$-family of reduced ∞-operads (Theorem 2.3.4.4). A precise formulation will make use of the following generalization of Definition 2.3.4.1:

Definition 2.3.4.2. We will say that an ∞-operad family $\mathcal{O}^{\otimes}$ is *reduced* if $\mathcal{O}^{\otimes}_{\langle 0 \rangle}$ is a Kan complex and, for each object $X \in \mathcal{O}^{\otimes}_{\langle 0 \rangle}$, the ∞-operad $\mathcal{O}^{\otimes}_{/X}$ is reduced.

Remark 2.3.4.3. Let $\mathcal{C}$ be a Kan complex, and let $q : \mathcal{O}^{\otimes} \to \mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a $\mathcal{C}$-family of ∞-operads. Every object $X \in \mathcal{O}^{\otimes}_{\langle 0 \rangle}$ is q-final, so that we have a trivial Kan fibration $\mathcal{O}^{\otimes}_{/X} \to \mathcal{C}_{/C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, where C denotes the image of X in $\mathcal{C}$. Since $\mathcal{C}$ is a Kan complex, the ∞-category $\mathcal{C}_{/C}$ is a contractible Kan complex, so that $\mathcal{O}^{\otimes}_{/X}$ is equivalent to the ∞-operad $\mathcal{O}^{\otimes}_C = \mathcal{O}^{\otimes} \times_{\mathcal{C}} \{C\}$.

It follows that $\mathcal{O}^{\otimes}$ is reduced (in the sense of Definition 2.3.4.2 if and only if $\mathcal{C}$ is reduced and each fiber $\mathcal{O}^{\otimes}_C$ is a reduced ∞-operad (in the sense of Definition 2.3.4.1). In particular, an ∞-operad is reduced if and only if it is reduced when regarded as a generalized ∞-operad.

We are now ready to state the main result of this section.

Theorem 2.3.4.4. *Let $\mathrm{Op}^{\mathrm{gn,rd}}_{\infty}$ denote the full subcategory of $\mathrm{Op}^{\mathrm{gn}}_{\infty}$ spanned by the reduced generalized ∞-operads. Then the assembly functor $\mathrm{Assem} : \mathrm{Op}^{\mathrm{gn}}_{\infty} \to \mathrm{Op}_{\infty}$ induces an equivalence from $\mathrm{Op}^{\mathrm{gn,rd}}_{\infty}$ to the full subcategory of Op_{∞} spanned by those unital ∞-operads $\mathcal{O}^{\otimes}$ such that the underlying Kan complex of $\mathcal{O}$ is a Kan complex.*

In other words, if $\mathcal{O}^{\otimes}$ is a unital ∞-operad such that $\mathcal{O}$ is a Kan complex, then $\mathcal{O}^{\otimes}$ can be obtained (in an essentially unique way) as the assembly of a family of reduced ∞-operads. The proof of Theorem 2.3.4.4 is based on the following assertion, which we will prove at the end of this section:

Proposition 2.3.4.5. *Let $\mathcal{O}'^{\otimes}$ be an ∞-operad and let $f : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ be a map of generalized ∞-operads. Assume that $\mathcal{O}^{\otimes}_{\langle 0 \rangle}$, $\mathcal{O}$, and $\mathcal{O}'$ are Kan complexes. Then:*

(1) *If f is a weak approximation to $\mathcal{O}'^{\otimes}$ which induces a homotopy equivalence $\mathcal{O} \to \mathcal{O}'$, then f exhibits $\mathcal{O}'^{\otimes}$ as an assembly of $\mathcal{O}^{\otimes}$.*

(2) *Assume that for each object $X \in \mathcal{O}^{\otimes}_{\langle 0 \rangle}$, the ∞-operad $\mathcal{O}^{\otimes}_{/X}$ is unital. If f exhibits $\mathcal{O}'^{\otimes}$ as an assembly of $\mathcal{O}^{\otimes}$, then f is an approximation to $\mathcal{O}'^{\otimes}$ and the underlying map $\mathcal{O} \to \mathcal{O}'$ is a homotopy equivalence of Kan complexes. Moreover, the ∞-operad $\mathcal{O}'^{\otimes}$ is also unital.*

The proofs of Theorem 2.3.4.4 and Proposition 2.3.4.5 will use some ideas from later in this book.

Proof of Theorem 2.3.4.4. It follows from Proposition 2.3.4.5 that the assembly functor Assem carries $\mathrm{Op}^{\mathrm{gn,rd}}_{\infty}$ into the full subcategory $\mathcal{X} \subseteq \mathrm{Op}_{\infty}$ spanned by those those unital ∞-operads $\mathcal{O}^{\otimes}$ such that $\mathcal{O}$ is a Kan complex. We next show that $\mathrm{Assem} : \mathrm{Op}^{\mathrm{gn,rd}}_{\infty} \to \mathcal{X}$ is essentially surjective. Let $\mathcal{O}^{\otimes}$ be such a unital ∞-operad whose underlying ∞-category is a Kan complex, and choose a homotopy equivalence $u_0 : \mathcal{O} \to S$ for some Kan complex S (for example, we can take $S = \mathcal{O}$ and u_0 to be the identity map). Let S^{II} be the ∞-operad defined in §2.4.3. Using Proposition 2.4.3.9, we can extend u_0 to an ∞-operad map $u : \mathcal{O}^{\otimes} \to S^{\mathrm{II}}$. Replacing

$\mathcal{O}^{\otimes}$ by an equivalent ∞-operad if necessary, we may suppose that u is a fibration of ∞-operads. Let $\mathcal{O}'^{\otimes}$ be the fiber product $\mathcal{O}^{\otimes} \times_{S^{\amalg}} (S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*))$. Then $\mathcal{O}'^{\otimes}$ is an S-family of ∞-operads equipped with a map $f : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ which induces an isomorphism $\mathcal{O}' \to \mathcal{O}$. The map f is a pullback of the approximation $S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to S^{\amalg}$ of Remark 2.4.3.6, and is therefore an approximation to $\mathcal{O}^{\otimes}$ (Remark 2.3.3.9). Invoking Proposition 2.3.4.5, we deduce that f exhibits $\mathcal{O}^{\otimes}$ as an assembly of $\mathcal{O}'^{\otimes}$, so that we have an equivalence $\mathrm{Assem}(\mathcal{O}'^{\otimes}) \simeq \mathcal{O}^{\otimes}$. To deduce the desired essential surjectivity, it suffices to show that $\mathcal{O}'^{\otimes}$ is reduced. In other words, we must show that for each $s \in S$, the ∞-operad $\mathcal{O}'^{\otimes}_s \simeq \mathcal{O}^{\otimes} \times_{S^{\amalg}} \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is reduced. This is clear: the underlying ∞-category $\mathcal{O}_s$ is given by the fiber of a trivial Kan fibration $f : \mathcal{O} \to S$, and $\mathcal{O}'^{\otimes}_s$ is unital because it is a homotopy fiber product of unital ∞-operads.

We now show that $\mathrm{Assem} : \mathrm{Op}^{\mathrm{gn,rd}}_{\infty} \to \mathrm{Op}_{\infty}$ is fully faithful. Let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be reduced generalized ∞-operads, and choose assembly maps $\mathcal{C}^{\otimes} \to \mathcal{C}'^{\otimes}$ and $\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$. We will show that the canonical map $\mathrm{Alg}_{\mathcal{C}}(\mathcal{D}) \to \mathrm{Alg}_{\mathcal{C}}(\mathcal{O}) \simeq \mathrm{Alg}_{\mathcal{C}'}(\mathcal{O})$ is an equivalence of ∞-categories. As above, we choose a Kan complex $S \simeq \mathcal{O}$ and a fibration of ∞-operads $\mathcal{O}^{\otimes} \to S^{\amalg}$, and define $\mathcal{O}'^{\otimes}$ to be the fiber product $(S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)) \times_{S^{\amalg}} \mathcal{O}^{\otimes}$. Using the equivalences $\mathrm{Alg}_{\mathcal{C}}(S^{\amalg}) \simeq \mathrm{Fun}(\mathcal{C}, S)$ and $\mathrm{Alg}_{\mathcal{C}}(S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)) \simeq \mathrm{Fun}(\mathcal{C}^{\otimes}_{\langle 0 \rangle}, S)$ provided by Propositions 2.4.3.16 and 2.3.2.9, we obtain a homotopy pullback diagram of ∞-categories

$$
\begin{array}{ccc}
\mathrm{Alg}_{\mathcal{C}}(\mathcal{O}') & \longrightarrow & \mathrm{Alg}_{\mathcal{C}}(\mathcal{O}) \\
\downarrow & & \downarrow \\
\mathrm{Fun}(\mathcal{C}^{\otimes}_{\langle 0 \rangle}, S) & \longrightarrow & \mathrm{Fun}(\mathcal{C}, S).
\end{array}
$$

Here the lower horizontal map is obtained by composing with the functor $\mathcal{C} = \mathcal{C}^{\otimes}_{\langle 1 \rangle} \to \mathcal{C}^{\otimes}_{\langle 0 \rangle}$ induced by the map $\langle 1 \rangle \to \langle 0 \rangle$ in $\mathcal{F}\mathrm{in}_*$. Since $\mathcal{C}$ is reduced, this map is an equivalence of ∞-categories, so the natural map $\mathrm{Alg}_{\mathcal{C}}(\mathcal{O}') \to \mathrm{Alg}_{\mathcal{C}}(\mathcal{O})$ is an equivalence. Similarly, we have an equivalence $\mathrm{Alg}_{\mathcal{D}}(\mathcal{O}') \to \mathrm{Alg}_{\mathcal{D}}(\mathcal{O})$. We may therefore assume that the assembly map $\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ factors through a map of generalized ∞-operads $\gamma : \mathcal{D}^{\otimes} \to \mathcal{O}'^{\otimes}$. To complete the proof, it will suffice to show that γ is an equivalence of generalized ∞-operads (and therefore induces an equivalence of ∞-categories $\mathrm{Alg}_{\mathcal{C}}(\mathcal{D}) \to \mathrm{Alg}_{\mathcal{C}}(\mathcal{O}') \simeq \mathrm{Alg}_{\mathcal{C}}(\mathcal{O})$).

Replacing $\mathcal{D}^{\otimes}$ by an equivalent generalized ∞-operad if necessary, we can assume that $\gamma : \mathcal{D}^{\otimes} \to \mathcal{O}'^{\otimes}$ is a categorical fibration, so that the composite map $\mathcal{D}^{\otimes} \to \mathcal{O}'^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ exhibits $\mathcal{D}$ as an S-family of ∞-operads. It will therefore suffice to show that for each $s \in S$, the induced map of fibers $\gamma_s : \mathcal{D}^{\otimes}_s \to \mathcal{O}'^{\otimes}_s$ is an equivalence of ∞-operads. For each $D \in \mathcal{D}^{\otimes}_s$ having an image $X \in \mathcal{O}^{\otimes}$, we have a commutative diagram

$$
\begin{array}{ccccc}
(\mathcal{D}^{\otimes}_s)^{\mathrm{act}}_{/D} & \longrightarrow & (\mathcal{D}^{\otimes})^{\mathrm{act}}_{/D} & \longrightarrow & (\mathcal{O}^{\otimes})^{\mathrm{act}}_{/X} \\
\downarrow & & \downarrow & & \downarrow \\
(\mathcal{O}'^{\otimes}_s)^{\mathrm{act}}_{/\gamma(D)} & \longrightarrow & (\mathcal{O}'^{\otimes})^{\mathrm{act}}_{/\gamma(D)} & \longrightarrow & (\mathcal{O}^{\otimes})^{\mathrm{act}}_{/X},
\end{array}
$$

where the superscript indicates that we consider the subcategory spanned by active morphisms. The horizontal maps in this diagram are categorical equivalences (by Proposition 2.3.4.5 and Corollary 2.3.3.17). It follows that the vertical maps are also categorical equivalences, so that γ_s is an approximation between reduced ∞-operads. It follows from Corollary 2.3.3.24 that γ_s is an equivalence of ∞-operads as desired. $\square$

We now turn to the proof of Proposition 2.3.4.5. We will need several preliminary results.

Lemma 2.3.4.6. *Let $f : X \to Y$ be a map of simplicial sets. If f is a weak homotopy equivalence and Y is a Kan complex, then f is left cofinal.*

Proof. The map f factors as a composition $X \xrightarrow{f'} X' \xrightarrow{f''} Y$, where f' is a categorical equivalence and f'' is a categorical fibration. Replacing f by f'', we can reduced to the case where f is a categorical fibration so

that X is an ∞-category. According to Theorem T.4.1.3.1, it suffices to show that for every vertex $y \in Y$, the fiber product $X \times_Y Y_{y/}$ is weakly contractible. Consider the pullback diagram

$$
\begin{array}{ccc}
X \times_Y Y_{y/} & \xrightarrow{f'} & Y_{y/} \\
\downarrow & & \downarrow{g} \\
X & \xrightarrow{f} & Y.
\end{array}
$$

The map g is a left fibration over a Kan complex, and therefore a Kan fibration (Lemma T.2.1.3.3). Since the usual model structure on simplicial sets is right proper, our diagram is a homotopy pullback square. Because f is a weak homotopy equivalence, we deduce that f' is a weak homotopy equivalence. Since $Y_{y/}$ is weakly contractible, we deduce that $X \times_Y Y_{y/}$ is weakly contractible, as desired.   □

Lemma 2.3.4.7. *Let $f : X \to Y$ be a weak homotopy equivalence of simplicial sets, let $\mathcal{C}$ be an ∞-category, and let $\overline{p} : Y^{\triangleright} \to \mathcal{C}$ be a colimit diagram. Suppose that $\overline{p}$ carries every edge of Y to an equivalence in $\mathcal{C}$. Then the composite map $X^{\triangleright} \to Y^{\triangleright} \to \mathcal{C}$ is a colimit diagram.*

Proof. Let $C \in \mathcal{C}$ be the image under $\overline{p}$ of the cone point of $Y^{\triangleright}$. Let $\mathcal{C}^{\simeq}$ be the largest Kan complex contained in $\mathcal{C}$, so that $\overline{p}$ induces a map $p : Y \to \mathcal{C}^{\simeq}$. Factor the map p as a composition

$$
Y \xrightarrow{p'} Z \xrightarrow{p''} \mathcal{C}^{\simeq},
$$

where p' is anodyne and p'' is a Kan fibration (so that Z is a Kan complex). Lemma 2.3.4.6 guarantees that the inclusion $Y \to Z$ is left cofinal and therefore right anodyne (Proposition T.4.1.1.3). Applying this observation to the lifting problem

$$
\begin{array}{ccc}
Y & \longrightarrow & \mathcal{C}_{/C} \\
\downarrow & \nearrow & \downarrow \\
Z & \xrightarrow[p'']{} & \mathcal{C},
\end{array}
$$

we deduce that $\overline{p}$ factors as a composition

$$
Y^{\triangleright} \to Z^{\triangleright} \xrightarrow{\overline{q}} \mathcal{C} .
$$

Since p' is left cofinal, the map $\overline{q}$ is a colimit diagram. Lemma 2.3.4.6 also guarantees that the composition $f \circ p' : X \to Z$ is left cofinal, so that

$$
X^{\triangleright} \to Z^{\triangleright} \xrightarrow{\overline{q}} \mathcal{C}
$$

is also a colimit diagram.   □

For the next statements, we will assume that the reader is familiar with the theory of free algebras that we discuss in §3.1.3.

Proposition 2.3.4.8. *Let $f : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ be a map between small ∞-operads, and let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Assume that $\mathcal{C}$ admits small colimits, and that the tensor product on $\mathcal{C}$ preserves small colimits in each variable, and let $F : \mathrm{Fun}(\mathcal{O}, \mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ and $F' : \mathrm{Fun}(\mathcal{O}', \mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$ be left adjoints to the forgetful functors (Example 3.1.3.6). The commutative diagram of forgetful functors*

$$
\begin{array}{ccc}
\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) & \xrightarrow{\theta} & \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \\
\downarrow & & \downarrow \\
\mathrm{Fun}(\mathcal{O}', \mathcal{C}) & \xrightarrow{\theta'} & \mathrm{Fun}(\mathcal{O}, \mathcal{C})
\end{array}
$$

induces a natural transformation $\alpha : F \circ \theta' \to \theta \circ F'$ from $\mathrm{Fun}(\mathcal{O}', \mathcal{C})$ to $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. Assume that $\mathcal{O}'$ is a Kan complex.

(1) *If f is a weak approximation to $\mathcal{O}'^{\otimes}$ then the natural transformation α is an equivalence.*

(2) *Conversely, suppose that α is an equivalence in the special case where $\mathcal{C} = \mathcal{S}$ (equipped with the Cartesian monoidal structure) and when evaluated on the constant functor $\mathcal{O}' \to \mathcal{C}$ taking the value Δ^0. Then f is an approximation to $\mathcal{O}'^{\otimes}$.*

Proof. Fix a map $A_0 \in \operatorname{Fun}(\mathcal{O}', \mathcal{C})$ and let $X \in \mathcal{O}$. Let $\mathcal{X}$ be the subcategory of $\mathcal{O}^{\otimes}_{/X}$ whose objects are active maps $Y \to X$ in $\mathcal{O}^{\otimes}$ and whose morphisms are maps which induce equivalences in $\operatorname{N}(\mathcal{F}in_*)$, and let $\mathcal{X}' \subseteq \mathcal{O}'^{\otimes}_{/\gamma(X)}$ be defined similarly. Then A_0 determines diagrams $\chi : \mathcal{X} \to \mathcal{C}$ and $\chi' : \mathcal{X}' \to \mathcal{C}$ (here χ is given by composing χ' with the map $\mathcal{X} \to \mathcal{X}'$ induced by γ). Using the characterization of free algebras given in §3.1.3, we deduce that $\alpha(A_0)(X) : (F \circ \theta')(A_0)(X) \to (\theta \circ F')(A_0)(X)$ is given by the evident map $\operatorname{colim}_{\mathcal{X}} \chi \to \operatorname{colim}_{\mathcal{X}'} \chi'$. Since $\mathcal{O}'$ is a Kan complex, A_0 carries every morphism in $\mathcal{O}'$ to an equivalence in $\mathcal{C}$, then χ' carries every morphism in $\mathcal{X}'$ to an equivalence in $\mathcal{C}$. If f is an approximation to $\mathcal{O}'^{\otimes}$, then the evident map $\mathcal{X} \to \mathcal{X}'$ is a weak homotopy equivalence (Corollary 2.3.3.16) that α is an equivalence by Lemma 2.3.4.7: this proves (1).

Conversely, suppose that the hypotheses of (2) are satisfied. Taking A_0 to be the constant functor taking the value $\Delta^0 \in \mathcal{S}$, we deduce from Corollary T.3.3.4.6 that the map $\mathcal{X} \to \mathcal{X}'$ is a weak homotopy equivalence for each $X \in \mathcal{O}$. From this it follows that f satisfies the criterion of Corollary 2.3.3.16 and is therefore an approximation to $\mathcal{O}'^{\otimes}$. $\qquad\square$

Proposition 2.3.4.9. *Let S be a Kan complex, let $\mathcal{O}^{\otimes} \to S \times \operatorname{N}(\mathcal{F}in_*)$ be an S-family of ∞-operads, and let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Suppose that, for each $s \in S$, the restriction functor $\operatorname{Alg}_{\mathcal{O}_s}(\mathcal{C}) \to \operatorname{Fun}(\mathcal{O}_s, \mathcal{C})$ admits a left adjoint F_s. Then:*

(1) *The restriction functor $\theta : \operatorname{Alg}_{\mathcal{O}}(\mathcal{C}) \to \operatorname{Fun}(\mathcal{O}, \mathcal{C})$ admits a left adjoint F.*

(2) *Let $A \in \operatorname{Alg}_{\mathcal{O}}(\mathcal{C})$, let $B \in \operatorname{Fun}(\mathcal{O}, \mathcal{C})$, and let $\alpha : B \to \theta(A)$ be a morphism in $\operatorname{Fun}(\mathcal{O}, \mathcal{C})$. Then the adjoint map $F(B) \to A$ is an equivalence in $\operatorname{Alg}_{\mathcal{O}}(\mathcal{C})$ if and only if, for each $s \in S$, the underlying map $F_s(B \mid \mathcal{O}_s) \to A \mid \mathcal{O}_s^{\otimes}$ is an equivalence in $\operatorname{Alg}_{\mathcal{O}_s}(\mathcal{C})$.*

Proof. Fix $B \in \operatorname{Fun}(\mathcal{O}, \mathcal{C})$. For every map of simplicial sets $\psi : T \to S$, let $\mathcal{O}_T = \mathcal{O} \times_S T$, $B_T = B \mid \mathcal{O}_T$, and $X(T)$ denote the full subcategory of $\operatorname{Alg}_{\mathcal{O}_T}(\mathcal{C}) \times_{\operatorname{Fun}(\mathcal{O}_T, \mathcal{C})} \operatorname{Fun}(\mathcal{O}_T, \mathcal{C})_{B_T/}$ spanned by those objects $(A_T \in \operatorname{Alg}_{\mathcal{O}_T}(\mathcal{C}), \phi : B_T \to A_T \mid \mathcal{O}_T)$ such that, for each vertex $t \in T$, the induced map $F_{\psi(t)}(B_T \mid \mathcal{O}_{\psi(t)}) \to A_T \mid \mathcal{O}_{\psi(t)}^{\otimes}$ is an equivalence. We claim that every inclusion of simplicial sets $i : T' \hookrightarrow T$ in $(\mathcal{S}et_\Delta)_{/S}$, the restriction map $X(T) \to X(T')$ is a trivial Kan fibration. The collection of maps i for which the conclusion holds is clearly weakly saturated; it therefore suffices to prove the claim in the case where i is an inclusion of the form $\partial \Delta^n \subset \Delta^n$. The proof proceeds by induction on n. The inductive hypothesis implies that the restriction map $X(\partial \Delta^n) \to X(\emptyset) \simeq \Delta^0$ is a trivial Kan fibration, so that $X(\partial \Delta^n)$ is a contractible Kan complex. The map $X(\Delta^n) \to X(\partial \Delta^n)$ is evidently a categorical fibration; it therefore suffices to show that it is a categorical equivalence. In other words, it suffices to show that $X(\Delta^n)$ is also a contractible Kan complex. Let $s \in S$ denote the image of the vertex $\{0\} \in \Delta^n$ in S. Since the inclusion $\mathcal{O}_s^{\otimes} \hookrightarrow \mathcal{O}_{\Delta^n}^{\otimes}$ is a categorical equivalence, it induces a categorical equivalence $X(\Delta^n) \to X(\{s\})$. We are therefore reduced to proving that $X(\{s\})$ is a contractible Kan complex, which is obvious.

The above argument shows that $X(S)$ is a contractible Kan complex; in particular, $X(S)$ is nonempty. Consequently, there exists a map $\phi : B \to \theta(A)$ satisfying the condition described in (2). We will prove (1) together with the "if" direction of (2) by showing that that ϕ induces a homotopy equivalence $\rho : \operatorname{Map}_{\operatorname{Alg}_{\mathcal{O}}(\mathcal{C})}(A, C) \to \operatorname{Map}_{\operatorname{Fun}(\mathcal{O}, \mathcal{C})}(B, \theta(C))$ for each $C \in \operatorname{Alg}_{\mathcal{O}}(\mathcal{C})$. The "only if" direction of (2) will then follow by the usual uniqueness argument. We proceed as before: for every map of simplicial sets $T \to S$, let $Y(T)$ denote the ∞-category $\operatorname{Alg}_{\mathcal{O}_T}(\mathcal{C})_{(A \mid \mathcal{O}_T^{\otimes})/} \times_{\operatorname{Fun}(\mathcal{O}_T, \mathcal{C})_{(A_T \mid \mathcal{O}_T)/}} \operatorname{Fun}(\mathcal{O}_T, \mathcal{C})_{\phi_T/}$ and $Y'(T) = \operatorname{Fun}(\mathcal{O}_T, \mathcal{C})_{(B \mid \mathcal{O}_T)/}$. The map ρ can be regarded as a pullback of the restriction map $Y(S) \to Y'(S)$. To complete the proof, it will suffice to show that $Y(S) \to Y'(S)$ is a trivial Kan fibration. We will prove the following stronger assertion: for every inclusion $T' \hookrightarrow T$ in $(\mathcal{S}et_\Delta)_{/S}$, the restriction map $\pi : Y(T) \to Y(T') \times_{Y'(T')} Y'(T)$ is a

trivial Kan fibration. As before, the collection of inclusions which satisfy this condition is weakly saturated, so we may reduce to the case where $T = \Delta^n$, $T' = \partial \Delta^n$, and the result holds for inclusions of simplicial sets having dimension $< n$. Moreover, since π is easily seen to be a categorical fibration, it suffices to show that π is a categorical equivalence. Using the inductive hypothesis, we deduce that $Y(T') \to Y'(T')$ is a trivial Kan fibration, so that the pullback map $Y(T') \times_{Y'(T')} Y'(T) \to Y'(T)$ is a categorical equivalence. By a two-out-of-three argument, we are reduced to proving that the restriction map $Y(T) \to Y'(T)$ is a categorical equivalence. If we define s to be the image of $\{0\} \subseteq \Delta^n \simeq T$ in S, then we have a commutative diagram

$$
\begin{array}{ccc}
Y(T) & \longrightarrow & Y'(T) \\
\downarrow & & \downarrow \\
Y(\{s\}) & \longrightarrow & Y'(\{s\})
\end{array}
$$

in which the vertical maps are categorical equivalences. We are therefore reduced to showing that $Y(\{s\}) \to Y'(\{s\})$ is a categorical equivalence, which is equivalent to the requirement that the map $F_s(B|\, \mathcal{O}_s) \to A|\, \mathcal{O}_s^{\otimes}$ be an equivalence in $\mathrm{Alg}_{\mathcal{O}_s}(\mathcal{C})$. $\square$

We conclude this section with the proof of Proposition 2.3.4.5.

Proof of Proposition 2.3.4.5. Assertion (1) is an immediate consequence of Theorem 2.3.3.23. We will prove (2). Assume that $f : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ exhibits $\mathcal{O}'^{\otimes}$ as an assembly of $\mathcal{O}^{\otimes}$ and that the ∞-operad $\mathcal{O}_{/X}^{\otimes}$ is reduced for each $X \in \mathcal{O}_{\langle 0 \rangle}^{\otimes}$. We wish to prove:

(a) The ∞-operad $\mathcal{O}'^{\otimes}$ is unital.

(c) The map f is an approximation to $\mathcal{O}'^{\otimes}$.

(b) The map f induces a homotopy equivalence of Kan complexes $\mathcal{O} \to \mathcal{O}'$.

It follows from Proposition 2.3.1.11 that for each $s \in S$, the induced map $\mathrm{Alg}_{\mathcal{O}_s}(\mathcal{O}'_*) \to \mathrm{Alg}_{\mathcal{O}_s}(\mathcal{O})$ is a trivial Kan fibration. Arguing as in Proposition 2.3.4.9, we deduce that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}'_*) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$ is a trivial Kan fibration. Since f exhibits $\mathcal{O}'^{\otimes}$ as an assembly of $\mathcal{O}^{\otimes}$, we deduce that the map $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{O}'_*) \to \mathrm{Alg}_{\mathcal{O}'}(\mathcal{O}')$ is an equivalence of ∞-categories, and therefore (since it is a categorical fibration) a trivial Kan fibration. In particular, the projection map $\mathcal{O}'^{\otimes}_* \to \mathcal{O}'^{\otimes}$ admits a section, so the final object of $\mathcal{O}'^{\otimes}$ is initial and $\mathcal{O}'^{\otimes}$ is also unital. This proves (a).

Let $\mathcal{C}$ be an arbitrary ∞-category, which we regard as the underlying ∞-category of the ∞-operad $\mathcal{C}^{\mathrm{II}}$. We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) & \longrightarrow & \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \\
\downarrow & & \downarrow \\
\mathrm{Fun}(\mathcal{O}', \mathcal{C}) & \longrightarrow & \mathrm{Fun}(\mathcal{O}, \mathcal{C})
\end{array}
$$

where the upper horizontal map is an equivalence and the vertical maps are equivalences by virtue of Proposition 2.4.3.16. It follows that the lower horizontal map is an equivalence. Allowing $\mathcal{C}$ to vary, we deduce that f induces an equivalence of ∞-categories $\mathcal{O} \to \mathcal{O}'$. This completes the proof of (b).

It remains to show f is an approximation to $\mathcal{O}'^{\otimes}$. According to Remark 2.3.3.19, it will suffice to show that for each $X \in \mathcal{O}_{\langle 0 \rangle}^{\otimes}$, the induced map $f_X : \mathcal{O}_{/X}^{\otimes} \to \mathcal{O}'^{\otimes}$ is an approximation to $\mathcal{O}'^{\otimes}$. Using Corollary 2.3.3.17, we are reduced to showing that f_X is a weak approximation to $\mathcal{O}'^{\otimes}$. We will show that the criterion of Proposition 2.3.4.8 is satisfied. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category such that $\mathcal{C}$ admits small

colimits and the tensor product $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable, and consider the commutative diagram of forgetful functors σ :

$$
\begin{array}{ccc}
\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) & \xrightarrow{\theta} & \mathrm{Alg}_{\mathcal{O}_{/X}}(\mathcal{C}) \\
\downarrow & & \downarrow \\
\mathrm{Fun}(\mathcal{O}', \mathcal{C}) & \xrightarrow{\theta'} & \mathrm{Fun}(\mathcal{O}_{/X}, \mathcal{C}),
\end{array}
$$

where the vertical maps have left adjoints $F_X : \mathrm{Fun}(\mathcal{O}_{/X}, \mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}_{/X}}(\mathcal{C})$ and $F' : \mathrm{Fun}(\mathcal{O}', \mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$. We wish to show that the natural transformation $\alpha : F_X \circ \theta' \to \theta \circ F'$ is an equivalence. Since f exhibits $\mathcal{O}'^{\otimes}$ as an assembly of $\mathcal{O}^{\otimes}$, the forgetful functor $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is an equivalence. Similarly, (b) implies that $\mathrm{Fun}(\mathcal{O}', \mathcal{C}) \to \mathrm{Fun}(\mathcal{O}, \mathcal{C})$ is an equivalence. We may therefore replace the diagram σ by the equivalent diagram

$$
\begin{array}{ccc}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) & \xrightarrow{\psi} & \mathrm{Alg}_{\mathcal{O}_{/X}}(\mathcal{C}) \\
\downarrow & & \downarrow \\
\mathrm{Fun}(\mathcal{O}, \mathcal{C}) & \xrightarrow{\psi'} & \mathrm{Fun}(\mathcal{O}_{/X}, \mathcal{C}),
\end{array}
$$

where the left vertical map has a left adjoint $F : \mathrm{Fun}(\mathcal{O}, \mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. We are therefore reduced to proving that the natural transformation $F_X \circ \psi' \to \psi \circ F$ is an equivalence, which is a special case of Proposition 2.3.4.9. $\qquad \square$

2.4 Products and Coproducts

Let $\mathcal{C}$ be a category which admits finite products. Then the product operation $(X, Y) \mapsto X \times Y$ is commutative and associative (up to canonical isomorphism), and has a unit given by the final object of $\mathcal{C}$. It follows that the formation of Cartesian products endows $\mathcal{C}$ with a symmetric monoidal structure, which we will call the *Cartesian symmetric monoidal structure* on $\mathcal{C}$. Similarly, if $\mathcal{C}$ admits finite coproducts, then the construction $(X, Y) \mapsto X \coprod Y$ endows $\mathcal{C}$ with another symmetric monoidal structure, which we call the *coCartesian symmetric monoidal structure*.

Our goal in this section is to study the ∞-categorical analogues of Cartesian and coCartesian symmetric monoidal structures. To this end, we introduce the following definition:

Definition 2.4.0.1. Let $\mathcal{C}$ be an ∞-category. We will say that a symmetric monoidal structure on $\mathcal{C}$ is *Cartesian* if the following conditions are satisfied:

(1) The unit object $1_{\mathcal{C}} \in \mathcal{C}$ is final.

(2) For every pair of objects $C, D \in \mathcal{C}$, the canonical maps

$$
C \simeq C \otimes 1_{\mathcal{C}} \leftarrow C \otimes D \to 1_{\mathcal{C}} \otimes D \simeq D
$$

exhibit $C \otimes D$ as a product of C and D in the ∞-category $\mathcal{C}$.

Dually, we will say that a symmetric monoidal structure on $\mathcal{C}$ is *coCartesian* if it satisfies the following pair of analogous conditions:

(1′) The unit object $1_{\mathcal{C}} \in \mathcal{C}$ is initial.

(2′) For every pair of objects $C, D \in \mathcal{C}$, the canonical maps

$$
C \simeq C \otimes 1_{\mathcal{C}} \to C \otimes D \leftarrow 1_{\mathcal{C}} \otimes D \simeq D
$$

exhibit $C \otimes D$ as a coproduct of C and D in the ∞-category $\mathcal{C}$.

It is natural to expect that if $\mathcal{C}$ is an ∞-category which admits finite products, then the formation of finite products is commutative and associative up to coherent equivalence: that is, $\mathcal{C}$ should admit a Cartesian symmetric monoidal structure. We will prove this result in §2.4.1 by means of an explicit construction (in fact, this Cartesian symmetric monoidal structure is unique in a strong sense: see Corollary 2.4.1.8). We can then apply to $\mathcal{C}$ all of the ideas introduced in §2.1; in particular, for any ∞-operad $\mathcal{O}^{\otimes}$, we can consider the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ of $\mathcal{O}$-algebra objects of $\mathcal{C}$. Since the Cartesian symmetric monoidal structure on $\mathcal{C}$ is entirely determined by the structure of the underlying ∞-category $\mathcal{C}$, it is natural to expect that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ admits a direct description which makes no reference to the theory of ∞-operads. In §2.4.2 we will provide such a description by introducing the notion of a $\mathcal{O}$-*monoid*. Using this notion, we will characterize the Cartesian symmetric monoidal structure on an ∞-category $\mathcal{C}$ by means of a universal mapping property (Proposition 2.4.2.5).

The theory of coCartesian symmetric monoidal structures should be regarded as dual to the theory of Cartesian symmetric monoidal structures: that is, we expect that giving a coCartesian symmetric monoidal structure on an ∞-category $\mathcal{C}$ is equivalent to giving a Cartesian symmetric monoidal structure on the ∞-category $\mathcal{C}^{op}$. However, this identification is somewhat subtle: our definition of symmetric monoidal ∞-category is not manifestly self-dual, so it is not immediately obvious that a symmetric monoidal structure on $\mathcal{C}^{op}$ determines a symmetric monoidal structure on $\mathcal{C}$ (this is nonetheless true; see Remark 2.4.2.7). Definition 2.0.0.7 encodes a symmetric monoidal structure on an ∞-category $\mathcal{C}$ by specifying maps of the form $X_1 \otimes \cdots \otimes X_n \to Y$; maps of the form $X \to Y_1 \otimes \cdots \otimes Y_n$ are more difficult to access. For this reason, we devote §2.4.3 to giving an explicit construction of a *coCartesian* symmetric monoidal structure on an ∞-category $\mathcal{C}$ which admits finite coproducts. Roughly speaking, the idea is to specify a symmetric monoidal structure in which giving morphism $X_1 \otimes \cdots \otimes X_n \to Y$ is equivalent to giving a collection of morphisms $\{X_i \to Y\}_{1 \leq i \leq n}$. This construction has the advantage of working in great generality: it yields an ∞-operad $\mathcal{C}^{\mathrm{II}}$ even in cases where the ∞-category $\mathcal{C}$ does not admit finite coproducts. We will apply these ideas in §2.4.4 to analyze the tensor product operation on ∞-operads described in §2.2.5.

As in the Cartesian case, it is natural to expect that if an ∞-category $\mathcal{C}$ is equipped with a coCartesian symmetric monoidal structure, then the theory of algebras in $\mathcal{C}$ can be formulated without reference to the theory of ∞-operads. This turns out to be true for a somewhat trivial reason: every object $C \in \mathcal{C}$ admits a unique commutative algebra structure, with multiplication given by the "fold" map $C \coprod C \to C$ (Corollary 2.4.3.10). In fact, this can be taken as a characterization of the coCartesian symmetric monoidal structure: we will show that the coCartesian symmetric monoidal structure on $\mathcal{C}$ is *universal* among symmetric monoidal ∞-categories $\mathcal{D}$ for which there exists a functor $\mathcal{C} \to \mathrm{CAlg}(\mathcal{D})$ (Theorem 2.4.3.18).

2.4.1   Cartesian Symmetric Monoidal Structures

Let $\mathcal{C}$ be an ∞-category which admits finite products. Our main goal in this section is to prove that $\mathcal{C}$ admits an essentially unique symmetric monoidal structure which is Cartesian in the sense of Definition 2.4.0.1. We begin by describing a useful mechanism for recognizing that a symmetric monoidal structure is Cartesian.

Definition 2.4.1.1. Let $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad. A *lax Cartesian structure* on $\mathcal{C}^{\otimes}$ is a functor $\pi : \mathcal{C}^{\otimes} \to \mathcal{D}$ satisfying the following condition:

(∗) Let C be an object of $\mathcal{C}^{\otimes}_{\langle n \rangle}$, and write $C \simeq C_1 \oplus \cdots \oplus C_n$, where each $C_i \in \mathcal{C}$. Then the canonical maps $\pi(C) \to \pi(C_i)$ exhibit $\pi(C)$ as a product $\prod_{1 \leq j \leq n} \pi(C_j)$ in the ∞-category $\mathcal{D}$.

We will say that π is a *weak Cartesian structure* if it is a lax Cartesian structure, $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category, and the following additional condition is satisfied:

(∗′) Let $f : C \to C'$ be a p-coCartesian morphism covering an active morphism $\alpha : \langle n \rangle \to \langle 1 \rangle$ in $\mathcal{F}\mathrm{in}_*$. Then $\pi(f)$ is an equivalence in $\mathcal{D}$.

We will say that a weak Cartesian structure π is a *Cartesian structure* if π restricts to an equivalence $\mathcal{C} \to \mathcal{D}$.

It follows immediately from the definitions that if $\mathcal{C}$ is a symmetric monoidal ∞-category and there exists a Cartesian structure $\mathcal{C}^{\otimes} \to \mathcal{D}$, then the symmetric monoidal structure on $\mathcal{C}$ is Cartesian. Consequently, to prove that an ∞-category $\mathcal{C}$ admits a Cartesian symmetric monoidal structure, it will suffice to construct a symmetric monoidal ∞-category $\mathcal{C}^{\times} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ together with a Cartesian structure $\pi : \mathcal{C}^{\times} \to \mathcal{C}$. This will require a few preliminaries.

Notation 2.4.1.2. We define a category $\Gamma^{\times}$ as follows:

(1) An object of $\Gamma^{\times}$ consists of an ordered pair $(\langle n \rangle, S)$, where $\langle n \rangle$ is an object of $\mathcal{F}\mathrm{in}_*$ and S is a subset of $\langle n \rangle^{\circ}$.

(2) A morphism from $(\langle n \rangle, S)$ to $(\langle n' \rangle, S')$ in $\Gamma^{\times}$ consists of a map $\alpha : \langle n \rangle \to \langle n' \rangle$ in $\mathcal{F}\mathrm{in}_*$ with the property that $\alpha^{-1} S' \subseteq S$.

We observe that the forgetful functor $\Gamma^{\times} \to \mathcal{F}\mathrm{in}_*$ is a Grothendieck fibration, so that the induced map of ∞-categories $\mathrm{N}(\Gamma^{\times}) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a Cartesian fibration (Remark T.2.4.2.2).

Remark 2.4.1.3. The forgetful functor $\Gamma^{\times} \to \mathcal{F}\mathrm{in}_*$ has a canonical section s, given by $s(\langle n \rangle) = (\langle n \rangle, \langle n \rangle^{\circ})$.

Construction 2.4.1.4. Let $\mathcal{C}$ be an ∞-category. We define a simplicial set $\widetilde{\mathcal{C}}^{\times}$ equipped with a map $\widetilde{\mathcal{C}}^{\times} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ by the following universal property: for every map of simplicial sets $K \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, we have a bijection
$$\mathrm{Hom}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(K, \widetilde{\mathcal{C}}^{\times}) \simeq \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(K \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\Gamma^{\times}), \mathcal{C}).$$

Fix $\langle n \rangle \in \mathcal{F}\mathrm{in}_*$. We observe that the fiber $\widetilde{\mathcal{C}}^{\times}_{\langle n \rangle}$ can be identified with the ∞-category of functors $f : \mathrm{N}(P)^{op} \to \mathcal{C}$, where P is the partially ordered set of subsets of $\langle n \rangle^{\circ}$. We let $\mathcal{C}^{\times}$ be the full simplicial subset of $\widetilde{\mathcal{C}}^{\times}$ spanned by those vertices which correspond to those functors f with the property that for every $S \subseteq \langle n \rangle^{\circ}$, the maps $f(S) \to f(\{j\})$ exhibit $f(S)$ as a product of the objects $\{f(\{j\})\}_{j \in S}$ in the ∞-category $\mathcal{C}$.

The fundamental properties of Construction 2.4.1.4 are summarized in the following result:

Proposition 2.4.1.5. *Let $\mathcal{C}$ be an ∞-category.*

(1) *The projection $p : \widetilde{\mathcal{C}}^{\times} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a coCartesian fibration.*

(2) *Let $\overline{\alpha} : f \to f'$ be a morphism of $\widetilde{\mathcal{C}}^{\times}$ whose image in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ corresponds to a map $\alpha : \langle n \rangle \to \langle n' \rangle$. Then $\overline{\alpha}$ is p-coCartesian if and only if, for every $S \subseteq \langle n' \rangle^{\circ}$, the induced map $f(\alpha^{-1}S) \to f'(S)$ is an equivalence in $\mathcal{C}$.*

(3) *The projection p restricts to a coCartesian fibration $\mathcal{C}^{\times} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ (with the same class of coCartesian morphisms).*

(4) *The projection $\mathcal{C}^{\times} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a symmetric monoidal ∞-category if and only if $\mathcal{C}$ admits finite products.*

(5) *Suppose that $\mathcal{C}$ admits finite products. Let $\pi : \mathcal{C}^{\times} \to \mathcal{C}$ be the map given by composition with the section $s : \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathrm{N}(\Gamma^{\times})$ defined in Remark 2.4.1.3. Then π is a Cartesian structure on $\mathcal{C}^{\times}$.*

Proof. Assertions (1) and (2) follow immediately from Corollary T.3.2.2.13, and (3) follows from (2) (since $\mathcal{C}^{\times}$ is stable under the pushforward functors associated to the coCartesian fibration p). We now prove (4). If $\mathcal{C}$ has no final object, then $\mathcal{C}^{\times}_{\langle 0 \rangle}$ is empty; consequently, we may assume without loss of generality that $\mathcal{C}$ has a final object. Then $\mathcal{C}^{\times}_{\langle 1 \rangle}$ is isomorphic to the ∞-category of diagrams $X \to Y$ in $\mathcal{C}$, where Y is final. It follows that π induces an equivalence $\mathcal{C}^{\times}_{\langle 1 \rangle} \simeq \mathcal{C}$. Consequently, $\mathcal{C}^{\times}$ is a symmetric monoidal ∞-category

if and only if, for each $n \geq 0$, the functors ρ_1^j determine an equivalence $\phi : \mathcal{C}_{\langle n \rangle}^\times \to \mathcal{C}^n$. Let P denote the partially ordered set of subsets of $\langle n \rangle^\circ$, and let $P_0 \subseteq P$ be the partially ordered set consisting of subsets which consist of a single element. Then $\mathcal{C}_{\langle n \rangle}^\times$ can be identified with the set of functors $f : \mathrm{N}(P)^{op} \to \mathcal{C}$ which are right Kan extensions of $f | \mathrm{N}(P_0)^{op}$, and ϕ can be identified with the restriction map determined by the inclusion $P_0 \subseteq P$. According to Proposition T.4.3.2.15, ϕ is fully faithful, and is essentially surjective if and only if every functor $f_0 : \mathrm{N}(P_0)^{op} \to \mathcal{C}$ admits a right Kan extension to $\mathrm{N}(P)^{op}$. Unwinding the definitions, we see that this is equivalent to the assertion that every finite collection of objects of $\mathcal{C}$ admits a product in $\mathcal{C}$. This completes the proof of (4). Assertion (5) follows immediately from (2) and the construction of $\mathcal{C}^\times$. $\square$

It follows from Proposition 2.4.1.5 that if $\mathcal{C}$ is an ∞-category which admits finite products, then $\mathcal{C}$ admits a Cartesian symmetric monoidal structure. Our next goal is to show that this Cartesian symmetric monoidal structure is unique up to equivalence. In other words, we claim that if $\mathcal{C}^\otimes$ is any Cartesian symmetric monoidal ∞-category, then there exists a symmetric monoidal equivalence $\mathcal{C}^\otimes \simeq \mathcal{C}^\times$ (extending the identity functor on $\mathcal{C}$). The proof will proceed in two steps:

(i) We will show that $\mathcal{C}^\otimes$ admits a Cartesian structure $\pi' : \mathcal{C}^\otimes \to \mathcal{C}$.

(ii) We will show that any Cartesian structure $\pi' : \mathcal{C}^\otimes \to \mathcal{C}$ is homotopic to a composition $\mathcal{C}^\otimes \xrightarrow{F} \mathcal{C}^\times \xrightarrow{\pi} \mathcal{C}$, where π is the Cartesian structure appearing in Proposition 2.4.1.5 and F is a symmetric monoidal functor (automatically an equivalence).

More precisely, we have the following pair of results, whose proofs will be given at the end of this section:

Proposition 2.4.1.6. *Let $p : \mathcal{C}^\otimes \to \mathrm{N}(\mathcal{F}in_*)$ be a Cartesian symmetric monoidal ∞-category and let $\mathcal{D}$ be another ∞-category which admits finite products. Let $\mathrm{Fun}^\times(\mathcal{C}^\otimes, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}^\otimes, \mathcal{D})$ spanned by the weak Cartesian structures, and let $\mathrm{Fun}^\times(\mathcal{C}, \mathcal{D})$ be the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which preserve finite products. The restriction map $\mathrm{Fun}^\times(\mathcal{C}^\otimes, \mathcal{D}) \to \mathrm{Fun}^\times(\mathcal{C}, \mathcal{D})$ is an equivalence of ∞-categories.*

Proposition 2.4.1.7. *Let $\mathcal{O}^\otimes$ be an ∞-operad, $\mathcal{D}$ an ∞-category which admits finite products, and $\pi : \mathcal{D}^\times \to \mathcal{D}$ the Cartesian structure of Proposition 2.4.1.5. Then composition with π induces a trivial Kan fibration*

$$\theta : \mathrm{Alg}_{\mathcal{O}}(\mathcal{D}) \to \mathrm{Fun}^{\mathrm{lax}}(\mathcal{O}^\otimes, \mathcal{D})$$

where $\mathrm{Fun}^{\mathrm{lax}}(\mathcal{O}^\otimes, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{O}^\otimes, \mathcal{D})$ spanned by the lax Cartesian structures. If $\mathcal{O}^\otimes$ is a symmetric monoidal ∞-category, then composition with π induces a trivial Kan fibration

$$\theta_0 : \mathrm{Fun}^\otimes(\mathcal{O}^\otimes, \mathcal{D}^\times) \to \mathrm{Fun}^\times(\mathcal{O}^\otimes, \mathcal{D})$$

where $\mathrm{Fun}^\times(\mathcal{O}^\otimes, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{O}^\otimes, \mathcal{D})$ spanned by the weak Cartesian structures.

Combining these results, we obtain the following:

Corollary 2.4.1.8. *Let $\mathcal{C}^\otimes$ be a Cartesian symmetric monoidal ∞-category whose underlying ∞-category $\mathcal{C}$ admits finite products, and let $\mathcal{D}$ be an ∞-category which admits finite products. Then:*

(1) *The restriction functor $\theta : \mathrm{Fun}^\otimes(\mathcal{C}^\otimes, \mathcal{D}^\times) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is fully faithful.*

(2) *The essential image of θ is the full subcategory $\mathrm{Fun}^\times(\mathcal{C}, \mathcal{D})$ spanned by those functors which preserve finite products.*

(3) *There exists a symmetric monoidal equivalence $\mathcal{C}^\otimes \simeq \mathcal{C}^\times$ whose restriction to the underlying ∞-category $\mathcal{C}$ is homotopic to the identity.*

Proof. To prove assertions (1) and (2), we note that θ factors as a composition

$$\mathrm{Fun}^{\otimes}(\mathcal{C}^{\otimes}, \mathcal{D}^{\times}) \xrightarrow{\theta'} \mathrm{Fun}^{\times}(\mathcal{C}^{\otimes}, \mathcal{D}) \xrightarrow{\theta''} \mathrm{Fun}^{\times}(\mathcal{C}, \mathcal{D}),$$

where θ' is the trivial Kan fibration of Proposition 2.4.1.7 and θ'' is the equivalence of ∞-categories of Proposition 2.4.1.6. Taking $\mathcal{C} = \mathcal{D}$, we deduce the existence of a symmetric monoidal functor $F : \mathcal{C}^{\otimes} \to \mathcal{C}^{\times}$ which is homotopic to the identity on $\mathcal{C}$. It follows from Remark 2.1.3.8 that F is an equivalence of symmetric monoidal ∞-categories. $\square$

Using Corollary 2.4.1.8, we can formulate an even stronger uniqueness claim for Cartesian symmetric monoidal structures. Let $\mathcal{C}\mathrm{at}_{\infty}^{\otimes}$ be the ∞-category of symmetric monoidal ∞-categories (see Variation 2.1.4.13), and let $\mathcal{C}\mathrm{at}_{\infty}^{\otimes,\times} \subseteq \mathcal{C}\mathrm{at}_{\infty}^{\otimes}$ be the full subcategory spanned by the Cartesian symmetric monoidal ∞-categories. Let $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Cart}} \subseteq \mathcal{C}\mathrm{at}_{\infty}$ denote the subcategory spanned by those ∞-categories $\mathcal{C}$ which admit finite products, and those functors which preserve finite products. Then:

Corollary 2.4.1.9. *The forgetful functor* $\theta : \mathcal{C}\mathrm{at}_{\infty}^{\otimes,\times} \to \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Cart}}$ *is an equivalence of ∞-categories.*

Proof. It follows from Proposition 2.4.1.5 that the functor θ is essentially surjective. To prove that θ is fully faithful, let us suppose that we are given a pair of Cartesian symmetric monoidal ∞-categories $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$; we wish to show that the map

$$\mathrm{Map}_{\mathcal{C}\mathrm{at}_{\infty}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) \to \mathrm{Map}_{\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Cart}}}(\mathcal{C}, \mathcal{D})$$

is a homotopy equivalence of Kan complexes. This follows immediately from Corollary 2.4.1.8. $\square$

Example 2.4.1.10. Let $\mathbf{A}$ be a simplicial model category. Suppose that the Cartesian monoidal structure on $\mathbf{A}$ is compatible with the model structure (in other words, that the final object of $\mathbf{A}$ is cofibrant, and that for any pair of cofibrations $i : A \to A'$, $j : B \to B'$, the induced map $i \wedge j : (A \times B') \coprod_{A \times B}(A' \times B) \to A' \times B'$ is a cofibration, trivial if either i or j is trivial). Since the collection of fibrant-cofibrant objects of $\mathbf{A}$ is stable under finite products, the construction $\{A_i\}_{1 \le i \le n} \mapsto \prod_i A_i$ determines a functor $\pi : \mathrm{N}(\mathbf{A}^{\circ})^{\otimes} \to \mathrm{N}(\mathbf{A}^{\circ})$, where $\mathrm{N}(\mathbf{A}^{\circ})^{\otimes}$ is the ∞-operad of Proposition 4.1.3.10. It is not difficult to see that π is a Cartesian structure on $\mathrm{N}(\mathbf{A}^{\circ})^{\otimes}$: that is, the symmetric monoidal structure on $\mathbf{A}$ determines a Cartesian symmetric monoidal structure on $\mathrm{N}(\mathbf{A}^{\circ})$ (which coincides with the symmetric monoidal structure given by Proposition 4.1.3.10).

We conclude this section by giving proofs of Propositions 2.4.1.6 and 2.4.1.7.

Proof of Proposition 2.4.1.6. We define a subcategory $\mathcal{I} \subseteq \mathcal{F}\mathrm{in}_* \times[1]$ as follows:

(a) Every object of $\mathcal{F}\mathrm{in}_* \times[1]$ belongs to $\mathcal{I}$.

(b) A morphism $(\langle n \rangle, i) \to (\langle n' \rangle, i')$ in $\mathcal{F}\mathrm{in}_* \times[1]$ belongs to $\mathcal{I}$ if and only if either $i' = 1$ or the induced map $\alpha : \langle n \rangle \to \langle n' \rangle$ satisfies $\alpha^{-1}\{*\} = *$.

Let $\mathcal{C}'$ denote the fiber product $\mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\mathcal{I})$, which we regard as a subcategory of $\mathcal{C}^{\otimes} \times \Delta^1$, and let $p' : \mathcal{C}' \to \mathrm{N}(\mathcal{I})$ denote the projection. Let $\mathcal{C}'_0$ and $\mathcal{C}'_1$ denote the intersections of $\mathcal{C}'$ with $\mathcal{C}^{\otimes} \times\{0\}$ and $\mathcal{C}^{\otimes} \times\{1\}$, respectively. We note that there is a canonical isomorphism $\mathcal{C}'_1 \simeq \mathcal{C}^{\otimes}$.

Let $\mathcal{E}$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}', \mathcal{D})$ spanned by those functors F which satisfy the following conditions:

(i) For every object $C \in \mathcal{C}^{\otimes}$, the induced map $F(C, 0) \to F(C, 1)$ is an equivalence in $\mathcal{D}$.

(ii) The restriction $F| \mathcal{C}'_1$ is a weak Cartesian structure on $\mathcal{C}^{\otimes}$.

It is clear that if (i) and (ii) are satisfied, then the restriction $F_0 = F| \mathcal{C}'_0$ satisfies the following additional conditions:

(iii) The restriction $F_0| \, \mathcal{C}^{\otimes}_{\langle 1 \rangle} \times \{0\}$ is a functor from $\mathcal{C}$ to $\mathcal{D}$ which preserves finite products.

(iv) For every p'-coCartesian morphism α in $\mathcal{C}'_0$, the induced map $F_0(\alpha)$ is an equivalence in $\mathcal{D}$.

Moreover, (i) is equivalent to the assertion that F is a right Kan extension of $F| \, \mathcal{C}'_1$. Proposition T.4.3.2.15 implies that the restriction map $r : \mathcal{E} \to \mathrm{Fun}^{\times}(\mathcal{C}^{\otimes}, \mathcal{D})$ induces a trivial Kan fibration onto its essential image. The map r has a section s, given by composition with the projection map $\mathcal{C}' \to \mathcal{C}^{\otimes}$. The restriction map $\mathrm{Fun}^{\times}(\mathcal{C}^{\otimes}, \mathcal{D}) \to \mathrm{Fun}^{\times}(\mathcal{C}, \mathcal{D})$ factors as a composition

$$\mathrm{Fun}^{\times}(\mathcal{C}^{\otimes}, \mathcal{D}) \xrightarrow{s} \mathcal{E} \xrightarrow{e} \mathrm{Fun}^{\times}(\mathcal{C}, \mathcal{D}),$$

where e is induced by composition with the inclusion $\mathcal{C} \subseteq \mathcal{C}'_0 \subseteq \mathcal{C}'$. Consequently, it will suffice to prove that e is an equivalence of ∞-categories.

Let $\mathcal{E}_0 \subseteq \mathrm{Fun}(\mathcal{C}'_0, \mathcal{D})$ be the full subcategory spanned by those functors which satisfy conditions (iii) and (iv). The map e factors as a composition

$$\mathcal{E} \xrightarrow{e'} \mathcal{E}_0 \xrightarrow{e''} \mathrm{Fun}^{\times}(\mathcal{C}, \mathcal{D}).$$

Consequently, it will suffice to show that e' and e'' are trivial Kan fibrations.

Let $f : \mathcal{C}'_0 \to \mathcal{D}$ be an arbitrary functor, and let $C \in \mathcal{C}^{\otimes}_{\langle n \rangle} \subseteq \mathcal{C}'_0$. There exists a unique map $\alpha : (\langle n \rangle, 0) \to (\langle 1 \rangle, 0)$ in $\mathcal{I}$; choose a p'-coCartesian morphism $\overline{\alpha} : C \to C'$ lifting α. We observe that C' is an initial object of $\mathcal{C} \times_{(\mathcal{C}'_0)_{/C'}} \times_{\mathcal{C}'_0} \mathcal{C}$. Consequently, f is a right Kan extension of $f| \, \mathcal{C}$ at C if and only if $f(\overline{\alpha})$ is an equivalence. It follows that f satisfies (iv) if and only if f is a right Kan extension of $f| \, \mathcal{C}$. The same argument (and Lemma T.4.3.2.7) shows that every functor $f_0 : \mathcal{C} \to \mathcal{D}$ admits a right Kan extension to $\mathcal{C}'_0$. Applying Proposition T.4.3.2.15, we deduce that e'' is a trivial Kan fibration.

It remains to show that e' is a trivial Kan fibration. In view of Proposition T.4.3.2.15, it will suffice to prove the following pair of assertions, for every functor $f \in \mathcal{E}_0$:

(1) There exist a functor $F : \mathcal{C}' \to \mathcal{D}$ which is a left Kan extension of $f = F| \, \mathcal{C}'_0$.

(2) An arbitrary functor $F : \mathcal{C}' \to \mathcal{D}$ which extends f is a left Kan extension of f if and only if F belongs to $\mathcal{E}$.

For every finite linearly ordered set J, let J^+ denote the disjoint union $J \coprod \{\infty\}$, where ∞ is a new element larger than every element of J. Let $(C, 1) \in \mathcal{C}^{\otimes}_{J_*} \times \{1\} \subseteq \mathcal{C}'$. Since there exists a final object $1_{\mathcal{C}} \in \mathcal{C}$, the ∞-category $\mathcal{C}'_0 \times_{\mathcal{C}'} \mathcal{C}'_{/C}$ also has a final object, given by the map $\alpha : (C', 0) \to (C, 1)$, where $C' \in \mathcal{C}^{\otimes}_{J_*^+}$ corresponds, under the equivalence

$$\mathcal{C}^{\otimes}_{J_*^+} \simeq \mathcal{C} \times \mathcal{C}^{\otimes}_{J_*},$$

to the pair $(1_{\mathcal{C}}, C)$. We now apply Lemma T.4.3.2.13 to deduce (1), together with the following analogue of (2):

($2'$) An arbitrary functor $F : \mathcal{C}' \to \mathcal{D}$ which extends f is a left Kan extension of f if and only if, for every morphism $\alpha : (C', 0) \to (C, 1)$ as above, the induced map $F(C', 0) \to F(C, 1)$ is an equivalence in $\mathcal{D}$.

To complete the proof, it will suffice to show that F satisfies the conditions stated in ($2'$) if and only if $F \in \mathcal{E}$. We first prove the "if" direction. Let $\alpha : (C', 0) \to (C, 1)$ be as above; we wish to prove that $F(\alpha) : F(C', 0) \to F(C, 1)$ is an equivalence in $\mathcal{D}$. The map α factors as a composition

$$(C', 0) \xrightarrow{\alpha'} (C', 1) \xrightarrow{\alpha''} (C, 1).$$

Condition (i) guarantees that $F(\alpha')$ is an equivalence. Condition (ii) guarantees that $F(C', 1)$ is equivalent to a product $F(1_{\mathcal{C}}, 1) \times F(C, 1)$, and that $F(\alpha'')$ can be identified with the projection onto the second factor.

Moreover, since $1_{\mathcal{C}}$ is a final object of $\mathcal{C}$, condition (ii) also guarantees that $F(1_{\mathcal{C}}, 1)$ is a final object of $\mathcal{D}$. It follows that $F(\alpha'')$ is an equivalence, so that $F(\alpha)$ is an equivalence as desired.

Now let us suppose that F satisfies the condition stated in $(2')$. We wish to prove that $F \in \mathcal{E}$. Here we must invoke our assumption that the monoidal structure on $\mathcal{C}$ is Cartesian. We begin by verifying condition (i). Let $C \in \mathcal{C}_{J_*}^{\otimes}$ for some finite linearly ordered set J, and let $\alpha : (C', 0) \to (C, 1)$ be defined as above. Let $\beta : (J_*, 0) \to (J_*^+, 0)$ be the morphism in $\mathcal{I}$ induced by the inclusion $J \subseteq J^+$. Choose a p'-coCartesian morphism $\overline{\beta} : (C, 0) \to (C'', 0)$ lifting β. Since the final object $1_{\mathcal{C}} \in \mathcal{C}$ is also the unit object of $\mathcal{C}$, we can identify C'' with C'. The composition $(C, 0) \xrightarrow{\overline{\beta}} (C', 1) \xrightarrow{\alpha} (C, 1)$ is homotopic to the canonical map $\gamma : (C, 0) \to (C, 1)$ appearing in the statement of (i). Condition (iv) guarantees that $F(\overline{\beta})$ is an equivalence, and $(2')$ guarantees that $F(\alpha)$ is an equivalence. Using the two-out-of-three property, we deduce that $F(\gamma)$ is an equivalence, so that F satisfies (i).

To prove that F satisfies (ii), we must verify two conditions:

(ii_0) If $\overline{\beta} : (C, 1) \to (D, 1)$ is a p'-coCartesian morphism in $\mathcal{C}'$, and the underlying morphism $\beta : \langle m \rangle \to \langle n \rangle$ satisfies $\beta^{-1}\{*\} = \{*\}$, then $F(\overline{\beta})$ is an equivalence.

(ii_1) Let $C \in \mathcal{C}_{\langle n \rangle}^{\otimes}$, and choose p-coCartesian morphisms $\gamma_i : C \to C_j$ covering the maps $\rho^j : \langle n \rangle \to \langle 1 \rangle$. Then the maps γ_i exhibit $F(C, 1)$ as a product $\prod_{1 \leq j \leq n} F(C_j, 1)$ in the ∞-category $\mathcal{D}$.

Condition (ii_0) follows immediately from (i) and (iv). To prove (ii_1), we consider the maps $\alpha : (C', 0) \to (C, 1)$ and $\alpha_j : (C'_j, 0) \to (C_j, 1)$ which appear in the statement of $(2')$. For each $1 \leq j \leq n$, we have a commutative diagram

$$
\begin{array}{ccc}
(C', 0) & \xrightarrow{\ \alpha\ } & (C, 1) \\
\downarrow{\scriptstyle \gamma'_j} & & \downarrow{\scriptstyle \gamma_j} \\
(C'_j, 0) & \xrightarrow{\ \alpha_j\ } & (C_j, 1).
\end{array}
$$

Condition $(2')$ guarantees that the maps $F(\alpha)$ and $F(\alpha_i)$ are equivalences in $\mathcal{D}$. Consequently, it will suffice to show that the maps $f(\gamma'_i)$ exhibit $f(C', 0)$ as a product $\prod_{j \in J} f(C'_j, 0)$ in $\mathcal{D}$. Let $f_0 = f | \mathcal{C}$. Using condition (iv), we obtain canonical equivalences

$$
f(C', 0) \simeq f_0(1_{\mathcal{C}} \otimes \bigotimes_{j \in J} C_j) \qquad f(C'_j, 0) \simeq f_0(1_{\mathcal{C}} \otimes C_j)
$$

Since condition (iii) guarantees that f_0 preserves products, it will suffice to show that the canonical map

$$
1_{\mathcal{C}} \otimes (\bigotimes_{1 \leq j \leq n} C_j) \to \bigotimes_{1 \leq j \leq n} (1_{\mathcal{C}} \otimes C_j)
$$

is an equivalence in the ∞-category $\mathcal{C}$. This follows easily from our assumption that the symmetric monoidal structure on $\mathcal{C}$ is Cartesian, using induction on n. $\qquad \square$

Proof of Proposition 2.4.1.7. Unwinding the definitions, we can identify $\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})$ with the full subcategory of $\mathrm{Fun}(\mathcal{O}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\Gamma^{\times}), \mathcal{D})$ spanned by those functors F which satisfy the following conditions:

(1) For every object $C \in \mathcal{O}_{\langle n \rangle}^{\otimes}$ and every subset $S \subseteq \langle n \rangle^{\circ}$, the functor F induces an equivalence

$$
F(C, S) \to \prod_{j \in S} F(C, \{j\})
$$

in the ∞-category $\mathcal{D}$.

(2) For every inert morphism $C \to C'$ in $\mathcal{O}^{\otimes}$ which covers $\langle n \rangle \to \langle n' \rangle$ and every subset $S \subseteq \langle n' \rangle^{\circ}$, the induced map $F(C, \alpha^{-1}S) \to F(C', S)$ is an equivalence in $\mathcal{D}$.

The functor $F' = \pi \circ F$ can be described by the formula $F'(C) = F(C, \langle n \rangle^\circ)$, for each $C \in \mathcal{O}_{\langle n \rangle}^\otimes$. In other words, F' can be identified with the restriction of F to the full subcategory $\mathcal{C} \subseteq \mathcal{O}^\otimes \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\Gamma^\times)$ spanned by objects of the form $(C, \langle n \rangle^\circ)$.

Let $X = (C, S)$ be an object of the fiber product $\mathcal{O}^\otimes \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\Gamma^\times)$. Here $C \in \mathcal{O}_{\langle n \rangle}^\otimes$ and $S \subseteq \langle n \rangle^\circ$. We claim that the ∞-category $\mathcal{C}_{X/}$ has an initial object. More precisely, if we choose a p-coCartesian morphism $\overline{\alpha} : C \to C'$ covering the map $\alpha : \langle n \rangle \to S_*$ given by the formula

$$\alpha(j) = \begin{cases} j & \text{if } j \in S \\ * & \text{otherwise,} \end{cases}$$

then the induced map $\widetilde{\alpha} : (C, S) \to (C', S)$ is an initial object of $\mathcal{C}_{X/}$. It follows that every functor $F' : \mathcal{C} \to \mathcal{D}$ admits a right Kan extension to $\mathcal{O}^\otimes \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\Gamma^\times)$, and that an arbitrary functor $F : \mathcal{O}^\otimes \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\Gamma^\times) \to \mathcal{D}$ is a right Kan extension of $F | \mathcal{C}$ if and only if $F(\widetilde{\alpha})$ is an equivalence, for every $\widetilde{\alpha}$ defined as above.

Let $\mathcal{E}$ be the full subcategory of $\mathrm{Fun}(\mathcal{O}^\otimes \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\Gamma^\times), \mathcal{D})$ spanned by those functors F which satisfy the following conditions:

(1') The restriction $F' = F | \mathcal{C}$ is a lax Cartesian structure on $\mathcal{O}^\otimes \simeq \mathcal{C}$.

(2') The functor F is a right Kan extension of F'.

Using Proposition T.4.3.2.15, we conclude that the restriction map $\mathcal{E} \to \mathrm{Fun}^{\mathrm{lax}}(\mathcal{O}^\otimes, \mathcal{D})$ is a trivial fibration of simplicial sets. To prove that θ is a trivial Kan fibration, it will suffice to show that conditions (1) and (2) are equivalent to conditions (1') and (2').

Suppose first that (1') and (2') are satisfied by a functor F. Condition then (1) follows easily; we will prove (2). Choose a map $C \to C'$ covering a inert morphism $\langle n \rangle \to \langle n' \rangle$ in $\mathcal{F}\mathrm{in}_*$, and let $S \subseteq \langle n' \rangle^\circ$. Define another inert morphism $\alpha : \langle n' \rangle \to S_*$ by the formula

$$\alpha(j) = \begin{cases} j & \text{if } j \in S \\ * & \text{otherwise,} \end{cases}$$

and choose a p-coCartesian morphism $C' \to C''$ lifting α. Condition (2') implies that the maps $F(C, \alpha^{-1}S) \to F(C'', S)$ and $F(C', S) \to F(C'', S)$ are equivalences in $\mathcal{D}$. Using the two-out-of-three property, we deduce that the map $F(C, \alpha^{-1}S) \to F(C', S)$ is likewise an equivalence in $\mathcal{D}$. This proves (2).

Now suppose that (1) and (2) are satisfied by F. The implication $(2) \Rightarrow (2')$ is obvious; it will therefore suffice to verify (1'). Let C be an object of $\mathcal{O}_{\langle n \rangle}^\otimes$, and choose p-coCartesian morphisms $g_j : C \to C_j$ covering the inert morphisms $\rho^j : \langle n \rangle \to \langle 1 \rangle$ for $1 \leq j \leq n$. We wish to show that the induced map $F(C, \langle n \rangle^\circ) \to \prod_{1 \leq j \leq n} F(C_j, \langle 1 \rangle^\circ)$ is an equivalence in $\mathcal{D}$, which follows immediately from (1) and (2'). This completes the proof that θ is a trivial Kan fibration.

Now suppose that $\mathcal{O}^\otimes$ is a symmetric monoidal ∞-category. To prove that θ_0 is a trivial Kan fibration, it will suffice to show that θ_0 is a pullback of θ. In other words, it will suffice to show that if $F : \mathcal{O}^\otimes \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\Gamma^\times) \to \mathcal{D}$ is a functor satisfying conditions (1) and (2), then $F | \mathcal{C}$ is a weak Cartesian structure on $\mathcal{O}^\otimes \simeq \mathcal{C}$ if and only if F determines a symmetric monoidal functor from $\mathcal{O}^\otimes$ into $\mathcal{D}^\times$. Let $q : \mathcal{D}^\times \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ denote the projection. Using the description of the class of q-coCartesian morphisms provided by Proposition 2.4.1.5, we see that the latter condition is equivalent to

(a) For every p-coCartesian morphism $\overline{\alpha} : C \to C'$ in $\mathcal{O}^\otimes$ covering a map $\alpha : \langle n \rangle \to \langle n' \rangle$ in $\mathcal{F}\mathrm{in}_*$, and every $S \subseteq \langle n' \rangle^\circ$, the induced map $F(C, \alpha^{-1}(S)) \to F(C, S)$ is an equivalence in $\mathcal{D}$.

Moreover, $F | \mathcal{C}$ is a weak Cartesian structure if and only if F satisfies the following:

(b) For each $n \geq 0$ and every p-coCartesian morphism $\overline{\beta} : C \to C'$ in $\mathcal{O}^\otimes$ lifting the map $\beta : \langle n \rangle \to \langle 1 \rangle$ such that $\beta^{-1}\{*\} = \{*\}$, the induced map $F(C, \langle n \rangle^\circ) \to F(C', \langle 1 \rangle^\circ)$ is an equivalence in $\mathcal{D}$.

It is clear that (a) implies (b). Conversely, suppose that (b) is satisfied, and let $\overline{\alpha}$ and $S \subseteq \langle n \rangle^\circ$ be as in the statement of (a). Choose p-coCartesian morphisms $\overline{\gamma} : C \to C_0$, $\overline{\gamma'} : C' \to C'_0$, $\overline{\beta} : C'_0 \to C''_0$ covering the maps $\gamma : \langle n \rangle \to (\alpha^{-1}S)_*$, $\gamma' : \langle n' \rangle \to S_*$, $\beta : \langle n' \rangle \to \langle 1 \rangle$ described by the formulas

$$\beta(j) = \begin{cases} 1 & \text{if } j \in S \\ * & \text{if } j = * \end{cases}$$

$$\gamma(j) = \begin{cases} j & \text{if } j \in \alpha^{-1}S \\ * & \text{otherwise} \end{cases}$$

$$\gamma'(j) = \begin{cases} 1 & \text{if } j \in S \\ * & \text{otherwise.} \end{cases}$$

We have a commutative diagram

$$
\begin{array}{ccc}
F(C, \alpha^{-1}S) & \longrightarrow & F(C', S) \\
\downarrow & & \downarrow \\
F(C_0, \alpha^{-1}S) \xrightarrow{\ g\ } & F(C'_0, S) \xrightarrow{\ g'\ } & F(C''_0, \{0\})
\end{array}
$$

Condition (b) implies that g' and $g' \circ g$ are equivalences, so that g is an equivalence by the two-out-of-three property. Condition $(2')$ implies that the vertical maps are equivalences, so that the upper horizontal map is also an equivalence, as desired. $\qquad\square$

2.4.2 Monoid Objects

At the beginning of this chapter, we reviewed the notion of a commutative monoid: that is, a set M equipped with a multiplication $M \times M \to M$ which is commutative, associative and unital. If $\mathcal{C}$ is a category which admits finite products, one can consider commutative monoids in $\mathcal{C}$: that is, objects $M \in \mathcal{C}$ equipped with unit and multiplication maps

$$* \to M \qquad M \times M \to M$$

satisfying the usual axioms, where $*$ denotes a final object of $\mathcal{C}$. In this section, we would like to generalize still further: if $\mathcal{C}$ is an ∞-category which admits finite products, then we should be able to define a new ∞-category $\mathrm{Mon}_{\mathrm{Comm}}(\mathcal{C})$ of commutative monoid objects of $\mathcal{C}$. Our definition will have the following features:

(a) If $\mathcal{C} = \mathrm{Cat}_\infty$ is the ∞-category of (small) ∞-categories, then a commutative monoid object of $\mathcal{C}$ is essentially the same thing as a symmetric monoidal ∞-category (in the sense of Definition 2.0.0.7).

(b) If we regard $\mathcal{C}$ as endowed with the Cartesian symmetric monoidal structure of §2.4.1, we have a canonical equivalence $\mathrm{Mon}_{\mathrm{Comm}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathrm{Comm}}(\mathcal{C})$.

In fact, (b) suggests that for any ∞-operad $\mathcal{O}^\otimes$, we can define an ∞-category $\mathrm{Mon}_\mathcal{O}(\mathcal{C})$ by the formula $\mathrm{Mon}_\mathcal{O}(\mathcal{C}) = \mathrm{Alg}_\mathcal{O}(\mathcal{C})$, where $\mathcal{C}$ is endowed with the Cartesian symmetric monoidal structure. However, for many purposes it is convenient to have a more direct description of $\mathrm{Mon}_\mathcal{O}(\mathcal{C})$ which does not make use of the theory of ∞-operads. We instead take our cue from (a). According to Definition 2.0.0.7, a symmetric monoidal ∞-category is a coCartesian fibration $\mathcal{C}^\otimes \to \mathrm{Comm}^\otimes$ satisfying certain conditions. Such a fibration is classified by a map $\chi : \mathrm{Comm}^\otimes \to \mathrm{Cat}_\infty$. This suggests the following definition:

Definition 2.4.2.1. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{O}^\otimes$ be an ∞-operad. A $\mathcal{O}$-*monoid* in $\mathcal{C}$ is a functor $M : \mathcal{O}^\otimes \to \mathcal{C}$ with the following property: for every object $X \in \mathcal{O}^\otimes_{\langle n \rangle}$ corresponding to a sequence of objects $\{X_i \in \mathcal{O}\}_{1 \leq i \leq n}$, the canonical maps $M(X) \to M(X_i)$ exhibit $M(X)$ as a product $\prod_{1 \leq i \leq n} M(X_i)$ in the ∞-category $\mathcal{C}$. We let $\mathrm{Mon}_\mathcal{O}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{O}^\otimes, \mathcal{C})$ spanned by the $\mathcal{O}$-monoids in $\mathcal{C}$.

Remark 2.4.2.2. In the special case where $\mathcal{O}^{\otimes}$ is the commutative ∞-operad, we will refer to $\mathcal{O}$-monoids in an ∞-category $\mathcal{C}$ as *commutative monoid objects* of $\mathcal{C}$. These objects might also be referred to as Γ-*objects* of $\mathcal{C}$; in the special case where $\mathcal{C}$ is the ∞-category of spaces, the theory of Γ objects is essentially equivalent to Segal's theory of Γ-spaces.

Remark 2.4.2.3. Let $\mathcal{C}$ be an ∞-category and let $F : \mathrm{Comm}^{\otimes} \simeq \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathcal{C}$ be a commutative monoid object of $\mathcal{C}$. It follows from Definition 2.4.2.1 that for each $n \geq 0$, the object $F(\langle n \rangle) \in \mathcal{C}$ can be identified with the n-fold product of $M = F(\langle 1 \rangle)$ with itself. The unique active morphism $\langle n \rangle \to \langle 1 \rangle$ then corresponds to a map $M^n \to M$. In particular, taking $n = 2$, we get a multiplication map $m : M \times M \simeq F(\langle 2 \rangle) \to F(\langle 1 \rangle) \simeq M$ in $\mathcal{C}$. It is not difficult to see that the multiplication m is commutative and associative up to homotopy: in fact, the existence of the functor F is an expression of the idea that m is commutative, associative, and unital, up to coherent homotopy.

The construction $F \mapsto M = F(\langle 1 \rangle)$ determines a forgetful functor $\mathrm{Mon}_{\mathrm{Comm}}(\mathcal{C}) \to \mathcal{C}$. We will often abuse notation by identifying M with F.

Example 2.4.2.4. Let $\mathcal{O}^{\otimes}$ be an ∞-operad. A functor $M : \mathcal{O}^{\otimes} \to \mathcal{C}\mathrm{at}_{\infty}$ is a $\mathcal{O}$-monoid in $\mathcal{C}\mathrm{at}_{\infty}$ if and only if the coCartesian fibration $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ classified by M is a $\mathcal{O}$-monoidal ∞-category.

We now compare the theory of $\mathcal{O}$-monoids with the theory of $\mathcal{O}$-algebras:

Proposition 2.4.2.5. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category, $\pi : \mathcal{C}^{\otimes} \to \mathcal{D}$ a Cartesian structure, and $\mathcal{O}^{\otimes}$ an ∞-operad. Then composition with π induces an equivalence of ∞-categories $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{D})$.*

Proof. As in the proof of Corollary 2.4.1.9, we may assume without loss of generality that $\mathcal{C}^{\otimes} = \mathcal{D}^{\times}$. We now apply Proposition 2.4.1.7 again to deduce that the map

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}^{\mathrm{lax}}(\mathcal{O}^{\otimes}, \mathcal{D}) = \mathrm{Mon}_{\mathcal{O}}(\mathcal{D})$$

is a trivial Kan fibration. $\square$

Remark 2.4.2.6. Let $\mathcal{O}^{\otimes}$ be an ∞-operad. Combining Proposition 2.4.2.5 with Example 2.4.2.4, we see that $\mathcal{O}$-monoidal ∞-categories can be identified with $\mathcal{O}$-algebra objects of $\mathcal{C}\mathrm{at}_{\infty}$ (where we endow the latter with the Cartesian monoidal structure). More precisely, we have a canonical equivalence of ∞-categories $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_{\infty}) \simeq \mathcal{C}\mathrm{at}_{\infty}^{\mathcal{O}}$, where $\mathcal{C}\mathrm{at}_{\infty}^{\mathcal{O}}$ denotes the subcategory of $(\mathrm{Op}_{\infty})_{/\mathcal{O}^{\otimes}}$ spanned by the $\mathcal{O}$-monoidal ∞-categories and $\mathcal{O}$-monoidal functors between them. This observation will play a vital role in Chapter 4.

Remark 2.4.2.7. Let $\mathcal{C}\mathrm{at}_{\infty}^{\Delta}$ denote the simplicial category whose objects are small ∞-categories, where $\mathrm{Map}_{\mathcal{C}\mathrm{at}_{\infty}^{\Delta}}(\mathcal{C}, \mathcal{D})$ is the Kan complex $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}$. Let $|\mathcal{C}\mathrm{at}_{\infty}^{\Delta}|$ denote the geometric realization of $\mathcal{C}\mathrm{at}_{\infty}^{\Delta}$: that is, the topological category with the same objects, but with morphisms given by $\mathrm{Map}_{|\mathcal{C}\mathrm{at}_{\infty}^{\Delta}|}(\mathcal{C}, \mathcal{D}) = |\mathrm{Map}_{\mathcal{C}\mathrm{at}_{\infty}^{\Delta}}(\mathcal{C}, \mathcal{D})|$. Let $\mathcal{C}\mathrm{at}_{\infty}'$ denote the nerve of the topological category $|\mathcal{C}\mathrm{at}_{\infty}^{\Delta}|$, so we have an equivalence of ∞-categories $\mathcal{C}\mathrm{at}_{\infty} = \mathrm{N}(\mathcal{C}\mathrm{at}_{\infty}^{\Delta}) \to \mathrm{N}(|\mathcal{C}\mathrm{at}_{\infty}^{\Delta}|) = \mathcal{C}\mathrm{at}_{\infty}'$. Using the existence of the canonical homeomorphism $|K| \simeq |K^{op}|$ for every simplicial set K, we deduce the existence of an involution R on the ∞-category $\mathcal{C}\mathrm{at}_{\infty}'$, which carries each ∞-category $\mathcal{C}$ to its opposite ∞-category $\mathcal{C}^{op}$.

The definition of a symmetric monoidal ∞-category is not manifestly self-dual. However, it is nevertheless true that any symmetric monoidal structure on an ∞-category $\mathcal{C}$ determines a symmetric monoidal structure on $\mathcal{C}^{op}$, which is unique up to contractible ambiguity. Roughly speaking, we can use Example 2.4.2.4 to identify symmetric monoidal ∞-categories $\mathcal{C}$ with commutative monoids $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathcal{C}\mathrm{at}_{\infty} \simeq \mathcal{C}\mathrm{at}_{\infty}'$. We can then obtain a new commutative monoid object by composing with the self-equivalence $R : \mathcal{C}\mathrm{at}_{\infty}' \to \mathcal{C}\mathrm{at}_{\infty}'$, which carries each ∞-category to its opposite. More informally: composition with the self-equivalence R allows us to pass between symmetric monoidal structures on an ∞-category $\mathcal{C}$ and symmetric monoidal structures on the opposite ∞-category $R(\mathcal{C}) = \mathcal{C}^{op}$.

2.4.3 CoCartesian Symmetric Monoidal Structures

Let $\mathcal{C}$ be an ∞-category which admits finite coproducts. Then the opposite ∞-category $\mathcal{C}^{op}$ admits finite products, and can therefore be endowed with a Cartesian symmetric monoidal structure, which is unique up to equivalence (see §2.4.1). Using Remark 2.4.2.7, we deduce that the ∞-category $\mathcal{C}$ inherits a symmetric monoidal structure, which is determined uniquely up to equivalence by the requirement that it be *coCartesian* (in the sense of Definition 2.4.0.1). Our goal in this section is to give an explicit construction of this symmetric monoidal structure on $\mathcal{C}$. More generally, we will show that *any* ∞-category $\mathcal{C}$ can be regarded as the underlying ∞-category of an ∞-operad $\mathcal{C}^{\amalg}$, where the morphism spaces are described informally by the formula

$$\mathrm{Mul}_{\mathcal{C}^{\amalg}}(C_1 \oplus \cdots \oplus C_n, D) \simeq \prod_{1 \leq i \leq n} \mathrm{Map}_{\mathcal{C}}(C_i, D).$$

We now define the ∞-operad $\mathcal{C}^{\amalg}$ more precisely.

Construction 2.4.3.1. We define a category Γ^* as follows:

(1) The objects of Γ^* are pairs $(\langle n \rangle, i)$ where $i \in \langle n \rangle^{\circ}$.

(2) A morphism in Γ^* from $(\langle m \rangle, i)$ to $(\langle n \rangle, j)$ is a map of pointed sets $\alpha : \langle m \rangle \to \langle n \rangle$ such that $\alpha(i) = j$.

Let $\mathcal{C}$ be any simplicial set. We define a new simplicial set $\mathcal{C}^{\amalg}$ equipped with a map $\mathcal{C}^{\amalg} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ so that the following universal property is satisfied: for every map of simplicial sets $K \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, we have a canonical bijection

$$\mathrm{Hom}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(K, \mathcal{C}^{\amalg}) \simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(K \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\Gamma^*), \mathcal{C}).$$

Remark 2.4.3.2. If $\mathcal{C}$ is a simplicial set, then each fiber $\mathcal{C}^{\amalg}_{\langle n \rangle} = \mathcal{C}^{\amalg} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \{\langle n \rangle\}$ can be identified with $\mathcal{C}^n$; we will henceforth invoke these identifications implicitly.

Proposition 2.4.3.3. *Let $\mathcal{C}$ be an ∞-category. Then the map $p : \mathcal{C}^{\amalg} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ of Construction 2.4.3.1 is an ∞-operad.*

Proof. We first show that p is an inner fibration of simplicial sets. Suppose we are given a lifting problem

$$\begin{array}{ccc} \Lambda^n_i & \xrightarrow{\ f_0\ } & \mathcal{C}^{\amalg} \\ \downarrow & \nearrow^{f} & \downarrow \\ \Delta^n & \longrightarrow & \mathrm{N}(\mathcal{F}\mathrm{in}_*) \end{array}$$

where $0 < i < n$. The lower horizontal map determines a sequence of maps $\langle k_0 \rangle \to \ldots \to \langle k_n \rangle$ in $\mathcal{F}\mathrm{in}_*$. Unwinding the definitions, we see that finding the desired extension f of f_0 is equivalent to the problem of solving a series of extension problems

$$\begin{array}{ccc} \Lambda^n_i & \xrightarrow{\ f_0^j\ } & \mathcal{C} \\ \downarrow & \nearrow^{f^j} & \\ \Delta^n & & \end{array}$$

indexed by those elements $j \in \langle k_0 \rangle^{\circ}$ whose image in $\langle k_n \rangle$ belongs to $\langle k_n \rangle^{\circ}$. These extensions exist by virtue of the assumption that $\mathcal{C}$ is an ∞-category. If $i = 0$ and the map $\langle k_0 \rangle \to \langle k_1 \rangle$ is inert, then the same argument applies: we conclude that the desired extension of f exists provided that $n \geq 2$ and f_0^j carries $\Delta^{\{0,1\}}$ to an equivalence in $\mathcal{C}$.

Unwinding the definitions, we see that an object of $\mathcal{C}^{\amalg}$ consists of an object $\langle n \rangle \in \mathcal{F}\mathrm{in}_*$ together with a sequence of objects $(C_1, \ldots, C_n)$ in $\mathcal{C}$. A morphism f from $(C_1, \ldots, C_m)$ to $(C'_1, \ldots, C'_n)$ in $\mathcal{C}^{\amalg}$ consists of a

map of pointed sets $\alpha : \langle m \rangle \to \langle n \rangle$ together with a sequence of morphisms $\{f_i : C_i \to C'_{\alpha(i)}\}_{i \in \alpha^{-1}\langle n \rangle^\circ}$. The above argument shows that f is p-coCartesian if α is inert and each of the maps f_i is an equivalence in $\mathcal{C}$. In particular (taking each f_i to be the identity map), we deduce that for every object $C \in \mathcal{C}^{\mathrm{II}}_{\langle m \rangle}$ and every inert morphism $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$, there exists a p-coCartesian morphism $C \to C'$ in $\mathcal{C}^{\mathrm{II}}$ lifting σ.

Let $C = (C_1, \ldots, C_n)$ be an object of $\mathcal{C}^{\mathrm{II}}_{\langle n \rangle}$ and choose p-coCartesian morphisms $C \to C'_i$ covering ρ^i for $1 \leq i \leq n$, corresponding to equivalences $g_i : C_i \simeq C'_i$ in $\mathcal{C}$. These morphisms determine a diagram $\overline{q} : \langle n \rangle^{\circ \triangleleft} \to \mathcal{C}^{\mathrm{II}}$; we must show that $\overline{q}$ is a p-limit diagram. To prove this, we must show that it is possible to solve lifting problems of the form

$$
\begin{array}{ccc}
\partial\,\Delta^m \star \langle n \rangle^\circ & \xrightarrow{\;f_0\;} & \mathcal{C}^{\mathrm{II}} \\
\downarrow & \overset{f}{\nearrow} & \downarrow \\
\Delta^m \star \langle n \rangle^\circ & \longrightarrow & \mathrm{N}(\mathcal{F}\mathrm{in}_*)
\end{array}
$$

provided that $f_0|(\{m\} \star \langle n \rangle^\circ)$ is given by $\overline{q}$. Unwinding the definitions, we see that this is equivalent to solving a collection of extension problems of the form

$$
\begin{array}{ccc}
\Lambda^{m+1}_{m+1} & \xrightarrow{\;f'_0\;} & \mathcal{C} \\
\downarrow & \overset{f'}{\nearrow} & \\
\Delta^{m+1}, &
\end{array}
$$

where f'_0 carries the final edge of Δ^{m+1} to one of the morphisms g_i. This is possible by virtue of our assumption that each g_i is an equivalence.

To complete the proof that $\mathcal{C}^{\mathrm{II}}$ is an ∞-operad, it suffices to show that for each $n \geq 0$, the functors $\rho^i_! : \mathcal{C}^{\mathrm{II}}_{\langle n \rangle} \to \mathcal{C}^{\mathrm{II}}_{\langle 1 \rangle}$ induce an equivalence $\theta : \mathcal{C}^{\mathrm{II}}_{\langle n \rangle} \to \prod_{1 \leq i \leq n} \mathcal{C}^{\mathrm{II}}_{\langle 1 \rangle}$. In fact, we have canonical isomorphisms of simplicial sets $\mathcal{C}^{\mathrm{II}}_{\langle n \rangle} \simeq \mathcal{C}^n$ which allow us to identify θ with $\mathrm{id}_{\mathcal{C}^n}$. $\qquad \square$

Remark 2.4.3.4. Unwinding the definitions, we deduce that a map $(C_1, \ldots, C_m) \to (C'_1, \ldots, C'_n)$ in $\mathcal{C}^{\mathrm{II}}$ covering a map $\alpha : \langle m \rangle \to \langle n \rangle$ is p-coCartesian if and only if for each $j \in \langle n \rangle^\circ$, the underlying maps $\{f_i : C_i \to C'_j\}_{\alpha(i)=j}$ exhibit C'_j as a coproduct of $\{C_i\}_{\alpha(i)=j}$ in the ∞-category $\mathcal{C}$. It follows that $\mathcal{C}^{\mathrm{II}}$ is a symmetric monoidal ∞-category if and only if $\mathcal{C}$ admits finite coproducts. If this condition is satisfied, then $\mathcal{C}^{\mathrm{II}}$ determines a coCartesian symmetric monoidal structure on $\mathcal{C}$ and is therefore determined by $\mathcal{C}$ up to essentially unique equivalence. We will see that the situation is similar even if $\mathcal{C}$ does not admit finite coproducts.

Example 2.4.3.5. The projection map $\mathrm{N}(\Gamma^*) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ induces a canonical map $\mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathcal{C}^{\mathrm{II}}$. If $\mathcal{C} = \Delta^0$, this map is an isomorphism (so that $\mathcal{C}^{\mathrm{II}}$ is the commutative ∞-operad $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$). For any ∞-operad $\mathcal{O}^\otimes$, we obtain a map $\mathcal{C} \times \mathcal{O}^\otimes \to \mathcal{C}^{\mathrm{II}} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathcal{O}^\otimes$ which determines a functor $\mathcal{C} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{A})$, where $\mathcal{A}^\otimes$ is the ∞-operad $\mathcal{C}^{\mathrm{II}} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathcal{O}^\otimes$.

Remark 2.4.3.6. Let $\mathcal{C}$ be an ∞-category, and let $\gamma : \mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathcal{C}^{\mathrm{II}}$ be the canonical map, where $\mathcal{C}^{\mathrm{II}}$. Then γ is an approximation to the ∞-operad $\mathcal{C}^{\mathrm{II}}$. Unwinding the definitions, this is equivalent to the observation that for every object $C \in \mathcal{C}$ and each $n \geq 0$, the ∞-category $\mathcal{C}^n_{/C}$ has a final object (given by $(C, C, \ldots, C)$).

Definition 2.4.3.7. We will say that an ∞-operad $\mathcal{O}^\otimes$ is *coCartesian* if it is equivalent to $\mathcal{C}^{\mathrm{II}}$, for some ∞-category $\mathcal{C}$.

The following result shows that a coCartesian ∞-operad $\mathcal{C}^\otimes$ is determined, in a very strong sense, by the underlying ∞-category $\mathcal{C}$:

Proposition 2.4.3.8. *Let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be coCartesian ∞-operads. Then the restriction functor $\mathrm{Alg}_{\mathcal{C}}(\mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is an equivalence of ∞-categories.*

We observe that for every ∞-category $\mathcal{C}$, the ∞-operad $\mathcal{C}^{\mathrm{II}}$ is unital. Consequently, Proposition 2.4.3.8 is a consequence of the following more general assertion:

Proposition 2.4.3.9. *Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad and let $\mathcal{C}^{\otimes}$ be a coCartesian ∞-operad. Then the restriction functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{O}, \mathcal{C})$ is an equivalence of ∞-categories.*

Corollary 2.4.3.10. *Let $\mathcal{C}$ be an ∞-category, which we regard as the underlying ∞-category of the ∞-operad $\mathcal{C}^{\mathrm{II}}$. Then the construction of Example 2.4.3.5 induces an equivalence of ∞-categories $\mathcal{C} \to \mathrm{CAlg}(\mathcal{C})$.*

Corollary 2.4.3.11. *Let $\mathrm{Op}_{\infty}^{\mathrm{II}}$ denote the full subcategory of Op_{∞} spanned by the coCartesian ∞-operads, and let $\theta : \mathrm{Op}_{\infty}^{\mathrm{II}} \to \mathcal{C}\mathrm{at}_{\infty}$ denote the forgetful functor (given on objects by $\mathcal{O}^{\otimes} \mapsto \mathcal{O}$). Then θ is an equivalence of ∞-categories.*

Proof. It follows from Proposition 2.4.3.16 that θ is fully faithful. Construction 2.4.3.1 shows that θ is essentially surjective. $\qquad\square$

Variant 2.4.3.12. Let $\mathcal{C}\mathrm{at}_{\infty}^{\otimes}$ denote the ∞-category of symmetric monoidal ∞-categories and let $\mathcal{C}\mathrm{at}_{\infty}^{\otimes,\mathrm{II}} \subseteq \mathcal{C}\mathrm{at}_{\infty}^{\otimes}$ denote the full subcategory spanned by the coCartesian symmetric monoidal ∞-categories. Let $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{coCart}}$ denote the subcategory of $\mathcal{C}\mathrm{at}_{\infty}$ spanned by those ∞-categories which admit finite coproducts and those functors which preserve finite coproducts. Then the restriction functor $\mathcal{C}\mathrm{at}_{\infty}^{\otimes,\mathrm{II}} \to \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{coCart}}$ is an equivalence of ∞-categories. This can be deduced either from the equivalence of Corollary 2.4.3.11 (by identifying $\mathcal{C}\mathrm{at}_{\infty}^{\otimes,\mathrm{II}}$ and $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{coCart}}$ with subcategories of $\mathrm{Op}_{\infty}^{\mathrm{II}}$ and $\mathcal{C}\mathrm{at}_{\infty}$) or from the equivalence of Corollary 2.4.1.9 (by passing to opposite ∞-categories; see Remark 2.4.2.7).

For later use, we formulate an even more general version of Proposition 2.4.3.8.

Definition 2.4.3.13. Let $\mathcal{C}$ be an ∞-category. We will say that a $\mathcal{C}$-family of ∞-operads $q : \mathcal{O}^{\otimes} \to \mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is *unital* if every object of $\mathcal{O}_{\langle 0 \rangle}^{\otimes} = \mathcal{O}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \{\langle 0 \rangle\}$ is q-initial.

Remark 2.4.3.14. In the special case where $\mathcal{C} = \Delta^0$, the notion of a unital $\mathcal{C}$-family of ∞-operads coincides with the notion of unital ∞-operad introduced in Definition 2.3.1.1. More generally, if $\mathcal{O}^{\otimes} \to \mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a unital $\mathcal{C}$-family of ∞-operads, then for each $C \in \mathcal{C}$ the fiber $\mathcal{O}_C^{\otimes}$ is a unital ∞-operad. If $\mathcal{C}$ is a Kan complex, then the converse is true as well: a $\mathcal{C}$-family of ∞-operads is unital if and only if each fiber is unital.

Remark 2.4.3.15. Let $\mathcal{O}^{\otimes}$ be a generalized ∞-operad. Then $\mathcal{O}^{\otimes}$ is categorically equivalent to a $\mathcal{C}$-family of ∞-operads $\mathcal{O}'^{\otimes} \to \mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, where $\mathcal{C} = \mathcal{O}_{\langle 0 \rangle}^{\otimes}$ (see Corollary 2.3.2.13), which is uniquely determined up to equivalence. We will say that $\mathcal{O}^{\otimes}$ is *unital* if $\mathcal{O}'^{\otimes}$ is a unital $\mathcal{C}$-family of ∞-operads, in the sense of Definition 2.4.3.13.

Proposition 2.4.3.9 is an immediate consequence of the following:

Proposition 2.4.3.16. *Let $\mathcal{O}^{\otimes}$ be a unital generalized ∞-operad and let $\mathcal{C}^{\otimes}$ be a coCartesian ∞-operad. Then the restriction functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{O}, \mathcal{C})$ is an equivalence of ∞-categories.*

Proof. We may assume without loss of generality that $\mathcal{C}^{\otimes} = \mathcal{C}^{\mathrm{II}}$ for some ∞-category $\mathcal{C}$. Let $\mathcal{D}$ denote the fiber product

$$\mathcal{O}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\Gamma^*).$$

By definition, a map $\mathcal{O}^{\otimes} \to \mathcal{C}^{\mathrm{II}}$ in $(\mathcal{S}\mathrm{et}_{\Delta})_{/\mathrm{N}(\mathcal{F}\mathrm{in}_*)}$ can be identified with a functor $A : \mathcal{D} \to \mathcal{C}$. Such a functor determines a map of generalized ∞-operads if and only if the following condition is satisfied:

($*$) Let α be a morphism in $\mathcal{D}$ whose image in $\mathcal{O}^{\otimes}$ is inert. Then $A(\alpha)$ is an equivalence in $\mathcal{C}$.

We can identify $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ with the full subcategory of $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ spanned by those functors which satisfy $(*)$.

We observe that the inverse image in $\mathcal{D}$ of $\langle 1 \rangle \in \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is canonically isomorphic to $\mathcal{O}$. Via this isomorphism, we will regard $\mathcal{O}$ as a full subcategory of $\mathcal{D}$. In view of Proposition T.4.3.2.15, it will suffice to prove the following:

(a) A functor $A : \mathcal{D} \to \mathcal{C}$ is a left Kan extension of $A| \mathcal{O}$ if and only if it satisfies condition $(*)$.

(b) Every functor $A_0 : \mathcal{O} \to \mathcal{C}$ admits an extension $A : \mathcal{D} \to \mathcal{C}$ satisfying the equivalent conditions of (a).

We can identify objects of $\mathcal{D}$ with pairs (X, i), where $X \in \mathcal{O}^{\otimes}_{\langle n \rangle}$ and $1 \le i \le n$. For every such pair, choose an inert morphism $X \to X_i$ lying over the map $\rho^i : \langle n \rangle \to \langle 1 \rangle$. We then have a morphism $f : (X, i) \to (X_i, 1)$ in $\mathcal{D}$. Using the assumption that $\mathcal{O}^{\otimes}$ is unital, we deduce that the map

$$\mathrm{Map}_{\mathcal{D}}(Y, (X, i)) \to \mathrm{Map}_{\mathcal{D}}(Y, (X_i, 1)) \simeq \mathrm{Map}_{\mathcal{O}}(Y, X_i)$$

is a homotopy equivalence for each $Y \in \mathcal{O} \subseteq \mathcal{D}$. In particular, we conclude that f admits a right homotopy inverse $g : (X_i, 1) \to (X, i)$. It follows that composition with g induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{D}}(Y, (X_i, 1)) \to \mathrm{Map}_{\mathcal{D}}(Y, (X, i))$$

for each $Y \in \mathcal{O}$. This implies that the inclusion $\mathcal{O} \subseteq \mathcal{D}$ admits a right adjoint G, given by $(X, i) \mapsto (X_i, 1)$. This immediately implies (b) (we can take $A = A_0 \circ G$) together with the following version of (a):

(a′) A functor $A : \mathcal{D} \to \mathcal{C}$ is a left Kan extension of $A| \mathcal{O}$ if and only if, for every object $(X, i) \in \mathcal{D}$, the map $A(g)$ is an equivalence in $\mathcal{C}$, where $g : (X_i, 1) \to (X, i)$ is defined as above.

Since g is a right homotopy inverse to the inert morphism $f : (X, i) \to (X_i, 1)$, assertion $(a′)$ can be reformulated as follows: a functor $A : \mathcal{D} \to \mathcal{C}$ is a left Kan extension of $A| \mathcal{O}$ if and only if the following condition is satisfied:

(∗′) Let $(X, i) \in \mathcal{D}$ be an object, and let $f : (X, i) \to (X_i, 1)$ be defined as above. Then $A(f)$ is an equivalence in $\mathcal{C}$.

To complete the proof, it will suffice to show that conditions $(*)$ and $(*′)$ are equivalent. The implication $(*) \Rightarrow (*′)$ is obvious. For the converse, suppose that $h : (Y, j) \to (X, i)$ is an arbitrary morphism in $\mathcal{D}$ whose image in $\mathcal{O}^{\otimes}$ is inert. We then have a commutative diagram

$$
\begin{array}{ccc}
(Y, j) & \xrightarrow{\quad f \quad} & (X, i) \\
 & {\scriptstyle f'} \searrow \qquad \swarrow {\scriptstyle f''} & \\
 & (X_i, 1) &
\end{array}
$$

Condition $(*′)$ guarantees that $A(f')$ and $A(f'')$ are equivalences, so that $A(f)$ is an equivalence by the two-out-of-three property.   □

Remark 2.4.3.17. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Then $\mathcal{C}^{\otimes}$ is coCartesian as a symmetric monoidal ∞-category (in the sense of Definition A.5.12) if and only if it is coCartesian as an ∞-operad (in the sense of Definition 2.4.3.7). This follows from the uniqueness of coCartesian symmetric monoidal structures (combine Remark 2.4.2.7 with Corollary 2.4.1.8), since $\mathcal{C}^{\mathrm{II}}$ satisfies the requirements of Definition A.5.12.

Let $\mathcal{C}$ be a category which admits finite coproducts. The construction $(X, Y) \mapsto X \coprod Y$ endows $\mathcal{C}$ with the structure of a symmetric monoidal category. For every object $C \in \mathcal{C}$, the codiagonal $C \coprod C \to C$ exhibits C as a commutative algebra object of $\mathcal{C}$. Corollary 2.4.3.10 can be regarded as an ∞-categorical analogue

of this assertion: it guarantees that the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ is an equivalence of ∞-categories for any coCartesian ∞-operad $\mathcal{C}^\otimes$, and therefore admits a homotopy inverse $\mathcal{C} \to \mathrm{CAlg}(\mathcal{C})$. Our goal in this section is to prove a converse: namely, we will show that if $\mathcal{D}^\otimes$ is an arbitrary ∞-operad, then every functor $\mathcal{C} \to \mathrm{CAlg}(\mathcal{D})$ is induced by a map of ∞-operads $\mathcal{C}^{\amalg} \to \mathcal{D}^\otimes$. This is a consequence of the following more general assertion:

Theorem 2.4.3.18. *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{O}^\otimes$ and $\mathcal{D}^\otimes$ be ∞-operads, and let $\mathcal{O}'^\otimes$ denote the fiber product $\mathcal{C}^{\amalg} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathcal{O}^\otimes$. Then the construction of Example 2.4.3.5 induces a trivial Kan fibration*

$$\theta : \mathrm{Alg}_{\mathcal{O}'}(\mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{D})).$$

In particular (taking $\mathcal{O}^\otimes$ to be the commutative ∞-operad), we have a trivial Kan fibration $\mathrm{Alg}_{\mathcal{C}}(\mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathrm{CAlg}(\mathcal{D}))$.

Proof. Note that the map $f : \mathcal{C} \times \mathcal{O}^\otimes \to \mathcal{C}^{\amalg} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathcal{O}^\otimes$ induces an isomorphism after passing to the fiber over the object $\mathbf{1} \in \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. According to Theorem 2.3.3.23, it will suffice to show that f is an approximation to $\mathcal{C}^{\amalg} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathcal{O}^\otimes$. This follows from Remarks 2.3.3.19 and 2.4.3.6. □

We conclude this section with a simple criterion which is useful for establishing that a symmetric monoidal structure on an ∞-category is coCartesian:

Proposition 2.4.3.19. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. The following conditions are equivalent:*

(1) *The symmetric monoidal structure on $\mathcal{C}$ is coCartesian.*

(2) *The induced symmetric monoidal structure on the homotopy category $\mathrm{h}\mathcal{C}$ is coCartesian.*

(3) *The unit object $1_{\mathcal{C}}$ is initial, and for each object $C \in \mathcal{C}$ there exists a codiagonal map $\delta_C : C \otimes C \to C$ satisfying the following conditions:*

    (*i*) *Let C be an object of $\mathcal{C}$ and let $u : 1_{\mathcal{C}} \to C$ be a map (automatically unique up to homotopy). Then the composition*

$$C \simeq C \otimes 1_{\mathcal{C}} \xrightarrow{\mathrm{id}\,\otimes u} C \otimes C \xrightarrow{\delta_C} C$$

    is homotopic to the identity.

    (*ii*) *For every morphism $f : C \to D$ in $\mathcal{C}$, the diagram*

$$
\begin{array}{ccc}
C \otimes C & \xrightarrow{f \otimes f} & D \otimes D \\
\downarrow{\scriptstyle \delta_C} & & \downarrow{\scriptstyle \delta_D} \\
C & \xrightarrow{\quad f \quad} & D
\end{array}
$$

    commutes up to homotopy.

    (*iii*) *Let C and D be objects of $\mathcal{C}$. Then the diagram*

$$
\begin{array}{ccc}
(C \otimes C) \otimes (D \otimes D) & \xrightarrow{\quad\sim\quad} & (C \otimes D) \otimes (C \otimes D) \\
& {\scriptstyle \delta_C \otimes \delta_D} \searrow \quad \swarrow {\scriptstyle \delta_{C \otimes D}} & \\
& C \otimes D &
\end{array}
$$

    commutes up to homotopy.

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are obvious. Let us suppose that (3) is satisfied. We wish to show that, for every pair of objects $C, D \in \mathcal{C}$, the maps

$$C \simeq C \otimes 1_{\mathcal{C}} \to C \otimes D \leftarrow 1_{\mathcal{C}} \otimes D \simeq D$$

exhibit $C \otimes D$ as a coproduct of C and D in $\mathcal{C}$. In other words, we must show that for every object $A \in \mathcal{C}$, the induced map

$$\phi : \mathrm{Map}_{\mathcal{C}}(C \otimes D, A) \to \mathrm{Map}_{\mathcal{C}}(C, A) \times \mathrm{Map}_{\mathcal{C}}(D, A)$$

is a homotopy equivalence. Let ψ denote the composition

$$\mathrm{Map}_{\mathcal{C}}(C, A) \otimes \mathrm{Map}_{\mathcal{C}}(D, A) \xrightarrow{\otimes} \mathrm{Map}_{\mathcal{C}}(C \otimes D, A \otimes A) \xrightarrow{\delta_A} \mathrm{Map}_{\mathcal{C}}(C \otimes D, A).$$

We claim that ψ is a homotopy inverse to ϕ. The existence of a homotopy $\psi \circ \phi \simeq \mathrm{id}$ follows from (i). We will show that $\psi \circ \phi$ is homotopic to the identity. In view of condition (ii), $\psi \circ \phi$ is homotopic to the map defined by composition with

$$C \otimes D \simeq (C \otimes 1_{\mathcal{C}}) \otimes (1_{\mathcal{C}} \otimes D) \to (C \otimes D) \otimes (C \otimes D) \xrightarrow{\delta_{C \otimes D}} C \otimes D.$$

It follows from (iii) and (i) that this map is homotopic to the identity. $\square$

2.4.4   Wreath Products

In §2.2.5, we saw that every pair of ∞-operads $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ admit a *tensor product* $\mathcal{O}''^{\otimes}$, which is well-defined up to equivalence. However, it can be very difficult to describe this tensor product directly. By definition, it is given by a fibrant replacement for the product $\mathcal{O}^{\otimes,\natural} \odot \mathcal{O}'^{\otimes,\natural}$ in the category $\mathcal{P}\mathrm{Op}_{\infty}$ of ∞-preoperads. This product is almost never itself fibrant, and the process of fibrant replacement is fairly inexplicit. Our goal in this section is to partially address this problem by introducing another construction: the *wreath product* $\mathcal{O}^{\otimes} \wr \mathcal{O}'^{\otimes}$ of a pair of ∞-operads $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$. This wreath product is an ∞-category which admits a forgetful functor $\mathcal{O}^{\otimes} \wr \mathcal{O}'^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, together with a distinguished class M of *inert* morphisms, so that $(\mathcal{O}^{\otimes} \wr \mathcal{O}'^{\otimes}, M)$ can be regarded as an ∞-preoperad. Our main result, Theorem 2.4.4.3, asserts that there is a weak equivalence of ∞-preoperads $\mathcal{O}^{\otimes,\natural} \times \mathcal{O}'^{\otimes,\natural} \to (\mathcal{O}^{\otimes} \wr \mathcal{O}'^{\otimes}, M)$. This is not really a complete answer, since the codomain $(\mathcal{O}^{\otimes} \wr \mathcal{O}'^{\otimes}, M)$ is still generally not fibrant. However, it is in many ways more convenient than the product $\mathcal{O}^{\otimes,\natural} \odot \mathcal{O}'^{\otimes,\natural}$, and will play a vital role in our analysis of tensor products of little cubes ∞-operads in §5.1.2.

Construction 2.4.4.1. If $\mathcal{C}$ is an ∞-category, we let $\mathcal{C}^{\mathrm{II}}$ be defined as in Construction 2.4.3.1. Note that if $\mathcal{C}$ is the nerve of a category $\mathcal{J}$, then $\mathcal{C}^{\mathrm{II}}$ can be identified with the nerve of the category $\mathcal{J}^{\mathrm{II}}$ defined as follows:

(i) The objects of $\mathcal{J}^{\mathrm{II}}$ are finite sequences $(J_1, \ldots, J_n)$ of objects in $\mathcal{J}$.

(ii) A morphism from $(I_1, \ldots, I_m)$ to $(J_1, \ldots, J_n)$ in $\mathcal{J}^{\mathrm{II}}$ consists of a map $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$ together with a collection of maps $\{I_i \to J_j\}_{\alpha(i)=j}$.

There is an evident functor $\mathcal{F}\mathrm{in}_*^{\mathrm{II}} \to \mathcal{F}\mathrm{in}_*$, given on objects by the formula

$$(\langle k_1 \rangle, \ldots, \langle k_n \rangle) \mapsto \langle k_1 + \cdots + k_n \rangle.$$

This functor induces a map $\Phi : \mathrm{N}(\mathcal{F}\mathrm{in}_*)^{\mathrm{II}} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

Let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be ∞-operads. We let $\mathcal{C}^{\otimes} \wr \mathcal{D}^{\otimes}$ denote the simplicial set

$$\mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} (\mathcal{D}^{\otimes})^{\mathrm{II}}.$$

We define a map of simplicial sets $\pi : \mathcal{C}^{\otimes} \wr \mathcal{D}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ by considering the composition

$$
\begin{aligned}
\mathcal{C}^{\otimes} \wr \mathcal{D}^{\otimes} &= \mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} (\mathcal{D}^{\otimes})^{\amalg} \\
&\to (\mathcal{D}^{\otimes})^{\amalg} \\
&\to \mathrm{N}(\mathcal{F}\mathrm{in}_*)^{\amalg} \\
&\xrightarrow{\Phi} \mathrm{N}(\mathcal{F}\mathrm{in}_*).
\end{aligned}
$$

We can identify a morphism f in $\mathcal{C}^{\otimes} \wr \mathcal{D}^{\otimes}$ with a map $g : (D_1, \ldots, D_m) \to (D'_1, \ldots, D'_n)$ in $(\mathcal{D}^{\otimes})^{\amalg}$ lying over $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, together with a map $h : C \to C'$ in $\mathcal{C}^{\otimes}$ lying over α. We will say that f is *inert* if h is an inert morphism in $\mathcal{D}^{\otimes}$ and g determines a set of inert morphisms $\{D_i \to D'_j\}_{\alpha(i)=j}$ in $\mathcal{D}^{\otimes}$. Note that the map π carries inert morphisms of $\mathcal{C}^{\otimes} \wr \mathcal{D}^{\otimes}$ to inert morphisms in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

Remark 2.4.4.2. Let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be ∞-operads. The map $\mathcal{D}^{\otimes} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to (\mathcal{D}^{\otimes})^{\amalg}$ of Example 2.4.3.5 induces a monomorphism of simplicial sets $\mathcal{C}^{\otimes} \times \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes} \wr \mathcal{D}^{\otimes}$.

The remainder of this section is devoted to proving the following technical result:

Theorem 2.4.4.3. *Let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be ∞-operads, and let M be the collection of inert morphisms in $\mathcal{C}^{\otimes} \wr \mathcal{D}^{\otimes}$. Then the inclusion $\mathcal{C}^{\otimes} \times \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes} \wr \mathcal{D}^{\otimes}$ of Remark 2.4.4.2 induces a weak equivalence of ∞-preoperads*

$$
\mathcal{C}^{\otimes,\natural} \odot \mathcal{D}^{\otimes,\natural} \to (\mathcal{C}^{\otimes} \wr \mathcal{D}^{\otimes}, M).
$$

Lemma 2.4.4.4. *Let S be a finite set (regarded as a discrete simplicial set), let v denote the cone point of $S^{\triangleleft}$, and suppose we are given coCartesian fibrations $p : X \to S^{\triangleleft}$ and $q : Y \to S^{\triangleleft}$ which induce categorical equivalences*

$$
X_v \simeq \prod_{s \in S} X_s \qquad Y_v \simeq \prod_{s \in S} Y_s.
$$

Let $\mathrm{Fun}'_{S^{\triangleleft}}(X, Y)$ denote the full subcategory of $\mathrm{Fun}_{S^{\triangleleft}}(X, Y)$ spanned by those maps which carry p-coCartesian morphisms to q-coCartesian morphisms. Then the restriction functor

$$
\mathrm{Fun}'_{S^{\triangleleft}}(X, Y) \to \mathrm{Fun}_S(X \times_{S^{\triangleleft}} S, Y \times_{S^{\triangleleft}} S)
$$

is a trivial Kan fibration.

Proof. In view of Proposition T.4.3.2.15, it will suffice to prove the following:

(1) A functor $F \in \mathrm{Fun}_{S^{\triangleleft}}(X, Y)$ belongs to $\mathrm{Fun}'_{S^{\triangleleft}}(X, Y)$ if and only if F is a q-right Kan extension of $F|X$.

(2) Every map $F_0 \in \mathrm{Fun}_S(X \times_{S^{\triangleleft}} S, Y \times_{S^{\triangleleft}} S)$ can be extended to a map $F \in \mathrm{Fun}_{S^{\triangleleft}}(X, Y)$ satisfying the equivalent conditions of (1).

To prove (1), consider an arbitrary object x of X_v, and choose p-coCartesian morphisms $f_s : x \to x_s$ to objects $x_s \in X_s$ for $s \in S$. We note that the inclusion $\{f_s\}_{s \in S} \hookrightarrow (X \times_{S^{\triangleleft}} S)_{x/}$ is right cofinal. It follows that a functor F as in (1) is a q-right Kan extension of F_0 at x if and only if the maps $F(f_s)$ exhibit $F(x)$ as a q-product of the objects $F_0(x_s)$. This is equivalent to the requirement that each $F(f_s)$ is q-coCartesian. This proves the "if" direction of (1); the converse follows from same argument together with the observation that every p-coCartesian morphism $f : x \to x_s$ in X can be completed to a collection of p-coCartesian morphisms $\{f_{s'} : x \to x_{s'}\}_{s' \in S}$.

To prove (2), it suffices (by Lemma T.4.3.2.13) to show that for every $x \in X_v$, the diagram $(X \times_{S^{\triangleleft}} S)_{x/} \to Y$ induced by F_0 can be extended to a q-limit diagram covering the projection map $(X \times_{S^{\triangleleft}} S)_{x/}^{\triangleleft} \to S^{\triangleleft}$. Let $\{f_s : x \to x_s\}_{s \in S}$ be as above. We must show that there exists an object $y \in Y_v$ equipped with morphisms $y \to F_0(x_s)$ for $s \in S$, which exhibit y as a q-product of $\{F_0(x_s)\}_{s \in S}$. It suffices to choose y to be any preimage of $\{F_0(x_s)\}_{s \in S}$ under the equivalence $Y_v \simeq \prod_{s \in S} Y_s$. $\square$

Lemma 2.4.4.5. *Let S be a finite set (regarded as a discrete simplicial set), let $n > 0$, and suppose we are given inner fibrations $p : X \to \Delta^n \star S$ and $q : Y \to \Delta^n \star S$. For every simplicial subset $K \subseteq \Delta^n$, let X_K denote the fiber product $X \times_{\Delta^n \star S} (K \star S)$, and define Y_K similarly. Assume that the maps $X_{\{n\}} \to S^\triangleleft$ and $Y_{\{n\}} \to S^\triangleleft$ satisfy the hypotheses of Lemma 2.4.4.4, and for $\{n\} \subseteq K$ define $\mathrm{Fun}'_{K \star S}(X_K, Y_K)$ to be the fiber product*

$$\mathrm{Fun}_{K \star S}(X_K, Y_K) \times_{\mathrm{Fun}_{\{n\} \star S}(X_{\{n\}}, Y_{\{n\}})} \mathrm{Fun}'_{\{n\} \star S}(X_{\{n\}}, Y_{\{n\}}).$$

Then the map

$$\theta : \mathrm{Fun}'_{\Delta^n \star S}(X, Y) \to \mathrm{Fun}'_{\partial \Delta^n \star S}(X_{\partial \Delta^n}, Y_{\partial \Delta^n})$$

is a trivial Kan fibration.

Proof. The proof proceeds by induction on n. We observe that θ is evidently a categorical fibration; to prove that it is a trivial Kan fibration, it will suffice to show that θ is a categorical equivalence. Let θ'' denote the composition

$$\mathrm{Fun}'_{\Delta^n \star S}(X, Y) \xrightarrow{\theta} \mathrm{Fun}'_{\partial \Delta^n \star S}(X_{\partial \Delta^n}, Y_{\partial \Delta^n}) \xrightarrow{\theta'} \mathrm{Fun}_{\Delta^{n-1} \star S}(X_{\Delta^{n-1}}, Y_{\Delta^{n-1}}).$$

By a two-out-of-three argument, it will suffice to show that θ' and θ'' are trivial Kan fibrations. The map θ' is a pullback of the composition

$$\mathrm{Fun}'_{K \star S}(X_K, Y_K) \xrightarrow{\phi} \mathrm{Fun}'_{\{n\} \star S}(X_{\{n\}}, Y_{\{n\}}) \xrightarrow{\phi'} \mathrm{Fun}_S(X_\emptyset, Y_\emptyset),$$

where $K = \Delta^{n-1} \coprod \{n\} \subseteq \Delta^n$. It follows from iterated application of the inductive hypothesis that ϕ is a trivial Kan fibration, and it follows from Lemma 2.4.4.4 that ϕ' is a trivial Kan fibration. Consequently, to complete the proof, it will suffice to show that θ'' is a trivial Kan fibration. In view of Proposition T.4.3.2.15, it will suffice to prove the following:

(1) A map $F \in \mathrm{Fun}_{\Delta^n \star S}(X, Y)$ is a q-right Kan extension of $F_0 = F|X_{\Delta^{n-1}}$ if and only if it belongs to $\mathrm{Fun}'_{\Delta^n \star S}(X, Y)$.

(2) Every map $F_0 \in \mathrm{Fun}_{\Delta^{n-1} \star S}(X_{\Delta^{n-1}}, Y_{\Delta^{n-1}})$ admits an extension $F \in \mathrm{Fun}_{\Delta^n \star S}(X, Y)$ satisfying the equivalent conditions of (1).

These assertions follow exactly as in the proof of Lemma 2.4.4.4. $\square$

Lemma 2.4.4.6. *Let $p : \mathcal{C} \to \Delta^n$ be a map of ∞-categories, let $0 < i < n$, and assume that for every object $X \in \mathcal{C}_{i-1}$ there exists a p-coCartesian morphism $f : X \to Y$, where $Y \in \mathcal{C}_i$. Then the inclusion $p^{-1}\Lambda^n_i \subseteq \mathcal{C}$ is a categorical equivalence.*

Proof. We first treat the special case $i = 1$. The proof proceeds by induction on n. Let S be the collection of all nondegenerate simplices in Δ^n which contain the vertices 0, 1, and at least one other vertex. For each $\sigma \in S$, let σ' be the simplex obtained from σ by removing the vertex 1. Choose an ordering $S = \{\sigma_1, \ldots, \sigma_m\}$ of S where the dimensions of the simplices σ_j are nonstrictly decreasing as a function of i (so that $\sigma_1 = \Delta^n$). For $0 \le j \le m$, let K_j denote the simplicial subset of Δ^n obtained by removing the simplices σ_k and σ'_k for $k \le j$. If we let n_j denote the dimension of σ_j, then we have a pushout diagram

$$
\begin{array}{ccc}
\Lambda^{n_j}_1 & \longrightarrow & K_j \\
\downarrow & & \downarrow \\
\Delta^{n_j} & \longrightarrow & K_{j-1}
\end{array}
$$

Applying the inductive hypothesis (and the left properness of the Joyal model structure), we deduce that the inclusion $K_j \times_{\Delta^n} \mathcal{C} \to K_{j-1} \times_{\Delta^n} \mathcal{C}$ is a categorical equivalence for $1 < j \le m$. Combining these facts, we deduce that the map $K_m \times_{\Delta^n} \mathcal{C} \subseteq K_1 \times_{\Delta^n} \mathcal{C} \simeq p^{-1}\Lambda^n_1$ is a categorical equivalence. By a two-out-of-three

argument, we are reduced to proving that the inclusion $K_m \times_{\Delta^n} \mathcal{C} \to \mathcal{C}$ is a categorical equivalence. Let $q : \Delta^n \to \Delta^2$ be the map given on vertices by the formula

$$q(k) = \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{if } k = 1 \\ 2 & \text{otherwise} \end{cases}$$

and observe that $K_m = q^{-1}\Lambda_1^2$. We may therefore replace p by $q \circ p$ and thereby reduce to the case $n = 2$.

Choose a map $h : \Delta^1 \times \mathcal{C}_0 \to \mathcal{C}$ which is a natural transformation from the identity map $\mathrm{id}_{\mathcal{C}_0}$ to a functor $F : \mathcal{C}_0 \to \mathcal{C}_1$, such that h carries $\Delta^1 \times \{X\}$ to a p-coCartesian morphism in $\mathcal{C}$ for each $X \in \mathcal{C}_0$. Let $\mathcal{D} = p^{-1}\Delta\{1, 2\}$. The natural transformation h induces maps

$$(\Delta^1 \times \mathcal{C}_0) \coprod_{\{1\} \times \mathcal{C}_0} \mathcal{C}_1 \to p^{-1}\Delta^{\{0,1\}}$$

$$(\Delta^1 \times \mathcal{C}_0) \coprod_{\{1\} \times \mathcal{C}_0} \mathcal{D} \to \mathcal{C}$$

and it follows from Proposition T.3.2.2.7 that these maps are categorical equivalences. Consider the diagram

$$(\Delta^1 \times \mathcal{C}_0) \coprod_{\{1\} \times \mathcal{C}_0} \mathcal{D}$$

$$p^{-1}\Lambda_1^2 \longrightarrow \mathcal{C}.$$

It follows from the above arguments (and the left properness of the Joyal model structure) that the diagonal maps are categorical equivalences, so that the horizontal map is a categorical equivalence by the two-out-of-three property. This completes the proof in the case $i = 1$.

We now treat the case $i > 1$. The proof again proceeds by induction on n. Let $q : \Delta^n \to \Delta^3$ be the map defined by the formula

$$q(k) = \begin{cases} 0 & \text{if } k < i - 1 \\ 1 & \text{if } k = i - 1 \\ 2 & \text{if } k = i \\ 3 & \text{otherwise.} \end{cases}$$

Let S denote the collection of all nondegenerate simplices σ of Δ^n such that the restriction $q|\sigma$ is surjective. For each $\sigma \in S$, let σ' denote the simplex obtained from σ by deleting the vertex i. Choosing an ordering $S = \{\sigma_1, \ldots, \sigma_m\}$ of S where the dimension of the simplex σ_j is a nondecreasing function of j (so that $\sigma_1 = \Delta^n$), and for $0 \leq j \leq m$ let K_j be the simplicial subset of Δ^n obtained by deleting σ_k and σ_k' for $k \leq j$. If we let n_j denote the dimension of σ_j, then we have pushout diagrams

$$\begin{array}{ccc} \Lambda_p^{n_j} & \longrightarrow & K_j \\ \downarrow & & \downarrow \\ \Delta^{n_j} & \longrightarrow & K_{j-1} \end{array}$$

where $1 < p < n_j$. Applying the inductive hypothesis and the left properness of the Joyal model structure, we deduce that $K_j \times_{\Delta^n} \mathcal{C} \to K_{j-1} \times_{\Delta^n} \mathcal{C}$ is a categorical equivalence for $1 < j \leq m$. It follows that the map

$$K_m \times_{\Delta^n} \mathcal{C} \to K_1 \times_{\Delta^n} \mathcal{C} = p^{-1}\Lambda_i^n$$

is a categorical equivalence. To complete the proof, it will suffice to show that the inclusion $K_m \times_{\Delta^n} \mathcal{C} \to \mathcal{C}$ is a categorical equivalence. We observe that $K_m = q^{-1}\Lambda_2^3$. We may therefore replace p by $q \circ p$ and reduce to the case where $n = 3$ and $i = 2$.

Applying Lemma 3.1.2.4, we can factor the map $p^{-1}\Lambda_2^3 \to \Delta^3$ as a composition

$$p^{-1}\Lambda_2^3 \xrightarrow{i} \mathcal{C}' \xrightarrow{p'} \Delta^3,$$

where $\mathcal{C}'$ is an ∞-category and i is a categorical equivalence which induces an isomorphism $p^{-1}\Lambda_2^3 \simeq p'^{-1}\Lambda_2^3$. In particular, i is a trivial cofibration with respect to the Joyal model structure, so there exists a solution to the following lifting property:

$$
\begin{array}{ccc}
p^{-1}\Lambda_2^3 & \xrightarrow{\ \ j\ \ } & \mathcal{C} \\
\downarrow{\scriptstyle i} & {\scriptstyle g}\ \nearrow & \downarrow \\
\mathcal{C}' & \xrightarrow{\quad\quad} & \Delta^3.
\end{array}
$$

Since the map g induces an isomorphism $\Lambda_2^3 \times_{\Delta^3} \mathcal{C}' \to \Lambda_2^3 \times_{\Delta^3} \mathcal{C}$, it is a categorical equivalence (it is bijective on vertices and induces isomorphisms $\mathrm{Hom}_{\mathcal{C}'}^{\mathrm{R}}(x,y) \to \mathrm{Hom}_{\mathcal{C}}^{\mathrm{R}}(g(x),g(y))$ for every pair of vertices $x, y \in \mathcal{C}'$, since Λ_2^3 contains every edge of Δ^3). It follows that $j = g \circ i$ is a categorical equivalence as well, which completes the proof. $\square$

Proof of Theorem 2.4.4.3. Let $\mathcal{E}^\otimes$ be an ∞-operad and let $\mathfrak{X}$ the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{C}^\otimes \wr \mathcal{D}^\otimes, \mathcal{E}^\otimes)$ spanned functors F which carry inert morphisms in $\mathcal{C}^\otimes \wr \mathcal{D}^\otimes$ to inert morphisms in $\mathcal{E}^\otimes$, and define $\mathcal{Y} \subseteq \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{C}^\otimes \times \mathcal{D}^\otimes, \mathcal{E}^\otimes)$ similarly. We will show that the restriction functor $\mathfrak{X} \to \mathcal{Y}$ is a trivial Kan fibration.

We now introduce a bit of terminology. Let σ be an n-simplex of $(\mathcal{D}^\otimes)^{\mathrm{II}}$ given by a sequence of morphisms

$$\sigma(0) \xrightarrow{\overline{\alpha}(1)} \sigma(1) \longrightarrow \cdots \xrightarrow{\overline{\alpha}(n)} \sigma(n)$$

and let

$$\langle k_0 \rangle \xrightarrow{\alpha(1)} \langle k_1 \rangle \longrightarrow \cdots \xrightarrow{\alpha(n)} \langle k_n \rangle$$

be the underlying n-simplex of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. We will say that σ is *quasidegenerate at* $\overline{\alpha}(i)$ if the following condition holds: whenever we are given $i_- < i \leq i_+$ and a sequence of integers $\{a_j \in \langle k_j \rangle^\circ\}_{i_- \leq j \leq i_+}$ satisfying $\alpha_j(a_{j-1}) = a_j$, the corresponding map $\Delta^{\{i_-, i_-+1, \ldots, i_+\}} \to \mathcal{D}^\otimes$ factors through the quotient map $\Delta^{\{i_-, i_-+1, \ldots, i_+\}} \to \Delta^{i_+-1-i_-}$ which identifies the vertices i and $i - 1$. If $n = 1$, we will simply say that σ is *quasidegenerate* if it is quasi-degenerate at $\overline{\alpha}(1)$. We will say that σ is *closed* if $k_n = 1$, and *open* otherwise. If σ is closed, we define the *tail length* of σ to be the largest integer m such that the maps $\alpha(k)$ are isomorphisms for $n - m < k \leq n$. We will denote the tail length of σ by $t(\sigma)$. We define the *break point* of a closed simplex σ to be smallest nonnegative integer m such that σ is quasi-degenerate at $\overline{\alpha}(k)$ and $\overline{\alpha}(k)$ is active for $m < k \leq n - t(\sigma)$. We will denote the break point of σ by $b(\sigma)$. Let $S = \coprod_{0 \leq i \leq n} \langle k_i \rangle^\circ$. We will say that an element $j \in \langle k_i \rangle^\circ \subseteq S$ is a *leaf* if $i = 0$ or if j does not lie in the image of the map $\alpha(i)$, and we will say that j is a *root* if $i = n$ or if $\alpha(i+1)(j) = *$. We define the *complexity* $c(\sigma)$ of σ to be $2l - r$, where l is the number of leaves of σ and r is the number of roots of σ. We will say that σ is *flat* if it belongs to the image of the embedding $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \mathcal{D}^\otimes \to (\mathcal{D}^\otimes)^{\mathrm{II}}$. Note that if σ is closed and $b(\sigma) = 0$, then σ is flat.

We now partition the nondegenerate, nonflat simplices of $(\mathcal{D}^\otimes)^{\mathrm{II}}$ into six groups:

(A) An n-dimensional nonflat nondegenerate simplex σ of $(\mathcal{D}^\otimes)^{\mathrm{II}}$ belongs to A if σ is closed and the map $\alpha(b(\sigma))$ is not inert.

(A') An n-dimensional nonflat nondegenerate simplex σ of $(\mathcal{D}^\otimes)^{\mathrm{II}}$ belongs to A' if σ is closed, $b(\sigma) < n - t(\sigma)$, and the map $\alpha(b(\sigma))$ is inert.

(B) An n-dimensional nonflat nondegenerate simplex σ of $(\mathcal{D}^\otimes)^{\mathrm{II}}$ belongs to B if σ is closed, $b(\sigma) = n - t(\sigma)$, the map $\alpha(b(\sigma))$ is inert, and σ is not quasidegenerate at $\overline{\alpha}(b(\sigma))$.

(B') An n-dimensional nonflat nondegenerate simplex σ of $(\mathcal{D}^{\otimes})^{\amalg}$ belongs to B if σ is closed, $b(\sigma) = n - t(\sigma) < n$, the map $\alpha(b(\sigma))$ is inert, and σ is quasidegenerate at $\overline{\alpha}(b(\sigma))$.

(C) An n-dimensional nonflat nondegenerate simplex σ of $(\mathcal{D}^{\otimes})^{\amalg}$ belongs to C if it is open.

(C') An n-dimensional nonflat nondegenerate simplex σ of $(\mathcal{D}^{\otimes})^{\amalg}$ belongs to C' is it is closed, $b(\sigma) = n - t(\sigma) = n$, the map $\alpha(b(\sigma))$ is inert, and σ is quasidegenerate at $\overline{\alpha}(b(\sigma))$.

If σ belongs to A', B', or C', then we define the *associate* $a(\sigma)$ of σ to be the face of σ opposite the $b(\sigma)$th vertex. Note that $a(\sigma)$ belongs to A if $\sigma \in A'$, B if $\sigma \in B'$, and C if $\sigma \in C'$. In this case, we will say that σ is an *associate* of $a(\sigma)$. We note that every simplex belonging to A or B has a unique associate, while a simplex σ of C has precisely k associates, where $\langle k \rangle$ is the image of the final vertex of σ in $N(\mathcal{F}in_*)$. Moreover, the associate of a simplex σ has the same complexity as σ.

For each $n \geq 0$, let $K(n) \subseteq (\mathcal{D}^{\otimes})^{\amalg}$ be the simplicial subset generated by those nondegenerate simplices which are either flat, have dimension $\leq n$, or have dimension $n + 1$ and belong to either A', B', or C'. We observe that $K(0)$ is generated by $\mathcal{D}^{\otimes} \times N(\mathcal{F}in_*)$ together with the collection of 1-simplices belonging to C'. Let $\mathfrak{X}(n)$ denote the full subcategory of $\operatorname{Map}_{N(\mathcal{F}in_*)}(\mathcal{C}^{\otimes} \times_{N(\mathcal{F}in_*)} K(n), \mathcal{E}^{\otimes})$ spanned by those maps F with the following properties:

(i) The restriction of F to $\mathcal{C}^{\otimes} \times \mathcal{D}^{\otimes}$ belongs to $\mathcal{Y}$.

(ii) Let f be an edge of $\mathcal{C}^{\otimes} \times_{N(\mathcal{F}in_*)} K(0)$ whose image in $\mathcal{C}^{\otimes}$ is inert and whose image in $K(0)$ belongs to C'. Then $F(f)$ is an inert morphism in $\mathcal{E}^{\otimes}$.

To complete the proof, it will suffice to show that the restriction maps

$$\mathfrak{X} \xrightarrow{\theta'} \mathfrak{X}(0) \xrightarrow{\theta''} \mathcal{Y}$$

are trivial Kan fibrations. For the map θ'', this follows from repeated application of Lemma 2.4.4.4. To prove that θ' is a trivial Kan fibration, we define

$$\mathfrak{X}(n) \subseteq \operatorname{Map}_{N(\mathcal{F}in_*)}(\mathcal{C}^{\otimes} \times_{N(\mathcal{F}in_*)} K(n), \mathcal{E}^{\otimes})$$

to be the full subcategory spanned by those functors F whose restriction to $\mathcal{C}^{\otimes} \times_{N(\mathcal{F}in_*)} K(0)$ belongs to $\mathfrak{X}(0)$. We will prove the following:

(a) A functor $F \in \operatorname{Fun}_{N(\mathcal{F}in_*)}(\mathcal{C}^{\otimes} \wr \mathcal{D}^{\otimes}, \mathcal{E}^{\otimes})$ carries inert morphisms to inert morphisms if and only if F satisfies conditions (i) and (ii). Consequently, the ∞-category $\mathfrak{X}$ can be identified with the inverse limit of the tower

$$\cdots \to \mathfrak{X}(2) \to \mathfrak{X}(1) \to \mathfrak{X}(0).$$

(b) For $n > 0$, the restriction map $\mathfrak{X}(n) \to \mathfrak{X}(n-1)$ is a trivial Kan fibration.

We first prove (a). The "only if" direction is obvious. For the converse, suppose that an object F of $\operatorname{Fun}_{N(\mathcal{F}in_*)}(\mathcal{C}^{\otimes} \wr \mathcal{D}^{\otimes}, \mathcal{E}^{\otimes})$ satisfies conditions (i) and (ii) above. We wish to prove that F preserves inert morphisms. Let $f : X \to X'$ be an inert morphism in $\mathcal{C}^{\otimes} \wr \mathcal{D}^{\otimes}$ covering the map $f_0 : (\langle k_1 \rangle, \dots, \langle k_m \rangle) \to (\langle k'_1 \rangle, \dots, \langle k'_{m'} \rangle)$ in $N(\mathcal{F}in_*)^{\amalg}$; we wish to prove that $F(f)$ is an inert morphism in $\mathcal{E}^{\otimes}$. If $m' = k'_1 = 1$, then f_0 factors as a composition of inert morphisms

$$(\langle k_1 \rangle, \dots, \langle k_m \rangle) \xrightarrow{f'_0} (\langle k_i \rangle) \xrightarrow{f''_0} (\langle 1 \rangle)$$

for some $i \in \langle m \rangle^{\circ}$, which we can lift to a factorization $f \simeq f'' \circ f'$ of f where f' is quasidegenerate. Condition (ii) guarantees that $F(f')$ is inert, and condition (i) guarantees that $F(f'')$ is inert. In the general case, we consider for each $j \in \langle k'_i \rangle^{\circ}$ an inert morphism $g_{i,j} : X' \to X''$ lifting the composite map

$$(\langle k'_1 \rangle, \dots, \langle k'_{m'} \rangle) \to (\langle k'_i \rangle) \to (\langle 1 \rangle).$$

The above argument shows that $F(g_{i,j})$ and $F(g_{i,j} \circ f)$ are inert morphisms in $\mathcal{E}^{\otimes}$. The argument of Remark 2.1.2.9 shows that $F(f)$ is inert, as desired.

We now prove (b). For each integer $c \geq 0$, let $K(n,c)$ denote the simplicial subset $K(n)$ spanned by those simplices which either belong to $K(n-1)$ or have complexity $\leq c$. Let $\mathfrak{X}(n,c)$ denote the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} K(n,c), \mathcal{E}^{\otimes})$ spanned by those maps F whose restriction to $K(0)$ satisfies conditions (i) and (ii). We have a tower of simplicial sets

$$\cdots \to \mathfrak{X}(n,2) \to \mathfrak{X}(n,1) \to \mathfrak{X}(n,0) \simeq \mathfrak{X}(n-1)$$

with whose inverse limit can be identified with $\mathfrak{X}(n)$. It will therefore suffice to show that for each $c > 0$, the restriction map $\mathfrak{X}(n,c) \to \mathfrak{X}(n,c-1)$ is a trivial Kan fibration.

We now further refine our filtration as follows. Let $K(n,c)_A$ denote the simplicial subset of $K(n,c)$ spanned by $K(n,c-1)$ together with those simplices of $K(n,c)$ which belong to A or A' and let $K(n,c)_B$ denote the simplicial subset of $K(n,c)$ spanned by $K(n,c-1)$ together with those simplices which belong to A, A', B, or B'. Let $\mathfrak{X}(n,c)_A$ denote the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} K(n,c)_A, \mathcal{E}^{\otimes})$ spanned by those maps F satisfying (i) and (ii), and define $\mathfrak{X}(n,c)_B$ similarly. To complete the proof, it will suffice to prove the following:

(A) The restriction map $\mathfrak{X}(n,c)_A \to \mathfrak{X}(n,c-1)$ is a trivial Kan fibration. To prove this, it suffices to show that the inclusion $\mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} K(n,c-1) \to \mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} K(n,c)_A$ is a categorical equivalence. Let $A_{n,c}$ denote the collection of all n-simplices belonging to A having complexity c. Choose a well-ordering of $A_{n,c}$ with the following properties:

– If $\sigma, \sigma' \in A_{n,c}$ and $t(\sigma) < t(\sigma')$, then $\sigma < \sigma'$.

– If $\sigma, \sigma' \in A_{n,c}$, $t(\sigma) = t(\sigma')$, and $b(\sigma) < b(\sigma')$, then $\sigma < \sigma'$.

For each $\sigma \in A_{n,c}$, let $K(n,c)_{\leq \sigma}$ denote the simplicial subset of $K(n,c)$ generated by $K(n,c-1)$, all simplices $\tau \leq \sigma$ in $A_{n,c}$, and all of the simplices in A' which are associated to simplices of the form $\tau \leq \sigma$. Define $K(n,c)_{<\sigma}$ similarly. Using transfinite induction on $A_{n,c}$, we are reduced to proving that for each $\sigma \in A_{n,c}$, the inclusion

$$i : \mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} K(n,c)_{<\sigma} \to \mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} K(n,c)_{\leq \sigma}$$

is a categorical equivalence. Let $\sigma' : \Delta^{n+1} \to (\mathcal{D}^{\otimes})^{\amalg}$ be the unique $(n+1)$-simplex of A' associated to σ. We observe that σ' determines a pushout diagram

$$
\begin{array}{ccc}
\Lambda^{n+1}_{b(\sigma')} & \longrightarrow & K(n,c)_{<\sigma} \\
\downarrow & & \downarrow \\
\Delta^{n+1} & \longrightarrow & K(n,c)_{\leq \sigma}.
\end{array}
$$

Consequently, the map i is a pushout of an inclusion

$$i' : \mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \Lambda^{n+1}_{b(\sigma')} \to \mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \Delta^{n+1}_{b(\sigma)}.$$

Since the Joyal model structure is left proper, it suffices to show that i' is a categorical equivalence, which follows from Lemma 2.4.4.6.

(B) The map $\mathfrak{X}(n,c)_B \to \mathfrak{X}(n,c)_A$ is a trivial Kan fibration. To prove this, it suffices to show that the inclusion $\mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} K(n,c)_A \subseteq \mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} K(n,c)_B$ is a categorical equivalence of simplicial sets. Let $B_{n,c}$ denote the collection of all n-simplices belonging to B having complexity c. Choose a well-ordering of $B_{n,c}$ such that the function $\sigma \mapsto t(\sigma)$ is nonstrictly decreasing. For each $\sigma \in B_{n,c}$, we let $K(n,c)_{\leq \sigma}$ be the simplicial subset of $K(n,c)$ generated by $K(n,c)_A$, those simplices τ of $B_{n,c}$

such that $\tau \leq \sigma$, and those simplices of B' which are associated to $\tau \leq \sigma \in B_{n,c}$. Let $K(n, c)_{<\sigma}$ be defined similarly. Using a induction on $B_{n,c}$, we can reduce to the problem of showing that each of the inclusions

$$\mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} K(n, c)_{<\sigma} \to \mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} K(n, c)_{\leq\sigma}$$

is a categorical equivalence. Let $\sigma' : \Delta^{n+1} \to (\mathcal{D}^{\otimes})^{\amalg}$ be the unique $(n+1)$-simplex of B' associated to σ. We observe that σ' determines a pushout diagram

$$\begin{array}{ccc} \Lambda_{b(\sigma')}^{n+1} & \longrightarrow & K(n, c)_{<\sigma} \\ \downarrow & & \downarrow \\ \Delta^{n+1} & \longrightarrow & K(n, c)_{\leq\sigma}. \end{array}$$

Consequently, the map i is a pushout of an inclusion

$$i' : \mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \Lambda_{b(\sigma')}^{n+1} \to \mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \Delta_{b(\sigma)}^{n+1}.$$

Since the Joyal model structure is left proper, it suffices to show that i' is a categorical equivalence, which follows from Lemma 2.4.4.6.

(C) The map $\mathcal{X}(n, c) \to \mathcal{X}(n, c)_B$ is a trivial Kan fibration. To prove this, let $C_{n,c}$ denote the subset of C consisting of n-dimensional simplices of complexity c, and choose a well-ordering of $C_{n,c}$. For each $\sigma \in C_{n,c}$, let $K(n, c)_{\leq\sigma}$ denote the simplicial subset of $K(n, c)$ generated by $K(n, c)_B$, those simplices $\tau \in C_{n,c}$ such that $\tau \leq \sigma$, and those simplices of C' which are associated to $\tau \in C_{n,c}$ with $\tau \leq \sigma$. Let $\mathcal{X}(n, c)_{\leq\sigma}$ be the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} K(n, c)_{\leq\sigma}, \mathcal{E}^{\otimes})$ spanned by those maps F satisfying (i) and (ii). We define $K(n, c)_{<\sigma}$ and $\mathcal{X}(n, c)_{<\sigma}$ similarly. Using transfinite induction on $C_{n,c}$, we are reduced to the problem of showing that for each $\sigma \in C_{n,c}$, the map $\psi : \mathcal{X}(n, c)_{\leq\sigma} \to \mathcal{X}(n, c)_{<\sigma}$ is a trivial Kan fibration.

Let $\langle k \rangle$ denote the image of the final vertex of σ in $\mathcal{F}\mathrm{in}_*$. For $1 \leq i \leq k$, let $\sigma_i \in C'$ denote the unique $(n+1)$-simplex associated to σ such that σ_i carries $\Delta^{\{n, n+1\}}$ to the morphism ρ^i in $\mathcal{F}\mathrm{in}_*$. The simplices $\{\sigma_i\}_{1 \leq i \leq k}$ determine a map of simplicial sets $\Delta^n \star \langle k \rangle^{\circ} \to K(n, c)_{\leq\sigma}$. We have a pushout diagram of simplicial sets

$$\begin{array}{ccc} (\partial \Delta^n) \star \langle k \rangle^{\circ} & \longrightarrow & K(n, c)_{<\sigma} \\ \downarrow & & \downarrow \\ \Delta^n \star \langle k \rangle^{\circ} & \longrightarrow & K(n, c)_{\leq\sigma}. \end{array}$$

The map ψ fits into a pullback diagram

$$\begin{array}{ccc} \mathcal{E}(n, c)_{\leq\sigma} & \longrightarrow & \mathrm{Fun}'_{\Delta^n \star \langle k \rangle^{\circ}}(\mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} (\Delta^n \star \langle k \rangle^{\circ}), \mathcal{E}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} (\Delta^n \star \langle k \rangle^{\circ})) \\ \downarrow{\psi} & & \downarrow{\psi'} \\ \mathcal{E}(n, c)_{<\sigma} & \longrightarrow & \mathrm{Fun}'_{\partial \Delta^n \star \langle k \rangle^{\circ}}(\mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} (\partial \Delta^n \star \langle k \rangle^{\circ}), \mathcal{E}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} (\partial \Delta^n \star \langle k \rangle^{\circ})) \end{array}$$

where ψ' denotes the trivial Kan fibration of Lemma 2.4.4.5.

$\square$

Chapter 3

Algebras and Modules over ∞-Operads

Let $\mathcal{O}^{\otimes}$ be an ∞-operad and let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. In §2.1, we introduced the definition of a $\mathcal{O}$-*algebra object of* $\mathcal{C}$ (that is, a map of ∞-operads from $\mathcal{O}^{\otimes}$ to $\mathcal{C}^{\otimes}$). The collection of $\mathcal{O}$-algebra objects of $\mathcal{C}$ is naturally organized into an ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. Our goal in this chapter is to study the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ in more detail.

Our starting point is the observation that for every object $X \in \mathcal{O}$, evaluation at X determines a forgetful functor $e : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$. In §3.1, we will show that (under some mild hypotheses) the forgetful functor e admits a left adjoint $\mathrm{Free} : \mathcal{C} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. This left adjoint carries an object $C \in \mathcal{C}$ to the *free algebra* $\mathrm{Free}(C)$ *generated by* C, which admits a relatively concrete description in terms of the tensor product on $\mathcal{C}$ and the structure of $\mathcal{O}^{\otimes}$. The adjoint functors

$$\mathcal{C} \underset{e}{\overset{\mathrm{Free}}{\rightleftarrows}} \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$$

allow us to reduce many questions about $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ to questions in the underlying ∞-category $\mathcal{C}$. In §3.2, we will use this device to show that the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is closed under a variety of categorical constructions (such as limits and colimits), assuming that $\mathcal{C}$ is sufficiently well-behaved.

The second half of this chapter is devoted to the theory of *modules* over algebra objects of a symmetric monoidal ∞-category $\mathcal{C}$. Suppose we are given a commutative algebra object $A \in \mathrm{CAlg}(\mathcal{C})$. We will define a new ∞-category $\mathrm{Mod}_A(\mathcal{C})$, whose objects we refer to as *A-module objects of* $\mathcal{C}$. Under some mild hypotheses, one can show that the ∞-category $\mathrm{Mod}_A(\mathcal{C})$ inherits a symmetric monoidal structure. More generally, we can attempt to associate to every algebra object $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ a fibration of ∞-operads $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$. We will explain the construction of $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ in §3.3. This construction will require an assumption on $\mathcal{O}^{\otimes}$: namely, that it be a *coherent ∞-operad* (Definition 3.3.1.9). The collection of coherent ∞-operads includes most of the examples which are of interest to us in this book: for example, the commutative ∞-operad $\mathrm{Comm}^{\otimes}$ of Example 2.1.1.18, the associative ∞-operad $\mathcal{A}\mathrm{ss}^{\otimes}$ we will study in Chapter 4, and the little cubes ∞-operads of Chapter 5 are all coherent. In §3.4, we will study the ∞-operads $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ in greater depth. For example, we give criteria which guarantee that $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ is a $\mathcal{O}$-monoidal ∞-category (in other words, that the tensor product on A-modules is well-defined), and study limits and colimits in the underlying ∞-category of $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$.

3.1 Free Algebras

Let $\mathcal{C}$ be a symmetric monoidal category. We let $\mathrm{CAlg}(\mathcal{C})$ denote the category of commutative algebra objects of $\mathcal{C}$: that is, objects $A \in \mathcal{C}$ equipped with a multiplication $m : A \otimes A \to A$ which is commutative,

associative, and unital. There is an evident forgetful functor $\theta : \mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$, which assigns to each commutative algebra its underlying object of $\mathcal{C}$. Assume that $\mathcal{C}$ admits small colimits, and that the tensor product operation $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves colimits separately in each variable. Then the forgetful functor θ admits a left adjoint. Moreover, this left adjoint admits an explicit description: it carries each object $C \in \mathcal{C}$ to the symmetric algebra

$$\mathrm{Sym}^* C = \coprod_n C^{\otimes n}/\Sigma_n.$$

Here Σ_n denotes the symmetric group on n letters, which acts on the nth tensor power $C^{\otimes n}$.

In this section, we will describe an analogous theory of free algebras in the ∞-categorical context. Our discussion will be more general in several respects:

- We will replace the symmetric monoidal category $\mathcal{C}$ by a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ (see Definition 2.0.0.7).

- The forgetful functor $\theta : \mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ can be identified with the restriction functor $\mathrm{Alg}_{\mathrm{Comm}}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Triv}}(\mathcal{C})$ induced by the inclusion of ∞-operads $\mathrm{Triv}^{\otimes} \subseteq \mathrm{Comm}^{\otimes}$, where $\mathrm{Triv}^{\otimes}$ is the trivial ∞-operad (Example 2.1.1.20) and $\mathrm{Comm}^{\otimes} = \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is the commutative ∞-operad (Example 2.1.1.18). More generally, we can consider the restriction functor $\theta : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$ induced by *any* map of ∞-operads $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$.

- If $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a $\mathcal{O}$-monoidal ∞-category, we can replace the ∞-categories $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ and $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$ with $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ and $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$, respectively.

We can now describe our main goal. Suppose that we are given a map of ∞-operads $f : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ and a $\mathcal{O}$-monoidal ∞-category $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$. Composition with f induces a forgetful functor $\theta : \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$. We will show that, under some mild hypotheses, the functor θ admits a left adjoint F. Moreover, if we assume that the ∞-operad $\mathcal{O}'^{\otimes}$ is sufficiently simple (for example, if it is the trivial ∞-operad $\mathrm{Triv}^{\otimes}$ of Example 2.1.1.20), then F admits a very explicit description (see Proposition 3.1.3.11).

The construction of the functor F uses a theory of operadic left Kan extensions, which we develop in §3.1.2. As in the non-operadic case, the theory of operadic left Kan extensions rests on a more basic theory of operadic colimit diagrams, which we discuss in §3.1.1. We will apply these ideas to the construction of free algebras in §3.1.3. Finally, in §3.1.4 we will establish a transitivity property of operadic left Kan extensions, which will play an important role in our discussion of tensor products of algebras in §5.1.5.

3.1.1   Operadic Colimit Diagrams

Let $\mathcal{C}$ be a symmetric monoidal category, and suppose we are given a sequence of maps

$$A_0 \to A_1 \to A_2 \to \cdots$$

in the category $\mathrm{CAlg}(\mathcal{C})$ of commutative algebra objects of $\mathcal{C}$. If $\mathcal{C}$ admits sequential colimits, then we can define the colimit $A = \varinjlim_i A_i$ as an object of $\mathcal{C}$. In many cases, the object $A \in \mathcal{C}$ can also be regarded as a commutative algebra object of $\mathcal{C}$. There are evident maps

$$A \otimes A \simeq (\varinjlim_i A_i) \otimes (\varinjlim_j A_j) \overset{\theta}{\leftarrow} \varinjlim_{i,j}(A_i \otimes A_j) \simeq \varinjlim_i(A_i \otimes A_i) \to \varinjlim_i A_i \simeq A.$$

If θ is an isomorphism, then we obtain a natural map $A \otimes A \to A$. We can guarantee that θ is an isomorphism by making an assumption like the following:

($*$) The category $\mathcal{C}$ admits small colimits. Moreover, for every object $C \in \mathcal{C}$, the functor $\bullet \mapsto C \otimes \bullet$ preserves small colimits.

Condition $(*)$ is satisfied in many cases. However, for many purposes it is unnecessarily restrictive. For example, it may be that $\mathcal{C}$ does not admit colimits in general, but that a particular sequence of commutative algebras $A_0 \to A_1 \to A_2 \to \cdots$ admits a colimit $A \in \mathcal{C}$. In this case, it is convenient to replace $(*)$ by the following *local* condition:

$(*')$ For every object $C \in \mathcal{C}$, the induced maps $A_i \otimes C \to A \otimes C$ exhibit $A \otimes C$ as a colimit of the sequence

$$A_0 \otimes C \to A_1 \otimes C \to A_2 \otimes C \to \cdots .$$

Condition $(*')$ follows immediately from condition $(*)$. Moreover, it is easy to see that condition $(*')$ guarantees that the map θ is an isomorphism, which allows us to endow the colimit $A \simeq \varinjlim A_i$ with the structure of a commutative algebra object of $\mathcal{C}$. We will say that a compatible collection of maps $A_i \to A$ *exhibits A as an operadic colimit of the sequence A_i* if condition $(*)$ is satisfied. This implies in particular that A is a colimit $\varinjlim_i A_i$ (take C to be the unit object of $\mathcal{C}$ in $(*)$).

Our goal in this section is to study an analogue of condition $(*')$. Our setting will be more general in several respects:

(a) Rather than working with symmetric monoidal categories, we work with colored operads. Note that condition $(*')$ has an immediate generalization to the setting of colored operads: rather than requiring that $A \otimes C \simeq \varinjlim A_i \otimes C$, we require that for every collection of objects $\{C_j \in \mathcal{C}\}_{j \in J}$ and every object $D \in \mathcal{C}$, the natural map

$$\mathrm{Mul}_{\mathcal{C}}(\{A\} \cup \{C_j\}_{j \in J}, D) \to \varprojlim_i \mathrm{Mul}_{\mathcal{C}}(\{A_i\} \cup \{C_j\}_{j \in J}, D)$$

is a bijection.

(b) We will work in the more general setting of ∞-operads, rather than ordinary colored operads.

(c) We will consider operadic colimits not just for sequences of objects in $\mathcal{C}$, but for more general diagrams in the ∞-category $\mathcal{C}^{\otimes}$.

(d) We will work not just with operadic colimit diagrams in a fixed ∞-operad $\mathcal{C}^{\otimes}$, but also with operadic colimits *relative* to a fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ (for a discussion of relative colimits in the non-operadic setting, we refer the reader to §T.4.3.1).

We can now describe the contents of this section. Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads. We will introduce below the definition of an *operadic q-colimit diagram* $K^{\triangleright} \to \mathcal{C}^{\otimes}$ (Definition 3.1.1.2). The remainder of this section is devoted to proving the basic facts about operadic q-colimit diagrams. In particular, we will prove an existence result for operadic q-colimit diagrams assuming that an appropriate analogue of condition $(*)$ is satisfied (Corollary 3.1.1.21).

Notation 3.1.1.1. Let $\mathcal{O}^{\otimes}$ be a generalized ∞-operad. We let $\mathcal{O}^{\otimes}_{\mathrm{act}}$ denote the subcategory of $\mathcal{O}^{\otimes}$ spanned by those morphisms whose images in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ are active.

Definition 3.1.1.2. Let $\mathcal{O}^{\otimes}$ be an ∞-operad and let $p : K \to \mathcal{O}^{\otimes}$ be a diagram. We let $\mathcal{O}_{p/}$ denote the ∞-category $\mathcal{O} \times_{\mathcal{O}^{\otimes}} \mathcal{O}^{\otimes}_{p/}$. If p factors through $\mathcal{O}^{\otimes}_{\mathrm{act}}$, we let $\mathcal{O}^{\otimes}_{p/} \subseteq \mathcal{O}_{p/}$ denote the ∞-category $\mathcal{O} \times_{\mathcal{O}^{\otimes}_{\mathrm{act}}} (\mathcal{O}^{\otimes}_{\mathrm{act}})_{p/}$.

Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, let $\overline{p} : K^{\triangleright} \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ be a diagram, and let $p = \overline{p}|K$. We will say that $\overline{p}$ is a *weak operadic q-colimit diagram* if the evident map

$$\psi : \mathcal{C}^{\mathrm{act}}_{\overline{p}/} \to \mathcal{C}^{\mathrm{act}}_{p/} \times_{\mathcal{O}^{\mathrm{act}}_{q\overline{p}/}} \mathcal{O}^{\mathrm{act}}_{q\overline{p}/}$$

is a trivial Kan fibration.

We say that an active diagram $p : K \to \mathcal{C}^{\otimes}$ is an *operadic q-colimit diagram* if the composite functor

$$K \xrightarrow{p} \mathcal{C}^{\otimes}_{\mathrm{act}} \xrightarrow{\bullet \oplus C} \mathcal{C}^{\otimes}_{\mathrm{act}}$$

is a weak operadic q-colimit diagram, for every object $C \in \mathcal{C}^{\otimes}$ (here the functor $\oplus$ is defined as in Remark 2.2.4.6).

Warning 3.1.1.3. Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads. A q-colimit diagram $K^{\triangleright} \to \mathcal{C}^{\otimes}$ need not be an operadic q-colimit diagram, and an operadic q-colimit diagram need not be a q-colimit diagram. However, these notions can be related in special cases: see Propositions 3.1.1.10 and 3.1.1.16 below.

Remark 3.1.1.4. Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be as in Definition 3.1.1.2, let $\overline{p} : K^{\triangleright} \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ be a weak operadic q-colimit diagram, and let $L \to K$ be a left cofinal map of simplicial sets. Then the induced map $L^{\triangleright} \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ is a weak operadic q-colimit diagram.

Remark 3.1.1.5. The map ψ appearing in Definition 3.1.1.2 is always a left fibration (Proposition T.2.1.2.1); consequently, it is a trivial Kan fibration if and only if it is a categorical equivalence (Corollary T.2.4.4.6).

Example 3.1.1.6. Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be as in Definition 3.1.1.2, let $K = \Delta^0$, and suppose that $\overline{p} : K^{\triangleright} \simeq \Delta^1 \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ corresponds to an equivalence in $\mathcal{C}^{\otimes}$. Then $\overline{p}$ is an operadic q-colimit diagram.

The remainder of this section is devoted to a series of results which are useful for working with operadic colimit diagrams.

Proposition 3.1.1.7. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads let $\overline{p} : K^{\triangleright} \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ be a diagram. The following conditions are equivalent:*

(1) *The map $\overline{p}$ is a weak operadic q-colimit diagram.*

(2) *For every $n > 0$ and every diagram*

$$
\begin{array}{ccc}
K \star \partial\,\Delta^n & \xrightarrow{f_0} & \mathcal{C}^{\otimes}_{\mathrm{act}} \\
\downarrow & \overset{f}{\nearrow} & \downarrow \\
K \star \Delta^n & \xrightarrow{\overline{f}} & \mathcal{O}^{\otimes}_{\mathrm{act}}
\end{array}
$$

such that the restriction of f_0 to $K \star \{0\}$ coincides with $\overline{p}$ and $f_0(n) \in \mathcal{C}$, there exists a dotted arrow f as indicated above.

Proof. The implication (2) $\Rightarrow$ (1) follows immediately from the definition. For the converse, suppose that (1) is satisfied. Let $r : \Delta^n \times \Delta^1 \to \Delta^n$ be the map which is given on vertices by the formula

$$r(i, j) = \begin{cases} i & \text{if } j = 0 \\ n & \text{if } j = 1. \end{cases}$$

Set

$$X = \partial\,\Delta^n \coprod_{\Lambda^n_n \times \{0\}} (\Lambda^n_n \times \Delta^1) \coprod_{\Lambda^n_n \times \{1\}} \Delta^n,$$

and regard X as a simplicial subset of $\Delta^n \times \Delta^1$; we observe that r carries X into $\partial\,\Delta^n$. Composing f_0 and $\overline{f}$ with r, we obtain a diagram

$$
\begin{array}{ccc}
X & \xrightarrow{g_0} & (\mathcal{C}^{\otimes}_{\mathrm{act}})_{f_0|K/} \\
\downarrow & \overset{g}{\nearrow} & \downarrow{q'} \\
\Delta^n \times \Delta^1) & \longrightarrow & (\mathcal{O}^{\otimes}_{\mathrm{act}})_{\overline{f}|K/}.
\end{array}
$$

To complete the proof, it will suffice to show that we can supply the dotted arrow in this diagram (we can then obtain f by restricting g to $\Delta^n \times \{0\}$). To prove this, we define a sequence of simplicial subsets

$$X = X(0) \subset X(1) \subset X(2) \subset \cdots \subset X(2n+1) = \Delta^n \times \Delta^1$$

and extend g_0 to a sequence of maps $g_i : X(i) \to (\mathcal{C}^{\otimes}_{\mathrm{act}})_{f_0|K/}$ compatible with the projection q'. The analysis proceeds as follows:

(i) For $1 \leq i \leq n$, we let $X(i)$ be the simplicial subset of $\Delta^n \times \Delta^1$ generated by $X(i-1)$ and the n-simplex $\sigma : \Delta^n \to \Delta^n \times \Delta^1$ given on vertices by the formula

$$\sigma(j) = \begin{cases} (j,0) & \text{if } j \leq n-i \\ (j-1,1) & \text{if } j > n-i. \end{cases}$$

We have a pushout diagram of simplicial sets

$$
\begin{array}{ccc}
\Lambda^n_{n-i} & \longrightarrow & X(i-1) \\
\downarrow & & \downarrow \\
\Delta^n & \longrightarrow & X(i).
\end{array}
$$

If $1 \leq i < n$, then the extension g_i of g_{i-1} exists because q' is an inner fibration. If $i = n$, then the desired extension exists by virtue of assumption (1).

(ii) If $i = n + i'$ for $0 < i' \leq n+1$, we let $X(i)$ be the simplicial subset of $\Delta^n \times \Delta^1$ generated by $X(i-1)$ and the $(n+1)$-simplex $\sigma : \Delta^{n+1} \to \Delta^n \times \Delta^1$ given on vertices by the formula

$$\sigma(j) = \begin{cases} (j,0) & \text{if } j < i' \\ (j-1,1) & \text{if } j \geq i'. \end{cases}$$

We have a pushout diagram of simplicial sets

$$
\begin{array}{ccc}
\Lambda^{n+1}_{i'} & \longrightarrow & X(i-1) \\
\downarrow & & \downarrow \\
\Delta^{n+1} & \longrightarrow & X(i).
\end{array}
$$

If $1 \leq i' \leq n$, then the extension g_i of g_{i-1} exists because q' is an inner fibration. If $i = n+1$, then the desired extension exists because g_{i-1} carries $\Delta^{\{n,n+1\}}$ to an equivalence in $(\mathcal{C}^{\otimes}_{\mathrm{act}})_{f_0|K/}$.

$\square$

Proposition 3.1.1.8. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, and suppose we are given a finite collection $\{\overline{p}_i : K_i^{\triangleright} \to \mathcal{C}^{\otimes}_{\mathrm{act}}\}_{i \in I}$ of operadic q-colimit diagrams. Let $K = \prod_{i \in I} K_i$, and let $\overline{p}$ denote the composition*

$$K^{\triangleright} \to \prod_{i \in I} K_i^{\triangleright} \to \prod_{i \in I} \mathcal{C}^{\otimes}_{\mathrm{act}} \overset{\oplus_I}{\to} \mathcal{C}^{\otimes}_{\mathrm{act}}.$$

Then $\overline{p}$ is an operadic q-colimit diagram.

The proof of Proposition 3.1.1.8 depends on the following lemma:

Lemma 3.1.1.9. *Let $q : \mathcal{C}^{\times} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, let $p' : K_0 \times K_1^{\triangleright} \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ be a map, and let $p = p'|K_0 \times K_1$. Suppose that, for each vertex v of K_0, the induced map $p'_v : K_1^{\triangleright} \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ is a weak operadic q-colimit diagram. Then the map*

$$\theta : \mathcal{C}^{\mathrm{act}}_{p'/} \to \mathcal{C}^{\mathrm{act}}_{p/} \times_{\mathcal{O}^{\mathrm{act}}_{qp/}} \mathcal{O}^{\mathrm{act}}_{qp'/}$$

is a trivial Kan fibration.

Proof. For each simplicial subset $A \subseteq K_0$, let L_A denote the pushout $(K_0 \times K_1) \coprod_{(A \times K_1)} (A \times K_1^{\triangleright})$, and let $p_A = p'|L_A$. For $A' \subseteq A$, let $\theta^A_{A'}$ denote the map

$$\mathcal{C}^{\mathrm{act}}_{p_A/} \to \mathcal{C}^{\mathrm{act}}_{p_{A'}/} \times_{\mathcal{O}^{\mathrm{act}}_{qp_{A'}/}} \mathcal{O}^{\mathrm{act}}_{qp_A/} \, .$$

We will prove that each of the maps $\theta^A_{A'}$ is a trivial Kan fibration. Taking $A = K_0$ and $A' = \emptyset$, this will imply the desired result.

Working simplex-by-simplex on A, we may assume without loss of generality that A is obtained from A' by adjoining a single nondegenerate n-simplex whose boundary already belongs to A'. Replacing K_0 by A, we may assume that $K_0 = A = \Delta^n$ and $A' = \partial \Delta^n$. Working by induction on n, we may assume that the map $\theta^{A'}_{\emptyset}$ is a trivial Kan fibration. Since the map $\theta^A_{A'}$ is a left fibration (Proposition T.2.1.2.1), it is a trivial Kan fibration if and only if it is a categorical fibration (Corollary T.2.4.4.6). Since $\theta^A_{\emptyset} = \theta^{A'}_{\emptyset} \circ \theta^A_{A'}$, we can use the two-out-of-three property to reduce to the problem of showing that $\theta^A_{\emptyset}$ is a categorical equivalence. Since $A = \Delta^n$, the inclusion $\{n\} \subseteq A$ is left cofinal so that $\theta^A_{\{n\}}$ is a trivial Kan fibration. It therefore suffices to show that $\theta^{\{n\}}_{\emptyset}$ is a trivial Kan fibration, which follows from our assumption that each p'_v is a weak operadic q-colimit diagram. $\square$

Proof of Proposition 3.1.1.8. If I is empty, the desired result follows from Example 3.1.1.6. If I is a singleton, the result is obvious. To handle the general case, we can use induction on the cardinality of I to reduce to the case where I consists of two elements. Let us therefore suppose that we are given operadic q-colimit diagrams $\overline{p}_0 : K_0^{\triangleright} \to \mathcal{C}^{\otimes}$ and $\overline{p}_1 : K_1^{\triangleright} \to \mathcal{C}^{\otimes}$; we wish to prove that the induced map $\overline{p} : (K_0 \times K_1)^{\triangleright} \to \mathcal{C}^{\otimes}$ is an operadic q-colimit diagram. Let $X \in \mathcal{C}^{\otimes}$; we must show that the composition

$$(K_0 \times K_1)^{\triangleright} \longrightarrow \mathcal{C}^{\otimes} \xrightarrow{\oplus X} \mathcal{C}^{\otimes}$$

is a weak operadic q-colimit diagram. Replacing $\overline{p}_0$ by the composition

$$K_0^{\triangleright} \longrightarrow \mathcal{C}^{\otimes} \xrightarrow{\oplus X} \mathcal{C}^{\otimes},$$

we can reduce to the case where X is trivial; it will therefore suffice to prove that $\overline{p}$ is a weak operadic q-colimit diagram.

Let L denote the simplicial set $K_0 \times K_1^{\triangleright}$. We have a commutative diagram of simplicial sets

$$
\begin{array}{ccccc}
K & \longrightarrow & L & \longleftarrow & K_0 \times \{v\} \\
\downarrow & & \downarrow & & \downarrow \\
K^{\triangleright} & \longrightarrow & L^{\triangleright} & \longleftarrow & K_0^{\triangleright} \times \{v\}
\end{array}
$$

where v denotes the cone point of $K_1^{\triangleright}$. Let $\overline{p}'$ denote the composition

$$L^{\triangleright} \to K_0^{\triangleright} \times K_1^{\triangleright} \to \mathcal{C}^{\otimes}_{\mathrm{act}} \times \mathcal{C}^{\otimes}_{\mathrm{act}} \xrightarrow{\oplus} \mathcal{C}^{\otimes}_{\mathrm{act}} \, .$$

Since $\overline{p}_0$ is an operadic q-colimit diagram, we deduce that $\overline{p}'|(K_0^{\triangleright} \times \{v\})$ is a weak operadic q-colimit diagram. Since the inclusion $K_0 \times \{v\} \subseteq L$ is left cofinal, we deduce that $\overline{p}'$ is a weak operadic q-colimit diagram

(Remark 3.1.1.4). The inclusion $K^{\triangleright} \subseteq L^{\triangleright}$ is left cofinal (since the cone point is left cofinal in both). Consequently, to prove that $\bar{p}$ is a weak operadic q-colimit diagram, it will suffice to show that the map

$$\mathcal{C}^{\mathrm{act}}_{\bar{p}'/} \to \mathcal{C}^{\mathrm{act}}_{p/} \times_{\mathcal{O}^{\mathrm{act}}_{qp/}} \mathcal{O}^{\mathrm{act}}_{q\bar{p}'/}$$

is a trivial Kan fibration, where $p = \bar{p}|K$. Since $\bar{p}'$ is a weak operadic q-colimit diagram, we are reduced to proving that

$$\mathcal{C}^{\mathrm{act}}_{p'/} \times_{\mathcal{O}^{\mathrm{act}}_{qp'/}} \mathcal{O}^{\mathrm{act}}_{q\bar{p}'/} \to \mathcal{C}^{\mathrm{act}}_{p/} \times_{\mathcal{O}^{\mathrm{act}}_{qp/}} \mathcal{O}^{\mathrm{act}}_{q\bar{p}'/}$$

is a trivial Kan fibration, where $p' = \bar{p}'|L$. This map is a pullback of $\mathcal{C}^{\mathrm{act}}_{p'/} \to \mathcal{C}^{\mathrm{act}}_{p/} \times_{\mathcal{O}^{\mathrm{act}}_{qp/}} \mathcal{O}^{\mathrm{act}}_{qp'/}$, which is a trivial Kan fibration by Lemma 3.1.1.9 (since $\bar{p}_1$ is an operadic q-colimit diagram). $\qquad\square$

Proposition 3.1.1.10. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads. Suppose we are given a finite collection of operadic q-colimit diagrams $\{\bar{p}_i : K_i^{\triangleright} \to \mathcal{C}^{\otimes}_{\mathrm{act}}\}_{i \in I}$, where each $\bar{p}_i$ carries the cone point of $K_i^{\triangleright}$ into $\mathcal{C} \subseteq \mathcal{C}^{\otimes}$. Let $K = \prod_{i \in I} K_i$, and let $\bar{p} : K^{\triangleright} \to \mathcal{C}^{\otimes}$ be defined as in Proposition 3.1.1.8. If each of the simplicial sets K_i is weakly contractible, then $\bar{p}$ is a q-colimit diagram.*

We will need a few preliminaries:

Lemma 3.1.1.11. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, let $X \in \mathcal{C}^{\otimes}$ be an object lying over $\langle n \rangle \in \mathcal{F}\mathrm{in}_*$, and choose inert morphisms $f_i : X \to X_i$ in $\mathcal{C}^{\otimes}$ covering $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \le i \le n$. Then the maps $\{f_i : X \to X_i\}_{1 \le i \le n}$ determine a q-limit diagram $\chi : \langle n \rangle^{\circ \triangleleft} \to \mathcal{C}^{\otimes}$.*

Proof. Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be the map which exhibits $\mathcal{O}^{\otimes}$ as an ∞-operad. Since each $q(f_i)$ is an inert morphism in $\mathcal{O}^{\otimes}$, we can invoke the definition of an ∞-operad to deduce that $q \circ \chi$ is a p-limit diagram in $\mathcal{O}^{\otimes}$. Similarly, we conclude that χ is a $p \circ q$-limit diagram in $\mathcal{C}^{\otimes}$. The desired result now follows from Proposition T.4.3.1.5. $\qquad\square$

Lemma 3.1.1.12. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, let $\bar{p} : K^{\triangleright} \to \mathcal{C}^{\otimes}$ be a diagram, and let $p = \bar{p}|K$. Then $\bar{p}$ is a q-colimit diagram if and only if the induced map $\phi_0 : \mathcal{C}_{\bar{p}/} \to \mathcal{C}_{p/} \times_{\mathcal{O}_{qp/}} \mathcal{O}_{q\bar{p}/}$ is a trivial Kan fibration.*

Proof. The "only if" direction is obvious. To prove the converse, suppose that ϕ_0 is a trivial Kan fibration, and consider the map $\phi : \mathcal{C}^{\otimes}_{\bar{p}/} \to \mathcal{C}^{\otimes}_{p/} \times_{\mathcal{O}^{\otimes}_{qp/}} \mathcal{O}^{\otimes}_{q\bar{p}/}$. We wish to prove that ϕ is a trivial Kan fibration. Since ϕ is a left fibration (Proposition T.2.1.2.1), it suffices to show that the fibers of ϕ are contractible (Lemma T.2.1.3.4). Let $\overline{X}$ be an object in the codomain of ϕ having image $X \in \mathcal{C}^{\otimes}$ and image $\langle n \rangle \in \mathcal{F}\mathrm{in}_*$. For $1 \le i \le n$, choose an inert morphism $X \to X_i$ in $\mathcal{C}^{\otimes}$ lying over $\rho^i : \langle n \rangle \to \langle 1 \rangle$. Since the projection $\mathcal{C}^{\otimes}_{p/} \times_{\mathcal{O}^{\otimes}_{qp/}} \mathcal{O}^{\otimes}_{q\bar{p}/} \to \mathcal{C}^{\otimes}$ is a left fibration (Proposition T.2.1.2.1), we can lift each of these inert morphisms in an essentially unique way to a map $\overline{X} \to \overline{X}_i$. Using Lemma 3.1.1.11, we deduce that the fiber $\phi^{-1}(X)$ is homotopy equivalent to the product $\prod_{1 \le i \le n} \phi^{-1}(X_i)$, and therefore contractible (since $\phi^{-1}(X_i) = \phi_0^{-1}(X_i)$ is a fiber of the trivial Kan fibration ϕ_0 for $1 \le i \le n$), as desired. $\qquad\square$

Proof of Proposition 3.1.1.10. In view of Lemma 3.1.1.12, it will suffice to show that the map $\phi_0 : \mathcal{C}_{\bar{p}/} \to \mathcal{C}_{p/} \times_{\mathcal{O}_{qp/}} \mathcal{O}_{q\bar{p}/}$ is a trivial Kan fibration. Since ϕ_0 is a left fibration (Proposition T.2.1.2.1), it will suffice to show that $\phi_0^{-1}\{\overline{X}\}$ is contractible for each vertex $\overline{X} \in \mathcal{C}_{p/} \times_{\mathcal{O}_{qp/}} \mathcal{O}_{q\bar{p}/}$ (Lemma T.2.1.3.4). Such a vertex determines a map $\alpha : I_* \to \langle 1 \rangle$; let $I_0 = \alpha^{-1}\{1\} \subseteq I$. Let $K' = \prod_{i \in I_0} K_i$, let $\bar{p}'$ denote the composite map

$$K'^{\triangleright} \to \prod_{i \in I_0} K_i^{\triangleright} \to \prod_{i \in I_0} \mathcal{C}^{\otimes} \xrightarrow{\oplus} \mathcal{C}^{\otimes},$$

and let $\bar{p}''$ denote the composition $K^{\triangleright} \to K'^{\triangleright} \xrightarrow{\bar{p}'} \mathcal{C}^{\otimes}$.

There is a canonical map $\overline{P} : \Delta^1 \times K^{\triangleright} \to \mathcal{C}^{\otimes}$ which is an inert natural transformation from $\overline{p}$ to $\overline{p}''$. Let $p'' = \overline{p}''|K$ and $P = \overline{P}|\Delta^1 \times K$, so we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{C}_{\overline{p}/} & \xrightarrow{\phi_0} & \mathcal{C}_{p/} \times_{\mathcal{O}_{qp/}} \mathcal{O}_{q\overline{p}/} \\
\uparrow & & \uparrow \\
\mathcal{C}_{\overline{P}/} & \longrightarrow & \mathcal{C}_{P/} \times_{\mathcal{O}_{qP/}} \mathcal{O}_{q\overline{P}/} \\
\downarrow & & \downarrow \\
\mathcal{C}_{\overline{p}''/} & \longrightarrow & \mathcal{C}_{p''/} \times_{\mathcal{O}_{qp''/}} \mathcal{O}_{q\overline{p}''/}
\end{array}
$$

which determines a homotopy equivalence $\phi_0^{-1}\overline{X}$ with a fiber of the map $\psi : \mathcal{C}^{\mathrm{act}}_{\overline{p}''/} \to \mathcal{C}^{\mathrm{act}}_{p''/} \times_{\mathcal{O}^{\mathrm{act}}_{qp''/}} \mathcal{O}^{\mathrm{act}}_{q\overline{p}''/}$. It will therefore suffice to show that the fibers of ψ are contractible. Since ψ is a left fibration (Proposition T.2.1.2.1), this is equivalent to the assertion that ψ is a categorical equivalence (Corollary T.2.4.4.6).

Because the simplicial sets $\{K_i\}_{i \in I - I_0}$ are weakly contractible, the projection map $K \to K'$ is left cofinal. Consequently, we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{C}^{\mathrm{act}}_{\overline{p}'/} & \xrightarrow{\psi'} & \mathcal{C}^{\mathrm{act}}_{p'/} \times_{\mathcal{O}^{\mathrm{act}}_{qp'/}} \mathcal{O}^{\mathrm{act}}_{q\overline{p}'/} \\
\downarrow & & \downarrow \\
\mathcal{C}^{\mathrm{act}}_{\overline{p}''/} & \xrightarrow{\psi} & \mathcal{C}^{\mathrm{act}}_{p''/} \times_{\mathcal{O}^{\mathrm{act}}_{qp''/}} \mathcal{O}^{\mathrm{act}}_{q\overline{p}''/}
\end{array}
$$

where the vertical maps are categorical equivalences. Using a two-out-of-three argument, we are reduced to proving that ψ' is a categorical equivalence. We conclude by observing that Proposition 3.1.1.8 guarantees that $\overline{p}'$ is an operadic q-colimit diagram, so that ψ' is a trivial Kan fibration. $\square$

Proposition 3.1.1.13. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads and let $K = \Delta^0$. The following conditions are equivalent:*

(1) *The map q is a coCartesian fibration of ∞-operads.*

(2) *For every map $p : K \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ and every extension $\overline{p}_0 : K^{\triangleright} \to \mathcal{O}^{\otimes}_{\mathrm{act}}$ of $q \circ p$ carrying the cone point of $K^{\triangleright}$ into $\mathcal{O}$, there exists an operadic q-colimit diagram $\overline{p} : K^{\triangleright} \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ which extends p and lifts $\overline{p}_0$.*

Lemma 3.1.1.14. *Let $\mathcal{O}^{\otimes}$ be an ∞-operad and let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a $\mathcal{O}$-monoidal ∞-category. Let $\overline{p} : \Delta^1 \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ classify an active q-coCartesian morphism $X \to Y$ in $\mathcal{C}^{\otimes}$. Then $\overline{p}$ is an operadic q-colimit diagram.*

Proof. Let $Z \in \mathcal{C}^{\otimes}$. Replacing p by its composition with the functor $\mathcal{C}^{\otimes} \xrightarrow{\oplus Z} \mathcal{C}^{\otimes}$ (and using Remark 2.2.4.8), we can reduce to showing that $\overline{p}$ is a weak operadic q-colimit diagram, which is clear. $\square$

Proof of Proposition 3.1.1.13. To prove that (1) implies (2), we observe that we can take $\overline{p}$ to be a q-coCartesian lift of $\overline{p}_0$ (by virtue of Lemma 3.1.1.14). Conversely, suppose that (2) is satisfied. Choose an object $\overline{X} \in \mathcal{C}^{\otimes}$, let $X = q(\overline{X})$, and suppose we are given a morphism $\alpha : X \to Z$ in $\mathcal{O}^{\otimes}$; we wish to prove that we can lift α to a q-coCartesian morphism $\overline{X} \to \overline{Z}$ in $\mathcal{C}^{\otimes}$. Using Proposition 2.1.2.4, we can factor α as a composition $X \xrightarrow{\alpha'} Y \xrightarrow{\alpha''} Z$ where α' is inert and α'' is active. Let α'_0 denote the image of α' in $\mathcal{F}\mathrm{in}_*$, and choose an inert morphism $\overline{\alpha}' : \overline{X} \to \overline{Y}$ in $\mathcal{C}^{\otimes}$ lifting α'_0. Then $q(\overline{\alpha}')$ is an inert lift of α'_0, so we may assume without loss of generality that $\overline{\alpha}'$ lifts α'. Proposition T.2.4.1.3 guarantees that $\overline{\alpha}'$ is q-coCartesian. Since the collection of q-coCartesian morphisms in $\mathcal{C}^{\otimes}$ is stable under composition (Proposition T.2.4.1.7), we may replace $\overline{X}$ by $\overline{Y}$ and thereby reduce to the case where the map α is active.

Let $\langle n \rangle$ denote the image of Z in $\mathcal{F}\mathrm{in}_*$. Since $\mathcal{C}^\otimes$ and $\mathcal{C}^\otimes$ are ∞-operads, we can identify $\overline{X}$ with a concatenation $\oplus_{1 \leq i \leq n} \overline{X}_i$ and α with $\oplus_{1 \leq i \leq n} \alpha_i$, where each $\alpha_i : X_i = q(\overline{X}_i) \to Z_i$ is an active morphism in $\mathcal{O}^\otimes$ where Z_i lies over $\langle 1 \rangle \in \mathcal{F}\mathrm{in}_*$. Assumption (2) guarantees that each α_i can be lifted to an operadic q-colimit diagram $\overline{\alpha}_i : \overline{X}_i \to \overline{Z}_i$ in $\mathcal{C}^\otimes$. It follows from Proposition 3.1.1.10 that the concatenation $\overline{\alpha} = \oplus_{1 \leq i \leq n} \overline{\alpha}_i$ is q-coCartesian. The map $q(\overline{\alpha})$ is equivalent to $\oplus_{1 \leq i \leq n} q(\overline{\alpha}_i) \simeq \alpha$. Since q is a categorical fibration, we can replace $\overline{\alpha}$ by an equivalent morphism if necessary to guarantee that $q(\overline{\alpha}) = \alpha$, which completes the proof of (1). $\qquad\square$

Our next two results, which are counterparts of Propositions T.4.3.1.9 and T.4.3.1.10, are useful for detecting the existence of operadic colimit diagrams:

Proposition 3.1.1.15. *Let* $q : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ *be a fibration of ∞-operads, let K be a simplicial set, and let* $\overline{h} : \Delta^1 \times K^\triangleright \to \mathcal{C}^\otimes_{\mathrm{act}}$ *be a natural transformation from* $\overline{h}_0 = \overline{h}|\{0\} \times K^\triangleright$ *to* $\overline{h}_1 = \overline{h}|\{1\} \times K^\triangleright$. *Suppose that*

(a) *For every vertex x of $K^\triangleright$, the restriction $\overline{h}|\Delta^1 \times \{x\}$ is a q-coCartesian edge of $\mathcal{C}^\otimes$.*

(b) *The composition $\Delta^1 \times \{v\} \subseteq \Delta^1 \times K^\triangleright \xrightarrow{\overline{h}} \mathcal{C}^\otimes \xrightarrow{q} \mathcal{O}^\otimes$ is an equivalence in $\mathcal{O}^\otimes$, where v denotes the cone point of $K^\triangleright$.*

Then:

(1) *The map $\overline{h}_0$ is a weak operadic q-colimit diagram if and only if $\overline{h}_1$ is a weak operadic q-colimit diagram.*

(2) *Assume that q is a coCartesian fibration. Then $\overline{h}_0$ is an operadic q-colimit diagram if and only if $\overline{h}_1$ is an operadic q-colimit diagram.*

Proof. Assertion (2) follows from (1) and Remark 2.2.4.8 (after composing with the functor $\mathcal{C}^\otimes \xrightarrow{\oplus X} \mathcal{C}^\otimes$ determined by an arbitrary object $X \in \mathcal{C}^\otimes$). It will therefore suffice to prove (1). Let $h = \overline{h}|\Delta^1 \times K$, $h_0 = h|\{0\} \times K$, and $h_1 = h|\{1\} \times K$. We have a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{C}^{\mathrm{act}}_{\overline{h}_0/} & \xleftarrow{\;\phi\;} & \mathcal{C}^{\mathrm{act}}_{\overline{h}/} & \longrightarrow & \mathcal{C}^{\mathrm{act}}_{\overline{h}_1/} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{C}^{\mathrm{act}}_{h_0/} \times_{\mathcal{O}^{\mathrm{act}}_{qh_0/}} \mathcal{O}^{\mathrm{act}}_{q\overline{h}_0/} & \xleftarrow{\;\psi\;} & \mathcal{C}^{\mathrm{act}}_{h/} \times_{\mathcal{O}^{\mathrm{act}}_{qh/}} \mathcal{O}^{\mathrm{act}}_{q\overline{h}/} & \longrightarrow & \mathcal{C}^{\mathrm{act}}_{h_1/} \times_{\mathcal{O}^{\mathrm{act}}_{qh_1/}} \mathcal{O}^{\mathrm{act}}_{q\overline{h}_1/} \, .
\end{array}
$$

According to Remark 3.1.1.5, it suffices to show that the left vertical map is a categorical equivalence if and only if the right vertical map is a categorical equivalence. Because the inclusions $\{1\} \times K \subseteq \Delta^1 \times K$ and $\{1\} \times K^\triangleright \subseteq \Delta^1 \times K^\triangleright$ are right anodyne, the horizontal maps on the right are trivial fibrations. Using a diagram chase, we are reduced to proving that the maps ϕ and ψ are categorical equivalences.

Let $f : x \to y$ denote the morphism $\mathcal{C}^\otimes$ obtained by restricting $\overline{h}$ to the cone point of $K^\triangleright$. The map ϕ fits into a commutative diagram

$$
\begin{array}{ccc}
\mathcal{C}^{\mathrm{act}}_{\overline{h}/} & \xrightarrow{\;\phi\;} & \mathcal{C}^{\mathrm{act}}_{h_0/} \\
\downarrow & & \downarrow \\
\mathcal{C}^{\mathrm{act}}_{f/} & \longrightarrow & \mathcal{C}^{\mathrm{act}}_{x/} \, .
\end{array}
$$

Since the inclusion of the cone point into $K^\triangleright$ is right anodyne, the vertical arrows are trivial fibrations. Moreover, hypotheses (1) and (2) guarantee that f is an equivalence in $\mathcal{C}^\otimes$, so that the map $\mathcal{C}^{\mathrm{act}}_{f/} \to \mathcal{C}^{\mathrm{act}}_{x/}$ is a trivial fibration. This proves that ϕ is a categorical equivalence.

The map ψ factors as a composition

$$
\mathcal{C}^{\mathrm{act}}_{h/} \times_{\mathcal{O}^{\mathrm{act}}_{qh/}} \mathcal{O}^{\mathrm{act}}_{q\overline{h}/} \xrightarrow{\;\psi'\;} \mathcal{C}^{\mathrm{act}}_{h_0/} \times_{\mathcal{O}^{\mathrm{act}}_{qh_0/}} \mathcal{O}^{\mathrm{act}}_{q\overline{h}/} \xrightarrow{\;\psi''\;} \mathcal{O}^{\mathrm{act}}_{h_0} \times_{\mathcal{O}^{\mathrm{act}}_{qh_0/}} \mathcal{O}^{\mathrm{act}}_{q\overline{h}_0/} \, .
$$

To complete the proof, it will suffice to show that ψ' and ψ'' are trivial fibrations of simplicial sets. We first observe that ψ' is a pullback of the map $\mathcal{C}^{\mathrm{act}}_{h/} \to \mathcal{C}^{\mathrm{act}}_{h_0/} \times_{\mathcal{O}^{\mathrm{act}}_{qh_0/}} \mathcal{O}^{\mathrm{act}}_{qh/}$, which is a trivial Kan fibration (Proposition T.3.1.1.11). The map ψ'' is a pullback of the left fibration $\psi''_0 : \mathcal{O}^{\mathrm{act}}_{q\bar{h}/} \to \mathcal{O}^{\mathrm{act}}_{q\bar{h}_0/}$. It therefore suffices to show that ψ''_0 is a categorical equivalence. To prove this, we consider the diagram

$$
\begin{array}{ccc}
\mathcal{O}^{\mathrm{act}}_{q\bar{h}/} & \xrightarrow{\psi''_0} & \mathcal{O}^{\mathrm{act}}_{q\bar{h}_0/} \\
\downarrow & & \downarrow \\
\mathcal{O}^{\mathrm{act}}_{q(f)/} & \xrightarrow{\psi''_1} & \mathcal{O}^{\mathrm{act}}_{q(x)/}\,.
\end{array}
$$

As above, we observe that the vertical arrows are trivial fibrations and that ψ''_1 is a trivial fibration (because the morphism $q(f)$ is an equivalence in $\mathcal{O}^\otimes$). It follows that ψ''_0 is a categorical equivalence, as desired. $\square$

In the case where $q : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ is a coCartesian fibration, Proposition 3.1.1.15 allows us to reduce the problem of testing whether a diagram $\bar{p} : K^\triangleright \to \mathcal{C}^\otimes_{\mathrm{act}}$ is an operadic q-colimit to the special case where $\bar{p}$ factors through $\mathcal{C}^\otimes_X$, for some $X \in \mathcal{O}^\otimes$. In this case, we can apply the following criterion:

Proposition 3.1.1.16. *Let $q : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a coCartesian fibration of ∞-operads, let $X \in \mathcal{O}^\otimes$ be an object, and let $\bar{p} : K^\triangleright \to \mathcal{C}^\otimes_X$ be a diagram. The following conditions are equivalent:*

(1) *The map $\bar{p}$ is an operadic q-colimit diagram.*

(2) *For every object $\overline{Y} \in \mathcal{C}^\otimes$ with image $Y \in \mathcal{O}^\otimes$, every object $Z \in \mathcal{O}$, and every active morphism $m : X \oplus Y \to Z$ in $\mathcal{O}$, the compositon*

$$
K^\triangleright \longrightarrow \mathcal{C}^\otimes_X \xrightarrow{\oplus \overline{Y}} \mathcal{C}^\otimes_{X \oplus Y} \xrightarrow{m_!} \mathcal{C}_Z
$$

is a colimit diagram in the ∞-category $\mathcal{C}_Z$.

Example 3.1.1.17. Let $q : \mathcal{C}^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ exhibit $\mathcal{C}$ as a symmetric monoidal ∞-category. Then a diagram $\bar{p} : K^\triangleright \to \mathcal{C}$ is an operadic q-colimit diagram if and only if, for every object $C \in \mathcal{C}$, the induced map $K^\triangleright \xrightarrow{\bar{p}} \mathcal{C} \xrightarrow{\otimes C} \mathcal{C}$ is a colimit diagram in $\mathcal{C}$. More informally: an operadic colimit diagram in $\mathcal{C}$ is a colimit diagram which remains a colimit diagram after tensoring with any object of $\mathcal{C}$.

Proof of Proposition 3.1.1.16. Replacing $\bar{p}$ by its image under the functor $\mathcal{C}^\otimes \xrightarrow{\oplus \overline{Y}} \mathcal{C}^\otimes$, we are reduced to proving the equivalence of the following pair of assertions:

$(1')$ The map $\bar{p}$ is a weak operadic q-colimit diagram.

$(2')$ For every object $Z \in \mathcal{O}$ and every active morphism $m : X \to Z$ in $\mathcal{O}^\otimes$, the functor $m_! : \mathcal{C}^\otimes_X \to \mathcal{C}_Z$ carries $\bar{p}$ to a colimit diagram in $\mathcal{C}_Z$.

Assertion $(1')$ is equivalent to the statement that the map $\theta : \mathcal{C}^{\mathrm{act}}_{\bar{p}/} \to \mathcal{C}^{\mathrm{act}}_{p/} \times_{\mathcal{O}^{\mathrm{act}}_{qp/}} \mathcal{O}^{\mathrm{act}}_{q\bar{p}/}$ is a trivial fibration of simplicial sets. Since θ is a left fibration (Proposition T.2.1.2.1), it will suffice to show that the fibers of θ are contractible. Consider an arbitrary vertex of $\mathcal{O}^{\mathrm{act}}_{q\bar{p}/}$ corresponding to a diagram $t : K \star \Delta^1 \to \mathcal{O}^\otimes_{\mathrm{act}}$. Since $K \star \Delta^1$ is categorically equivalent to $(K \star \{0\}) \coprod_{\{0\}} \Delta^1$ and $t|K \star \{0\}$ is constant, we may assume without loss of generality that t factors as a composition

$$
K \star \Delta^1 \to \Delta^1 \xrightarrow{m} \mathcal{O}^\otimes_{\mathrm{act}}\,.
$$

Here $m : X \to Z$ can be identified with an active morphism in $\mathcal{O}^\otimes$. It will therefore suffice to show that the following assertions are equivalent, where $m : X \to Z$ is *fixed*:

(1″) The map $\mathcal{C}^{\mathrm{act}}_{\overline{p}/} \times_{\mathcal{O}^{\mathrm{act}}_{q\overline{p}/}} \{t\} \to \mathcal{C}^{\mathrm{act}}_{p/} \times_{\mathcal{O}^{\mathrm{act}}_{qp/}} \{t_0\}$ is a homotopy equivalence, where $t \in \mathcal{O}^{\otimes}_{q\overline{p}/}$ is determined by m as above and t_0 is the image of t in $\mathcal{O}^{\mathrm{act}}_{qp/}$.

(2″) The map $m_! : \mathcal{C}^{\otimes}_X \to \mathcal{C}_Z$ carries $\overline{p}$ to a colimit diagram in $\mathcal{C}_Z$.

This equivalence results from the observation that the fibers of the maps $\mathcal{C}^{\mathrm{act}}_{\overline{p}/} \to \mathcal{O}^{\mathrm{act}}_{q\overline{p}/}$ and $\mathcal{C}^{\mathrm{act}}_{p/} \to \mathcal{O}^{\mathrm{act}}_{qp/}$ are equivalent to the fibers of the maps $\mathcal{C}^{\mathrm{act}}_{m_!\overline{p}/} \to \mathcal{O}^{\mathrm{act}}_{qm_!\overline{p}/}$ and $\mathcal{C}^{\mathrm{act}}_{m_!p/} \to \mathcal{O}^{\mathrm{act}}_{qm_!p/}$. $\square$

We now establish a general criterion for the existence of operadic colimit diagrams.

Definition 3.1.1.18. Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads and let K be a simplicial set. We will say that q is *compatible with K-indexed colimits* if the following conditions are satisfied:

(1) For every object $X \in \mathcal{O}$, the ∞-category $\mathcal{C}_X$ admits K-indexed colimits.

(2) For every operation $f \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{1 \leq i \leq n}, Y)$, the functor $\otimes_t : \prod_{1 \leq i \leq n} \mathcal{C}_{X_i} \to \mathcal{C}_Y$ of Remark 2.1.2.16 preserves K-indexed colimits separately in each variable.

Variant 3.1.1.19. If $\mathcal{K}$ is some collection of simplicial sets, we will say that a coCartesian fibration of ∞-operads $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is *compatible with $\mathcal{K}$-indexed colimits* if it compatible with K-indexed colimits, for every $K \in \mathcal{K}$. For example, we may speak of q being compatible with filtered colimits, sifted colimits, coproducts, or κ-small colimits where κ is some regular cardinal.

Proposition 3.1.1.20. *Let K be a simplicial set, let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads which is compatible with K-indexed colimits, let $p : K \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ be a diagram, and let $\overline{p}_0 : K^{\triangleright} \to \mathcal{O}^{\otimes}_{\mathrm{act}}$ be an extension of qp which carries the cone point of $K^{\triangleright}$ to an object $X \in \mathcal{O}$. Then there exists an operadic q-colimit diagram $\overline{p} : K^{\triangleright} \to \mathcal{C}^{\otimes}$ which extends p and lifts $\overline{p}_0$.*

Proof. Let $p_0 = qp$. The map $\overline{p}_0$ determines a natural transformation $\alpha : p_0 \to \underline{X}$ of diagrams $K \to \mathcal{O}^{\otimes}$, where $\underline{X}$ denotes the constant diagram taking the value X. Choose a q-coCartesian natural transformation $\overline{\alpha} : p \to p'$ lifting α. Since $\mathcal{C}_X$ admits K-indexed colimits, we can extend $\overline{p}'$ to a colimit diagram $\overline{p}' : K^{\triangleright} \to \mathcal{C}_X$. The compatibility of q with K-indexed colimits and Proposition 3.1.1.16 imply that $\overline{p}'$ is an operadic q-colimit diagram. Let $C \in \mathcal{C}_X$ be the image under $\overline{p}'$ of the cone point of $K^{\triangleright}$, so we can regard $\overline{p}'$ as a diagram $K \to \mathcal{C}^{\otimes}_{/C}$ which lifts p'. Using the assumption that $\mathcal{C}^{\otimes}_{/C} \to \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{O}^{\otimes}_{/X}$ is a right fibration, we can choose a transformation $\overline{p} \to \overline{p}'$ lifting $\overline{\alpha} : p \to p'$. It follows from Proposition 3.1.1.15 that $\overline{p}$ is an extension of p with the desired properties. $\square$

Corollary 3.1.1.21. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads and let κ be a regular cardinal. The following conditions are equivalent:*

(1) *The map q is a coCartesian fibration of ∞-operads which is compatible with κ-small colimits.*

(2) *For every κ-small simplicial set K and every diagram*

$$
\begin{array}{ccc}
K & \xrightarrow{\ p\ } & \mathcal{C}^{\otimes}_{\mathrm{act}} \\
\downarrow & \overline{p} \nearrow & \downarrow \\
K^{\triangleright} & \xrightarrow{\ \overline{p}_0\ } & \mathcal{O}^{\otimes}
\end{array}
$$

such that $\overline{p}_0$ carries the cone point of $K^{\triangleright}$ into $\mathcal{O}$, there exists an extension $\overline{p}$ of p as indicated in the diagram, which is an operadic q-colimit diagram.

Proof. The implication $(1) \Rightarrow (2)$ is immediate from Proposition 3.1.1.20. Assume that (2) is satisfied. Taking $K = \Delta^0$ and applying Proposition 3.1.1.13, we deduce that q is a coCartesian fibration of ∞-operads. Applying (2) in the case where the map $\bar{p}_0$ takes some constant value $X \in \mathcal{O}$, we deduce that every κ-small diagram $K \to \mathcal{C}_X^{\otimes}$ can be extended to an operadic q-colimit diagram. This operadic q-colimit diagram is in particular a colimit diagram, so that $\mathcal{C}_X^{\otimes}$ admits κ-small colimits. Moreover, the uniqueness properties of colimits show that every κ-small colimit diagram $\bar{p} : K^{\triangleright} \to \mathcal{C}_X^{\otimes}$ is a q-operadic colimit diagram. Combining this with the criterion of Proposition 3.1.1.16, we conclude that the q is compatible with κ-small colimits. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.1.2   Operadic Left Kan Extensions

In §3.1.1, we introduced the theory of operadic colimit diagrams: this is an analogue of the usual theory of colimits to the setting of ∞-operads. The theory of colimits can be regarded as a special case of the theory of left Kan extensions: if K and $\mathcal{C}$ are ∞-categories, then a map $\bar{p} : K^{\triangleright} \to \mathcal{C}$ is a colimit diagram if and only if $\bar{p}$ is a left Kan extension of $p = \bar{p}|K$. The theory of left Kan extensions also has a counterpart in the setting of ∞-operads, which we will call *operadic left Kan extensions.* Our goal in this section is to introduce this counterpart (Definition 3.1.2.2) and to prove a basic existence result (Theorem 3.1.2.3). The results of this section will play a crucial role in our discussion of free algebras in §3.1.3.

Definition 3.1.2.1. A *correspondence of ∞-operads* is a Δ^1-family of ∞-operads $p : \mathcal{M}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^1$. In this case, we will say that $\mathcal{M}^{\otimes}$ is a *correspondence* from the ∞-operad $\mathcal{A}^{\otimes} = \mathcal{M}^{\otimes} \times_{\Delta^1} \{0\}$ to the ∞-operad $\mathcal{B}^{\otimes} = \mathcal{M}^{\otimes} \times_{\Delta^1} \{1\}$.

Definition 3.1.2.2. Let $\mathcal{M}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^1$ be a correspondence from an ∞-operad $\mathcal{A}^{\otimes}$ to another ∞-operad $\mathcal{B}^{\otimes}$, let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, and let $\overline{F} : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes}$ be a map of generalized ∞-operads. We will say that $\overline{F}$ is an *operadic q-left Kan extension* of $F = \overline{F}|\mathcal{A}^{\otimes}$ if the following condition is satisfied, for every object $B \in \mathcal{B}^{\otimes}$:

$(*)$ Let $K = (\mathcal{M}_{\mathrm{act}}^{\otimes})_{/B} \times_{\mathcal{M}^{\otimes}} \mathcal{A}^{\otimes}$. Then the composite map

$$K^{\triangleright} \to (\mathcal{M}^{\otimes})_{/B}^{\triangleright} \to \mathcal{M}^{\otimes} \xrightarrow{\overline{F}} \mathcal{C}^{\otimes}$$

is an operadic q-colimit diagram.

In the situation of Definition 3.1.2.2, the value of $\overline{F}$ on any object $B \in \mathcal{B}$ is determined up to equivalence by requirement $(*)$. Consequently, it is natural to expect that we can reconstruct $\overline{F}$ from its restriction $F = \overline{F}|\mathcal{A}^{\otimes}$ together with the composite map $q\overline{F} : \mathcal{M}^{\otimes} \to \mathcal{O}^{\otimes}$. More precisely, we have the following result concerning the existence and uniqueness of $\overline{F}$:

Theorem 3.1.2.3. *Let $n \geq 1$, let $p : \mathcal{M}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^n$ be a Δ^n-family of ∞-operads, and let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads. Suppose we are given a commutative diagram of ∞-operad family maps*

$$
\begin{array}{ccc}
\mathcal{M}^{\otimes} \times_{\Delta^n} \Lambda_0^n & \xrightarrow{f_0} & \mathcal{C}^{\otimes} \\
\downarrow & \nearrow{\scriptstyle f} & \downarrow{\scriptstyle q} \\
\mathcal{M}^{\otimes} & \xrightarrow{g} & \mathcal{O}^{\otimes}.
\end{array}
$$

Then

(A) *Suppose that $n = 1$. The following conditions are equivalent:*

    (a) *There exists a dotted arrow f as indicated in the diagram which is an operadic q-left Kan extension of f_0 (in particular, such that f is a map of ∞-operad families).*

    (b) *For every object $B \in \mathcal{M} \times_{\Delta^n} \{1\}$, the diagram*

$$(\mathcal{M}^{\otimes}_{\mathrm{act}})_{/B} \times_{\Delta^n} \{0\} \to \mathcal{M}^{\otimes} \times_{\Delta^n} \{0\} \xrightarrow{f_0} \mathcal{C}^{\otimes}$$

    can be extended to an operadic q-colimit diagram lifting the map

$$((\mathcal{M}^{\otimes}_{\mathrm{act}})_{/B} \times_{\Delta^n} \{0\})^{\triangleright} \to \mathcal{M}^{\otimes} \xrightarrow{g} \mathcal{O}^{\otimes}.$$

(B) *Suppose that $n > 1$, and that the restriction of f_0 to $\mathcal{M}^{\otimes} \times_{\Delta^n} \Delta^{\{0,1\}}$ is an operadic q-left Kan extension of $f_0|(\mathcal{M}^{\otimes} \times_{\Delta^n} \{0\})$. Then there exists a dotted arrow f as indicated in the diagram (automatically a map of generalized ∞-operads).*

    The remainder of this section is devoted to the proof of Theorem 3.1.2.3. Though the proof is somewhat complicated, the details of our argument will not be needed for subsequent applications; consequently, readers are invited to skip the remainder of this section and proceed directly to the applications which are presented in §3.1.3.

    The proof of Theorem 3.1.2.3 will require the following Lemma.

Lemma 3.1.2.4. *Suppose we are given an inner fibration of simplicial sets $p : \mathcal{C} \to \Lambda^3_2$ satisfying the following condition: for every object $X \in \mathcal{C}_1$, there exists a p-coCartesian morphism $f : X \to Y$ where $Y \in \mathcal{C}_2$. Then there exists a pullback diagram*

$$
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & \mathcal{C}' \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\
\Lambda^3_2 & \longrightarrow & \Delta^3
\end{array}
$$

where $\mathcal{C}'$ is an ∞-category and the map $\mathcal{C} \hookrightarrow \mathcal{C}'$ is a categorical equivalence.

Proof. We will construct a sequence of categorical equivalences

$$\mathcal{C} = \mathcal{C}(0) \subseteq \mathcal{C}(1) \subseteq \cdots$$

in $(\mathrm{Set}_{\Delta})_{/\Delta^3}$ with the following properties:

    (i) If $0 < i < n$ and $\Lambda^n_i \to \mathcal{C}(m)$ is a map, then the induced map $\Lambda^n_i \to \mathcal{C}(m+1)$ can be extended to an n-simplex of $\mathcal{C}(m+1)$.

    (ii) For each $m \geq 0$, the inclusion $\mathcal{C} \subseteq \mathcal{C}(m) \times_{\Delta^3} \Lambda^3_2$ is an isomorphism of simplicial sets.

Our construction proceeds by induction on m. Assume that $\mathcal{C}(m)$ has been constructed, and let A denote the collection of all maps $\alpha : \Lambda^n_i \to \mathcal{C}(m)$ for $0 < i < n$. We will prove that, for each $\alpha \in A$, there exists a categorical equivalence $\mathcal{C}(m) \subseteq \mathcal{C}(m, \alpha)$ in $(\mathrm{Set}_{\Delta})_{/\Delta^3}$ with the following properties:

    (i') The composite map $\Lambda^n_i \xrightarrow{\alpha} \mathcal{C}(m) \subseteq \mathcal{C}(m, \alpha)$ can be extended to an n-simplex of $\mathcal{C}(m, \alpha)$.

    (ii') The map $\mathcal{C} \to \mathcal{C}(m, \alpha) \times_{\Delta^3} \Lambda^3_2$ is an isomorphism of simplicial sets.

Assuming that we can satisfy these conditions, we can complete the proof by defining $\mathcal{C}(m+1)$ to be the amalgamation $\coprod_{\alpha \in A} \mathcal{C}(m, \alpha)$ in the category $(\mathrm{Set}_{\Delta})_{\mathcal{C}(m)//\Delta^3}$.

    Suppose now that $\alpha : \Lambda^n_i \to \mathcal{C}(m)$ is given. If the composite map $\alpha_0 : \Lambda^n_i \xrightarrow{\alpha} \mathcal{C}(m) \to \Delta^3$ factors through Λ^3_2, then we can use the assumption that p is an inner fibration to extend α to an n-simplex of $\mathcal{C}$. In this case, we can satisfy requirements (i') and (ii') by setting $\mathcal{C}(m, \alpha) = \mathcal{C}(m)$. We may therefore assume without loss of generality that the image of α_0 contains $\Delta^{\{0,1,3\}} \subseteq \Delta^3$. In particular, we see that $\alpha_0(0)$ and $\alpha_0(n) = 3$.

We observe that α_0 extends uniquely to a map $\overline{\alpha}_0 : \Delta^n \to \Delta^3$. The pushout $\mathcal{C}(m) \coprod_{\Lambda_i^n} \Delta^n$ evidently satisfies condition (i'). It satisfies condition (ii') unless $\overline{\alpha}_0$ carries $\Delta^{\{0,\ldots,i-1,i+1,\ldots,n\}}$ into Λ_2^3. If condition (ii') is satisfied, we can set $\mathcal{C}(m,\alpha) = \mathcal{C}(m) \coprod_{\Lambda_i^n} \Delta^n$; otherwise, we have $\overline{\alpha}_0^{-1}\{1\} = \{i\}$.

Let $A = \overline{\alpha}_0^{-1}\{0\} \subseteq \Delta^n$, $B = \overline{\alpha}_0^{-1}\{2\} \subseteq \Delta^n$, and $C = \overline{\alpha}_0^{-1}\{3\} \subseteq \Delta^n$, so we have a canonical isomorphism $\Delta^n \simeq A \star \{x\} \star B \star C$ where x corresponds to the ith vertex of Δ^n. Let β denote the restriction of α to $A \star \{x\} \star B$ and γ the restriction of α to $\{x\} \star B \star C$. Let $X \in \mathcal{C}_1$ denote the image of x in $\mathcal{C}_1$, and choose a p-coCartesian morphism $f : X \to Y$ in $\mathcal{C}$ where $Y \in \mathcal{C}_2$. Since f is p-coCartesian, we can choose a map $\overline{\gamma} : \{x\} \star \{y\} \star B \star C \to \mathcal{C}$ which is compatible with f and γ. Let $\overline{\gamma}_0 = \overline{\gamma}|\{x\} \star \{y\} \star B$, and choose a map $\overline{\beta} : A \star \{x\} \star \{y\} \star B \to \mathcal{C}$ compatible with $\overline{\gamma}_0$ and β. The restrictions of $\overline{\beta}$ and $\overline{\gamma}$ determine a map $\delta : (A \star \{y\} \star B) \coprod_{\{y\}\star B}(\{y\} \star B \star C) \to \mathcal{C}$. Using the fact that p is an inner fibration, we can extend δ to a map $\overline{\delta} : A \star \{y\} \star B \star C \to \mathcal{C}$.

We define simplicial subsets

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq A \star \{x\} \star \{y\} \star B \star C$$

as follows:

(a) The simplicial subset $K_0 \subseteq A \star \{x\} \star \{y\} \star B \star C$ is generated by $A \star \{x\} \star \{y\} \star B$ and $\{x\} \star \{y\} \star B \star C$.

(b) The simplicial subset $K_1 \subseteq A \star \{x\} \star \{y\} \star B \star C$ is generated by K_0 together with $\Lambda_i^n \subseteq \Delta^n \simeq A \star \{x\} \star B \star C$. Since the inclusions $\Lambda_i^n \subseteq \Delta^n$ and $(A \star \{x\} \star B) \coprod_{\{x\}\star B}(\{x\} \star B \star C) \subseteq A \star \{x\} \star B \star C$ are inner anodyne, the inclusion

$$i : (A \star \{x\} \star B) \coprod_{\{x\}\star B} (\{x\} \star B \star C) \subseteq \Lambda_i^n$$

is a categorical equivalence. The inclusion $K_0 \subseteq K_1$ is a pushout of i, and therefore also a categorical equivalence.

(c) The simplicial subset $K_2 \subseteq A \star \{x\} \star \{y\} \star B \star C$ is generated by K_1 together with $A \star \{y\} \star B \star C$. The inclusion $K_1 \subseteq K_2$ is a pushout of the inclusion $(A \star \{y\} \star B) \coprod_{\{y\}\star B}(\{y\} \star B \star C) \subseteq A \star \{y\} \star B \star C$, and therefore a categorical equivalence.

The inclusion $K_0 \subseteq A \star \{x\} \star \{y\} \star B \star C$ is evidently a categorical equivalence. It follows from a two-out-of-three argument that the inclusion $K_2 \subseteq A \star \{x\} \star \{y\} \star B \star C$ is also a categorical equivalence. The maps $\overline{\delta}, \overline{\beta}, \overline{\beta}'$, and α determine a map $K_2 \to \mathcal{C}(m)$. We now define $\mathcal{C}(m,\alpha)$ to be the pushout $\mathcal{C}(m) \coprod_{K_2}(A \star \{x\} \star \{y\} \star B \star C)$. It is not difficult to verify that $\mathcal{C}(m,\alpha)$ has the desired properties. $\qquad\square$

Lemma 3.1.2.5. *Let $\mathcal{C}$ be an ∞-category containing a full subcategory $\mathcal{C}^0$, and let $\sigma : \Delta^n \to \mathcal{C}$ be a nondegenerate simplex which does not intersect $\mathcal{C}^0$. Let $K \subseteq \mathcal{C}$ be the simplicial subset spanned by those vertices $\tau : \Delta^m \to \mathcal{C}$ with the following property:*

(∗) *There exists a map $\Delta^m \to \Delta^1$ such that $\tau_0 = \tau|(\Delta^m \times_{\Delta^1} \{0\})$ factors through $\mathcal{C}^0$ and $\tau_1 = \tau|(\Delta^m \times_{\Delta^1} \{1\})$ factors through σ.*

Let K_0 be the simplicial subset of K spanned by those simplices τ for which there exists a decomposition as in (∗), where τ_1 factors through $\partial \sigma : \partial \Delta^n \to \mathcal{C}$. Let $\mathcal{C}^0_{/\sigma} = \mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/\sigma}$. Then the evident map

$$i : K_0 \coprod_{\mathcal{C}^0_{/\sigma} \star \partial \Delta^n} (\mathcal{C}^0_{/\sigma} \star \Delta^n) \to K$$

is a trivial cofibration (with respect to the Joyal model structure).

Proof. Let S be the collection of all simplices $\sigma' : \Delta^{n'} \to \mathcal{C}$ which factor as a composition $\Delta^{n'} \to \Delta^n \xrightarrow{\sigma} \mathcal{C}$, where the map $\Delta^{n'} \to \Delta^n$ is surjective. Choose an well-ordering of S such that $\sigma' < \sigma''$ if the dimension of σ' is smaller than the dimension of σ'' (so that σ is the least element of S). Let α denote the order type of S, and for $\beta < \alpha$ let σ'_β denote the corresponding element of S. For $\beta \leq \alpha$, let K_β be the simplicial subset of $\mathcal{C}$ spanned by the nondegenerate simplices $\tau : \Delta^m \to \mathcal{C}$ for which there exists a map $\Delta^m \to \Delta^1$ such that τ_0 factors through $\mathcal{C}^0$ and τ_1 either factors through $\partial\sigma$ or is isomorphic to σ'_γ for some $\gamma < \beta$ (here τ_0 and τ_1 are defined as in $(*)$). Then $K_\alpha = K$ and $K_1 = K_0 \coprod_{\mathcal{C}^0_{/\sigma} \star \partial \Delta^n} (\mathcal{C}^0_{/\sigma} \star \Delta^n)$. It will therefore suffice to show that the inclusion $K_1 \hookrightarrow K_\beta$ is a categorical equivalence for $1 \leq \beta \leq \alpha$. We proceed by induction on β. The case $\beta = 1$ is trivial, and if β is a nonzero limit ordinal then the assertion follows immediately from the inductive hypothesis since $K_\beta \simeq \varinjlim_{\gamma < \beta} K_\gamma$. It will therefore suffice to show that each of the inclusions $K_\beta \hookrightarrow K_{\beta+1}$ is a categorical equivalence. Let $\sigma' = \sigma'_\beta$, so that we can identify σ' with a surjective map $\epsilon : \Delta^{n'} \to \Delta^n$.

Let $\mathcal{J}$ be the category whose objects are factorizations

$$\Delta^{n'} \xrightarrow{\epsilon'} \Delta^{n''} \xrightarrow{\epsilon''} \Delta^n$$

of ϵ such that ϵ' is surjective. For each object $J = (\epsilon', \epsilon'') \in \mathcal{J}$ we let σ_J denote the composite map $\Delta^{n''} \xrightarrow{\epsilon''} \Delta^n \xrightarrow{\sigma} \mathcal{C}$. Let $\mathcal{J}_0$ denote the full subcategory of $\mathcal{J}$ spanned by those objects where $n'' < n'$. There is an evident injective map $\mathcal{C}^0_{/\sigma_J} \to \mathcal{C}^0_{/\sigma'}$ for each $J \in \mathcal{J}_0$; let us denote the union of their images by $X \subseteq \mathcal{C}^0_{/\sigma'}$. Unwinding the definitions, we note that the inclusion $K_\beta \hookrightarrow K_{\beta+1}$ is a pushout of the inclusion map

$$j : (X \star \Delta^{n'}) \coprod_{X \star \partial \Delta^{n'}} (\mathcal{C}^0_{/\sigma'} \star \partial \Delta^{n'}) \hookrightarrow \mathcal{C}^0_{/\sigma} \star \Delta^{n'}.$$

It will therefore suffice to show that j is categorical equivalence: that is, that the diagram

$$
\begin{array}{ccc}
X \star \partial \Delta^n & \longrightarrow & X \star \Delta^n \\
\downarrow & & \downarrow \\
\mathcal{C}^0_{/\sigma'} \star \partial \Delta^n & \longrightarrow & \mathcal{C}^0_{/\sigma'} \star \Delta^n
\end{array}
$$

is a homotopy pushout diagram (with respect to the Joyal model structure). In fact, we claim that the vertical maps are both categorical equivalences. To prove this, it suffices to show that the inclusion $X \hookrightarrow \mathcal{C}^0_{/\sigma'}$ is a categorical equivalence. Note that X is a colimit of the cofibrant diagram $\{\mathcal{C}^0_{/\sigma_J}\}_{J \in \mathcal{J}_0^{op}}$. It will therefore suffice to show that $\mathcal{C}^0_{/\sigma'}$ is a homotopy colimit of the diagram $\{\mathcal{C}^0_{/\sigma_J}\}_{J \in \mathcal{J}_0^{op}}$. This diagram is essentially constant: for each $J \in \mathcal{J}_0$, the canonical map $\mathcal{C}^0_{/\sigma_J} \to \mathcal{C}^0_{/\sigma'}$ is a categorical equivalence, since for $C = \sigma_J(0) = \sigma'(0) \in \mathcal{C}$ we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{C}^0_{/\sigma_J} & \longrightarrow & \mathcal{C}^0_{/\sigma'} \\
& \searrow \quad \swarrow & \\
& \mathcal{C}^0_{/C} &
\end{array}
$$

where the vertical maps are trivial Kan fibrations. Consequently, it will suffice to observe that the simplicial set $N(\mathcal{J}_0)$ is weakly contractible. This is clear, since $\mathcal{J}_0$ has a final object (given by the factorization $\epsilon = \mathrm{id}_{\Delta^n} \circ \epsilon$). $\square$

Proof of Theorem 3.1.2.3. The implication $(a) \Rightarrow (b)$ in the case (A) is obvious. Let us therefore assume either that (b) is satisfied (if $n = 1$) or that $f_0|(\mathcal{M}^\otimes \times_{\Delta^n} \Delta^{\{0,1\}})$ is an operadic q-left Kan extension of $f_0|(\mathcal{M}^\otimes \times_{\Delta^n} \{0\})$ (if $n > 1$). To complete the proof, we need to construct a functor $f : \mathcal{M}^\otimes \to \mathcal{C}^\otimes$ satisfying

some natural conditions. The construction is somewhat elaborate, and will require us to introduce some terminology.

Let α be a morphism in $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^{\{1,\ldots,n\}}$, having image $\alpha_0 : \langle m \rangle \to \langle n \rangle$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. We will say that α is *active* if the morphism α_0 is active. We will say that α is *strongly inert* if the morphism α_0 is inert, the image of α in $\Delta^{\{1,\ldots,n\}}$ is degenerate, and the injective map $\alpha_0^{-1} : \langle n \rangle^\circ \to \langle m \rangle^\circ$ is order preserving. We will say that α is *neutral* if it is neither active nor strongly inert. Note that the collections of active and strongly inert morphisms are closed under composition. Moreover, every morphism α can be written *uniquely* as a composition $\alpha'' \circ \alpha'$, where α' is strongly inert and α'' is active.

Let σ be an m-simplex of $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^{\{1,\ldots,n\}}$, corresponding to a chain of morphisms

$$(\langle k_0 \rangle, e_0) \xrightarrow{\alpha_\sigma(1)} (\langle k_1 \rangle, e_1) \xrightarrow{\alpha_\sigma(2)} \cdots \xrightarrow{\alpha_\sigma(m)} (\langle k_m \rangle, e_m).$$

We will say that σ is *closed* if $k_m = 1$; otherwise we will say that σ is *open*. We will say that σ is *new* if the projection map $\sigma \to \Delta^{\{1,\ldots,n\}}$ is surjective.

We now partition the collection of nondegenerate new simplices σ of $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^{\{1,\ldots,n\}}$ into five groups:

$(G'_{(1)})$ A nondegenerate new simplex σ of $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^{\{1,\ldots,n\}}$ belongs to $G_{(1)}$ if it is closed and each of the maps $\alpha_\sigma(i)$ is active.

$(G_{(2)}$ A nondegenerate new m-simplex σ of $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^{\{1,\ldots,n\}}$ belongs to $G_{(3)}$ if there exist integers $0 < j < k \le m$ such that $\alpha_\sigma(i)$ is strongly inert for $i = j$, active for $j < i \le k$, and strongly inert for $k < i \le m$.

$(G'_{(2)}$ A nondegenerate new m-simplex σ of $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^{\{1,\ldots,n\}}$ belongs to $G'_{(3)}$ if there exist integers $0 < j \le k \le m$ such that $\alpha_\sigma(i)$ is neutral for $i = j$, active for $j < i \le k$, and strongly inert for $k < i \le m$.

$(G_{(3)}$ A nondegenerate new m-simplex σ of $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^{\{1,\ldots,n\}}$ belongs to $G_{(2)}$ if it is closed and there exists $0 \le j < m$ such that $\alpha_\sigma(i)$ is active for $i \le j$ and strongly inert for $i > j$.

$(G'_{(3)}$ A nondegenerate new m-simplex σ of $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^{\{1,\ldots,n\}}$ belongs to $G'_{(2)}$ if it is open and there exists $0 \le j \le m$ such that $\alpha_\sigma(i)$ is active for $i \le j$ and strongly inert for $i > j$.

If $\sigma : \Delta^m \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^{\{1,\ldots,m\}}$ is a simplex belonging to $G_{(2)}$, we define the *associate* $\sigma' \in G'_{(2)}$ of σ to be the $(m-1)$-simplex obtained by restricting σ to $\Delta^{\{0,\ldots,m-1\}}$. If σ belongs to $G_{(3)}$, we define its *associate* $\sigma' \in G'_{(3)}$ to be the $(m-1)$-simplex given by the restriction $\sigma|\Delta^{\{0,\ldots,j-1,j+1,\ldots,m\}}$, where j is defined as above (note that j is uniquely determined). In either of these cases, we will also say that σ is an *associate* of σ'. Note that every simplex of $G'_{(2)}$ is the associate of a unique simplex belonging to $G_{(2)}$, and that a simplex σ of $G'_{(3)}$ with final vertex $(\langle k \rangle, e)$ is associated to exactly k simplices of $G_{(3)}$, obtained by concatenating σ with all possible strongly inert morphisms $(\langle k \rangle, e) \to (\langle 1 \rangle, e)$.

For each integer $m \ge 0$, let $F(m)$ denote simplicial subset of $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^{\{1,\ldots,n\}}$ spanned by the nondegenerate simplices which either belong to $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \partial \Delta^{\{1,\ldots,n\}}$, have dimension less than m, or have dimension m and belong to $G_{(2)}$ or $G_{(3)}$. Let $\overline{F}(m)$ be the simplicial subset of $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^n$ spanned by those simplices whose intersection with $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^{\{1,\ldots,n\}}$ belongs to $F(m)$, and let $\mathcal{M}^\otimes_{(m)}$ denote the inverse image of $\overline{F}(m)$ in $\mathcal{M}^\otimes$. We observe that $\mathcal{M}^\otimes_{(0)} = \mathcal{M}^\otimes \times_{\Delta^n} \Lambda^n_0$. To complete the proof, it will suffice to show that $f_0 : \mathcal{M}^\otimes_{(0)} \to \mathcal{C}^\otimes$ can be extended to a sequence of maps $f_m : \mathcal{M}^\otimes_{(m)} \to \mathcal{C}^\otimes$ which fit into commutative diagrams

$$\begin{array}{ccc} \mathcal{M}^\otimes_{(m-1)} & \xrightarrow{f_{m-1}} & \mathcal{C}^\otimes \\ \downarrow & {\scriptstyle f_m} \nearrow & \downarrow {\scriptstyle q} \\ \mathcal{M}^\otimes_{(m)} & \longrightarrow & \mathcal{O}^\otimes \end{array}$$

and such that f_1 has the following special properties in the case $n = 1$:

(i) For each object $B \in \mathcal{M} \times_{\Delta^n} \{1\}$, the map

$$((\mathcal{M}^{\otimes}_{\mathrm{act}})_{/B} \times_{\Delta^n} \{0\})^{\triangleright} \to \mathcal{M}^{\otimes}_{(1)} \xrightarrow{f_1} \mathcal{C}^{\otimes}$$

is an operadic q-colimit diagram.

(ii) For every inert morphism $e : M' \to M$ in $\mathcal{M}^{\otimes} \times_{\Delta^n} \{1\}$ such that $M \in \mathcal{M} = \mathcal{M}^{\otimes}_{\langle 1 \rangle}$, the functor f_1 carries e to an inert morphism in $\mathcal{C}^{\otimes}$.

Fix $m > 0$, and suppose that f_{m-1} has already been constructed. Our construction proceeds in several steps.

(1) Let $F'(m) \subseteq F(m)$ denote the simplicial subset of $F(m)$ spanned by those nondegenerate simplices belonging to either $F(m-1)$ or $G'_{(1)}$. Let $\overline{F'(m)}$ denote the simplicial subset of $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^n$ spanned by those simplices whose intersection with $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^{\{1,\dots,n\}}$ belongs to $F(m)'$, and let $\mathcal{M}'^{\otimes}_{(m)}$ denote the inverse image of $\overline{F'(m)}$ in $\mathcal{M}^{\otimes}$. We first prove the existence of a solution to the lifting problem

$$\begin{array}{ccc}
\mathcal{M}^{\otimes}_{(m-1)} & \xrightarrow{f_{m-1}} & \mathcal{C}^{\otimes} \\
\downarrow & {\scriptstyle f'_m} \nearrow & \downarrow {\scriptstyle q} \\
\mathcal{M}'^{\otimes}_{(m)} & \longrightarrow & \mathcal{O}^{\otimes}
\end{array}$$

such that, when $m = n = 1$, the map f'_m satisfies condition (i).

Let $\{\sigma_a\}_{a \in A}$ be the collection of all nondegenerate simplices of $\mathcal{M}^{\otimes} \times_{\Delta^n} \Delta^{\{1,\dots,n\}}$ whose image in $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^{\{1,\dots,n\}}$ has dimension $(m-1)$ and belongs to the class $G'_{(1)}$. Choose a well-ordering of the set A such that the dimensions of the simplices σ_a form a (nonstrictly) increasing function of a. For each $a \in A$, let $\mathcal{M}^{\otimes}_{\leq a}$ denote the simplicial subset of $\mathcal{M}^{\otimes}$ spanned by $\mathcal{M}^{\otimes}_{(m-1)}$ together with those simplices whose intersection with $\mathcal{M}^{\otimes} \times_{\Delta^n} \Delta^{\{1,\dots,n\}}$ factors through $\sigma_{a'}$ for some $a' \leq a$, and define $\mathcal{M}^{\otimes}_{<a}$ similarly. We construct a compatible family of maps $f^{\leq a} : \mathcal{M}^{\otimes}_{\leq a} \to \mathcal{C}^{\otimes}$ extending f_{m-1}, using transfinite induction on a. Assume that $f^{\leq a'}$ has been constructed for $a' < a$; these maps can be amalgamated to obtain a map $f^{<a} : \mathcal{M}^{\otimes}_{<a} \to \mathcal{C}^{\otimes}$. Let $X = \{0\} \times_{\Delta^n} \mathcal{M}^{\otimes}_{/\sigma_a}$. Lemma 3.1.2.5 implies that we have a homotopy pushout diagram of simplicial sets

$$\begin{array}{ccc}
X \star \partial \sigma_a & \longrightarrow & \mathcal{M}^{\otimes}_{<a} \\
\downarrow & & \downarrow \\
X \star \sigma_a & \longrightarrow & \mathcal{M}^{\otimes}_{\leq a} .
\end{array}$$

It will therefore suffice to extend the composition

$$g_0 : X \star \partial \sigma_a \longrightarrow \mathcal{M}^{\otimes}_{<a} \xrightarrow{f^{<a}} \mathcal{C}^{\otimes}$$

to a map $g : X \star \sigma_a \to \mathcal{C}^{\otimes}$ which is compatible with the projection to $\mathcal{O}^{\otimes}$.

We first treat the special case where the simplex σ_a is zero-dimensional (in which case we must have $m = n = 1$). We can identify σ_a with an object $B \in \mathcal{M}^{\otimes}$. Let $X_0 = (\mathcal{M}^{\otimes}_{\mathrm{act}})_{/B} \times_{\Delta^n} \{0\} \subseteq X$. Using assumption (b), we can choose a map $g_1 : X_0^{\triangleright} \to \mathcal{C}^{\otimes}$ compatible with the projection to $\mathcal{O}^{\otimes}$ which is an operadic q-colimit diagram. We note that X_0 is a localization of X so that the inclusion $X_0 \subseteq X$ is left cofinal; it follows that g_1 can be extended to a map $X^{\triangleright} \to \mathcal{C}^{\otimes}$ with the desired properties. Note that our particular construction of g guarantees that the map f_m will satisfy condition (i).

Now suppose that σ_a is a simplex of positive dimension. We again let X_0 denote the simplicial subset of X spanned by those vertices of X which correspond to diagrams $\sigma_a^\lhd \to \mathcal{C}^\otimes$ which project to a sequence of active morphisms in $N(\mathcal{F}\mathrm{in}_*)$. The inclusion $X_0 \subseteq X$ admits a left adjoint and is therefore left cofinal; it follows that the induced map $\mathcal{C}^\otimes_{(f_0|X)/} \to \mathcal{C}^\otimes_{(f_0|X_0)/} \times_{\mathcal{O}^\otimes_{(qf_0|X_0)/}} \mathcal{O}^\otimes_{(qf_0|X)/}$ is a trivial Kan fibration. It therefore suffices to show that the restriction $g'_0 = g_0|(X_0 \star \partial \sigma_a)$ can be extended to a map $g' : X_0 \star \sigma_a \to \mathcal{C}^\otimes$ compatible with the projection to $\mathcal{O}^\otimes$. In view of Proposition 3.1.1.7, it will suffice to show that the restriction $g'_0|(X_0 \star \{B\})$ is a operadic q-colimit diagram. Since the inclusion $\{B\} \subseteq \sigma_a$ is left anodyne, the projection map $X_0 \to \{0\} \times_{\Delta^n} (\mathcal{M}^\otimes_{\mathrm{act}})_{/B}$ is a trivial Kan fibration. It will therefore suffice to show that f_{m-1} induces an operadic q-colimit diagram $\delta : \{0\} \times_{\Delta^n} (\mathcal{M}^\otimes_{\mathrm{act}})^\rhd_{/B} \to \mathcal{C}^\otimes$.

Let $\langle p \rangle$ be the image of B in $N(\mathcal{F}\mathrm{in}_*)$. The desired result follows immediately from (i) if $p = 1$, so assume that $p \neq 1$. Then the image of σ_a in $N(\mathcal{F}\mathrm{in}_*)$ has dimension ≥ 1, so that $m \geq 2$ and therefore f_{m-1} is defined on $\mathcal{M}^\otimes_{(1)}$. For $1 \leq i \leq p$, choose an inert morphism $B \to B_i$ in $\mathcal{B}^\otimes$ covering the map $\rho^i : \langle p \rangle \to \langle 1 \rangle$, and let $\delta_i : \{0\} \times (\mathcal{M}^\otimes_{\mathrm{act}})^\rhd_{/B_i} \to \mathcal{C}^\otimes$ be the map induced by f_{m-1}. Condition (i) guarantees that each δ_i is an operadic q-colimit diagram. Using the fact that $\mathcal{M}^\otimes$ is a Δ^n-family of ∞-operads, we deduce that the maps $B \to B_i$ induce an equivalence

$$\{0\} \times_{\Delta^n} (\mathcal{M}^\otimes_{\mathrm{act}})_{/B} \simeq \prod_{1 \leq i \leq p} (\{0\} \times_{\Delta^n} \mathcal{M}^\otimes_{\mathrm{act}})_{/B_i}.$$

Using the fact that f_{m-1} satisfies (ii), we see that under this equivalence we can identify δ with the diagram

$$(\prod_{1 \leq i \leq p} (\{0\} \times_{\Delta^n} \mathcal{M}^\otimes_{\mathrm{act}})^\rhd_{/B_i} \to \prod_{1 \leq i \leq p} (\mathcal{M}^\otimes_{\mathrm{act}})^\rhd_{/B_i} \overset{\oplus \delta_i}{\to} \mathcal{C}^\otimes$$

Proposition 3.1.1.8 now implies that δ is an operadic q-colimit diagram, as desired.

(2) Let $F''(m)$ denote the simplicial subset of $N(\mathcal{F}\mathrm{in}_*) \times \Delta^{\{1,\ldots,n\}}$ spanned by $F'(m)$ together with those $(m-1)$-simplices σ belonging to $G'_{(3)}$ which have no associates (so that the final vertex of σ is equal to $(\langle 0 \rangle, n)$). Let $\overline{F''(m)}$ denote the collection of those simplices of $N(\mathcal{F}\mathrm{in}_*) \times \Delta^n$ whose intersection with $N(\mathcal{F}\mathrm{in}_*) \times \Delta^{\{1,\ldots,n\}}$ belongs to $F''(m)$, and let $\mathcal{M}''^\otimes_{(m)}$ denote the inverse image of $\overline{F''(m)}$ in $\mathcal{M}^\otimes$. We next show that f'_m can be extended to a map $f''_m : \mathcal{M}''^\otimes_{(m)} \to \mathcal{C}^\otimes$ (compatible with the projection to $\mathcal{O}^\otimes$).

We proceed as in the first step. Let $\{\sigma_a\}_{a \in A}$ be the collection of all nondegenerate simplices of $\mathcal{M}^\otimes \times_{\Delta^n} \Delta^{\{1,\ldots,n\}}$ whose image in $N(\mathcal{F}\mathrm{in}_*) \times \Delta^{\{1,\ldots,n\}}$ has dimension $(m-1)$ and belongs to the class $G'_{(3)}$ and has no associates. Choose a well-ordering of the set A such that the dimensions of the simplices σ_a form a (nonstrictly) increasing function of a. For each $a \in A$, let $\mathcal{M}^\otimes_{\leq a}$ denote the simplicial subset of $\mathcal{M}^\otimes$ spanned by $\mathcal{M}'^\otimes_{(m)}$ together with those simplices whose intersection with $\mathcal{M}^\otimes \times_{\Delta^n} \Delta^{\{1,\ldots,n\}}$ factors through $\sigma_{a'}$ for some $a' \leq a$, and define $\mathcal{M}^\otimes_{<a}$ similarly. We construct a compatible family of maps $f^{\leq a} : \mathcal{M}^\otimes_{\leq a} \to \mathcal{C}^\otimes$ extending f'_m, using transfinite induction on a. Assume that $f^{\leq a'}$ has been constructed for $a' < a$; these maps can be amalgamated to obtain a map $f^{<a} : \mathcal{M}^\otimes_{<a} \to \mathcal{C}^\otimes$. Let $X = \{0\} \times_{\Delta^n} \mathcal{M}^\otimes_{/\sigma_a}$. Lemma 3.1.2.5 implies that we have a homotopy pushout diagram of simplicial sets

$$
\begin{array}{ccc}
X \star \partial \sigma_a & \longrightarrow & \mathcal{M}^\otimes_{<a} \\
\downarrow & & \downarrow \\
X \star \sigma_a & \longrightarrow & \mathcal{M}^\otimes_{\leq a}.
\end{array}
$$

It will therefore suffice to extend the composition

$$g_0 : X \star \partial \sigma_a \longrightarrow \mathcal{M}^\otimes_{<a} \overset{f^{<a}}{\longrightarrow} \mathcal{C}^\otimes$$

to a map $g : X \star \sigma_a \to \mathcal{C}^{\otimes}$ which is compatible with the projection to $\mathcal{O}^{\otimes}$. In other words, it suffices to solve a lifting problem of the form

$$
\begin{array}{ccc}
\partial\,\sigma_a & \longrightarrow & \mathcal{C}^{\otimes}_{X/} \\
\downarrow & \nearrow & \downarrow \\
\sigma_a & \longrightarrow & \mathcal{O}^{\otimes}_{X/} .
\end{array}
$$

The existence of the desired lifting follows from the fact that the upper vertical map carries the final vertex of σ_a to a q-final vertex of $\mathcal{C}^{\otimes}$.

(3) We now complete the induction by showing that f_m'' can be extended to a map $f_m : \mathcal{M}^{\otimes}_{(m)} \to \mathcal{C}^{\otimes}$ (satisfying condition (ii) in the case $m = n = 1$). The construction of this extension will require a somewhat intricate induction. Let $\{\sigma_a'\}_{a\in A}$ be the collection of all $(m-1)$-simplices of $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^{\{1,\dots,n\}}$ which belong to either $G_{(2)}'$ or $G_{(3)}'$ and admit at least one associate. Let $a \in A$, and let σ denote an associate of the simplex σ_a', given by a sequence of morphisms

$$
(\langle k_0 \rangle, e_0) \xrightarrow{\alpha_\sigma(1)} (\langle k_1 \rangle, e_1) \xrightarrow{\alpha_\sigma(2)} \cdots \xrightarrow{\alpha_\sigma(m)} (\langle k_m \rangle, e_m).
$$

We define nonnegative integers $u(a)$, $v(a)$, and $w(a)$ as follows:

- $u(a)$ is the number of integers $1 \le i \le m$ such that $\alpha_\sigma(i)$ is neutral.

- $v(a)$ is the number of integers $1 \le i \le m$ such that $\alpha_\sigma(i)$ is active.

- $w(a)$ is the number of pairs of integers $1 \le i < j \le m$ such that $\alpha_\sigma(i)$ is active and $\alpha_\sigma(j)$ is strictly inert.

Note that these integers are independent of the choice of σ (in the case where $\sigma_a' \in G_{(3)}'$, so that σ_a' has more than one associate). Choose a well-ordering on the set A with the following properties:

- If $a, b \in A$ satisfy $u(a) < u(b)$, then $a < b$.

- If $a, b \in A$ satisfy $u(a) = u(b)$ and $v(a) < v(b)$, then $a < b$.

- Let $a, b \in A$, and let σ and τ denote associates of σ_a' and σ_b', respectively. If $u(a) = u(b)$, $v(a) = v(b)$, σ is closed, and τ is open, then $a < b$.

- Let $a, b \in A$, and let σ and τ denote associates of σ_a' and σ_b', respectively. If $u(a) = u(b)$, $v(a) = v(b)$, σ and τ are either both open or both closed, and $w(a) < w(b)$, then $a < b$.

For each $a \in A$, let $F_{\le a}$ denote the simplicial subset of $F(m)$ given by the union of $F''(m)$, all of the simplices σ_b' for $b \le a$, and all of the associates of the simplices σ_b' for $b \le a$. We let $\overline{F}_{\le a}$ denote the simplicial subset of $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^n$ spanned by those simplices whose intersection with $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^{\{1,\dots,n\}}$ belongs to $F_{\le a}$, and $\mathcal{M}^{\otimes}_{\le a}$ the inverse image of $\overline{F}_{\le a}$ in $\mathcal{M}^{\otimes}$. Define $F_{<a}$, $\overline{F}_{<a}$, and $\mathcal{M}^{\otimes}_{<a}$ similarly. We will construct f_m as the amalgamation of a compatible family of maps $f_{\le a} : \mathcal{M}^{\otimes}_{\le a} \to \mathcal{C}^{\otimes}$ extending f_{m-1}. The construction proceeds by transfinite recursion. Assume that $f_{\le b}$ has been constructed for $b < a$, so that the maps $\{f_{\le b}\}_{b<a}$ determine a map $f_{<a} : \mathcal{M}^{\otimes}_{<a} \to \mathcal{C}^{\otimes}$. We wish to extend $f_{<a}$ to a map $f_{\le a}$ (compatible with the projection to $\mathcal{O}^{\otimes}$). There are several cases to consider.

Suppose first that σ_a' belongs to $G_{(2)}'$. In this case, we will prove the existence of the desired extension by showing that the inclusion $\mathcal{M}^{\otimes}_{<a} \hookrightarrow \mathcal{M}^{\otimes}_{\le a}$ is a categorical equivalence of simplicial sets, so that the existence of the map $f_{\le a}$ follows from the fact that q is a categorical fibration of simplicial sets. The simplex σ_a' has a unique associate, which we will denote by σ_a. Note that σ_a' is a face of σ_a, and all of

the other faces of σ_a belong to $F_{<a}$. Let $\Lambda \subseteq \sigma_a$ denote the inner horn opposite to the face σ_a'. Let $Y = \{0\} \times_{\Delta^n} (N(\mathcal{F}\mathrm{in}_*) \times \Delta^n)_{/\sigma_a}$, so that we have a pushout diagram of simplicial sets

$$\begin{array}{ccc} Y \star \Lambda & \longrightarrow & \overline{F}_{<a} \\ \downarrow & & \downarrow \\ Y \star \sigma_a & \longrightarrow & \overline{F}_{\leq a}. \end{array}$$

It will therefore suffice to show that the inclusion

$$(Y \star \Lambda) \times_{N(\mathcal{F}\mathrm{in}_*) \times \Delta^n} \mathcal{M}^{\otimes} \hookrightarrow (Y \star \sigma_a) \times_{N(\mathcal{F}\mathrm{in}_*) \times \Delta^n} \mathcal{M}^{\otimes}$$

is a categorical equivalence. In fact, we will prove the following more general claim: for every map of simplicial sets $Y' \to Y$, the map

$$\eta_{Y'} : (Y' \star \Lambda) \times_{N(\mathcal{F}\mathrm{in}_*) \times \Delta^n} \mathcal{M}^{\otimes} \hookrightarrow (Y' \star \sigma_a) \times_{N(\mathcal{F}\mathrm{in}_*) \times \Delta^n} \mathcal{M}^{\otimes}$$

is a categorical equivalence. Since the construction $Y' \mapsto \eta_{Y'}$ commutes with filtered colimits, we may assume that Y' has finitely many nondegenerate simplices. We proceed by induction on the dimension of Y', and on the number of nondegenerate simplices of Y'. If Y' is empty, then the desired result follows immediately from Lemma 2.4.4.6. If Y' is nonempty, we have a pushout diagram of simplicial sets

$$\begin{array}{ccc} \partial \Delta^p & \longrightarrow & Y_0' \\ \downarrow & & \downarrow \\ \Delta^p & \longrightarrow & Y'. \end{array}$$

In this case, $\eta_{Y'}$ factors as a composition

$$(Y' \star \Lambda) \times_{N(\mathcal{F}\mathrm{in}_*) \times \Delta^n} \mathcal{M}^{\otimes} \xrightarrow{\eta'} ((Y' \star \Lambda) \amalg_{Y_0' \star \Lambda} (Y_0' \star \sigma_a)) \times_{N(\mathcal{F}\mathrm{in}_*) \times \Delta^n} \mathcal{M}^{\otimes}$$
$$\xrightarrow{\eta''} (Y' \star \sigma_a) \times_{N(\mathcal{F}\mathrm{in}_*) \times \Delta^n} \mathcal{M}^{\otimes}$$

The map η' is a pushout of $\eta_{Y_0'}$, and therefore a categorical equivalence by the inductive hypothesis. The map η'' is a pushout of the inclusion

$$(\Delta^p \star \Lambda) \amalg_{\partial \Delta^p \star \Lambda} (\partial \Delta^p \star \sigma_a) \times_{N(\mathcal{F}\mathrm{in}_*) \times \Delta^n} \mathcal{M}^{\otimes} \hookrightarrow (\Delta^p \star \sigma_a) \times_{N(\mathcal{F}\mathrm{in}_*) \times \Delta^n} \mathcal{M}^{\otimes},$$

which is categorical equivalence by virtue of Lemma 2.4.4.6.

We now treat the case where $\sigma_a' \in G'_{(3)}$. Let $\{\tau_b\}_{b \in B}$ be the collection of all nondegenerate simplices τ of $\mathcal{M}^{\otimes}$ with the property that $\tau \times_{\Delta^n} \Delta^{\{1,\dots,n\}}$ maps surjectively to the simplex σ_a' of $N(\mathcal{F}\mathrm{in}_*) \times \Delta^n$. Choose a well-ordering of B such that the dimension of the simplex τ_b is a (nonstrictly) increasing function of b. For each $b \in B$, let $\mathcal{N}_{\leq b} \subseteq \mathcal{M}^{\otimes}$ denote the simplicial subset generated by $\mathcal{M}^{\otimes}_{<a}$ together with those simplices $\tau : \Delta^p \to \mathcal{M}^{\otimes}$ such that for some $p' < p$, the restriction $\tau|\Delta^{\{0,\dots,p'\}}$ factors through a simplex $\tau_{b'}$ for $b' \leq b$, the map $\tau(p') \to \tau(p'+1)$ is an inert morphism in $\mathcal{D}^{\otimes}$, and $\tau|\Delta^{\{p'+1,\dots,p\}}$ factors through $\mathcal{D}$. Define $\mathcal{N}_{<b}$ similarly. We will construction $f_{\leq a}$ as the amalgamation of a compatible family of maps $g_{\leq b} : \mathcal{N}_{\leq b} \to \mathcal{C}^{\otimes}$. The construction proceeds by transfinite induction on b. Assume that $g_{\leq b'}$ has been constructed for $b' < b$, so that the set of maps $\{g_{\leq b'}\}_{b' < b}$ determines a map $g_{<b} : \mathcal{N}_{<b} \to \mathcal{C}^{\otimes}$. We wish to show that there is a solution to the lifting problem

$$\begin{array}{ccc} \mathcal{N}_{<b} & \xrightarrow{g_{<b}} & \mathcal{C}^{\otimes} \\ \downarrow & \nearrow^{g_{\leq b}} & \downarrow q \\ \mathcal{N}_{\leq b} & \longrightarrow & \mathcal{O}^{\otimes}. \end{array}$$

The final vertex of τ_b is given by $(\langle k \rangle, n)$ for some $k \geq 2$. For $1 \leq i \leq k$, we have a strongly inert morphism $(\langle k \rangle, n) \rightarrow (\langle 1 \rangle, n)$. Taken together, these morphisms determine a map $\tau_b \star \{1, \ldots, k\} \rightarrow N(\mathcal{F}\mathrm{in}_*) \times \Delta^n$. Let $\mathfrak{X} = \mathcal{M}^{\otimes} \times_{N(\mathcal{F}\mathrm{in}_*)} (\tau_b \star \{1, \ldots, k\})$. Note that τ_b lifts to a simplex $\overline{\tau}_b$ in $\mathfrak{X}$. For $1 \leq i \leq k$ let $\mathfrak{X}_i$ denote the fiber of $\mathfrak{X}$ over the vertex $i \in \{1, \ldots, k\}$, let $\mathfrak{X}^0 = \bigcup_{1 \leq i \leq k} \mathfrak{X}_i$, and set $\mathfrak{X}^0_{\overline{\tau}_b/} = \mathfrak{X}^0 \times_{\mathfrak{X}} \mathfrak{X}_{\overline{\tau}_b/}$. Using Lemma 3.1.2.5, we obtain a homotopy pushout diagram

$$
\begin{array}{ccc}
\partial \overline{\tau}_b \star \mathfrak{X}^0_{\overline{\tau}_b/} & \longrightarrow & \mathcal{N}_{<b} \\
\downarrow & & \downarrow \\
\overline{\tau}_b \star \mathfrak{X}^0_{\overline{\tau}_b/} & \longrightarrow & \mathcal{N}_{\leq b} \, .
\end{array}
$$

According to Proposition T.A.2.3.1, it will suffice to solve the associated lifting problem

$$
\begin{array}{ccc}
\partial \tau_b \star \mathfrak{X}^0_{\overline{\tau}_b/} & \xrightarrow{g'_0} & \mathcal{C}^{\otimes} \\
\downarrow & \nearrow^{g'} & \downarrow q \\
\tau_b \star \mathfrak{X}^0_{\overline{\tau}_b/} & \longrightarrow & \mathcal{O}^{\otimes} \, .
\end{array}
$$

Note that the ∞-category $\mathfrak{X}^0_{\overline{\tau}_b/}$ decomposes naturally as a disjoint union $\coprod_{1 \leq i \leq k} X_i$, where $X_i = \mathfrak{X}_i \times_{\mathfrak{X}} \mathfrak{X}_{\overline{\tau}_b/}$. Each of the ∞-categories X_i has an initial object $\overline{B}_i$, given by any map $\tau_b \star \{D_i\} \rightarrow \mathcal{C}^{\otimes}$ which induces an inert morphism $D_b \rightarrow D_i$ covering $\rho^i : \langle k \rangle \rightarrow \langle 1 \rangle$. Let $h : \mathfrak{X}^0_{\overline{\tau}_b/} \rightarrow \mathcal{C}^{\otimes}$ be the map induced by g'_0, and let h' be the restriction of h to the discrete simplicial set $X' = \{\overline{B}_i\}_{1 \leq i \leq k}$. Since inclusion $X' \rightarrow \mathfrak{X}^0_{\overline{\tau}_b/}$ is left anodyne, we have a trivial Kan fibration $\mathcal{C}^{\otimes}_{/h} \rightarrow \mathcal{C}^{\otimes}_{/h'} \times_{\mathcal{O}^{\otimes}_{/qh'}} \mathcal{O}^{\otimes}_{/qh}$. We are therefore reduced to the problem of solving the a lifting problem of the form

$$
\begin{array}{ccc}
\partial \overline{\tau}_b \star X' & \xrightarrow{g''_0} & \mathcal{C}^{\otimes} \\
\downarrow & \nearrow^{g''} & \downarrow q \\
\overline{\tau}_b \star X' & \xrightarrow{\overline{g}''} & \mathcal{O}^{\otimes} \, .
\end{array}
$$

If the dimension of $\overline{\tau}_b$ is positive, then it suffices to check that g''_0 carries $\{D_b\} \star X'$ to a q-limit diagram in $\mathcal{C}^{\otimes}$. Let q' denote the canonical map $\mathcal{O}^{\otimes} \rightarrow N(\mathcal{F}\mathrm{in}_*)$. In view of Proposition T.4.3.1.5, it suffices to show that g''_0 carries $\{D_b\} \star X'$ to a $(q' \circ q)$-limit diagram and that $q \circ g''_0$ carries $\{D_b\} \star X'$ to a q'-limit diagram. The first of these assertions follows from (ii) and from the fact that $\mathcal{C}^{\otimes}$ is an ∞-operad, and the second follows by the same argument (since q is a map of ∞-operads).

It remains to treat the case where $\overline{\tau}_b$ is zero dimensional (in which case we must have $m = n = 1$). Since $\mathcal{C}^{\otimes}$ is an ∞-operad, we can solve the lifting problem depicted in the diagram

$$
\begin{array}{ccc}
\partial \overline{\tau}_b \star X' & \xrightarrow{g''_0} & \mathcal{C}^{\otimes} \\
\downarrow & \nearrow^{j} & \downarrow q' \circ q \\
\overline{\tau}_b \star X' & \longrightarrow & N(\mathcal{F}\mathrm{in}_*)
\end{array}
$$

in such a way that j carries edges of $\overline{\tau}_b \star X'$ to inert morphisms in $\mathcal{C}^{\otimes}$. Since q is an ∞-operad map, it follows that $q \circ j$ has the same property. Since $\mathcal{O}^{\otimes}$ is an ∞-operad, we conclude that $q \circ j$ and $\overline{g}''$ are both q'-limit diagrams in $\mathcal{O}^{\otimes}$ extending $q \circ g''_0$, and therefore homotopic via a homotopy which is

fixed on X' and compatible with the projection to $N(\mathcal{F}in_*)$. Since q is a categorical fibration, we can lift this equivalence to a homotopy $j \simeq g''$, where $g'' : \overline{\tau}_b \star X' \to \mathcal{C}^{\otimes}$ is the desired extension of g_0''. We note that this construction ensures that condition (ii) is satisfied.

$\square$

3.1.3   Construction of Free Algebras

Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, and suppose we are given ∞-operad maps $\mathcal{A}^{\otimes} \xrightarrow{i} \mathcal{B}^{\otimes} \xrightarrow{j} \mathcal{O}^{\otimes}$. Composition with i induces a forgetful functor $\theta : \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{C})$. Our goal in this section is to show that, under suitable conditions, the functor θ admits a left adjoint. This left adjoint can be described informally as carrying an algebra $F \in \mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{C})$ to the free $\mathcal{B}$-algebra generated by F.

Our first step is to describe the structure of free algebras more explicitly. Suppose we are given an algebra object $F' \in \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C})$ and a map of $\mathcal{A}$-algebras $F \to \theta(F')$. What can we say about the object $F'(B) \in \mathcal{C}$, where $B \in \mathcal{B}$ is some object? For every $A \in \mathcal{A}^{\otimes}$ and every active morphism $i(A) \to B$ in $\mathcal{A}^{\otimes}$, we obtain a composite map

$$F(A) \to \theta(F')(A) = F'(i(A)) \to F'(B)$$

in the ∞-operad $\mathcal{C}^{\otimes}$. We will say that F' is freely generated by F if each $F'(B) \in \mathcal{C}$ is universal with respect to the existence of these maps (for each $B \in \mathcal{B}$). More precisely, we have the following definition:

Definition 3.1.3.1. Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $\mathcal{A}^{\otimes} \xrightarrow{i} \mathcal{B}^{\otimes} \xrightarrow{j} \mathcal{O}^{\otimes}$ be as above, let $A \in \mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{C})$ and $F \in \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C})$, and let $f : F \to \theta(F')$ be a morphism of $\mathcal{A}$-algebra objects in $\mathcal{C}$.

For every object $B \in \mathcal{B}$, we let $(\mathcal{A}_{\mathrm{act}}^{\otimes})_{/B}$ denote the fiber product $\mathcal{A}^{\otimes} \times_{\mathcal{B}^{\otimes}} (\mathcal{B}_{\mathrm{act}}^{\otimes})_{/B}$. The maps F and F' induce maps $\alpha, \alpha' : (\mathcal{A}_{\mathrm{act}}^{\otimes})_{/B} \to \mathcal{C}_{\mathrm{act}}^{\otimes}$, f determines a natural transformation $g : \alpha \to \alpha'$. We note that α' lifts to a map $\overline{\alpha}' : (\mathcal{A}_{\mathrm{act}}^{\otimes})_{/B} \to (\mathcal{C}_{\mathrm{act}}^{\otimes})_{/F'(B)}$; since the projection $(\mathcal{C}_{\mathrm{act}}^{\otimes})_{/F'(B)} \to \mathcal{C}_{\mathrm{act}}^{\otimes} \times_{\mathcal{O}_{\mathrm{act}}^{\otimes}} (\mathcal{O}_{\mathrm{act}}^{\otimes})_{/qF'(B)}$ is a right fibration, we can lift g (in an essentially unique fashion) to a natural transformation $\overline{g} : \overline{\alpha} \to \overline{\alpha}'$ which is compatible with the projection to $\mathcal{O}^{\otimes}$.

We will say that f *exhibits F' as a q-free $\mathcal{B}$-algebra generated by F* if the following condition is satisfied, for every object $B \in \mathcal{B}$:

(∗) The map $\overline{\alpha}$ above determines an operadic q-colimit diagram $(\mathcal{A}_{\mathrm{act}}^{\otimes})_{/B}^{\triangleright} \to \mathcal{C}^{\otimes}$.

The following result guarantees that free algebras have the expected universal property (and are therefore unique up to equivalence, whenever they exist):

Proposition 3.1.3.2. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, and suppose we are given maps of ∞-operads $\mathcal{A}^{\otimes} \to \mathcal{B}^{\otimes} \to \mathcal{O}^{\otimes}$. Let $\theta : \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{C})$ denote the forgetful functor, let $F \in \mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{C})$, let $F' \in \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C})$, and let $f : F \to \theta(F')$ be a map which exhibits F' as a q-free $\mathcal{B}$-algebra generated by F. For every $F'' \in \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C})$, composition with f induces a homotopy equivalence*

$$\gamma : \mathrm{Map}_{\mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C})}(F', F'') \to \mathrm{Map}_{\mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{C})}(F, \theta(F'')).$$

We also have the following general existence result for free algebras:

Proposition 3.1.3.3. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, and suppose we are given maps of ∞-operads $\mathcal{A}^{\otimes} \to \mathcal{B}^{\otimes} \to \mathcal{O}^{\otimes}$. Let $\theta : \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{C})$ denote the forgetful functor and let $F \in \mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{C})$. The following conditions are equivalent:*

(1) *There exists $F' \in \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C})$ and a map $F \to \theta(F')$ which exhibits F' as a q-free $\mathcal{B}$-algebra generated by F.*

(2) *For every object $B \in \mathcal{B}$, the induced map*

$$(\mathcal{A}_{\mathrm{act}}^{\otimes})_{/B} \to \mathcal{A}_{\mathrm{act}}^{\otimes} \xrightarrow{F} \mathcal{C}_{\mathrm{act}}^{\otimes}$$

can be extended to an operadic q-colimit diagram lying over the composition

$$(\mathcal{A}_{\mathrm{act}}^{\otimes})_{/B}^{\triangleright} \to \mathcal{B}_{\mathrm{act}}^{\otimes} \to \mathcal{O}_{\mathrm{act}}^{\otimes} .$$

Let us postpone the proofs of Proposition 3.1.3.2 and 3.1.3.3 for the moment, and study some applications.

Corollary 3.1.3.4. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, and suppose we are given maps of ∞-operads $\mathcal{A}^{\otimes} \to \mathcal{B}^{\otimes} \to \mathcal{O}^{\otimes}$. Assume that the following condition is satisfied:*

(∗) *For every object $B \in \mathcal{B}$ and every $F \in \mathrm{Alg}_{\mathcal{A} / \mathcal{O}}(\mathcal{C})$, the diagram*

$$(\mathcal{A}_{\mathrm{act}}^{\otimes})_{/B} \to \mathcal{A}_{\mathrm{act}}^{\otimes} \xrightarrow{F} \mathcal{C}_{\mathrm{act}}^{\otimes}$$

can be extended to an operadic q-colimit diagram lifting the composition

$$(\mathcal{A}_{\mathrm{act}}^{\otimes})_{/B}^{\triangleright} \to (\mathcal{B}_{\mathrm{act}}^{\otimes})_{/B}^{\triangleright} \to \mathcal{B}_{\mathrm{act}}^{\otimes} \to \mathcal{O}_{\mathrm{act}}^{\otimes} .$$

Then the forgetful functor $\theta : \mathrm{Alg}_{\mathcal{B} / \mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{A} / \mathcal{O}}(\mathcal{C})$ admits a left adjoint, which carries each $F \in \mathrm{Alg}_{\mathcal{A} / \mathcal{O}}(\mathcal{C})$ to a q-free $\mathcal{B}$-algebra generated by F.

Proof. Combine Propositions 3.1.3.3, 3.1.3.2, and T.5.2.2.12. □

Corollary 3.1.3.5. *Let κ be an uncountable regular cardinal, let $\mathcal{O}^{\otimes}$ be an ∞-operad, and let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be coCartesian fibration of ∞-operads which is compatible with κ-small colimits (see Variant 3.1.1.19). Suppose we are given maps of ∞-operads $\mathcal{A}^{\otimes} \to \mathcal{B}^{\otimes} \to \mathcal{O}^{\otimes}$, where $\mathcal{A}^{\otimes}$ and $\mathcal{B}^{\otimes}$ are essentially κ-small. Then the forgetful functor $\mathrm{Alg}_{\mathcal{B} / \mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{A} / \mathcal{O}}(\mathcal{C})$ admits a left adjoint, which carries each $\mathcal{O}'$-algebra F in $\mathcal{C}$ to a q-free $\mathcal{O}$-algebra generated by F.*

Proof. Combine Proposition 3.1.1.20 with Corollary 3.1.3.4. □

Example 3.1.3.6. Let $\mathcal{O}^{\otimes}$ be a κ-small ∞-operad, let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads which is compatible with κ-small colimits. Applying Corollary 3.1.3.5 in the case where $\mathcal{B}^{\otimes} = \mathcal{O}^{\otimes}$ and $\mathcal{A}^{\otimes} = \mathcal{O}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{Triv}^{\otimes}$ (and using Example 2.1.3.5), we deduce that the forgetful functor $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ admits a left adjoint.

To apply Corollary 3.1.3.5 in practice, it it convenient to have a more explicit description of free algebras. To obtain such a description, we will specialize to the case where the ∞-operad $\mathcal{A}^{\otimes}$ is trivial.

Construction 3.1.3.7. Let $\mathcal{O}^{\otimes}$ be an ∞-operad, let $X, Y \in \mathcal{O}$ be objects, and let $f : \mathrm{Triv}^{\otimes} \to \mathcal{O}^{\otimes}$ be a map of ∞-operads such that $f(\langle 1 \rangle) = X$ (such a map exists and is unique up to equivalence, by Remark 2.1.3.6).

Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads and let $C \in \mathcal{C}_X$ be an object. Using Example 2.1.3.5, we conclude that C determines an essentially unique Triv-algebra $\overline{C} \in \mathrm{Alg}_{\mathrm{Triv} / \mathcal{O}}(\mathcal{C})$ such that $\overline{C}(\langle 1 \rangle) = C$.

For each $n \geq 0$, let $\mathcal{P}(n)$ denote the full subcategory of $\mathrm{Triv}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{O}_{/Y}^{\otimes}$ spanned by the active morphisms $f(\langle n \rangle) \to Y$; we observe that $\mathcal{P}(n)$ is a Kan complex and that the fibers of the canonical map $q : \mathcal{P}(n) \to \mathrm{N}(\Sigma_n)$ can be identified with the space of n-ary operations $\mathrm{Mul}_{\mathcal{O}}(\{X\}_{1 \leq i \leq n}, Y)$. By construction, we have a canonical map $h : \mathcal{P}(n) \times \Delta^1 \to \mathcal{O}^{\otimes}$ which we regard as a natural transformation from $h_0 = f \circ q$ to the constant map $h_1 : \mathcal{P}(n) \to \{Y\}$. Since p is a coCartesian fibration, we can choose a p-coCartesian natural transformation $\overline{h} : \overline{C} \circ q \to \overline{h}_1$, for some map $\overline{h}_1 : \mathcal{P}(n) \to \mathcal{C}_Y$. We let $\mathrm{Sym}_{\mathcal{O},Y}^n(C)$ denote a colimit of the diagram $\overline{h}_1$, if such a colimit exists. We observe that $\mathrm{Sym}_{\mathcal{O},Y}^n(C) \in \mathcal{C}_Y$ is well-defined up to equivalence (in fact, up to a contractible space of choices).

Notation 3.1.3.8. In the special case $X = Y$, we will simply write $\mathrm{Sym}^n_{\mathcal{O}}(C)$ for $\mathrm{Sym}^n_{\mathcal{O},Y}(C)$. When $\mathcal{O}^{\otimes}$ is the commutative ∞-operad $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ (so that $X = Y = \langle 1 \rangle$) we will denote $\mathrm{Sym}^n_{\mathcal{O}}(C)$ by $\mathrm{Sym}^n(C)$.

Remark 3.1.3.9. In the situation of Construction 3.1.3.7, suppose that $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ is a $\mathcal{O}$-algebra and that we are equipped with a map $\overline{C} \to A \circ f$ of $\mathcal{T}\mathrm{riv}$-algebras (in view of Example 2.1.3.5, this is equivalent to giving a map $C \to A(X)$ in $\mathcal{C}_X$). This map induces a natural transformation from $\overline{h}_1$ to the constant map $\mathcal{P}(n) \to \{A(Y)\} \subseteq \mathcal{C}$, which we can identify with a map $\mathrm{Sym}^n_{\mathcal{O},Y}(C) \to A(Y)$ (provided that the left side is well-defined).

Definition 3.1.3.10. Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads, let $X \in \mathcal{O}$, let $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$, and suppose we are given a morphism $f : C \to A(X)$ in $\mathcal{C}_X$. Using Remark 2.1.3.6 and Example 2.1.3.5, we can extend (in an essentially unique way) X to an ∞-operad map $\mathcal{T}\mathrm{riv} \to \mathcal{O}$, C to an object $\overline{C} \in \mathrm{Alg}_{\mathcal{T}\mathrm{riv}/\mathcal{O}}(\mathcal{C})$, and f to a map of $\mathcal{T}\mathrm{riv}$-algebras $\overline{f} : \overline{C} \to A|\mathcal{T}\mathrm{riv}$. We will say that f *exhibits A as a free $\mathcal{O}$-algebra generated by C* if $\overline{f}$ exhibits A as a q-free $\mathcal{O}$-algebra generated by $\overline{C}$.

Proposition 3.1.3.11. *Let κ be an uncountable regular cardinal, $\mathcal{O}^{\otimes}$ a κ-small ∞-operad, and $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ a coCartesian fibration of ∞-operads which is compatible with κ-small colimits. Let $X \in \mathcal{O}$ and let $C \in \mathcal{C}_X$. Then:*

(i) *There exists an algebra $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ and a map $C \to A(X)$ which exhibits A as a free $\mathcal{O}$-algebra generated by X.*

(ii) *An arbitrary map $f : C \to A(X)$ as in (i) exhibits A as a free $\mathcal{O}$-algebra generated by X if and only if, for every object $Y \in \mathcal{O}$, the maps $\mathrm{Sym}^n_{\mathcal{O},Y}(C) \to A(Y)$ of Remark 3.1.3.9 exhibit $A(Y)$ as a coproduct $\coprod_{n \geq 0} \mathrm{Sym}^n_{\mathcal{O},Y}(C)$.*

Proof. Assertion (i) is a special case of Corollary 3.1.3.5. Assertion (ii) follows by combining the criterion of Corollary 3.1.3.5 with Propositions 3.1.1.15 and 3.1.1.16. $\qquad\square$

Example 3.1.3.12. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Assume that the underlying ∞-category $\mathcal{C}$ admits countable colimits, and that for each $X \in \mathcal{C}$ the functor $Y \mapsto X \otimes Y$ preserves countable colimits. Then the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint, which is given informally by the formula

$$C \mapsto \coprod_{n \geq 0} \mathrm{Sym}^n(C).$$

The following observation allows us to reformulate Propositions 3.1.3.2 and 3.1.3.3 in terms of the theory of operadic left Kan extensions developed in §3.1.2:

Remark 3.1.3.13. Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$, $\mathcal{A}^{\otimes} \xrightarrow{i} \mathcal{B}^{\otimes} \xrightarrow{j} \mathcal{O}^{\otimes}$, $\theta : \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{C})$, and $f : F \to \theta(F')$ be as in Definition 3.1.3.1. The maps f, F, and F' determine a map

$$h : (\mathcal{A}^{\otimes} \times \Delta^1) \coprod_{\mathcal{A}^{\otimes} \times \{1\}} \mathcal{B}^{\otimes} \to \mathcal{C}^{\otimes} \times \Delta^1.$$

Choose a factorization of h as a composition

$$(\mathcal{A}^{\otimes} \times \Delta^1) \coprod_{\mathcal{A}^{\otimes} \times \{1\}} \mathcal{B}^{\otimes} \xrightarrow{h'} \mathcal{M}^{\otimes} \xrightarrow{h''} \mathcal{C}^{\otimes} \times \Delta^1$$

where h' is a categorical equivalence and $\mathcal{M}^{\otimes}$ is an ∞-category; we note that the composite map $\mathcal{M}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^1$ is exhibits $\mathcal{M}^{\otimes}$ as a correspondence of ∞-operads. Unwinding the definitions, we see that f exhibits F' as a q-free $\mathcal{B}$-algebra generated by F if and only if the map h'' is an operadic q-left Kan extension.

We will also need the following general observation concerning operadic left Kan extensions:

Lemma 3.1.3.14. *Let $\mathcal{M}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^1$ be a correspondence between ∞-operads $\mathcal{A}^{\otimes}$ and $\mathcal{B}^{\otimes}$, and let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads. Suppose that $n > 0$ and we are given a diagram*

$$
\begin{array}{ccc}
(\mathcal{A}^{\otimes} \times \Delta^n) \coprod_{\mathcal{A}^{\otimes} \times \partial \Delta^n} (\mathcal{M}^{\otimes} \times \partial \Delta^n) & \xrightarrow{\;f_0\;} & \mathcal{C}^{\otimes} \\
\downarrow & \overset{f}{\nearrow} & \downarrow \\
\mathcal{M}^{\otimes} \times \Delta^n & \longrightarrow & \mathcal{O}^{\otimes}
\end{array}
$$

where f is a map of generalized ∞-operads, the restriction of f_0 to $\mathcal{A}^{\otimes} \times \Delta^n$ is a map of generalized ∞-operads, the restriction of f_0 to $\mathcal{M}^{\otimes} \times \sigma$ is a map of generalized ∞-operads for every simplex $\sigma \subseteq \partial \Delta^n$, and the restriction of f_0 to $\mathcal{M}^{\otimes} \times \{0\}$ is an operadic q-left Kan extension. Then there exists a functor f as indicated in the diagram, which is a map of generalized ∞-operads.

Proof. Let $K(0) = (\Delta^n \times \{0\}) \coprod_{\partial \Delta^n \times \{0\}} (\partial \Delta^n \times \Delta^1)$, which we identify with a simplicial subset of $\Delta^n \times \Delta^1$. We define a sequence of simplicial subsets

$$K(0) \subseteq K(1) \subseteq \ldots \subseteq K(n+1) = \Delta^n \times \Delta^1$$

so that each $K(i+1)$ is obtained from $K(i)$ by adjoining the image of the simplex $\sigma_i : \Delta^{n+1} \to \Delta^n \times \Delta^1$ which is given on vertices by the formula

$$
\sigma_i(j) = \begin{cases} (j, 0) & \text{if } j \leq n - i \\ (j-1, 1) & \text{otherwise.} \end{cases}
$$

We construct a compatible family of maps $f_i : K(i) \times_{\Delta^1} \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes}$ extending f_0 using induction on i. Assuming f_i has been constructed, to build f_{i+1} it suffices to solve the lifting problem presented in the following diagram:

$$
\begin{array}{ccc}
\Lambda^{n+1}_{n-i} \times_{\Delta^1} \mathcal{M}^{\otimes} & \xrightarrow{\;j\;} & \mathcal{C}^{\otimes} \\
\downarrow & \nearrow & \downarrow \\
\Delta^{n+1} \times_{\Delta^1} \mathcal{M}^{\otimes} & \longrightarrow & \mathcal{O}^{\otimes}.
\end{array}
$$

If $i < n$, then Lemma 2.4.4.6 guarantees that j is a categorical equivalence, so the dotted arrow exists by virtue of the fact that q is a categorical fibration. If $i = n$, then the lifting problem admits a solution by Theorem 3.1.2.3. $\qquad\square$

Notation 3.1.3.15. Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, let $\mathcal{M}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^1$ be a correspondence between ∞-operads, and suppose we are given a map of generalized ∞-operads $\mathcal{M}^{\otimes} \to \mathcal{O}^{\otimes}$. We let $\mathrm{Alg}_{\mathcal{M}/\mathcal{O}}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{M}^{\otimes}, \mathcal{C}^{\otimes})$ spanned by the maps of generalized ∞-operads.

We are now ready to give the proofs of Proposition 3.1.3.2 and 3.1.3.3.

Proof of Proposition 3.1.3.2. The triple (F, F', f) determines a map $h : (\mathcal{A}^{\otimes} \times \Delta^1) \coprod_{\mathcal{A}^{\otimes} \times \{1\}} \mathcal{B}^{\otimes} \to \mathcal{C}^{\otimes} \times \Delta^1$. Choose a factorization of h as a composition

$$(\mathcal{A}^{\otimes} \times \Delta^1) \coprod_{\mathcal{A}^{\otimes} \times \{1\}} \mathcal{B}^{\otimes} \xrightarrow{h'} \mathcal{M}^{\otimes} \xrightarrow{h''} \mathcal{C}^{\otimes} \times \Delta^1$$

where h' is a categorical equivalence and $\mathcal{M}^{\otimes}$ is a correspondence of ∞-operads from $\mathcal{A}^{\otimes}$ to $\mathcal{B}^{\otimes}$. Let $\mathrm{Alg}'_{\mathcal{M}/\mathcal{O}}(\mathcal{C})$ denote the full subcategory of $\mathrm{Alg}_{\mathcal{M}/\mathcal{O}}(\mathcal{C})$ spanned by the functors $\mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes}$ which are operadic

q-left Kan extensions, and let $\mathrm{Alg}''_{\mathcal{M}/\mathcal{O}}(\mathcal{C})$ denote the full subcategory of $\mathrm{Alg}_{\mathcal{M}/\mathcal{O}}(\mathcal{C})$ spanned by those functors which are q-right Kan extensions of their restrictions to $\mathcal{B}^{\otimes}$. Proposition T.4.3.2.15 guarantees that the restriction map $\mathrm{Alg}''_{\mathcal{M}/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C})$ is a trivial Kan fibration; let $s : \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}''_{\mathcal{M}/\mathcal{O}}(\mathcal{C})$ be a section. We observe that the functor θ is equivalent to the composition of s with the restriction map $\mathrm{Alg}_{\mathcal{M}/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{C})$. Let $\overline{F} \in \mathrm{Alg}_{\mathcal{M}/\mathcal{O}}(\mathcal{C})$ be the object determined by h'' so that we have a homotopy commutative diagram

$$
\begin{array}{ccc}
& \mathrm{Map}_{\mathrm{Alg}_{\mathcal{M}/\mathcal{O}}(\mathcal{C})}(\overline{F}, s(F'')) & \\
{}^{\beta'}\swarrow & & \searrow^{\beta} \\
\mathrm{Map}_{\mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C})}(F', F'') \longrightarrow & & \mathrm{Map}_{\mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{C})}(F, s(F'')|\mathcal{A}^{\otimes});
\end{array}
$$

where γ can be identified with the bottom horizontal map. By the two-out-of-three property, it will suffice to show that β and β' are trivial Kan fibrations. For the map β, this follows from the observation that $s(F'')$ is a q-right Kan extension of F''. For the map β', we apply Lemma 3.1.3.14 (together with the observation that $\overline{F}$ is an operadic q-left Kan extension, by virtue of Remark 3.1.3.13). $\square$

Proof of Proposition 3.1.3.3. The implication $(1) \Rightarrow (2)$ is obvious. For the converse, we use the small object argument to choose a factorization of the map $h : (\mathcal{A}^{\otimes} \times \Delta^1) \coprod_{\mathcal{A}^{\otimes} \times \{1\}} \mathcal{B}^{\otimes} \to \mathcal{O}^{\otimes} \times \Delta^1$ as a composition

$$
(\mathcal{A}^{\otimes} \times \Delta^1) \coprod_{\mathcal{A}^{\otimes} \times \{1\}} \mathcal{B}^{\otimes} \xrightarrow{h'} \mathcal{M}^{\otimes} \xrightarrow{h''} \mathcal{O}^{\otimes} \times \Delta^1
$$

where h' is inner anodyne and $\mathcal{M}^{\otimes}$ is a correspondence of ∞-operads from $\mathcal{A}^{\otimes}$ to $\mathcal{B}^{\otimes}$. Using assumption (2) and Theorem 3.1.2.3, we can solve the lifting problem depicted in the diagram

$$
\begin{array}{ccc}
\mathcal{A}^{\otimes} & \longrightarrow & \mathcal{C}^{\otimes} \\
\downarrow & {}^{G}\nearrow & \downarrow \\
\mathcal{M}^{\otimes} & \longrightarrow & \mathcal{O}^{\otimes}
\end{array} .
$$

in such a way that G is an operadic q-left Kan extension. Composing G with h', we obtain an object $F' \in \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C})$ and a natural transformation $f : F \to \theta(F')$. It follows from Remark 3.1.3.13 that f exhibits F' as a q-free $\mathcal{B}$-algebra generated by F. $\square$

3.1.4 Transitivity of Operadic Left Kan Extensions

Our goal in this section is to prove the following transitivity formula for operadic left Kan extensions:

Theorem 3.1.4.1. *Let $\mathcal{M}^{\otimes} \to \Delta^2 \times \mathrm{N}(\mathrm{Fin}_*)$ be a Δ^2-family of ∞-operads (Definition 2.3.2.10). Let $q : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a fibration of ∞-operads, and let $A : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes}$ be a map of generalized ∞-operads. Assume that $A|(\mathcal{M}^{\otimes} \times_{\Delta^2} \Delta^{\{0,1\}})$ and $A|(\mathcal{M}^{\otimes} \times_{\Delta^2} \Delta^{\{1,2\}})$ are operadic q-left Kan extensions, and that the map $\mathcal{M}^{\otimes} \to \Delta^2$ is a flat categorical fibration (see Definition B.3.8). Then $A|(\mathcal{M}^{\otimes} \times_{\Delta^2} \Delta^{\{0,2\}})$ is an operadic q-left Kan extension.*

Theorem 3.1.4.1 has the following consequence:

Corollary 3.1.4.2. *Let $\mathcal{M}^{\otimes} \to \Delta^2 \times \mathrm{N}(\mathrm{Fin}_*)$ be a Δ^2-family of ∞-operads, $q : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathrm{Fin}_*)$ a symmetric monoidal ∞-category, and κ an uncountable regular cardinal. Assume that:*

(i) The ∞-category $\mathcal{M}^{\otimes}$ is essentially κ-small.

(*ii*) *The ∞-category $\mathcal{C}$ admits κ-small colimits, and the tensor product on $\mathcal{C}$ preserves κ-small colimits separately in each variable.*

(*iii*) *The projection map $\mathcal{M}^{\otimes} \to \Delta^2$ is a flat categorical fibration.*

For $i \in \{0, 1, 2\}$, let $\mathcal{M}_i^{\otimes}$ denote the fiber $\mathcal{M}^{\otimes} \times_{\Delta^2} \{i\}$. Let $f_{0,1} : \mathrm{Alg}_{\mathcal{M}_0}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{M}_1}(\mathcal{C})$, $f_{1,2} : \mathrm{Alg}_{\mathcal{M}_1}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{M}_2}(\mathcal{C})$, and $f_{0,2} : \mathrm{Alg}_{\mathcal{M}_0}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{M}_2}(\mathcal{C})$ be the functors given by operadic q-left Kan extension (see below). Then there is a canonical equivalence of functors $f_{0,2} \simeq f_{1,2} \circ f_{0,1}$.

Proof. For $0 \leq i \leq j \leq 2$, let $\mathrm{Alg}_{i,j}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{M}^{\otimes} \times_{\Delta^2} \Delta^{\{i,j\}}), \mathcal{C}^{\otimes})$ spanned by those ∞-operad family maps which are operadic q-left Kan extensions, where $q : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ denotes the projection. Using Lemma 3.1.3.14, Theorem 3.1.2.3, and Proposition 3.1.1.20, we see that conditions (*i*) and (*ii*) guarantee that the restriction map $r : \mathrm{Alg}_{i,j}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{M}_i}(\mathcal{C})$ is a trivial Kan fibration. The map $f_{i,j}$ is defined to be the composition

$$\mathrm{Alg}_{\mathcal{M}_i}(\mathcal{C}) \xrightarrow{s} \mathrm{Alg}_{i,j}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{M}_j}(\mathcal{C}),$$

where s is a section of r. Consequently, the composition $f_{1,2} \circ f_{0,1}$ can be defined as a composition

$$\mathrm{Alg}_{\mathcal{M}_0}(\mathcal{C}) \xrightarrow{s'} \mathrm{Alg}_{0,1}(\mathcal{C}) \times_{\mathrm{Alg}_{\mathcal{M}_1}(\mathcal{C})} \mathrm{Alg}_{1,2}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{M}_2}(\mathcal{C}),$$

where s' is a section of the trivial Kan fibration $\mathrm{Alg}_{0,1}(\mathcal{C}) \times_{\mathrm{Alg}_{\mathcal{M}_1}(\mathcal{C})} \mathrm{Alg}_{1,2}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{M}_0}(\mathcal{C})$.

Let $\mathrm{Alg}_{0,1,2}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{M}^{\otimes}, \mathcal{C}^{\otimes})$ spanned by the ∞-operad family maps whose restrictions to $\mathcal{M}^{\otimes} \times_{\Delta^2} \Delta^{\{0,1\}}$ and $\mathcal{M}^{\otimes} \times_{\Delta^2} \Delta^{\{1,2\}}$ are operadic q-left Kan extensions. Condition (*iii*) guarantees that the inclusion $\mathcal{M}^{\otimes} \times_{\Delta^2} \Lambda_1^2 \subseteq \mathcal{M}^{\otimes}$ is a categorical equivalence, so that the restriction maps

$$\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{M}^{\otimes}, \mathcal{C}^{\otimes}) \to \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{M}^{\otimes} \times_{\Delta^2} \Lambda_1^2, \mathcal{C}^{\otimes})$$

$$\mathrm{Alg}_{0,1,2}(\mathcal{C}) \to \mathrm{Alg}_{0,1}(\mathcal{C}) \times_{\mathrm{Alg}_{\mathcal{M}_1}(\mathcal{C})} \mathrm{Alg}_{1,2}(\mathcal{C})$$

are trivial Kan fibrations. It follows that the restriction map $r'' : \mathrm{Alg}_{0,1,2}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{M}_0}(\mathcal{C})$ is a trivial Kan fibration admitting a section s'', and that $f_{1,2} \circ f_{0,1}$ can be identified with the composition

$$\mathrm{Alg}_{\mathcal{M}_0}(\mathcal{C}) \xrightarrow{s''} \mathrm{Alg}_{0,1,2}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{M}_2}(\mathcal{C}).$$

Using Theorem 3.1.4.1, we deduce that the restriction map $\mathrm{Alg}_{0,1,2}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{M}_2}(\mathcal{C})$ factors as a composition

$$\mathrm{Alg}_{0,1,2}(\mathcal{C}) \xrightarrow{\theta} \mathrm{Alg}_{0,2}(\mathcal{C}) \xrightarrow{\theta'} \mathrm{Alg}_{\mathcal{M}_2}(\mathcal{C}).$$

The composition $\theta \circ s''$ is a section of the trivial Kan fibration $\mathrm{Alg}_{0,2}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{M}_0}(\mathcal{C})$, so that $f_{0,2}$ can be identified with the composition $\theta' \circ (\theta \circ s'') \simeq f_{1,2} \circ f_{0,1}$ as desired. $\square$

The proof of Theorem 3.1.4.1 rests on a more basic transitivity property of operadic colimit diagrams. To state this property, we need to introduce a bit of terminology. Let $q : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a fibration of ∞-operads, and let $p : K \diamond \Delta^0 \to \mathcal{C}^{\otimes}$ be a map of simplicial sets which carries each edge of $K \diamond \Delta^0$ to an active morphism in $\mathcal{C}^{\otimes}$. Since the map $K \diamond \Delta^0 \to K^{\triangleright}$ is a categorical equivalence (Proposition T.4.2.1.2), there exists a map $p' : K^{\triangleright} \to \mathcal{C}^{\otimes}$ such that p is homotopic to the composition $K \diamond \Delta^0 \to K^{\triangleright} \xrightarrow{p'} \mathcal{C}^{\otimes}$. Moreover, the map p' is unique up to homotopy. We will say that p is a *(weak) operadic q-colimit diagram* if p' is a (weak) operadic q-colimit diagram, in the sense of Definition 3.1.1.2.

Lemma 3.1.4.3. *Let $X \to S$ be a coCartesian fibration of simplicial sets, and let $q : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a fibration of ∞-operads. Let*

$$\theta : (X \diamond_S S) = (X \times \Delta^1) \coprod_{X \times \{1\}} S \to \mathcal{C}^{\otimes}$$

be a map satisfying the following conditions:

(i) *The map θ carries every edge in $X \diamond_S S$ to an active morphism in $\mathcal{C}^{\otimes}$.*

(ii) *For every vertex $s \in S$, the induced map $\theta_s : X_s \diamond \Delta^0 \to \mathcal{C}^{\otimes}$ is a weak operadic q-colimit diagram.*

Let $\theta_0 = \theta|X$. Let $\mathcal{C}_{\theta/}^{\mathrm{act}}$ denote the full subcategory of $\mathcal{C}_{\theta/}^{\otimes} \times_{\mathcal{C}^{\otimes}} \mathcal{C}$ spanned by those objects which correspond to maps $\overline{\theta} : (X \diamond_S S)^{\triangleright} \to \mathcal{C}^{\otimes}$ which carry every edge of $(X \diamond_S S)^{\triangleright}$ to an inert morphism of $\mathcal{C}^{\otimes}$, and define $\mathcal{C}_{\theta_0/}^{\mathrm{act}}$, $\mathcal{D}_{q\theta/}^{\mathrm{act}}$, and $\mathcal{D}_{q\theta_0/}^{\mathrm{act}}$ similarly. Then:

(1) *The map $\mathcal{C}_{\theta/}^{\mathrm{act}} \to \mathcal{C}_{\theta_0/}^{\mathrm{act}} \times_{\mathcal{D}_{q\theta_0/}^{\mathrm{act}}} \mathcal{D}_{q\theta/}^{\mathrm{act}}$ is a trivial Kan fibration.*

(2) *Let $\overline{\theta} : (X \diamond_S S)^{\triangleright} \to \mathcal{C}^{\otimes}$ be an extension of θ which carries each edge of $(X \diamond_S S)^{\triangleright}$ to an active morphism in $\mathcal{C}^{\otimes}$. Then $\overline{\theta}$ is a weak operadic q-colimit diagram if and only if $\overline{\theta}_0 = \overline{\theta}|X^{\triangleright}$ is a weak operadic q-colimit diagram.*

(3) *Assume that each θ_s is an operadic q-colimit diagram, and let $\overline{\theta}$ be as in (2). Then $\overline{\theta}$ is an operadic q-colimit diagram if and only if $\overline{\theta}_0$ is an operadic q-colimit diagram.*

Proof. Assertion (2) follows immediately from (1), and assertion (3) follows from (2) after replacing θ by the composite functor

$$X \diamond_S S \xrightarrow{\theta} \mathcal{C}^{\otimes} \xrightarrow{\oplus Y} \mathcal{C}^{\otimes},$$

where Y denotes an arbitrary object of $\mathcal{C}^{\otimes}$. It will therefore suffice to prove (1). For every map of simplicial sets $K \to S$, let θ_K denote the induced map $X \diamond_S K \to \mathcal{C}^{\otimes}$. We will prove more generally that for $K' \subseteq K$, the induced map

$$\psi_{K',K} : \mathcal{C}_{\theta_K/}^{\mathrm{act}} \to \mathcal{C}_{\theta_{K'}/}^{\mathrm{act}} \times_{\mathcal{D}_{q\theta_{K'}/}^{\mathrm{act}}} \mathcal{D}_{q\theta_K/}^{\mathrm{act}}$$

is a trivial Kan fibration. We proceed by induction on the (possibly infinite) dimension n of K. If K is empty, the result is obvious. Otherwise, working simplex-by-simplex, we can assume that K is obtained from K' by adjoining a single nondegenerate m-simplex σ whose boundary already belongs to K'. Replacing K by σ, we may assume that $K = \Delta^m$ and $K' = \partial \Delta^m$. If $m = 0$, then the desired result follows from assumption (ii). Assume therefore that $m > 0$.

   Because $\theta_{K',K}$ is clearly a categorical fibration (even a left fibration), to prove that $\theta_{K',K}$ is a trivial Kan fibration it suffices to show that $\theta_{K',K}$ is a categorical equivalence. Since $m \leq n$, K' has dimension $< n$, so the inductive hypothesis guarantees that $\psi_{\{m\},K'}$ is a trivial Kan fibration. The map $\psi_{\{m\},K}$ is a composition of $\psi_{K',K}$ with a pullback of $\psi_{\{m\},K'}$. Using a two-out-of-three argument, we are reduced to proving that $\psi_{\{m\},K}$ is a categorical equivalence. For this, it suffices to show that the inclusion $f : X \diamond_S \{m\} \to X \diamond_S \Delta^m$ is left cofinal.

   Let $X' = X \times_S \Delta^m$. The map f is a pushout of the inclusion

$$f' : X' \coprod_{X'_m} (X'_m \diamond \{m\}) \hookrightarrow X' \diamond_{\Delta^m} \Delta^m.$$

It will therefore suffice to show that f' is left cofinal. We have a commutative diagram

The map g'' is a pushout of the inclusion $X' \times \{1\} \subseteq X' \times \Delta^1$, and therefore left cofinal; the same argument shows that g is left cofinal. The map f'' is obviously left cofinal. The map g' is a pushout of the inclusion $X'_m \subseteq X'$, which is left cofinal because $\{m\}$ is a final object of Δ^m and the map $X' \to \Delta^m$ is a coCartesian fibration. It now follows from Proposition T.4.1.1.3 that f' is left cofinal, as required. $\square$

Proof of Theorem 3.1.4.1. Fix an object $Z \in \mathcal{M}_2^\otimes$, and let $\mathcal{Z}$ denote the full subcategory of $\mathcal{M}_{/Z}^\otimes$ whose objects are active morphisms $X \to Z$ where $X \in \mathcal{M}_0^\otimes$. We wish to prove that the composite map

$$\phi : \mathcal{Z}^\triangleright \to (\mathcal{M}_{/Z}^\otimes)^\triangleright \to \mathcal{M}^\otimes \xrightarrow{A} \mathcal{C}^\otimes$$

is an operadic q-colimit diagram. Let $\overline{\mathcal{Z}}$ denote the subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{M}_{/Z}^\otimes)$ whose objects are diagrams of active morphisms
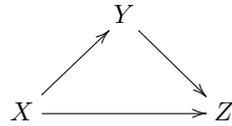
in $\mathcal{M}^\otimes$ such that $X \in \mathcal{M}_0^\otimes$ and $Y \in \mathcal{M}_1^\otimes$. Evaluation at $\{0\}$ induces a Cartesian fibration $\psi : \overline{\mathcal{Z}} \to \mathcal{Z}$. Let Z' be an object of $\mathcal{Z}$, corresponding to an active morphism $X \to Z$ in $\mathcal{M}^\otimes$. Then the fiber $\psi^{-1}\{Z'\}$ is a localization of the ∞-category $(\mathcal{M}_{/Z}^\otimes)^{X/}$, which is equivalent to $(\mathcal{M}^\otimes)_{X//Z}$ and therefore weakly contractible (since $\mathcal{M}^\otimes \to \Delta^2$ is flat). Note that the map $\psi' : \overline{\mathcal{Z}} \times_\mathcal{Z} \mathcal{Z}_{Z'/} \to \mathcal{Z}_{Z'/}$ is a Cartesian fibration (Proposition T.2.4.3.2). Since $\mathcal{Z}_{Z'/}$ has an initial object $\mathrm{id}_{Z'}$, the weakly contractible simplicial set $\psi^{-1}\{Z'\} \simeq \psi'^{-1}\{\mathrm{id}_{Z'}\}$ is weakly homotopy equivalent to $\overline{\mathcal{Z}} \times_\mathcal{Z} \mathcal{Z}_{Z'/}$. Applying Theorem T.4.1.3.1, we deduce that ψ is left cofinal. Consequently, it will suffice to show that $\phi \circ \psi : \overline{\mathcal{Z}}^\triangleright \to \mathcal{C}^\otimes$ is an operadic q-colimit diagram.

Let $\mathcal{Y}$ denote the full subcategory of $\mathcal{M}_{/Z}^\otimes$ spanned by active morphisms $Y \to Z$ where $Y \in \mathcal{M}_1^\otimes$. Evaluation at $\{1\}$ induces a coCartesian fibration $\rho : \overline{\mathcal{Z}} \to \mathcal{Y}$. We observe that there is a canonical map $\overline{\mathcal{Z}} \diamond_\mathcal{Y} \mathcal{Y} \to \mathcal{M}_{/Z}^\otimes$, which determines a map

$$\theta : (\overline{\mathcal{Z}} \diamond_\mathcal{Y} \mathcal{Y})^\triangleright \to \mathcal{C}^\otimes$$

extending $\phi \circ \psi$. Fix an object $Y' \in \mathcal{Y}$, corresponding to an active morphism $Y \to Z$ in $\mathcal{M}^\otimes$. Then θ induces a map $\theta_{Y'} : \rho^{-1}\{Y'\} \diamond \Delta^0 \to \mathcal{C}^\otimes$. We claim that $\theta_{Y'}$ is an operadic q-colimit diagram. To prove this, let $\mathcal{X}(Y)$ denote the full subcategory of $(\mathcal{M}^\otimes)^{/Y}$ spanned by the active morphisms $X \to Y$, and define $\mathcal{X}'(Y) \subseteq (\mathcal{M}^\otimes)_{/Y}$ similarly. The map $\theta_{Y'}$ factors through a map

$$\theta'_{Y'} : \mathcal{X}(Y) \diamond \Delta^0 \to \mathcal{C}^\otimes.$$

Since $\mathcal{M}_{/Z}^\otimes \to \mathcal{M}^\otimes$ is a left fibration, the map $\rho^{-1}\{Y'\} \to \mathcal{X}(Y)$ is a trivial Kan fibration; it therefore suffices to show that $\theta'_{Y'}$ is an operadic q-colimit diagram. Since the evident map $\mathcal{X}'(Y) \to \mathcal{X}(Y)$ is a categorical equivalence (Proposition T.4.2.1.5), it suffices to show that the induced map $\mathcal{X}'(Y) \diamond \Delta^0 \to \mathcal{C}^\otimes$ is an operadic q-colimit diagram, which is equivalent to the requirement that the composite map

$$\mathcal{X}'(Y)^\triangleright \to (\mathcal{M}_{/Y}^\otimes)^\triangleright \to \mathcal{M}^\otimes \to \mathcal{C}^\otimes$$

is an operadic q-colimit diagram. This follows from our assumption that $A|(\mathcal{M}^\otimes \times_{\Delta^2} \Delta^{\{0,1\}})$ is an operadic q-left Kan extension.

Since $A|(\mathcal{M}^\otimes \times_{\Delta^2} \Delta^{\{0,1\}})$, the restriction of θ to $\mathcal{Y}^\triangleright$ is an operadic q-colimit diagram. The inclusion $\mathcal{Y} \to \overline{\mathcal{Z}} \diamond_\mathcal{Y} \mathcal{Y}$ is a pushout of the inclusion $\overline{\mathcal{Z}} \times \{1\} \subseteq \overline{\mathcal{Z}} \times \Delta^1$, and therefore left cofinal. It follows that θ itself is an operadic q-colimit diagram. Invoking Lemma 3.1.4.3, we conclude that $\phi \circ \psi$ is an operadic q-colimit diagram, as desired. $\square$

3.2   Limits and Colimits of Algebras

Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category and $\mathcal{O}^{\otimes}$ an arbitrary ∞-operad. In §2.1.3, we introduced the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ of $\mathcal{O}$-algebra objects of $\mathcal{C}$. Our goal in this section is to study these ∞-categories in more detail. In particular, we will study conditions which guarantee the existence (and allow for the computation of) limits and colimits in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. We begin in §3.2.2 with the study of limits in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. This is fairly straightforward: the basic result is that limits in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ can usually be computed in the underlying ∞-category $\mathcal{C}$ (Proposition 3.2.2.1).

The study of colimits is much more involved. First of all, we do not expect colimits in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ to be computed in the underlying ∞-category $\mathcal{C}$ in general. This is often true for colimits of a special type (for example, colimits of diagrams indexed by *sifted* simplicial sets), provided that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ behaves well with respect to colimits. The case of general colimits is more difficult. For example, if A and B are objects of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$, then it is in general difficult to describe the coproduct $A \coprod B$ explicitly. We will sidestep this issue using the theory of free algebras developed in §3.1. Though it is difficult to construct colimits of algebras in general, it is often much easier to construct colimits of free algebras, since the free algebra functor preserves colimits (being a left adjoint). In §3.2.3 we will exploit this observation to construct general colimits in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$: the basic idea is to resolve arbitrary algebras with free algebras. Although this strategy leads to a fairly general existence result (Corollary 3.2.3.3), it is somewhat unsatisfying because it does not usually give a direct description of the colimit of a diagram $K \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. However, we can often say more for specific choices of the ∞-operad $\mathcal{O}^{\otimes}$. sometimes do better for specific choices of the underlying ∞-operad. For example, if $\mathcal{O}^{\otimes}$ is the commutative ∞-operad $\mathrm{Comm}^{\otimes} = \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, then it is easy to construct finite coproducts in the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) = \mathrm{CAlg}(\mathcal{C})$: these are simply given by tensor products in the underlying ∞-category $\mathcal{C}$. We will prove this in §3.2.4 (Proposition 3.2.4.7), after giving a general discussion of tensor products of algebras. For coproducts of *empty* collections, we can be more general: there is an easy construction for initial objects of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ whenever the ∞-operad $\mathcal{O}^{\otimes}$ is unital. We will describe this construction in §3.2.1.

3.2.1   Unit Objects and Trivial Algebras

Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad, and let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads. For each object $X \in \mathcal{O}$, there is an essentially unique morphism $0 \to X$ (where 0 denotes the zero object of $\mathcal{O}^{\otimes}$), which determines a functor $\mathcal{C}_0^{\otimes} \to \mathcal{C}_X$. Since $\mathcal{C}_0^{\otimes}$ is a contractible Kan complex, we can identify this functor with an object of $\mathbf{1}_X \in \mathcal{C}_X$. We will refer to the object $\mathbf{1}_X$ as a *unit object* of $\mathcal{C}_X$. Our goal in this section is to study the basic features of these unit objects. We begin by formulating a slightly more general definition, which makes sense even if we do not assume that p is a coCartesian fibration.

Definition 3.2.1.1. Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, let $X \in \mathcal{O}^{\otimes}$, and let $f : C_0 \to \mathbf{1}_X$ be a morphism in $\mathcal{C}^{\otimes}$, where $\mathbf{1}_X \in \mathcal{C}_X$. We will say that f *exhibits $\mathbf{1}_X$ as an X-unit object* if the following conditions are satisfied:

(1) The object C_0 belongs to $\mathcal{C}_{\langle 0 \rangle}^{\otimes}$.

(2) The morphism f is given by an operadic p-colimit diagram $\Delta^1 \simeq (\Delta^0)^{\triangleright} \to \mathcal{C}^{\otimes}$.

More generally, we will say that an arbitrary morphism $C_0 \to C$ in $\mathcal{C}^{\otimes}$ *exhibits C as a unit object* if, for every inert morphism $C \to C'$ with $C' \in \mathcal{C}$, the composite map $C_0 \to C'$ exhibits C' as a $p(C')$-unit object.

Suppose that $\mathcal{O}^{\otimes}$ is a unital ∞-operad. We will say that a fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ *has unit objects* if, for every object $X \in \mathcal{O}^{\otimes}$, there exists a morphism $f : C_0 \to \mathbf{1}_X$ in $\mathcal{C}^{\otimes}$ which exhibits $\mathbf{1}_X$ as an X-unit object.

Remark 3.2.1.2. The terminology of Definition 3.2.1.1 is slightly abusive: the condition that a morphism $f : C_0 \to C$ exhibits C as a unit object of $\mathcal{C}^{\otimes}$ depends not only on the ∞-operad $\mathcal{C}^{\otimes}$, but also on the ∞-operad fibration $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$.

Remark 3.2.1.3. Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads. If $X \in \mathcal{O}$ and $f : C_0 \to C$ is a morphism in $\mathcal{C}^{\otimes}$ which exhibits C as a unit object, then f is p-coCartesian: this follows immediately from Proposition 3.1.1.10.

Remark 3.2.1.4. Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, and assume that $\mathcal{O}^{\otimes}$ is unital. Let $\mathfrak{X}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C}^{\otimes})$ spanned by those morphisms $f : C_0 \to C$ which exhibit C as a unit object. Then the composite map

$$\phi : \mathfrak{X} \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C}^{\otimes}) \to \mathrm{Fun}(\{1\}, \mathcal{C}^{\otimes}) \xrightarrow{p} \mathcal{O}^{\otimes}$$

induces a trivial Kan fibration onto a full subcategory of $\mathcal{O}^{\otimes}$. To see this, we let $\mathfrak{X}'$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C}^{\otimes})$ spanned by those morphisms $f : C_0 \to C$ which are p-coCartesian and satisfy $C_0 \in \mathcal{C}^{\otimes}_{\langle 0 \rangle}$. Proposition T.4.3.2.15 implies that the map

$$\mathfrak{X}' \to \mathcal{C}^{\otimes}_{\langle 0 \rangle} \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$$

is a trivial Kan fibration onto a full subcategory of the fiber product $\mathfrak{X}'' = \mathcal{C}^{\otimes}_{\langle 0 \rangle} \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$. Since $\mathcal{C}^{\otimes}_{\langle 0 \rangle} \to \mathcal{O}^{\otimes}_{\langle 0 \rangle}$ is a categorical fibration between contractible Kan complexes, it is a trivial Kan fibration; it follows that the induced map $\mathfrak{X}'' \to \mathcal{O}^{\otimes}_{\langle 0 \rangle} \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$ is also a trivial Kan fibration. The target of this map can be identified with the ∞-category $\mathcal{O}^{\otimes}_{*}$ of pointed objects of $\mathcal{O}^{\otimes}$, and the forgetful functor $\mathcal{O}^{\otimes}_{*} \to \mathcal{O}^{\otimes}$ is a trivial Kan fibration because $\mathcal{O}^{\otimes}$ is unital. The desired result now follows by observing that ϕ is given by the composition

$$\mathfrak{X} \subseteq \mathfrak{X}' \to \mathfrak{X}'' \to \mathcal{O}^{\otimes}_{*} \to \mathcal{O}^{\otimes} \, .$$

We can summarize our discussion as follows: if $f : C_0 \to C$ is a morphism in $\mathcal{C}^{\otimes}$ which exhibits C as a unit object, then f is determined (up to canonical equivalence) by the object $p(C) \in \mathcal{O}^{\otimes}$. We observe that $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ has unit objects if and only if the functor ϕ above is essentially surjective. In this case the inclusion $\mathfrak{X} \to \mathfrak{X}'$ must also be essentially surjective: in other words, a morphism $f : C_0 \to C$ exhibits C as a unit object if and only if f is p-coCartesian and $C_0 \in \mathcal{C}^{\otimes}_{\langle 0 \rangle}$.

Remark 3.2.1.5. . It is easy to see that the essential image of the functor ϕ of Remark 3.2.1.4 is stable under the concatenation functor $\oplus$ of Remark 2.2.4.6. Consequently, to show that a fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ has units objects, it suffices to show that for every object $X \in \mathcal{O}$ there exists a map $f : C_0 \to \mathbf{1}_X$ which exhibits $\mathbf{1}_X$ as an X-unit object of $\mathcal{C}$.

Example 3.2.1.6. Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads, where $\mathcal{O}^{\otimes}$ is unital. Then p has unit objects. This follows immediately from Proposition 3.1.1.20 and Remark 3.2.1.5.

Definition 3.2.1.7. Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, and assume that $\mathcal{O}^{\otimes}$ is unital. We will say that an algebra object $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ is *trivial* if, for every object $X \in \mathcal{O}^{\otimes}$, the induced map $A(0) \to A(X)$ exhibits $A(X)$ as a unit object; here 0 denotes the zero object of $\mathcal{O}^{\otimes}$.

When trivial algebra objects exist, they are precisely the initial objects of $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$:

Proposition 3.2.1.8. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, where $\mathcal{O}^{\otimes}$ is unital. Assume that p has unit objects. The following conditions on a $\mathcal{O}$-algebra object $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ are equivalent:*

(1) *The object A is an initial object of $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$.*

(2) *The functor A is a p-left Kan extension of $A|\,\mathcal{O}^{\otimes}_{\langle 0 \rangle}$.*

(3) *The algebra object A is trivial.*

Corollary 3.2.1.9. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Then the ∞-category $\mathrm{CAlg}(\mathcal{C})$ has an initial object. Moreover, a commutative algebra object A of $\mathcal{C}$ is an initial object of $\mathrm{CAlg}(\mathcal{C})$ if and only if the unit map $\mathbf{1} \to A$ is an equivalence in $\mathcal{C}$.*

The proof of Proposition 3.2.1.8 depends on the following:

Lemma 3.2.1.10. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, where $\mathcal{O}^{\otimes}$ is unital. The following conditions are equivalent:*

(1) *The fibration p has unit objects.*

(2) *There exists a trivial $\mathcal{O}$-algebra object $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$.*

Proof. The implication $(2) \Rightarrow (1)$ follows immediately from Remark 3.2.1.5. Conversely, suppose that (1) is satisfied. Choose an arbitrary section s of the trivial Kan fibration $\mathcal{C}^{\otimes}_{\langle 0 \rangle} \to \mathcal{O}^{\otimes}_{\langle 0 \rangle}$. Since p has units, Lemma T.4.3.2.13 (and Remark 3.2.1.3) imply that s admits a p-left Kan extension $A : \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$. For every morphism $f : X \to Y$ in $\mathcal{O}^{\otimes}$, we have a commutative diagram

$$
\begin{array}{ccc}
& X & \\
{\scriptstyle g}\nearrow & & \searrow{\scriptstyle f} \\
0 \xrightarrow{\quad h \quad} & & Y
\end{array}
$$

in $\mathcal{O}^{\otimes}$. Since $A(g)$ and $A(h)$ are p-coCartesian morphisms of $\mathcal{C}^{\otimes}$, Proposition T.2.4.1.7 implies that $A(f)$ is p-coCartesian. It follows in particular that A preserves inert morphisms, so that $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$. Using Remark 3.2.1.4, we conclude that A is trivial. $\qquad\square$

Proof of Proposition 3.2.1.8. The implication $(2) \Rightarrow (3)$ follows from the proof of Lemma 3.2.1.10, and $(3) \Rightarrow (2)$ follows immediately from Remark 3.2.1.3. The implication $(2) \Rightarrow (1)$ follows from Proposition T.4.3.2.17 (since the ∞-category of sections of the trivial Kan fibration $\mathcal{C}^{\otimes}_{\langle 0 \rangle} \to \mathcal{O}^{\otimes}_{\langle 0 \rangle}$ is a contractible Kan complex). The reverse implication $(1) \Rightarrow (2)$ follows from the uniqueness of initial objects up to equivalence, since Lemma 3.2.1.10 guarantees the existence of an object $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ satisfying (3) (and therefore (2) and (1) as well). $\qquad\square$

3.2.2   Limits of Algebras

Let $\mathcal{C}$ be a symmetric monoidal category, and let A and B be commutative algebra objects of $\mathcal{C}$. Assume that the objects $A, B \in \mathcal{C}$ admit a product $A \times B \in \mathcal{C}$. The diagram

$$ B \leftarrow B \otimes B \leftarrow (A \times B) \otimes (A \times B) \to A \otimes A \to A $$

determines a multiplication map $(A \times B) \otimes (A \times B) \to A \times B$. This multiplication determines a commutative algebra structure on $A \times B$. In fact, $A \times B$ can be identified with the product of A and B both in the underlying category $\mathcal{C}$ and in the category $\mathrm{CAlg}(\mathcal{C})$ of commutative algebra objects of $\mathcal{C}$.

In this section, we will generalize the above observation in several ways:

(a) We work in the setting of ∞-categories, rather than ordinary categories.

(b) In place of commutative algebra objects, we will study $\mathcal{O}$-algebra objects for an arbitrary ∞-operad $\mathcal{O}^{\otimes}$.

(c) We work with limits indexed by arbitrary diagrams, rather than merely Cartesian products.

(d) Rather than considering limits in a fixed symmetric monoidal ∞-category $\mathcal{C}$, we consider *relative* limits with respect to a fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$.

We can state our main result as follows:

Proposition 3.2.2.1. *Let $\mathcal{O}^{\otimes}$ be an ∞-operad, let $p : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a fibration of ∞-operads, and suppose we are given a commutative diagram*

$$
\begin{array}{ccc}
K & \xrightarrow{\ f\ } & \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \\
\downarrow & & \downarrow{\scriptstyle q} \\
K^{\triangleleft} & \xrightarrow{\ g\ } & \mathrm{Alg}_{\mathcal{O}}(\mathcal{D}).
\end{array}
$$

Assume that for every object $X \in \mathcal{O}$, the induced diagram

$$
\begin{array}{ccc}
K & \xrightarrow{\ f_X\ } & \mathcal{C} \\
\downarrow {\scriptstyle \overline{f}_X} \nearrow & & \downarrow \\
K^{\triangleleft} & \longrightarrow & \mathcal{D}
\end{array}
$$

admits an extension as indicated, where $\overline{f}_X$ is a p-limit diagram. Then:

(1) *There exists an extension $\overline{f} : K^{\triangleleft} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ of f which is compatible with g, such that $\overline{f}$ is a q-limit diagram.*

(2) *Let $\overline{f} : K^{\triangleleft} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ be an arbitrary extension of f which is compatible with g. Then $\overline{f}$ is a q-limit diagram if and only if for every object $X \in \mathcal{O}$, the induced map $\overline{f}_X : K^{\triangleleft} \to \mathcal{C}$ is a p-limit diagram.*

Warning 3.2.2.2. Let $\overline{f} : K^{\triangleleft} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ be as in part (2) of Proposition 3.2.2.1, and let $X \in \mathcal{O}$. Although the map $\overline{f}_X$ takes values in $\mathcal{C} \subseteq \mathcal{C}^{\otimes}$, the condition that $\overline{f}_X$ be a p-limit diagram is generally *stronger* than the condition that $\overline{f}_X$ be a p_0-limit diagram, where $p_0 : \mathcal{C} \to \mathcal{D}$ denotes the restriction of p.

In spite of Warning 3.2.2.2, the criterion of Proposition 3.2.2.1 can be simplified if we are willing to restrict our attention to the setting of ∞-operads admitting a coCartesian fibration to $\mathcal{O}^{\otimes}$.

Corollary 3.2.2.3. *Suppose we are given a commutative diagram*

$$
\begin{array}{ccc}
\mathcal{C}^{\otimes} & \xrightarrow{\quad p \quad} & \mathcal{D}^{\otimes} \\
 & \searrow \quad \swarrow & \\
 & \mathcal{O}^{\otimes} &
\end{array}
$$

where p is a fibration of ∞-operads and the vertical maps are coCartesian fibrations of ∞-operads. Suppose given a commutative diagram

$$
\begin{array}{ccc}
K & \xrightarrow{\ f\ } & \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \\
\downarrow & & \downarrow{\scriptstyle q} \\
K^{\triangleleft} & \xrightarrow{\ g\ } & \mathrm{Alg}_{/\mathcal{O}}(\mathcal{D})
\end{array}
$$

such that, for every object $X \in \mathcal{O}$, the induced diagram

$$
\begin{array}{ccc}
K & \xrightarrow{\ f_X\ } & \mathcal{C}_X \\
\downarrow {\scriptstyle \overline{f}_X} \nearrow & & \downarrow{\scriptstyle p_X} \\
K^{\triangleleft} & \longrightarrow & \mathcal{D}_X
\end{array}
$$

admits an extension as indicated, where $\overline{f}_X$ is a p_X-limit diagram. Then:

(1) *There exists an extension $\overline{f} : K^{\triangleleft} \to \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C})$ of f which is compatible with g, such that $\overline{f}$ is a q-limit diagram.*

(2) *Let $\overline{f} : K^{\triangleleft} \to \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C})$ be an arbitrary extension of f which is compatible with g. Then $\overline{f}$ is a q-limit diagram if and only if for every object $X \in \mathcal{O}$, the induced map $K^{\triangleleft} \to \mathcal{C}$ is a p_X-limit diagram.*

Proof. Combine Proposition 3.2.2.1 with Corollary T.4.3.1.15. $\square$

Passing to the case $\mathcal{D}^{\otimes} = \mathcal{O}^{\otimes}$, we obtain the following result:

Corollary 3.2.2.4. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads, and let $q : K \to \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C})$ be a diagram. Suppose that, for every object $X \in \mathcal{O}$, the induced diagram $q_X : K \to \mathcal{C}_X$ admits a limit. Then:*

(1) *The diagram $q : K \to \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C})$ admits a limit.*

(2) *An extension $\overline{q} : K^{\triangleleft} \to \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C})$ of q is a limit diagram if and only if the induced map $\overline{q}_X : K^{\triangleleft} \to \mathcal{C}_X$ is a limit diagram for each $X \in \mathcal{O}$.*

Corollary 3.2.2.5. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads and let K be a simplicial set. Suppose that, for every object $X \in \mathcal{O}^{\otimes}$, the fiber $\mathcal{C}_X$ admits K-indexed limits. Then:*

(1) *The ∞-category $\mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C})$ admits K-indexed limits.*

(2) *An arbitrary diagram $K^{\triangleleft} \to \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C})$ is a limit diagram if and only the composite diagram*

$$K^{\triangleleft} \to \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C}) \to \mathcal{C}_X$$

is a limit, for each $X \in \mathcal{O}$.

In particular, for each $X \in \mathcal{O}$ the evaluation functor $\mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C}) \to \mathcal{C}_X$ preserves K-indexed limits.

We now turn to the proof of Proposition 3.2.2.1. We need some preliminaries.

Lemma 3.2.2.6. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, and let $\gamma : A \to A'$ be a morphism in $\mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C})$. The following conditions are equivalent:*

(1) *The morphism γ is an equivalence in $\mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C})$.*

(2) *For every object $X \in \mathcal{O}$, the morphism $\gamma(X) : A(X) \to A'(X)$ is an equivalence in $\mathcal{C}$.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious. Conversely, suppose that (2) is satisfied. Let $X \in \mathcal{O}^{\otimes}_{\langle n \rangle}$; we wish to prove that $\gamma(X)$ is an equivalence in $\mathcal{C}^{\otimes}_{\langle n \rangle}$. Since $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category, it suffices to show that for every $1 \leq j \leq n$, the image of $\gamma(X)$ under the functor $\rho^j_! : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}$ is an equivalence. Since A and A' are maps of ∞-operads, this morphism can be identified with $\gamma(X_j)$ where X_j is the image of X under the corresponding functor $\mathcal{O}^{\otimes}_{\langle n \rangle} \to \mathcal{O}$. The desired result now follows immediately from (2). $\square$

Lemma 3.2.2.7. *Let $p : \mathcal{D} \to \mathcal{C}$ be a right fibration of ∞-categories, let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory, and let $\mathcal{D}^0 = \mathcal{C}^0 \times_{\mathcal{C}} \mathcal{D}$. Let $q : X \to S$ be a categorical fibration of simplicial sets, and let $F : \mathcal{C} \to X$ be a map which is a q-left Kan extension of $F|\,\mathcal{C}^0$. Then $F \circ p$ is a q-left Kan extension of $F \circ p|\,\mathcal{D}^0$.*

Proof. Let D be an object of $\mathcal{D}$, $C = p(D)$, and define

$$\mathcal{C}^0_{/C} = \mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/C} \quad \mathcal{D}^0_{/D} = \mathcal{D}^0 \times_{\mathcal{D}} \mathcal{D}_{/D}\,.$$

We wish to show that the composition

$$(\mathcal{D}^0_{/D})^{\triangleright} \to (\mathcal{C}^0_{/C})^{\triangleright} \to \mathcal{C} \xrightarrow{F} X$$

is a q-colimit diagram. Since F is a q-left Kan extension of $F|\,\mathcal{C}^0$, it will suffice to show that the map $\phi_0 : \mathcal{D}^0_{/D} \to \mathcal{C}^0_{/C}$ is a trivial Kan fibration. The map ϕ_0 is a pullback of the map $\phi : \mathcal{D}_{/D} \to \mathcal{C}_{/C}$, which is a trivial Kan fibration by Proposition T.2.1.2.5. $\square$

Lemma 3.2.2.8. *Let $p : X \to S$ and $q : Y \to Z$ be maps of simplicial sets. Assume that q is a categorical fibration, and that p is a flat categorical fibration.*

Define new simplicial sets Y' and Z' equipped with maps $Y' \to S$, $Z' \to S$ via the formulas

$$\mathrm{Hom}_S(K, Y') \simeq \mathrm{Hom}(X \times_S K, Y)$$

$$\mathrm{Hom}_S(K, Z') \simeq \mathrm{Hom}(X \times_S K, Z).$$

Let $\mathcal{C}'$ be an ∞-category equipped with a functor $f : \mathcal{C}' \to Y'$, and let $\mathcal{C}$ be a full subcategory of $\mathcal{C}'$. Then:

(1) *Composition with q determines a categorical fibration $q' : Y' \to Z'$.*

(2) *Let $F : X \times_S \mathcal{C}' \to Y$ be the map classified by f, and suppose that F is a q-left Kan extension of $F | X \times_S \mathcal{C}$. Then f is a q'-left Kan extension of $f | \mathcal{C}$.*

Proof. We first prove (1). We wish to show that q' has the right lifting property with respect to every inclusion $i : A \to B$ of simplicial sets which is a categorical equivalence. For this, it suffices to show that q has the right lifting property with respect to every inclusion of the form $i' : X \times_S A \to X \times_S B$. Since q is a categorical fibration, it suffices to prove that i' is a categorical equivalence, which follows from Corollary B.3.15.

We now prove (2). Let C be an object of $\mathcal{C}'$, and let $\mathcal{C}_{/C}$ denote the fiber product $\mathcal{C} \times_{\mathcal{C}'} \mathcal{C}'_{/C}$. We wish to show that the composition

$$\mathcal{C}^{\triangleright}_{/C} \to \mathcal{C}' \xrightarrow{f} Y'$$

is a q'-colimit diagram. Replacing $\mathcal{C} \subseteq \mathcal{C}'$ by the inclusion $\mathcal{C}_{/C} \subseteq \mathcal{C}^{\triangleright}_{/C}$ (and applying Lemma 3.2.2.7), we can reduce to the case $\mathcal{C}' = \mathcal{C}^{\triangleright}$.

Let $n > 0$, and suppose we are given a diagram

$$
\begin{array}{ccc}
\mathcal{C} \star \partial \Delta^n & \xrightarrow{f'} & Y' \\
\downarrow & \nearrow & \downarrow q' \\
\mathcal{C} \star \Delta^n & \xrightarrow{g} & Z',
\end{array}
$$

where $f' | \mathcal{C} \star \{0\}$ coincides with f. We wish to show that there exists a dotted arrow, as indicated in the diagram. Composing g with the map $Z' \to S$, we obtain a map $\mathcal{C} \star \Delta^n \to S$. Let $\mathcal{D}$ denote the fiber product $X \times_S (\mathcal{C} \star \Delta^n)$, and let $\mathcal{D}^0 = X \times_S (\mathcal{C} \star \partial \Delta^n)$. Unwinding the definitions, we are reduced to solving a lifting problem depicted in the diagram

$$
\begin{array}{ccc}
\mathcal{D}^0 & \xrightarrow{F_0} & Y \\
\downarrow & \nearrow & \downarrow q \\
\mathcal{D} & \xrightarrow{G} & Z.
\end{array}
$$

Let $\mathcal{D}'$ denote the inverse image of Δ^n in $\mathcal{D}$, and let $\mathcal{D}'_{\partial} = \mathcal{D}' \times_{\Delta^n} \partial \Delta^n$. Let A be the collection of all (possibly degenerate) simplices of $\mathcal{D}'$ which do not belong to $\mathcal{D}'_{\partial}$. For every nondegenerate simplex $\sigma : \Delta^m \to \mathcal{D}$ which does not belong to $\mathcal{D}^0$, let i be the smallest integer such that $\sigma(i) \in \mathcal{D}'$ and set $r(\sigma) = \sigma | \Delta^{\{i, i+1, \ldots, m\}}$. Note that $r(\sigma)$ is an element of A (though $r(\sigma)$ may be degenerate).

Choose a well ordering of A with the following property: if $\sigma, \tau \in A$ are simplices such that the dimension of σ is smaller than the dimension of τ, then $\sigma < \tau$. Let α be the order type of the well-ordering of A, so we have an order-preserving bijection $\{\beta < \alpha\} \simeq A$ given by $\beta \mapsto \sigma_\beta$. For $0 \leq \beta \leq \alpha$, let $\mathcal{D}^\beta$ denote the simplicial subset of $\mathcal{D}$ spanned by $\mathcal{D}^0$ together with those nondegenerate simplices σ such that $r(\sigma) = \sigma_\gamma$ for some $\gamma < \beta$. Note that when $\beta = 0$, this agrees with our previous definition of $\mathcal{D}^0$. We will construct a

compatible family of maps $F_\beta : \mathcal{D}^\beta \to Y$ which extend F_0 and satisfy $q \circ F_\beta = G | \mathcal{D}^\beta$. Taking $F = F_\alpha$, we obtain a proof of the desired result.

It remains to construct the maps F_β. We proceed by induction on β. If β is a limit ordinal, we take $F_\beta = \bigcup_{\gamma < \beta} F_\gamma$. It therefore suffices to treat the case of successor ordinals. Assume therefore that F_β has been defined; we wish to construct the map $F_{\beta+1}$. Let $\sigma = \sigma_\beta : \Delta^m \to \mathcal{D}'$ be the corresponding simplex. Note that the induced map $\Delta^m \to \Delta^n$ is surjective, so we automatically have $m > 0$. We first treat the case where the simplex σ is nondegenerate. Let $\mathcal{E}$ denote the fiber product $\mathcal{D}_{/\sigma} \times_{\mathcal{C} \star \Delta^n} \mathcal{C}$, so that we have a pushout diagram of simplicial sets

$$
\begin{array}{ccc}
\mathcal{E} \star \partial \sigma & \longrightarrow & \mathcal{D}^\beta \\
\uparrow & & \downarrow \\
\mathcal{E} \star \sigma & \longrightarrow & \mathcal{D}^{\beta+1}
\end{array} .
$$

We are therefore reduced to solving the lifting problem depicted in the diagram

$$
\begin{array}{ccc}
\mathcal{E} \star \partial \sigma & \xrightarrow{\; h \;} & Y \\
\uparrow & \nearrow & \downarrow q \\
\mathcal{E} \star \sigma & \longrightarrow & Z.
\end{array}
$$

Let h' denote the restriction of h to $\mathcal{E} \star \{v\}$, where v is the initial vertex of σ. Our assumption on F_0 guarantees that h' is a q-colimit diagram, so that the above lifting problem admits a solution.

We now treat the case where the simplex σ is degenerate. In this case, we will prove the existence of $F_{\beta+1}$ by showing that the inclusion $i : \mathcal{D}^\beta \to \mathcal{D}^{\beta+1}$ is a categorical equivalence. Let $\mathcal{J}$ be the category whose objects are diagrams

$$
\begin{array}{ccc}
 & \Delta^{m'} & \\
\;\;\nearrow^{u} & & \searrow^{\sigma'} \\
\Delta^m & \xrightarrow{\quad \sigma \quad} & \mathcal{D}'
\end{array}
$$

where u is surjective and $m' < m$. Let $H : \mathcal{J}^{op} \to \mathrm{Set}_\Delta$ be the functor which carries the above diagram to the simplicial set $\mathcal{D}_{/\sigma'} \times_{\mathcal{C} \star \Delta^n} \mathcal{C}$. We note that i is a pushout of the inclusion

$$
(\varinjlim(H) \star \sigma) \coprod_{\varinjlim(H) \star \partial \sigma} (\mathcal{E} \star \partial \sigma) \to \mathcal{E} \star \sigma.
$$

Consequently, to prove that i is a categorical equivalence, it suffices to show that the diagram

$$
\begin{array}{ccc}
\varinjlim(H) \star \partial \sigma & \longrightarrow & \mathcal{E} \star \partial \sigma \\
\downarrow & & \\
\varinjlim(H) \star \sigma & \longrightarrow & \mathcal{E} \star \sigma.
\end{array}
$$

We claim that the horizontal arrows in this diagram are categorical equivalences. To prove this, it suffices to verify that the canonical map $j : \varinjlim(H) \to \mathcal{E}$ is a categorical equivalence. Since H is a projectively cofibrant diagram, it suffices to show that j exhibits $\mathcal{E}$ as a homotopy colimit of H (with respect to the Joyal model structure). Note that the category $\mathcal{J}$ has a final object, corresponding to a diagram in which the map $\sigma' : \Delta^{m'} \to \mathcal{D}'$ is a nondegenerate simplex of $\mathcal{D}'$. Consequently, it will suffice to H is weakly equivalent to

the constant diagram taking the value $\mathcal{E}$. In other words, we are reduced to proving that for every diagram

$$
\begin{array}{ccc}
 & \Delta^{m'} & \\
{\scriptstyle u}\nearrow & & \searrow{\scriptstyle \sigma'} \\
\Delta^m & \xrightarrow{\quad\sigma\quad} & \mathcal{D}'
\end{array}
$$

belonging to $\mathcal{C}$, the induced map

$$
\mathcal{D}_{/\sigma'} \times_{\mathcal{C} \star \Delta^n} \mathcal{C} \to \mathcal{D}_{/\sigma} \times_{\mathcal{C} \star \Delta^n} \mathcal{C}
$$

is a categorical equivalence. This is clear, since both sides are equivalent to $\mathcal{D}_{/v} \times_{\mathcal{C} \star \Delta^n} \mathcal{C}$, where v is the initial vertex of σ. □

Lemma 3.2.2.9. *Let $p : \mathcal{C} \to \mathcal{D}$ be a categorical fibration of ∞-categories, let $\mathcal{E}$ and K be simplicial sets, and suppose given a diagram*

$$
\begin{array}{ccc}
K \times \mathcal{E} & \xrightarrow{\;f\;} & \mathcal{C} \\
\Big\uparrow & {\overline{f}}\nearrow & \Big\downarrow{\scriptstyle p} \\
K^{\triangleleft} \times \mathcal{E} & \longrightarrow & \mathcal{D}.
\end{array}
$$

Suppose further that for each vertex E of $\mathcal{E}$, there exists an extension $\overline{f}_E : K^{\triangleleft} \to \mathcal{C}$ of f_E which is compatible with the above diagram and is a p-limit. Then:

(1) *There exists a map $\overline{f} : K^{\triangleleft} \times \mathcal{E} \to \mathcal{C}$ rendering the above diagram commutative, with the property that for each vertex E of $\mathcal{E}$, the induced map $\overline{f}_E : K^{\triangleleft} \to \mathcal{C}$ is a p-limit diagram.*

(2) *Let $\overline{f} : K^{\triangleleft} \times \mathcal{E} \to \mathcal{C}$ be an arbitrary map which renders the above diagram commutative. Then $\overline{f}$ satisfies the condition of (1) if and only if the adjoint map $K^{\triangleleft} \to \mathrm{Fun}(\mathcal{E}, \mathcal{C})$ is a $p^{\mathcal{E}}$-limit diagram, where $p^{\mathcal{E}} : \mathrm{Fun}(\mathcal{E}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{E}, \mathcal{D})$ is given by composition with p.*

Proof. Without loss of generality, we may suppose that $\mathcal{E}$ and K are ∞-categories. Let ∞ denote the cone point of $K^{\triangleleft}$. For each object $E \in \mathcal{E}$, the inclusion $K \times \{\mathrm{id}_E\} \subseteq K \times \mathcal{E}_{E/} \simeq (K \times \mathcal{E})_{(\infty, E)/}$ is left anodyne. Consequently, $\overline{f}$ satisfies (1) if and only if $\overline{f}$ is a p-right Kan extension of f'. The existence of $\overline{f}$ follows from Lemma T.4.3.2.13.

The "only if" direction of (2) follows immediately from Lemma 3.2.2.8. The converse follows from the uniqueness of limits (up to equivalence). □

Proof of Proposition 3.2.2.1. We first establish the following:

(∗) Let $h : K^{\triangleleft} \to \mathcal{C}^{\otimes}_{\langle n \rangle}$ be a diagram, and for $1 \leq i \leq n$ let h_i denote the image of h under the functor $\rho^i_! : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}$. If each h_i is a p-limit diagram, then h is a p-limit diagram.

To prove (∗), we observe that there are natural transformations $h \to h_i$ which together determine a map $H : K^{\triangleleft} \times \langle n \rangle^{\circ \triangleleft} \to \mathcal{C}^{\otimes}$. Since the restriction of H to $K^{\triangleleft} \times \{i\}$ is a p-limit diagram for $1 \leq i \leq n$ and the restriction of H to $\{v\} \times \langle n \rangle^{\circ \triangleleft}$ is a p-limit diagram for all vertices v in $K^{\triangleleft}$ (Remark 2.1.2.11), we deduce from Lemma T.5.5.2.3 that h is a p-limit diagram as desired.

We now prove the "if" direction of (2). Suppose that $\overline{f} : K^{\triangleleft} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ induces a p-limit diagram $\overline{f}_X : K^{\triangleleft} \to \mathcal{C}^{\otimes}$ for each $X \in \mathcal{O}$. We claim that the same assertion holds for each $X \in \mathcal{O}^{\otimes}$. To prove this, let $\langle n \rangle$ denote the image of X in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, and choose inert morphisms $X \to X_i$ in $\mathcal{O}^{\otimes}$ lifting $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \leq i \leq n$. The image of $\overline{f}_X$ under $\rho^i_!$ can be identified with $\overline{f}_{X_i}$, and is therefore a p-limit diagram; the desired result now follows from (∗). Applying Lemma 3.2.2.9, we deduce that the diagram $\overline{f}$ is a q'-limit diagram, where q' denotes the projection $\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes}) \to \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{O}^{\otimes}, \mathcal{D}^{\otimes})$. Passing to the full subcategories

$$
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \subseteq \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes}) \qquad \mathrm{Alg}_{\mathcal{O}}(\mathcal{D}) \subseteq \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{O}^{\otimes}, \mathcal{D}^{\otimes})
$$

we deduce that $\overline{f}$ is also a q-limit diagram, as desired.

We now prove (1). Choose a categorical equivalence $i : K \to K'$, where K' is an ∞-category and i is a monomorphism. Since $\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})$ is an ∞-category and q is a categorical fibration, we can assume that f and g factor through compatible maps $f' : K' \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ and $K'^{\triangleleft} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{D})$. Using Proposition T.A.2.3.1, we deduce that the extension $\overline{f}$ exists if and only if there is an analogous extension of f'. We are therefore free to replace K by K' and reduce to the case where K is an ∞-category. The map f classifies a functor $F : K \times \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ and the map g classifies a functor $G : K^{\triangleleft} \times \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$. We first claim:

($*'$) There exists an extension $\overline{F} : K^{\triangleleft} \times \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ of F lying over G such that $\overline{F}$ is a p-right Kan extension of F.

Let v denote the cone point of $K^{\triangleleft}$. According to Lemma T.4.3.2.13, it suffices to show that for each object $X \in \mathcal{O}^{\otimes}$, the induced diagram $(K \times \mathcal{O}^{\otimes})_{(v,X)/} \simeq K \times \mathcal{O}^{\otimes}_{X/} \to \mathcal{C}^{\otimes}$ can be extended to a p-limit diagram compatible with G. Since the inclusion $K \times \{\mathrm{id}_X\} \hookrightarrow K \times \mathcal{O}^{\otimes}_{X/}$ is right cofinal, it suffices to show that we can complete the diagram

$$\begin{array}{ccc} K & \xrightarrow{f_X} & \mathcal{C}^{\otimes} \\ \downarrow & \nearrow^{\overline{f}_X} & \downarrow \\ K^{\triangleleft} & \xrightarrow{g_X} & \mathcal{D}^{\otimes} \end{array}$$

so that $\overline{f}_X$ is a p-limit diagram. Let $\langle n \rangle$ denote the image of X in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Since $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are ∞-operads, we have a homotopy commutative diagram

$$\begin{array}{ccc} \mathcal{C}^{\otimes}_{\langle n \rangle} & \longrightarrow & \mathcal{C}^n \\ \downarrow & & \downarrow \\ \mathcal{D}^{\otimes}_{\langle n \rangle} & \longrightarrow & \mathcal{D}^n \end{array}$$

where the horizontal morphisms are categorical equivalences. Using Proposition T.A.2.3.1 and our assumption on f, we deduce that f_X admits an extension $\overline{f}_X$ (compatible with g_X) whose images under the functors $\rho^i_!$ are p-limit diagrams in $\mathcal{C}$. It follows from ($*$) that $\overline{f}_X$ is a p-limit diagram, which completes the verification of ($*'$). Moreover, the proof shows that an extension $\overline{F}$ of F (compatible with G) is a p-right Kan extension if and only if, for each $X \in \mathcal{O}^{\otimes}_{\langle n \rangle}$, the functor $\rho^i_! : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}$ carries $\overline{F}|(K^{\triangleleft} \times \{X\})$ to a p-limit diagram in $\mathcal{C}^{\otimes}$.

Let $s : \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ be the functor obtained by restricting $\overline{F}$ to the cone point of $K^{\triangleleft}$. Then s preserves inert morphisms lying over the maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$ and is therefore a map of ∞-operads (Remark 2.1.2.9). It follows that $\overline{F}$ determines an extension $\overline{f} : K^{\triangleleft} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ of f lifting g. The first part of the proof shows that $\overline{f}$ is a q-limit diagram, which completes the proof of (1).

We conclude by proving the "only if" direction of (2). Let $\overline{f}$ be a q-left Kan extension of f lying over g. The proof of (1) shows that we can choose another q-left Kan extension $\overline{f}'$ of f lying over g such that $\overline{f}'_X : K^{\triangleleft} \to \mathcal{C}^{\otimes}$ is a p-limit diagram for each $X \in \mathcal{O}$. It follows that $\overline{f}$ and $\overline{f}'$ are equivalent, so that each $\overline{f}_X$ is also a p-limit diagram. □

3.2.3 Colimits of Algebras

Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads. Our goal in this section is to construct colimits in the ∞-category $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$, given suitable hypotheses on p. Our strategy is as follows: we begin by constructing *sifted* colimits in $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ (Proposition 3.2.3.1), which are given by forming colimits of the underlying objects in $\mathcal{C}$. We will then combine this construction with existence results for free algebras (see §3.1.3) to deduce the existence of general colimits (Corollary 3.2.3.3).

Our first main result can be stated as follows:

Proposition 3.2.3.1. *Let K be a sifted simplicial set and let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-categories which is compatible with K-indexed colimits. Then:*

(1) *The ∞-category $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$ of sections of p admits K-indexed colimits.*

(2) *A map $\overline{f} : K^{\triangleright} \to \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$ is a colimit diagram if and only if, for each $X \in \mathcal{O}^{\otimes}$, the induced diagram $\overline{f}_X : K^{\triangleright} \to \mathcal{C}_X^{\otimes}$ is a colimit diagram.*

(3) *The full subcategories*
$$\mathrm{Fun}_{\mathcal{O}}^{\otimes}(\mathcal{O}, \mathcal{C}) \subseteq \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \subseteq \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$$
are stable under K-indexed colimits.

(4) *A map $\overline{f} : K^{\triangleright} \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ is a colimit diagram if and only if, for each $X \in \mathcal{O}$, the induced diagram $\overline{f}_X : K^{\triangleright} \to \mathcal{C}_X$ is a colimit diagram.*

We will give the proof of Proposition 3.2.3.1 at the end of this section.

Corollary 3.2.3.2. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category, and let K be a sifted simplicial set such that the symmetric monoidal structure on $\mathcal{C}$ is compatible with K-indexed colimits. Then:*

(1) *The ∞-category $\mathrm{CAlg}(\mathcal{C})$ admits K-indexed colimits.*

(2) *The forgetful functor $\theta : \mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ detects K-indexed colimits. More precisely, a map $\overline{q} : K^{\triangleright} \to \mathrm{CAlg}(\mathcal{C})$ is a colimit diagram if and only if $\theta \circ \overline{q}$ is a colimit diagram.*

Proof. Apply Proposition 3.2.3.1 in the case where $\mathcal{O}^{\otimes}$ is the commutative ∞-operad. $\square$

We now treat the case of more general colimits.

Corollary 3.2.3.3. *Let κ be an uncountable regular cardinal and let $\mathcal{O}^{\otimes}$ be an essentially κ-small ∞-operad. Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads which is compatible with κ-small colimits. Then $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ admits κ-small colimits.*

Proof. Without loss of generality we may assume that $\mathcal{O}^{\otimes}$ is κ-small. In view of Corollary T.4.2.3.11 and Lemma 1.3.3.10, arbitrary κ-small colimits in $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ can be built from κ-small filtered colimits, geometric realizations of simplicial objects, and finite coproducts. Since $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ admits κ-small sifted colimits (Corollary 3.2.3.2), it will suffice to prove that $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ admits finite coproducts.

According to Example 3.1.3.6, the forgetful functor $G : \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ admits a left adjoint which we will denote by F. Let us say that an object of $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ is *free* if it belongs to the essential image of F. Since $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ admits κ-small colimits and F preserves κ-small colimits (Proposition T.5.2.3.5), a finite collection of objects $\{A^i\}$ of $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ admits a coproduct whenever each A^i is free. We now claim the following:

($\ast$) For every object $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$, there exists a simplicial object $A_{\bullet}$ of $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ such that A is equivalent to the geometric realization $|A_{\bullet}|$ and each A_n is free.

This follows from Proposition 6.2.2.12, since the forgetful functor G is conservative (Lemma 3.2.2.6) and commutes with geometric realizations (Corollary 3.2.3.2).

Now let $\{A^i\}$ be an arbitrary finite collection of objects of $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$; we wish to prove that A^i admits a coproduct in $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$. According to ($\ast$), each A^i can be obtained as the geometric realization of a simplicial object $A_{\bullet}^i$ of $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$, where each A_n^i is free. It follows that the diagrams $A_{\bullet}^i$ admit a coproduct $A_{\bullet} : \mathrm{N}(\mathbf{\Delta}^{op}) \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$. Using Lemma T.5.5.2.3, we conclude that a geometric realization of $A_{\bullet}$ (which exists by virtue of Corollary 3.2.3.2) is a coproduct for the collection A^i. $\square$

We next give a criterion for establishing that an ∞-category of algebras $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ is presentable.

Lemma 3.2.3.4. *Let $\mathcal{O}^{\otimes}$ be an essentially small ∞-operad, and let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads. The following conditions are equivalent:*

(1) *For each $X \in \mathcal{O}$, the fiber $\mathcal{C}_X$ is accessible, and p is compatible with κ-filtered colimits for κ sufficiently large.*

(2) *Each fiber of p is accessible, and for every morphism $f : X \to Y$ in $\mathcal{O}^{\otimes}$, the associated functor $f_! : \mathcal{C}_X^{\otimes} \to \mathcal{C}_Y^{\otimes}$ is accessible.*

Proof. The implication $(2) \Rightarrow (1)$ is obvious. Conversely, suppose that (1) is satisfied. For each $X \in \mathcal{O}_{\langle m \rangle}^{\otimes}$, choose inert morphisms $X \to X_i$ lying over $\rho^i : \langle m \rangle \to \langle 1 \rangle$ for $1 \leq i \leq m$. Proposition 2.1.2.12 implies that $\mathcal{C}_X^{\otimes}$ is equivalent to the product $\prod_{1 \leq i \leq m} \mathcal{C}_{X_i}$, which is accessible by virtue of (1) and Lemma T.5.4.7.2. Now suppose that $f : X \to Y$ is a morphism in $\mathcal{O}^{\otimes}$; we wish to prove that the functor $f_! : \mathcal{C}_X^{\otimes} \to \mathcal{C}_Y^{\otimes}$ is accessible. This follows from (1) and Lemma 3.2.3.7. $\qquad\square$

Corollary 3.2.3.5. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads, where $\mathcal{O}^{\otimes}$ is essentially small. Assume that for each $X \in \mathcal{O}$, the fiber $\mathcal{C}_X$ is accessible.*

(1) *If p is compatible with κ-filtered colimits for κ sufficiently large, $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ is an accessible ∞-category.*

(2) *Suppose that each fiber $\mathcal{C}_X$ is presentable and that p is compatible with small colimits. Then $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ is a presentable ∞-category.*

In particular, if $\mathcal{C}$ is a presentable ∞-category equipped with a symmetric monoidal structure, and the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves colimits separately in each variable, then $\mathrm{CAlg}(\mathcal{C})$ is again presentable.

Proof. Assertion (1) follows from Lemma 3.2.3.4 and Proposition T.5.4.7.11. To prove (2), we combine (1) with Corollary 3.2.3.3. $\qquad\square$

We now turn to the proof of Proposition 3.2.3.1. We will need the following generalization of Proposition T.5.5.8.6:

Lemma 3.2.3.6. *Let K be a sifted simplicial set, let $\{\mathcal{C}_i\}_{1 \leq i \leq n}$ be a finite collection of ∞-categories which admit K-indexed colimits, let $\mathcal{D}$ be another ∞-category which admits K-indexed colimits, and let $F : \prod_{1 \leq i \leq n} \mathcal{C}_i \to \mathcal{D}$ be a functor. Suppose that F preserves K-indexed colimits separately in each variable: that is, if we are given $1 \leq i \leq n$ and objects $\{C_j \in \mathcal{C}_j\}_{j \neq i}$, then the restriction of F to $\{C_1\} \times \cdots \times \{C_{i-1}\} \times \mathcal{C}_i \times \{C_{i+1}\} \times \cdots \times \{C_n\}$ preserves K-indexed colimits. Then F preserves K-indexed colimits.*

Proof. Choose $S \subseteq \{1, \ldots, n\}$, and consider the following assertion:

$(\ast_S)$ Fix objects $\{C_i \in \mathcal{C}_i\}_{i \notin S}$. Then the restriction of F to $\prod_{i \in S} \mathcal{C}_i \times \prod_{i \notin S} \{C_i\}$ preserves K-indexed colimits.

We will prove $(\ast_S)$ using induction on the cardinality of S. Taking $S = \{1, \ldots, n\}$, we can deduce that F preserves K-indexed colimits and complete the proof.

If S is empty, then $(\ast_S)$ follows from Corollary T.4.4.4.10, since K is weakly contractible (Proposition T.5.5.8.7). If S contains a single element, then $(\ast_S)$ follows from the assumption that F preserves K-indexed colimits separately in each variable. We may therefore assume that S has at least two elements, so we can write $S = S' \cup S''$ where S' and S'' are disjoint nonempty subsets of S. We now observe that $(\ast_S)$ follows from $(\ast_{S'})$, $(\ast_{S''})$, and Proposition T.5.5.8.6. $\qquad\square$

Lemma 3.2.3.7. *Let K be a sifted simplicial set, let $\mathcal{O}^{\otimes}$ be an ∞-operad, and let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads which is compatible with K-indexed colimits (Definition 3.1.1.18). For each morphism $f : X \to Y$ in $\mathcal{O}^{\otimes}$, the associated functor $f_! : \mathcal{C}_X^{\otimes} \to \mathcal{C}_Y^{\otimes}$ preserves K-indexed colimits.*

Proof. Factor f as a composition $X \xrightarrow{f'} Z \xrightarrow{f''} Y$, where f' is inert and f'' is active. Using Proposition 2.1.2.12, we deduce that $\mathcal{C}_X^\otimes$ is equivalent to a product $\mathcal{C}_Z^\otimes \times \mathcal{C}_{Z'}^\otimes$, and that $f'_!$ can be identified with the projection onto the first factor. Since $\mathcal{C}_Z^\otimes$ and $\mathcal{C}_{Z'}^\otimes$ both admit K-indexed colimits, this projection preserves K-indexed colimits. We may therefore replace X by Z and thereby reduce to the case where f is active.

Let $\langle n \rangle$ denote the image of Y in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, and choose inert morphisms $g_i : Y \to Y_i$ lying over $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \leq i \leq n$. Applying Proposition 2.1.2.12 again, we deduce that the functors $(g_i)_!$ exhibit $\mathcal{C}_Y^\otimes$ as equivalent to the product $\prod_{1 \leq i \leq n} \mathcal{C}_{Y_i}^\otimes$. It will therefore suffice to show that each of the functors $(g_i \circ f)_!$ preserve K-indexed colimits. Replacing Y by Y_i, we can reduce to the case $n = 1$.

Let $\langle m \rangle$ denote the image of X in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and choose inert morphisms $h_j : X \to X_j$ lying over $\rho^j : \langle m \rangle \to \langle 1 \rangle$ for $1 \leq j \leq m$. Invoking Proposition 2.1.2.12 again, we obtain an equivalence $\mathcal{C}_X^\otimes \simeq \prod_{1 \leq j \leq m} \mathcal{C}_{X_i}^\otimes$. It will therefore suffice to show that the composite map

$$\phi : \prod_{1 \leq j \leq m} \mathcal{C}_{X_i}^\otimes \simeq \mathcal{C}_X^\otimes \xrightarrow{f_!} \mathcal{C}_Y^\otimes$$

preserves K-indexed colimits. Since p is compatible with K-indexed colimits, we conclude that ϕ preserves K-indexed colmits separately in each variable. Since K is sifted, the desired result now follows from Lemma 3.2.3.6. $\qquad\square$

Proof of Proposition 3.2.3.1. Assertions (1), (2), and (3) follow immediately from Lemma 3.2.3.7. The "only if" direction of (4) follows immediately from (2). To prove the converse, suppose that $\overline{f} : K^\triangleleft \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ has the property that $\overline{f}_X$ is a colimit diagram in $\mathcal{C}_X$ for each $X \in \mathcal{O}$. We wish to prove that the analogous assertion holds for any $X \in \mathcal{O}_{\langle n \rangle}^\otimes$. Choose inert morphisms $g(i) : X \to X_i$ lying over $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \leq i \leq n$. According to Proposition 2.1.2.12, the functors $g(i)_!$ induce an equivalence $\mathcal{C}_X^\otimes \simeq \prod_{1 \leq i \leq n} \mathcal{C}_{X_i}$. It will therefore suffice to prove that each composition $g(i)_! \circ \overline{f}_X$ is a colimit diagram $K^\triangleright \to \mathcal{C}_{X_i}$. This follows from the observation that $g(i)_! \circ \overline{f}_X \simeq \overline{f}_{X_i}$. $\qquad\square$

3.2.4   Tensor Products of Commutative Algebras

The proof of the existence for colimits in $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ given in Corollary 3.2.3.3 is rather indirect. In general, this seems to be necessary: there is no simple formula which describes the coproduct of a pair of associative algebras, for example. However, for commutative algebras we can be much more explicit: coproducts can be computed by forming the tensor product of the underlying algebras (see Proposition 3.2.4.7 below). To formulate this statement precisely, we first need to discuss tensor products of algebras over an ∞-operad.

Construction 3.2.4.1. Let $f : \mathcal{O}^\otimes \times \mathcal{O}'^\otimes \to \mathcal{O}''^\otimes$ be a bifunctor of ∞-operads, and let $q : \mathcal{C}^\otimes \to \mathcal{O}''^\otimes$ be a fibration of ∞-operads. We define a map of simplicial sets $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^\otimes \to \mathcal{O}^\otimes$ by the following universal property: for every map of simplicial sets $K \to \mathcal{O}^\otimes$, there is a canonical bijection between $\mathrm{Hom}_{(\mathrm{Set}_\Delta)_{/\mathcal{O}^\otimes}}(K, \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^\otimes)$ and the set of commutative diagrams

$$
\begin{array}{ccc}
K \times \mathcal{O}'^\otimes & \xrightarrow{\;F\;} & \mathcal{C}^\otimes \\
\downarrow & & \downarrow \\
\mathcal{O}^\otimes \times \mathcal{O}'^\otimes & \longrightarrow & \mathcal{O}''^\otimes
\end{array}
$$

such that F has the following property: for every vertex $v \in K$ and every inert morphism α in $\mathcal{O}'^\otimes$, $F(\mathrm{id}_v, \alpha)$ is an inert morphism in $\mathcal{C}^\otimes$.

Remark 3.2.4.2. Let $f : \mathcal{O}^\otimes \times \mathcal{O}'^\otimes \to \mathcal{O}''^\otimes$ and $q : \mathcal{C}^\otimes \to \mathcal{O}''^\otimes$ be as in Construction 3.2.4.1. For every object $X \in \mathcal{O}^\otimes$, the restriction of f to $\{X\} \times \mathcal{O}'^\otimes$ determines a map of ∞-operads $\mathcal{O}'^\otimes \to \mathcal{O}''^\otimes$, and we have a canonical isomorphism of the fiber $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})_X^\otimes \simeq \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C})^\otimes \times_{\mathcal{O}^\otimes} \{X\}$ with the ∞-category $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}''}(\mathcal{C}) \subseteq \mathrm{Fun}_{\mathcal{O}''^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)$.

The crucial properties of Construction 3.2.4.1 can be summarized as follows:

Proposition 3.2.4.3. *Suppose we are given a bifunctor of ∞-operads $\mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ and a fibration of ∞-operads $q : \mathcal{C}^{\otimes} \to \mathcal{O}''^{\otimes}$. Then:*

(1) *The map $p : \mathrm{Alg}_{\mathcal{O}' / \mathcal{O}''}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ is a fibration of ∞-operads.*

(2) *A morphism α in $\mathrm{Alg}_{\mathcal{O}' / \mathcal{O}''}(\mathcal{C})^{\otimes}$ is inert if and only if $p(\alpha)$ is an inert morphism in $\mathcal{O}'$ and, for every object $X \in \mathcal{O}'$, the image $\alpha(X)$ is an inert morphism in $\mathcal{C}^{\otimes}$.*

(3) *Suppose that q is a coCartesian fibration of ∞-operads. Then p is a coCartesian fibration of ∞-operads.*

(4) *Assume that q is a coCartesian fibration of ∞-operads. Then a morphism $\alpha \in \mathrm{Alg}_{\mathcal{O}' / \mathcal{O}''}(\mathcal{C})^{\otimes}$ is p-coCartesian if and only if, for every $X \in \mathcal{O}'$, the image $\alpha(X)$ is a q-coCartesian morphism in $\mathcal{C}^{\otimes}$.*

We will give the proof of Proposition 3.2.4.3 at the end of this section.

Example 3.2.4.4. Let $\mathcal{O}^{\otimes}$ be an arbitrary ∞-operad. Note that there is a unique bifunctor of ∞-operads $\mathrm{Comm}^{\otimes} \times \mathcal{O}^{\otimes} \to \mathrm{Comm}^{\otimes}$. In particular, for every ∞-operad $\mathcal{C}^{\otimes}$, Construction 3.2.4.1 produces a fibration of ∞-operads $\mathrm{Alg}_{\mathcal{O} / \mathrm{Comm}}(\mathcal{C})^{\otimes} \to \mathrm{Comm}^{\otimes}$. This is equivalent to specifying the underlying ∞-operad $\mathrm{Alg}_{\mathcal{O} / \mathrm{Comm}}(\mathcal{C})^{\otimes}$, which we will denote by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes}$. The underlying ∞-category of this ∞-operad is the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ of $\mathcal{O}$-algebra objects of $\mathcal{C}$ (Remark 3.2.4.2).

Note that for every object $X \in \mathcal{O}$, we have an evaluation functor $e_X : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$. The description of inert morphisms in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ provided by Proposition 3.2.4.3 shows that e_X is a map of ∞-operads. If $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category, then Proposition 3.2.4.3 implies that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ is also a symmetric monoidal ∞-category, and that the evaluation functors $e_X : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$ are symmetric monoidal functors. We can summarize the situation informally as follows: if $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category and $\mathcal{O}^{\otimes}$ is any ∞-operad, then the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ inherits a symmetric monoidal structure, given by pointwise tensor product.

Example 3.2.4.5. The functor $\wedge : \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a bifunctor of ∞-operads. Moreover, $\wedge$ induces a weak equivalence of ∞-preoperads $\mathrm{N}(\mathcal{F}\mathrm{in}_*)^{\natural} \odot \mathrm{N}(\mathcal{F}\mathrm{in}_*)^{\natural} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)^{\natural}$. To prove this, it suffices to show that for any ∞-operad $\mathcal{C}$, the functor $\wedge$ induces an equivalence of ∞-categories $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathrm{CAlg}(\mathcal{C}))$. This follows from Corollary 2.4.3.10, since the ∞-operad $\mathrm{CAlg}(\mathcal{C})^{\otimes}$ is coCartesian (Proposition 3.2.4.10).

Let Op_{∞} denote the ∞-category of ∞-operads, and let $V : \mathrm{Op}_{\infty} \to \mathrm{Op}_{\infty}$ be induced by the left Quillen functor $\bullet \odot \mathrm{Comm}^{\otimes \otimes, \natural}$ from $\mathcal{P}\mathrm{Op}_{\infty}$ to itself. It follows from Example 3.2.4.5 and Proposition T.5.2.7.4 that V is a localization functor on Op_{∞}. The following result characterizes the essential image of V:

Proposition 3.2.4.6. *Let $\mathcal{C}^{\otimes}$ be an ∞-operad. The following conditions are equivalent:*

(1) *The ∞-operad $\mathcal{C}^{\otimes}$ lies in the essential image of the localization functor V defined above: in other words, there exists another ∞-operad $\mathcal{D}^{\otimes}$ and a bifunctor of ∞-operads $\theta : \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ which induces a weak equivalence of ∞-preoperads.*

(2) *The ∞-operad $\mathcal{C}^{\otimes}$ is coCartesian.*

Proof. Suppose first that (1) is satisfied. We will prove that $\mathcal{C}^{\otimes}$ is equivalent to $\mathcal{D}^{\mathrm{II}}$. Let $\theta_0 : \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \mathcal{D} \to \mathcal{C}^{\otimes}$ be the restriction of θ. In view of Theorem 2.4.3.18, it will suffice to show that for every ∞-operad $\mathcal{E}^{\otimes}$, composition with θ_0 induces an equivalence of ∞-categories $\mathrm{Alg}_{\mathcal{C}}(\mathcal{E}) \to \mathrm{Fun}(\mathcal{D}, \mathrm{CAlg}(\mathcal{E}))$. This map factors as a composition

$$\mathrm{Alg}_{\mathcal{C}}(\mathcal{E}) \overset{\phi}{\to} \mathrm{Alg}_{\mathcal{D}}(\mathrm{CAlg}(\mathcal{E})) \overset{\phi'}{\to} \mathrm{Fun}(\mathcal{D}, \mathrm{CAlg}(\mathcal{E})).$$

Our assumption that θ induces a weak equivalence of ∞-preoperads guarantees that ϕ is an equivalence of ∞-categories. To prove that ϕ' is a weak equivalence, it suffices to show that $\mathrm{CAlg}(\mathcal{E})^{\otimes}$ is a coCartesian ∞-operad (Proposition 2.4.3.9), which follows from Proposition 3.2.4.10.

Conversely, suppose that $\mathcal{C}^{\otimes}$ is coCartesian. We will prove that the canonical map $\mathcal{C}^{\otimes} \times \{\langle 1 \rangle\} \to \mathcal{C}^{\otimes} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ induces a weak equivalence of ∞-preoperads $\mathcal{C}^{\otimes,\natural} \simeq \mathcal{C}^{\otimes,\natural} \odot \mathrm{N}(\mathcal{F}\mathrm{in}_*)^{\natural}$. Unwinding the definitions, it suffices to show that for every ∞-operad $\mathcal{D}^{\otimes}$, the canonical map

$$\mathrm{Alg}_{\mathcal{C}}(\mathrm{CAlg}(\mathcal{D})) \to \mathrm{Alg}_{\mathcal{C}}(\mathcal{D})$$

is an equivalence of ∞-categories. Invoking Theorem 2.4.3.18, we can reduce to proving that the forgetful functor $\psi : \mathrm{Fun}(\mathcal{C}, \mathrm{CAlg}(\mathrm{CAlg}(\mathcal{D}))) \to \mathrm{Fun}(\mathcal{C}, \mathrm{CAlg}(\mathcal{D}))$ is an equivalence of ∞-categories. This is clear, since ψ is a left inverse to the functor induced by the equivalence $\mathrm{CAlg}(\mathcal{D}) \to \mathrm{CAlg}(\mathrm{CAlg}(\mathcal{D}))$ of Example 3.2.4.5. $\square$

We are now ready to construct coproducts in an ∞-category $\mathrm{CAlg}(\mathcal{C})$, where $\mathcal{C}^{\otimes}$ is any symmetric monoidal ∞-category.

Proposition 3.2.4.7. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then the symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{C})$ provided by Example 3.2.4.4 is coCartesian.*

Corollary 3.2.4.8. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then the ∞-category $\mathrm{CAlg}(\mathcal{C})$ admits finite coproducts.*

Proof of Proposition 3.2.4.7. We will show that the symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{C})$ satisfies criterion (3) of Proposition 2.4.3.19. We first show that the unit object of $\mathrm{CAlg}(\mathcal{C})$ is initial. It follows from Example 3.2.4.4 that the forgetful functor $\theta : \mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ can be promoted to a symmetric monoidal functor. According to Corollary 3.2.1.9, the ∞-category $\mathrm{CAlg}(\mathcal{C})$ has an initial object A_0, which is characterized by the requirement that the unit map $\mathbf{1} \to A_0$ is an equivalence. Composing A_0 with the bifunctor of ∞-operads $\wedge : \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ of Notation 2.2.5.1, we obtain trivial algebra $A_0' \in \mathrm{CAlg}(\mathrm{CAlg}(\mathcal{C}))$. In particular, the unit map $\mathbf{1}_{\mathrm{CAlg}(\mathcal{C})} \to A_0$ for A_0' is an equivalence, which proves that the unit of $\mathrm{CAlg}(\mathcal{C})$ is an initial object of $\mathrm{CAlg}(\mathcal{C})$.

It remains to show that we can produce a collection of codiagonal map $\{\delta_A : A \otimes A \to A\}_{A \in \mathrm{CAlg}(\mathcal{C})}$, satisfying the axioms (i), (ii), and (iii) of Proposition 2.4.3.19. For this, we observe that composition with the bifunctor $\wedge$ of Notation 2.2.5.1 induces a functor $l : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathrm{CAlg}(\mathcal{C}))$. In particular, for every object $A \in \mathrm{CAlg}(\mathcal{C})$, the multiplication on $l(A)$ induces a map $\delta_A : A \otimes A \to A$ in the ∞-category $\mathrm{CAlg}(\mathcal{C})$. It is readily verified that these maps possess the desired properties. $\square$

Remark 3.2.4.9. Let $\mathcal{C}$ be an ∞-category which admits finite coproducts, and let $\mathcal{D}^{\otimes}$ be a symmetric monoidal ∞-category. According to Theorem 2.4.3.18, an ∞-operad map $F : \mathcal{C}^{\mathrm{II}} \to \mathcal{D}^{\otimes}$ is classified up to equivalence by the induced functor $f : \mathcal{C} \to \mathrm{CAlg}(\mathcal{D})$. We note that F is a symmetric monoidal functor if and only if for every finite collection of objects $C_i \in \mathcal{C}$, the induced map $\theta : \otimes f(C_i) \to f(\coprod C_i)$ is an equivalence in $\mathrm{CAlg}(\mathcal{D})$. According to Proposition 3.2.4.7, we can identify the domain of θ with the coproduct of the commutative algebra obejcts $f(C_i)$. Consequently, we conclude that F is symmetric monoidal if and only if f preserves finite coproducts.

The following generalization of Proposition 3.2.4.7 is valid for an arbitrary ∞-operad $\mathcal{C}^{\otimes}$.

Proposition 3.2.4.10. *Let $\mathcal{C}^{\otimes}$ be an ∞-operad. Then the ∞-operad $\mathrm{CAlg}(\mathcal{C})^{\otimes}$ of Example 3.2.4.4 is coCartesian.*

Proof. Consider the functor $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathrm{CAlg}(\mathcal{C}))$ appearing in the proof of Proposition 3.2.4.7. According to Theorem 2.4.3.18, this map determines an ∞-operad map $\theta : \mathrm{CAlg}(\mathcal{C})^{\mathrm{II}} \to \mathrm{CAlg}(\mathcal{C})^{\otimes}$ which is well-defined up to homotopy. We wish to prove that θ is an equivalence. We first observe that θ induces an equivalence between the underlying ∞-categories, and is therefore essentially surjective. To prove that

θ is fully faithful, choose a fully faithful map of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ where $\mathcal{D}^{\otimes}$ is a symmetric monoidal ∞-category (Remark 2.2.4.10). We have a homotopy commutative diagram

$$
\begin{array}{ccc}
\mathrm{CAlg}(\mathcal{C})^{\mathrm{II}} & \xrightarrow{\ \theta\ } & \mathrm{CAlg}(\mathcal{C})^{\otimes} \\
\downarrow & & \downarrow \\
\mathrm{CAlg}(\mathcal{D})^{\mathrm{II}} & \xrightarrow{\ \theta'\ } & \mathrm{CAlg}(\mathcal{D})^{\otimes}
\end{array}
$$

where the vertical maps are fully faithful. Consequently, it will suffice to show that θ' is an equivalence of symmetric monoidal ∞-categories. In view of Remark 2.1.3.8, it will suffice to show that θ' is a symmetric monoidal functor. This is equivalent to the assertion that the composite functor $\mathrm{CAlg}(\mathcal{D})^{\mathrm{II}} \to \mathrm{CAlg}(\mathcal{D})^{\otimes} \to \mathcal{D}^{\otimes}$ is symmetric monoidal (Proposition 3.2.4.3), which follows from Remark 3.2.4.9 because the underlying functor $\mathrm{CAlg}(\mathcal{D}) \to \mathrm{CAlg}(\mathcal{D})$ is equivalent to the identity (and therefore preserves finite coproducts). $\qquad\square$

We conclude this section with the proof of Proposition 3.2.4.3.

Proof of Proposition 3.2.4.3. We use the terminology of Appendix B.4. For every ∞-operad $\mathcal{O}^{\otimes}$, let $\mathfrak{P}_{\mathcal{O}}$ denote the categorical pattern $(M, T, \{p_\alpha : \Lambda_0^2 \to \mathcal{O}^{\otimes}\}_{\alpha \in A})$ on $\mathcal{O}^{\otimes}$, where M is the collection of inert morphisms in $\mathcal{O}^{\otimes}$, T is the collection of all 2-simplices in $\mathcal{O}^{\otimes}$, and A parametrizes all diagrams of inert morphisms $X_0 \leftarrow X \to X_1$ which lie over diagrams $\langle p \rangle \leftarrow \langle n \rangle \to \langle q \rangle$ in $\mathcal{F}\mathrm{in}_*$ which induce a bijection $\langle p \rangle^{\circ} \coprod \langle q \rangle^{\circ} \to \langle n \rangle^{\circ}$.

If $f : \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{O}''^{\otimes}$ is a bifunctor of ∞-operads, then the construction $\overline{X} \mapsto \overline{X} \times \mathcal{O}'^{\otimes,\natural}$ determines functor $F : (\mathcal{S}\mathrm{et}_\Delta^+)_{/\mathfrak{P}_{\mathcal{O}}} \to (\mathcal{S}\mathrm{et}_\Delta^+)_{/\mathfrak{P}_{\mathcal{O}''}}$, which admits a right adjoint $G : (\mathcal{S}\mathrm{et}_\Delta^+)_{/\mathfrak{P}_{\mathcal{O}''}} \to (\mathcal{S}\mathrm{et}_\Delta^+)_{/\mathfrak{P}_{\mathcal{O}}}$. Unwinding the definitions, we see that (1) and (2) are equivalent to the following assertion:

($*$) The functor G carries fibrant objects of $(\mathcal{S}\mathrm{et}_\Delta^+)_{/\mathfrak{P}_{\mathcal{O}''}}$ to fibrant objects of $(\mathcal{S}\mathrm{et}_\Delta^+)_{/\mathfrak{P}_{\mathcal{O}''}}$.

To prove ($*$), it suffices to show that G is a right Quillen functor or equivalently that F is a left Quillen functor. We will prove a stronger assertion: the product map $(\mathcal{S}\mathrm{et}_\Delta^+)_{/\mathfrak{P}_{\mathcal{O}}} \times (\mathcal{S}\mathrm{et}_\Delta^+)_{/\mathfrak{P}_{\mathcal{O}'}} \to (\mathcal{S}\mathrm{et}_\Delta^+)_{/\mathfrak{P}_{\mathcal{O}''}}$ is a left Quillen bifunctor. This is a special case of Remark B.2.5.

The proofs of (3) and (4) are similar, after replacing the categorical pattern $\mathfrak{P}_{\mathcal{O}}$ with $\mathfrak{P}'_{\mathcal{O}} = (M', T, \{p_\alpha : \Lambda_0^2 \to \mathcal{O}^{\otimes}\}_{\alpha \in A})$ where M' is the collection of all edges in $\mathcal{O}^{\otimes}$ (and similarly replacing $\mathfrak{P}_{\mathcal{O}''}$ by $\mathfrak{P}'_{\mathcal{O}''}$). $\qquad\square$

3.3   Modules over ∞-Operads

Let $\mathcal{C}$ be a symmetric monoidal category, and suppose that A is an algebra object of $\mathcal{C}$: that is, an object of $\mathcal{C}$ equipped with an associative (and unital) multiplication $m : A \otimes A \to A$. It then makes sense to consider left A-modules in $\mathcal{C}$: that is, objects $M \in \mathcal{C}$ equipped with a map $a : A \otimes M \to M$ such that the following diagrams commute

$$
\begin{array}{ccc}
A \otimes A \otimes M & \xrightarrow{m \otimes \mathrm{id}_M} & A \otimes M \\
{\scriptstyle \mathrm{id}_A \otimes a}\downarrow & & \downarrow{\scriptstyle a} \\
A \otimes M & \xrightarrow{\quad a \quad} & M
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbf{1} \otimes M & \longrightarrow & A \otimes M \\
 & \searrow \qquad \swarrow{\scriptstyle a} & \\
 & M. &
\end{array}
$$

There is also a corresponding theory of right A-modules in $\mathcal{C}$. If the multiplication on A is commutative, then we can identify left modules with right modules and speak simply of *A-modules in* $\mathcal{C}$. Moreover, a new phenomenon occurs: in many cases, the category $\mathrm{Mod}_A(\mathcal{C})$ of A-modules in $\mathcal{C}$ inherits the structure of a symmetric monoidal category with respect to the relative tensor product over A: namely, we can define $M \otimes_A N$ to be the coequalizer of the diagram

$$
M \otimes A \otimes N \rightrightarrows M \otimes N
$$

determined by the actions of A on M and N (this construction is not always sensible: we must assume that the relevant coequalizers exist, and that they behave well with respect to the symmetric monoidal structure on $\mathcal{C}$).

If we do not assume that A is commutative, then we can often still make sense of the relative tensor product $M \otimes_A N$ provided that M is a right A-module and N is a left A-module. However, this tensor product is merely an object of $\mathcal{C}$: it does not inherit any action of the algebra A. We can remedy this situation by assuming that M and N are *bimodules* for the algebra A: in this case, we can use the right module structure on M and the left module structure on N to define the tensor product $M \otimes_A N$, and this tensor product inherits the structure of a bimodule using the left module structure on M and the right module structure on N. In good cases, this definition endows the category of A-bimodules in $\mathcal{C}$ with the structure of a monoidal category (which is not symmetric in general).

We can summarize the above discussion as follows: if A is an object of $\mathcal{C}$ equipped with some algebraic structure (in this case, the structure of either a commutative or an associative algebra), then we can construct a new category of A-modules (in the associative case, we consider bimodules rather than left or right modules). This category is equipped with the same sort of algebraic structure as the original algebra A (if A is a commutative algebra, the category of A-modules inherits a tensor product which is commutative and associative up to isomorphism; in the case where A is associative, the tensor product of bimodules $\otimes_A$ is merely associative up to isomorphism).

Our goal here is to describe an analogous picture for algebras of a more general nature: namely, $\mathcal{O}$-algebra objects of $\mathcal{C}$ for an arbitrary fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$. We have seen that the collection of such algebras can itself be organized into an ∞-category $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$. If $\mathcal{O}$ is unital, then we can associate to every object $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ a new ∞-category $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$, whose objects we will refer to as *A-module objects of* $\mathcal{C}$. We will lay out the general definitions in this section, and study important special cases (notably, the cases where $\mathcal{O}^{\otimes}$ is the commutative or associative ∞-operad) in §3.4.

There is very little we can say about the ∞-categories $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$ of module objects in the case where $\mathcal{O}^{\otimes}$ is a general unital ∞-operad. We have therefore adopted to work instead in the more restrictive setting of *coherent* ∞-operads. We will give the definition of coherence in §3.3.1 and give an alternate (more technical) characterization in §3.3.2. In §3.3.3, we will use show that if $\mathcal{O}$ is coherent, then every $\mathcal{O}$-algebra object $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ determines a fibration of ∞-operads $p : \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$.

Remark 3.3.0.1. In good cases (which we will discuss in §3.4, one can show that p is a coCartesian fibration of ∞-operads: in other words, the theory of A-modules inherits a tensor structure with exactly the same commutativity and associativity properties as the multiplication on A itself (both are governed by the structure of the ∞-operad $\mathcal{O}^{\otimes}$). This picture was suggested by John Francis.

3.3.1 Coherent ∞-Operads

In this section, we introduce the notion of a *coherent* ∞-operad. Roughly speaking, the coherence of an ∞-operad $\mathcal{O}^{\otimes}$ is a condition which guarantees the existence of a reasonable theory of tensor products for modules over $\mathcal{O}$-algebras. Our goal here is to introduce the definition of coherence, provide a few key examples (see Examples 3.3.1.12, 3.3.1.13), and prove a somewhat technical result (Theorem 3.3.2.2) which will play an important role when we develop the theory of modules in §3.3.3.

We begin by sketching the basic idea. Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad, and fix an active morphism $f : X \to Y$ in $\mathcal{O}^{\otimes}$. An *extension* of f consists of an object $X_0 \in \mathcal{O}$ together with an active morphism $f^+ : X \oplus X_0 \to Y$ such that $f^+|X \simeq f$; here the hypothesis that $\mathcal{O}^{\otimes}$ is unital guarantees that there is an essentially unique section to the projection $X \oplus X_0 \to X$, so that the restriction $f^+|X$ is well-defined. The collection of extensions of f can be organized into an ∞-category, which we will denote by $\mathrm{Ext}(f)$ (see Definition 3.3.1.4 below for a precise definition).

If $g : Y \to Z$ is another active morphism, then there are canonical maps $\mathrm{Ext}(f) \to \mathrm{Ext}(g \circ f) \leftarrow \mathrm{Ext}(g)$, well-defined up to homotopy. In particular, we have canonical maps $\mathrm{Ext}(f) \leftarrow \mathrm{Ext}(\mathrm{id}_Y) \to \mathrm{Ext}(g)$ which fit

into a (homotopy coherent) diagram

$$\begin{array}{ccc}
 & \text{Ext}(\text{id}_Y) & \\
 \swarrow & & \searrow \\
\text{Ext}(f) & & \text{Ext}(g) \\
 \searrow & & \swarrow \\
 & \text{Ext}(g \circ f). &
\end{array}$$
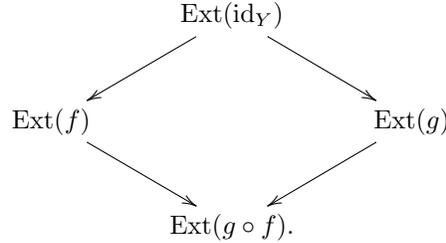
If we assume that $\mathcal{O}$ is a Kan complex, then the ∞-categories appearing in this diagram are all Kan complexes. We will say that $\mathcal{O}^\otimes$ is *coherent* if this diagram is a homotopy pushout square, for every pair of active morphisms f and g in $\mathcal{O}^\otimes$.

We now make these ideas more precise. The first step is to give a careful definition of the ∞-category $\text{Ext}(f)$.

Definition 3.3.1.1. We will say that a morphism $\alpha : \langle m \rangle \to \langle n \rangle$ in $\text{N}(\mathcal{F}\text{in}_*)$ is *semi-inert* if $\alpha^{-1}\{i\}$ has at most one element, for each $i \in \langle n \rangle^\circ$. We will say that α is *null* if α carries $\langle m \rangle$ to the base point of $\langle n \rangle$.

Let $p : \mathcal{O}^\otimes \to \text{N}(\mathcal{F}\text{in}_*)$ be an ∞-operad, and let $f : X \to Y$ be a morphism in $\mathcal{O}^\otimes$. We will say that f is *semi-inert* if the following conditions are satisfied:

(1) The image $p(f)$ is a semi-inert morphism in $\text{N}(\mathcal{F}\text{in}_*)$.

(2) For every inert morphism $g : Y \to Z$ in $\mathcal{O}^\otimes$, if $p(g \circ f)$ is an inert morphism in $\text{N}(\mathcal{F}\text{in}_*)$, then $g \circ f$ is an inert morphism in $\mathcal{O}^\otimes$.

We will say that f is *null* if its image in $\text{N}(\mathcal{F}\text{in}_*)$ is null.

Remark 3.3.1.2. Let $\mathcal{O}^\otimes$ be an ∞-operad. We can think of $\mathcal{O}^\otimes$ as an ∞-category whose objects are finite sequences of objects $(X_1, \ldots, X_m)$ of $\mathcal{O}$. Let $f : (X_1, \ldots, X_m) \to (Y_1, \ldots, Y_n)$ be a morphism in $\mathcal{O}^\otimes$, corresponding to a map $\alpha : \langle m \rangle \to \langle n \rangle$ and a collection of maps

$$\phi_j \in \text{Mul}_{\mathcal{O}}(\{X_i\}_{\alpha(i)=j}, Y_j).$$

Then f is semi-inert if and only for every $j \in \langle n \rangle^\circ$, exactly one of the following conditions holds:

(a) The set $\alpha^{-1}\{j\}$ is empty.

(b) The set $\alpha^{-1}\{j\}$ contains exactly one element i, and the map ϕ_i is an equivalence between X_i and Y_j in the ∞-category $\mathcal{O}$.

The morphism f is inert if (b) holds for each $j \in \langle n \rangle^\circ$, and null if (a) holds for each $j \in \langle n \rangle^\circ$. It follows that every null morphism in $\mathcal{O}^\otimes$ is semi-inert.

Note that $\mathcal{O}^\otimes$ is a unital ∞-operad, then the maps ϕ_j are unique up to homotopy (in fact, up to a contractible space of choices) when $\alpha^{-1}\{j\}$ is empty.

Remark 3.3.1.3. Let $\mathcal{O}^\otimes$ be an ∞-operad, and suppose we are given a commutative diagram

$$\begin{array}{ccc}
 & Y & \\
{\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} \\
X & \xrightarrow{\quad h \quad} & Z
\end{array}$$
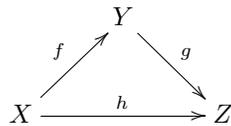
in $\mathcal{O}^\otimes$, where f is inert. Then g is semi-inert if and only if h is semi-inert; this follows immediately from Remark 3.3.1.2.

Definition 3.3.1.4. Let $q : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a unital ∞-operad, and let $\sigma : \Delta^n \to \mathcal{O}^{\otimes}_{\mathrm{act}}$ be an n-simplex of $\mathcal{O}^{\otimes}$ corresponding to a composable chain $X_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} X_n$ of active morphisms in $\mathcal{O}^{\otimes}$.

If $S \subseteq [n]$ is a downward-closed subset, we let $\mathrm{Ext}(\sigma, S)$ denote the full subcategory of $\mathrm{Fun}(\Delta^n, \mathcal{O}^{\otimes})_{\sigma/}$ spanned by those diagrams

$$
\begin{array}{ccc}
X_0 \xrightarrow{\;\; f_1 \;\;} \cdots \xrightarrow{\;\; f_n \;\;} X_n \\
\downarrow{\scriptstyle g_0} \qquad\qquad\qquad \downarrow{\scriptstyle g_n} \\
X'_0 \xrightarrow{\;\; f'_1 \;\;} \cdots \xrightarrow{\;\; f'_n \;\;} X'_n
\end{array}
$$

with the following properties:

(a) If $i \notin S$, the map g_i is an equivalence.

(b) If $i \in S$, the map g_i is semi-inert and $q(g_i)$ is an inclusion $\langle n_i \rangle \to \langle n_i + 1 \rangle$ which omits a single value $a_i \in \langle n_i + 1 \rangle$.

(c) If $0 < i \in S$, then the map $q(f_i)$ carries $a_{i-1} \in q(X'_{i-1})$ to $a_i \in q(X'_i)$.

(d) Each of the maps f'_i is active.

If $f : \Delta^1 \to \mathcal{O}^{\otimes}_{\mathrm{act}}$ is an active morphism in $\mathcal{O}^{\otimes}$, we will denote $\mathrm{Ext}(f, \{0\})$ by $\mathrm{Ext}(f)$.

Remark 3.3.1.5. Let $\mathrm{Ext}(\sigma, S)$ be as in Definition 3.3.1.4. If $\mathcal{O}$ is a Kan complex, then it is easy to see that every morphism in $\mathrm{Ext}(\sigma, S)$ is an equivalence, so that $\mathrm{Ext}(\sigma, S)$ is also a Kan complex.

Remark 3.3.1.6. Let $\sigma : \Delta^n \to \mathcal{O}^{\otimes}_{\mathrm{act}}$ correspond to a sequence of active morphisms

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n$$

and let $S \subseteq [n]$. For every morphism $j : \Delta^m \to \Delta^n$, composition with j induces a restriction map

$$\mathrm{Ext}(\sigma, S) \to \mathrm{Ext}(\sigma \circ j, j^{-1}(S)).$$

In particular, if S has a largest element $i < n$, then we obtain a canonical map $\mathrm{Ext}(\sigma, S) \to \mathrm{Ext}(f_{i+1})$. If $\mathcal{O}$ is a Kan complex, then this map is a trivial Kan fibration.

Remark 3.3.1.7. Let $q : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad, let $\sigma : \Delta^n \to \mathcal{O}^{\otimes}_{\mathrm{act}}$ correspond to a sequence of active morphisms

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n,$$

let $\langle m \rangle = q(X_n)$, and let S be a nonempty proper subset of $[n]$. Then $\mathrm{Ext}(\sigma, S)$ decomposes naturally as a disjoint union $\coprod_{1 \le i \le m} \mathrm{Ext}(\sigma, S)_i$, where $\mathrm{Ext}(\sigma, S)_i$ denotes the full subcategory of $\mathrm{Ext}(\sigma, S)$ spanned by those diagrams

$$
\begin{array}{ccc}
X_0 \xrightarrow{\;\; f_1 \;\;} \cdots \xrightarrow{\quad} X_n \\
\downarrow{\scriptstyle g_0} \qquad\qquad\qquad \downarrow{\scriptstyle g_n} \\
X'_0 \xrightarrow{\;\; f'_1 \;\;} \cdots \xrightarrow{\quad} X'_n
\end{array}
$$

where $q(f'_n \cdots f'_1)$ carries the unique element of $q(X'_0) - q(X_0)$ to $i \in \langle m \rangle \simeq q(X'_n)$. In this case, the diagram σ is equivalent to an amalgamation $\bigoplus_{1 \le i \le m} \sigma_i$, and we have canonical equivalences $\mathrm{Ext}(\sigma, S)_i \simeq \mathrm{Ext}(\sigma_i, S)$.

Remark 3.3.1.8. Let $f : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ be a map of unital ∞-operads, where $\mathcal{O}$ and $\mathcal{O}'$ are Kan complexes. Let σ be an n-simplex of $\mathcal{O}^{\otimes}_{\mathrm{act}}$ and let S be a downward-closed subset of $[n]$. If f is an approximation to $\mathcal{O}'^{\otimes}$ (see Definition 2.3.3.6), then it induces a homotopy equivalence $\mathrm{Ext}(\sigma, S) \to \mathrm{Ext}(f(\sigma), S)$. This follows from Remark 3.3.1.6 and Corollary 2.3.3.17.

We are now ready to give the definition of a coherent ∞-operad.

Definition 3.3.1.9. Let $\mathcal{O}^{\otimes}$ be an ∞-operad. We will say that $\mathcal{O}$ is *coherent* if the following conditions are satisfied:

(1) The ∞-operad $\mathcal{O}^{\otimes}$ is unital.

(2) The ∞-category $\mathcal{O}$ is a Kan complex.

(3) Suppose we are given a degenerate 3-simplex σ :

$$
\begin{array}{ccc}
 & Y \xrightarrow{\quad g \quad} Z \\
\end{array}
$$

with f (from X to Y), id_Y (from Y to Y), g (from Y to Z), f (from X to Y)
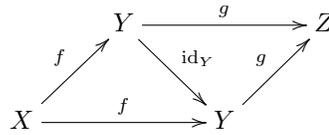
in $\mathcal{O}^{\otimes}$, where f and g are active. Then the diagram

$$
\begin{array}{ccc}
\mathrm{Ext}(\sigma, \{0,1\}) & \longrightarrow & \mathrm{Ext}(\sigma|\Delta^{\{0,1,3\}}, \{0,1\}) \\
\downarrow & & \downarrow \\
\mathrm{Ext}(\sigma|\Delta^{\{0,2,3\}}, \{0\}) & \longrightarrow & \mathrm{Ext}(\sigma|\Delta^{\{0,3\}}, \{0\})
\end{array}
$$
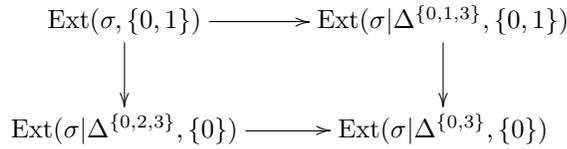
is a homotopy pushout square.

Remark 3.3.1.10. Requirement (3) of Definition 3.3.1.9 can be stated more informally as follows: for every pair of active morphisms $f : X \to Y$ and $g : Y \to Z$, the (homotopy coherent) diagram of Kan complexes

$$
\begin{array}{ccc}
\mathrm{Ext}(\mathrm{id}_Y) & \longrightarrow & \mathrm{Ext}(g) \\
\downarrow & & \downarrow \\
\mathrm{Ext}(f) & \longrightarrow & \mathrm{Ext}(gf)
\end{array}
$$
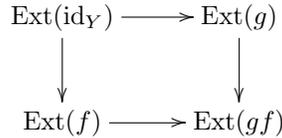
is a homotopy pushout square.

Remark 3.3.1.11. In Definition 3.3.1.9, it is sufficient to require that condition (3) hold in cases where the object Z belongs to $\mathcal{O}$: this is an immediate consequence of Remark 3.3.1.7.

Example 3.3.1.12. The commutative ∞-operad $\mathrm{Comm}^{\otimes} = \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is coherent. Unwinding the definitions, we note that for every active morphism $f : \langle m \rangle \to \langle n \rangle$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, the Kan complex $\mathrm{Ext}(f)$ is canonically homotopy equivalent to $\langle n \rangle^{\circ} = \{1, \dots, n\}$, regarded as a discrete simplicial set. Since $\mathrm{Comm}^{\otimes}$ is unital and the underlying ∞-category $\mathrm{Comm} \simeq \Delta^0$ is a Kan complex, we are reduced to proving that for every pair of active morphisms $f : \langle l \rangle \to \langle m \rangle$ and $g : \langle m \rangle \to \langle n \rangle$, the diagram

$$
\begin{array}{ccc}
\langle m \rangle^{\circ} & \longrightarrow & \langle n \rangle^{\circ} \\
\downarrow & & \downarrow \\
\langle m \rangle^{\circ} & \longrightarrow & \langle n \rangle^{\circ}
\end{array}
$$
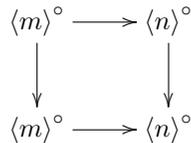
is a homotopy pushout square, which is obvious.

Example 3.3.1.13. The ∞-operad $\mathbb{E}_0^{\otimes}$ is coherent. It is clear that $\mathbb{E}_0^{\otimes}$ is unital and that the underlying ∞-category $\mathbb{E}_0 \simeq \Delta^0$ is a Kan complex. It therefore suffices to prove that for every pair of active morphisms $f : \langle l \rangle \to \langle m \rangle$ and $g : \langle m \rangle \to \langle n \rangle$ in $\mathbb{E}_0^{\otimes}$, the induced diagram

$$
\begin{array}{ccc}
\mathrm{Ext}(\mathrm{id}_Y) & \longrightarrow & \mathrm{Ext}(g) \\
\downarrow & & \downarrow \\
\mathrm{Ext}(f) & \longrightarrow & \mathrm{Ext}(gf).
\end{array}
$$

is a homotopy pushout square. This follows from the observation that the above diagram can be identified with the commutative square of finite sets

$$
\begin{array}{ccc}
\emptyset & \longrightarrow & \langle n \rangle - g(\langle m \rangle) \\
\downarrow & & \downarrow \\
\langle m \rangle - f(\langle l \rangle) & \longrightarrow & \langle n \rangle - gf(\langle l \rangle).
\end{array}
$$

Remark 3.3.1.14. We will later show that the little cubes operads $\mathbb{E}_k^{\otimes}$ are coherent for every integer k (Theorem 5.1.1.1). This observation recovers Example 3.3.1.13 in the case $k = 0$ and Example 3.3.1.12 in the limiting case $k \to \infty$. We will give a more direct treatment of the case $k = 1$ when we discuss associative algebras in §4.1 (see Proposition 4.1.1.16).

Proposition 3.3.1.15. *Let $f : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ be a map between unital ∞-operads. Assume that both $\mathcal{O}$ and $\mathcal{O}'$ are Kan complexes and that f is an approximation to $\mathcal{O}'^{\otimes}$ (see Definition 2.3.3.6). If $\mathcal{O}'^{\otimes}$ is coherent, then $\mathcal{O}^{\otimes}$ is coherent. The converse holds if the underlying map $\pi_0 \mathcal{O} \to \pi_0 \mathcal{O}'$ is surjective.*

Remark 3.3.1.16. Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad such that $\mathcal{O}$ is a Kan complex. According to Theorem 2.3.4.4, the ∞-operad $\mathcal{O}^{\otimes}$ can be obtained as the assembly of a $\mathcal{O}$-family of reduced unital ∞-operads $\mathcal{O}'^{\otimes} \to \mathcal{O} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Proposition 3.3.1.15 (and Proposition 2.3.4.5) imply that $\mathcal{O}^{\otimes}$ is coherent if and only if, for each $X \in \mathcal{O}$, the ∞-operad $\mathcal{O}_X'^{\otimes}$ is coherent. In §3.3.3, we will see that there is a good theory of modules associated to $\mathcal{O}$-algebras and to $\mathcal{O}_X'$-algebras, for each $X \in \mathcal{O}$. One can show that these module theories are closely related to one another. To describe the relationship, suppose that $\mathcal{C}$ is another ∞-operad and $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is a $\mathcal{O}$-algebra object of $\mathcal{C}$, corresponding to a family of $\mathcal{O}_X'^{\otimes}$-algebra objects $\{A_X \in \mathrm{Alg}_{\mathcal{O}_X'}(\mathcal{C})\}_{X \in \mathcal{O}}$. Then giving an A-module object $M \in \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$ is equivalent to giving a family $\{M_X \in \mathrm{Mod}_{A_X}^{\mathcal{O}_X'}(\mathcal{C})\}_{X \in \mathcal{O}'}$. We leave the precise formulation to the reader.

Proof of Proposition 3.3.1.15. We may assume without loss of generality that f is a categorical fibration. The first assertion follows immediately Remark 3.3.1.8. To prove the second, it will suffice (by virtue of Remark 3.3.1.8) to show that every 3-simplex $\sigma : \Delta^3 \to \mathcal{O}_{\mathrm{act}}'^{\otimes}$ can be lifted to a 3-simplex of $\mathcal{O}_{\mathrm{fl}}^{\otimes}$. Let $Z = \sigma(3) \in \mathcal{O}'$. Since f is a categorical fibration, the induced map $\mathcal{O} \to \mathcal{O}'$ is a Kan fibration. Since this Kan fibration induces a surjection on connected components, it is surjective on vertices and we may write $Z = f(\overline{Z})$ for some $\overline{Z} \in \mathcal{O}$. Corollary 2.3.3.17 guarantees that the induced map $f' : (\mathcal{O}_{/\overline{Z}}^{\otimes})^{\mathrm{act}} \to (\mathcal{O}_{/Z}'^{\otimes})^{\mathrm{act}}$ is a categorical equivalence. Since f is a categorical fibration, f' is also a categorical fibration and therefore a trivial Kan fibration. We can interpret the 3-simplex σ as a 2-simplex $\tau : \Delta^2 \to (\mathcal{O}_{/Z}'^{\otimes})^{\mathrm{act}}$, which can be lifted to a 2-simplex $\overline{\tau} : \Delta^2 \to (\mathcal{O}_{/\overline{Z}}^{\otimes})^{\mathrm{act}}$. This map determines a 3-simplex of $\mathcal{O}_{\mathrm{act}}^{\otimes}$ lifting σ, as desired. $\square$

3.3.2 A Coherence Criterion

In §3.3.1, we introduced the notion of a *coherent* ∞-operad. In this section, we will describe characterize coherent ∞-operads as those ∞-operads which satisfy a certain technical condition (see Theorem 3.3.2.2). This condition will play an important role in the construction of ∞-categories of modules which we carry out in §3.3.3.

Notation 3.3.2.1. Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad. We let $\mathcal{K}_{\mathcal{O}}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$ spanned by the semi-inert morphisms of $\mathcal{O}^{\otimes}$. For $i \in \{0, 1\}$ we let $e_i : \mathcal{K}_{\mathcal{O}} \to \mathcal{O}^{\otimes}$ be the map given by evaluation on i. We will say that a morphism in $\mathcal{K}_{\mathcal{O}}$ is *inert* if its images under e_0 and e_1 are inert morphisms in $\mathcal{O}^{\otimes}$.

We can now state our main result:

Theorem 3.3.2.2. *Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad such that $\mathcal{O}$ is a Kan complex. The following conditions are equivalent:*

(1) *The ∞-operad $\mathcal{O}^{\otimes}$ is coherent.*

(2) *The evaluation map $e_0 : \mathcal{K}_{\mathcal{O}} \to \mathcal{O}^{\otimes}$ is a flat categorical fibration (see Definition B.3.8).*

The remainder of this section is devoted to the proof of Theorem 3.3.2.2. Our first step is to introduce an apparently weaker version of condition (2).

Definition 3.3.2.3. Let $q : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a unital ∞-operad, and let $m \geq 0$ be an integer. We will say that a morphism $f : X \to X'$ in $\mathcal{O}^{\otimes}$ is *m-semi-inert* if f is semi-inert and the underlying map $q(f) : \langle n \rangle \to \langle n' \rangle$ is such that the cardinality of the set $\langle n' \rangle - f(\langle n \rangle)$ is less than or equal to m.

By definition, an ∞-operad $\mathcal{O}^{\otimes}$ satisfies condition (2) of Theorem 3.3.2.2 if, for eveyr map $\Delta^2 \to \mathcal{O}^{\otimes}$ corresponding to a diagram $X \to Y \to Z$ and every morphism $\overline{X} \to \overline{Z}$ in $\mathcal{K}_{\mathcal{O}}$ lifting the underlying map $X \to Z$, the ∞-category $(\mathcal{K}_{\mathcal{O}})_{\overline{X}//\overline{Z}} \times_{\mathcal{O}^{\otimes}_{X//Z}} \{Y\}$ is weakly contractible. We will say that $\mathcal{O}^{\otimes}$ is *m-coherent* if this condition holds whenever $\overline{X}$ is an m-semi-inert morphism in $\mathcal{O}^{\otimes}$. We note that $\mathcal{K}_{\mathcal{O}} \xrightarrow{e_0} \mathcal{O}^{\otimes}$ is a flat categorical fibration if and only if $\mathcal{O}^{\otimes}$ is m-coherent for all $m \geq 0$.

Lemma 3.3.2.4. *Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad. The following conditions are equivalent:*

(1) *The ∞-operad $\mathcal{O}^{\otimes}$ is m-coherent.*

(2) *Consider a diagram σ :*

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow{\scriptstyle f} & & \\
X' & &
\end{array}
$$

in $\mathcal{O}^{\otimes}$, where f is m-semi-inert. Let $\mathcal{A}[\sigma]$ denote the full subcategory of $\mathcal{O}^{\otimes}_{\sigma/}$ spanned by those commutative diagrams

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
X' & \longrightarrow & Y'
\end{array}
$$

where g is semi-inert. Then the inclusion $\mathcal{A}[\sigma] \subseteq \mathcal{O}^{\otimes}_{\sigma/}$ is right cofinal.

(3) *Let σ be as in (2), and let Z' be an object of $\mathcal{O}^{\otimes}_{\sigma/}$, and let $\mathcal{A}[\sigma]_{/Z'}$ denote the full subcategory of $\mathcal{O}^{\otimes}_{\sigma//Z'}$ spanned by those diagrams*

$$
\begin{array}{ccccc}
X & \longrightarrow & Y & & \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & \searrow & \\
X' & \longrightarrow & Y' & \longrightarrow & Z'
\end{array}
$$

such that g is semi-inert. Then $\mathcal{A}[\sigma]_{/Z'}$ is a weakly contractible simplicial set.

Proof. The equivalence of (2) and (3) follows immediately from Theorem T.4.1.3.1. We next prove that $(1) \Rightarrow (3)$. We can extend σ and Z to a commutative diagram

$$
\begin{array}{ccccc}
X & \longrightarrow & Y & \longrightarrow & Z \\
\downarrow{\scriptstyle f} & & & & \downarrow{\scriptstyle h} \\
X' & & \longrightarrow & & Z'
\end{array}
$$

where h is semi-inert (for example, we can take $Z = Z'$ and $h = \mathrm{id}_{Z'}$. The upper line of this diagram determines a diagram $\Delta^2 \to \mathcal{O}^\otimes$. Let $\mathcal{K}$ denote the fiber product $\mathcal{K}_{\mathcal{O}} \times_{\mathcal{O}^\otimes} \Delta^2$. The maps f and h determine objects of $\mathcal{K} \times_{\Delta^2}\{0\}$ and $\mathcal{K} \times_{\Delta^2}\{2\}$, which we will denote by $\overline{X}$ and $\overline{Z}$. Since $\mathcal{O}^\otimes$ is m-coherent, the ∞-category $\mathcal{K}_{\overline{X}//\overline{Z}} \times_{\Delta^2}\{1\}$ is weakly contractible. We now observe that there is a trivial Kan fibration $\psi : \mathcal{K}_{\overline{X}//\overline{Z}} \to \mathcal{A}[\sigma]_{/Z'}$, so that $\mathcal{A}[\sigma]_{/Z'}$ is likewise weakly contractible.

Now suppose that (3) is satisfied. We wish to prove that $\mathcal{O}^\otimes$ is m-coherent. Fix a map $\Delta^2 \to \mathcal{O}^\otimes$, and let $\mathcal{K}$ be the fiber product $\mathcal{K}_{\mathcal{O}} \times_{\mathcal{O}^\otimes} \Delta^2$. Suppose we are given objects $\overline{X} \in \mathcal{K} \times_{\Delta^2}\{0\}$ and $\overline{Z} \in \mathcal{K} \times_{\Delta^2}\{2\}$; we wish to prove that if $\overline{X}$ is m-semi-inert, then $\mathcal{K}_{\overline{X}//\{Z\}} \times_{\Delta^2}\{1\}$ is weakly contractible. The above data determines a commutative diagram

$$
\begin{array}{ccccc}
X & \longrightarrow & Y & \longrightarrow & Z \\
\downarrow{\scriptstyle f} & & & & \downarrow{\scriptstyle h} \\
X' & & \longrightarrow & & Z'
\end{array}
$$

in $\mathcal{O}^\otimes$. If we let σ denote the left part of this diagram, then we can define a simplicial set $\mathcal{A}[\sigma]_{/Z'} \subseteq (\mathcal{O}^\otimes_{\sigma/})_{/Z'}$ as in (3), which is weakly contractible by assumption. Once again, we have a trivial Kan fibration $\psi : \mathcal{K}_{\overline{X}//\overline{Z}} \to \mathcal{A}[\sigma]_{/Z'}$, so that $\mathcal{K}_{\overline{X}//\overline{Z}}$ is weakly contractible as desired. $\qquad\square$

Remark 3.3.2.5. Let $\mathcal{O}^\otimes$, σ, and Z' be as in the part (3) of Lemma 3.3.2.4. Let $\mathcal{A}[\sigma]'_{/Z}$ denote the full subcategory of $\mathcal{A}[\sigma]_{/Z}$ spanned by those diagrams

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
X' & \longrightarrow & Y' \xrightarrow{\ j\ } Z'
\end{array}
$$

for which the map j is active. The inclusion $\mathcal{A}[\sigma]'_{/Z} \subseteq \mathcal{A}[\sigma]_{/Z}$ admits a left adjoint, and is therefore a weak homotopy equivalence. Consequently, condition (3) of Lemma 3.3.2.4 is equivalent to the requirement that $\mathcal{A}[\sigma]'_{/Z}$ is weakly contractible.

Remark 3.3.2.6. In the situation of Lemma 3.3.2.4, let $Z' \in \mathcal{O}^\otimes$ and let $\mathcal{B}$ denote the full subcategory of $\mathcal{O}^\otimes_{/Z'}$ spanned by the active morphisms $W \to Z'$. The inclusion $\mathcal{B} \subseteq \mathcal{O}^\otimes_{/Z'}$ admits a left adjoint L. For any diagram σ :

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \\
\downarrow & & \\
X' & &
\end{array}
$$

in $\mathcal{O}^\otimes_{/Z}$, L induces a functor $\mathcal{A}[\sigma]_{/Z'} \to \mathcal{A}[L\sigma]_{/Z'}$, which restricts to an equivalence $\mathcal{A}[\sigma]'_{/Z'} \to \mathcal{A}[L\sigma]'_{/Z'}$, where $\mathcal{A}[\sigma]'_{/Z'}$ and $\mathcal{A}[L\sigma]'_{/Z'}$ are defined as in Remark 3.3.2.5. Consequently, to verify condition (3) of Lemma 3.3.2.4, we are free to replace σ by $L\sigma$ and thereby reduce to the case where the maps $X \to Z'$, $Y \to Z'$, and $X' \to Z'$ are active.

Remark 3.3.2.7. Let $Z' \in \mathcal{O}^{\otimes}$ be as in Lemma 3.3.2.4, and choose an equivalence $Z' \simeq \bigoplus Z'_i$, where $Z'_i \in \mathcal{O}$. Let $\mathcal{B} \subseteq \mathcal{O}^{\otimes}_{/Z'}$ be defined as in Remark 3.3.2.6, and let $\mathcal{B}_i \subseteq \mathcal{O}^{\otimes}_{/Z'_i}$ be defined similarly. Every diagram $\sigma : \Lambda^2_0 \to \mathcal{B}$ can be identified with an amalgamation $\bigoplus_i \sigma_i$ of diagrams $\sigma_i : \Lambda^2_0 \to \mathcal{B}_i$. We observe that $\mathcal{A}[\sigma]'_{/Z'}$ is equivalent to the product of the ∞-categories $\mathcal{A}[\sigma_i]'_{/Z'_i}$. Consequently, to verify that $\mathcal{A}[\sigma_i]'_{/Z'}$ is weakly contractible, we may replace Z' by Z'_i and thereby reduce to the case where $Z' \in \mathcal{O}$.

Lemma 3.3.2.8. *Let $q : X \to S$ be an inner fibration of ∞-categories. Suppose that the following conditions are satisfied:*

(a) *The inner fibration q is flat.*

(b) *Each fiber X_s of q is weakly contractible.*

(c) *For every vertex $x \in X$, the induced map $X_{x/} \to S_{q(x)/}$ has weakly contractible fibers.*

Then for every map of simplicial sets $S' \to S$, the pullback map $X \times_S S' \to S'$ is weak homotopy equivalence. In particular, q is a weak homotopy equivalence.

Proof. We will show more generally that for every map of simplicial sets $S' \to S$, the induced map $q_{S'} : X \times_S S' \to S'$ is a weak homotopy equivalence. Since the collection of weak homotopy equivalences is stable under filtered colimits, we may suppose that S' is finite. We now work by induction on the dimension n of S' and the number of simplices of S' of maximal dimension. If S' is empty the result is obvious; otherwise we have a homotopy pushout diagram

$$
\begin{array}{ccc}
\partial \Delta^n & \longrightarrow & \Delta^n \\
\downarrow & & \downarrow \\
S'_0 & \longrightarrow & S'.
\end{array}
$$

By the inductive hypothesis, the maps $q_{S'_0}$ and $q_{\partial \Delta^n}$ are weak homotopy equivalences. Since $q_{S'}$ is a homotopy pushout of the morphisms $q_{S'_0}$ with q_{Δ^n} over $q_{\partial \Delta^n}$, we can reduce to proving that q_{Δ^n} is a weak homotopy equivalence. Note that assumption (a) guarantees that q_{Δ^n} is a flat categorical fibration. If $n > 1$, then we have a commutative diagram

$$
\begin{array}{ccc}
X \times_S \Lambda^n_1 & \longrightarrow & X \times_S \Delta^n \\
\downarrow{\scriptstyle q_{\Lambda^n_1}} & & \downarrow{\scriptstyle q_{\Delta^n}} \\
\Lambda^n_1 & \longrightarrow & \Delta^n
\end{array}
$$

where the upper horizontal map is a categorical equivalence (Corollary B.3.15) and therefore a weak homotopy equivalence; the lower horizontal map is obviously a weak homotopy equivalence. Since $q_{\Lambda^n_1}$ is a weak homotopy equivalence by the inductive hypothesis, we conclude that q_{Δ^n} is a weak homotopy equivalence.

It remains to treat the cases where $n \leq 1$. If $n = 0$, the desired result follows from (b). Suppose finally that $n = 1$. Let $X' = X \times_S \Delta^1$; we wish to prove that X' is weakly contractible. Let X'_0 and X'_1 denote the fibers of the map q_{Δ^1}, and let $Y = \mathrm{Fun}_{\Delta^1}(\Delta^1, X')$. According to Proposition B.3.17, the natural map

$$
X'_0 \coprod_{Y \times \{0\}} (Y \times \Delta^1) \coprod_{Y \times \{1\}} X'_1 \to X'
$$

is a categorical equivalence. Since X'_0 and X'_1 are weakly contractible, we are reduced to showing that Y is weakly contractible. Let $p : Y \to X'_0$ be the map given by evaluation at $\{0\}$. Let $x' \in X'_0$, and let x denote its image in x; the fiber $p^{-1}\{x'\}$ is isomorphic to $(X')^{x'/} \times_{\Delta^1} \{1\}$ and therefore categorically equivalent to $X'_{x'/} \times_{\Delta^1} \{1\} \simeq X_{x/} \times_{S_{q(x)/}} \{f\}$, where f denotes the edge $\Delta^1 \to S$ under consideration. Assumption (c) guarantees that $p^{-1}\{x'\}$ is weakly contractible. Since p is a Cartesian fibration, Lemma T.4.1.3.2 guarantees that p is left cofinal and therefore a weak homotopy equivalence. Since X'_0 is weakly contractible by (b), we deduce that Y is weakly contractible as desired. $\square$

Example 3.3.2.9. Let $q : X \to S$ be a flat inner fibration of simplicial sets, and let $f : x \to y$ be an edge of X. Then the induced map $X_{x//y} \to S_{q(x)//q(y)}$ satisfies the hypotheses of Lemma 3.3.2.8 (see Proposition B.3.13), and is therefore a categorical equivalence.

Proposition 3.3.2.10. *Let* $q : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *be a unital* ∞-*operad. The following conditions are equivalent:*

(1) *The evaluation map* $e_0 : \mathcal{K}_{\mathcal{O}} \to \mathcal{O}^{\otimes}$ *is a flat categorical fibration.*

(2) *Let* $Z \in \mathcal{O}$, *and suppose we are given a diagram* σ :

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow{\scriptstyle f} & & \\
X' & &
\end{array}
$$

in $\mathcal{O}^{\otimes}_{/Z}$ *where* f *is semi-inert and the maps* $X \to Z$, $Y \to Z$, *and* $X' \to Z$ *are active. Let* $\mathcal{B}[\sigma, Z]$ *denote the full subcategory of* $\mathcal{O}^{\otimes}_{\sigma///Z}$ *spanned by those diagrams*

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
X' & \longrightarrow & Y'
\end{array}
$$

in $\mathcal{O}^{\otimes}$ *where the map* $Y' \to Z$ *is active, the map* g *is semi-inert, and the map* $q(X') \coprod_{q(X)} q(Y) \to q(Y')$ *is a surjective map of pointed finite sets. Then* $\mathcal{B}[\sigma, Z]$ *is weakly contractible.*

(3) *Condition* (2) *holds in the special case where* f *is required to be 1-semi-inert.*

Proof. In the situation of (2), let $\mathcal{A}[\sigma]'_{/Z}$ be defined as in Remark 3.3.2.5. There is a canonical inclusion $\mathcal{B}[\sigma, Z] \subseteq \mathcal{A}[\sigma]'_{/Z}$. Our assumption that $\mathcal{O}^{\otimes}$ is unital implies that this inclusion admits a right adjoint, and is therefore a weak homotopy equivalence. The equivalence $(1) \Leftrightarrow (2)$ now follows by combining Lemma 3.3.2.4 with Remarks 3.3.2.5, 3.3.2.6, and 3.3.2.7. The same argument shows that (3) is equivalent to the condition that $\mathcal{O}^{\otimes}$ is 1-coherent. The implication $(2) \Rightarrow (3)$ is obvious; we will complete the proof by showing that $(3) \Rightarrow (2)$.

Let (σ, Z) be as in (2); we wish to show that $\mathcal{B}[\sigma, Z]$ is weakly contractible. The image $q(f)$ is a semi-inert morphism $\langle n \rangle \to \langle n + m \rangle$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, for some $m \geq 0$. If $m = 0$, then $\mathcal{B}[\sigma, Z]$ has an initial object and there is nothing to prove. We assume therefore that $m > 0$, so that (since $\mathcal{O}^{\otimes}$ is unital) f admits a factorization

$$
X \xrightarrow{f'} X_0 \xrightarrow{f''} X'
$$

such that $q(f')$ is an inclusion $\langle n \rangle \hookrightarrow \langle n + m - 1 \rangle$ and $q(f'')$ is an inclusion $\langle n + m - 1 \rangle \hookrightarrow \langle n + m \rangle$. Let $\tau : \Delta^1 \coprod_{\{0\}} \Delta^2 \to \mathcal{O}^{\otimes}_{/Z}$ be the diagram given by $Y \leftarrow X \to X_0 \to X'$, and let τ_0 be the restriction of τ to $\Delta^1 \coprod_{\{0\}} \Delta^1$. Let $\mathcal{C}$ denote the ∞-category

$$
\mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes}_{\tau_0///Z}) \times_{\mathcal{O}^{\otimes}_{\tau_0///Z}} \mathcal{O}^{\otimes}_{\tau///Z}
$$

whose objects are commutative diagrams

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g'} \\
X_0 & \longrightarrow & Y_0 \\
\downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle g''} \\
X' & \longrightarrow & Y' & \longrightarrow Z
\end{array}
$$

in $\mathcal{O}^{\otimes}$. Let $\mathcal{C}_0$ denote the full subcategory of $\mathcal{C}$ spanned by those diagrams where the maps g' and g'' are semi-inert, the maps $Y_0 \to Z$ and $Y' \to Z$ are active, and the maps $q(Y) \coprod_{q(X)} q(X_0) \to q(Y_0)$ and $q(Y_0) \coprod_{q(X_0)} q(X') \to q(Y')$ are surjective. There are evident forgetful functors $\mathcal{B}[\sigma], Z] \xleftarrow{\phi} \mathcal{C}_0 \xrightarrow{\psi} \mathcal{B}[\tau_0, Z]$. The map ϕ admits a right adjoint and is therefore a weak homotopy equivalence. The simplicial set $\mathcal{B}[\tau_0, Z]$, and therefore weakly contractible by the inductive hypothesis. To complete the proof, it will suffice to show that ψ is a weak homotopy equivalence. For this, we will show that ψ satisfies the hypotheses of Lemma 3.3.2.8:

(a) The map ψ is a flat inner fibration. Fix a diagram

$$
\begin{array}{ccc}
 & B & \\
\nearrow & & \searrow \\
A & \xrightarrow{\quad j \quad} & C
\end{array}
$$

in $\mathcal{A}[\tau_0]'_{/Z}$ and a morphism $\bar{j} : \overline{A} \to \overline{C}$ in $\mathcal{C}_0$ lifting j; we wish to show that the ∞-category

$$
\mathcal{Z} = (\mathcal{C}_0)_{\overline{A}//\overline{C}} \times_{(\mathcal{B}[\tau_0, Z])_{A//C}} \{B\}
$$

is weakly contractible. We have a commutative diagram

$$
\begin{array}{ccccccc}
X & \longrightarrow & Y & & & & \\
\downarrow & & \downarrow & & & & \\
X_0 & \longrightarrow & Y_A & \longrightarrow & Y_B & \longrightarrow & Y_C \\
\downarrow & & \downarrow & & & & \downarrow \\
X' & \longrightarrow & Y'_A & \longrightarrow & & Y'_C & \longrightarrow & Z
\end{array}
$$

in $\mathcal{O}^{\otimes}$. Restricting our attention to the rectangle in the lower right, we obtain a commutative diagram

$$
\begin{array}{ccc}
 & Y_B & \\
\nearrow & & \searrow \\
Y_A & \xrightarrow{\quad j_0 \quad} & Y_C
\end{array}
$$

in $\mathcal{O}^{\otimes}$ and a morphism $\bar{j}_0 : \overline{Y}_A \to \overline{Y}_C$ in $\mathcal{K}_{\mathcal{O}}$ lifting j. Let $\mathcal{Z}_0 = (\mathcal{K}_{\mathcal{O}})_{\overline{Y}_A//\overline{Y}_C} \times_{\mathcal{O}^{\otimes}_{Y_A//Y_C}} \{Y_B\}$ be the ∞-category whose objects are diagrams

$$
\begin{array}{ccccc}
Y_A & \longrightarrow & Y_B & \longrightarrow & Y_C \\
\downarrow {\scriptstyle \alpha} & & \downarrow & & \downarrow \\
Y'_A & \longrightarrow & Y'_B & \longrightarrow & Y'_C
\end{array}
$$

in $\mathcal{O}^{\otimes}$. Let $\mathcal{Z}_1$ be the full subcategory of $\mathcal{Z}_0$ spanned by those diagrams for which the map $Y'_B \to Y'_C$ is active, and let $\mathcal{Z}_2$ be the full subcategory of $\mathcal{Z}_1$ spanned by those maps for which $q(Y_B) \coprod_{q(Y_A)} q(Y'_A) \to q(Y'_B)$ is surjective. Since the map $q(Y_A) \coprod_{q(X_0)} q(X') \to q(Y'_A)$ is surjective and f'' is 1-semi-inert, we deduce that α is 1-semi-inert. Condition (3) guarantees that $\mathcal{O}^{\otimes}$ is 1-coherent, so the ∞-category $\mathcal{Z}_0$ is weakly contractible. The inclusion $\mathcal{Z}_1 \subseteq \mathcal{Z}_0$ admits a left adjoint, and the inclusion $\mathcal{Z}_2 \subseteq \mathcal{Z}_1$ admits a right adjoint. It follows that both of these inclusions are weak homotopy equivalences, so that $\mathcal{Z}_2$ is weakly contractible. There is an evident restriction map $\mathcal{Z} \to \mathcal{Z}_2$, which is easily shown to be a trivial Kan fibration. It follows that $\mathcal{Z}$ is weakly contractible as desired.

(b) The fibers of ψ are weakly contractible. To prove this, we observe an object $b \in \mathcal{B}[\tau_0, Z]$ determines a commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X_0 & \longrightarrow & Y_0 \\
\downarrow & & \\
X' & &
\end{array}
$$

in $\mathcal{O}^{\otimes}_{/Z}$. If we let σ' denote the lower part of this diagram, then we have a trivial Kan fibration $\psi^{-1}(b) \to \mathcal{B}[\sigma', Z]$, which is weakly contractible by virtue of (3).

(c) For every object $c \in \mathcal{C}_0$ and every morphism $\beta : \psi(c) \to b$ in $\mathcal{B}[\tau_0, Z]$, the ∞-category

$$
\mathcal{Y} = (\mathcal{C}_0)_{c/} \times_{\mathcal{B}[\tau_0, Z]_{\psi(c)/}} \{\beta\}
$$

is weakly contractible. The pair (c, β) determines a diagram

$$
\begin{array}{ccccc}
X & \longrightarrow & Y & & \\
\downarrow & & \downarrow & & \\
X_0 & \longrightarrow & Y_0 & \longrightarrow & Y_1 \\
\downarrow & & \downarrow{\scriptstyle g''} & & \\
X' & \longrightarrow & Y' & &
\end{array}
$$

in $\mathcal{O}^{\otimes}_{/Z}$. Let σ'' denote the lower right corner of this diagram. Then we have a trivial Kan fibration $\mathcal{Y} \to \mathcal{B}[\sigma'', Z]$. Since the map $q(Y_0) \coprod_{q(X_0)} q(X') \to q(Y')$ is surjective, we deduce that g'' is 1-semi-inert, so that $\mathcal{B}[\sigma'', Z]$ is weakly contractible by virtue of (3).

$\square$

Proof of Theorem 3.3.2.2. In view of Proposition 3.3.2.10, we can rephrase Theorem 3.3.2.2 as follows: if $\mathcal{O}^{\otimes}$ is a unital ∞-operad and $\mathcal{O}$ is a Kan complex, then $\mathcal{O}^{\otimes}$ is coherent if and only if it is 1-coherent.

Fix a pair of active morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{O}^{\otimes}$, where $Z \in \mathcal{O}$. Let $\sigma : \Delta^3 \to \mathcal{O}^{\otimes}$ be as in the formulation of condition (3), and consider the diagram

$$
\begin{array}{ccc}
\mathrm{Ext}(\sigma, \{0,1\}) & \longrightarrow & \mathrm{Ext}(\sigma|\Delta^{\{0,1,3\}}, \{0,1\}) \\
\downarrow & & \downarrow \\
\mathrm{Ext}(\sigma|\Delta^{\{0,2,3\}}, \{0\}) & \longrightarrow & \mathrm{Ext}(\sigma|\Delta^{\{0,3\}}, \{0\}).
\end{array}
$$

Each of the maps in this diagram is a Kan fibration between Kan complexes. Consequently, $\mathcal{O}^{\otimes}$ is 1-coherent if and only if, for each vertex v of $\mathrm{Ext}(\sigma|\Delta^{\{0,3\}}, \{0\})$, the induced diagram of fibers

$$
\mathrm{Ext}(\sigma|\Delta^{\{0,1,3\}}, \{0,1\})_v \leftarrow \mathrm{Ext}(\sigma, \{0,1\})_v \to \mathrm{Ext}(\sigma|\Delta^{\{0,2,3\}}, \{0\})_v
$$

has a contractible homotopy pushout. Without loss of generality, we may assume that v determines a diagram

$$
\begin{array}{ccc}
X & \longrightarrow Y \longrightarrow & Z \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Z} \\
X' & \longrightarrow & Z
\end{array}
$$

in $\mathcal{O}^\otimes$, where the left vertical map is 1-semi-inert. Let τ denote the induced diagram $X' \leftarrow X \to Y$ in $\mathcal{O}^\otimes_{/Z}$, and let $\mathcal{B}[\tau, Z]$ be the ∞-category defined in Proposition 3.3.2.10. Let $\mathcal{B}[\tau, Z]_1$ denote the full subcategory of $\mathcal{B}[\tau, Z]$ spanned by those objects which correspond to diagrams

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X' & \longrightarrow & Y'
\end{array}
$$

where the right vertical map is an equivalence. Then there is a unique map $p : \mathcal{B}[\tau, Z] \to \Delta^1$ such that $p^{-1}\{1\} \simeq \mathcal{B}[\tau, Z]_1$. Let $\mathcal{B}[\tau, Z]_0 = p^{-1}\{0\}$, and let $\mathcal{Z} = \mathrm{Fun}_{\Delta^1}(\Delta^1, \mathcal{B}[\tau, Z])$ be the ∞-category of sections of p. We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Ext}(\sigma | \Delta^{\{0,1,3\}}, \{0,1\})_v & \longleftarrow \mathrm{Ext}(\sigma, \{0,1\})_v \longrightarrow & \mathrm{Ext}(\sigma | \Delta^{\{0,2,3\}}, \{0\})_v \\
\downarrow & \downarrow & \downarrow \\
\mathcal{B}[\tau, Z]_0 \longleftarrow & \mathcal{Z} & \longrightarrow \mathcal{B}[\tau, Z]_1.
\end{array}
$$

in which the vertical maps are trivial Kan fibrations. Consequently, condition $\mathcal{O}^\otimes$ is 1-coherent if and only if each homotopy pushout

$$
\mathcal{B}[\tau, Z]_0 \coprod_{\mathcal{Z} \times \{0\}} (\mathcal{Z} \times \Delta^1) \coprod_{\mathcal{Z} \times \{1\}} \mathcal{B}[\tau, Z]_1
$$

is weakly contractible. According to Proposition B.3.17, this homotopy pushout is categorically equivalent to $\mathcal{B}[\tau, Z]$, so that the 1-coherence of $\mathcal{O}^\otimes$ is equivalent to the coherence of $\mathcal{O}^\otimes$. $\square$

3.3.3   Module Objects

Let $\mathcal{O}^\otimes$ be a coherent ∞-operad and suppose we are given a fibration of ∞-operads $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$. In this section, we will explain how to associate to each algebra object $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ a new ∞-operad $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^\otimes$ of A-modules in $\mathcal{C}$. The construction of $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^\otimes$ (as a simplicial set with a map to $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$) is fairly straightforward; the bulk of our work will be in proving that $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^\otimes$ is actually an ∞-operad (Theorem 3.3.3.9).

Construction 3.3.3.1. Let $\mathcal{O}^\otimes$ be a unital ∞-operad and let $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a map of generalized ∞-operads. We let $\mathcal{K}_\mathcal{O} \subseteq \mathrm{Fun}(\Delta^1, \mathcal{O}^\otimes)$ denote the full subcategory spanned by the semi-inert morphisms in $\mathcal{O}^\otimes$ (Notation 3.3.2.1), and $e_i : \mathcal{K}_\mathcal{O} \to \mathcal{O}^\otimes$ the evaluation functor for $i \in \{0, 1\}$. We define a simplicial set $\widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^\otimes$ equipped with a map $q : \widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^\otimes \to \mathcal{O}^\otimes$ so that the following universal property is satisfied: for every map of simplicial sets $X \to \mathcal{O}^\otimes$, there is a canonical bijection

$$
\mathrm{Fun}_{\mathcal{O}^\otimes}(X, \widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^\otimes) \simeq \mathrm{Fun}_{\mathrm{Fun}(\{1\}, \mathcal{O}^\otimes)}(X \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^\otimes)} \mathcal{K}_\mathcal{O}, \mathcal{C}^\otimes).
$$

We let $\overline{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^\otimes$ denote the full simplicial subset of $\widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^\otimes$ spanned by those vertices $\overline{v}$ with the following property: if we let $v = q(\overline{v}) \in \mathcal{O}^\otimes$, then $\overline{v}$ determines a functor

$$
\{v\} \times_{\mathcal{O}^\otimes} \mathcal{K}_\mathcal{O} \to \mathcal{C}^\otimes
$$

which carries inert morphisms to inert morphisms (recall that a morphism in $\alpha : \mathcal{K}_\mathcal{O}$ is *inert* if its images $e_0(\alpha)$ and $e_1(\alpha)$ are inert).

Notation 3.3.3.2. Let $\mathcal{O}^\otimes$ be an ∞-operad. We let $\mathcal{K}_\mathcal{O}^0$ denote the full subcategory of $\mathcal{K}_\mathcal{O}$ spanned by the null morphisms $f : X \to Y$ in $\mathcal{O}^\otimes$. For $i = 0, 1$, we let e_i^0 denote the restriction to $\mathcal{K}_\mathcal{O}^0$ of the evaluation map $e_i : \mathcal{K}_\mathcal{O} \to \mathcal{O}^\otimes$.

Lemma 3.3.3.3. *Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad. Then the maps e_0^0 and e_1^0 determine a trivial Kan fibration $\theta : \mathcal{K}_{\mathcal{O}}^0 \to \mathcal{O}^{\otimes} \times \mathcal{O}^{\otimes}$.*

Proof. Since θ is evidently a categorical fibration, it will suffice to show that θ is a categorical equivalence. Corollary T.2.4.7.12 implies that evaluation at $\{0\}$ induces a Cartesian fibration $p : \operatorname{Fun}(\Delta^1, \mathcal{O}^{\otimes}) \to \mathcal{O}^{\otimes}$. If f is a null morphism in $\mathcal{O}^{\otimes}$, then so is $f \circ g$ for every morphism g in $\mathcal{O}^{\otimes}$; it follows that if $\alpha : f \to f'$ is a p-Cartesian morphism in $\operatorname{Fun}(\Delta^1, \mathcal{O}^{\otimes})$ and $f' \in \mathcal{K}_{\mathcal{O}}^0$, then $f \in \mathcal{K}_{\mathcal{O}}^0$. Consequently, p restricts to a Cartesian fibration $e_0^0 : \mathcal{K}_{\mathcal{O}}^0 \to \mathcal{O}^{\otimes}$. Moreover, a morphism α in $\mathcal{K}_{\mathcal{O}}^0$ is e_0^0-Cartesian if and only if $e_1^0(\alpha)$ is an equivalence. Consequently, θ fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{K}_{\mathcal{O}}^0 & \xrightarrow{\theta} & \mathcal{O}^{\otimes} \times \mathcal{O}^{\otimes} \\ & {\scriptstyle e_0^0} \searrow & \downarrow {\scriptstyle \pi} \\ & & \mathcal{O}^{\otimes} \end{array}$$

and carries e_0^0-Cartesian morphisms to π-Cartesian morphisms. According to Corollary T.2.4.4.4, it will suffice to show that for every object $X \in \mathcal{O}^{\otimes}$, the map θ induces a categorical equivalence $\theta_X : (\mathcal{O}^{\otimes})_0^{X/} \to \mathcal{O}^{\otimes}$, where $(\mathcal{O}^{\otimes})_0^{X/}$ is the full subcategory of $(\mathcal{O}^{\otimes})^{X/}$ spanned by the null morphisms $X \to Y$.

The map θ_X is obtained by restricting the left fibration $q : (\mathcal{O}^{\otimes})^{X/} \to \mathcal{O}^{\otimes}$. We observe that if f is a null morphism in $\mathcal{O}^{\otimes}$, then so is every composition of the form $g \circ f$. It follows that if $\alpha : f \to f'$ is a morphism in $(\mathcal{O}^{\otimes})^{X/}$ such that $f \in (\mathcal{O}^{\otimes})_0^{X/}$, then $f' \in (\mathcal{O}^{\otimes})_0^{X/}$, so the map q restricts to a left fibration $\theta_X : (\mathcal{O}^{\otimes})_0^{X/} \to \mathcal{O}^{\otimes}$. Since the fibers of θ_X are contractible (Lemma 3.3.3.11), Lemma T.2.1.3.4 implies that θ_X is a trivial Kan fibration. $\square$

Corollary 3.3.3.4. *Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad. The projection map $e_0^0 : \mathcal{K}_{\mathcal{O}}^0 \to \mathcal{O}^{\otimes}$ is a flat categorical fibration.*

Notation 3.3.3.5. Let $\mathcal{O}^{\otimes}$ be an ∞-operad and let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a $\mathcal{O}$-operad family. We define a simplicial set $\widetilde{\operatorname{Alg}}_{/\mathcal{O}}(\mathcal{C})$ equipped with a map $\widetilde{\operatorname{Alg}}_{/\mathcal{O}}(\mathcal{C}) \to \mathcal{O}^{\otimes}$ so that the following universal property is satisfied: for every simplicial set X with a map $X \to \mathcal{O}^{\otimes}$, we have a canonical bijection

$$\operatorname{Fun}_{\mathcal{O}^{\otimes}}(X, \widetilde{\operatorname{Alg}}_{/\mathcal{O}}(\mathcal{C})) \simeq \operatorname{Fun}_{\operatorname{Fun}(\{1\}, \mathcal{O}^{\otimes})}(X \times_{\operatorname{Fun}(\{0\}, \mathcal{O}^{\otimes})} \mathcal{K}_{\mathcal{O}}^0, \mathcal{C}^{\otimes}).$$

We observe that for every object $Y \in \mathcal{O}^{\otimes}$, a vertex of the fiber $\widetilde{\operatorname{Alg}}_{/\mathcal{O}}(\mathcal{C}) \times_{\mathcal{O}^{\otimes}} \{Y\}$ can be identified with a functor $F : \{Y\} \times_{\operatorname{Fun}(\{0\}, \mathcal{O}^{\otimes})} \mathcal{K}_{\mathcal{O}}^0 \to \mathcal{C}^{\otimes}$. We let $^{\mathrm{p}}\operatorname{Alg}_{/\mathcal{O}}(\mathcal{C})$ denote the full simplicial subset of $\widetilde{\operatorname{Alg}}_{/\mathcal{O}}(\mathcal{C})$ spanned by those vertices for which the functor F preserves inert morphisms.

Remark 3.3.3.6. Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad. Since the map $e_1^0 : \mathcal{K}_{\mathcal{O}}^0 \to \mathcal{O}^{\otimes}$ is a trivial Kan fibration (Lemma 3.3.3.3), we conclude that $\phi : \mathcal{O}^{\otimes} \times \operatorname{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes}) \to \widetilde{\operatorname{Alg}}_{/\mathcal{O}}(\mathcal{C})$ induces a categorical equivalence $\mathcal{O}^{\otimes} \times \operatorname{Alg}_{/\mathcal{O}}(\mathcal{C}) \to {}^{\mathrm{p}}\operatorname{Alg}_{/\mathcal{O}}(\mathcal{C})$.

Remark 3.3.3.7. Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad and $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ a $\mathcal{O}$-operad family. The map $\theta : \mathcal{K}_{\mathcal{O}}^0 \to \mathcal{O}^{\otimes} \times \mathcal{O}^{\otimes}$ of Lemma 3.3.3.3 induces a map $\phi : \mathcal{O}^{\otimes} \times \operatorname{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes}) \to \widetilde{\operatorname{Alg}}_{/\mathcal{O}}(\mathcal{C})$. Unwinding the definitions, we see that $\phi^{-1} \, {}^{\mathrm{p}}\operatorname{Alg}_{/\mathcal{O}}(\mathcal{C})$ can be identified with the product $\mathcal{O}^{\otimes} \times \operatorname{Alg}_{/\mathcal{O}}(\mathcal{C})$.

Definition 3.3.3.8. Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad and $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ a fibration of generalized ∞-operads. We let $\operatorname{Mod}^{\mathcal{O}}(\mathcal{C})^{\otimes}$ denote the fiber product

$$\overline{\operatorname{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes} \times_{^{\mathrm{p}}\operatorname{Alg}_{/\mathcal{O}}(\mathcal{C})} (\mathcal{O}^{\otimes} \times \operatorname{Alg}_{/\mathcal{O}}(\mathcal{C})).$$

For every algebra object $A \in \operatorname{Alg}_{/\mathcal{O}}(\mathcal{C})$, we let $\operatorname{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ denote the fiber product

$$\overline{\operatorname{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes} \times_{^{\mathrm{p}}\operatorname{Alg}_{/\mathcal{O}}(\mathcal{C})} \{A\} \simeq \operatorname{Mod}^{\mathcal{O}}(\mathcal{C})^{\otimes} \times_{\operatorname{Alg}_{/\mathcal{O}}(\mathcal{C})} \{A\}.$$

We will refer to $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ as the ∞-*operad of* $\mathcal{O}$-*module objects over* A.

We can now state the main result of this section as follows:

Theorem 3.3.3.9. *Let* $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ *be a fibration of* ∞-*operads, where* $\mathcal{O}^{\otimes}$ *is coherent, and let* $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$. *Then the induced map* $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ *is a fibration of* ∞-*operads.*

The main ingredient in the proof of Theorem 3.3.3.9 is the following result, which we will prove at the end of this section.

Proposition 3.3.3.10. *Let* $\mathcal{O}^{\otimes}$ *be a coherent* ∞-*operad, and* $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ *be a map of generalized* ∞-*operads. Then the induced map* $q : \overline{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ *is again a map of generalized* ∞-*operads. Moreover, a morphism* f *in* $\overline{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes}$ *is inert if and only if it satisfies the following conditions:*

(1) *The morphism* $f_0 = q(f) : X \to Y$ *is inert in* $\mathcal{O}^{\otimes}$.

(2) *Let* $F : \mathcal{K}_{\mathcal{O}} \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} \Delta^1 \to \mathcal{C}^{\otimes}$ *be the functor classified by* f. *For every* e_0-*Cartesian morphism* $\widetilde{f}$ *of* $\mathcal{K}_{\mathcal{O}}$ *lifting* f_0 *(in other words, for every lift* $\widetilde{f}$ *of* f_0 *such that* $e_1(\widetilde{f})$ *is an equivalence: see part (4) of the proof of Proposition 3.3.3.18), the morphism* $F(\widetilde{f})$ *of* $\mathcal{C}^{\otimes}$ *is inert.*

We will obtain Theorem 3.3.3.9 by combining Proposition 3.3.3.10 with a few elementary observations.

Lemma 3.3.3.11. *Let* $p : \mathcal{O}^{\otimes}$ *be a unital* ∞-*operad. Let* X *and* Y *be objects of* $\mathcal{O}^{\otimes}$ *lying over objects* $\langle m \rangle, \langle n \rangle \in \mathcal{F}\mathrm{in}_*$, *and let* $\mathrm{Map}_{\mathcal{O}^{\otimes}}^0(X, Y)$ *denote the summand of* $\mathrm{Map}_{\mathcal{O}^{\otimes}}(X, Y)$ *corresponding to those maps which cover the null morphism* $\langle m \rangle \to \{*\} \to \langle n \rangle$ *in* $\mathcal{F}\mathrm{in}_*$. *Then* $\mathrm{Map}_{\mathcal{O}^{\otimes}}^0(X, Y)$ *is contractible.*

Proof. Choose an inert morphism $f : X \to Z$ covering the map $\langle m \rangle \to \{*\}$ in $\mathcal{F}\mathrm{in}_*$. We observe that composition with f induces a homotopy equivalence $\mathrm{Map}_{\mathcal{O}^{\otimes}}(Z, Y) \to \mathrm{Map}_{\mathcal{O}^{\otimes}}^0(X, Y)$. Since $\mathcal{O}^{\otimes}$ is unital, the space $\mathrm{Map}_{\mathcal{O}^{\otimes}}(Z, Y)$ is contractible. $\qquad\square$

Lemma 3.3.3.12. *Let* $\mathcal{O}^{\otimes}$ *be a unital* ∞-*operad and let* $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ *be a* $\mathcal{O}$-*operad family. Then the map* $\widetilde{\mathrm{Alg}}_{/\mathcal{O}}(\mathcal{C}) \to \mathcal{O}_0^{\otimes}$ *is a categorical fibration. In particular,* $\widetilde{\mathrm{Alg}}_{/\mathcal{O}}(\mathcal{C})$ *is an* ∞-*category.*

Proof. The simplicial set $\widetilde{\mathrm{Alg}}_{/\mathcal{O}}(\mathcal{C})$ can be described as $(e_0^0)_*(\mathcal{K}_{\mathcal{O}}^0 \times_{\mathrm{Fun}(\{1\}, \mathcal{O}^{\otimes})} \mathcal{C}^{\otimes})$. The desired result now follows from Proposition B.4.5 and Corollary 3.3.3.4. $\qquad\square$

Remark 3.3.3.13. In the situation of Lemma 3.3.3.12, the simplicial set $^{\mathrm{p}}\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ is a full simplicial subset of an ∞-category, and therefore also an ∞-category. Since $^{\mathrm{p}}\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ is evidently stable under equivalence in $\widetilde{\mathrm{Alg}}_{/\mathcal{O}}(\mathcal{C})$, we conclude that the map $^{\mathrm{p}}\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathcal{O}^{\otimes}$ is also a categorical fibration.

Lemma 3.3.3.14. *Let* $\mathcal{O}^{\otimes}$ *be a coherent* ∞-*operad, and let* $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ *be a* $\mathcal{O}$-*operad family. Then the inclusion* $\mathcal{K}_{\mathcal{O}}^0 \to \mathcal{K}_{\mathcal{O}}$ *induces a categorical fibration* $\overline{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes} \to {}^{\mathrm{p}}\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$.

Proof. Since $\overline{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes}$ and $^{\mathrm{p}}\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ are full subcategories stable under equivalence in $\widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes}$ and $\widetilde{\mathrm{Alg}}_{/\mathcal{O}}(\mathcal{C})$, it suffices to show that the map $\widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \widetilde{\mathrm{Alg}}_{/\mathcal{O}}(\mathcal{C})$ has the right lifting property with respect to every trivial cofibration $A \to B$ with respect to the Joyal model structure on $(\mathcal{S}\mathrm{et}_{\Delta})_{/\mathcal{O}^{\otimes}}$. Unwinding the definitions, we are required to provide solutions to lifting problems of the form

$$
\begin{array}{ccc}
(A \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} \mathcal{K}_{\mathcal{O}}) \coprod_{A \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} \mathcal{K}_{\mathcal{O}}^0} (B \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} \mathcal{K}_{\mathcal{O}}^0) & \longrightarrow & \mathcal{C}^{\otimes} \\
\downarrow {\scriptstyle i} & \nearrow & \downarrow {\scriptstyle q} \\
B \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} \mathcal{K}_{\mathcal{O}} & \longrightarrow & \mathrm{Fun}(\{1\}, \mathcal{O}^{\otimes}).
\end{array}
$$

Since q is a categorical fibration, it suffices to prove that the monomorphism i is a categorical equivalence of simplicial sets. In other words, we need to show that the diagram

$$
\begin{array}{ccc}
A \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^\otimes)} \mathcal{K}_\mathcal{O}^0 & \longrightarrow & A \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^\otimes)} \mathcal{K}_\mathcal{O} \\
\downarrow & & \downarrow \\
B \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^\otimes)} \mathcal{K}_\mathcal{O}^0 & \longrightarrow & B \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^\otimes)} \mathcal{K}_\mathcal{O}
\end{array}
$$

is a homotopy pushout square (with respect to the Joyal model structure). To prove this, it suffices to show that the vertical maps are categorical equivalences. Since $A \to B$ is a categorical equivalence, this follows from Corollary B.3.15, since the restriction maps $e_0^0 : \mathcal{K}_\mathcal{O}^0 \to \mathcal{O}^\otimes$ and $e_0 : \mathcal{K}_\mathcal{O} \to \mathcal{O}_0^\otimes$ are flat categorical fibrations (Corollary 3.3.3.4). $\qquad\square$

Remark 3.3.3.15. Let $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be as in Lemma 3.3.3.14. Since every morphism $f : X \to Y$ in $\mathcal{O}^\otimes$ with $X \in \mathcal{O}_{\langle 0 \rangle}^\otimes$ is automatically null, the categorical fibration $\overline{\mathrm{Mod}}^\mathcal{O}(\mathcal{C})^\otimes \to {}^{\mathrm{p}}\mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C})$ induces an isomorphism $\overline{\mathrm{Mod}}^\mathcal{O}(\mathcal{C})_{\langle 0 \rangle}^\otimes \to {}^{\mathrm{p}}\mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C})_{\langle 0 \rangle}$.

Remark 3.3.3.16. In the situation of Definition 3.3.3.8, consider the pullback diagram

$$
\begin{array}{ccc}
\mathrm{Mod}^\mathcal{O}(\mathcal{C})^\otimes & \xrightarrow{\;\;j\;\;} & \overline{\mathrm{Mod}}^\mathcal{O}(\mathcal{C})^\otimes \\
{\scriptstyle \theta'}\downarrow & & \downarrow{\scriptstyle \theta} \\
\mathcal{O}^\otimes \times \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C}) & \xrightarrow{\;\;j'\;\;} & {}^{\mathrm{p}}\mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C}).
\end{array}
$$

Since θ is a categorical fibration and $\mathcal{O}^\otimes \times \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C})$ is an ∞-category, this diagram is a homotopy pullback square (with respect to the Joyal model structure). Remark 3.3.3.6 implies that j' is a categorical equivalence, so j is also a categorical equivalence. Using Proposition 3.3.3.10, we deduce that $\mathrm{Mod}^\mathcal{O}(\mathcal{C})^\otimes \to \mathcal{O}^\otimes \times \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C})$ is a fibration of generalized ∞-operads.

Proof of Theorem 3.3.3.9. It follows from Remark 3.3.3.16 that the map $\mathrm{Mod}_A^\mathcal{O}(\mathcal{C})^\otimes \to \mathcal{O}^\otimes$ is a fibration of generalized ∞-operads. To complete the proof, it will suffice to show that $\mathrm{Mod}_A^\mathcal{O}(\mathcal{C})^\otimes$ is itself an ∞-operad. According to Proposition 2.3.2.5, this is equivalent to the assertion that $\mathrm{Mod}_A^\mathcal{O}(\mathcal{C})_{\langle 0 \rangle}^\otimes$ is a contractible Kan complex. This is clear, since Remark 3.3.3.15 implies that θ' induces an isomorphism $\mathrm{Mod}^\mathcal{O}(\mathcal{C})_{\langle 0 \rangle}^\otimes \to \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C})$. $\qquad\square$

We now turn to the proof of Proposition 3.3.3.10.

Lemma 3.3.3.17. *Let $\mathcal{O}^\otimes$ be an ∞-operad, let $\overline{X} : X \to X'$ be an object of $\mathrm{Fun}(\Delta^1, \mathcal{O}^\otimes)$, and let $e_0 : \mathrm{Fun}(\Delta^1, \mathcal{O}^\otimes) \to \mathcal{O}^\otimes$ be given by evaluation at 0.*

(1) *For every inert morphism $f_0 : X \to Y$ in $\mathcal{O}^\otimes$, there exists an e_0-coCartesian morphism $f : \overline{X} \to \overline{Y}$ in $\mathrm{Fun}(\Delta^1, \mathcal{O}^\otimes)$ lifting f_0.*

(2) *An arbitrary morphism $f : \overline{X} \to \overline{Y}$ in $\mathrm{Fun}(\Delta^1, \mathcal{O}^\otimes)$ lifting f_0 is e_0-coCartesian if and only if the following conditions are satisfied:*

   (i) *The image of f in $\mathrm{Fun}(\{1\}, \mathcal{O}^\otimes)$ is inert.*

(ii) Let $\alpha : \langle m \rangle \to \langle m' \rangle$ be the morphism in $\mathfrak{F}\mathrm{in}_*$ determined by $\overline{X}$, and let $\beta : \langle n \rangle \to \langle n' \rangle$ be the morphism in $\mathfrak{F}\mathrm{in}_*$ determined by $\overline{Y}$. Then f induces a diagram

$$
\begin{array}{ccc}
\langle m \rangle & \xrightarrow{\ \delta\ } & \langle n \rangle \\
{\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\
\langle m' \rangle & \xrightarrow{\ \gamma\ } & \langle n' \rangle
\end{array}
$$

with the property that $\gamma^{-1}\{*\} = \alpha(\delta^{-1}\{*\})$.

(3) If $\overline{X} \in \mathcal{K}_{\mathcal{O}}$ and f is e_0-coCarteian, then $\overline{Y} \in \mathcal{K}_{\mathcal{O}}$ (the map f is then automatically coCartesian with respect to the restriction $e_0 | \mathcal{K}_{\mathcal{O}}$).

Proof. We note that the "only if" direction of (2) follows from the "if" direction together with (1), since a coCartesian lift f of f_0 is determined uniquely up to homotopy. We first treat the case where $\mathcal{O}^{\otimes} = \mathrm{N}(\mathfrak{F}\mathrm{in}_*)$. Then we can identify $\overline{X}$ with α with β, and f_0 with an inert morphism $\delta : \langle m \rangle \to \langle n \rangle$. Choose a map $\gamma : \langle m' \rangle \to \langle n' \rangle$ which identifies $\langle n' \rangle$ with the finite pointed set obtained from $\langle m' \rangle$ by collapsing $\alpha(\delta^{-1}\{*\})$ to a point, so that we have a commutative diagram

$$
\begin{array}{ccc}
\langle m \rangle & \xrightarrow{\ \delta\ } & \langle n \rangle \\
{\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\
\langle m' \rangle & \xrightarrow{\ \gamma\ } & \langle n' \rangle
\end{array}
$$

as in the statement of (ii). By construction, the map γ is inert. The above diagram is evidently a pushout square in $\mathfrak{F}\mathrm{in}_*$ and so therefore corresponds to a morphism morphism f in $\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathfrak{F}\mathrm{in}_*))$ which is coCartesian with respect to the projection $\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathfrak{F}\mathrm{in}_*))$, and therefore with respect to the projection $e_0 : \mathcal{K}_{\mathrm{N}(\mathfrak{F}\mathrm{in}_*)} \to \mathrm{N}(\mathfrak{F}\mathrm{in}_*)$. This completes the proof of (1) and the "if" direction of (2); assertion (3) follows from the observation that β is semi-inert whenever α is semi-inert.

We now prove the "if" direction of (2) in the general case. Since f_0 is inert, we conclude (Proposition T.2.4.1.3) that f is e_0-coCartesian if and only if it is coCartesian with respect to the composition

$$
\mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes}) \overset{e_0}{\to} \mathcal{O}^{\otimes} \to \mathrm{N}(\mathfrak{F}\mathrm{in}_*).
$$

This map admits another factorization

$$
\mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes}) \overset{p}{\to} \mathrm{Fun}(\Delta^1, \mathrm{N}(\mathfrak{F}\mathrm{in}_*)) \overset{q}{\to} \mathrm{N}(\mathfrak{F}\mathrm{in}_*).
$$

The first part of the proof shows that $p(f)$ is q-coCartesian. Since f_0 is inert, condition (i) and Lemma 3.2.2.9 guarantee that f is p-coCartesian. Applying Proposition T.2.4.1.3, we deduce that f is $(q \circ p)$-coCartesian as desired.

To prove (1), we first choose a diagram

$$
\begin{array}{ccc}
\langle m \rangle & \xrightarrow{\ \delta\ } & \langle n \rangle \\
{\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\
\langle m' \rangle & \xrightarrow{\ \gamma\ } & \langle n' \rangle
\end{array}
$$

satisfying the hypotheses of (2). We can identify this diagram with a morphism $\overline{f}$ in $\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathfrak{F}\mathrm{in}_*))$. We will show that it is possible to choose a map $f : \overline{X} \to \overline{Y}$ satisfying (i) and (ii), which lifts both f_0 and $\overline{f}$.

Using the assumption that $\mathcal{O}^\otimes$ is an ∞-operad, we can choose an inert morphism $f_1 : X' \to Y'$ lifting γ. Using the fact that the projection $\mathcal{O}^\otimes \to N(\mathcal{F}in_*)$ is an inner fibration, we obtain a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ f_0\ } & Y \\
\downarrow{\scriptstyle \overline{X}} & \searrow & \vdots \\
X' & \xrightarrow{\ f_1\ } & Y'
\end{array}
$$

in $\mathcal{O}^\otimes$. To complete the proof, it suffices to show that we can complete the diagram by filling an appropriate horn $\Lambda_0^2 \subseteq \Delta^2$ to supply the dotted arrow. The existence of this arrow follows from the fact that f_0 is inert, and therefore coCartesian with respect to the projection $\mathcal{O}^\otimes \to N(\mathcal{F}in_*)$.

We now prove (3). Let $f : \overline{X} \to \overline{Y}$ be a morphism corresponding to a diagram as above, where $\overline{X}$ is semi-inert; we wish to show that $\overline{Y}$ is semi-inert. The first part of the proof shows that the image of $\overline{Y}$ in $N(\mathcal{F}in_*)$ is semi-inert. It will therefore suffice to show that for any inert morphism $g : Y' \to Z$, if the image of $g \circ \overline{Y}$ in $N(\mathcal{F}in_*)$ is inert, then $g \circ \overline{Y}$ is inert. We note that $g \circ \overline{Y} \circ f_0 \simeq g \circ f_1 \circ \overline{X}$ has inert image in $N(\mathcal{F}in_*)$; since $\overline{X}$ is semi-inert we deduce that $g \circ \overline{Y} \circ f_0$ is inert. Since f_0 is inert, we deduce from Proposition T.2.4.1.7 that $g \circ \overline{Y}$ is inert as desired. $\qquad\square$

For the remainder of this section, we will assume that the reader is familiar with the language of categorical patterns developed in Appendix B.4. Let $\mathfrak{P}$ denote the categorical pattern $(M, T, \{\sigma_\alpha : \Delta^1 \times \Delta^1 \to \mathcal{O}^\otimes\}_{\alpha \in A})$ on $\mathcal{O}^\otimes$ where M is the collection of inert morphisms in $\mathcal{O}^\otimes$, T is the collection of all 2-simplices in $\mathcal{O}^\otimes$, and A is the collection of all diagrams of inert morphisms in $\mathcal{O}^\otimes$ which cover a diagram

$$
\begin{array}{ccc}
\langle m \rangle & \longrightarrow & \langle n \rangle \\
\downarrow & & \downarrow \\
\langle m' \rangle & \longrightarrow & \langle n' \rangle
\end{array}
$$

in $\mathcal{F}in_*$ which induces a bijection $\langle m' \rangle^\circ \coprod_{\langle n' \rangle^\circ} \langle n \rangle^\circ \to \langle m \rangle^\circ$. We will regard $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}} = (\mathcal{S}et_\Delta^+)_{/\mathcal{O}^\otimes}$ as endowed with the model structure of Theorem B.0.19. Let M' denote the collection of inert morphisms in $\mathcal{K}_\mathcal{O}$, so that the construction $\overline{X} \mapsto \overline{X} \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^\otimes)^\natural} \mathcal{K}_\mathcal{O}, M')$ determines a functor $F : (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}} \to (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$. The functor F admits a right adjoint G. By construction, if $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ is a fibration of generalized ∞-operads, then $G(\mathcal{C}^{\otimes,\natural})$ can be identified with the pair $(\overline{\mathrm{Mod}}^\mathcal{O}(\mathcal{C}), M'')$, where M'' is the collection of edges in $\overline{\mathrm{Mod}}^\mathcal{O}(\mathcal{C})$ satisfying condition (2) of Proposition 3.3.3.10. Unwinding the definitions, we see that an object $\overline{X} \in (\mathcal{S}et_\Delta^+)_{/\mathcal{O}^\otimes}$ is fibrant if and only if the underlying map $X \to \mathcal{O}^\otimes$ is a map of generalized ∞-operads and the collection of marked edges of X coincides with the collection of inert morphisms in X. We can therefore reformulate Proposition 3.3.3.10 as follows:

(∗) The functor $G : (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}} \to (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ carries fibrant objects to fibrant objects.

This is an immediate consequence of the following stronger assertion:

Proposition 3.3.3.18. *Let $\mathcal{O}^\otimes$ be a coherent ∞-operad, let M_0 denote the collection of all inert morphisms in $\mathcal{O}^\otimes$, and let M denote the collection of all inert morphisms in $\mathcal{K}_\mathcal{O}$. Then the construction*

$$
\overline{X} \mapsto \overline{X} \times_{\mathcal{O}^{\otimes,\natural}} (\mathcal{K}_\mathcal{O}, M')
$$

determines a left Quillen functor from $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ to itself.

Proof. It will suffice to show that the map $\mathcal{K}_\mathcal{O} \to \mathcal{O}^\otimes \times \mathcal{O}^\otimes$ satisfies the hypotheses of Theorem B.4.2:

(1) The evaluation map $e_0 : \mathcal{K}_\mathcal{O} \to \mathcal{O}^\otimes$ is a flat categorical fibration. This follows from our assumption that $\mathcal{O}^\otimes$ is coherent.

(2) The collections of inert morphisms in $\mathcal{O}^{\otimes}$ and $\mathcal{K}_{\mathcal{O}}$ contain all equivalences and are closed under composition. This assertion is clear from the definitions.

(3) For every 2-simplex σ of $\mathcal{K}_{\mathcal{O}}$, if $e_0(\sigma)$ is thin, then $e_1(\sigma)$ thin. Moreover, a 2-simplex $\Delta^2 \to \mathcal{K}_{\mathcal{O}}$ is thin if its restriction to $\Delta^{\{0,1\}}$ is thin. These assertions are clear, since the categorical pattern $\mathfrak{P}_{\mathcal{O}^{\otimes}}^{\mathrm{fam}}$ designates every 2-simplex as thin.

(4) For every inert edge $\Delta^1 \to \mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})$, the induced map $\mathcal{K}_{\mathcal{O}} \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} \Delta^1 \to \Delta^1$ is a Cartesian fibration. To prove this, it suffices to show that if $\overline{X}$ is an object of $\mathcal{K}_{\mathcal{O}}$ corresponding to a semi-inert morphism $X \to X'$ in $\mathcal{O}^{\otimes}$, and $f : Y \to X$ is an inert morphism in $\mathcal{O}^{\otimes}$, then we can lift f to an e_0-Cartesian morphism $\overline{f} : \overline{Y} \to \overline{X}$. According to Corollary T.2.4.7.12, we can lift f to a morphism $\overline{f} : \overline{Y} \to \overline{X}$ in $\mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$ which is e'_0-coCartesian, where $e'_0 : \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes}) \to \mathcal{O}_0^{\otimes}$ is given by evaluation at 0. To complete the proof, it suffices to show that $\overline{Y}$ belongs to $\mathcal{K}_{\mathcal{O}}$. We can identify $\overline{f}$ with a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\ f\ } & X \\
{\scriptstyle \overline{Y}}\downarrow & & \downarrow{\scriptstyle \overline{X}} \\
Y' & \xrightarrow{\ f'\ } & X'
\end{array}
$$

in $\mathcal{O}^{\otimes}$; we wish to prove that the morphism $\overline{Y}$ is semi-inert. Since $\overline{f}$ is e'_0-Cartesian, Corollary T.2.4.7.12 implies that f' is an equivalence. We can therefore identify $\overline{Y}$ with $f' \circ \overline{Y} \simeq \overline{X} \circ f$, which is inert by virtue of Remark 3.3.1.3. This completes the verification of condition (4) and establishes the following criterion: if $\overline{f}$ is a morphism in $\mathcal{K}_{\mathcal{O}}$ such that $f = e_0(\overline{f})$ is inert, the following conditions are equivalent:

   (i) The map $\overline{f}$ is locally e_0-Cartesian.

   (ii) The map $\overline{f}$ is e_0-Cartesian.

   (iii) The morphism $e_1(\overline{f})$ is an equivalence in $\mathcal{O}_1^{\otimes}$.

(5) Let $p : \Delta^1 \times \Delta^1 \to \mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})$ be one of the diagrams specified in the definition of the categorical pattern $\mathfrak{P}$. Then the restriction of e_0 determines a coCartesian fibration $(\Delta^1 \times \Delta^1) \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} \mathcal{K}_{\mathcal{O}} \to \Delta^1 \times \Delta^1$. This follows immediately from Lemma 3.3.3.17.

(6) Let $p : \Delta^1 \times \Delta^1 \to \mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})$ be one of the diagrams specified in the definition of the categorical pattern $\mathfrak{P}^{\mathrm{fam}}$ and let s be a coCartesian section of the projection $(\Delta^1 \times \Delta^1) \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} \mathcal{K}_{\mathcal{O}} \to \Delta^1 \times \Delta^1$. Then the composite map

$$
q : \Delta^1 \times \Delta^1 \xrightarrow{s} (\Delta^1 \times \Delta^1) \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} \mathcal{K}_{\mathcal{O}} \xrightarrow{e_1} \mathrm{Fun}(\{1\}, \mathcal{O}^{\otimes})
$$

is one of the diagrams specified in the definition of the categorical pattern $\mathfrak{P}_{\mathrm{Fun}(\{1\}, \mathcal{O}^{\otimes})}^{\mathrm{fam}}$. It follows from Lemma 3.3.3.17 that q carries each morphism in $\Delta^1 \times \Delta^1$ to an inert morphism in $\mathrm{Fun}(\{1\}, \mathcal{O}^{\otimes})$. Unwinding the definitions (and using the criterion provided by Lemma 3.3.3.17), we are reduced to verifying the following simple combinatorial fact: given a semi-inert morphism $\langle m \rangle \to \langle k \rangle$ in $\mathcal{F}\mathrm{in}_*$ and a commutative diagram of inert morphisms

$$
\begin{array}{ccc}
\langle m \rangle & \longrightarrow & \langle n \rangle \\
\downarrow & & \downarrow \\
\langle m' \rangle & \longrightarrow & \langle n' \rangle
\end{array}
$$

which induces a bijection $\langle n \rangle^\circ \amalg_{\langle n' \rangle^\circ} \langle m' \rangle^\circ \to \langle m \rangle^\circ$, the induced diagram

$$
\begin{array}{ccc}
\langle k \rangle & \longrightarrow & \langle n \rangle \amalg_{\langle m \rangle} \langle k \rangle \\
\downarrow & & \downarrow \\
\langle m' \rangle \amalg_{\langle m \rangle} \langle k \rangle & \longrightarrow & \langle n' \rangle \amalg_{\langle m \rangle} \langle k \rangle
\end{array}
$$

has the same property.

(7) Suppose we are given a commutative diagram

$$
\begin{array}{ccc}
 & Y & \\
{}^{f}\nearrow & & \searrow {}^{g} \\
X & \xrightarrow{\;\;h\;\;} & Z
\end{array}
$$

in $\mathcal{K}_\mathcal{O}$, where g is locally e_0-Cartesian, $e_0(g)$ is inert, and $e_0(f)$ is an equivalence. We must show that f is inert if and only if h is inert. Consider the underlying diagram in $\mathcal{O}^\otimes$

$$
\begin{array}{ccccc}
X_0 & \xrightarrow{\;f_0\;} & Y_0 & \xrightarrow{\;g_0\;} & Z_0 \\
\downarrow & & \downarrow & & \downarrow \\
X_1 & \xrightarrow{\;f_1\;} & Y_1 & \xrightarrow{\;g_1\;} & Z_1.
\end{array}
$$

Since f_0 is an equivalence and g_0 is inert, $h_0 = g_0 \circ f_0$ is inert. It will therefore suffice to prove that f_1 is inert if and only if $h_1 = g_1 \circ f_1$ is inert. For this, it sufficess to show that g_1 is an equivalence in $\mathcal{O}^\otimes$. This follows from the proof of (4), since g is assumed to be locally e_0-Cartesian.

(8) Suppose we are given a commutative diagram

$$
\begin{array}{ccc}
 & Y & \\
{}^{f}\nearrow & & \searrow {}^{g} \\
X & \xrightarrow{\;\;h\;\;} & Z
\end{array}
$$

in $\mathcal{K}_\mathcal{O}$ where f is e_0-coCartesian, $e_0(f)$ is inert, and $e_0(g)$ is an equivalence. We must show that g is inert if and only if h is inert. Consider the underlying diagram in $\mathcal{O}^\otimes$

$$
\begin{array}{ccccc}
X_0 & \xrightarrow{\;f_0\;} & Y_0 & \xrightarrow{\;g_0\;} & Z_0 \\
\downarrow & & \downarrow & & \downarrow \\
X_1 & \xrightarrow{\;f_1\;} & Y_1 & \xrightarrow{\;g_1\;} & Z_1.
\end{array}
$$

Since f_0 and f_1 are inert (Lemma 3.3.3.17), Propositions T.5.2.8.6 and 2.1.2.4 guarantee that g_0 is inert if and only if h_0 is inert, and that g_1 is inert if and only if h_1 is inert. Combining these facts, we conclude that g is inert if and only if h is inert.

$\square$

We close this section by describing the structure of the module ∞-categories $\mathrm{Mod}_A^\mathcal{O}(\mathcal{C})$ in the simplest case, where $\mathcal{O}^\otimes$ is the ∞-operad $\mathbb{E}_0^\otimes$ of Example 2.1.1.19. Recall that $\mathbb{E}_0^\otimes$ is coherent (Example 3.3.1.13).

Proposition 3.3.3.19. *Let $q : \mathcal{C}^{\otimes} \to \mathbb{E}_0^{\otimes}$ be a fibration of ∞-operads. Then the canonical map*

$$\overline{\mathcal{M}\mathrm{od}}^{\mathbb{E}_0}(\mathcal{C}) \to {}^{\mathrm{P}}\mathrm{Alg}_{/\mathbb{E}_0}(\mathcal{C}) \times \mathcal{C} = \mathrm{Alg}_{/\mathbb{E}_0}(\mathcal{C}) \times \mathcal{C}$$

is a trivial Kan fibration. In particular, for every $\mathbb{E}_0$-algebra object A of $\mathcal{C}$, the forgetful functor

$$\mathrm{Mod}_A^{\mathbb{E}_0}(\mathcal{C}) \to \mathcal{C}$$

is an equivalence of ∞-categories.

Proof. There is a canonical isomorphism of simplicial sets $\mathcal{K}_{\mathbb{E}_0} \times_{\mathcal{O}^{\otimes}} \{\langle 1 \rangle\} \simeq \mathbb{E}_0^{\otimes} \times \Delta^1$. Let $\mathcal{A}$ be the full subcategory of $\mathbb{E}_0^{\otimes} \times \Delta^1$ spanned by $(\langle 0 \rangle, 0)$ together with the objects $(\langle n \rangle, 1)$ for $n \geq 0$. Unwinding the definitions, we can identify $\mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\mathbb{E}_0 \times \Delta^1, \mathcal{C}^{\otimes})$ spanned by those functors F which satisfy the following conditions:

   (i) The restriction $F | \mathbb{E}_0^{\otimes} \times \{1\}$ is an $\mathbb{E}_0$-algebra object $\mathcal{C}^{\otimes}$.

   (ii) The functor F is a q-right Kan extension of $F | \mathcal{A}$.

Using Proposition T.4.3.2.15, we deduce that the restriction functor

$$\overline{\mathcal{M}\mathrm{od}}^{\mathbb{E}_0}(\mathcal{C}) \to \mathrm{Fun}_{\mathbb{E}_0}(\mathcal{A}, \mathcal{C}^{\otimes}) \times_{\mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\mathbb{E}_0^{\otimes} \times \{1\}, \mathcal{C})} \mathrm{Alg}_{/\mathbb{E}_0}(\mathcal{C})$$

is a trivial Kan fibration. It therefore suffices to show that the map

$$\mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\mathcal{A}, \mathcal{C}^{\otimes}) \to \mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\mathbb{E}_0^{\otimes} \times \{1\}, \mathcal{C}) \times \mathcal{C}$$

is a trivial Kan fibration. Since $\mathbb{E}_0^{\otimes}$ contains $\langle 0 \rangle$ as an initial object and $\mathcal{A}$ is isomorphic to the cone $(\mathbb{E}_0^{\otimes})^{\triangleleft}$, the inclusion

$$\mathcal{A}_0 = (\{\langle 0 \rangle\} \times \Delta^1) \coprod_{\{(\langle 0 \rangle, 1)\}} (\mathbb{E}_0^{\otimes} \times \{1\}) \subseteq \mathcal{A}$$

is a categorical equivalence. Since q is a categorical fibration, the restriction map

$$\mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\mathcal{A}, \mathcal{C}^{\otimes}) \to \mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\mathcal{A}_0, \mathcal{C}^{\otimes})$$

is a trivial Kan fibration. We are therefore reduced to proving that the restriction map

$$\phi : \mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\mathcal{A}_0, \mathcal{C}^{\otimes}) \to \mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\mathbb{E}_0^{\otimes} \times \{1\}, \mathcal{C}^{\otimes}) \times \mathcal{C}$$

is a trivial Kan fibration. The map ϕ is a pullback of the evaluation map

$$\mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\Delta^1, \mathcal{C}^{\otimes}) \to \mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\{0\}, \mathcal{C}^{\otimes}) = \mathcal{C},$$

which is a trivial Kan fibration by virtue of Proposition T.4.3.2.15 (since every object of $\mathcal{C}_{\langle 0 \rangle}^{\otimes}$ is q-final). $\qquad\square$

3.4   General Features of Module ∞-Categories

Let A be a commutative ring. Then the relative tensor product functor $(M, N) \mapsto M \otimes_A N$ endows the category Mod_A of A-modules with the structure of a symmetric monoidal category. This symmetric monoidal structure has the following features:

   (1) The category $\mathrm{CAlg}(\mathrm{Mod}_A)$ of commutative algebra objects of Mod_A is equivalent to the category of A-algebras: that is, the category whose objects are commutative rings B equipped with a ring homomorphism $A \to B$.

(2) Let $\mathcal{A}b$ denote the category of abelian groups. There is an evident forgetful functor $\theta : \mathrm{Mod}_A \to \mathcal{A}b$. When A is the ring of integers $\mathbf{Z}$, the functor θ is an equivalence of categories.

(3) The category Mod_A admits small limits and colimits. Moreover, the forgetful functor $\theta : \mathrm{Mod}_A \to \mathcal{A}b$ preserves all small limits and colimits.

Our goal in this section is to prove analogues of assertions (1), (2) and (3) in the ∞-categorical context. We will replace the symmetric monoidal category of abelian groups by an arbitrary fibration of ∞-operads $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ with $\mathcal{O}^{\otimes}$ coherent, the commutative ring A by an algebra object $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$, and the symmetric monoidal category Mod_A of A-modules by the ∞-operad fibration $q' : \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ defined in §3.3.3. We will prove the following:

(1′) There is an equivalence of ∞-categories $\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})) \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})_{A/}$ (Corollary 3.4.1.7).

(2′) There is a forgetful functor $\theta : \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$. If A is a trivial $\mathcal{O}$-algebra (Definition 3.2.1.7), then θ is an equivalence of ∞-operads (Proposition 3.4.2.1).

(3′) Let $X \in \mathcal{O}$ be an object such that the fiber $\mathcal{C}_X$ admits K-indexed limits, for some simplicial set K. Then the ∞-category $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})_X$ admits K-indexed limits. Moreover, θ restricts to a functor $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})_X \to \mathcal{C}_X$ which preserves K-indexed limits (Corollary 3.4.3.6).

We will prove assertions (1′) and (2′) in §3.4.1 and §3.4.2, respectively. Assertion (3′) will be deduced from a more general assertion concerning limits *relative* to the forgetful functor $q' : \mathrm{Mod}^{\mathcal{O}}(\mathcal{C}) \to \mathcal{O}^{\otimes} \times \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ (Theorem 3.4.3.1) which we prove in §3.4.3. There is an analogous statement for relative colimits (Theorem 3.4.4.3), which we will prove in §3.4.4. However, both the statement and the proof are considerably more involved: we must assume not only that the relevant colimits exist in the underlying ∞-category $\mathcal{C}$, but that they are *operadic* colimits in the sense of §3.1.1. Nevertheless, we will be able to use Theorem 3.4.4.3 to establish an analogue of (3′) for colimit diagrams, under some mild assumptions on the fibration $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ (see, for example, Theorem 3.4.4.2).

3.4.1 Algebra Objects of ∞-Categories of Modules

Let $\mathcal{C}$ be a symmetric monoidal category, let A be a commutative algebra object of $\mathcal{C}$, and let $\mathcal{D} = \mathrm{Mod}_A(\mathcal{C})$ be the category of A-modules in $\mathcal{C}$. Under some mild hypotheses, the category $\mathcal{D}$ inherits the structure of a symmetric monoidal category. Moreover, one can show the following:

(1) The forgetful functor $\theta : \mathcal{D} \to \mathcal{C}$ induces an equivalence of categories from the category of commutative algebra objects $\mathrm{CAlg}(\mathcal{D})$ to the category $\mathrm{CAlg}(\mathcal{C})_{A/}$ of commutative algebra objects $A' \in \mathrm{CAlg}(\mathcal{D})$ equipped with a map $A \to A'$.

(2) Given a commutative algebra object $B \in \mathrm{CAlg}(\mathcal{D})$, the category of B-modules in $\mathcal{D}$ is equivalent to the category of $\theta(B)$-modules in $\mathcal{C}$.

Our goal in this section is to obtain ∞-categorical analogues of the above statements for algebras over an arbitrary coherent ∞-operad (Corollaries 3.4.1.7 and 3.4.1.9). Before we can state our results, we need to introduce a bit of terminology.

Notation 3.4.1.1. Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad, and let $\mathcal{K}_{\mathcal{O}} \subseteq \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$ be defined as in Notation 3.3.2.1. We let $\mathcal{O}_*^{\otimes}$ denote the ∞-category of pointed objects of $\mathcal{O}^{\otimes}$: that is, the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$ spanned by those morphisms $X \to Y$ such that X is a final object of $\mathcal{O}^{\otimes}$ (which is equivalent to the requirement that $p(X) = \langle 0 \rangle$). If $\mathcal{O}^{\otimes}$ is unital, then a diagram $\Delta^1 \to \mathcal{O}^{\otimes}$ belongs to $\mathcal{O}_*^{\otimes} \subseteq \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$ is and only if it is a left Kan extension of its restriction to $\{1\}$. In this case, Proposition T.4.3.2.15 implies that evaluation at $\{1\}$ induces a trivial Kan fibration $e : \mathcal{O}_*^{\otimes} \to \mathcal{O}^{\otimes}$. We let $s : \mathcal{O}^{\otimes} \to \mathcal{O}_*^{\otimes}$ denote a section of e, and regard s as a functor from $\mathcal{O}^{\otimes}$ to $\mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$. We observe that there is a canonical natural transformation

$s \to \delta$, where δ is the diagonal embedding $\mathcal{O}^\otimes \to \mathrm{Fun}(\Delta^1, \mathcal{O}^\otimes)$. We regard this natural transformation as defining a map

$$\gamma_{\mathcal{O}\otimes} : \mathcal{O}^\otimes \times \Delta^1 \to \mathcal{K}_\mathcal{O} .$$

Remark 3.4.1.2. More informally, the map $\gamma_{\mathcal{O}\otimes}$ can be described as follows. If $X \in \mathcal{O}^\otimes$, then

$$\gamma_{\mathcal{O}\otimes}(X, i) = \begin{cases} \mathrm{id}_X \in \mathcal{K}_\mathcal{O} & \text{if } i = 1 \\ (f : 0 \to X) \in \mathcal{K}_\mathcal{O} & \text{if } i = 0. \end{cases}$$

Here 0 denotes a zero object of $\mathcal{O}^\otimes$.

Let $\mathcal{O}^\otimes$ be a coherent ∞-operad and let $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a fibration of generalized ∞-operads. Unwinding the definitions, we see that giving an $\mathcal{O}$-algebra in $\overline{\mathcal{M}\mathrm{od}}^\mathcal{O}(\mathcal{C})^\otimes$ is equivalent to giving a commutative diagram of simplicial sets

$$\begin{array}{ccc} \mathcal{K}_\mathcal{O} & \xrightarrow{\quad f \quad} & \mathcal{C}^\otimes \\ & {\scriptstyle e_1} \searrow & \downarrow \\ & & \mathcal{O}^\otimes . \end{array}$$

such that f preserves inert morphisms. Composing with the map $\gamma_\mathcal{O}$ of Notation 3.4.1.1, we obtain a map

$$\mathrm{Alg}_{/\mathcal{O}}(\overline{\mathcal{M}\mathrm{od}}^\mathcal{O}(\mathcal{C})) \to \mathrm{Fun}_{\mathcal{O}\otimes}(\mathcal{O}^\otimes \times \Delta^1, \mathcal{C}^\otimes).$$

Our main results can be stated as follows:

Proposition 3.4.1.3. *Let $\mathcal{O}^\otimes$ be a coherent ∞-operad, and let $q : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a fibration of ∞-operads. Then the construction above determines a categorical equivalence*

$$\mathrm{Alg}_{/\mathcal{O}}(\overline{\mathcal{M}\mathrm{od}}^\mathcal{O}(\mathcal{C})) \to \mathrm{Fun}(\Delta^1, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})).$$

Proposition 3.4.1.4. *Let $\mathcal{O}^\otimes$ be a coherent ∞-operad, let $p : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a fibration of ∞-operads, and let $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$. Then the forgetful functor $\mathrm{Mod}_A^\mathcal{O}(\mathcal{C})^\otimes \to \mathcal{C}^\otimes$ induces a homotopy pullback diagram of ∞-categories*

$$\begin{array}{ccc} \mathrm{Mod}^\mathcal{O}(\mathrm{Mod}_A^\mathcal{O}(\mathcal{C}))^\otimes & \longrightarrow & \mathrm{Mod}^\mathcal{O}(\mathcal{C})^\otimes \\ \downarrow & & \downarrow \\ \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}_A^\mathcal{O}(\mathcal{C})) \times \mathcal{O}^\otimes & \longrightarrow & \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \times \mathcal{O}^\otimes . \end{array}$$

We defer the proofs of Proposition 3.4.1.3 and 3.4.1.4 until the end of this section.

Corollary 3.4.1.5. *Let $\mathcal{O}^\otimes$ be a coherent ∞-operad and let $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a fibration of ∞-operads. The composition*

$$\theta : \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}^\mathcal{O}(\mathcal{C})) \to \mathrm{Alg}_{/\mathcal{O}}(\overline{\mathcal{M}\mathrm{od}}^\mathcal{O}(\mathcal{C})) \to \mathrm{Fun}(\Delta^1, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}))$$

is an equivalence of ∞-categories.

Proof. Combine Proposition 3.4.1.3 with Remark 3.3.3.16. $\square$

Remark 3.4.1.6. Let $\theta : \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}^\mathcal{O}(\mathcal{C})) \to \mathrm{Fun}(\Delta^1, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}))$ be the categorical equivalence of Corollary 3.4.1.5. Composing with the map $\mathrm{Fun}(\Delta^1, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})) \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ given by evaluation at $\{0\}$, we obtain a map $\theta_0 : \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}^\mathcal{O}(\mathcal{C})) \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$. Unwinding the definitions, we see that θ_0 factors as a composition

$$\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}^\mathcal{O}(\mathcal{C})) \xrightarrow{\theta_0'} \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \times \mathcal{O}) \xrightarrow{\theta_0''} \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}),$$

where θ_0'' is induced by composition with the diagonal embedding $\mathcal{O}^\otimes \to \mathcal{O}^\otimes \times \mathcal{O}^\otimes$. In particular, if A is a $\mathcal{O}$-algebra object of $\mathcal{C}$, then the the restriction of θ_0 to $\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}))$ is a constant map taking the value $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$.

Corollary 3.4.1.7. *Let $\mathcal{O}^\otimes$ be a coherent ∞-operad, let $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a fibration of ∞-operads, and let $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ be a $\mathcal{O}$-algebra object of $\mathcal{C}$. Then the categorical equivalence θ of Corollary 3.4.1.5 restricts to a categorical equivalence*

$$\theta_A : \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})) \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{A/}.$$

Proof. Remark 3.4.1.6 guarantees that the restriction of θ carries $\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}))$ into

$$\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{A/} \subseteq \mathrm{Fun}(\Delta^1, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{A/}).$$

Consider the diagram

$$
\begin{array}{ccccc}
\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})) & \longrightarrow & \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}^{\mathcal{O}}(\mathcal{C})) & \xrightarrow{\theta} & \mathrm{Fun}(\Delta^1, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})) \\
\downarrow & & \downarrow & & \downarrow \\
\Delta^0 & \longrightarrow & \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \times \mathcal{O}) & \xrightarrow{\theta'} & \mathrm{Fun}(\{0\}, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})).
\end{array}
$$

The left square is a homotopy pullback, since it is a pullback square between fibrant objects in which the vertical maps are categorical fibrations. The right square is a homotopy pullback since both of the horizontal arrows are categorical equivalences (θ is a categorical equivalence by virtue of Corollary 3.4.1.5, and θ' is a categorical equivalence since it is left inverse to a categorical equivalence). It follows that the outer square is also a homotopy pullback, which is equivalent to the assertion that θ_A is a categorical equivalence (Proposition T.3.3.1.3). $\square$

Corollary 3.4.1.8. *Let $\mathcal{O}^\otimes$ be a coherent ∞-operad, $p : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ a fibration of ∞-operads, and $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ an algebra object. Then there is a canonical equivalence of ∞-categories*

$$\mathrm{Mod}^{\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}))^\otimes \to \mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^\otimes \times_{\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})} \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{A/}.$$

Proof. Combine Proposition 3.4.1.4 with Corollary 3.4.1.7. $\square$

Corollary 3.4.1.9. *Let $\mathcal{O}^\otimes$ be a coherent ∞-operad, $p : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ a fibration of ∞-operads. Let $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$, let $\overline{B} \in \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}))$, and let $B \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ be the algebra object determined by $\overline{B}$. Then there is a canonical equivalence of ∞-operads*

$$\mathrm{Mod}_{\overline{B}}^{\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}))^\otimes \to \mathrm{Mod}_B^{\mathcal{O}}(\mathcal{C})^\otimes.$$

We now turn to the proof of Propositions 3.4.1.3 and 3.4.1.4. We will need a few preliminary results.

Lemma 3.4.1.10. *Suppose we are given a commutative diagram of ∞-categories*

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
& {}_p \searrow \quad \swarrow {}_q & \\
& \mathcal{E} &
\end{array}
$$

where p and q are Cartesian fibrations and the map F carries p-Cartesian morphisms to q-Cartesian morphisms. Let $D \in \mathcal{D}$ be an object, let $E = q(D)$, and let $\mathcal{C}_{D/} = \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$. Then:

(1) *The induced map $p' : \mathcal{C}_{D/} \to \mathcal{E}_{E/}$ is a Cartesian fibration.*

(2) *A morphism f in $\mathcal{C}_{D/}$ is p'-Cartesian if and only if its image in $\mathcal{C}$ is p-Cartesian.*

Proof. Let us say that a morphism in $\mathcal{C}_{D/}$ is *special* if its image in $\mathcal{C}$ is p-Cartesian. We first prove the "if" direction of (2) by showing that every special morphism of $\mathcal{C}_{D/}$ is p-Cartesian; we will simultaneously show that p' is an inner fibration. For this, we must show that every lifting problem of the form

$$
\begin{array}{ccc}
\Lambda_i^n & \xrightarrow{g_0} & \mathcal{C}_{D/} \\
\downarrow & {\scriptstyle g}\nearrow & \downarrow \\
\Delta^n & \longrightarrow & \mathcal{E}_{E/}
\end{array}
$$

admits a solution, provided that $n \geq 2$ and either $0 < i < n$ or $i = n$ and g_0 carries $\Delta^{\{n-1,n\}}$ to a special morphism e in $\mathcal{C}_{D/}$. To prove this, we first use the fact that p is an inner fibration (together with the observation that the image e_0 of e in $\mathcal{C}$ is p-Cartesian when $i = n$) to solve the associated lifting problem

$$
\begin{array}{ccc}
\Lambda_i^n & \xrightarrow{g_0'} & \mathcal{C} \\
\downarrow & {\scriptstyle g'}\nearrow & \downarrow \\
\Delta^n & \longrightarrow & \mathcal{E} \, .
\end{array}
$$

To extend this to solution of our original lifting problem, we are required to solve another lifting problem of the form

$$
\begin{array}{ccc}
\Lambda_{i+1}^{n+1} & \xrightarrow{g_0''} & \mathcal{D} \\
\downarrow & {\scriptstyle g''}\nearrow & \downarrow {\scriptstyle q} \\
\Delta^{n+1} & \longrightarrow & \mathcal{E} \, .
\end{array}
$$

If $i < n$, the desired solution exists by virtue of our assumption that q is an inner fibration. If $i = n$, then it suffices to observe that $g_0''(\Delta^{\{n,n+1\}}) = F(e_0)$ is a q-Cartesian morphism in $\mathcal{D}$.

To prove (1), it will suffice to show that for every object $\overline{C} \in \mathcal{C}_{D/}$ and every morphism $\overline{f}_0 : \overline{E}' \to p'(\overline{C})$ in $\mathcal{E}_{E/}$, there exists a special morphism $\overline{f}$ in $\mathcal{C}_{D/}$ with $p'(\overline{f}) = \overline{f}_0$. We can identify $\overline{C}$ with an object $C \in \mathcal{C}$ together with a morphism $\alpha : D \to F(C)$ in $\mathcal{D}$, and we can identify $\overline{f}_0$ with a 2-simplex

$$
\begin{array}{ccc}
 & E' & \\
 \nearrow & & \searrow {\scriptstyle f_0} \\
E & \xrightarrow{\quad q(\alpha) \quad} & p(C)
\end{array}
$$

in $\mathcal{E}$. Since p is a Cartesian fibration, we can choose a p-Cartesian morphism $f : C' \to C$ with $p(f) = f_0$. In order to lift f to a special morphism $\overline{f} : \overline{C}' \to \overline{C}$, it suffices to complete the diagram

$$
\begin{array}{ccc}
 & F(C') & \\
 \nearrow & & \searrow {\scriptstyle F(f_0)} \\
D & \xrightarrow{\quad \alpha \quad} & F(C)
\end{array}
$$

to a 2-simplex of $\mathcal{D}$. This is possibly by virtue of our assumption that $F(f_0)$ is q-Cartesian.

To complete the proof of (2), it will suffice to show that every p'-Cartesian morphism $\overline{f}' : \overline{C}'' \to \overline{C}$ of $\mathcal{C}_{D/}$ is special. The proof of (1) shows that there exists a special morphism $\overline{f} : \overline{C}' \to \overline{C}$ with $p'(\overline{f}) = p'(\overline{f}')$. Since $\overline{f}$ is also p'-Cartesian, it is equivalent to $\overline{f}'$, so that $\overline{f}'$ is also special. $\qquad\square$

Lemma 3.4.1.11. *Suppose given a commutative diagram of ∞-categories*

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\quad F \quad} & \mathcal{D} \\
 & {}_{p}\searrow \quad \swarrow_{q} & \\
 & \mathcal{E} &
\end{array}
$$

where p and q are Cartesian fibrations, and the map F carries p-Cartesian morphisms to q-Cartesian morphisms. Suppose furthermore that for every object $E \in \mathcal{E}$, the induced functor $F_E : \mathcal{C}_E \to \mathcal{D}_E$ is left cofinal. Then F is left cofinal.

Proof. In view of Theorem T.4.1.3.1, it suffices to show that for each object $D \in \mathcal{D}$, the simplicial set $\mathcal{C}_{D/} = \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$ is weakly contractible. Let E denote the image of D in $\mathcal{E}$; we observe that $\mathcal{C}_{D/}$ comes equipped with a map $p' : \mathcal{C}_{D/} \to \mathcal{E}_{E/}$. Moreover, the fiber of p' over the initial object $\mathrm{id}_E \in \mathcal{E}_{E/}$ can be identified with $\mathcal{C}_E \times_{\mathcal{D}_E} (\mathcal{D}_E)_{D/}$, which is weakly contractible by virtue of our assumption that F_E is left cofinal (Theorem T.4.1.3.1). To prove that $\mathcal{C}_{D/}$ is contractible, it will suffice to show that the inclusion $i : p'^{-1}\{\mathrm{id}_E\} \hookrightarrow \mathcal{C}_{D/}$ is a weak homotopy equivalence. We will prove something slightly stronger: the inclusion i is right cofinal. Since the inclusion $\{\mathrm{id}_E\} \to \mathcal{E}_{E/}$ is evidently right cofinal, it will suffice to show that p' is a Cartesian fibration (Proposition T.4.1.2.15). This follows from Lemma 3.4.1.10. $\qquad\square$

Lemma 3.4.1.12. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, and let $\alpha : \langle n \rangle \to \langle m \rangle$ be an inert morphism in $\mathfrak{Fin}_*$. Let σ denote the diagram*

$$
\begin{array}{ccc}
\langle n \rangle & \xrightarrow{\quad \alpha \quad} & \langle m \rangle \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
\langle n \rangle & \xrightarrow{\quad \alpha \quad} & \langle m \rangle
\end{array}
$$

in $\mathrm{N}(\mathfrak{Fin}_*)$, let $K \simeq \Lambda_2^2$ be the full subcategory of $\Delta^1 \times \Delta^1$ obtained by omitting the initial vertex, and let $\sigma_0 = \sigma|K$. Suppose that $\overline{\sigma}_0 : K \to \mathcal{C}^{\otimes}$ is a diagram lifting σ_0, corresponding to a commutative diagram

$$
\begin{array}{ccc}
 & & X \\
 & & \downarrow \\
Y' & \xrightarrow{\quad \overline{\alpha}' \quad} & X'
\end{array}
$$

where $\overline{\alpha}'$ is inert. Then:

(1) *Let $\overline{\sigma} : \Delta^1 \times \Delta^1 \to \mathcal{C}^{\otimes}$ be an extension of $\overline{\sigma}_0$ lifting σ, corresponding to a commutative diagram*

$$
\begin{array}{ccc}
Y & \xrightarrow{\quad \overline{\alpha} \quad} & X \\
{\scriptstyle \overline{\beta}}\downarrow & & \downarrow \\
Y' & \xrightarrow{\quad \overline{\alpha}' \quad} & X'
\end{array}
$$

in $\mathcal{C}^{\otimes}$. Then σ is a q-limit diagram if and only if it satisfies the following conditions:

   (i) *The map $\overline{\alpha}$ is inert.*

   (ii) *Let $\gamma : \langle n \rangle \to \langle k \rangle$ be an inert morphism in $\mathrm{N}(\mathfrak{Fin}_*)$ such that $\alpha^{-1}\langle m \rangle^{\circ} \subseteq \gamma^{-1}\{*\}$, and let $\overline{\gamma} : Y' \to Z$ be an inert morphism in $\mathcal{C}^{\otimes}$ lifting γ. Then $\overline{\gamma} \circ \overline{\beta} : Y \to Z$ is an inert morphism in $\mathcal{C}^{\otimes}$.*

(2) *There exists an extension $\overline{\sigma}$ of $\overline{\sigma}_0$ lying over σ which satisfies conditions (i) and (ii) of (1).*

Proof. This is a special case of Lemma 3.4.3.15.

$\square$

Proof of Proposition 3.4.1.3. We define a simplicial set $\mathcal{M}$ equipped with a map $p : \mathcal{M} \to \Delta^1$ so that the following universal property is satisfied: for every map of simplicial sets $K \to \Delta^1$, the set $\text{Hom}_{(\text{Set}_\Delta)_{/\Delta^1}}(K, \mathcal{M})$ can be identified with the collection of all commutative diagrams

$$
\begin{array}{ccc}
K \times_{\Delta^1} \{1\} & \longrightarrow & \mathcal{O}^\otimes \times \Delta^1 \\
\downarrow & & \downarrow{\scriptstyle \gamma_\mathcal{O}} \\
K & \longrightarrow & \mathcal{K}_\mathcal{O} \, .
\end{array}
$$

The map p is a Cartesian fibration, associated to the functor $\gamma_\mathcal{O}$ from $\mathcal{M}_1 \simeq \mathcal{O}^\otimes \times \Delta^1$ to $\mathcal{M}_0 \simeq \mathcal{K}_\mathcal{O}$. We observe that $\mathcal{M}$ is equipped with a functor $\mathcal{M} \to \mathcal{O}^\otimes$, whose restriction to $\mathcal{M}_1 \simeq \mathcal{O}^\otimes \times \Delta^1$ is given by projection onto the first factor and whose restriction to $\mathcal{M}_0 \simeq \mathcal{K}_\mathcal{O}$ is given by evaluation at $\{1\}$.

We let $\mathcal{X}$ denote the full subcategory of $\text{Fun}_{\mathcal{O}^\otimes}(\mathcal{M}, \mathcal{C}^\otimes)$ spanned by those functors F satisfying the following pair of conditions:

(i) The functor F is a q-left Kan extension of $F|\mathcal{M}_0$.

(ii) The restriction $F|\mathcal{M}_0 \in \text{Fun}_{\mathcal{O}^\otimes}(\mathcal{K}_\mathcal{O}, \mathcal{C}^\otimes)$ belongs to $\text{Alg}_{/\mathcal{O}}(\overline{\text{Mod}}^\mathcal{O}(\mathcal{C}))$.

Since p is a Cartesian fibration, condition (i) can be reformulated as follows:

(i') For every p-Cartesian morphism f in $\mathcal{M}$, the image $F(f)$ is a q-coCartesian morphism in $\mathcal{C}^\otimes$. Since the image of f in $\mathcal{O}^\otimes$ is an equivalence, this is equivalent to the requirement that $F(f)$ is an equivalence in $\mathcal{C}^\otimes$.

Using Proposition T.4.3.2.15, we deduce that the restriction map $\mathcal{X} \to \text{Alg}_{/\mathcal{O}}(\overline{\text{Mod}}^\mathcal{O}(\mathcal{C}))$ is a trivial Kan fibration. This restriction map has a section s, given by composition with the natural retraction $r : \mathcal{M} \to \mathcal{M}_0$. It follows that s is a categorical equivalence, and that every object $F \in \mathcal{X}$ is equivalent $(F|\mathcal{M}_0) \circ r$. We deduce that restriction to $\mathcal{M}_1 \subseteq \mathcal{M}$ induces a functor $\theta' : \mathcal{X} \to \text{Fun}(\Delta^1, \text{Alg}_{/\mathcal{O}}(\mathcal{C}))$. We have a commutative diagram

$$
\begin{array}{ccc}
 & \mathcal{X} & \\
{\scriptstyle s}\nearrow & & \searrow{\scriptstyle \theta'} \\
\text{Alg}_{/\mathcal{O}}(\overline{\text{Mod}}^\mathcal{O}(\mathcal{C})) & \xrightarrow{\quad \theta \quad} & \text{Fun}(\Delta^1, \text{Alg}_{/\mathcal{O}}(\mathcal{C})).
\end{array}
$$

To complete the proof, it will suffice to show that θ' is a categorical equivalence. We will show that θ' is a trivial Kan fibration. In view of Proposition T.4.3.2.15, it will suffice to prove the following:

(a) A functor $F \in \text{Fun}_{\mathcal{O}^\otimes}(\mathcal{M}, \mathcal{C}^\otimes)$ belongs to $\mathcal{X}$ if and only if $F_1 = F|\mathcal{M}_1 \in \text{Fun}(\Delta^1, \text{Alg}_{/\mathcal{O}}(\mathcal{C}))$ and F is a q-right Kan extension of F_1.

(b) Every object $F_1 \in \text{Fun}_{\mathcal{O}^\otimes}(\mathcal{M}_1, \mathcal{C}^\otimes)$ belonging to $\text{Fun}(\Delta^1, \text{Alg}_{/\mathcal{O}}(\mathcal{C}))$ admits a q-right Kan extension of $F \in \text{Fun}_{\mathcal{O}^\otimes}(\mathcal{M}, \mathcal{C}^\otimes)$.

To prove these claims, we will need a criterion for detecting whether a functor $F \in \text{Fun}_{\mathcal{O}^\otimes}(\mathcal{M}, \mathcal{C}^\otimes)$ is a q-right Kan extension of $F_1 = F|\mathcal{M}_1 \in \text{Fun}(\Delta^1, \text{Alg}_{/\mathcal{O}}(\mathcal{C}))$ at an object $\overline{X} \in \mathcal{M}_0$. Let $\overline{X}$ correspond to a semi-inert morphism $\overline{\alpha} : X' \to X$ in $\mathcal{O}^\otimes$, covering a morphism $\alpha : \langle m \rangle \to \langle n \rangle$ in $\text{N}(\mathcal{F}\text{in}_*)$. Let $\mathcal{D}$ denote the

∞-category $(\mathcal{O}^{\otimes} \times \Delta^1) \times_{\mathcal{M}} \mathcal{M}_{\overline{X}/}$, so that $\mathcal{D}$ is equipped with a projection $\mathcal{D} \to \Delta^1$; we let $\mathcal{D}_0$ and $\mathcal{D}_1$ denote the fibers of this map. Form a pushout diagram

$$
\begin{array}{ccc}
\langle m \rangle & \xrightarrow{\alpha} & \langle n \rangle \\
\downarrow & & \downarrow{\scriptstyle \beta} \\
\langle 0 \rangle & \longrightarrow & \langle k \rangle
\end{array}
$$

in $\mathcal{F}\mathrm{in}_*$, and choose an inert morphism $\overline{\beta} : X' \to Y$ in $\mathcal{O}^{\otimes}$ lying over β. Let $\overline{Y}$ denote the image of Y under our chosen section $s : \mathcal{O}^{\otimes} \to \mathcal{O}_*^{\otimes}$, so we can identify $\overline{Y}$ with a morphism $0 \to Y$ in $\mathcal{O}^{\otimes}$, where 0 is a zero object of $\mathcal{O}^{\otimes}$. We can therefore lift $\overline{\beta}$ to a morphism $\widetilde{\beta} : \overline{X} \to \overline{Y}$ in $\mathcal{K}_{\mathcal{O}}$, corresponding to a commutative diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{\overline{\alpha}} & X \\
\downarrow & & \downarrow{\scriptstyle \overline{\beta}} \\
0 & \longrightarrow & Y.
\end{array}
$$

The pair $(Y, \widetilde{\beta})$ can be identified with an object of $\mathcal{N}_0$, which we will denote by $\widetilde{Y}$. We claim that $\widetilde{Y}$ is an initial object of $\mathcal{N}_0$. Unwinding the definitions, this is equivalent to the following assertion: for every object $A \in \mathcal{O}^{\otimes}$, composition with $\widetilde{\beta}$ induces a homotopy equivalence

$$
\phi : \mathrm{Map}_{\mathcal{O}^{\otimes}}(Y, A) \to \mathrm{Map}_{\mathcal{K}_{\mathcal{O}}}(\overline{X}, s(A)).
$$

To prove this, we observe that ϕ factors as a composition

$$
\mathrm{Map}_{\mathcal{O}^{\otimes}}(Y, A) \xrightarrow{\phi'} \mathrm{Map}_{\mathcal{K}_{\mathcal{O}}}(\overline{Y}, s(A)) \xrightarrow{\phi''} \mathrm{Map}_{\mathcal{K}_{\mathcal{O}}}(\overline{X}, s(A)).
$$

The map ϕ' is a homotopy equivalence because s is a categorical equivalence, and the map ϕ'' is a homotopy equivalence because $\overline{\beta}$ is coCartesian with respect to the projection $\mathcal{K}_{\mathcal{O}} \to \mathcal{K}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}$.

We have an evident natural transformation $\widetilde{\gamma} : \overline{X} \to \mathrm{id}_X$ in $\mathcal{K}_{\mathcal{O}}$. The pair $(X, \widetilde{\gamma})$ determines an object $\widetilde{Z} \in \mathcal{N}_1$. We claim that $\widetilde{Z}$ is an initial object of $\mathcal{N}_1$. Unwinding the definitions, we see that this is equivalent to the assertion that for every object $A \in \mathcal{O}^{\otimes}$, composition with $\widetilde{\gamma}$ induces a homotopy equivalence

$$
\psi : \mathrm{Map}_{\mathcal{O}^{\otimes}}(X, A) \to \mathrm{Map}_{\mathcal{K}_{\mathcal{O}}}(\overline{X}, \delta(A)),
$$

where $\delta : \mathcal{O}^{\otimes} \to \mathcal{K}_{\mathcal{O}}$ is the diagonal embedding. To prove this, we factor ψ as a composition

$$
\mathrm{Map}_{\mathcal{O}^{\otimes}}(X, A) \xrightarrow{\psi'} \mathrm{Map}_{\mathcal{K}_{\mathcal{O}}}(\delta(X), \delta(A)) \xrightarrow{\psi''} \mathrm{Map}_{\mathcal{K}_{\mathcal{O}}}(\overline{X}, \delta(A)).
$$

The map ψ' is a homotopy equivalence since δ is fully faithful, and the map ψ'' is a homotopy equivalence by virtue of Corollary T.5.2.8.18 (applied to the trivial factorization system on $\mathcal{O}^{\otimes}$).

Since $\mathcal{N} \to \mathcal{O}^{\otimes} \times \Delta^1$ is a left fibration, we can lift the map $(Y, 0) \to (Y, 1)$ to a map $e : \widetilde{Y} \to \widetilde{Y}'$ in $\mathcal{N}$. Since $\widetilde{Z}$ is an initial object of $\mathcal{N}_1$, we can choose a map $e' : \widetilde{Z} \to \widetilde{Y}$ in $\mathcal{N}_1$. Let $C \simeq \Lambda_2^2$ denote the full subcategory of $\Delta^1 \times \Delta^1$ obtained by omitting the final vertex, so that e and e' together determine a map of simplicial sets $C \to \mathcal{N}$. Applying the dual of Lemma 3.4.1.11 to the diagram

$$
\begin{array}{ccc}
C & \longrightarrow & \mathcal{N} \\
& \searrow \quad \swarrow & \\
& \Delta^1,
\end{array}
$$

we deduce that $C \to \mathcal{N}$ is right cofinal. We therefore arrive at the following:

($*$) A functor $F \in \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{M}, \mathcal{C}^{\otimes})$ is a q-right Kan extension of $F_1 = F | \mathcal{M}_1 \in \mathrm{Fun}(\Delta^1, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}))$ at an object $\overline{X} \in \mathcal{K}_{\mathcal{O}}$ if and only if the induced diagram

$$
\begin{array}{ccc}
F(\overline{X}) & \longrightarrow & F_1(X,1) \\
\downarrow & & \downarrow \\
F_1(Y,0) & \longrightarrow & F_1(Y,1)
\end{array}
$$

is a q-limit diagram.

Moreover, Lemma T.4.3.2.13 yields the following:

($*'$) A functor $F_1 \in \mathrm{Fun}(\Delta^1, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}))$ admits a q-right Kan extension $F \in \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{M}, \mathcal{C}^{\otimes})$ if and only if, for every object $\overline{X} \in \mathcal{K}_{\mathcal{O}}$, the diagram

$$
\begin{array}{ccc}
& & F_1(X,1) \\
& & \downarrow \\
F_1(Y,0) & \longrightarrow & F_1(Y,1)
\end{array}
$$

can be extended to a q-limit diagram lying over the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\;\mathrm{id}\;} & X \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\;\mathrm{id}\;} & Y.
\end{array}
$$

Assertion (b) follows immediately from ($*'$) together with Lemma 3.4.1.12. Combining assertion ($*$) with Lemma 3.4.1.12, we deduce that a functor $F \in \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{M}, \mathcal{C}^{\otimes})$ is a q-right Kan extension of $F_1 = F | \mathcal{M}_1 \in \mathrm{Fun}(\Delta^1, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}))$ if and only if the following conditions are satisfied:

(i') The restriction F_1 belongs to $\mathrm{Fun}(\Delta^1, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}))$. That is, F carries every inert morphism in $\mathcal{O}^{\otimes} \times \{j\} \subseteq \mathcal{M}_1$ to an inert morphism in $\mathcal{C}^{\otimes}$, for $j \in \{0,1\}$.

(ii') Let $\overline{X}$ be as above. Then the induced morphism $F(\overline{X}) \to F(Y,0)$ is inert.

(iii') Let $\overline{X}$ be as above, and suppose that we are given an inert morphism $\alpha' : \langle n \rangle \to \langle l \rangle$ such that the composite map $\alpha' \circ \alpha : \langle m \rangle \to \langle n' \rangle$ is surjective together with an inert morphism $\overline{\alpha}' : X \to X''$ lifting α'. Then the composite map $F(\overline{X}) \to F(X,1) \to F(X'',1)$ is an inert morphism in $\mathcal{C}^{\otimes}$.

To complete the proof, it will suffice to show that a functor $F \in \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{M}, \mathcal{C}^{\otimes})$ satisfies conditions (i) and (ii) if and only if it satisfies conditions (i'), (ii'), and (iii').

Suppose first that F satisfies (i) and (ii). We have already seen that F must also satisfy (i'). To prove (ii'), we observe that the map $F(\overline{X}) \to F(Y,0)$ factors as a composition

$$ F(\overline{X}) \to F(\overline{Y}) \to F(Y,0). $$

The first map is inert because F satisfies (ii) and $\widetilde{\beta} : \overline{X} \to \overline{Y}$ is an inert morphism in $\mathcal{K}_{\mathcal{O}}$. The second morphism is inert by virtue of assumption (i). Now suppose that we are given an inert morphism $X \to X''$ as in (iii'). We have a commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow & & \downarrow \\
X'' & \xrightarrow{\;\mathrm{id}\;} & X''
\end{array}
$$

corresponding to an inert morphism $\overline{X} \to \delta(X'')$ in $\mathcal{K}_{\mathcal{O}}$, and the map $F(\overline{X}) \to F(X'', 1)$ factors as a composition

$$F(\overline{X}) \to F(\delta(X'')) \to F(X'', 1).$$

The first of these maps is inert by virtue of assumption (ii), and the second by virtue of assumption (i).

Now suppose that F satisfies (i'), (ii'), and (iii'); we wish to show that F satisfies (i) and (ii). To prove (i), we must show that for every object $X \in \mathcal{O}^{\otimes}$, the morphisms $F(s(X)) \to F(X, 0)$ and $F(\delta(X)) \to F(X, 1)$ are inert in $\mathcal{C}$. The first of these assertions is a special case of (ii'), and the second is a special case of (iii'). To prove (ii), consider an arbitrary inert morphism $\overline{\beta} : \overline{X} \to \overline{Y}$ in $\mathcal{O}^{\otimes}$, corresponding to a commutative diagram σ:

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \beta} \\
Y' & \longrightarrow & Y
\end{array}
$$

in the ∞-category $\mathcal{O}^{\otimes}$. We wish to show that $F(\overline{\beta})$ is an inert morphism in $\mathcal{C}^{\otimes}$. Let $\beta_0 : \langle n \rangle \to \langle k \rangle$ be the image of β in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, and let $(\beta_0)_! : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\otimes}_{\langle k \rangle}$ denote the induced functor. Then $F(\overline{\beta})$ factors as a composition $F(\overline{X}) \xrightarrow{\epsilon} (\beta_0)_! F(\overline{X}) \xrightarrow{\epsilon'} F(\overline{Y})$, where ϵ is inert; we wish to prove that ϵ' is an equivalence in $\mathcal{C}^{\otimes}_{\langle k \rangle}$. Since $\mathcal{C}^{\otimes}$ is an ∞-operad, it will suffice to show that $\rho_!^j \epsilon'$ is an equivalence in $\mathcal{C}^{\otimes}_{\langle 1 \rangle}$ for $1 \le j \le k$. Let

$$
\begin{array}{ccc}
\langle n \rangle & \longrightarrow & \langle m \rangle \\
\downarrow & & \downarrow{\scriptstyle \beta_0} \\
\langle k' \rangle & \longrightarrow & \langle k \rangle
\end{array}
$$

denote the image of σ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. This diagram admits a unique extension

$$
\begin{array}{ccc}
\langle n \rangle & \longrightarrow & \langle m \rangle \\
\downarrow & & \downarrow{\scriptstyle \beta_0} \\
\langle k' \rangle & \xrightarrow{\chi} & \langle k \rangle \\
\downarrow & & \downarrow{\scriptstyle \rho^j} \\
\langle t \rangle & \xrightarrow{\chi'} & \langle 1 \rangle
\end{array}
$$

where the vertical morphisms are inert, the integer t is equal to 1 and χ' is an isomorphism if j lies in the image of $\langle n \rangle \to \langle k \rangle$, and $t = 0$ otherwise. We can lift this diagram to a commutative triangle

$$
\begin{array}{ccc}
\overline{X} & \xrightarrow{\overline{\beta}} & \overline{Y} \\
& {\scriptstyle \overline{\beta}'}\searrow \quad \nearrow{\scriptstyle \overline{\beta}''} & \\
& \overline{Z} &
\end{array}
$$

of inert morphisms in $\mathcal{K}_{\mathcal{O}}$. If $F(\overline{\beta}'')$ is inert, then we can identify $F(\overline{\beta}')$ with the composition

$$F(\overline{X}) \longrightarrow (\rho^j \circ \beta_0)_! F(\overline{X}) \xrightarrow{\rho_!^j \epsilon'} F(\overline{Z}),$$

so that $\rho_!^j \epsilon'$ is an equivalence in $\mathcal{C}^{\otimes}_{\langle 1 \rangle}$ if and only if $F(\overline{\beta}')$ is inert. We are therefore reduced to proving that $F(\overline{\beta}')$ and $F(\overline{\beta}'')$ are inert. Replacing $\overline{\beta}$ by $\overline{\beta}'$ or $\overline{\beta}''$, we may reduce to the where either χ is an isomorphism or $k' = 0$.

If $k' = 0$, then we can identify $\overline{Y}$ with $s(Y)$ and condition (i) guarantees that $F(\overline{Y}) \to F(Y, 0)$ is an equivalence. It therefore suffices to show that the composite map $F(\overline{X}) \to F(Y, 0)$ is inert. Since the collection of inert morphisms in $\mathcal{C}^{\otimes}$ is stable under composition, this follows from (i') and (ii'). If χ is an isomorphism, then we can identify $\overline{Y}$ with $s(Y)$, and condition (i) guarantees that $F(\overline{Y}) \to F(Y, 1)$ is an equivalence. It therefore suffices to show that the composite map $F(\overline{X}) \to F(Y, 1)$ is inert. This follows from (i') and (iii'), again because the collection of inert morphisms in $\mathcal{C}^{\otimes}$ is stable under composition. This completes the verification of condition (ii) and the proof of Proposition 3.4.1.3. $\square$

Proof of Proposition 3.4.1.4. Let $\mathcal{K} \subseteq \mathrm{Fun}(\Lambda_1^2, \mathcal{O}^{\otimes})$ be the full subcategory spanned by those diagrams $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ in $\mathcal{O}^{\otimes}$ where α and β are semi-inert, and let $e_i : \mathcal{K} \to \mathcal{O}^{\otimes}$ be the map given by evaluation at the vertex $\{i\}$ for $0 \leq i \leq 2$. We will say that a morphism in $\mathcal{K}$ is *inert* if its image under each e_i is an inert morphism in $\mathcal{O}^{\otimes}$. If S is a full subcategory of $\mathcal{K}$, we $\overline{\mathcal{X}}(S)$ denote the simplicial set $(e_0|S)_*(e_2|S)^* \mathcal{C}^{\otimes}$: that is, $\overline{\mathcal{X}}(S)$ is a simplicial set equipped with a map $\overline{\mathcal{X}}(S) \to \mathcal{O}^{\otimes}$ characterized by the following universal property: for any map of simplicial sets $K \to \mathcal{O}^{\otimes} \simeq \mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})$, we have a canonical bijection

$$\mathrm{Hom}_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})}(K, \overline{\mathcal{X}}(S)) = \mathrm{Hom}_{\mathrm{Fun}(\{2\}, \mathcal{O}^{\otimes})}(K \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} S, \mathcal{C}^{\otimes}).$$

We let $\mathcal{X}(S)$ denote the full simplicial subset of $\overline{\mathcal{X}}(S)$ spanned by those vertices which classify functors carrying inert morphisms in S to inert morphisms in $\mathcal{C}^{\otimes}$.

Let $\mathcal{K}_1$ denote the full subcategory of $\mathcal{K}$ spanned by those diagrams $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ where β is an equivalence, and let $\mathcal{K}_{01}$ denote the full subcategory of $\mathcal{K}_1$ spanned by those diagrams where α is null. We have a canonical embedding $j : \mathcal{K}_{\mathcal{O}} \to \mathcal{K}_1$ which carries $\alpha : X \to Y$ to the diagram $X \xrightarrow{\alpha} Y \xrightarrow{\mathrm{id}_Y} Y$. Note that this embedding restricts to an embedding $j_0 : \mathcal{K}_{\mathcal{O}}^0 \hookrightarrow \mathcal{K}_{01}$. Composition with these embeddings gives rise to a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^{\otimes} & \longrightarrow & \overline{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes} & \longrightarrow & \mathcal{X}(\mathcal{K}_1) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \times \mathcal{O}^{\otimes} & \longrightarrow & {}^{\mathrm{p}}\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) & \longrightarrow & \mathcal{X}(\mathcal{K}_{01}).
\end{array}
$$

The left horizontal maps are categorical equivalences (Remark 3.3.3.6). The right horizontal maps are categorical equivalences because j and j_0 admit simplicial homotopy inverses (given by restriction along the inclusion $\Delta^{\{0,1\}} \subseteq \Lambda_1^2$). Consequently, we are reduced to proving that the diagram

$$
\begin{array}{ccc}
\mathrm{Mod}^{\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}))^{\otimes} & \longrightarrow & \mathcal{X}(\mathcal{K}_1) \\
\downarrow & & \downarrow \\
\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})) \times \mathcal{O}^{\otimes} & \longrightarrow & \mathcal{X}(\mathcal{K}_{01})
\end{array}
$$

is a homotopy pullback square.

Let $\mathcal{K}_0$ denote the full subcategory of $\mathcal{K}$ spanned by those diagrams $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ for which α is null, and consider the diagram

$$
\begin{array}{ccccc}
\mathrm{Mod}^{\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}))^{\otimes} & \longrightarrow & \mathcal{X}(\mathcal{K}) & \longrightarrow & \mathcal{X}(\mathcal{K}_1) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})) \times \mathcal{O}^{\otimes} & \longrightarrow & \mathcal{X}(\mathcal{K}_0) & \longrightarrow & \mathcal{X}(\mathcal{K}_{01}).
\end{array}
$$

To complete the proof, it will suffice to show that both of the squares appearing in this diagram are homotopy pullback squares.

We first treat the square on the left. Consider the diagram

$$
\begin{array}{ccccc}
\mathrm{Mod}^{\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})) & \longrightarrow & \overline{\mathcal{M}\mathrm{od}}^{\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})) & \longrightarrow & \mathfrak{X}(\mathcal{K}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})) \times \mathcal{O}^{\otimes} & \longrightarrow & {}^{\mathrm{p}}\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})) & \longrightarrow & \mathfrak{X}(\mathcal{K}_0).
\end{array}
$$

Since the left horizontal maps are categorical equivalences (Remark 3.3.3.6), it suffices to show that the right square is homotopy Cartesian. Let $\mathcal{K}_2$ be the full subcategory of $\mathcal{K}$ spanned by those diagrams $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ for which β is null, and let $\mathcal{K}_{02}$ denote the full subcategory of $\mathcal{K}$ spanned by those diagrams where α and β are both null. The algebra $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ determines a vertex v of $\mathfrak{X}(\mathcal{K}_2)$ (and therefore, by restriction, a vertex v' of $\mathfrak{X}(\mathcal{K}_{02})$). We have a commutative diagram

$$
\begin{array}{ccccc}
\overline{\mathcal{M}\mathrm{od}}^{\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})) & \longrightarrow & \mathfrak{X}(\mathcal{K}) & \xrightarrow{\theta} & \mathfrak{X}(\mathcal{K}_2) \\
\downarrow & & \downarrow & & \downarrow{\phi} \\
{}^{\mathrm{p}}\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})) & \longrightarrow & \mathfrak{X}(\mathcal{K}_0) & \xrightarrow{\theta'} & \mathfrak{X}(\mathcal{K}_{02})
\end{array}
$$

where the horizontal maps are fiber sequences (where the fibers are taken over the vertices v and v', respectively). To show that the left square is a homotopy pullback, it suffices to prove the following:

(i) The maps θ and θ' are categorical fibrations of ∞-categories.

(ii) The map ϕ is a categorical equivalence.

To prove (i), we first show that the simplicial sets $\mathfrak{X}(\mathcal{K})$, $\mathfrak{X}(\mathcal{K}_0)$, $\mathfrak{X}(\mathcal{K}_2)$, and $\mathfrak{X}(\mathcal{K}_{02})$ are ∞-categories. In view of Proposition B.4.5, it will suffice to show that the maps

$$
e_0 : \mathcal{K} \to \mathcal{O}^{\otimes} \qquad e_0^0 : \mathcal{K}_0 \to \mathcal{O}^{\otimes}
$$

$$
e_0^2 : \mathcal{K}_2 \to \mathcal{O}^{\otimes} \qquad e_0^{02} : \mathcal{K}_{02} \to \mathcal{O}^{\otimes}
$$

are flat categorical fibrations. The map e_0 can be written as a

$$
\mathcal{K} \xrightarrow{e_0'} \mathcal{K}_{\mathcal{O}} \xrightarrow{e_0''} \mathcal{O}^{\otimes}
$$

where e_0' is given by restriction along the inclusion $\Delta^{\{0,1\}} \subseteq \Lambda_1^2$ and e_0'' is given by evaluation at $\{0\}$. The map e_0'' is a flat categorical fibration by virtue of our assumption that $\mathcal{O}^{\otimes}$ is coherent. The map e_0' is a pullback of e_0'', and therefore also flat. Applying Corollary B.3.16, we deduce that e_0 is flat. The proofs in the other cases three are similar: the only additional ingredient that is required is the observation that evaluation at 0 induces a flat categorical fibration $\mathcal{K}_{\mathcal{O}}^0 \to \mathcal{O}^{\otimes}$, which follows from Lemma 3.3.3.3.

To complete the proof of (i), we will show that θ and θ' are categorical fibrations. We will give the proof for the map θ; the case of θ' is handled similarly. We wish to show that θ has the right lifting property with respect to every trivial cofibration $A \to B$ in $(\mathrm{Set}_{\Delta})_{/\mathcal{O}^{\otimes}}$. Unwinding the definitions, we are required to provide solutions to lifting problems of the form

$$
\begin{array}{ccc}
(A \times_{\mathcal{O}^{\otimes}} \mathcal{K}) \coprod_{A \times_{\mathcal{O}^{\otimes}} \mathcal{K}_2} (B \times_{\mathcal{O}^{\otimes}} \mathcal{K}_2) & \longrightarrow & \mathcal{C}^{\otimes} \\
\downarrow{i} & & \downarrow{p} \\
B \times_{\mathcal{O}^{\otimes}} \mathcal{K} & \longrightarrow & \mathcal{O}^{\otimes}.
\end{array}
$$

Since p is a categorical fibration, it suffices to prove that the monomorphism i is a categorical equivalence of simplicial sets. In other words, we need to show that the diagram

$$
\begin{array}{ccc}
A \times_{\mathcal{O}^{\otimes}} \mathcal{K}_2 & \longrightarrow & A \times_{\mathcal{O}^{\otimes}} \mathcal{K} \\
\downarrow & & \downarrow \\
B \times_{\mathcal{O}^{\otimes}} \mathcal{K}_2 & \longrightarrow & B \times_{\mathcal{O}^{\otimes}} \mathcal{K}
\end{array}
$$

is a homotopy pushout square (with respect to the Joyal model structure). To prove this, it suffices to show that the vertical maps are categorical equivalences. Since $A \to B$ is a categorical equivalence, this follows from Corollary B.3.15 (since the maps e_0 and e_0^2 are flat categorical fibrations).

We now prove (ii). Let $\mathcal{K}_3$ denote the full subcategory of $\mathcal{K}$ spanned by those diagrams $X \xrightarrow{\alpha} Y \xrightarrow{\alpha} Z$ in $\mathcal{O}^{\otimes}$ where $Y \in \mathcal{O}^{\otimes}_{\langle 0 \rangle}$. We have a commutative diagram

$$
\begin{array}{ccc}
\mathfrak{X}(\mathcal{K}_0) & \xrightarrow{\quad \theta \quad} & \mathfrak{X}(\mathcal{K}_{02}) \\
& \searrow \qquad \swarrow & \\
& \mathfrak{X}(\mathcal{K}_3). &
\end{array}
$$

Consequently, to show that θ is a categorical equivalence, it suffices to show that the diagonal maps in this diagram are categorical equivalences. We will show that $\mathfrak{X}(\mathcal{K}_0) \to \mathfrak{X}(\mathcal{K}_3)$ is a categorical equivalence; the proof for $\mathfrak{X}(\mathcal{K}_{02}) \to \mathfrak{X}(\mathcal{K}_3)$ is similar. Let $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ be an object $K \in \mathcal{K}_0$, and choose a morphism $\gamma : Y \to Y_0$ where $Y_0 \in \mathcal{O}^{\otimes}$. Since β is null, it factors through γ, and we obtain a commutative diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathrm{id}} \\
X & \xrightarrow{\gamma \circ \alpha} & Y_0 & \longrightarrow & Z.
\end{array}
$$

We can interpret this diagram as a morphism $\bar{\gamma} : K \to K_0$ in $\mathcal{K}_0$. It is not difficult to see that this $\bar{\gamma}$ exhibits K_0 as a $\mathcal{K}_3$-localization of K. Consequently, the construction $K \mapsto K_0$ can be made into a functor $L : \mathcal{K}_0 \to \mathcal{K}_3$, equipped with a natural transformation $t : \mathrm{id} \to L$. Without loss of generality, we can assume that L and t commute with the evaluation maps e_0 and e_2. Composition with L determines a map $\mathfrak{X}(\mathcal{K}_3) \to \mathfrak{X}(\mathcal{K}_0)$, and the transformation t exhibits this map as a homotopy inverse to the restriction map $\mathfrak{X}(\mathcal{K}_0) \to \mathfrak{X}(\mathcal{K}_3)$. This completes the proof of (ii).

It remains to show that the diagram

$$
\begin{array}{ccc}
\mathfrak{X}(\mathcal{K}) & \longrightarrow & \mathfrak{X}(\mathcal{K}_1) \\
\downarrow & & \downarrow{\scriptstyle \psi} \\
\mathfrak{X}(\mathcal{K}_0) & \longrightarrow & \mathfrak{X}(\mathcal{K}_{01})
\end{array}
$$

is a homotopy pullback square. We first claim that ψ is a categorical equivalence of ∞-categories. The proof is similar to the proof of (i): the only nontrivial point is to verify that the restriction maps

$$
e_0^1 : \mathcal{K}_1 \to \mathcal{O}^{\otimes} \qquad e_0^{01} : \mathcal{K}_{01} \to \mathcal{O}^{\otimes}
$$

are flat categorical fibrations. We will give the proof for e_0^1; the proof for e_0^{01} is similar. We can write e_0^1 as a composition $\mathcal{K}_1 \to \mathcal{K}_{\mathcal{O}} \to \mathcal{O}^{\otimes}$, where the second map is a flat categorical fibration by virtue of our assumption that $\mathcal{O}^{\otimes}$ is coherent. The first map is a pullback of the restriction map $\mathrm{Fun}^0(\Delta^1, \mathcal{O}^{\otimes}) \, \mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})$, where

$\mathrm{Fun}^0(\Delta^1, \mathcal{O}^\otimes)$ denotes the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{O}^\otimes)$ spanned by the equivalences in $\mathcal{O}^\otimes$, and therefore a trivial Kan fibration (and, in particular, a flat categorical fibration). Applying Corollary B.3.16, we conclude that e_0^1 is a flat categorical fibration, as desired.

Since ψ is a categorical fibration of ∞-categories and $\mathfrak{X}(\mathcal{K}_0)$ is an ∞-category, we have a homotopy pullback diagram

$$
\begin{array}{ccc}
\mathfrak{X}(\mathcal{K}_0) \times_{\mathfrak{X}(\mathcal{K}_{01})} \mathfrak{X}(\mathcal{K}_1) & \longrightarrow & \mathfrak{X}(\mathcal{K}_1) \\
\downarrow & & \downarrow \\
\mathfrak{X}(\mathcal{K}_0) & \longrightarrow & \mathfrak{X}(\mathcal{K}_{01}).
\end{array}
$$

To complete the proof, it will suffice to show that the restriction map $\tau : \mathfrak{X}(\mathcal{K}) \to \mathfrak{X}(\mathcal{K}_0) \times_{\mathfrak{X}(\mathcal{K}_{01})} \mathfrak{X}(\mathcal{K}_1)$ is a categorical equivalence. We will show that τ is a trivial Kan fibration. Note that the evaluation map $e_0 : \mathcal{K} \to \mathcal{O}^\otimes$ is a Cartesian fibration; moreover, if $K \to K'$ is an e_0-Cartesian moprhism in $\mathcal{K}$ and $K' \in \mathcal{K}_0 \coprod_{\mathcal{K}_{01}} \mathcal{K}_1$, then $K \in \mathcal{K}_0 \coprod_{\mathcal{K}_{01}} \mathcal{K}_1$. It follows that e_0 restricts to a Cartesian fibration $\mathcal{K}_0 \coprod_{\mathcal{K}_{01}} \mathcal{K}_1 \to \mathcal{O}^\otimes$. In view of Lemma 3.4.2.2, the map τ will be a trivial Kan fibration provided that the following pair of assertions holds:

(a) Let $F \in \overline{\mathfrak{X}}(\mathcal{K})$ be an object lying over $X \in \mathcal{O}^\otimes$ which we will identify with a functor $\{X\} \times_{\mathcal{O}^\otimes} \mathcal{K} \to \mathcal{C}^\otimes$. Let $F_0 = F|(\{X\} \times_{\mathcal{O}^\otimes} (\mathcal{K}_0 \coprod_{\mathcal{K}_{01}} \mathcal{K}_1)$, and assume that $F_0 \in \mathfrak{X}(\mathcal{K}_0) \times_{\mathfrak{X}(\mathcal{K}_{01})} \mathfrak{X}(\mathcal{K}_1)$. Then $F \in \mathfrak{X}(\mathcal{K})$ if and only if F is a p-right Kan extension of F_0.

(b) Let $F_0 \in \mathfrak{X}(\mathcal{K}_0) \times_{\mathfrak{X}(\mathcal{K}_{01})} \mathfrak{X}(\mathcal{K}_1)$. Then there exists an extension F of F_0 which satisfies the equivalent conditions of (a).

To prove these assertions, let us consider an object $X \in \mathcal{O}^\otimes$ and an object $F_0 \in \mathfrak{X}(\mathcal{K}_0) \times_{\mathfrak{X}(\mathcal{K}_{01})} \mathfrak{X}(\mathcal{K}_1)$ lying over X. Let $\mathcal{D} = \{X\} \times_{\mathcal{O}^\otimes} \mathcal{K}$ denote the ∞-category of diagrams $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ in $\mathcal{O}^\otimes$, and define full subcategories $\mathcal{D}_0$, $\mathcal{D}_1$, and $\mathcal{D}_{01}$ similarly. Let K be an object of $\mathcal{D}$, corresponding to a decomposition $X \simeq X_0 \oplus X_1 \oplus X_2$ and a diagram

$$X_0 \oplus X_1 \oplus X_2 \to X_0 \oplus X_1 \oplus Y_0 \oplus Y_1 \to X_0 \oplus Y_0 \oplus Z$$

of semi-inert morphisms in $\mathcal{O}^\otimes$. We have a commutative diagram

$$
\begin{array}{ccc}
K & \longrightarrow & K_1 \\
\downarrow & & \downarrow \\
K_0 & \longrightarrow & K_{01}
\end{array}
$$

in $\mathcal{D}$, where $K_0 \in \mathcal{D}_0$ represents the diagram $X_0 \oplus X_1 \oplus X_2 \to Y_0 \oplus Y_1 \to Y_0 \oplus Z$, $K_1 \in \mathcal{D}_1$ represents the diagram $X_0 \oplus X_1 \oplus X_2 \to X_0 \oplus Y_0 \oplus Z \simeq X_0 \oplus Y_0 \oplus Z$, and $K_{01} \in \mathcal{D}_{01}$ represents the diagram $X_0 \oplus X_1 \oplus X_2 \to Y_0 \oplus Z \to Y_0 \oplus Z$. This diagram exhibits K_0, K_1, and K_{01} as initial objects of $(\mathcal{D}_0)_{K/}$, $(\mathcal{D}_1)_{K/}$, and $(\mathcal{D}_{01})_{K/}$, respectively. Applying Theorem T.4.1.3.1, we conclude that the induced map $\Lambda_2^2 \to (\mathcal{D}_0 \coprod_{\mathcal{D}_{01}} \mathcal{D}_1)_{K/}$ is right cofinal, so that an extension F of F_0 is a p-right Kan extension of F_0 at K if and only if the diagram

$$
\begin{array}{ccc}
F(K) & \longrightarrow & F_0(K_1) \\
\downarrow & & \downarrow \\
F_0(K_0) & \longrightarrow & F_0(K_{01})
\end{array}
$$

is a p-limit diagram in $\mathcal{C}^\otimes$. Choose equivalences $F_0(K_0) = y_0 \oplus z$, $F_0(K_1) = x_0' \oplus y_0' \oplus z'$, and $F_0(K_{01}) = y_0'' \oplus z''$. If we let $K' \in \mathcal{D}$ denote the object corresponding to the diagram $X_0 \oplus X_1 \oplus X_2 \to Y_0 \simeq Y_0$ and apply our

assumption that $F_0|\,\mathcal{D}_0 \in \mathcal{X}(\mathcal{K}_0)$ to the commutative diagram

$$K_0 \longrightarrow K_{01}$$
$$\searrow \qquad \swarrow$$
$$K',$$

then we deduce that $F_0(K_0) \to F_0(K_{01})$ induces an equivalence $y_0 \simeq y_0''$. Similarly, the assumption that $F_0|\,\mathcal{D}_1 \in \mathcal{X}(\mathcal{K}_1)$ guarantees that $F_0(K_1) \to F_0(K_{01})$ is inert, so that $y_0' \simeq y_0''$ and $z' \simeq z''$. It follows from Lemma 3.4.3.15 that the diagram $F_0(K_1) \to F_0(K_{01}) \leftarrow F_0(K_0)$ admits a p-limit (covering the evident map $\Delta^1 \times \Delta^1 \to \mathcal{O}^\otimes$), so that $F_0|(\mathcal{D}_0 \coprod_{\mathcal{D}_{01}} \mathcal{D}_1)_{K/}$ also admits a p-limit (covering the map $(\mathcal{D}_0 \coprod_{\mathcal{D}_{01}} \mathcal{D}_1)_{K/}^{\triangleleft} \to \mathcal{O}^\otimes$)); assertion (b) now follows from Lemma T.4.3.2.13. Moreover, the criterion of Lemma 3.4.3.15 gives the following version of (a):

(a') An extension $F \in \overline{\mathcal{X}}(\mathcal{K})$ of F_0 is a p-right Kan extension of F_0 at K if and only if, for every object $K \in \mathcal{D}$ as above, the maps $F(K) \to F_0(K_0)$ and $F(K) \to F_0(K_1)$ induce an equivalence $F(K) \simeq x_0' \oplus y_0 \oplus z$ in $\mathcal{C}^\otimes$.

To complete the proof, it will suffice to show that the criterion of (a') holds if and only if $F \in \mathcal{X}(\mathcal{K})$. We first prove the "if" direction. Fix $K \in \mathcal{D}$, so that we have an equivalence $F(K) \simeq \overline{x}_0 \oplus \overline{y}_0 \oplus \overline{z}$. Since $K \to K_0$ is inert, the assumption that $F \in \mathcal{X}(\mathcal{K})$ implies that $F(K) \to F(K_0)$ is inert, so that $\overline{y}_0 \simeq y_0$ and $\overline{z} \simeq z$. Let $K'' \in \mathcal{D}$ be the diagram $X_0 \oplus X_1 \oplus X_2 \to Y_0 \simeq Y_0$, so that we have a commutative diagram

$$K \longrightarrow K_1$$
$$\searrow \qquad \swarrow$$
$$K''$$

in which the diagonal maps are inert. It follows that the morphisms $F(K) \to F(K'') \leftarrow F(K_1)$ are inert, so that the map $F(K) \to F(K_1)$ induces an equivalence $\overline{y}_0 \simeq y_0'$.

We now prove the "only if" direction. Assume that $F \in \overline{\mathcal{X}}(\mathcal{K})$ is an extension of F_0 which satisfies the criterion given in (a'); we will show that F carries inert morphisms in $\mathcal{D} \subseteq \mathcal{K}$ to inert morphisms in $\mathcal{C}^\otimes$. Let $K \to L$ be an inert morphism in $\mathcal{D}$, where K is as above and L corresponds to a diagram $X \to Y' \to Z'$; we wish to prove that the induced map $F(K) \to F(L)$ is inert. Let $\langle n \rangle$ denote the image of Z' in $N(\mathcal{F}in_*)$, and choose inert morphisms $Z' \to Z_i'$ lying over $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \le i \le n$. Each of the induced maps $Y' \to Z_i$ factors as a composition $Y' \to Y_i' \to Z_i$, where the first map is inert and the second is active. Let $L_i \in \mathcal{D}$ denote the diagram $X \to Y_i' \to Z_i'$. To show that $F(K) \to F(L)$ is inert, it will suffice to show that the maps $F(K) \to F(L_i) \leftarrow F(L)$ are inert for $1 \le i \le n$. Replacing L by L_i (and possibly replacing K by L), we may reduce to the case where $n = 1$ and the map $Y' \to Z'$ is active. There are two cases to consider:

- The map $Y' \to Z'$ is an equivalence. In this case, the map $K \to L$ factors as a composition $K \to K_1 \to L$. Since $F_0|\,\mathcal{D}_1 \in \mathcal{X}(\mathcal{K}_1)$, the map $F(K_1) \to F(L)$ is inert. Consequently, the assertion that $F(K) \to F(L)$ is inert follows our assumption that $F(K) \to F(K_1)$ induces an equivalence $\overline{y}_0 \to y_0'$.

- The map $Y' \to Z'$ is null (so that $Y' \in \mathcal{O}_{\langle 0 \rangle}^\otimes$). In this case, the map $K \to L$ factors as a composition $K \to K_0 \to L$. Assumption (a') guarantees that $F(K) \to F(K_0)$ is inert, and the assumption that $F_0|\,\mathcal{K}_0 \in \mathcal{X}(\mathcal{K}_0)$ guarantees that $F(K_0) \to F(L)$ is inert.

$\square$

3.4.2 Modules over Trivial Algebras

Let $\mathcal{C}$ be a symmetric monoidal category. Let $\mathbf{1}$ denote the unit object of $\mathcal{C}$, so that for every object $C \in \mathcal{C}$ we have a canonical isomorphism $u_C : \mathbf{1} \otimes C \to C$. In particular, $u_{\mathbf{1}}$ gives a multiplication $\mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$ which exhibits $\mathbf{1}$ as a commutative algebra object of $\mathcal{C}$. Moreover, for every object $C \in \mathcal{C}$, u_C exhibits C as a module over the commutative algebra $\mathbf{1}$. In fact, this construction determines a functor $\mathcal{C} \to \mathrm{Mod}_{\mathbf{1}}(\mathcal{C})$, which is homotopy inverse to the forgetful functor $\theta : \mathrm{Mod}_{\mathbf{1}}(\mathcal{C}) \to \mathcal{C}$. It follows that θ is an equivalence of categories: that is, every object of $C \in \mathcal{C}$ admits a unique action of the commutative algebra $\mathbf{1}$.

Our goal in this section is to prove the following result, which can be regarded as an ∞-categorical generalization of the above discussion:

Proposition 3.4.2.1. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads and assume that $\mathcal{O}^{\otimes}$ is coherent. Let A be a trivial $\mathcal{O}$-algebra object of $\mathcal{C}$. Then the forgetful functor $\theta : \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$ is an equivalence of ∞-operads.*

The of Proposition 3.4.2.1 will require a few preliminary results.

Lemma 3.4.2.2. *Let $p : X \to S$ be an inner fibration of simplicial sets. Let X^0 be a full simplicial subset of X, and assume that the restriction map $p^0 = p|X^0$ is a coCartesian fibration. Let $q : Y \to Z$ be a categorical fibration of simplicial sets. Define a simplicial sets A and B equipped with a maps $A, B \to S$ so that the following universal property is satisfied: for every map of simplicial sets $K \to S$, we have bijections*

$$\mathrm{Hom}_{(\mathrm{Set}_\Delta)_{/S}}(K, A) \simeq \mathrm{Fun}(X \times_S K, Y)$$

$$\mathrm{Hom}_{(\mathrm{Set}_\Delta)_{/S}}(K, B) \simeq \mathrm{Fun}(X^0 \times_S K, Y) \times_{\mathrm{Fun}(X^0 \times_S K, Z)} \mathrm{Fun}(X \times_S K, Z).$$

Let $\phi : A \to B$ be the restriction map. Let A' denote the full simplicial subset of A spanned by those vertices corresponding to maps $f : X_s \to Y$ such that f is a q-left Kan extension of $f|X_s^0$, and let B' denote the full simplicial subset of B spanned by those vertices of the form $\phi(f)$ where $f \in A'$. Then ϕ induces a trivial Kan fibration $\phi' : A' \to B'$.

Proof. For every map of simplicial sets $T \to S$, let $F(T) = \mathrm{Map}_S(T, A')$ and let $G(T) = \mathrm{Map}_S(T, B')$. If $T_0 \subseteq T$ is a simplicial subset, we have a restriction map $\theta_{T_0, T} : F(T) \to G(T) \times_{G(T_0)} F(T_0)$. To prove that ϕ' is a trivial Kan fibration, it will suffice to show that $\theta_{T_0, T}$ is surjective on vertices whenever $T = \Delta^n$ and $T_0 = \partial \Delta^n$. We will complete the proof by showing that $\theta_{T_0, T}$ is a trivial Kan fibration whenever T has only finitely many simplices.

The proof proceeds by induction on the dimension of T (if T is empty, the result is trivial). Assume first that $T = \Delta^n$. If $T_0 = T$ there is nothing to prove. Otherwise, we may assume that T_0 has dimension smaller than n. Using the fact that q is a categorical fibration, we deduce that $\theta_{T_0, T}$ is a categorical fibration. It therefore suffices to show that $\theta_{T_0, T}$ is a categorical equivalence. We have a commutative diagram

$$
\begin{array}{ccc}
F(T) & \longrightarrow & G(T) \times_{G(T_0)} F(T_0) \\
& \searrow{\scriptstyle \psi} \quad \swarrow{\scriptstyle \psi'} & \\
& G(T). &
\end{array}
$$

The inductive hypothesis (applied to the inclusion $\emptyset \subseteq T_0$) guarantees that ψ' is a trivial Kan fibration. It will therefore suffice to show that ψ is a trivial Kan fibration. In view of Proposition T.4.3.2.15, it will suffice to prove the following assertions:

(a) Let $F : \Delta^n \times_S X \to Y$ be a functor. Then F is a q-left Kan extension of $F|(\Delta^n \times_S X^0)$ if and only if $F|(\{i\} \times_S X)$ is a q-left Kan extension of $F|(\{i\} \times_S X^0)$ for $0 \leq i \leq n$.

(b) Suppose we are given a commutative diagram

$$
\begin{array}{ccc}
\Delta^n \times_S X^0 & \xrightarrow{\ f\ } & Y \\
\downarrow & \overset{F}{\nearrow} & \downarrow{\scriptstyle q} \\
\Delta^n \times_S X & \xrightarrow{\ g\ } & Z
\end{array}
$$

such that for $0 \le i \le n$, there exists a q-left Kan extension $F_i : \{i\} \times_S X \to Y$ of $f|\{i\} \times_S X^0$ which is compatible with g. Then there exists a dotted arrow F as indicated satisfying the condition described in (a).

Assertion (a) follows from the observation that for $x \in \{i\} \times_S X$, the assumption that p^0 is a coCartesian fibration guarantees that $X^0 \times_X (\{i\} \times_S X)_{/x}$ is left cofinal in $X^0 \times_X (\Delta^n \times_S X)_{/x}$. Assertion ($b$) follows from the same observation together with Lemma T.4.3.2.13.

We now complete the proof by considering the case where T is not a simplex. We use induction on the number k of simplices of T' which do not belong to T. If $k = 0$, then $T' = T$ and there is nothing to prove. If $k = 1$, then there is a pushout diagram

$$
\begin{array}{ccc}
\partial\,\Delta^n & \longrightarrow & T_0 \\
\downarrow & & \downarrow \\
\Delta^n & \longrightarrow & T.
\end{array}
$$

It follows that $\theta_{T_0,T}$ is a pullback of the map $\theta_{\partial\,\Delta^n,\Delta^n}$, and we are reduced to the case where T is a simplex. If $k > 1$, then we have nontrivial inclusions $T_0 \subset T_1 \subset T$. Using the inductive hypothesis, we conclude that $\theta_{T_1,T}$ and θ_{T_0,T_1} are trivial Kan fibrations. The desired result follows from the observation that $\theta_{T_0,T}$ can be obtained by composing $\theta_{T_1,T}$ with a pullback of the morphism θ_{T_0,T_1}. $\qquad\square$

Lemma 3.4.2.3. *Let $p : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a fibration of ∞-operads. Assume that $\mathcal{O}^\otimes$ is unital and that p has unit objects. Let $C \in \mathcal{C}^\otimes$ and let $\alpha : p(C) \to Y$ be a semi-inert morphism in $\mathcal{O}^\otimes$. Then α can be lifted to a p-coCartesian morphism $\overline{\alpha} : C \to \overline{Y}$ in $\mathcal{C}^\otimes$.*

Proof. The map α can be factored as the composition of an inert morphism and an active morphism. We may therefore reduce to the case where α is either active or inert. If α is inert, we can choose $\overline{\alpha}$ to be an inert morphism lifting α. Assume therefore that α is active. Write $C \simeq C_1 \oplus \ldots \oplus C_m$ (using the notation of Remark 2.2.4.6), and write $Y = p(C_1) \oplus \ldots \oplus C_m \oplus Y_1 \oplus \ldots \oplus Y_n$. Since $\mathcal{O}^\otimes$ is unital, we may assume that α has the form $\mathrm{id}_{p(C_1)} \oplus \ldots \oplus \mathrm{id}_{p(C_m)} \oplus \alpha_1 \oplus \ldots \oplus \alpha_n$, where each $\alpha_i : 0 \to Y_i$ is a morphism with $0 \in \mathcal{O}^\otimes_{\langle 0 \rangle}$. Since p has units, we can lift each α_i to a morphism $\overline{\alpha}_i : \overline{0}_i \to \overline{Y}_i$ which exhibits $\overline{Y}_i$ as a p-unit. Let $\overline{\alpha} = \mathrm{id}_{C_1} \oplus \ldots \oplus \mathrm{id}_{C_m} \oplus \overline{\alpha}_1 \oplus \ldots \oplus \overline{\alpha}_n$. It follows from Proposition 3.1.1.10 that $\overline{\alpha}$ is p-coCartesian. $\qquad\square$

Proof of Proposition 3.4.2.1. We may assume that p has unit objects (otherwise the assertion is vacuous). Let $\phi : \mathcal{O}^\otimes \times \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to {}^{\mathrm{P}}\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ be the equivalence of Remark 3.3.3.6, and let $\mathcal{X} \subseteq {}^{\mathrm{P}}\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ denote the essential image of the full subcategory spanned by those pairs (X, A) where A is trivial. Let $\mathcal{X}'$ denote the fiber product $\mathcal{X} \times_{{}^{\mathrm{P}}\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})} \overline{\mathcal{M}\mathrm{od}}^{\mathcal{O}}(\mathcal{C})^\otimes$. Since Proposition 3.2.1.8 implies that trivial $\mathcal{O}$-algebras form a contractible Kan complex, the inclusion $\mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C})^\otimes \subseteq \mathcal{X}'$ is a categorical equivalence. It will therefore suffice to show that composition with the diagonal map $\delta : \mathcal{O}^\otimes \to \mathcal{K}_{\mathcal{O}}$ induces a categorical equivalence $\mathcal{X}' \to \mathcal{C}^\otimes$.

Let $\mathcal{K}^1_{\mathcal{O}}$ denote the essential image of δ, and define a simplicial set $\mathcal{Y}$ equipped with a map $\mathcal{Y} \to \mathcal{O}^\otimes$ so that the following universal property is satisfied: for every map of simplicial sets $K \to \mathcal{O}^\otimes$, we have a canonical bijection

$$
\mathrm{Hom}_{(\mathrm{Set}_\Delta)_{/\mathcal{O}^\otimes}}(K, \mathcal{Y}) \simeq \mathrm{Hom}_{(\mathrm{Set}_\Delta)_{/\mathcal{O}^\otimes}}(K \times_{\mathrm{Fun}(\{0\},\mathcal{O}^\otimes)} \mathcal{K}^1_{\mathcal{O}}, \mathcal{C}^\otimes).
$$

Since δ is fully faithful, it induces a categorical equivalence $\mathcal{O}^\otimes \to \mathcal{K}^1_{\mathcal{O}}$. It follows that the canonical map $\mathcal{Y} \to \mathcal{C}^\otimes$ is a categorical equivalence.

We have a commutative diagram

$$
\begin{array}{ccc}
\mathfrak{X}' & \xrightarrow{\;\;\theta'\;\;} & \mathcal{Y} \\
& \searrow \quad \swarrow & \\
& \mathcal{C}^{\otimes}. &
\end{array}
$$

Consequently, it will suffice to show that θ' is a categorical equivalence. We will prove that θ' is a trivial Kan fibration.

Define a simplicial set $\mathcal{D}$ equipped with a map $\mathcal{D} \to \mathcal{O}^{\otimes}$ so that the following universal property is satisfied: for every map of simplicial sets $K \to \mathcal{O}^{\otimes}$, we have a canonical bijection

$$
\mathrm{Hom}_{(\mathrm{Set}_{\Delta})_{/\mathcal{O}^{\otimes}}}(K, \mathcal{D}) \simeq \mathrm{Hom}_{(\mathrm{Set}_{\Delta})_{/\mathcal{O}^{\otimes}}}(K \times_{\mathrm{Fun}(\{0\}, \mathcal{O}^{\otimes})} \mathcal{K}_{\mathcal{O}}, \mathcal{C}^{\otimes}).
$$

For each $X \in \mathcal{O}^{\otimes}$, let $\mathcal{E}_X$ denote the full subcategory of $(\mathcal{O}^{\otimes})^{X/}$ spanned by the semi-inert morphisms $X \to Y$ in $\mathcal{O}^{\otimes}$, and let $\mathcal{E}_X^1$ denote the full subcategory of $(\mathcal{O}^{\otimes})^{X/}$ spanned by the equivalences $X \to Y$ in $\mathcal{O}^{\otimes}$. An object of $\mathcal{D}$ can be identified with a pair (X, F), where $X \in \mathcal{O}^{\otimes}$ and $F : A_X \to \mathcal{C}^{\otimes}$ is a functor. We will prove the following:

(a) The full subcategory $\mathfrak{X}' \subseteq \mathcal{D}$ is spanned by those pairs (X, F) where $F : \mathcal{E}_X \to \mathcal{C}^{\otimes}$ is a q-left Kan extension of $F|\,\mathcal{E}_X^1$.

(b) For every $X \in \mathcal{O}^{\otimes}$ and every functor $f \in \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{E}_X^1, \mathcal{C}^{\otimes})$, there exists a q-left Kan extension $F \in \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{E}_X, \mathcal{C}^{\otimes})$ of f.

Assuming that (a) and (b) are satisfied, the fact that the restriction functor $\mathfrak{X}' \to \mathcal{Y}$ is a trivial Kan fibration will follow immediately from Lemma 3.4.2.2.

Note that for $X \in \mathcal{O}^{\otimes}$, we can identify $\mathcal{E}_X^1$ with the full subcategory of $\mathcal{E}_X$ spanned by the initial objects. Consequently, a functor $f \in \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{E}_X^1, \mathcal{C}^{\otimes})$ as in (b) is determined up to equivalence by $f(\mathrm{id}_X) \in \mathcal{C}_X^{\otimes}$. Using Lemma T.4.3.2.13, we deduce that f admits a q-left Kan extension $F \in \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{E}_X, \mathcal{C}^{\otimes})$ if and only if every semi-inert morphism $X \to Y$ in $\mathcal{O}^{\otimes}$ can be lifted to a q-coCartesian morphism $f(\mathrm{id}_X) \to \overline{Y}$ in $\mathcal{C}^{\otimes}$. Assertion (b) now follows from Lemma 3.4.2.3.

We now prove (a). Suppose first that F is a q-left Kan extension of $F|\,\mathcal{E}_X^1$. The proof of (b) shows that $F(u)$ is q-coCartesian for every morphism $u : Y \to Z$ in $\mathcal{E}_X$ such that Y is an initial object of $\mathcal{E}_X$. Since every morphism u in $\mathcal{E}_X$ fits into a commutative diagram

$$
\begin{array}{ccc}
& Y & \\
& \nearrow \quad \searrow{\scriptstyle u} & \\
\mathrm{id}_X & \xrightarrow{\hspace{3cm}} & Z,
\end{array}
$$

Proposition T.2.4.1.7 guarantees that F carries *every* morphism in $\mathcal{E}_X$ to a q-coCartesian morphism in $\mathcal{C}^{\otimes}$. In particular, F carries inert morphisms in $\mathcal{E}_X$ to inert morphisms in $\mathcal{C}^{\otimes}$, and therefore belongs to $\overline{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes} \times_{\mathcal{O}^{\otimes}} \{X\}$. Let $\mathcal{E}_X^0$ denote the full subcategory of $\mathcal{E}_X$ spanned by the null morphisms $X \to Y$ in $\mathcal{O}^{\otimes}$, and let $s : \mathcal{O}^{\otimes} \to \mathcal{E}_X^0$ denote a section to the trivial Kan fibration $\mathcal{E}_X^0 \to \mathcal{O}^{\otimes}$. To prove that $F \in \mathfrak{X}'$, it suffices to show that the composition

$$
\mathcal{O}^{\otimes} \xrightarrow{s} \mathcal{E}_X^0 \subseteq \mathcal{E}_X \xrightarrow{F} \mathcal{C}^{\otimes}
$$

is a trivial $\mathcal{O}$-algebra. Since this composition carries every morphism in $\mathcal{O}^{\otimes}$ to a q-coCartesian morphism in $\mathcal{C}^{\otimes}$, it is an $\mathcal{O}$-algebra: the triviality now follows from Remark 3.2.1.4.

To complete the proof of (a), let us suppose that $F \in \mathfrak{X}'$; we wish to show that F is a q-left Kan extension of $f = F|\,\mathcal{E}_X^1$. Using (b), we deduce that f admits a q-left Kan extension $F' \in \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{E}_X, \mathcal{C}^{\otimes})$. Let $\alpha : F' \to F$ be a natural transformation which is the identity on f; we wish to prove that α is an equivalence. Fix an object $\overline{Y} \in \mathcal{E}_X$, corresponding to a semi-inert morphism $X \to Y$ in $\mathcal{O}^{\otimes}$. Let $\langle n \rangle \in \mathrm{N}(\mathfrak{Fin}_*)$ denote

the image of Y, and choose inert morphisms $Y \to Y_i$ lifting the maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$. Let $\overline{Y}_i$ denote the composition of $\overline{Y}$ with the map $Y \to Y_i$ for $1 \leq i \leq n$. Since F and F' both preserve inert morphisms and $\mathcal{C}^{\otimes}$ is an ∞-operad, it suffices to prove that $\alpha_{\overline{Y}_i} : F'(\overline{Y}_i) \to F(\overline{Y}_i)$ is an equivalence for $1 \leq i \leq n$. We may therefore replace Y by Y_i and reduce to the case where $Y \in \mathcal{O}$. In this case, the semi-inert morphism $\overline{Y}$ is either null or inert.

If the map $\overline{Y} : X \to Y$ is null, then $\overline{Y} \in \mathcal{E}_X^0$. Since $F \circ s$ and $F' \circ s$ both determine trivial $\mathcal{O}$-algebra objects, the induced natural transformation $F' \circ s \to F \circ s$ is an equivalence (Proposition 3.2.1.8). It follows that the natural transformation $F'| \mathcal{E}_X^0 \to F| \mathcal{E}_X^0$ is an equivalence, so that $F'(\overline{Y}) \simeq F(\overline{Y})$.

If the map $\overline{Y} : X \to Y$ is inert, then we have an inert morphism $u : \mathrm{id}_X \to \overline{Y}$ in $\mathcal{E}_X$. Since F and F' both preserve inert morphisms, it suffices to show that the map $F'(\mathrm{id}_X) \to F(\mathrm{id}_X)$ is an equivalence. This is clear, since $\mathrm{id}_X \in \mathcal{E}_X^1$. $\square$

3.4.3   Limits of Modules

Let $\mathcal{C}$ be a symmetric monoidal category and let A be a commutative algebra object of $\mathcal{C}$. Suppose we are given a diagram $\{M_\alpha\}$ in the category of A-modules, and let $M = \varprojlim M_\alpha$ be the limit of this diagram in the category $\mathcal{C}$. The collection of maps

$$A \otimes M \to A \otimes M_\alpha \to M_\alpha$$

determines a map $A \otimes M \to M$, which endows M with the structure of an A-module. Moreover, we can regard M also as a limit of the diagram $\{M_\alpha\}$ in the category of A-modules.

Our goal in this section is to prove an analogous result in the ∞-categorical setting, for algebras over an arbitrary coherent ∞-operad. We can state our main result as follows:

Theorem 3.4.3.1. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, where $\mathcal{O}^{\otimes}$ is coherent. Suppose we are given a commutative diagram*

such that the underlying map $K^{\triangleleft} \to \mathcal{O}^{\otimes}$ takes some constant value $X \in \mathcal{O}$, and the lifting problem

admits a solution, where $\overline{p}$ is a q-limit diagram. Then:

(1) *There exists a map $\overline{p}$ making the original diagram commute, such that $\delta \circ \overline{p}$ is a q-limit diagram in $\mathcal{C}^{\otimes}$ (here $\delta : \mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$ denotes the map given by composition with the diagonal embedding $\mathcal{O}^{\otimes} \to \mathcal{K}_{\mathcal{O}} \subseteq \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$).*

(2) *Let $\overline{p}$ be an arbitrary map making the above diagram commute. Then $\overline{p}$ is a ψ-limit diagram if and only if $\delta \circ \overline{p}$ is a q-limit diagram.*

Theorem 3.4.3.1 has a number of consequences. First, it allows us to describe limits in an ∞-category of modules:

Corollary 3.4.3.2. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, where $\mathcal{O}^{\otimes}$ is coherent, and let $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ and $X \in \mathcal{O}$. Suppose we are given a diagram $p : K \to \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})_X^{\otimes}$ such that the induced map $p' : K \to \mathcal{C}_X^{\otimes}$ can be extended to a q-limit diagram $\overline{p}' : K^{\triangleleft} \to \mathcal{C}_X^{\otimes}$. Then:*

(1) *There exists an extension $\overline{p} : K^{\triangleleft} \to \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})_X^{\otimes}$ of p such that the induced map $K^{\triangleleft} \to \mathcal{C}^{\otimes}$ is a q-limit diagram.*

(2) *Let $\overline{p} : K^{\triangleleft} \to \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})_X^{\otimes}$ be an arbitrary extension of p. Then $\overline{p}$ is a limit diagram if and only if it induces a q-limit diagram $K^{\triangleleft} \to \mathcal{C}^{\otimes}$.*

We can also use Theorem 3.4.3.1 to describe the relationships between the ∞-categories $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$ as the algebra A varies.

Corollary 3.4.3.3. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, where $\mathcal{O}^{\otimes}$ is coherent. Then for each $X \in \mathcal{O}$, the functor $\phi : \mathrm{Mod}^{\mathcal{O}}(\mathcal{C})_X^{\otimes} \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ is a Cartesian fibration. Moreover, a morphism f in $\mathrm{Mod}^{\mathcal{O}}(\mathcal{C})_X$ is ϕ-Cartesian if and only if its image in $\mathcal{C}_X^{\otimes}$ is an equivalence.*

More informally: if we are given a morphism $A \to B$ in $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$, then there is an evident forgetful functor from B-modules to A-modules, which does not change the underlying object of $\mathcal{C}$.

Proof. Apply Corollary 3.4.3.2 in the case $K = \Delta^0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Corollary 3.4.3.4. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, where $\mathcal{O}^{\otimes}$ is coherent. Then:*

(1) *The functor $\phi : \mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ is a Cartesian fibration.*

(2) *A morphism $f \in \mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^{\otimes}$ is ϕ-Cartesian if and only if its image in $\mathcal{C}^{\otimes}$ is an equivalence.*

Proof. Let $M \in \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ and let $\langle n \rangle$ denote its image in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Suppose we are given a morphism $f_0 : A' \to A$ in $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$; we will to construct a ϕ-Cartesian morphism $f : M' \to M$ lifting f_0.

Choose inert morphisms $g_i : M \to M_i$ in $\mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^{\otimes}$ lying over $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \leq i \leq n$. These maps determine a diagram $F : \langle n \rangle^{\circ \triangleleft} \to \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$. Let X_i denote the image of M_i in $\mathcal{O}$, and let $\phi_i : \mathrm{Mod}^{\mathcal{O}}(\mathcal{C})_{X_i} \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ be the restriction of ϕ. Using Corollary 3.4.3.3, we can choose ϕ_i-Cartesian morphisms $f_i : M_i' \to M_i$ in $\mathrm{Mod}^{\mathcal{O}}(\mathcal{C})_{X_i}$ lying over f_0, whose images in $\mathcal{C}_{X_i}$ are equivalences. Since $q : \mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \times \mathcal{O}^{\otimes}$ is a $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$-family of ∞-operads, we can choose a q-limit diagram $F' : \langle n \rangle^{\circ \triangleleft} \to \mathrm{Mod}_{A'}^{\mathcal{O}}(\mathcal{C})^{\otimes}$ with $F'(i) = M_i'$ for $1 \leq i \leq n$, where F' carries the cone point of $\langle n \rangle^{\circ \triangleleft}$ to $M' \in \mathrm{Mod}_{A'}^{\mathcal{O}}(\mathcal{C})_X^{\otimes}$. Using the fact that F is a q-limit diagram, we get a natural transformation of functors $F' \to F$, which we may view as a diagram $H : \langle n \rangle^{\circ \triangleleft} \times \Delta^1 \to \mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^{\otimes}$.

Let v denote the cone point of $\langle n \rangle^{\circ \triangleleft}$, and let $f = H|\{v\} \times \Delta^1$. Since each composition

$$\{i\} \times \Delta^1 \xrightarrow{H} \mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$$

is an equivalence for $1 \leq i \leq n$, the assumption that $\mathcal{C}^{\otimes}$ is an ∞-operad guarantees also that the image of f in $\mathcal{C}^{\otimes}$ is an equivalence. We will prove that f is a ϕ-coCartesian lift of f_0. In fact, we will prove the slightly stronger assertion that f is q-Cartesian. Since the inclusion $\{v\} \subseteq \langle n \rangle^{\circ \triangleleft}$ is right cofinal, it will suffice to show that $H|(\langle n \rangle^{\circ \triangleleft} \times \{1\})^{\triangleleft}$ is a q-limit diagram. Since $H|(\langle n \rangle^{\circ \triangleleft} \times \{1\})$ is a q-right Kan extension of $H|(\langle n \rangle^{\circ} \times \{1\})$, it will suffice to show that the restriction $H|(\langle n \rangle^{\circ} \times \{1\})^{\triangleleft}$ is a q-limit diagram (Lemma T.4.3.2.7). Note that $H|(\langle n \rangle^{\circ} \times \Delta^1)$ is a q-right Kan extension of $H|(\langle n \rangle^{\circ} \times \{1\})$ (this follows from the construction, since the maps f_i are ϕ_i-Cartesian and therefore also q-Cartesian, by virtue of Theorem 3.4.3.1). Using Lemma T.4.3.2.7 again, we are reduced to showing that $H|(\langle n \rangle^{\circ} \times \Delta^1)^{\triangleleft}$ is a q-limit diagram. Since the inclusion $\langle n \rangle^{\circ} \times \{0\} \subseteq \langle n \rangle^{\circ} \times \Delta^1$ is right cofinal, it suffices to show that $F' = H|(\langle n \rangle^{\circ \triangleleft} \times \{0\})$ is a q-limit diagram, which follows from our assumption.

The above argument shows that for every $M \in \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ and every morphism $f_0 : A' \to A$ in $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$, there exists a ϕ-Cartesian morphism $f : M' \to M$ lifting f_0 whose image in $\mathcal{C}^{\otimes}$ is an equivalence. This immediately implies (1), and the "only if" direction of (2) follows from the uniqueness properties of Cartesian

morphisms. To prove the "if" direction of (2), suppose that $g : M'' \to M$ is a lift of f_0 whose image in $\mathcal{C}^{\otimes}$ is an equivalence, and let $f : M' \to M$ be as above. Since f is ϕ-Cartesian, we have a commutative diagram

$$
\begin{array}{ccc}
 & M' & \\
{\scriptstyle h}\nearrow & & \searrow{\scriptstyle f} \\
M'' & \xrightarrow{\;\;g\;\;} & M;
\end{array}
$$

to prove that g is ϕ-Cartesian it will suffice to show that h is an equivalence. Since $\mathrm{Mod}_{A'}^{\mathcal{O}}(\mathcal{C})^{\otimes}$ is an ∞-operad, it suffices to show that each of the maps $h_i = \rho_!^i(h)$ is an equivalence in $\mathrm{Mod}_{A'}^{\mathcal{O}}(\mathcal{C})$, for $1 \leq i \leq n$. This follows from Corollary 3.4.3.3, since each h_i maps to an equivalence in $\mathcal{C}$. $\square$

Corollary 3.4.3.5. *Let $\mathcal{O}^{\otimes}$ be a coherent ∞-operad, and let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a $\mathcal{O}$-monoidal ∞-category. Let $X \in \mathcal{O}$, and suppose we are given a commutative diagram*

$$
\begin{array}{ccc}
K & \xrightarrow{\;p\;} & \mathrm{Mod}^{\mathcal{O}}(\mathcal{C})_X^{\otimes} \\
\downarrow & {\scriptstyle \bar{p}}\nearrow & \downarrow{\scriptstyle \psi_X} \\
K^{\triangleleft} & \xrightarrow{\;\bar{p}_0\;} & \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})
\end{array}
$$

such that the induced diagram $K \to \mathcal{C}_X^{\otimes}$ admits a limit. Then there extension $\bar{p}$ of p (as indicated in the diagram) which is a ψ_X-limit diagram. Moreover, an arbitrary extension $\bar{p}$ of p (as in the diagram) is a ψ_X-limit if and only if it induces a limit diagram $K^{\triangleleft} \to \mathcal{C}_X^{\otimes}$.

Proof. Combine Corollary 3.4.3.2 with Corollary T.4.3.1.15. $\square$

Corollary 3.4.3.6. *Let $\mathcal{O}^{\otimes}$ be a coherent ∞-operad, let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads, and let $X \in \mathcal{O}$. Assume that the ∞-category $\mathcal{C}_X^{\otimes}$ admits K-indexed limits, for some simplicial set K. Then:*

(1) *For every algebra object $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$, the ∞-category $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})_X^{\otimes}$ admits K-indexed limits.*

(2) *A functor $\bar{p} : K^{\triangleleft} \to \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})_X^{\otimes}$ is a limit diagram if and only if it induces a limit diagram $K^{\triangleleft} \to \mathcal{C}_X^{\otimes}$.*

We now turn to the proof of Theorem 3.4.3.1. First, choose an inner anodyne map $K \to K'$, where K' is an ∞-category. Since $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \times \mathcal{O}^{\otimes}$ is an ∞-category and ψ is a categorical fibration, we can extend our commutative diagram as indicated:

$$
\begin{array}{ccccc}
K & \longrightarrow & K' & \longrightarrow & \mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^{\otimes} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \psi} \\
K^{\triangleleft} & \longrightarrow & K'^{\triangleleft} & \longrightarrow & \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \times \mathcal{O}^{\otimes}.
\end{array}
$$

Using Proposition T.A.2.3.1, we see that it suffices to prove Theorem 3.4.3.1 after replacing K by K'. We may therefore assume that K is an ∞-category. In this case, the desired result is a consequence of the following:

Proposition 3.4.3.7. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, where $\mathcal{O}^{\otimes}$ is coherent. Let K be an ∞-category. Suppose we are given a commutative diagram*

$$
\begin{array}{ccc}
K & \xrightarrow{\;p\;} & \overline{\mathcal{M}\mathrm{od}}^{\mathcal{O}}(\mathcal{C})^{\otimes} \\
\downarrow & {\scriptstyle \bar{p}}\nearrow & \downarrow{\scriptstyle \psi} \\
K^{\triangleleft} & \xrightarrow{\;\bar{p}_0\;} & {}^{\mathrm{p}}\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}),
\end{array}
$$

where the induced diagram $K^{\triangleleft} \to \mathcal{O}^{\otimes}$ is the constant map taking some value $C \in \mathcal{O}$. Assume that:

(∗) The induced lifting problem

$$
\begin{array}{ccc}
K & \xrightarrow{p'} & \mathcal{C}^{\otimes} \\
\downarrow & \nearrow{\scriptstyle \overline{p}'} & \downarrow{\scriptstyle q} \\
K^{\triangleleft} & \longrightarrow & \mathcal{O}^{\otimes}
\end{array}
$$

admits a solution, where $\overline{p}$ is a q-limit diagram.

Then:

(1) There exists a map $\overline{p}$ making the original diagram commute, such that $\delta \circ \overline{p}$ is a q-limit diagram in $\mathcal{C}^{\otimes}$ (here $\delta : \overline{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$ denotes the map given by composition with the diagonal embedding $\mathcal{O}^{\otimes} \to \mathcal{K}_{\mathcal{O}} \subseteq \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$).

(2) Let $\overline{p}$ be an arbitrary map making the above diagram commute. Then $\overline{p}$ is a ψ-limit diagram if and only if $\delta \circ \overline{p}$ is a q-limit diagram.

The proof of Proposition 3.4.3.7 will require some preliminaries. We first need the following somewhat more elaborate version of Proposition B.4.12:

Proposition 3.4.3.8. *Suppose we are given a diagram of ∞-categories $X \xrightarrow{\phi} Y \xrightarrow{\pi} Z$ where π is a flat categorical fibration and ϕ is a categorical fibration. Let $Y' \subseteq Y$ be a full subcategory, let $X' = X \times_Y Y'$, let $\pi' = \pi|Y'$, and let $\psi : \pi_* X \to \pi'_* X'$ be the canonical map. (See Notation B.4.4.) Let K be an ∞-category and $\overline{p}_0 : K^{\triangleleft} \to \pi'_* X'$ a diagram. Assume that the following conditions are satisfied:*

(i) *The full subcategory $Y' \times_Z K^{\triangleleft} \subseteq Y \times_Z K^{\triangleleft}$ is a cosieve on Y.*

(ii) *For every object $y \in Y'$ and every morphism $f : z \to \pi(y)$ in Z, there exists a π-Cartesian morphism $\overline{f} : \overline{z} \to y$ in Y' such that $\pi(\overline{f}) = f$.*

(iii) *Let π'' denote the projection map $K^{\triangleleft} \times_Z Y \to K^{\triangleleft}$. Then π'' is a coCartesian fibration.*

(iv) *Let v denote the cone point of $K^{\triangleleft}$, let $\mathcal{C} = \pi''^{-1}\{v\}$, and let $\mathcal{C}' = \mathcal{C} \times_Y Y'$. Then $\mathcal{C}'$ is a localization of $\mathcal{C}$.*

Condition (iii) implies that there is a map $\delta' : K^{\triangleleft} \times \mathcal{C} \to K^{\triangleleft} \times_Z Y$ which is the identity on $\{v\} \times \mathcal{C}$ and carries carries $e \times \{C\}$ to a π''-coCartesian edge of $K^{\triangleleft} \times_Z Y$, for each edge e of $K^{\triangleleft}$ and each object C of $\mathcal{C}$. Condition (iv) implies that there is a map $\delta'' : \mathcal{C} \times \Delta^1 \to \mathcal{C}$ such that $\delta''|\mathcal{C} \times \{0\} = \mathrm{id}_{\mathcal{C}}$ and $\delta''|\{C\} \times \Delta^1$ exhibits $\delta''(C, 1)$ as a $\mathcal{C}'$-localization of C, for each $C \in \mathcal{C}$. Let δ denote the composition

$$
K^{\triangleleft} \times \mathcal{C} \times \Delta^1 \xrightarrow{\delta''} K^{\triangleleft} \times \mathcal{C} \xrightarrow{\delta'} K^{\triangleleft} \times_Z Y.
$$

Then:

(1) *Let $\overline{p} : K^{\triangleleft} \to \pi_* X$ be a map lifting $\overline{p}_0$, corresponding to a functor $\overline{F} : K^{\triangleleft} \times_Z Y \to X$. Suppose that for each $C \in \mathcal{C}$, the induced map*

$$
K^{\triangleleft} \times \{C\} \times \Delta^1 \hookrightarrow K^{\triangleleft} \times \mathcal{C} \xrightarrow{\delta} K^{\triangleleft} \times_Z Y \xrightarrow{\overline{F}} X
$$

is a ϕ-limit diagram. Then $\overline{p}$ is a ψ-limit diagram.

(2) *Suppose that $p : K^\lhd \to \pi_* X$ is a map lifting $p_0 = \bar{p}_0 | K$, corresponding to a functor $F : (K^\lhd \times_Z Y') \coprod_{K \times_Z Y'} (K \times_Z Y) \to X$. Assume furthermore that for each $C \in \mathcal{C}$, the induced map*

$$(K^\lhd \times \{C\} \times \{1\}) \coprod_{K \times \{C\} \times \{1\}} (K \times \{C\} \times \Delta^1) \quad \to \quad (K^\lhd \times \mathcal{C} \times \{1\}) \coprod_{K \times \mathcal{C} \times \{1\}} (K \times \mathcal{C} \times \Delta^1)$$

$$\xrightarrow{\delta} \quad (K^\lhd \times_Z Y') \coprod_{K \times_Z Y'} (K \times_Z Y)$$

$$\xrightarrow{F} \quad X$$

can be extended to a ψ-limit diagram lifting the map

$$K^\lhd \times \{C\} \times \Delta^1 \hookrightarrow K^\lhd \times \mathcal{C} \times \Delta^1 \xrightarrow{\delta} K^\lhd \times_Z Y \xrightarrow{\pi''} Y.$$

Then there exists an extension $\bar{p} : K^\lhd \to \pi_* X$ of p lifting $\bar{p}_0$ which satisfies condition (1).

Proof. Let $W = K^\lhd \times_Y Z$ and let W_0 denote the coproduct $(K^\lhd \times_Z Y') \coprod_{K \times_Z Y'} (K \times_Z Y)$; condition (i) allows us to identify W_0 with a full subcategory of W. Let $\bar{p} : K^\lhd \to \pi_* X$ satisfy the condition described in (1), corresponding to a functor $\overline{F} : W \to X$. In view of assumptions (i), (ii), and Proposition B.4.9, it will suffice to show that $\overline{F}$ is a ϕ-right Kan extension of $F = \overline{F} | W_0$. Pick an object $C \in \mathcal{C}$; we wish to show that $\overline{F}$ is a ϕ-right Kan extension of F at C. In other words, we wish to show that the map

$$(W_0 \times_W W_{C/})^\lhd \to W \xrightarrow{\overline{F}} X$$

is a ϕ-limit diagram. Restricting δ, we obtain a map $K^\lhd \times \{C\} \times \Delta^1 \to W$, which we can identify with a map

$$s : (K^\lhd \times \{C\} \times \{1\}) \coprod_{K \times \{C\} \times \{1\}} (K \times \{C\} \times \Delta^1) \to W' \times_W W_{C/}.$$

Since $\bar{p}$ satisfies (1), it will suffice to show that s is right cofinal. We have a commutative diagram

$$(K^\lhd \times \{C\} \times \{1\}) \coprod_{K \times \{C\} \times \{1\}} (K \times \{C\} \times \Delta^1) \xrightarrow{\quad s \quad} W' \times_W W_{C/}$$

$$\theta \searrow \qquad \swarrow \theta'$$

$$K^\lhd$$

The map θ is evidently a coCartesian fibration, and θ' is a coCartesian fibration by virtue of assumptions (i) and (iii). Moreover, the map s carries θ-coCartesian edges to θ'-coCartesian edges. Invoking Lemma 8.1.2.6, we are reduced to showing that for each vertex k of $K^\lhd$, the map of fibers s_k is right cofinal. If $k = v$ is the cone point of $K^\lhd$, then we are required to show that s carries $\{v\} \times \{C\} \times \{1\}$ to an initial object of $\mathcal{C}'_{C/}$: this follows from the definition of δ'. If $k \neq v$, then we are required to show that s carries $K^\lhd \times \{C\} \times \{0\}$ to an initial object of $W_{C/} \times_{K^\lhd} \{k\}$, which follows from our assumption that δ carries $\{v\}^\lhd \times \{C\} \times \{0\}$ to a π''-coCartesian edge of W. This completes the proof of (1).

We now prove (2). The diagram p gives rise to a map $F : W_0 \to X$ fitting into a commutative diagram

$$\begin{array}{ccc} W_0 & \xrightarrow{F} & X \\ \downarrow & \nearrow^{\overline{F}} & \downarrow \phi \\ W & \longrightarrow & Y. \end{array}$$

The above argument shows that a dotted arrow $\overline{F}$ as indicated will correspond to a map $\bar{p} : K^\lhd \to \pi_* X$ satisfying (1) if and only if $\overline{F}$ is a ϕ-right Kan extension of F. In view of Lemma T.4.3.2.13, the existence of such an extension is equivalent to the requirement that for each $C \in \mathcal{C}$, the diagram

$$W_0 \times_W W_{C/} \to W_0 \xrightarrow{F} X$$

can be extended to a ϕ-limit diagram lifting the map

$$(W_0 \times_W W_{C/})^{\triangleleft} \to W \to Y.$$

This follows from the hypothesis of part (2) together with the right cofinality of the map s considered in the proof of (1). □

Definition 3.4.3.9. Let $\langle n \rangle$ be an object of $\mathcal{F}\mathrm{in}_*$. A *splitting* of $\langle n \rangle$ is a pair of inert morphisms $\alpha : \langle n \rangle \to \langle n_0 \rangle$, $\beta : \langle n \rangle \to \langle n_1 \rangle$ with the property that the map $(\alpha^{-1} \coprod \beta^{-1}) : \langle n_0 \rangle^{\circ} \coprod \langle n_1 \rangle^{\circ} \to \langle n \rangle^{\circ}$ is a bijection.

More generally, let K be a simplicial set. A *splitting* of a diagram $p : K \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a pair of natural transformations $\alpha : p \to p_0$, $\beta : p \to p_1$ with the following property: for every vertex k of K, the morphisms $\alpha_k : p(k) \to p_0(k)$ and $\beta_k : p(k) \to p_1(k)$ determine a splitting of $p(k)$.

We will say that a natural transformation $\alpha : p \to p_0$ of diagrams $p, p_0 : K \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *splits* if there exists another natural transformation $\beta : p \to p_1$ which gives a splitting of p.

Remark 3.4.3.10. Let $\alpha : p \to p_0$ be a natural transformation of diagrams $p, p_0 : K \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. If α splits, then the natural transformation $\beta : p \to p_1$ which provides the splitting of p is well-defined up to (unique) equivalence. Moreover, a bit of elementary combinatorics shows that α splits if and only if it satisfies the following conditions:

(1) The natural transformation α is *inert*: that is, for each vertex $k \in K$, the map $\alpha_k : p(k) \to p_0(k)$ is an inert morphism in $\mathcal{F}\mathrm{in}_*$.

(2) For every edge $e : x \to x'$ in K, consider the diagram

$$
\begin{array}{ccc}
\langle n \rangle & \xrightarrow{\alpha_x} & \langle n_0 \rangle \\
{\scriptstyle p(e)} \downarrow & & \downarrow {\scriptstyle p_0(e)} \\
\langle m \rangle & \xrightarrow{\alpha_{x'}} & \langle m_0 \rangle
\end{array}
$$

in $\mathcal{F}\mathrm{in}_*$ obtained by applying α to e. Then $p(e)$ carries $(\alpha_x^{-1} \langle n_0 \rangle^{\circ})_* \subseteq \langle n \rangle$ into $(\alpha_x'^{-1} \langle m_0 \rangle^{\circ})_* \subseteq \langle m \rangle$.

Definition 3.4.3.11. Let $q : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad. We will say that a natural transformation $\alpha : p \to p_0$ of diagrams $p, p_0 : K \to \mathcal{O}^{\otimes}$ is *inert* if the induced map $\alpha_k : p(k) \to p_0(k)$ is an inert morphism in $\mathcal{O}^{\otimes}$ for every vertex $k \in K$.

A *splitting* of $p : K \to \mathcal{O}^{\otimes}$ is a pair of inert natural transformations $\alpha : p \to p_0$, $\beta : p \to p_1$ such that the induced transformations $q \circ p_0 \leftarrow q \circ p \to q \circ p_1$ determine a splitting of $q \circ p : K \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, in the sense of Definition 3.4.3.9.

We will say that an inert natural transformation $\alpha : p \to p_0$ is *split* if there exists another inert natural transformation $\beta : p \to p_1$ such that α and β are a splitting of p. In this case, we will say that β is a *complement* to α.

Lemma 3.4.3.12. *Let $q : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and let $\alpha : p \to p_0$ be an inert natural transformation of diagrams $p, p_0 : K \to \mathcal{O}^{\otimes}$. The following conditions are equivalent:*

(1) *The natural transformation α is split: that is, there exists a complement $\beta : p \to p_1$ to α.*

(2) *The natural transformation α induces a split natural transformation $\overline{\alpha} : q \circ p \to q \circ p_0$.*

Moreover, if these conditions are satisfied, then β is determined uniquely up to equivalence.

Proof. The implication $(1) \Rightarrow (2)$ is clear: if $\beta : p \to p_1$ is a complement to α, then the induced transformations $q \circ p_0 \leftarrow q \circ p \to q \circ p_1$ form a splitting of $q \circ p : K \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Conversely, suppose that $q \circ p$ is split, and choose a complement $\overline{\beta} : q \circ p \to \overline{p}_1$ to $\overline{\alpha}$. Then $\overline{\beta}$ is inert, so we can choose a q-coCartesian lift $\beta : p \to p_1$ of $\overline{\beta}$ which is a complement to α. The uniqueness of β follows from the observation that $\overline{\beta}$ and its q-coCartesian lift are both well-defined up to equivalence. □

Lemma 3.4.3.13. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads. Let $\alpha : X \to X_0$ and $\beta : X \to X_1$ be morphisms in $\mathcal{O}^{\otimes}$ which determine a splitting of X, and suppose that $\overline{X}_0$ and $\overline{X}_1$ are objects of $\mathcal{C}^{\otimes}$ lying over X_0 and X_1, respectively. Then:*

(1) *Let $\overline{\alpha} : \overline{X} \to \overline{X}_0$ and $\overline{\beta} : \overline{X} \to \overline{X}_1$ be morphisms in $\mathcal{C}^{\otimes}$ lying over α and β. Then $\overline{\alpha}$ and $\overline{\beta}$ determine a splitting of $\overline{X}$ if and only if they exhibit $\overline{X}$ as a q-product of $\overline{X}_0$ and $\overline{X}_1$.*

(2) *There exist morphisms $\overline{\alpha} : \overline{X} \to \overline{X}_0$ and $\overline{\beta} : \overline{X} \to \overline{X}_1$ satisfying the equivalent conditions of (1).*

Proof. We will prove (2) and the "if" direction of (1); the "only if" direction follows from (2) together with the uniqueness properties of q-limit diagrams. We begin with (1). Choose a diagram $\sigma : \Delta^1 \times \Delta 1 \to \mathcal{C}^{\otimes}$

$$
\begin{array}{ccc}
\overline{X} & \xrightarrow{\overline{\alpha}} & \overline{X}_0 \\
{\scriptstyle\overline{\beta}}\downarrow & & \downarrow \\
\overline{X}_1 & \longrightarrow & 0,
\end{array}
$$

where 0 is a final object of $\mathcal{C}^{\otimes}$ (in other words, 0 lies in $\mathcal{C}^{\otimes}_{\langle 0 \rangle}$). Let $K \simeq \Lambda_0^2$ denote the full subcategory of $\Delta^1 \times \Delta^1$ obtained by removing the final object. Since 0 is final in $\mathcal{C}^{\otimes}$ and $q(0)$ is final in $\mathcal{O}^{\otimes}$, we deduce that 0 is a a q-final object of $\mathcal{C}^{\otimes}$ (Proposition T.4.3.1.5), so that σ is a q-right Kan extension of $\sigma|K$. It follows from Proposition 2.3.2.5 that the σ is a q-limit diagram. Applying Lemma T.4.3.2.7, we deduce that $\sigma|K$ is a q-limit, so that σ exhibits $\overline{X}$ as a q-product of $\overline{X}_0$ and $\overline{X}_1$.

We now prove (2). Let 0 be an object of $\mathcal{C}^{\otimes}_{\langle 0 \rangle}$. Since 0 is a final object of $\mathcal{C}^{\otimes}$, we can find morphisms (automatically inert) $\gamma : \overline{X}_0 \to 0$ and $\delta : \overline{X}_1 \to 0$ in $\mathcal{C}^{\otimes}$. Since $q(0)$ is a final object of $\mathcal{O}^{\otimes}$, we can find a commutative square

$$
\begin{array}{ccc}
X & \xrightarrow{\alpha} & X_0 \\
{\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle q(\gamma)} \\
X_1 & \xrightarrow{q(\delta)} & q(0)
\end{array}
$$

in $\mathcal{O}^{\otimes}$. Using Proposition 2.3.2.5, deduce the existence of a q-limit diagram σ :

$$
\begin{array}{ccc}
\overline{X} & \xrightarrow{\overline{\alpha}} & \overline{X}_0 \\
{\scriptstyle\overline{\beta}}\downarrow & & \downarrow{\scriptstyle\gamma} \\
\overline{X}_1 & \xrightarrow{\delta} & 0
\end{array}
$$

in $\mathcal{C}^{\otimes}$, where $\overline{\alpha}$ and $\overline{\beta}$ are inert and therefore determine a splitting of $\overline{X}$. $\square$

Corollary 3.4.3.14. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, let $\mathcal{X}$ be the full subcategory of $\mathrm{Fun}(\Lambda_0^2, \mathcal{C}^{\otimes})$ be the full subcategory spanned by those diagrams $X_0 \leftarrow X \to X_1$ which determine a splitting of X, and let $\mathcal{Y} \subseteq \mathrm{Fun}(\Lambda_0^2, \mathcal{O}^{\otimes})$ be defined similarly. Then the canonical map*

$$
\mathcal{X} \to \mathcal{Y} \times_{\mathrm{Fun}(\{1\}, \mathcal{O}^{\otimes}) \times \mathrm{Fun}(\{2\}, \mathcal{O}^{\otimes})} (\mathrm{Fun}(\{1\}, \mathcal{C}^{\otimes}) \times \mathrm{Fun}(\{2\}, \mathcal{C}^{\otimes}))
$$

is a trivial Kan fibration.

Proof. Combine Lemma 3.4.3.13 with Proposition T.4.3.2.15. $\square$

Lemma 3.4.3.15. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads. Suppose we are given a split natural transformation $\alpha : p \to p_0$ of diagrams $p, p_0 : K^{\triangleleft} \to \mathcal{O}^{\otimes}$. Let $\overline{p}_0 : K^{\triangleleft} \to \mathcal{C}^{\otimes}$ be a diagram lifting p_0, let $\overline{p}' : K \to \mathcal{C}^{\otimes}$ be a diagram lifting $p' = p|K$, and let $\overline{\alpha}' : \overline{p}' \to \overline{p}_0|K$ be a natural transformation lifting $\alpha' = \alpha|(\Delta^1 \times K)$. Suppose that the following condition is satisfied:*

($*$) *Let $\beta : p \to p_1$ be a complement to α, let $\beta' = \beta|(\Delta^1 \times K)$, and let $\overline{\beta}' : \overline{p}' \to \overline{p}_1'$ be a q-coCartesian natural transformation lifting β' (so that $\overline{\beta}'$ is a complement to $\overline{\alpha}'$). Then $\overline{p}_1'$ can be extended to a q-limit diagram $\overline{p}_1 : K^\triangleleft \to \mathcal{C}^{\otimes}$ such that $q \circ \overline{p}_1 = p_1$.*

Then:

(1) *Let $\overline{\alpha} : \overline{p} \to \overline{p}_0$ be a natural transformation of diagrams $\overline{p}, \overline{p}_0 : K^\triangleleft \to \mathcal{C}^{\otimes}$ which extends $\overline{\alpha}'$ and lies over α. The following conditions are equivalent:*

  (*i*) *The map of simplicial sets $\overline{\alpha} : \Delta^1 \times K^\triangleleft \to \mathcal{C}^{\otimes}$ is a q-limit diagram.*

  (*ii*) *The natural transformation $\overline{\alpha}$ is inert (and therefore split), and if $\overline{\beta} : \overline{p} \to \overline{p}_1$ is a complement to $\overline{\alpha}$, then $\overline{p}_1$ is a q-limit diagram.*

(2) *There exists a natural transformation $\overline{\alpha} : \overline{p} \to \overline{p}_0$ satisfying the equivalent conditions of (1).*

Proof. We first prove the implication $(ii) \Rightarrow (i)$ of assertion (1). Choose a complement $\overline{\beta} : \overline{p} \to \overline{p}_1$ to $\overline{\alpha}$, so that $\overline{\alpha}$ and $\overline{\beta}$ together determine a map $\overline{F} : \Lambda_0^2 \times K^\triangleleft \to \mathcal{C}^{\otimes}$ with $\overline{F}|\Delta^{\{0,1\}} \times K^\triangleleft = \overline{\alpha}$ and $\overline{F}|\Delta^{\{0,2\}} \times K^\triangleleft = \overline{\beta}$. Using the small object argument, we can choose an inner anodyne map $K \to K'$ which is bijective on vertices, where K' is an ∞-category. Since $\mathcal{C}^{\otimes}$ is an ∞-category, the map $\overline{F}$ factors as a composition $K^\triangleleft \times \Lambda_0^2 \to K'^\triangleleft \times \Lambda_0^2 \to \mathcal{C}^{\otimes}$. We may therefore replace K by K' and thereby reduce to the case where K is an ∞-category.

The inclusion $i : K \times \{0\} \subseteq K \times \Delta^{\{0,2\}}$ is left anodyne, so that i is right cofinal. It will therefore suffice to show that the restriction $\overline{F}_0$ of $\overline{F}$ to $(K^\triangleleft \times \Delta^{\{0,1\}}) \coprod_{K \times \{0\}} (K \times \Delta^{\{0,2\}})$ is a q-limit diagram. Since $\overline{p}_1$ is a q-limit diagram, $\overline{F}$ is a q-right Kan extension of $\overline{F}_0$; according to Lemma T.4.3.2.7 it will suffice to prove that $\overline{F}$ is a q-limit diagram.

Let v denote the cone point of $K^\triangleleft$. Let $\mathcal{D}$ be the full subcategory of $K^\triangleleft \times \Lambda_0^2$ spanned by $K^\triangleleft \times \{1\}$, $K^\triangleleft \times \{2\}$, and $(v, 0)$. Using Lemma 3.4.3.13, we deduce that $\overline{F}$ is a q-right Kan extension of $F|\mathcal{D}$. Using Lemma T.4.3.2.7 again, we are reduced to proving that $\overline{F}|\mathcal{D}$ is a q-limit diagram. Since the inclusion

$$\{(v,1)\} \coprod \{(v,2)\} \subseteq (K^\triangleleft \times \{1\}) \coprod (K^\triangleleft \times \{2\})$$

is right cofinal, it suffices to show that $F|\{v\} \times \Lambda_0^2$ is a q-limit diagram, which follows from Lemma 3.4.3.13. This completes the verification of condition (i).

We now prove (2). Choose a complement $\beta : p \to p_1$ to the split natural transformation α, let $\beta' = \beta|\Delta^1 \times K$, and choose an q-coCartesian natural transformation $\overline{\beta}' : \overline{p}' \to \overline{p}_1'$ lifting β'. Invoking assumption ($*$), we can extend $\overline{p}_1'$ to a q-limit diagram $\overline{p}_1 : K^\triangleleft \to \mathcal{C}^{\otimes}$ such that $q \circ \overline{p}_1 = p_1$. The maps $\overline{p}_0, \overline{p}_1, \overline{\alpha}'$ and $\overline{\beta}'$ can be amalgamated to give a map

$$F : (\Lambda_0^2 \times K) \coprod_{(\{1,2\}) \times K} (\{1,2\} \times K^\triangleleft) \to \mathcal{C}^{\otimes} .$$

Using Corollary 3.4.3.14, we can extend F to a map $\overline{F} : \Lambda_0^2 \times K^\triangleleft \to \mathcal{C}^{\otimes}$ corresponding to a pair of morphisms $\overline{\alpha} : \overline{p} \to \overline{p}_0$ and $\overline{\beta} : \overline{p} \to \overline{p}_1$ having the desired properties.

The implication $(i) \Rightarrow (ii)$ of (1) now follows from (2), together with the uniqueness properties of q-limit diagrams. $\qquad\square$

Proof of Proposition 3.4.3.7. We first treat the case where K is an ∞-category. Let $Y = \mathcal{K}_\mathcal{O}$ and $Y' = \mathcal{K}_\mathcal{O}^0 \subseteq Y$. Let $\pi : Y \to \mathcal{O}^{\otimes}$ be the map given by evaluation at $\{0\}$, and let $\pi' = \pi|Y'$. Our assumption that $\mathcal{O}^{\otimes}$ is coherent guarantees that π is a flat categorical fibration. Let $X = Y \times_{\mathrm{Fun}(\{1\}, \mathcal{O}^{\otimes})} \mathcal{C}^{\otimes}$, and let $X' = X \times_Y Y'$. The map $\psi : \overline{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes} \to {}^{\mathrm{P}}\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ can be identified with a restriction of the map $\pi_* X \to \pi_*' X'$. We are

given a diagram

$$
\begin{array}{ccc}
K & \xrightarrow{\ p\ } & \pi_* X \\
\downarrow & \overline{p}\ \nearrow \quad \nearrow & \downarrow \\
K^\triangleleft & \xrightarrow{\ \overline{p}_0\ } & \pi'_* X'.
\end{array}
$$

We claim that this situation satisfies the hypotheses of Proposition 3.4.3.8:

(i) The full subcategory $Y' \times_{\mathcal{O}^\otimes} K^\triangleleft$ is a cosieve on $Y \times_{\mathcal{O}^\otimes} K^\triangleleft$. Since the map $K^\triangleleft \to \mathcal{O}^\otimes$ is constant taking some value $C \in \mathcal{O}$, it will suffice to show that the $Y' \times_{\mathcal{O}^\otimes} \{C\}$ is a cosieve on $Y \times_{\mathcal{O}^\otimes} \{C\}$. Unwinding the definitions, this amounts to the following assertion: given a commutative diagram

$$
\begin{array}{ccc}
C & \longrightarrow & D \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
C & \longrightarrow & D'
\end{array}
$$

in $\mathcal{O}^\otimes$, if the upper horizontal map is null then the lower horizontal map is null. This is clear, since the collection of null morphisms in $\mathcal{O}^\otimes$ is closed under composition with other morphisms.

(ii) For every object $y \in Y'$ and every morphism $f : z \to \pi(y)$ in $\mathcal{O}$, there exists a π-Cartesian morphism $\overline{f} : \overline{z} \to y$ in Y' such that $\pi(\overline{f}) = f$. We can identify y with a semi-inert morphism $y_0 \to y_1$ in $\mathcal{O}^\otimes$, and f with a morphism $z \to y_0$ in $\mathcal{O}^\otimes$. Using Corollary T.2.4.7.12, we see that the morphism $\overline{f}$ can be taken to correspond to the commutative diagram

$$
\begin{array}{ccc}
z & \longrightarrow & y_1 \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
y_0 & \longrightarrow & y_1
\end{array}
$$

in $\mathcal{O}^\otimes$: our assumption that $y_0 \to y_1$ is null guarantees that the composite map $z \to y$ is also null.

(iii) Let π'' denote the projection map $K^\triangleleft \times_{\mathcal{O}^\otimes} Y \to K^\triangleleft$. Then π'' is a coCartesian fibration. This is clear, since π'' is a pullback of the coCartesian fibration $(\mathcal{K}_\mathcal{O} \times_{\mathcal{O}^\otimes} \{C\}) \to \Delta^0$.

(iv) Let v denote the cone point of $K^\triangleleft$ and let $\mathcal{D} = \pi''^{-1}\{v\}$. Then $\mathcal{D}' = \mathcal{D} \times_Y Y'$ is a localization of $\mathcal{D}$. We can identify an object of $\mathcal{D}$ with a semi-inert morphism $f : C \to C'$ in $\mathcal{O}^\otimes$. We wish to prove that for any such object f, there exists a morphism $f \to g$ in $\mathcal{D}$ which exhibits g as a $\mathcal{D}'$-localization of f. Let $f_0 : \langle 1 \rangle \to \langle k \rangle$ denote the underlying morphism in $\mathcal{F}\mathrm{in}_*$. If f_0 is null, then $f \in \mathcal{D}'$ and there is nothing to prove. Otherwise, $f_0(1) = i$ for some $1 \le i \le k$. Choose an inert map $h_0 : \langle k \rangle \to \langle k - 1 \rangle$ such that $h_0(i) = *$, and choose an inert morphism $h : C' \to D$ in $\mathcal{O}^\otimes$ lifting h_0. We then have a commutative diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\ f\ } & C' \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle h} \\
C & \xrightarrow{\ g\ } & D
\end{array}
$$

in $\mathcal{O}^\otimes$, corresponding to a map $\alpha : f \to g$ in $\mathcal{D}$; by construction, g is null so that $g \in \mathcal{D}'$. We claim that α exhibits g as a $\mathcal{D}'$-localization of $\mathcal{C}$. To prove this, choose any object $g' : C \to D'$ in $\mathcal{D}'$; we wish to show that composition with α induces a homotopy equivalence

$$
\mathrm{Map}_\mathcal{D}(g, g') \simeq \mathrm{Map}_{(\mathcal{O}^\otimes)^{C/}}(g, g') \to \mathrm{Map}_{(\mathcal{O}^\otimes)^{C/}}(f, g') \simeq \mathrm{Map}_\mathcal{D}(f, g).
$$

Since the projection map

$$(\mathcal{O}^{\otimes})^{C/} \to \mathcal{O}^{\otimes}$$

is a left fibration, it will suffice to show that the map $\mathrm{Map}_{\mathcal{O}^{\otimes}}(D, D') \to \mathrm{Map}^0_{\mathcal{O}^{\otimes}}(C', D')$, where the superscript indicates that we consider only morphisms $C' \to D'$ such that the underlying map $\langle k \rangle \to \langle k' \rangle$ carries i to the base point $* \in \langle k' \rangle$. Since h is inert, this follows from the observation that composition with h_0 induces an injection $\mathrm{Hom}_{\mathcal{F}\mathrm{in}_*}(\langle k-1 \rangle, \langle k' \rangle) \to \mathrm{Hom}_{\mathcal{F}\mathrm{in}_*}(\langle k \rangle, \langle k' \rangle)$ whose image consists of those maps which carry i to the base point.

Fix an object of $\mathcal{D}$ corresponding to a semi-inert morphism $f : C \to C'$ in $\mathcal{O}^{\otimes}$, and let $\alpha : f \to g$ be a map in $\mathcal{D}$ which exhibits g as a $\mathcal{D}'$-localization of f (as in the proof of (iv)). Using the maps p and $\bar{p}_0$, we get commutative diagram

$$
\begin{array}{ccc}
(K \times \Delta^1) \coprod_{K \times \{1\}} (K^{\triangleleft} \times \{1\}) & \longrightarrow & \mathcal{C}^{\otimes} \\
\downarrow & \overset{\theta}{\nearrow} & \downarrow q \\
K^{\triangleleft} \times \Delta^1 & \longrightarrow & \mathcal{O}^{\otimes}.
\end{array}
$$

To apply Proposition 3.4.3.8, we must know that every such diagram admits an extension as indicated, where θ is a q-limit. This follows from Lemma 3.4.3.15 and assumption $(*)$. Moreover, we obtain the following criterion for testing whether θ is a q-limit diagram:

$(*')$ Let $\theta : K^{\triangleleft} \times \Delta^1 \to \mathcal{C}^{\otimes}$ be as above, and view θ as a natural transformation $d \to d_0$ of diagrams $d, d_0 : K^{\triangleleft} \to \mathcal{C}^{\otimes}$. Then θ is a q-limit diagram if and only if it is an inert (and therefore split) natural transformation, and admits a complement $d \to d_1$ where $d_1 : K^{\triangleleft} \to \mathcal{C}^{\otimes}$ is a q-limit diagram.

Applying Proposition 3.4.3.8, we obtain the following:

(a) There exists a solution to the lifting problem

$$
\begin{array}{ccc}
K & \overset{p}{\longrightarrow} & \widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes} \\
\downarrow & \overset{\bar{p}}{\nearrow} & \downarrow \overline{\psi} \\
K^{\triangleleft} & \overset{\bar{p}_0}{\longrightarrow} & \widetilde{\mathrm{Alg}}_{/\mathcal{O}}(\mathcal{C}),
\end{array}
$$

where $\bar{p}$ is an $\overline{\psi}$-limit diagram.

(b) An arbitrary extension $\bar{p}$ as above is an $\overline{\psi}$-limit diagram if and only if the following condition is satisfied:

$(*'')$ For every object $f : C \to C'$ in $\mathcal{D}$ and every morphism $\alpha : f \to g$ in $\mathcal{D}$ which exhibits g as a $\mathcal{D}'$-localization of f, if $\theta : K^{\triangleleft} \times \Delta^1 \to \mathcal{C}^{\otimes}$ is defined as above, then θ is a split natural transformation of diagrams $d, d_0 : K^{\triangleleft} \to \mathcal{C}^{\otimes}$ and admits a complement $d \to d_1$ where $d_1 : K^{\triangleleft} \to \mathcal{C}^{\otimes}$ is a q-limit diagram.

To complete the proof, we must show that condition $(*'')$ is equivalent to the following pair of assertions:

(I) The map $\bar{p}$ carries $K^{\triangleleft}$ into the full subcategory $\overline{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes} \subseteq \widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes}$. (Since we know already that p has this property, it suffices to check that $\bar{p}$ carries the cone point v of $K^{\triangleleft}$ into $\overline{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes}$).

(II) The composite map

$$\bar{p} : K^{\triangleleft} \to \widetilde{\mathrm{Mod}}^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$$

is a q-limit diagram. Here the second map is induced by composition with the diagonal embedding $\mathcal{O}^{\otimes} \hookrightarrow \mathcal{K}_{\mathcal{O}}$.

Assume first that condition $(*'')$ is satisfied by $\overline{p} : K^{\triangleleft} \to \widetilde{\mathcal{M}\mathrm{od}}^{\mathcal{O}}(\mathcal{C})^{\otimes}$; we will prove that $\overline{p}$ also satisfies (I) and (II). We can identify $\overline{p}$ with a map $\overline{P} : K^{\triangleleft} \times \mathcal{D} \to \mathcal{C}^{\otimes}$. We first prove (II). Fix an object $f \in \mathcal{D}$ corresponding to an equivalence $C \to C'$ in $\mathcal{O}^{\otimes}$; we will show that $\overline{P}_f = \overline{P}|K^{\triangleleft} \times \{f\}$ is a q-limit diagram. Choose a morphism $f \to g$ which exhibits g as a $\mathcal{D}'$-localization of f, and let $\theta : \overline{P}_f \to \overline{P}_g$ be the induced natural transformation as in $(*'')$. Let $\theta' : \overline{P}_f \to d_1$ be a complement to θ. Since $C \simeq C'$, $\overline{P}_g$ takes values in $\mathcal{C}^{\otimes}_{\langle 0 \rangle}$, so θ' is an equivalence of diagrams. Condition $(*'')$ implies that d_1 is a q-limit diagram, so that $\overline{P}_f$ is a q-limit diagram.

To prove (I), we must show that for every morphism $\alpha : f \to f'$ in $\mathcal{D}$ whose image in $\mathcal{K}_{\mathcal{O}}$ is inert, the induced map $\overline{P}(v, f) \to \overline{P}(v, f')$ is inert in $\mathcal{C}^{\otimes}$. There are several cases to consider:

$(I1)$ The map f belongs to $\mathcal{D}'$. Then $f' \in \mathcal{D}'$ and the desired result follows from our assumption that $\overline{p}_0$ factors through $^{\mathrm{P}}\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$.

$(I2)$ The map f does not belong to $\mathcal{D}'$, but f' does. Then α factors as a composition

$$f \xrightarrow{\alpha'} g \xrightarrow{\alpha''} f',$$

where α' exhibits g as a $\mathcal{D}'$-localization of f. Since the composition of inert morphisms in $\mathcal{C}^{\otimes}$ is inert and $g \in \mathcal{D}'$, we can apply $(I1)$ to reduce to the case where $\alpha = \alpha'$. In this case, the desired result follows immediately from $(*'')$.

$(I3)$ The map f' is an equivalence in $\mathcal{O}^{\otimes}$. Let $\overline{\alpha} : \overline{P}_f \to \overline{P}_{f'}$ be the natural transformation induced by α; it will suffice to show that this natural transformation in inert. Let $\beta : f \to g$ be a map in $\mathcal{D}$ which exhibits g as a $\mathcal{D}'$-localization of f. Then β induces a natural transformation $\theta : \overline{P}_f \to \overline{P}_g$. Using $(*'')$, we can choose a complement $\theta' : \overline{P}_f \to d_1$ to θ. Since θ' is a q-coCartesian transformation of diagrams, we obtain a factorization of $\overline{\alpha}$ as a composition

$$\overline{P}_f \xrightarrow{\theta'} d_1 \xrightarrow{\gamma} \overline{P}_{f'}.$$

We wish to prove that γ is an equivalence. Since d_1 is a q-limit diagram (by virtue of $(*'')$) and $\overline{P}_{f'}$ is a q-limit diagram (by virtue of (I)), it will suffice to show that γ induces an equivalence $d_1|K \to \overline{P}_{f'}|K$. This follows from the fact that p factors through $\overline{\mathcal{M}\mathrm{od}}^{\mathcal{O}}(\mathcal{C})^{\otimes}$.

$(I4)$ The map f' does not belong to $\mathcal{D}'$. Let us identify f' with a semi-inert morphism $C \to C'$ in $\mathcal{O}^{\otimes}$, lying over an injective map $j : \langle 1 \rangle \hookrightarrow \langle k \rangle$ in $\mathcal{F}\mathrm{in}_*$. Choose a splitting $C'_0 \leftarrow C' \to C'_1$ of C' corresponding to the decomposition $\langle k \rangle^{\circ} \simeq \langle 1 \rangle^{\circ} \coprod \langle k - 1 \rangle^{\circ}$ induced by j. This splitting can be lifted to a pair of morphisms $f' \to f'_0$ and $f' \to f'_1$ in $\mathcal{D}$. Using $(I2)$ and $(I3)$, we deduce that the maps $\overline{P}(v, f') \to \overline{P}(v, f'_0)$ and $\overline{P}(v, f') \to \overline{P}(v, f'_1)$ are inert. Since $\mathcal{C}^{\otimes}$ is an ∞-operad, to prove that the map $\overline{P}(v, f) \to \overline{P}(v, f')$ is inert, it will suffices to show that the composite maps $\overline{P}(v, f) \to \overline{P}(v, f'_0)$ and $\overline{P}(v, f) \to \overline{P}(v, f'_1)$ are inert. In other words, we may replace f' by f'_0 or f'_1 and thereby reduce to the cases $(I2)$ and $(I3)$.

Now suppose that conditions (I) and (II) are satisfied; we will prove $(*'')$. Fix an object f in $\mathcal{D}$, let $\alpha : f \to g$ be a map which exhibits g as a $\mathcal{D}'$-localization of f, let $\theta : \overline{P}_f \to \overline{P}_g$ be the induced natural transformation. Our construction of α together with assumption (I) guarantees that θ is split; let $\theta' : \overline{P}_f \to d_1$ be a complement to θ. We wish to prove that d_1 is a q-limit diagram. If $f \in \mathcal{D}'$, then d_1 takes values in $\mathcal{C}^{\otimes}_{\langle 0 \rangle}$ and the result is obvious. We may therefore assume that $f : C \to C'$ induces an injective map $\langle 1 \rangle \to \langle k \rangle$ in $\mathcal{F}\mathrm{in}_*$; choose a splitting $C'_0 \leftarrow C' \to C'_1$ corresponding the decomposition $\langle k \rangle^{\circ} \simeq \langle 1 \rangle^{\circ} \coprod \langle k - 1 \rangle^{\circ}$. This splitting lifts to a pair of maps $\beta_0 : f \to f_0$, $\beta_1 : f \to f_1$ in $\mathcal{D}$, and we can identify β_1 with $\alpha : f \to g$. Using assumption (I), we see that β_0 induces a transformation $\overline{P}_f \to \overline{P}_{f_0}$ which is a complement to θ. We are therefore reduced to showing that $\overline{P}_{f_0}$ is a q-limit diagram. This follows from (II), since $f_0 : C \to C'_0$ is an equivalence in $\mathcal{C}$ and therefore equivalent (in $\mathcal{D}$) to the identity map id_C. $\square$

3.4.4 Colimits of Modules

Let $\mathcal{C}$ be a symmetric monoidal category, let A be a commutative algebra object of $\mathcal{C}$, let $\{M_\alpha\}$ be a diagram in the category of A-modules, and let $M = \varinjlim M_\alpha$ be a colimit of this diagram in the underling category $\mathcal{C}$. For each index α, we have a canonical map $A \otimes M_\alpha \to M_\alpha \to M$. These maps together determine a morphism $\varinjlim(A \otimes M_\alpha) \to M$. If the tensor product with A preserves colimits, then we can identify the domain of this map with $A \otimes M$, and the object $M \in \mathcal{C}$ inherits the structure of an A-module (which is then a colimit for the diagram $\{M_\alpha\}$ in the category of A-modules).

Our goal in this section is to obtain an ∞-categorical generalization of the above discussion. We first formalize the idea that "tensor products commute with colimits".

Definition 3.4.4.1. Let $\mathcal{O}^\otimes$ be an ∞-operad. We will say that a fibration of ∞-operads $q : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ is a *presentable $\mathcal{O}$-monoidal ∞-category* if the following conditions are satisfied:

(1) The functor q is a coCartesian fibration of ∞-operads.

(2) The coCartesian fibration q is compatible with small colimits (Definition 3.1.1.18).

(3) For each $X \in \mathcal{O}$, the fiber $\mathcal{C}_X^\otimes$ is a presentable ∞-category.

Theorem 3.4.4.2. *Let $\mathcal{O}^\otimes$ be a small coherent ∞-operad, and let $q : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a presentable $\mathcal{O}$-monoidal ∞-category. Let $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ be a $\mathcal{O}$-algebra object of $\mathcal{C}$. Then the induced map $\psi : \mathrm{Mod}_A^\mathcal{O}(\mathcal{C})^\otimes \to \mathcal{O}^\otimes$ exhibits $\mathrm{Mod}_A^\mathcal{O}(\mathcal{C})^\otimes$ as a presentable $\mathcal{O}$-monoidal ∞-category.*

We will deduce Theorem 3.4.4.2 from the a more general result, which can be used to construct colimits in ∞-categories of module objects in a wider variety of situations. The statement is somewhat complicated, since the idea that "tensor product with A preserves colimits" needs to be formulated using the theory of operadic colimit diagrams described in §3.1.3.

Theorem 3.4.4.3. *Let $q : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a fibration of ∞-operads, where $\mathcal{O}^\otimes$ is coherent. Let K be an ∞-category and let $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ be a $\mathcal{O}$-algebra object of $\mathcal{C}$. Suppose we are given a commutative diagram*

$$
\begin{array}{ccc}
K & \xrightarrow{\;p\;} & \mathrm{Mod}_A^\mathcal{O}(\mathcal{C})^\otimes \\
\downarrow & \overset{\overline{p}}{\nearrow} & \downarrow{\scriptstyle \psi} \\
K^\triangleright & \xrightarrow{\quad} & \mathcal{O}^\otimes .
\end{array}
$$

Let $\overline{\mathcal{D}} = K^\triangleright \times_{\mathcal{O}^\otimes} \mathcal{K}_\mathcal{O}$ and let $\mathcal{D} = K \times_{\mathcal{O}^\otimes} \mathcal{K}_\mathcal{O} \subseteq \overline{\mathcal{D}}$, so that p classifies a diagram $F : \mathcal{D} \to \mathcal{C}^\otimes$. Assume the following:

(i) *The induced map $K^\triangleright \to \mathcal{O}^\otimes$ factors through $\mathcal{O}_{\mathrm{act}}^\otimes$, and carries the cone point of $K^\triangleright$ to an object $X \in \mathcal{O}$.*

(ii) *Let $D = (v, \mathrm{id}_X) \in \overline{\mathcal{D}}$. Let $\mathcal{D}_{/D}^{\mathrm{act}}$ denote the full subcategory of $\mathcal{D} \times_{\overline{\mathcal{D}}} \overline{\mathcal{D}}_{/D}$ spanned by those morphisms $D' \to D$ in $\overline{\mathcal{D}}$ which induce diagrams*

$$
\begin{array}{ccc}
X' & \longrightarrow & Y' \\
\downarrow & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{\;\mathrm{id}\;} & X
\end{array}
$$

in $\mathcal{O}^\otimes$, where f is active. Then the diagram

$$
\mathcal{D}_{/D}^{\mathrm{act}} \to \mathcal{D} \xrightarrow{\;F\;} \mathcal{C}^\otimes
$$

can be extended to a q-operadic colimit diagram $(\mathcal{D}_{/D}^{\mathrm{act}})^\triangleright \to \mathcal{C}^\otimes$ lying over the composite map

$$
(\mathcal{D}_{/D}^{\mathrm{act}})^\triangleright \to \overline{\mathcal{D}}_{/D}^{\triangleright} \to \overline{\mathcal{D}} \to K^\triangleright \to \mathcal{O}^\otimes .
$$

Then:

(1) *Let $\overline{p}$ be an extension of p as indicated in the above diagram, corresponding to a map $\overline{F} : \overline{\mathcal{D}} \to \mathcal{C}^{\otimes}$.
   Then $\overline{p}$ is an operadic ψ-colimit diagram if and only if the following condition is satisfied:*

   (∗) *For every object (v, id_X) as in (ii), the map*

$$(\mathcal{D}_{/D}^{\mathrm{act}})^{\triangleright} \to \overline{\mathcal{D}}_{/D}^{\triangleright} \to \overline{\mathcal{D}} \xrightarrow{\overline{F}} \mathcal{C}^{\otimes}$$

   is an operadic q-colimit diagram.

(2) *There exists an extension $\overline{p}$ of p satisfying condition $(∗)$.*

The proof of Theorem 3.4.4.3 is rather technical, and will be given at the end of this section.

Corollary 3.4.4.4. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, where $\mathcal{O}^{\otimes}$ is coherent. Let $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$.
Let $f : M_0 \to M$ be a morphism in $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ be a morphism where $M_0 \in \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})_{\langle 0 \rangle}^{\otimes}$ and $M \in \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$.
The following conditions are equivalent:*

(1) *The morphism f is classified by an operadic q-colimit diagram*

$$\Delta^1 \to \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes},$$

   where $q : \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ denotes the projection.

(2) *Let $F : \mathcal{K}_{\mathcal{O}} \times_{\mathcal{O}^{\otimes}} \Delta^1 \to \mathcal{C}^{\otimes}$ be the map corresponding to f. Then F induces an equivalence $F(q(f)) \to F(\mathrm{id}_{q(M)})$.*

Moreover, for every $X \in \mathcal{O}^{\otimes}$, there exists a morphism $f : M_0 \to M$ satisfying the above conditions, with $q(M) = X$.

Proof. Apply Theorem 3.4.4.3 together with the observation that the ∞-category $\mathcal{D}_{/D}^{\mathrm{act}}$ has a final object. □

Example 3.4.4.5. Corollary 3.4.4.4 implies that if $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a fibration of ∞-operads and $\mathcal{O}^{\otimes}$ is coherent, then $\mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ has units.

Corollary 3.4.4.6. *Let κ be an uncountable regular cardinal. Let $\mathcal{O}^{\otimes}$ be a κ-small coherent ∞-operad, and let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a $\mathcal{O}$-monoidal ∞-category which is compatible with κ-small colimits. Let $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ be a $\mathcal{O}$-algebra object of $\mathcal{C}$. Then:*

(1) *The map $\psi : \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ is a coCartesian fibration of ∞-operads which is compatible with κ-small colimits.*

(2) *For each object $X \in \mathcal{O}$, consider the induced functor $\phi : \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})_X^{\otimes} \to \mathcal{C}_X^{\otimes}$. Let K be a κ-small simplicial set and let $\overline{p} : K^{\triangleright} \to \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})_X^{\otimes}$ be a map. Then $\overline{p}$ is a colimit diagram if and only if $\phi \circ \overline{p}$ is a colimit diagram.*

Proof. Assertion (1) follows immediately from Theorem 3.4.4.3 and Corollary 3.1.1.21. We will prove (2). Without loss of generality, we may assume that K is an ∞-category. Let $\mathcal{D} = \mathcal{K}_{\mathcal{O}} \times_{\mathcal{O}^{\otimes}} \{X\}$ denote the full subcategory of $(\mathcal{O}^{\otimes})^{X/}$ spanned by the semi-inert morphisms $X \to Y$ in $\mathcal{O}^{\otimes}$, so that we can identify $\overline{p}$ with a functor $F : \mathcal{D} \times K^{\triangleright} \to \mathcal{C}^{\otimes}$. Let $p = \overline{p}|K$. It follows from (1) (and Corollary 3.1.1.21) that p can be extended to an operadic ψ-colimit diagram in $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})_X^{\otimes}$, and any such diagram is automatically a colimit diagram. From the uniqueness properties of colimit diagrams, we deduce that $\overline{p}$ is a colimit diagram if and only if it is an operadic ψ-colimit diagram. In view of Theorem 3.4.4.3, this is true if and only if F satisfies the following condition:

$(\ast)$ Let $D = \mathrm{id}_X \in \mathcal{D}$. Then the diagram

$$\mathcal{D}_{/D} \times K^{\triangleright} \to \mathcal{D} \times K^{\triangleright} \xrightarrow{F} \mathcal{C}^{\otimes}$$

is an operadic q-colimit diagram.

Since the inclusion $\{\mathrm{id}_D\} \hookrightarrow \mathcal{D}_{/D}$ is left cofinal, condition $(\ast)$ is equivalent to the requirement that $\phi \circ \overline{p} = F|\{D\} \times K^{\triangleright}$ is an operadic q-colimit diagram. Since $\phi \circ p$ can be extended to an operadic q-colimit diagram in $\mathcal{C}_X^{\otimes}$ (Corollary 3.1.1.21) and any such diagram is automatically a colimit diagram in $\mathcal{C}_X^{\otimes}$, the uniqueness properties of colimit diagrams show that $(\ast)$ is equivalent to the requirement that $\phi \circ \overline{p}$ is a colimit diagram in $\mathcal{C}_X^{\otimes}$. $\qquad\square$

Example 3.4.4.7. Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, where $\mathcal{O}^{\otimes}$ is coherent. Let $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ be a $\mathcal{O}$-algebra object of $\mathcal{C}$. Then the composition $\mathcal{K}_{\mathcal{O}} \xrightarrow{e_1} \mathcal{O}^{\otimes} \xrightarrow{A} \mathcal{C}^{\otimes}$ determines an object in $\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}))$, which we will denote by $\overline{A}$ (it is a preimage of the identity map id_A under the equivalence $\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})) \simeq \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{A/}$ of Corollary 3.4.1.7). We can informally summarize the situation by saying that any algebra object $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ can be viewed as a module over itself.

Let $0 \in \mathcal{O}_{\langle 0 \rangle}^{\otimes}$ be a zero object of $\mathcal{O}^{\otimes}$ and let $X \in \mathcal{O}$ be any object. Then $\overline{A}(0)$ is a zero object of $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$, and $\overline{A}(X)$ is an object of $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})_X^{\otimes}$. Any choice of map $0 \to X$ in $\mathcal{O}^{\otimes}$ induces a map $\eta_X : \overline{A}(0) \to \overline{A}(X)$, which is given by an edge $\overline{p} : \Delta^1 \to \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$. We claim that $\overline{p}$ is an operadic ψ-colimit diagram, where $\psi : \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ denotes the projection. In view of Theorem 3.4.4.3, it suffices to prove that $\overline{p}$ induces an operadic q-colimit diagram

$$\theta : ((\mathcal{O}^{\otimes})^{0/} \times_{\mathcal{D}} \mathcal{D}_{/D})^{\triangleright} \to \mathcal{C}^{\otimes},$$

where $\mathcal{D} = \mathcal{K}_{\mathcal{O}} \times_{\mathcal{O}^{\otimes}} \Delta^1$ and D is the object of $\mathcal{D}$ determined by the pair $(\mathrm{id}_X, 1)$. We observe that the fiber product $(\mathcal{O}^{\otimes})^{0/} \times_{\mathcal{D}} \mathcal{D}_{/D}$ contains a final object C, corresponding to the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ X & \xrightarrow{\mathrm{id}} & X \end{array}$$

in $\mathcal{O}^{\otimes}$. It therefore suffices to show that the restriction $\theta_0 = \theta|\{C\}^{\triangleright} \to \mathcal{C}^{\otimes}$ is an operadic q-colimit diagram (Remark 3.1.1.4). This is clear, since θ_0 corresponds to the identity morphism $\mathrm{id} : A(X) \to A(X)$ in $\mathcal{C}^{\otimes}$.

We can summarize the situation informally as follows: for every $X \in \mathcal{O}$, the map η_X exhibits $\overline{A}(X) \in \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})_X^{\otimes}$ as a "unit object" with respect to the $\mathcal{O}$-operad structure on $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$.

We can deduce Theorem 3.4.4.2 from Theorem 3.4.4.3:

Proof of Theorem 3.4.4.2. In view of Corollary 3.4.4.6, it will suffice to show that for each $X \in \mathcal{O}$, the fiber $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})_X^{\otimes}$ is an accessible ∞-category. Let $\mathcal{D}$ denote the full subcategory of $(\mathcal{O}^{\otimes})^{X/}$ spanned by the semi-inert morphisms $f : X \to Y$, let $\mathcal{D}_0 \subseteq \mathcal{D}$ be the full subcategory spanned by those objects for which f is an equivalence, and let $A' : \mathcal{D}_0 \to \mathcal{C}^{\otimes}$. We will say that a morphism in $\mathcal{D}$ is inert if its image in $\mathcal{O}^{\otimes}$ is inert. We observe that $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})_X^{\otimes}$ can be identified with a fiber of the restriction functor

$$\phi : \mathrm{Fun}_{\mathcal{O}^{\otimes}}^0(\mathcal{D}, \mathcal{C}^{\otimes}) \to \mathrm{Fun}_{\mathcal{O}^{\otimes}}^0(\mathcal{D}_0, \mathcal{C}^{\otimes}),$$

where the superscript 0 indicates that we consider only those functors which carry inert morphisms in $\mathcal{D}$ (or $\mathcal{D}_0$) to inert morphisms in $\mathcal{C}^{\otimes}$. It follows from Corollary T.5.4.7.17 that the domain and codomain of ϕ are accessible ∞-categories and that ϕ is an accessible functor. Invoking Proposition T.5.4.6.6, we deduce that $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})_X^{\otimes}$ is accessible as desired. $\qquad\square$

We now turn to the proof of Theorem 3.4.4.3. We will treat assertions (1) and (2) separately. In both cases, our basic strategy is similar to that of Theorem 3.1.2.3.

Proof of Part (2) *of Theorem 3.4.4.3.* Let $\overline{\mathcal{D}}^0$ denote the inverse image of $\mathcal{K}_{\mathcal{O}}^0$ in $\overline{\mathcal{D}}$, and let $\mathcal{D}^0 = \overline{\mathcal{D}}^0 \cap \mathcal{D}$. Let $\mathcal{D}'$ denote the full subcategory of $\overline{\mathcal{D}}$ spanned by $\mathcal{D}$ together with $\overline{\mathcal{D}}^0$. Note that there is a unique map $z : \mathcal{D}' \to \Delta^2$ such that $z^{-1}\Delta^{\{0,1\}} \simeq \mathcal{D}$ and $z^{-1}\Delta^{\{1,2\}} = \overline{\mathcal{D}}^0$. The map z is a coCartesian fibration, and therefore flat. The algebra A and the map F determine a map $\Lambda_1^2 \times_{\Delta^2} \mathcal{D}' \to \mathcal{C}^{\otimes}$. Using the fact that q is a categorical fibration and that the inclusion $\Lambda_1^2 \times_{\Delta^2} \mathcal{D}' \subseteq \mathcal{D}'$ is a categorical equivalence (Proposition B.3.2), we can find a map $F_0 \in \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{D}', \mathcal{C}^{\otimes})$ compatible with F and A. To complete the proof, we wish to prove that F_0 can be extended to a map $\overline{F} \in \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\overline{\mathcal{D}}, \mathcal{C}^{\otimes})$ satisfying $(*)$ together with the following condition (which guarantees that $\overline{F}$ encodes a diagram $\overline{p} : K^{\triangleright} \to \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$):

($\star$) Let $\alpha : D \to D'$ be a morphism in $\overline{\mathcal{D}}$ lying over the cone point of $K^{\triangleleft}$ whose image in $\mathcal{K}_{\mathcal{O}}$ is inert. Then $\overline{F}'(\alpha)$ is an inert morphism of $\mathcal{C}^{\otimes}$.

Note that, because $\mathcal{C}^{\otimes}$ is an ∞-operad, it suffices to verify condition $(\star)$ when the object $D' \in \overline{\mathcal{D}}$ lies over $\langle 1 \rangle \in \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

Let S denote the full subcategory of $\overline{\mathcal{D}}$ spanned by those objects which lie over the cone point of $K^{\triangleright}$, and let $S^0 = S \times_{\mathcal{K}_{\mathcal{O}}} \mathcal{K}_{\mathcal{O}}^0$. Let $\mathcal{J}$ denote the category $(\mathcal{F}\mathrm{in}_*)_{\langle 1 \rangle /}$ of pointed objects of $\mathcal{F}\mathrm{in}_*$. There is an evident forgetful functor $S \to \mathrm{N}(\mathcal{J})$, given by the map

$$S \subseteq \{X\} \times_{\mathcal{O}^{\otimes}} \mathcal{K}_{\mathcal{O}} \subseteq (\mathcal{O}^{\otimes})^{X/} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)^{\langle 1 \rangle /} \simeq \mathrm{N}(\mathcal{J}).$$

We will say that a morphism α in $\mathcal{J}$ is *active* or *inert* if its image in $\mathcal{F}\mathrm{in}_*$ is active or inert, respectively; otherwise, we will say that α is *neutral*. Let σ be an m-simplex of $\mathrm{N}(\mathcal{J})$, corresponding to a chain of morphisms

$$\langle 1 \rangle \xrightarrow{\alpha(0)} (\langle k_0 \rangle) \xrightarrow{\alpha(1)} (\langle k_1 \rangle) \xrightarrow{\alpha(2)} \cdots \xrightarrow{\alpha(m)} (\langle k_m \rangle).$$

in the category $\mathcal{F}\mathrm{in}_*$. We will say that σ is *new* if it is nondegenerate and the map $\alpha(0)$ is not null. We let J_σ denote the collection of integers $j \in \{1, \ldots, m\}$ for which the map $\alpha(j)$ is not an isomorphism. We will denote the cardinality of J_σ by $l(\sigma)$ and refer to it as the *length* of σ (note that this length is generally smaller than m). For $1 \le d \le l(\sigma)$, we let j_d^σ denote the dth element of J_σ and set $\alpha_d^\sigma = \alpha(j_d^\sigma)$. We will say that σ is *closed* if $k_m = 1$; otherwise we will say that σ is *open*. We now partition the collection of new simplices σ of E into eleven groups, as in the proof of Theorem 3.1.2.3:

$(G'_{(1)})$ A new simplex σ of $\mathrm{N}(\mathcal{J})$ belongs to $G'_{(1)}$ if it is a closed and the maps α_i^σ are active for $1 \le i \le l(\sigma)$.

$(G_{(2)})$ A new simplex σ of $\mathrm{N}(\mathcal{J})$ belongs to $G_{(2)}$ if σ is closed and there exists $1 \le k < l(\sigma)$ such that α_k^σ is inert, while α_j^σ is active for $k < j \le l(\sigma)$.

$(G'_{(2)})$ A new simplex σ of $\mathrm{N}(\mathcal{J})$ belongs to $G'_{(2)}$ if σ is closed and there exists $1 \le k \le l(\sigma)$ such that α_k^σ is neutral while the maps α_j^σ are active for $k < j \le l(\sigma)$.

$(G_{(3)})$ A new simplex σ of $\mathrm{N}(\mathcal{J})$ belongs to $G_{(3)}$ if σ is closed and there exists $1 \le k < l(\sigma) - 1$ such that α_k^σ is inert, the maps α_j^σ are active for $k < j < l(\sigma)$, and $\alpha_{l(\sigma)}^\sigma$ is inert.

$(G'_{(3)})$ A new simplex σ of $\mathrm{N}(\mathcal{J})$ belongs to $G'_{(3)}$ if σ is closed and there exists $1 \le k < l(\sigma)$ such that the map α_k^σ is neutral, the maps α_j^σ are active for $k < j < l(\sigma)$, and $\alpha_{l(\sigma)}^\sigma$ is inert.

$(G_{(4)})$ A new simplex σ of $\mathrm{N}(\mathcal{J})$ belongs to $G_{(4)}$ if it is a closed, the maps α_i^σ are active for $1 \le i < l(\sigma)$, and the map $\alpha_{l(\sigma)}^\sigma$ is inert.

$(G'_{(4)})$ A new simplex σ of $\mathrm{N}(\mathcal{J})$ belongs to $G'_{(4)}$ if it is an open and the maps α_i^σ are active for $1 \le i \le l(\sigma)$.

$(G_{(5)})$ A new simplex σ of N($\mathcal{J}$) belongs to $G_{(5)}$ if σ is an open and there exists $1 \leq k < l(\sigma)$ such that α_k^σ is inert and α_j^σ is active for $k < j \leq l(\sigma)$.

$(G'_{(5)})$ A new simplex σ of N($\mathcal{J}$) belongs to $G'_{(5)}$ if σ is an open and there exists $1 \leq k \leq l(\sigma)$ such that α_k^σ is neutral and the maps α_j^σ are active for $k < j \leq l(\sigma)$.

$(G_{(6)})$ A new simplex σ of N($\mathcal{J}$) belongs to $G_{(6)}$ if it is closed, has length ≥ 2, and the maps $\alpha_{l(\sigma)-1}^\sigma$ and $\alpha_{l(\sigma)}^\sigma$ are both inert.

$(G'_{(6)})$ A new simplex σ of N($\mathcal{J}$) belongs to $G'_{(6)}$ if it is open, has length at least 1, and the map $\alpha_{l(\sigma)}^\sigma$ is inert.

For each integer $m \geq 0$, we let N($\mathcal{J}$)$_m$ denote the simplicial subset spanned by those simplices which are either not new, have length $\leq m$, or have length m and belong to one of the groups $G_{(i)}$ for $2 \leq i \leq 6$. Let $S(m)$ denote the inverse image $S \times_{\mathrm{N}(\mathcal{J})} \mathrm{N}(\mathcal{J})_m$ and let $\overline{\mathcal{D}}(m)$ denote the simplicial subset of $\overline{\mathcal{D}}$ spanned by those simplices whose intersection with S belongs to $S(m)$. Then $\overline{\mathcal{D}}(0) = \mathcal{D}'$ is the domain of the map F_0. We will complete the proof by extending F_0 to a compatible sequence of maps $F_m \in \mathrm{Fun}_{\mathcal{O}^\otimes}(\overline{\mathcal{E}}(m), \mathcal{C}^\otimes)$, where F_1 satisfies conditions $(*)$ and $(\star)$.

Let us now fix $m > 0$ and assume that F_{m-1} has already been constructed. We define a filtration

$$\mathrm{N}(\mathcal{J})_{m-1} = K(0) \subseteq K(1) \subseteq K(2) \subseteq K(3) \subseteq K(4) \subseteq K(5) \subseteq K(6) = \mathrm{N}(\mathcal{J})_m$$

as follows:

- We let $K(1)$ denote the simplicial subset of N($\mathcal{J}$) spanned by those simplices which either belong to $K(0)$ or have length $(m-1)$ and belong to $G'_{(1)}$.

- For $2 \leq i \leq 6$, we let $K(i)$ be the simplicial subset of N($\mathcal{J}$) spanned by those simplices which either belong to $K(i-1)$, have length m and belong to $G_{(i)}$, or have length $m-1$ and belong to $G'_{(i)}$.

For $0 \leq i \leq 6$, we let $\overline{K}(i)$ denote the simplicial subset of $\overline{\mathcal{E}}$ spanned by those simplices whose intersection with E belongs to the inverse image of $K(i)$. We will define maps $f^i : \overline{K}(i) \to \mathcal{C}^\otimes$ with $f^0 = F_{m-1}$. The construction now proceeds in six steps:

(1) Assume that $f^0 = F_{m-1}$ has been constructed; we wish to define f^1. Let $\{\sigma_a\}_{a \in A}$ be the collection of all simplices of S whose image in N($\mathcal{J}$) have length $(m-1)$ and belong to $G'_{(1)}$. Choose a well-ordering of the set A such that the dimensions of the simplices σ_a form a (nonstrictly) increasing function of a. For each $a \in A$, let $\overline{\mathcal{D}}_{\leq a}$ denote the simplicial subset of $\overline{\mathcal{D}}$ spanned by those simplices which either belong to $\overline{K}(0)$ or whose intersection with E is contained in $\sigma_{a'}$ for some $a' \leq a$, and define $\overline{\mathcal{D}}_{<a}$ similarly. We construct a compatible family of maps $f^{\leq a} \in \mathrm{Fun}_{\mathcal{O}^\otimes}(\overline{\mathcal{D}}_{\leq a}, \mathcal{C}^\otimes)$ extending f^0, using transfinite induction on a. Assume that $f^{\leq a'}$ has been constructed for $a' < a$; these maps can be amalgamated to obtain a map $f^{<a} \in \mathrm{Fun}_{\mathcal{O}^\otimes}(\overline{\mathcal{D}}_{<a}, \mathcal{C}^\otimes)$. Let $Z = \mathcal{D} \times_{\overline{\mathcal{D}}} \overline{\mathcal{D}}_{/\sigma_a}$. Lemma 3.1.2.5 implies that the diagram

$$\begin{CD} Z \star \partial\,\sigma_a @>>> \overline{\mathcal{D}}_{<a} \\ @VVV @VVV \\ Z \star \sigma_a @>>> \overline{\mathcal{D}}_{\leq a}. \end{CD}$$

is a homotopy pushout square. Since q is a categorical fibration, it will suffice to extend the composition

$$g_0 : Z \star \partial\,\sigma_a \to \overline{\mathcal{D}}_{<a} \overset{f^{<a}}{\to} \mathcal{C}^\otimes$$

to a map $g \in \mathrm{Fun}_{\mathcal{O}^\otimes}(Z \star \sigma_a, \mathcal{C}^\otimes)$. We first treat the special case where the simplex σ_a is zero-dimensional (in which case we must have $m = 1$). We can identify σ_a with an object $D \in \overline{\mathcal{D}}$. Since σ_a is new

and closed, D is equivalent to (v, id_X). Let $Z_0 = \mathcal{D} \times_{\overline{\mathcal{D}}} \overline{\mathcal{D}}_{/D}^{\mathrm{act}}$. It follows from assumption (ii) that $g_0 | Z_0$ can be can be extended to an operadic q-colimit diagram (compatible with the projection to $\mathcal{O}^{\otimes}$). Since Z_0 is a localizatin of Z, the inclusion $Z_0 \subseteq Z$ is left cofinal; we can therefore extend g_0 to a map $g \in \mathrm{Fun}_{\mathcal{O}^{\otimes}}(Z \star \sigma_a, \mathcal{C}^{\otimes})$. whose restriction to $Z_0^{\triangleright}$ is an operadic q-colimit diagram. Moreover, this construction guarantees that f^1 will satisfy condition $(*)$ for the object D.

Now suppose that σ_a is a simplex of positive dimension. We again let Z_0 denote the simplicial subset of Z spanned by those vertices which correspond to diagrams $\sigma_a^{\triangleleft} \to \mathcal{C}^{\otimes}$ which project to a sequence of active morphisms in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. The inclusion $Z_0 \subseteq Z$ admits a left adjoint and is therefore left cofinal; it follows that the induced map $\mathcal{C}_{(F_{m-1}|Z)/}^{\otimes} \to \mathcal{C}_{(F_{m-1}|Z_0)/}^{\otimes} \times_{\mathcal{O}_{(qF_{m-1}|Z_0)/}^{\otimes}} \mathcal{O}_{(qF_{m-1}|Z)/}^{\otimes}$ is a trivial Kan fibration. It therefore suffices to show that the restriction $g_0' = g_0 | (Z_0 \star \partial \sigma_a)$ can be extended to a map $g' \in \mathrm{Fun}_{\mathcal{O}^{\otimes}}(Z_0 \star \sigma_a, \mathcal{C}^{\otimes})$. Let D denote the initial vertex of σ. In view of Proposition 3.1.1.7, it will suffice to show that the restriction $g_0' | (Z_0 \star \{D\})$ is an operadic q-colimit diagram. Since the inclusion $\{D\} \subseteq \sigma_a$ is left anodyne, the projection map $Z_0 \to \mathcal{D}_{/D}^{\mathrm{act}} = \mathcal{D} \times_{\overline{\mathcal{D}}} \overline{\mathcal{D}}_{/D}^{\mathrm{act}}$ is a trivial Kan fibration. It will therefore suffice to show that F_{m-1} induces an operadic q-colimit diagram $\delta : (\mathcal{D}_{/D}^{\mathrm{act}})^{\triangleright} \to \mathcal{C}^{\otimes}$.

The object $D \in \overline{\mathcal{D}}$ determines a semi-inert morphism $\gamma : X \to Y$ in $\mathcal{O}^{\otimes}$. Since σ_a is new, this morphism is not null and therefore determines an equivalence $Y \simeq Y_0 \oplus X$. Let $\mathfrak{X}$ denote the full subcategory of $\mathcal{D}_{/D}^{\mathrm{act}}$ spanned by those objects corresponding to diagrams

$$
\begin{array}{ccc}
X' & \longrightarrow & Y' \\
\downarrow & & \downarrow{\scriptstyle \phi} \\
X & \longrightarrow & Y_0 \oplus X
\end{array}
$$

where the active morphism ϕ exhibits Y' as a sum $Y_0' \oplus Y_1'$, where $Y_0' \simeq Y_0$ and $Y_1' \to X$ is an active morphism. The inclusion $\mathfrak{X} \subseteq \mathcal{D}_{/D}^{\mathrm{act}}$ admits a left adjoint, and is therefore left cofinal. Let $D_0 = (v, \mathrm{id}_X) \in \overline{\mathcal{D}}$, so that the operation $\bullet \mapsto Y_0 \oplus \bullet$ determines an equivalence $\mathcal{D}_{/D_0}^{\mathrm{act}} \to \mathfrak{X}$. It therefore suffices to show that the composite map

$$
(\mathcal{E}_{/D_0}^{\mathrm{act}})^{\triangleright} \xrightarrow{Y_0 \oplus} \mathfrak{X}^{\triangleright} \subseteq (\mathcal{D}_{/D}^{\mathrm{act}})^{\triangleright} \xrightarrow{\delta} \mathcal{C}^{\otimes}
$$

is an operadic q-colimit diagram. Using condition $(\star)$, we can identify this map with the composition

$$
(\mathcal{D}_{/D_0}^{\mathrm{act}})^{\triangleright} \longrightarrow \mathcal{C}^{\otimes} \xrightarrow{A(Y_0) \oplus} \mathcal{C}^{\otimes},
$$

which is an operadic q-colimit diagram by virtue of $(*)$.

(2) We now assume that f^1 has been constructed. Since q is a categorical fibration, to produce the desired extension f^2 of f^1 it is sufficient to show that the inclusion $\overline{K}(1) \subseteq \overline{K}(2)$ is a categorical equivalence. For each simplex σ of $\mathrm{N}(\mathcal{J})$ of length m belonging to $G_{(2)}$, let $k(\sigma) < m$ be the integer such that $\alpha_{k(\sigma)}^{\sigma}$ is inert while α_j^{σ} is active for $k(\sigma) < j \leq m$. We will say that σ is *good* if $\alpha_{k(\sigma)}^{\sigma}$ induces a map $\langle p \rangle \to \langle p' \rangle$ in $\mathcal{F}\mathrm{in}_*$ whose restriction to $(\alpha_k^{\sigma})^{-1} \langle p' \rangle^{\circ}$ is order preserving. Let $J = K^{\triangleright} \times_{\mathrm{Fun}(\{0\}, \mathrm{N}(\mathcal{F}\mathrm{in}_*))} \mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{F}\mathrm{in}_*))$, so that we can identify $\mathrm{N}(\mathcal{J})$ with the simplicial subset $J \times_{K^{\triangleright}} \{v\}$. Let $\{\sigma_a\}_{a \in A}$ be the collection of all nondegenerate simplices of J such that the intersection $\sigma_a' = \sigma_a \cap \mathrm{N}(\mathcal{J})$ is nonempty and good. For each $a \in A$, let $k_a = k(\sigma_a')$ and $j_a = j_k^{\sigma_a'}$. Choose a well-ordering of A with the following properties:

 – The map $a \mapsto k_a$ is a (nonstrictly) increasing function of $a \in A$.

 – For each integer k, the dimension of the simplex σ_a is a nonstrictly increasing function of $a \in A_k = \{a \in A : k_a = k\}$.

 – Fix integers $k, d \geq 0$, and let $A_{k,d}$ be the collection of elements $a \in A_k$ such that σ_a has dimension d. The map $a \mapsto j_a$ is a nonstrictly increasing function on $A_{k,d}$.

Let $J_{<0}$ be the collection of those simplices of J whose intersection with $\mathrm{N}(\mathcal{J})$ belongs to $K(1)$. For each $a \in A$, let $J_{\leq a}$ denote the simplicial subset of J generated by J_0 together with the simplices the simplices $\{\sigma_{a'}\}_{a' \leq a}$, and define $J_{<a}$ similarly. The inclusion $\overline{K}(1) \subseteq \overline{K}(2)$ can be obtained as a transfinite composition of inclusions

$$i_a : \overline{\mathcal{D}} \times_J J_{<a} \to \overline{\mathcal{D}} \times_J J_{\leq a}.$$

Each i_a is a pushout of an inclusion $i'_a : \overline{\mathcal{D}} \times_J \sigma^0_a \overline{\mathcal{D}} \times_J \sigma_a$, where $\sigma^0_a \subseteq \sigma_a$ denotes the inner horn obtained by removing the interior of σ_a together with the face opposite the j_ath vertex of σ'_a, and therefore a categorical equivalence by Lemma 2.4.4.6.

(3) To find the desired extension f^3 of f^2, it suffices to show that the inclusion $\overline{K}(2) \subseteq \overline{K}(3)$ is a categorical equivalence. This follows from the argument given in step (2).

(4) Let $\{\sigma_a\}_{a \in A}$ denote the collection of all nondegenerate simplices σ of $\overline{\mathcal{D}}$ with the property that $\sigma \cap S$ is nonempty and projects to a simplex of $\mathrm{N}(\mathcal{J})$ of length $m-1$ which belongs to G'_4. Choose a well-ordering of A having the property that the dimensions of the simplices σ_a form a (nonstrictly) increasing function of a. For each $a \in A$ let $D_a \in \overline{\mathcal{D}}$ denote the final vertex of σ_a and let Z_a denote the full subcategory of $S \times_{\overline{\mathcal{D}}} \overline{\mathcal{D}}_{\sigma_a/}$ spanned by those objects for which the underlying map $D_a \to D$ induces an inert morphism $\langle m \rangle \to \langle 1 \rangle$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. We have a canonical map $t_a : \sigma_a \star Z_a \to \overline{\mathcal{D}}$. For each $a \in A$, let $\overline{\mathcal{E}}_{\leq a} \subseteq \overline{\mathcal{D}}$ denote the simplicial subset generated by $\overline{K}(3)$ together with the image of $t_{a'}$ for all $a' \leq a$, and define $\overline{\mathcal{D}}_{<a}$ similarly. Then $\overline{K}(4) = \bigcup_{a \in A} \overline{\mathcal{E}}_{\leq a}$ and that for each $a \in A$ Lemma 3.1.2.5 implies that we have a homotopy pushout diagram of simplicial sets

$$
\begin{array}{ccc}
\partial \sigma_a \star Z_a & \longrightarrow & \overline{\mathcal{D}}_{<a} \\
\downarrow & & \downarrow \\
\sigma_a \star Z_a & \longrightarrow & \overline{\mathcal{D}}_{\leq a}.
\end{array}
$$

To construct f^4, we are reduced to the problem of solving a sequence of extension problems of the form

$$
\begin{array}{ccc}
\partial \sigma_a \star Z_a & \overset{g_0}{\longrightarrow} & \mathcal{C}^{\otimes} \\
\downarrow & \overset{g}{\nearrow} & \downarrow q \\
\sigma_a \star Z_a & \longrightarrow & \mathcal{O}^{\otimes}.
\end{array}
$$

Note that Z_a decomposes a disjoint union of ∞-categories $\coprod_{1 \leq i \leq m} Z_{a,i}$ (where $\langle m \rangle$ denotes the image of D_a in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$). Each of the ∞-categories $Z_{a,i}$ has an initial object B_i, given by any map $\sigma_a \star \{D_i\} \to \mathcal{C}^{\otimes}$ which induces an inert morphism $D_a \to D_i$ covering $\rho^i : \langle m \rangle \to \langle 1 \rangle$. Let $h : Z_a \to \mathcal{C}^{\otimes}$ be the map induced by g_0, and let h' be the restriction of h to the discrete simplicial set $Z'_a = \{B_i\}_{1 \leq i \leq m}$. Since the inclusion $Z'_a \to Z_a$ is left anodyne, we have a trivial Kan fibration $\mathcal{C}^{\otimes}_{/h} \to \mathcal{C}^{\otimes}_{/h'} \times_{\mathcal{O}^{\otimes}_{/qh'}} \mathcal{O}^{\otimes}_{/qh}$. Unwinding the definitions, we are reduced to the lifting probelm depicted in the diagram

$$
\begin{array}{ccc}
\partial \sigma_a \star Z'_a & \overset{g'_0}{\longrightarrow} & \mathcal{C}^{\otimes} \\
\downarrow & \overset{g'}{\nearrow} & \downarrow q \\
\sigma_a \star Z'_a & \overset{\overline{g}'}{\longrightarrow} & \mathcal{O}^{\otimes}.
\end{array}
$$

If the dimension of σ_a is positive, then it suffices to show that g_0' carries $\{D_a\} \star Z_a'$ to a q-limit diagram in $\mathcal{C}^{\otimes}$. Let q' denote the canonical map $\mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. In view of Proposition T.4.3.1.5, it suffices to show that g_0' carries $\{D_a\} \star Z_a'$ to a $(q' \circ q)$-limit diagram and that $q \circ g_0'$ carries $\{D_a\} \star Z_a'$ to a q'-limit diagram. The first of these assertions follows from $(\star)$ and from the fact that $\mathcal{C}^{\otimes}$ is an ∞-operad, and the second follows by the same argument (since q is a map of ∞-operads).

It remains to treat the case where σ_a is zero dimensional (in which case we have $m = 1$). Since $\mathcal{C}^{\otimes}$ is an ∞-operad, we can solve the lifting problem depicted in the diagram

$$
\begin{array}{ccc}
\partial\,\sigma_a \star Z_a' & \xrightarrow{\;g_0'\;} & \mathcal{C}^{\otimes} \\
\downarrow & \overset{g''}{\nearrow} & \downarrow{\scriptstyle q' \circ q} \\
\sigma_a \star Z_a' & \longrightarrow & \mathrm{N}(\mathcal{F}\mathrm{in}_*)
\end{array}
$$

in such a way that g'' carries edges of $\sigma_a \star Z_a'$ to inert morphisms in $\mathcal{C}^{\otimes}$. Since q is an ∞-operad map, it follows that $q \circ g''$ has the same property. Since $\mathcal{O}^{\otimes}$ is an ∞-operad, we conclude that $q \circ g''$ and $\overline{g}'$ are both q'-limit diagrams in $\mathcal{O}^{\otimes}$ extending $q \circ g_0'$, and therefore equivalent to one another via an equivalence which is fixed on Z_a' and compatible with the projection to $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Since q is a categorical fibration, we can lift this equivalence to an equivalence $g'' \simeq g'$, where $g' : \sigma_a \star Z_a' \to \mathcal{C}^{\otimes}$ is the desired extension of g_0'. We note that this construction ensures that condition $(\star)$ is satisfied.

(5) To find the desired extension f^5 of f^4, it suffices to show that the inclusion $\overline{K}(4) \subseteq \overline{K}(5)$ is a categorical equivalence. This again follows from the argument given in step (2).

(6) The verification that f^5 can be extended to a map $f^6 : \overline{K}(6) \to \mathcal{C}^{\otimes}$ proceeds as in step (4), but is slightly easier (since $G_{(6)}'$ contains no 0-simplices).

$\square$

Proof of Part (1) *of Theorem 3.4.4.3.* We wish to show that a diagram $\overline{p} : K^{\triangleright} \to \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ satisfies condition $(*)$ if and only if it is an operadic ψ-colimit diagram. We will prove the "only if" direction; the converse will follow from assertion (2) together with the uniqueness properties of operadic colimit diagrams. Fix an object $\overline{Y} \in \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ lying over $Y \in \mathcal{O}^{\otimes}$ and let $\overline{p}_{\overline{Y}} : K^{\triangleright} \to \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$; given by the composition

$$
K^{\triangleright} \xrightarrow{\;\overline{p}\;} \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes} \xrightarrow{\;\oplus\overline{Y}\;} \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes};
$$

we must show that $\overline{p}_{\overline{Y}}$ is a weak ψ-operadic colimit diagram. Unwinding the definitions, we must show that for $n > 0$, every lifting problem of the form

$$
\begin{array}{ccc}
K \star \partial\,\Delta^n & \xrightarrow{\;p'\;} & \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})_{\mathrm{act}}^{\otimes} \\
\downarrow & \overset{}{\nearrow} & \downarrow \\
K \star \Delta^n & \xrightarrow{\;\overline{p}_0'\;} & \mathcal{O}_{\mathrm{act}}^{\otimes}
\end{array}
$$

admits a solution, provided that $p'|K \star \{0\} = \overline{p}_{\overline{Y}}$ and $\overline{p}_0'$ carries $\Delta^{\{1,\ldots,n\}}$ into $\mathcal{O} \subseteq \mathcal{O}_{\mathrm{act}}^{\otimes}$.

Let $\overline{\mathcal{T}}$ denote the fiber product $(K \star \Delta^n) \times_{\mathcal{O}^{\otimes}} \mathcal{K}_{\mathcal{O}}$, let $\overline{\mathcal{T}}^0 = (K \star \Delta^n) \times_{\mathcal{O}^{\otimes}} \mathcal{K}_{\mathcal{O}}$. Let $\overline{\mathcal{T}}^1$ denote the full subcategory of $\overline{\mathcal{T}} = (K \star \{0\}) \times_{\mathcal{O}^{\otimes}} \mathcal{K}_{\mathcal{O}}$ spanned by $\mathcal{T} = K \times_{\mathcal{O}^{\otimes}} \mathcal{K}_{\mathcal{O}}$ together with those vertices of $\{0\} \times_{\mathcal{O}^{\otimes}} \mathcal{K}_{\mathcal{O}}$ which correspond to semi-inert morphisms $X \oplus Y \to Y'$ in $\mathcal{O}^{\otimes}$ such that the underlying map in $\mathcal{F}\mathrm{in}_*$ is not injective. Let $\mathcal{T}'$ denote the full subcategory of $\overline{\mathcal{T}}$ spanned by $\overline{\mathcal{T}}^1$ together with $\overline{\mathcal{T}}^0$. Note that there is a unique map $z : \mathcal{T}' \to \Delta^2$ such that $z^{-1}\Delta^{\{0,1\}} = \overline{\mathcal{T}}^1$ and $z^{-1}\Delta^{\{1,2\}} = \overline{\mathcal{T}}^0$. The map z is a coCartesian fibration and therefore flat. The algebra A and the map $\overline{p}$ determine a map $\Lambda_1^2 \times_{\Delta^2} \mathcal{T}' \to \mathcal{C}^{\otimes}$. Using the fact that

q is a categorical fibration and that the inclusion $\Lambda_1^2 \times_{\Delta^2} \mathcal{T}' \subseteq \mathcal{T}'$ is a categorical equivalence (Proposition B.3.2), we can find a map $F' \in \mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{T}', \mathcal{C}^\otimes)$ compatible with F and A. Amalgamating F_0 with the map determined by p', we obtain a map $F'' \in \mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{T}'', \mathcal{C}^\otimes)$ where

$$\mathcal{T}'' = ((K \star \partial \Delta^n) \times_{\mathcal{O}^\otimes} \mathcal{K}_{\mathcal{O}}) \coprod_{\mathcal{T}} \mathcal{T}' \subseteq \overline{\mathcal{T}}$$

To complete the proof, we wish to prove that F'' can be extended to a map $\overline{F} \in \mathrm{Fun}_{\mathcal{O}^\otimes}(\overline{\mathcal{T}}, \mathcal{C}^\otimes)$.

Let T denote the fiber product $\overline{\mathcal{T}} \times_{K \star \Delta^n} \Delta^n$ and let let $T^0 = T \times_{\mathcal{K}_{\mathcal{O}}} \mathcal{K}_{\mathcal{O}}^0$. The diagram $K \star \Delta^n \to \mathcal{O}^\otimes$ determines a map $\Delta^n \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, corresponding to a composable sequence of active morphisms

$$\langle j + 1 \rangle \to \langle 1 \rangle \simeq \langle 1 \rangle \simeq \cdots \simeq \langle 1 \rangle.$$

Here $\langle j \rangle$ corresponds to the image of Y in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Let $\mathcal{J}$ denote the fiber product category $[n] \times_{\mathrm{Fun}(\{0\}, \mathcal{F}\mathrm{in}_*)} \mathrm{Fun}([1], \mathcal{F}\mathrm{in}_*)$. There is an evident forgetful functor $T \to \mathrm{N}(\mathcal{J})$, whose second projection is given by the composition

$$T \subseteq \Delta^n \times_{\mathcal{O}^\otimes} \mathcal{K}_{\mathcal{O}} \to \mathrm{Fun}(\Delta^1, \mathcal{O}^\otimes) \to \mathrm{N}(\mathrm{Fun}([1], \mathcal{J})).$$

We will say that a morphism α in $\mathcal{J}$ is *active* or *inert* if its image in $\mathcal{F}\mathrm{in}_*$ is active or inert, respectively; otherwise, we will say that α is *neutral*. Let σ be an m-simplex of $\mathrm{N}(\mathcal{J})$. We will say that σ is *new* if σ is nondegenerate, the map $\sigma \to \Delta^n$ is surjective, and the semi-inert morphism $\langle j + 1 \rangle \to \langle k_0 \rangle$ given by the first vertex of σ is injective. Every such simplex determines a chain of morphisms

$$\langle k_0 \rangle \xrightarrow{\alpha(1)} \langle k_1 \rangle \xrightarrow{\alpha(2)} \cdots \xrightarrow{\alpha(m)} \langle k_m \rangle.$$

in the category $\mathcal{F}\mathrm{in}_*$. We let J_σ denote the collection of integers $j \in \{1, \ldots, m\}$ for which the map $\alpha(j)$ is not an isomorphism. We will denote the cardinality of J_σ by $l(\sigma)$ and refer to it as the *length* of σ (note that this length is generally smaller than m). For $1 \le d \le l(\sigma)$, we let j_d^σ denote the dth element of J_σ and set $\alpha_d^\sigma = \alpha(j_d^\sigma)$. We will say that σ is *closed* if $k_m = 1$; otherwise we will say that σ is *open*. As in the proof of assertion (2) (and the proof of Theorem 3.1.2.3), we partition the new simplices of $\mathrm{N}(\mathcal{J})$ into eleven groups $\{G_{(i)}\}_{2 \le i \le 6}$, $\{G'_{(i)}\}_{1 \le i \le 6}$.

For each integer $m \ge 0$, we let $\mathrm{N}(\mathcal{J})_m$ denote the simplicial subset spanned by those simplices which are either not new, have length $\le m$, or have length m and belong to one of the groups $G_{(i)}$ for $2 \le i \le 6$. Let $T(m)$ denote the inverse image $T \times_{\mathrm{N}(\mathcal{J})} \mathrm{N}(\mathcal{J})_m$ and let $\overline{\mathcal{T}}(m)$ denote the simplicial subset of $\overline{\mathcal{T}}$ spanned by those simplices whose intersection with E belongs to $E(m)$. Then $\overline{\mathcal{T}}(0) = \mathcal{T}''$ is the domain of the map $F'' = F_0''$. We will complete the proof by extending F_0'' to a compatible sequence of maps $F_m'' \in \mathrm{Fun}_{\mathcal{O}^\otimes}(\overline{\mathcal{T}}(m), \mathcal{C}^\otimes)$.

Fix $m > 0$ and assume that F_{m-1}'' has already been constructed. We define a filtration

$$\mathrm{N}(\mathcal{J})_{m-1} = K(0) \subseteq K(1) \subseteq K(2) \subseteq K(3) \subseteq K(4) \subseteq K(5) \subseteq K(6) = \mathrm{N}(\mathcal{J})_m$$

as follows:

- We let $K(1)$ denote the simplicial subset of $\mathrm{N}(\mathcal{J})$ spanned by those simplices which either belong to $K(0)$ or have length $(m - 1)$ and belong to $G'_{(1)}$.

- For $2 \le i \le 6$, we let $K(i)$ be the simplicial subset of $\mathrm{N}(\mathcal{J})$ spanned by those simplices which either belong to $K(i - 1)$, have length m and belong to $G_{(i)}$, or have length $m - 1$ and belong to $G'_{(i)}$.

For $0 \le i \le 6$, we let $\overline{K}(i)$ denote the simplicial subset of $\overline{\mathcal{E}}$ spanned by those simplices whose intersection with E belongs to the inverse image of $K(i)$. We will define maps $f^i : \overline{K}(i) \to \mathcal{C}^\otimes$ with $f^0 = F_{m-1}''$. We will explain how to construct f^1 from f^0; the remaining steps can be handled as in our proof of part (2).

Assume that $f^0 = F_{m-1}''$ has been constructed; we wish to define f^1. Let $\{\sigma_a\}_{a \in A}$ be the collection of all simplices of E whose image in $\mathrm{N}(\mathcal{J})$ have length $(m-1)$ and belong to $G'_{(1)}$. Choose a well-ordering of the set

A such that the dimensions of the simplices σ_a form a (nonstrictly) increasing function of a. For each $a \in A$, let $\overline{\mathcal{T}}_{\leq a}$ denote the simplicial subset of $\overline{\mathcal{T}}$ spanned by those simplices which either belong to $\overline{K}(0)$ or whose intersection with E is contained in $\sigma_{a'}$ for some $a' \leq a$, and define $\overline{\mathcal{T}}_{<a}$ similarly. We construct a compatible family of maps $f^{\leq a} \in \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\overline{\mathcal{T}}_{\leq a}, \mathcal{C}^{\otimes})$ extending f^0, using transfinite induction on a. Assume that $f^{\leq a'}$ has been constructed for $a' < a$; these maps can be amalgamated to obtain a map $f^{<a} \in \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\overline{\mathcal{T}}_{<a}, \mathcal{C}^{\otimes})$. Let $\mathcal{T} = \overline{\mathcal{T}} \times_{K \star \Delta^n} K$, and let $Z = \mathcal{T} \times_{\overline{\mathcal{T}}} \overline{\mathcal{T}}_{/\sigma_a}$. Lemma 3.1.2.5 implies that we have a homotopy pushout diagram of simplicial sets

$$
\begin{array}{ccc}
Z \star \partial \sigma_a & \longrightarrow & \overline{\mathcal{T}}_{<a} \\
\downarrow & & \downarrow \\
Z \star \sigma_a & \longrightarrow & \overline{\mathcal{T}}_{\leq a},
\end{array}
$$

where the vertical maps are cofibrations. It will therefore suffice to extend the composition

$$g_0 : Z \star \partial \sigma_a \to \overline{\mathcal{T}}_{<a} \stackrel{f^{<a}}{\to} \mathcal{C}^{\otimes}$$

to a map $g \in \mathrm{Fun}_{\mathcal{O}^{\otimes}}(Z \star \sigma_a, \mathcal{C}^{\otimes})$.

Note that σ_a necessarily has positive dimension (since the map $\sigma_a \to \Delta^n$ is surjective). Let Z_0 denote the simplicial subset of Z spanned by those vertices which correspond to diagrams $\sigma_a^{\triangleleft} \to \mathcal{C}^{\otimes}$ which project to a sequence of active morphisms in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. The inclusion $Z_0 \subseteq Z$ admits a left adjoint and is therefore left cofinal; it follows that the induced map

$$\mathcal{C}^{\otimes}_{(F_{m-1}|Z)/} \to \mathcal{C}^{\otimes}_{(F_{m-1}|Z_0)/} \times_{\mathcal{O}^{\otimes}_{(qF_{m-1}|Z_0)/}} \mathcal{O}^{\otimes}_{(qF_{m-1}|Z)/}$$

is a trivial Kan fibration. It therefore suffices to show that the restriction $g_0' = g_0|(Z_0 \star \partial \sigma_a)$ can be extended to a map $g' \in \mathrm{Fun}_{\mathcal{O}^{\otimes}}(Z_0 \star \sigma_a, \mathcal{C}^{\otimes})$. Let D denote the initial vertex of σ. In view of Proposition 3.1.1.7, it will suffice to show that the restriction $g_0'|(Z_0 \star \{D\})$ is an operadic q-colimit diagram. Since the inclusion $\{D\} \subseteq \sigma_a$ is left anodyne, the projection map $Z_0 \to \mathcal{T}_{/D}^{\mathrm{act}} = \mathcal{T} \times_{\overline{\mathcal{T}}} \overline{\mathcal{T}}_{/D}^{\mathrm{act}}$ is a trivial Kan fibration. It will therefore suffice to show that F_{m-1} induces an operadic q-colimit diagram $\delta : (\mathcal{T}_{/D}^{\mathrm{act}})^{\triangleright} \to \mathcal{C}^{\otimes}$.

The object $D \in \overline{\mathcal{T}}$ determines a semi-inert morphism $\gamma : X \oplus Y \to Y'$ in $\mathcal{O}^{\otimes}$. Since σ_a is new, the underlying morphism in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is injective, so that γ induces an equivalence $Y' \simeq X \oplus Y \oplus Y''$, for some $Y'' \in \mathcal{O}^{\otimes}$. Let $\mathcal{X}$ denote the full subcategory of $\mathcal{T}_{/D}^{\mathrm{act}}$ spanned by those objects corresponding to diagrams

$$
\begin{array}{ccc}
X_0 \oplus Y & \longrightarrow & Y_0 \\
\downarrow & & \downarrow \phi \\
X \oplus Y & \longrightarrow & X \oplus Y \oplus Y''
\end{array}
$$

where the active morphism ϕ exhibits Y_0 as a sum $Y_0' \oplus Y_0''$, where $Y_0' \to X$ is an active morphism and $Y_0'' \simeq Y \oplus Y''$. The inclusion $\mathcal{X} \subseteq \mathcal{T}_{/D}^{\mathrm{act}}$ admits a left adjoint, and is therefore left cofinal.

Let $D_0 \in \overline{\mathcal{D}} = (K \star \{0\}) \times_{\mathcal{O}^{\otimes}} \mathcal{K}_{\mathcal{O}}$ be the object corresponding to id_X. The operation $\bullet \mapsto \bullet \oplus Y \oplus Y''$ determines an equivalence of ∞-categories $\mathcal{D}_{/D_0}^{\mathrm{act}} \to \mathcal{X}$. It therefore suffices to show that the composite map

$$(\mathcal{D}_{/D_0}^{\mathrm{act}})^{\triangleright} \to \mathcal{X}^{\triangleright} \subseteq (\mathcal{T}_{/D}^{\mathrm{act}})^{\triangleright} \stackrel{\delta}{\to} \mathcal{C}^{\otimes}$$

is an operadic q-colimit diagram. This composition can be identified with the map

$$(\mathcal{D}_{/D_0}^{\mathrm{act}})^{\triangleright} \longrightarrow \mathcal{C}^{\otimes} \stackrel{A(Y \oplus Y'')}{\longrightarrow} \mathcal{C}^{\otimes},$$

which is an operadic q-colimit diagram by virtue of our assumption that $\overline{p}$ satisfies condition $(*)$. $\square$

Chapter 4

Associative Algebras and Their Modules

Let $\mathcal{C}$ be a symmetric monoidal ∞-category. For any ∞-operad $\mathcal{O}^{\otimes}$, we can consider the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ of $\mathcal{O}$-algebra objects of $\mathcal{C}$ (Definition 2.1.2.7), which we studied in Chapter 3. In this chapter, we will specialize the general theory to obtain a notion of *associative algebra* in $\mathcal{C}$, which we study in depth. We begin in §4.1 by introducing the *associative ∞-operad*, which we denote by $\mathcal{A}\mathrm{ss}^{\otimes}$. By definition, an associative algebra object of a symmetric monoidal ∞-category $\mathcal{C}$ is a map of ∞-operads $\mathcal{A}\mathrm{ss}^{\otimes} \to \mathcal{C}^{\otimes}$.

Many basic facts about associative algebras can be deduced from the general formalism developed in Chapters 2 and 3. However, there are some facets of the theory which are special to the associative case. For example, if A is an associative algebra object of a symmetric monoidal ∞-category $\mathcal{C}$, then there is an associated theory of *left modules* over A (and a formally dual theory of *right modules* over A). The collection of all left A-modules is naturally organized into an ∞-category, which we denote by $\mathrm{LMod}_A(\mathcal{C})$. We will undertake a detailed study of this ∞-category in §4.2.

If A is an associative algebra object of an ∞-category $\mathcal{C}$, we can use the general formalism of §3.3 to obtain a different notion of A-module: namely, an $\mathcal{A}\mathrm{ss}$-module over A (see Definition 3.3.3.8). The collection of $\mathcal{A}\mathrm{ss}$-modules over A form an ∞-category $\mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C})$. There is a forgetful functor $\mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C}) \to \mathrm{LMod}_A(\mathcal{C})$, which is usually not an equivalence of ∞-categories: every object $M \in \mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C})$ is equipped not only with a left action of the algebra A, but also with a (commuting) *right* action of A. In §4.3, we will describe the situation formally by introducing the notion of a *bimodule object* of $\mathcal{C}$. For every pair of associative algebra objects A and B, we will define an ∞-category ${}_A\mathrm{BMod}_B(\mathcal{C})$, which is equipped with forgetful functors

$$\mathrm{LMod}_A(\mathcal{C}) \leftarrow {}_A\mathrm{BMod}_B(\mathcal{C}) \to \mathrm{RMod}_B(\mathcal{C}).$$

We will show that there is a canonical equivalence of ∞-categories $\theta : \mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C}) \simeq {}_A\mathrm{BMod}_A(\mathcal{C})$ (Theorem 4.3.4.28). In fact, we will prove something a little stronger: under some mild hypotheses, the functor θ is an equivalence of monoidal categories. Here the tensor product on $\mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C})$ is determined by the general formalism of Chapter 3, and the tensor product on ${}_A\mathrm{BMod}_A(\mathcal{C})$ is given as a special case of a more general *relative tensor product*

$$ {}_A\mathrm{BMod}_B(\mathcal{C}) \times {}_B\mathrm{BMod}_C(\mathcal{C}) \to {}_A\mathrm{BMod}_C(\mathcal{C}) $$

which we will study in §4.3.

Suppose now that A is a *commutative* algebra object of $\mathcal{C}$. In this case, the general formalism of Chapter 3 determines an ∞-category $\mathrm{Mod}_A(\mathcal{C}) = \mathrm{Mod}_A^{\mathrm{CRing}}(\mathcal{C})$ of CRing-modules over A. The map of ∞-operads $\mathcal{A}\mathrm{ss}^{\otimes} \to \mathrm{Comm}^{\otimes}$ determines a forgetful functor $\mathrm{Mod}_A(\mathcal{C}) \to \mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C})$, which fits into a commutative

diagram

$$
\begin{array}{c}
\mathrm{Mod}_A(\mathcal{C}) \\
\downarrow \\
\mathrm{Mod}_A^{\mathcal{A}ss}(\mathcal{C}) \\
\downarrow \sim \\
\mathrm{LMod}_A(\mathcal{C}) \longleftarrow {}_A\mathrm{BMod}_A(\mathcal{C}) \longrightarrow \mathrm{RMod}_A(\mathcal{C}).
\end{array}
$$

In §4.4, we will show that the diagonal arrows in this diagram are categorical equivalences (Proposition 4.4.1.4). In other words, if A is a commutative algebra object of $\mathcal{C}$, then the ∞-categories of left, right, and CRing modules over A are canonically equivalence. Under some mild hypotheses, each of these ∞-categories admits a symmetric monoidal structure, given by the relative tensor product over A (Theorem 4.4.2.1).

4.1   Associative Algebras

Let $\mathcal{C}$ be a monoidal category. Then one can consider *associative algebra objects* of $\mathcal{C}$: that is, objects $A \in \mathcal{C}$ equipped with a unit map $u : \mathbf{1} \to A$ and a multiplication map $m : A \times A \to A$ such that the diagrams

$$
\begin{array}{ccc}
\mathbf{1} \otimes A \xrightarrow{\;u \otimes \mathrm{id}\;} A \otimes A & & A \otimes \mathbf{1} \xrightarrow{\;\mathrm{id}\, \otimes u\;} A \otimes A \\
\searrow \quad \swarrow m & & \searrow \quad \swarrow m \\
A & & A
\end{array}
$$

$$
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{\;m \otimes \mathrm{id}\;} & A \otimes A \\
\downarrow{\scriptstyle \mathrm{id}\, \otimes m} & & \downarrow{\scriptstyle m} \\
A \otimes A & \xrightarrow{\quad m \quad} & A
\end{array}
$$

commute. Our goal in this section is to generalize the theory of associative algebras to the ∞-categorical setting.

The theory of associative algebras has a natural formulation in the language of operads. If $\mathcal{C}$ is a symmetric monoidal category, then giving an associative algebra object of $\mathcal{C}$ is equivalent to giving a map of colored operads $\mathbf{Ass} \to \mathcal{C}$, where $\mathbf{Ass}$ is the associative operad (Definition 4.1.1.1). In §4.1.1, we will obtain a theory of associative algebras in the ∞-categorical setting by using the construction of Example 2.1.1.21 to convert $\mathbf{Ass}$ into an ∞-operad $\mathcal{A}ss^{\otimes}$. We can then define an associative algebra object of a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ to be a map of ∞-operads $\mathcal{A}ss^{\otimes} \to \mathcal{C}^{\otimes}$.

The advantage of the approach taken in §4.1.1 is that it allows us to apply the general theory developed in Chapters 2 and 3 to the study of associative algebras. One disadvantage is that the simplicial set $\mathcal{A}ss^{\otimes}$ is somewhat complicated. In §4.1.2, we will remedy the situation by showing that, for many purposes, the ∞-operad $\mathcal{A}ss^{\otimes}$ can be replaced by the ∞-category $\mathrm{N}(\mathbf{\Delta})^{op}$: that is, associative algebras can be efficiently encoded as certain kinds of simplicial objects.

The theory of model categories provides a rich source of examples of monoidal ∞-categories. In §4.1.3, we will show that every monoidal model category $\mathbf{A}$ determines a monoidal ∞-category whose underlying ∞-category (if $\mathbf{A}$ is simplicial) can be identified with $\mathrm{N}(\mathbf{A}^{\mathrm{o}})$. In this context, there is a close relationship between associative algebra objects of the ∞-category $\mathrm{N}(\mathbf{A}^{\mathrm{o}})$ and (strict) associative algebra objects of the ordinary category $\mathbf{A}$, which we will discuss in §4.1.4.

4.1.1 The ∞-Operad $\mathcal{A}\mathrm{ss}^\otimes$

As a first step towards building an ∞-categorical theory of associative algebras, let us recast the classical theory of associative algebras in the language of operads.

Definition 4.1.1.1. e define a colored operad **Ass** (see Definition 2.1.1.1) as follows:

- The colored operad **Ass** has a single object, which we will denote by $*$.

- For every finite set I, the set $\mathrm{Mul}_{\mathbf{Ass}}(\{*\}_{i \in I}, *)$ can be identified with the set of linear orderings on I.

- Suppose we are given a map of finite sets $\alpha : I \to J$ together with operations $\phi_j \in \mathrm{Mul}_{\mathbf{Ass}}(\{*\}_{\alpha(i)=j}, *)$ and $\psi \in \mathrm{Mul}_{\mathbf{Ass}}(\{*\}_{j \in J}, *)$. We will identify each ϕ_j with a linear ordering $\preceq_j$ on $\alpha^{-1}\{j\}$ and ψ with a linear ordering $\preceq'$ on J. The composition of ψ with $\{\phi_j\}$ corresponds to the linear ordering $\preceq$ on I which is defined as follows: $i \leq i'$ if either $\alpha(i) \prec' \alpha(i')$ or $\alpha(i) = j = \alpha(i')$ and $i \preceq_j i'$.

We will refer to **Ass** as the *associative operad*.

Remark 4.1.1.2. Let $\mathcal{C}$ be a symmetric monoidal category, and let $F : \mathbf{Ass} \to \mathcal{C}$ be a map of colored operads. Then $F(*)$ is an object $A \in \mathcal{C}$. Given any linear ordering of a finite set I, the corresponding element of $\mathrm{Mul}_{\mathbf{Ass}}(\{*\}_{i \in I}, *)$ determines a map $A^{\otimes I} \to A$ in $\mathcal{C}$. In particular, taking I to be empty, we obtain a map $u : \mathbf{1} \to A$ (where $\mathbf{1}$ denotes the unit object of $\mathcal{C}$), and taking $I = \{1, 2\}$ (with its usual ordering) we obtain a map $m : A \otimes A \to A$. It is not difficult to see that these maps exhibit A as an associative algebra in $\mathcal{C}$. Conversely, if A is any associative algebra in $\mathcal{C}$, then we can associate to every finite linearly ordered set I a map $A^{\otimes I} \to A$; this construction determines a map of colored operads $\mathbf{Ass} \to \mathcal{C}$. We can informally summarize the above discussion by saying that the theory of associative algebras is *controlled* by the colored operad **Ass**.

Definition 4.1.1.3. We let $\mathbf{Ass}^\otimes$ denote the category obtained by applying Construction 2.1.1.7 to the colored operad **Ass**, and $\mathcal{A}\mathrm{ss}^\otimes$ denote the ∞-operad $\mathrm{N}(\mathbf{Ass}^\otimes)$ (see Example 2.1.1.21). We will refer to $\mathcal{A}\mathrm{ss}^\otimes$ as the *associative ∞-operad*.

Remark 4.1.1.4. Unwinding the definitions, we can describe $\mathbf{Ass}^\otimes$ as follows:

- The objects of $\mathbf{Ass}^\otimes$ are the objects of $\mathcal{F}\mathrm{in}_*$.

- Given a pair of objects $\langle m \rangle, \langle n \rangle \in \mathcal{F}\mathrm{in}_*$, a morphism from $\langle m \rangle$ to $\langle n \rangle$ in $\mathbf{Ass}^\otimes$ consists of a pair $(\alpha, \{\preceq_i\}_{1 \leq i \leq n})$, where $\alpha : \langle m \rangle \to \langle n \rangle$ is a map of pointed finite sets and $\leq_i$ is a linear ordering on the inverse image $f^{-1}\{i\} \subseteq \langle m \rangle^\circ$ for $1 \leq i \leq n$.

- The composition of a pair of morphisms

$$(f, \{\preceq_i\}_{1 \leq i \leq n}) : \langle m \rangle \to \langle n \rangle \qquad (g, \{\preceq'_j\}_{1 \leq j \leq p}) : \langle n \rangle \to \langle p \rangle$$

  is the pair $(g \circ f, \{\preceq''_j\}_{1 \leq j \leq p})$, where each $\leq''_j$ is the lexicographical ordering characterized by the property that for $a, b \in \langle m \rangle^\circ$ such that $(g \circ f)(a) = (g \circ f)(b) = j$, we have $a \preceq''_j b$ if and only if $f(a) \preceq'_j f(b)$ and $a \preceq_i b$ if $f(a) = f(b) = i$.

Remark 4.1.1.5. Following the conventions of Chapter 2, we let $\mathcal{A}\mathrm{ss}$ denote the fiber $\mathcal{A}\mathrm{ss}^\otimes \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \{\langle 1 \rangle\}$. As a simplicial set, $\mathcal{A}\mathrm{ss}$ is isomorphic to the 0-simplex Δ^0. However, the notation $\mathcal{A}\mathrm{ss}$ emphasizes the role of this simplicial set as the underlying ∞-category for the ∞-operad $\mathcal{A}\mathrm{ss}^\otimes$.

Definition 4.1.1.6. A *planar ∞-operad* is a fibration of ∞-operads $\mathcal{C}^\otimes \to \mathcal{A}\mathrm{ss}^\otimes$. If $\mathcal{C}^\otimes \to \mathcal{A}\mathrm{ss}^\otimes$ is a planar ∞-operad, we will often denote the ∞-category $\mathrm{Alg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C})$ by $\mathrm{Alg}(\mathcal{C})$; we will refer to $\mathrm{Alg}(\mathcal{C})$ as the *∞-category of associative algebra objects of* $\mathcal{C}$.

Remark 4.1.1.7. In the situation of Definition 4.1.1.6, we will often abuse terminology and simply refer to $\mathcal{C}^\otimes$ as a planar ∞-operad. However, there is some danger of ambiguity: see Remark 4.1.1.8 below.

Remark 4.1.1.8. The colored operad **Ass** is equipped with a canonical involution σ, which carries the unique object $* \in$ **Ass** to itself and carries an operation $\phi \in \mathrm{Mul}_{\mathbf{Ass}}(\{*\}_{i \in I}, *)$ corresponding to a linear order $\preceq$ on I to the operation ϕ^σ which corresponds to the opposite ordering $\succeq$. This involution σ induces an involution on the ∞-operad $\mathcal{A}\mathrm{ss}^\otimes$.

If $\mathcal{C}^\otimes \to \mathcal{A}\mathrm{ss}^\otimes$ is a planar ∞-operad, then the composite map $\mathcal{C}^\otimes \to \mathcal{A}\mathrm{ss}^\otimes \xrightarrow{\sigma} \mathcal{A}\mathrm{ss}^\otimes$ is another planar ∞-operad, which we will denote by $\mathcal{C}^\otimes_{\mathrm{rev}}$. We will refer to $\mathcal{C}^\otimes_{\mathrm{rev}}$ as the *reverse* of $\mathcal{C}^\otimes$; note that the underlying ∞-category $\mathcal{C}_{\mathrm{rev}}$ of $\mathcal{C}^\otimes_{\mathrm{rev}}$ is canonically isomorphic with $\mathcal{C}$. Composition with σ induces an isomorphism $\mathrm{Alg}(\mathcal{C}) \simeq \mathrm{Alg}(\mathcal{C}_{\mathrm{rev}})$. If $A \in \mathrm{Alg}(\mathcal{C})$, we will denote to the image of A under this isomorphism by A^{rev}, and refer to it as the *opposite algebra* of A.

If $\mathcal{D}^\otimes$ is an ∞-operad and $\mathcal{C}^\otimes = \mathcal{D}^\otimes \times_{\mathrm{Comm}^\otimes} \mathcal{A}\mathrm{ss}^\otimes$ is the underlying planar ∞-operad, then there is a canonical isomorphism $\mathcal{C}^\otimes_{\mathrm{rev}} \simeq \mathcal{C}^\otimes$. It follows that passage to the opposite algebra can be regarded as a functor from the ∞-category $\mathrm{Alg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C}) = \mathrm{Alg}_{\mathcal{A}\mathrm{ss}}(\mathcal{D})$ to itself.

Notation 4.1.1.9. Let $\mathcal{C}^\otimes$ be a generalized ∞-operad. We let $\mathrm{Alg}(\mathcal{C})$ denote the ∞-category $\mathrm{Alg}_{\mathcal{A}\mathrm{ss}}(\mathcal{C})$ of $\mathcal{A}\mathrm{ss}$-algebra objects of $\mathcal{C}$. If $\mathcal{C}^\otimes$ is an ∞-operad, then $\mathrm{Alg}(\mathcal{C})$ can be identified with $\mathrm{Alg}(\mathcal{D})$, where $\mathcal{D}^\otimes \simeq \mathcal{C}^\otimes \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathcal{A}\mathrm{ss}^\otimes$ is the planar ∞-operad associated to $\mathcal{C}^\otimes$. This notation is potentially ambiguous: if $\mathcal{C}^\otimes$ is already equipped with a fibration of ∞-operads $q : \mathcal{C}^\otimes \to \mathcal{A}\mathrm{ss}^\otimes$, then the notation $\mathrm{Alg}(\mathcal{C})$ has two different meanings, depending on whether we regard $\mathcal{C}^\otimes$ as a planar ∞-operad via q, or we ignore q and simply regard $\mathcal{C}^\otimes$ as an ∞-operad. However, the ambiguity is slight: the ∞-category $\mathrm{Alg}_{\mathcal{A}\mathrm{ss}}(\mathcal{A}\mathrm{ss})$ has only two objects, corresponding to the identity map $\mathcal{A}\mathrm{ss}^\otimes \to \mathcal{A}\mathrm{ss}^\otimes$ and the reversal involution of Remark 4.1.1.8. Consequently, if $\mathcal{C}^\otimes$ is a planar ∞-operad with an underlying ∞-operad which we denote by $\mathcal{C}^\otimes_0$, then $\mathrm{Alg}(\mathcal{C}_0) \simeq \mathrm{Alg}(\mathcal{C}) \coprod \mathrm{Alg}(\mathcal{C}_{\mathrm{rev}})$.

Definition 4.1.1.10. A *monoidal ∞-category* is a coCartesian fibration of ∞-operads $\mathcal{C}^\otimes \to \mathcal{A}\mathrm{ss}^\otimes$.

Remark 4.1.1.11. In the situation of Definition 4.1.1.10, we will often abuse terminology by referring to $\mathcal{C}^\otimes$ or the fiber $\mathcal{C} = \mathcal{C}^\otimes \times_{\mathcal{A}\mathrm{ss}^\otimes} \{\langle 1 \rangle\}$ as a monoidal ∞-category.

Remark 4.1.1.12. Let $\mathcal{C}^\otimes \to \mathcal{A}\mathrm{ss}^\otimes$ be a monoidal ∞-category. For each $n \geq 0$, the fiber $\mathcal{C}^\otimes_{\langle n \rangle} \simeq \mathcal{C}^\otimes \times_{\mathcal{A}\mathrm{ss}^\otimes} \{\langle n \rangle\}$ is canonically equivalent to $\mathcal{C}^n$. For every choice of linear ordering on the set $\{1, \ldots, n\}$, the corresponding map $\langle n \rangle \to \langle 1 \rangle$ in $\mathcal{A}\mathrm{ss}^\otimes$ induces a functor $\mathcal{C}^n \to \mathcal{C}$. In particular, taking $n = 0$ we obtain an object $\mathbf{1} \in \mathcal{C}$, which we call the *unit object*, and taking $n = 2$ (and the ordering on $\{1, 2\}$ to be the standard ordering) we obtain a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. It is not difficult to see that the tensor product $\otimes$ is associative (with unit $\mathbf{1}$) up to homotopy; in particular, the homotopy category $h\mathcal{C}$ inherits a monoidal structure (in the usual sense).

Remark 4.1.1.13. Let $\mathcal{C}$ be a monoidal ∞-category. Evaluation on the object $\langle 1 \rangle \in \mathcal{A}\mathrm{ss}^\otimes$ determines a forgetful functor $\theta : \mathrm{Alg}(\mathcal{C}) \to \mathcal{C}$. We will often abuse notation by identifying an algebra object $A \in \mathrm{Alg}(\mathcal{C})$ with its image $\theta(A) \in \mathcal{C}$. For each $n \geq 0$, every choice of ordering on the set $\{1, \ldots, n\}$ determines a map $\langle n \rangle \to \langle 1 \rangle$ in $\mathcal{A}\mathrm{ss}^\otimes$, which induces a morphism $\theta(A)^{\otimes n} \to \theta(A)$ in $\mathcal{C}$. In particular, taking $n = 2$ and the standard ordering of the set $\{1, 2\}$, we obtain a multiplication $m : \theta(A) \otimes \theta(A) \to \theta(A)$. It is not difficult to see that this multiplication map is associative and unital up to homotopy; in particular, it endows $\theta(A)$ with the structure of an associative algebra object of the monoidal category $h\mathcal{C}$.

Recall that if V is a vector space over a field k, then the free associative k-algebra generated by V can be identified with the tensor algebra $\bigoplus_{n \geq 0} V^{\otimes n}$. Using the results of §3.1, we can obtain an analogous description of free associative algebras in a general monoidal ∞-category $\mathcal{C}$. Note that $\mathcal{A}\mathrm{ss}$ has a unique object $\langle 1 \rangle$, and that the spaces $\mathcal{P}(n)$ appearing in Construction 3.1.3.7 are contractible. It follows that for each $C \in \mathcal{C}$, the objects $\mathrm{Sym}^n_{\mathcal{A}\mathrm{ss}}(C)$ are automatically well-defined and given by the tensor power $C^{\otimes n}$. Applying Proposition 3.1.3.11 to this situation, we obtain:

Proposition 4.1.1.14. *Let $\mathcal{C}$ be a monoidal ∞-category. Assume that $\mathcal{C}$ admits countable colimits and that the tensor product $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves countable colimits separately in each variable. Then the forgetful*

functor $\theta : \mathrm{Alg}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint Fr. *Moreover, given an object $A \in \mathrm{Alg}(\mathcal{C})$ and an object $C \in \mathcal{C}$, a morphism $u : C \to \theta(A)$ is adjoint to an equivalence $\mathrm{Fr}(C) \to A$ if and only if the maps $C^{\otimes n} \to \theta(A)^{\otimes n} \to \theta(A)$ determined by u and the multiplication on A induce an equivalence $\coprod_{n \geq 0} C^{\otimes n} \to \theta(A)$.*

Remark 4.1.1.15. Proposition 4.1.1.14 admits a slight refinement: it is sufficient to assume that $\mathcal{C}$ admits countable coproducts (and that the tensor product functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves countable coproducts separately in each variable), rather than arbitrary countable colimits.

Proposition 4.1.1.16. *The ∞-operad $\mathcal{A}\mathrm{ss}^{\otimes}$ is coherent (see Definition 3.3.1.9).*

Consequently, to every associative algebra object A of a monoidal ∞-category $\mathcal{C}$, we can associate an ∞-category of modules $\mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C})$. In §4.3, we will see that the objects of $\mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C})$ can be understood as A-A bimodule objects of $\mathcal{C}$ (Theorem 4.3.4.28).

Proof. It is easy to see that the final object $\langle 0 \rangle$ of $\mathcal{A}\mathrm{ss}^{\otimes}$ is also initial, since the empty set admits a unique linear ordering. Consequently, $\mathcal{A}\mathrm{ss}^{\otimes}$ is unital. The underlying ∞-category $\mathcal{A}\mathrm{ss} \simeq \Delta^0$ is obviously a Kan complex. To complete the proof, we must show that $\mathcal{A}\mathrm{ss}^{\otimes}$ satisfies condition (3) of Definition 3.3.1.9. To prove this, we need to introduce a bit of notation. For every linearly ordered set, let $C(S)$ denote the set of *cuts* in S: that is, decompositions $S = S_- \cup S_+$, where S_- and S_+ are subsets of S satisfying $s < t$ whenever $s \in S_-$, $t \in S_+$. Let $f : \langle m \rangle \to \langle n \rangle$ be an active morphism in $\mathcal{A}\mathrm{ss}^{\otimes}$, corresponding to a map of sets $f : \langle m \rangle^{\circ} \to \langle n \rangle^{\circ}$ and a linear ordering on each fiber $f^{-1}\{i\}$, $1 \leq i \leq n$. Unwinding the definitions, we deduce that $\mathrm{Ext}(f)$ is homotopy equivalent to the disjoint union $\coprod_{1 \leq i \leq n} C(f^{-1}\{i\})$ (regarded as a discrete simplicial set).

Since $\mathcal{A}\mathrm{ss}^{\otimes}$ is obviously unital and the underlying ∞-category $\mathcal{A}\mathrm{ss} \simeq \Delta^0$ is a Kan complex, to prove that $\mathcal{A}\mathrm{ss}^{\otimes}$ is coherent if will suffice (by Remark 3.3.1.11) to show that for every pair of active morphisms $f : \langle m \rangle \to \langle n \rangle$ and $g : \langle n \rangle \to \langle 1 \rangle$ in $\mathcal{A}\mathrm{ss}^{\otimes}$, the diagram

$$
\begin{array}{ccc}
\mathrm{Ext}(\mathrm{id}_Y) & \longrightarrow & \mathrm{Ext}(g) \\
\downarrow & & \downarrow \\
\mathrm{Ext}(f) & \longrightarrow & \mathrm{Ext}(gf)
\end{array}
$$

is a homotopy pushout square. The map g determines a linear ordering on $\langle n \rangle^{\circ}$ and the composition gf determines a linear ordering on $\langle m \rangle^{\circ}$. Unwinding the definitions, we must show that the

$$
\begin{array}{ccc}
\coprod_{1 \leq j \leq n} C(\{j\}) & \longrightarrow & C(\langle n \rangle^{\circ}) \\
\downarrow & & \downarrow \\
\coprod_{1 \leq j \leq n} C(f^{-1}\{j\}) & \longrightarrow & C(\langle m \rangle^{\circ})
\end{array}
$$

is a homotopy pushout square (of Kan complexes). This is a consequence of the following general observation:

($*$) Let $\alpha : X \to Y$ be a map of sets, and suppose we are given a finite sequence of elements

$$x_1, \ldots, x_k, x_1', \ldots, x_k' \in X$$

whose images in Y are disjoint. Let X' denote the quotient of X obtained by identifying x_i with x_i' for $1 \leq i \leq k$, and let Y' be defined similarly. Then the diagram of sets

$$
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow{\scriptstyle \alpha} & & \downarrow \\
Y & \longrightarrow & Y'
\end{array}
$$

is a homotopy pushout square (of Kan complexes).

□

Definition 4.1.1.17. Let $\mathcal{C}$ be a monoidal ∞-category. We will say that $\mathcal{C}$ is *left closed* if, for each $C \in \mathcal{C}$, the functor $D \mapsto C \otimes D$ admits a right adjoint. Similarly, we will say that $\mathcal{C}$ is right closed if, for each $C \in \mathcal{C}$, the functor $D \mapsto D \otimes C$ admits a right adjoint. We will say that $\mathcal{C}$ is *closed* if it is both left closed and right closed.

Remark 4.1.1.18. In view of Proposition T.5.2.2.12, the condition that a monoidal ∞-category $\mathcal{C}$ be closed can be checked at the level of the ($\mathcal{H}$-enriched) homotopy category of $\mathcal{C}$, with its induced monoidal structure. More precisely, $\mathcal{C}$ is *right closed* if and only if, for every pair of objects $C, D \in \mathcal{C}$, there exists another object D^C and a map $D^C \otimes C \to D$ with the following universal property: for every $E \in \mathcal{C}$, the induced map $\mathrm{Map}_{\mathcal{C}}(E, D^C) \to \mathrm{Map}(E \otimes C, D)$ is a homotopy equivalence. In this case, the construction $D \mapsto D^C$ determines a right adjoint to the functor $E \mapsto E \otimes C$.

Remark 4.1.1.19. Let $\mathcal{C}$ monoidal ∞-category. We say that an object $C \in \mathcal{C}$ is *invertible* if it is an invertible object of the homotopy category h$\mathcal{C}$: that is, if there exists an object $D \in \mathcal{C}$ and equivalences

$$C \otimes D \simeq 1_{\mathcal{C}} \simeq D \otimes C.$$

Let $\mathcal{C}^0 \subseteq \mathcal{C}$ denote the full subcategory spanned by the invertible objects. It is easy to see that the hypotheses of Proposition 2.2.1.1 are satisfied, so that $\mathcal{C}^0$ inherits the structure of a monoidal ∞-category.

We conclude this section with the following observation:

Proposition 4.1.1.20. *Let* $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *be an* ∞*-operad. Then* $\mathcal{C}^{\otimes}$ *is a symmetric monoidal* ∞*-category if and only if the induced map* $p' : \mathcal{A}ss^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathcal{C}^{\otimes} \to \mathcal{A}ss^{\otimes}$ *is a monoidal* ∞*-category.*

Proof. The "only if" direction is obvious. For the converse, suppose that p' is a coCartesian fibration; we wish to prove that p is a coCartesian fibration. According to Corollary T.2.4.2.10, it will suffice to show that for every 2-simplex $\sigma : \Delta^2 \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, the pullback map $\Delta^2 \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathcal{C}^{\otimes} \to \Delta^2$ is a coCartesian fibration. Since p' is a coCartesian fibration, it will suffice to show that σ factors through $\mathcal{A}ss^{\otimes}$. This is clear: if σ corresponds to a pair of maps $\langle l \rangle \xrightarrow{\alpha} \langle m \rangle \xrightarrow{\beta} \langle n \rangle$, then a lifting of σ is determined by a choice of linear ordering on each of the fibers $\alpha^{-1}\{i\}$ and each of the fibers $\beta^{-1}\{j\}$. □

4.1.2 Simplicial Models for Associative Algebras

Let $\mathcal{C}$ be an ordinary category which admits finite products. A *monoid object* of $\mathcal{C}$ is an object $M \in \mathcal{C}$, equipped with maps

$$* \to M, \quad M \times M \to M$$

which satisfy the usual associativity and unit conditions; here $*$ denotes a final object of $\mathcal{C}$. Equivalently, we can define a monoid object of $\mathcal{C}$ to be a contravariant functor from $\mathcal{C}$ to the category of monoids, such that the underlying functor $\mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$ is representable by an object $M \in \mathcal{C}$.

Example 4.1.2.1. If $\mathcal{C}$ is the category of sets, then a monoid object of $\mathcal{C}$ is simply a monoid M. We can identify the monoid M with a category $\mathcal{D}_M$, having only a single object E with $\mathrm{Hom}_{\mathcal{D}_M}(E, E) = M$. The nerve $\mathrm{N}(\mathcal{D}_M)$ is a simplicial set, which is typically denoted by BM and called the *classifying space* of M. Concretely, the set of n-simplices of BM can be identified with an n-fold product of M with itself, and the face and degeneracy operations on BM encode the multiplication and unit operations on M. The functor $M \mapsto BM$ is a fully faithful embedding of the category of monoids into the category of simplicial sets. Moreover, a simplicial set X is isomorphic to the classifying space of a monoid if and only if, for each $n \geq 0$, the natural map $X([n]) \to X(\{0,1\}) \times \ldots \times X(\{n-1,n\})$ is a bijection. In this case, the underlying monoid is given by $X([1])$, with unit determined by the degeneracy map $* \simeq X([0]) \to X([1])$ and multiplication by the face map $X([1]) \times X([1]) \simeq X([2]) \xrightarrow{d_1} X([1])$.

It follows from Example 4.1.2.1 that we can identify monoids in an arbitrary category $\mathcal{C}$ with certain simplicial objects of $\mathcal{C}$. This observation allows us to generalize the notion of a monoid to higher category theory.

Definition 4.1.2.2. Let $\mathcal{C}$ be an ∞-category. A *monoid object* of $\mathcal{C}$ is a simplicial object $X : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{C}$ with the property that, for each $n \geq 0$, the collection of maps $X([n]) \to X(\{i, i+1\})$ exhibits $X([n])$ as a product $X(\{0, 1\}) \times \ldots \times X(\{n-1, n\})$. We let $\mathrm{Mon}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{C})$ spanned by the monoid objects of $\mathcal{C}$.

Remark 4.1.2.3. Let X be a monoid object of an ∞-category $\mathcal{C}$. We will sometimes abuse terminology and refer to $X([1]) \in \mathcal{C}$ as a monoid object of $\mathcal{C}$; note that $X([1])$ "controls" the simplicial object X in the sense that $X([n]) \simeq X([1])^n$.

Example 4.1.2.4. Let $\mathcal{C}$ be an ∞-category. Every group object of $\mathcal{C}$ (see Definition T.7.2.2.1) is a monoid object of $\mathcal{C}$.

In §2.4.2, we introduced the notion of a $\mathcal{O}$-monoid object of an ∞-category $\mathcal{C}$, where $\mathcal{O}^{\otimes}$ is an arbitrary ∞-operad (Definition 2.4.2.1). Our next goal is to show that when $\mathcal{O}^{\otimes} = \mathcal{A}\mathrm{ss}^{\otimes}$, then this recovers the theory of monoids given by Definition 4.1.2.2. To formulate this more precisely, we need to introduce a bit of notation.

Construction 4.1.2.5. Let $\mathbf{\Delta}$ denote the category of combinatorial simplices. Given an object $[n] \in \mathbf{\Delta}$, we define a *cut* in $[n]$ to be an equivalence relation on $[n]$ with at most two equivalence classes, each of which is a convex subset of $[n]$. The collection of all cuts in $[n]$ forms a set $\mathrm{Cut}([n])$. There is a canonical bijection $\langle n \rangle \simeq \mathrm{Cut}([n])$, which carries the base point $* \in \langle n \rangle$ to the trivial equivalence relation on $[n]$ (that is, the equivalence relation with only one equivalence class) and an element $i \in \langle n \rangle^{\circ}$ to the equivalence relation given by the partition $[n] = \{0, \ldots, i-1\} \cup \{i, i+1, \ldots, n\}$.

We can regard the construction $[n] \mapsto \mathrm{Cut}([n])$ as a functor $\mathrm{Cut} : \mathbf{\Delta}^{op} \to \mathbf{Ass}^{\otimes}$, where $\mathbf{Ass}^{\otimes}$ is the functor defined in Definition 4.1.1.3. Namely, if we are given a map $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$, then there is an induced map $\alpha' : \mathrm{Cut}([n]) \to \mathrm{Cut}([m])$, which obviously preserves base points. Moreover, if $x \in \mathrm{Cut}([m])$ is an equivalence relation with two equivalence classes, then every element of $\alpha'^{-1}(x)$ corresponds to a decomposition $[n] = S \cup T$ of $[n]$ into nonempty subsets $S, T \subseteq [n]$ such that $s < t$ for $s \in S$, $t \in T$. We therefore obtain a linear ordering on $\alpha'^{-1}(x)$, with $(S, T) \leq (S', T')$ if $S \subseteq S'$.

In more explicit terms, the functor $\mathrm{Cut} : \mathbf{\Delta}^{op} \to \mathbf{Ass}^{\otimes}$ can be described as follows:

(1) For each $n \geq 0$, we have $\mathrm{Cut}([n]) = \langle n \rangle$.

(2) Given a morphism $\alpha : [n] \to [m]$ in $\mathbf{\Delta}$, the associated morphism $\mathrm{Cut}(\alpha) : \langle m \rangle \to \langle n \rangle$ is given by the formula
$$\mathrm{Cut}(\alpha)(i) = \begin{cases} j & \text{if } (\exists j)[\alpha(j-1) < i \leq \alpha(j)] \\ * & \text{otherwise.} \end{cases}$$
where we endow each $\mathrm{Cut}(\alpha)^{-1}\{j\}$ with the linear ordering induced by its inclusion into $\langle n \rangle^{\circ}$.

The functor Cut induces a map of simplicial sets $\mathrm{N}(\mathbf{\Delta})^{op} \to \mathrm{N}(\mathbf{Ass}^{\otimes}) = \mathcal{A}\mathrm{ss}^{\otimes}$, which we will also denote by Cut.

Our comparison result can now be stated as follows:

Proposition 4.1.2.6. *For every ∞-category $\mathcal{C}$ which admits finite products, composition with the functor* $\mathrm{Cut} : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{A}\mathrm{ss}^{\otimes}$ *of Construction 4.1.2.5 induces an equivalence of ∞-categories $\theta : \mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}) \to \mathrm{Mon}(\mathcal{C})$.*

We will deduce Proposition 4.1.2.6 from a much more general comparison result. To state it, we need a definition:

Definition 4.1.2.7. Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad, let $f : \mathcal{E} \to \mathcal{O}^{\otimes}$ be a weak approximation to $\mathcal{O}^{\otimes}$ (see Definition 2.3.3.6). Choose a factorization of f as a composition

$$\mathcal{E} \xrightarrow{f'} \mathcal{E}' \xrightarrow{f''} \mathcal{O}^{\otimes}$$

where f' is a categorical equivalence and f'' is a categorical fibration. Let $\mathcal{C}$ be an arbitrary ∞-category. We will say that a functor $M : \mathcal{E} \to \mathcal{C}$ is a $\mathcal{E}$-*monoid object* of $\mathcal{C}$ if it satisfies the following condition:

($*$) Let $E \in \mathcal{E}$ be an object with $(p \circ f)(E) = \langle n \rangle$, and choose maps $\alpha_i : E \to E_i$ covering the maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$ such that such that $f'(\alpha_i)$ is locally $(p \circ f'')$-coCartesian for $1 \le i \le n$. Then the maps $M(\alpha_i)$ exhibit $M(E)$ as a product $\prod_{1 \le i \le n} M(E_i)$ in the ∞-category $\mathcal{C}$.

We let $\mathrm{Mon}_{\mathcal{E}}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{E}, \mathcal{C})$ spanned by the $\mathcal{E}$-monoid objects.

Remark 4.1.2.8. In the situation of Definition 4.1.2.7, the condition that a functor $M : \mathcal{E} \to \mathcal{C}$ be a $\mathcal{E}$-monoid depends only on the underlying map $p' : \mathcal{E} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, and not on the particular presentation of $\mathcal{E}$ as an approximation to an ∞-operad $\mathcal{O}^{\otimes}$ or on the factorization $f \simeq f'' \circ f'$.

Remark 4.1.2.9. In the situation of Definition 4.1.2.7, assume that $\mathcal{E} = \mathcal{O}'^{\otimes}$ is an ∞-operad. It follows from Remark 2.1.2.9 that $\mathrm{Mon}_{\mathcal{E}}(\mathcal{C})$ and $\mathrm{Mon}_{\mathcal{O}'}(\mathcal{C})$ coincide (as subsets of $\mathrm{Fun}(\mathcal{E}, \mathcal{C})$).

We will deduce Proposition 4.1.2.6 from the following pair of assertions (see also Remark 4.1.2.12):

Proposition 4.1.2.10. *The functor* $\mathrm{Cut} : \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathcal{A}\mathrm{ss}^{\otimes}$ *is an approximation to* $\mathcal{A}\mathrm{ss}^{\otimes}$.

Proposition 4.1.2.11. *Let* $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *be an* ∞-*operad and let* $f : \mathcal{E} \to \mathcal{O}^{\otimes}$ *be a weak approximation to* $\mathcal{O}^{\otimes}$. *Assume that the induced map* $\mathcal{E} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \{\langle 1 \rangle\} \to \mathcal{O}$ *is an equivalence of* ∞-*categories. Then, for any* ∞-*category* $\mathcal{C}$ *which admits finite products, composition with* f *induces an equivalence of* ∞-*categories* $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Mon}_{\mathcal{E}}(\mathcal{C})$.

Proof of Proposition 4.1.2.10. We define a category $\overline{\boldsymbol{\Delta}}^{op}$ as follows: the objects of $\overline{\boldsymbol{\Delta}}^{op}$ are triples $([n], \langle m \rangle, \alpha)$ where $[n] \in \boldsymbol{\Delta}^{op}$, $\langle m \rangle \in \mathbf{Ass}^{\otimes}$, $\alpha : \mathrm{Cut}([n]) \simeq \langle m \rangle$ is an isomorphism in $\mathbf{Ass}^{\otimes}$ (the existence of which implies that $m = n$). The construction $[n] \mapsto ([n], \mathrm{Cut}([n]), \mathrm{id})$ determines an equivalence of categories $\boldsymbol{\Delta}^{op} \to \overline{\boldsymbol{\Delta}}^{op}$, and the construction $([n], \langle m \rangle, \alpha) \mapsto \langle m \rangle$ determines a categorical fibration $\theta : \mathrm{N}(\overline{\boldsymbol{\Delta}})^{op} \to \mathcal{A}\mathrm{ss}^{\otimes}$. We will show that θ is approximation to $\mathcal{A}\mathrm{ss}^{\otimes}$.

Let θ_0 denote the composite map $\mathrm{N}(\overline{\boldsymbol{\Delta}})^{op} \xrightarrow{\theta} \mathcal{A}\mathrm{ss}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Note that the fiber $\theta_0^{-1}\{\langle 1 \rangle\}$ is isomorphic to Δ^0; in particular, it contains a single vertex $([1], \langle 1 \rangle, \mathrm{id})$. For every object $([n], \langle m \rangle, \alpha)$ in $\mathrm{N}(\overline{\boldsymbol{\Delta}})^{op}$, the map

$$\mathrm{Hom}_{\boldsymbol{\Delta}}([1], [n]) \simeq \mathrm{Hom}_{\overline{\boldsymbol{\Delta}}^{op}}(([n], \langle m \rangle, \alpha), ([1], \langle 1 \rangle, \mathrm{id})) \to \mathrm{Hom}_{\mathcal{F}\mathrm{in}_*}(\langle m \rangle, \langle 1 \rangle)$$

is injective. It follows that every morphism $([n], \langle m \rangle, \alpha) \to ([1], \langle 1 \rangle, \mathrm{id})$ in $\mathrm{N}(\overline{\boldsymbol{\Delta}})^{op}$ is locally θ_0-coCartesian. Moreover, for each $1 \le i \le m$, there is a unique morphism $([n], \langle m \rangle, \alpha) \to ([1], \langle 1 \rangle, \mathrm{id})$ covering the map $\rho^i : \langle m \rangle \to \langle 1 \rangle$, which induces the map of linearly ordered sets $[1] \simeq \{\alpha^{-1}(i) - 1, \alpha^{-1}(i)\} \subseteq [n]$. It follows that condition (1) of Definition 2.3.3.6 is satisfied.

To verify condition (2) of Definition 2.3.3.6, let us suppose we are given an object $([m], \langle m \rangle, \alpha) \in \mathrm{N}(\overline{\boldsymbol{\Delta}})^{op}$ and an active morphism $\beta : \langle m' \rangle \to \langle m \rangle$ in $\mathcal{A}\mathrm{ss}^{\otimes}$; we wish to show that β can be lifted to a θ-Cartesian morphism $\overline{\beta}$ in $\mathrm{N}(\overline{\boldsymbol{\Delta}})^{op}$. Permuting the elements of $\langle m \rangle^{\circ}$ if necessary, we may suppose that $\alpha = \mathrm{id}$. For $1 \le i \le m$, let k_i be the cardinality of $\beta^{-1}\{i\}$. Permuting the elements of $\langle m' \rangle^{\circ}$ if necessary, we may assume that $\beta^{-1}\{i\} = \{k_1 + \cdots + k_{i-1} + 1, k_1 + \cdots + k_{i-1} + 2, \ldots, k_1 + \cdots + k_{i-1} + k_i\}$ equipped with its usual ordering. In this case, the map of linearly ordered sets $[m] \to [m']$ given by $i \mapsto k_1 + \cdots + k_i$ determines a map $\overline{\beta} : ([m'], \langle m' \rangle, \mathrm{id}) \to ([m], \langle m \rangle, \mathrm{id})$ lifting the map β. A simple calculation shows that $\overline{\beta}$ is θ-Cartesian. $\square$

Remark 4.1.2.12. Let $\mathrm{Cut} : \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathcal{A}\mathrm{ss}^{\otimes}$ be the approximation of Proposition 4.1.2.10. The proof given above shows that the morphisms $\alpha_i : E \to E_i$ appearing in condition ($*$) of Definition 4.1.2.7 are precisely those maps of $\mathrm{N}(\boldsymbol{\Delta})^{op}$ which correspond to inclusions $[1] \simeq \{i, i+1\} \hookrightarrow [n]$ in $\boldsymbol{\Delta}$. It follows

that for any ∞-category $\mathcal{C}$, the ∞-categories $\mathrm{Mon}(\mathcal{C})$ and $\mathrm{Mon}_{\mathrm{N}(\mathbf{\Delta})^{op}}(\mathcal{C})$ coincide (as full subcategories of $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{C})$. Consequently, Proposition 4.1.2.6 is an immediate consequence of Propositions 4.1.2.10 and 4.1.2.11.

Proof of Proposition 4.1.2.11. We employ the same strategy as in the proof of Theorem 2.3.3.23, but the details are slightly easier. We may assume without of generality that the weak approximation $\mathcal{E} \to \mathcal{O}^{\otimes}$ is a categorical fibration. Choose a Cartesian fibration $u : \mathcal{M} \to \Delta^1$ associated to the functor f, so that we have isomorphisms $\mathcal{O}^{\otimes} \simeq \mathcal{M} \times_{\Delta^1} \{0\}$, $\mathcal{C} \simeq \mathcal{M} \times_{\Delta^1} \{1\}$, and choose a retraction r from $\mathcal{M}$ onto $\mathcal{O}^{\otimes}$ such that $r | \mathcal{C} = f$. Let $\mathcal{X}$ denote the full subcategory of $\mathrm{Fun}(\mathcal{M}, \mathcal{C})$ spanned by those functors $F : \mathcal{M} \to \mathcal{C}$ satisfying the following conditions:

(*i*) The restriction $F | \mathcal{O}^{\otimes}$ belongs to $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$.

(*ii*) For every u-Cartesian morphism α in $\mathcal{M}$, the image $F(\alpha)$ is an equivalence in $\mathcal{C}$.

Condition (*ii*) is equivalent to the requirement that F be a left Kan extension of $F | \mathcal{O}^{\otimes}$. Using Proposition T.4.3.2.15, we conclude that the restriction functor $\mathcal{X} \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ is a trivial Kan fibration. Composition with r determines a section s of this trivial Kan fibration. Let $\psi : \mathcal{X} \to \mathrm{Fun}(\mathcal{E}, \mathcal{C})$ be the other restriction functor. Then the forgetful functor $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{E}, \mathcal{C})$ is given by the composition $\psi \circ s$. It will therefore suffice to show that ψ determines an equivalence from $\mathcal{X}$ onto $\mathrm{Mon}_{\mathcal{E}}(\mathcal{C})$. In view of Proposition T.4.3.2.15, it will suffice to verify the following:

(*a*) Let $F_0 \in \mathrm{Mon}_{\mathcal{E}}(\mathcal{C})$. Then there exists a functor $F \in \mathrm{Fun}(\mathcal{M}, \mathcal{C})$ which is a right Kan extension of F_0.

(*b*) A functor $F \in \mathrm{Fun}(\mathcal{M}, \mathcal{C})$ belongs to $\mathcal{X}$ if and only if F is a right Kan extension of $F_0 = F | \mathcal{E}$, and $F_0 \in \mathrm{Mon}_{\mathcal{E}}(\mathcal{C})$.

We begin by proving (*a*). Fix an object $X \in \mathcal{O}^{\otimes}$, let $\mathcal{E}_{X/}$ denote the fiber product $\mathcal{M}_{X/} \times_{\mathcal{M}} \mathcal{E}$, and let $F_X = F_0 | \mathcal{E}_{X/}$. According to Lemma T.4.3.2.13, it will suffice to show that the functor F_X can be extended to a limit diagram $\mathcal{E}_{X/}^{\triangleleft} \to \mathcal{C}$. Let $\mathcal{E}_{X/}'$ denote the full subcategory of $\mathcal{E}_{X/}$ spanned by those morphisms $X \to C$ in $\mathcal{M}$ which correspond to inert morphisms $X \to f(C)$ in $\mathcal{O}^{\otimes}$. Since f is a weak approximation to $\mathcal{O}^{\otimes}$, Theorem T.4.1.3.1 implies that the inclusion $\mathcal{E}_{X/}' \hookrightarrow \mathcal{E}_{X/}$ is right cofinal. It will therefore suffice to show that the restriction $F_X' = F_X | \mathcal{E}_{X/}'$ can be extended to a limit diagram $\mathcal{E}_{X/}'^{\triangleleft} \to \mathcal{C}$.

Let $\langle n \rangle = p(X)$, and let $\mathcal{E}_{X/}''$ denote the full subcategory of $\mathcal{E}_{X/}'$ corresponding to inert morphisms $X \to f(C)$ for which $(p \circ f)(C) = \langle 1 \rangle$. We claim that F_X' is a right Kan extension of $F_X'' = F | \mathcal{E}_{X/}''$. To prove this, let us choose an arbitrary object of $\mathcal{E}_{X/}'$, given by a map $\alpha : X \to C$ in $\mathcal{M}$. The fiber product $\mathcal{E}_{X/}'' \times_{\mathcal{E}_{X/}'} (\mathcal{E}_{X/}')_{\alpha/}$ can be identified with the full subcategory of $\mathcal{M}_{\alpha/}$ spanned by those diagrams $X \xrightarrow{\alpha} C \xrightarrow{\beta} C'$ such that $(p \circ f)(\beta)$ has the form $\rho^i : \langle n \rangle \to \langle 1 \rangle$, for some $1 \le i \le n$. In particular, this ∞-category is a disjoint union of full subcategories $\{\mathcal{D}(i)\}_{1 \le i \le n}$, where each $\mathcal{D}(i)$ is equivalent to the full subcategory of $\mathcal{E}_{C/}$ spanned by morphisms $C \to C'$ covering the map ρ^i. Our assumption that f is a weak approximation to $\mathcal{O}^{\otimes}$ guarantees that each of these ∞-categories has a final object, given by a locally $(p \circ f)$-coCartesian morphism $C \to C_i$ in $\mathcal{C}$. It will therefore suffice to show that $F_0(C)$ is a product of the objects $\{F_0(C_i)\}_{1 \le i \le n}$. Since $\mathcal{C}$ is an ∞-operad, we are reduced to proving that each of the maps $F_0(C) \to F_0(C_i)$ is inert, which follows from our assumption that $F_0 \in \mathrm{Mon}_{\mathcal{E}}(\mathcal{C})$.

Using Lemma T.4.3.2.7, we are reduced to proving that the diagram F_X'' can be extended to a limit diagram $\mathcal{E}''_{X/}^{\triangleleft} \to \mathcal{C}$. For $1 \le i \le n$, let $\mathcal{E}(i)_{X/}''$ denote the full subcategory of $\mathcal{E}_{X/}''$ spanned by those objects for which the underlying morphism $X \to C$ covers $\rho^i : p(X) \simeq \langle n \rangle \to \langle 1 \rangle$. Then $\mathcal{E}_{X/}''$ is the disjoint union of the full subcategories $\{\mathcal{E}(i)_{X/}''\}$. Let $\mathcal{O}(i)$ denote the full subcategory of $\mathcal{O}_{X/}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{\langle n \rangle/}} \{\rho^i\}$, so that we have a left fibration of simplicial sets $\mathcal{O}(i) \to \mathcal{O}$ and a categorical equivalence $\mathcal{E}(i)_{X/}'' \simeq \mathcal{O}(i) \times_{\mathcal{O}^{\otimes}} \mathcal{E}$. Choose inert morphisms $X \to X_i$ in $\mathcal{O}^{\otimes}$ for $1 \le i \le n$, so that each X_i determines an initial object of $\mathcal{O}(i)$. If f induces a categorical equivalence $\mathcal{E} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \{\langle 1 \rangle\} \to \mathcal{O}$, then we can write $X_i \simeq f(C_i)$ for some $C_i \in \mathcal{E}$,

and that the induced map $X \to C_i$ can be identified with a final object of $\mathcal{E}(i)''_{X/}$. Consequently, we are reduced to proving the existence of a product for the set of objects $\{F_0(C_i)\}_{1 \leq i \leq n}$, which follows from our assumption that $\mathcal{C}$ admits finite products. This completes the proof of (a). Moreover, it yields the following version of (b):

(b') Let $F \in \mathrm{Fun}(\mathcal{M}, \mathcal{C})$ be such that $F_0 = F | \mathcal{E} \in \mathrm{Mon}_{\mathcal{E}}(\mathcal{C})$. Then F is a right Kan extension of F_0 if and only if, for every object $X \in \mathcal{O}^{\otimes}_{\langle n \rangle}$, if we choose $C_i \in \mathcal{E}$ and maps $\alpha_i : X \to C_i$ in $\mathcal{M}$ corresponding to inert morphisms $X \to f(C_i)$ in $\mathcal{O}^{\otimes}$ covering $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \leq i \leq n$, then the inducedmap $F(X) \to \prod_{1 \leq i \leq n} F_0(C_i)$ is an equivalence in $\mathcal{C}$.

We now prove (b). Assume first that $F \in \mathcal{X}$. Then $F_0 = F | \mathcal{E}$ is equivalent to the functor $(F | \mathcal{O}^{\otimes}) \circ f$. It follows immediately that $F_0 \in \mathrm{Mon}_{\mathcal{E}}(\mathcal{C})$. Criterion (b') immediately implies that F is a right Kan extension of F_0. This proves the "only if" direction.

For the converse, assume that $F_0 \in \mathrm{Mon}_{\mathcal{E}}(\mathcal{C})$ and that F is a right Kan extension of F_0. We wish to prove that $F \in \mathcal{X}$. We first verify that F satisfies (ii). Pick an object $C \in \mathcal{E}$ and choose locally $(p \circ f)$-coCartesian morphisms $\alpha_i : C \to C_i$ for $1 \leq i \leq n$. Let $X = f(C)$; we wish to show that the induced map $F(X) \to F(C)$ is an equivalence in $\mathcal{C}$. Since $F_0 \in \mathrm{Mon}_{\mathcal{E}}(\mathcal{C})$, we can identify $F(C)$ with the product $\prod_{1 \leq i \leq n} F(C_i)$, so that the desired assertion follows immediately from (b').

To complete the proof, we must show that $F | \mathcal{O}^{\otimes}$ belongs to $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$. Let $X \in \mathcal{O}^{\otimes}_{\langle n \rangle}$ and choose inert morphisms $\alpha_i : X \to X_i$ covering $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \leq i \leq n$; we wish to show that the induced map $F(X) \to \prod_{1 \leq i \leq n} F(C_i)$ is an equivalence. We may assume without loss of generality that $X_i = f(C_i)$ for some objects $\{C_i \in \mathcal{E}\}_{1 \leq i \leq n}$. Condition (ii) implies that F induces an equivalence $F(X_i) \to F(C_i)$ for $1 \leq i \leq n$. We are therefore reduced to showing that the map $F(X) \to \prod_{1 \leq i \leq n} F(C_i)$ is an equivalence in $\mathcal{C}$, which follows from criterion (b'). $\square$

Combining Proposition 4.1.2.6 with Proposition 2.4.2.5, we conclude that if $\mathcal{C}$ is an ∞-category equipped with a Cartesian symmetric monoidal structure, then there is a canonical equivalence $\mathrm{Alg}_{\mathcal{A}ss}(\mathcal{C}) \to \mathrm{Mon}(\mathcal{C})$. Using Theorem 2.3.3.23, we can obtain an analogous statement for an arbitrary monoidal ∞-category $\mathcal{C}$. To formulate it, we need a bit of notation.

Definition 4.1.2.13. We will say that a morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$ is *inert* if the induced map $\mathrm{Cut}([n]) \to \mathrm{Cut}([m])$ is an inert morphism in $\mathcal{A}ss^{\otimes}$. More directly, $\alpha : [m] \to [n]$ is inert if it induces an isomorphism from $[m]$ onto a convex subset $\{i, i+1, \ldots, j-1, j\} \subseteq [n]$.

Definition 4.1.2.14. Let $\mathcal{C}^{\otimes} \to \mathcal{A}ss^{\otimes}$ be a planar ∞-operad. We let $^{\Delta}\mathrm{Alg}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{A}ss^{\otimes}}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{C}^{\otimes})$ spanned by those functors $F : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{C}^{\otimes}$ which carry inert morphisms of $\mathrm{N}(\mathbf{\Delta})^{op}$ to inert morphisms of $\mathcal{C}^{\otimes}$.

The following result is an immediate consequence of Theorem 2.3.3.23 and Proposition 4.1.2.10:

Proposition 4.1.2.15. *Let $\mathcal{C}^{\otimes}$ be a planar ∞-operad. Then composition with the functor $\mathrm{Cut} : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{A}ss^{\otimes}$ of Construction 4.1.2.5 induces an equivalence of ∞-categories $\mathrm{Alg}(\mathcal{C}) \to {}^{\Delta}\mathrm{Alg}(\mathcal{C})$.*

Remark 4.1.2.16. We will generally abuse terminology by referring to objects of $^{\Delta}\mathrm{Alg}(\mathcal{C})$ as *associative algebra objects of* $\mathcal{C}$. This abuse is justified by Proposition 4.1.2.15.

4.1.3   Monoidal Model Categories

Let $\mathbf{A}$ be a monoidal category which is equipped with a model structure. Recall that $\mathbf{A}$ is said to be a *monoidal model category* if the unit object $\mathbf{1} \in \mathbf{A}$ is cofibrant, the monoidal structure on $\mathbf{A}$ is closed, and the tensor product functor $\otimes : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ is a left Quillen bifunctor. (For a more leisurely introduction to the theory of monoidal model categories, we refer the reader to [72] or to the appendix of [97]). Our goal in this section is to extract from $\mathbf{A}$ a monoidal ∞-category $\mathcal{C}^{\otimes}$. In the case where $\mathbf{A}$ is simplicial, the

∞-category $\mathcal{C}$ can be described as the homotopy coherent nerve $N(\mathbf{A}^\circ)$, where $\mathbf{A}^\circ$ is the collection of fibrant-cofibrant objects of $\mathbf{A}$. However, the assumption that $\mathbf{A}$ is simplicial is inconvenient for many applications (for example, in the case where $\mathbf{A}$ is the category of chain complexes of vector spaces over some field k). In the general case, we can describe $\mathcal{C}$ instead as the the ∞-category obtained from $\mathbf{A}$ by formally inverting the collection of weak equivalences. We begin with a few general remarks about this procedure.

Construction 4.1.3.1. Let $\mathcal{C}$ be an ∞-category. We define a *system* on $\mathcal{C}$ to be a collection of morphisms $W \subseteq \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^1, \mathcal{C})$ which is stable under homotopy, composition, and contains all equivalences. In other words, a system on $\mathcal{C}$ is a subcategory of the homotopy category $h\mathcal{C}$, which contains all objects and isomorphisms in $h\mathcal{C}$. The collection of systems on $\mathcal{C}$ forms a partially ordered set $\mathrm{Sys}(\mathcal{C})$. We regard the construction $\mathcal{C} \mapsto N(\mathrm{Sys}(\mathcal{C}))$ as a functor $\mathcal{C}\mathrm{at}_\infty^{op} \to \mathcal{C}\mathrm{at}_\infty$, which classifies a Cartesian fibration $q : \mathcal{W}\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$. Unwinding the definitions, we can identify the objects of $\mathcal{W}\mathcal{C}\mathrm{at}_\infty$ with pairs $(\mathcal{C}, W)$, where $\mathcal{C}$ is an ∞-category and W is a system on $\mathcal{C}$. Unwinding the definitions, we see that for objects $(\mathcal{C}, W), (\mathcal{C}', W') \in \mathcal{W}\mathcal{C}\mathrm{at}_\infty$, the mapping space

$$\mathrm{Map}_{\mathcal{W}\mathcal{C}\mathrm{at}_\infty}((\mathcal{C}, W), (\mathcal{C}', W'))$$

can be described as the summand of $\mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{C}, \mathcal{C}')$ spanned by those functors $f : \mathcal{C} \to \mathcal{C}'$ such that $f(W) \subseteq W'$.

The Cartesian fibration q admits a section G, given by the formula $G(\mathcal{C}) = (\mathcal{C}, W)$ where W is the collection of all equivalences in $\mathcal{C}$.

Proposition 4.1.3.2. *Let $G : \mathcal{C}\mathrm{at}_\infty \to \mathcal{W}\mathcal{C}\mathrm{at}_\infty$ be the functor described in Construction 4.1.3.1. Then G admits a left adjoint. Moreover, this left adjoint commutes with finite products.*

Notation 4.1.3.3. In the situation of Proposition 4.1.3.2, we will denote the left adjoint to G by $(\mathcal{C}, W) \mapsto \mathcal{C}[W^{-1}]$.

Proof. Let $(\mathcal{C}, W)$ be an object of $\mathcal{W}\mathcal{C}\mathrm{at}_\infty$, and regard $(\mathcal{C}, W)$ as an object of the category Set_Δ^+ of marked simplicial sets. Choose a marked equivalence $i : (\mathcal{C}, W) \to (\mathcal{C}', W')$, where $(\mathcal{C}', W')$ is a fibrant object of Set_Δ^+. Then $(\mathcal{C}', W') = G(\mathcal{C}')$. It follows immediately from the definitions that for any ∞-category $\mathcal{D}$, composition with i induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}(\mathcal{C}', \mathcal{D}) \to \mathrm{Map}_{\mathcal{W}\mathcal{C}\mathrm{at}_\infty}((\mathcal{C}, W), G(\mathcal{D}))$. The assertion regarding products follows from Proposition T.3.1.4.2. $\square$

In what follows, let us regard $\mathcal{W}\mathcal{C}\mathrm{at}_\infty$ and $\mathcal{C}\mathrm{at}_\infty$ as endowed with the Cartesian symmetric monoidal structures. For any ∞-operad $\mathcal{O}$, composition with the functor G induces a fully faithful embedding $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) \simeq \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty) \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{W}\mathcal{C}\mathrm{at}_\infty) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{W}\mathcal{C}\mathrm{at}_\infty)$. It follows from Propositions 4.1.3.2 2.2.1.9 that this fully faithful embedding admits a left adjoint. Combining this observation with Example 2.4.2.4, we obtain the following result:

Proposition 4.1.3.4. *Let $\mathcal{C}^\otimes$ be a (symmetric) monoidal ∞-category, and let W be a collection of morphisms in $\mathcal{C}$. Assume that for every object $C \in \mathcal{C}$ and every morphism $f : D \to D'$ in W, the induced maps*

$$C \otimes D \to C \otimes D' \qquad D \otimes C \to D' \otimes C$$

also belong to W, so that $(\mathcal{C}, W)$ can be regarded as a (commutative) monoid object of $\mathcal{W}\mathcal{C}\mathrm{at}_\infty$.

(1) *There exists a (symmetric) monoidal functor $F : \mathcal{C}^\otimes \to \mathcal{C}'^\otimes$ with the following universal property: for every (symmetric) monoidal ∞-category $\mathcal{D}^\otimes$, composition with F induces a fully faithful embedding from the ∞-category of (symmetric) monoidal functors from $\mathcal{C}'^\otimes$ to $\mathcal{D}^\otimes$ to the ∞-category of (symmetric) monoidal functors from $\mathcal{C}^\otimes$ to $\mathcal{D}^\otimes$. Moreover, the essential image of this embedding consists of those symmetric monoidal functors $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ which carry each morphism in W to an equivalence in $\mathcal{D}$.*

(2) *The underlying ∞-category of $\mathcal{C}'^\otimes$ can be identified with the ∞-category $\mathcal{C}[W^{-1}]$ of Notation 4.1.3.3.*

Remark 4.1.3.5. In the situation of Proposition 4.1.3.4, the (symmetric) monoidal functor $F : \mathcal{C}^{\otimes} \to \mathcal{C}'^{\otimes}$ is characterized up to equivalence by property (2). Indeed, suppose we are given another symmetric monoidal functor $G : \mathcal{C}^{\otimes} \to \mathcal{C}''^{\otimes}$ satisfying (2). Using (1), we conclude that G is equivalent to a composition

$$\mathcal{C}^{\otimes} \xrightarrow{F} \mathcal{C}'^{\otimes} \xrightarrow{G'} \mathcal{C}''^{\otimes}.$$

Then G' induces an equivalence $\mathcal{C}' \simeq \mathcal{C}''$ (both are equivalent to $\mathcal{C}[W^{-1}]$), so that G' is an equivalence of (symmetric) monoidal ∞-categories by Remark 2.1.3.8.

Example 4.1.3.6. Let $\mathbf{A}$ be a (symmetric) monoidal model category. Then the full subcategory $\mathbf{A}^c \subseteq \mathbf{A}$ inherits a (symmetric) monoidal structure. Moreover, the collection of weak equivalences in $\mathbf{A}^c$ is stable under (left and right) tensor product by objects of $\mathbf{A}^c$. It follows that the underlying ∞-category $\mathbf{A}^c[W^{-1}]$ of $\mathbf{A}$ inherits a (symmetric) monoidal structure. We will refer to the (symmetric) monoidal ∞-category $\mathbf{A}^c[W^{-1}]$ as the *underlying (symmetric) monoidal ∞-category of $\mathbf{A}$.*

Our final goal in this section is to give an explicit description of the monoidal structure on the underlying ∞-category of a monoidal model category $\mathbf{A}$, in the special case where $\mathbf{A}$ is equipped with a simplicial structure.

Definition 4.1.3.7. Let $\mathcal{C}$ be simplicial category. We will say that a monoidal structure on $\mathcal{C}$ is *weakly compatible* with the simplicial structure on $\mathcal{C}$ provided that the operation $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is endowed with the structure of a simplicial functor, which is compatible with associativity and unit constraints of $\mathcal{C}$. We will say that a symmetric monoidal structure on $\mathcal{C}$ is *weakly compatible* with the simplicial structure on $\mathcal{C}$ if the underlying monoidal category is weakly compatible with the simplicial structure on $\mathcal{C}$ and, in addition, the symmetry constraint $\eta_{X,Y} : X \otimes Y \to Y \otimes X$ is a natural transformation of simplicial functors.

Suppose furthermore that the monoidal structure on $\mathcal{C}$ is *closed*: that is, for every pair objects $X, Y \in \mathcal{C}$, there exist an exponential objects $X^Y, {}^Y X \in \mathcal{C}$ and evaluation map

$$e : X^Y \otimes Y \to X \qquad e' : Y \otimes {}^Y X \to X$$

which induce bijections

$$\mathrm{Hom}_{\mathcal{C}}(Z, X^Y) \to \mathrm{Hom}_{\mathcal{C}}(Z \otimes Y \to X) \qquad \mathrm{Hom}_{\mathcal{C}}(Z, {}^Y X) \to \mathrm{Hom}_{\mathcal{C}}(Y \otimes Z, X)$$

for every object $Z \in \mathcal{C}$. We will say that a (symmetric) monoidal structure on $\mathcal{C}$ is *compatible* with a simplicial structure on $\mathcal{C}$ if it is weakly compatible, and the maps e and e' induce isomorphisms of simplicial sets

$$\mathrm{Map}_{\mathcal{C}}(Z, X^Y) \to \mathrm{Map}_{\mathcal{C}}(Z \otimes Y, X) \qquad \mathrm{Map}_{\mathcal{C}}(Z, {}^Y X) \to \mathrm{Map}_{\mathcal{C}}(Y \otimes Z, X)$$

for every $Z \in \mathcal{C}$.

Definition 4.1.3.8. A *simplicial (symmetric) monoidal model category* is a (symmetric) monoidal model category which is also equipped with the structure of a simplicial model category, where the simplicial structure and the (symmetric) monoidal structure are compatible in the sense of Definition 4.1.3.7.

Remark 4.1.3.9. In the symmetric case, one can reformulate Definition 4.1.3.8 as follows. Let $\mathbf{A}$ be a symmetric monoidal model category. Then a compatible simplicial structure on $\mathbf{A}$ can be identified with symmetric monoidal left Quillen functor $\psi : \mathcal{S}\mathrm{et}_{\Delta} \to \mathbf{A}$. Given such a functor, $\mathbf{A}$ inherits the structure of a simplicial category, where the mapping spaces are characterized by the existence of a natural bijection

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta}}(K, \mathrm{Map}_{\mathbf{A}}(A, B)) \simeq \mathrm{Hom}_{\mathbf{A}}(\psi(K) \otimes A, B).$$

Proposition 4.1.3.10. *Let $\mathbf{A}$ be a simplicial symmetric monoidal model category, and let $\mathbf{A}^o$ denote the full subcategory spanned by the fibrant-cofibrant objects. We regard $\mathbf{A}^o$ as a simplicial colored operad via the formula*

$$\mathrm{Mul}_{\mathbf{A}^o}(\{X_i\}, Y) = \mathrm{Map}_{\mathbf{A}}(\bigotimes_i X_i, Y).$$

Then the operadic nerve $\mathrm{N}^{\otimes}(\mathbf{A}^o)$ is a symmetric monoidal ∞-category.

Remark 4.1.3.11. In the situation of Proposition 4.1.3.10, the simplicial colored operad $\mathbf{A}^\circ$ is fibrant: that is, each of the mapping spaces $\mathrm{Mul}_{\mathbf{A}^\circ}(\{X_i\}, Y) = \mathrm{Map}_{\mathbf{A}}(\bigotimes_i X_i, Y)$ is a Kan complex. This follows from the fact that Y is fibrant and $\bigotimes_i X_i$ is cofibrant (being a tensor product of cofibrant objects of $\mathbf{A}$). Invoking Proposition 2.1.1.27, we deduce that $p : \mathrm{N}^\otimes(\mathbf{A}^\circ) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is an ∞-operad.

Before giving the proof of Proposition 4.1.3.10, it will be convenient to formulate a stronger assertion which implies not only that $\mathrm{N}^\otimes(\mathbf{A}^\circ)$ is symmetric monoidal, but that it can be identified with the underlying symmetric monoidal ∞-category of $\mathbf{A}$ (in the sense of Example 4.1.3.6). To state this result, we need to introduce an elaboration on Construction 1.3.4.18.

Construction 4.1.3.12. Let $\mathbf{A}$ be a simplicial symmetric monoidal model category. We define a simplicial category $\mathcal{M}^\otimes$ as follows:

(1) An object of $\mathcal{M}^\otimes$ is a sequence $(i, A_1, \ldots, A_n)$, where $(A_1, \ldots, A_n)$ is a finite sequence of cofibrant objects of $\mathbf{A}$ and $i \in \{0, 1\}$. We further assume that if $i = 1$, then each A_j is a fibrant object of $\mathbf{A}$.

(2) Suppose we are given objects $(i, A_1, \ldots, A_n)$ and $(i', A_1', \ldots, A_{n'}')$ in $\mathcal{M}^\otimes$. Then

$$\mathrm{Map}_{\mathcal{M}}((i, A_1, \ldots, A_n), (i', A_1', \ldots, A_{n'}')) = \begin{cases} \coprod_\alpha \prod_{1 \leq j' \leq n'} \mathrm{Map}_{\mathbf{A}}(\bigotimes_{\alpha(j)=j'} A_j, A_{j'}') & \text{if } i' = 1 \\ \coprod_\alpha \prod_{1 \leq j' \leq n'} \mathrm{Hom}_{\mathbf{A}}(\bigotimes_{\alpha(j)=j'} A_j, A_{j'}') & \text{if } i = i' = 0 \\ \emptyset & \text{if } i' < i. \end{cases}$$

Here the disjoint unions are taken over all morphisms $\alpha : \langle n \rangle \to \langle n' \rangle$ in the category $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

Remark 4.1.3.13. In the situation of Construction 4.1.3.12, the fiber $\mathcal{M}^\otimes \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \{\langle 1 \rangle\}$ can be identified with the simplicial category $\mathcal{M}$ introduced in Construction 1.3.4.18.

Remark 4.1.3.14. Let $\mathbf{A}$ be a simplicial symmetric monoidal model category, and let $\mathcal{M}^\otimes$ be as in Construction 4.1.3.12. The construction $(i, A_1, \ldots, A_n) \mapsto (i, \langle n \rangle)$ determines a forgetful functor $\mathrm{N}(\mathcal{M}) \to \Delta^1 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. We have canonical isomorphisms

$$\mathrm{N}(\mathcal{M}) \times_{\Delta^1} \{0\} \simeq \mathrm{N}^\otimes(\mathbf{A}^c) \qquad \mathrm{N}(\mathcal{M}) \times_{\Delta^1} \{1\} \simeq \mathrm{N}^\otimes(\mathbf{A}^\circ).$$

Proposition 4.1.3.15. *Let $\mathbf{A}$ be a simplicial symmetric monoidal model category, and let $\mathcal{M}^\otimes$ be defined as in Construction 4.1.3.12. Then the induced map $p : \mathrm{N}(\mathcal{M}^\otimes) \to \Delta^1 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a coCartesian fibration.*

Proposition 4.1.3.15 immediately implies Proposition 4.1.3.10 (by passing to the fiber over $\{1\} \in \Delta^1$ and invoking Remark 4.1.3.14). Moreover, it shows that $\mathrm{N}(\mathcal{M}^\otimes)$ is the correspondence associated to a symmetric monoidal functor $\mathrm{N}^\otimes(\mathbf{A}^c) \to \mathrm{N}^\otimes(\mathbf{A}^\circ)$. Combining this observation with Remark 4.1.3.5 and Theorem 1.3.4.20, we obtain the following:

Corollary 4.1.3.16. *Let $\mathbf{A}$ be a simplicial symmetric monoidal model category. Then the symmetric monoidal functor $\mathrm{N}^\otimes(\mathbf{A}^c) \to \mathrm{N}^\otimes(\mathbf{A}^\circ)$ constructed above exhibits $\mathrm{N}^\otimes(\mathbf{A}^\circ)$ as the underlying symmetric monoidal ∞-category of $\mathbf{A}$, in the sense of Example 4.1.3.6.*

Proof of Proposition 4.1.3.15. Let $(i, C_1, \ldots, C_n)$ be an object of $\mathcal{M}^\otimes$, and let $\alpha : (i, \langle n \rangle) \to (i', \langle n' \rangle)$ be a morphism in $\Delta^1 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Fix $j' \in \langle n' \rangle^\circ$. If $i = i' = 0$, let $\eta_{j'}$ be the identity map from $D_{j'} = \bigotimes_{\alpha(j)=j'} C_j$ to itself. Otherwise, choose a trivial cofibration

$$\eta_{j'} : \bigotimes_{\alpha(j)=j'} C_j \to D_{j'},$$

where $D_{j'} \in \mathbf{A}$ is fibrant. Together these determine a map $\overline{\alpha} : (i, C_1, \ldots, C_n) \to (i', D_1, \ldots, D_{n'})$ in $\mathcal{M}^\otimes$ lying over α. We claim that $\overline{\alpha}$ is p-coCartesian. To prove this, we must show that for every morphism

$\beta : \langle n' \rangle \to \langle n'' \rangle$ in $\mathcal{A}\mathrm{ss}^{\otimes}$, every $i' \le i'' \le 1$, and every object $(i'', E_1, \ldots, E_{n''}) \in \mathcal{M}^{\otimes}$, the induced map

$$\mathrm{Map}_{\mathcal{M}^{\otimes}}((i', D_1, \ldots, D_{n'}), (i'', E_1, \ldots, E_{n''})) \times_{\mathrm{Hom}_{\mathcal{F}\mathrm{in}_*}(\langle n' \rangle, \langle n'' \rangle)} \{\beta\}$$

$$\downarrow$$

$$\mathrm{Map}_{\mathcal{M}^{\otimes}}((i, C_1, \ldots, C_n), (i'', E_1, \ldots, E_{n''}))$$

determines a homotopy equivalence onto the summand of $\mathrm{Map}_{\mathcal{M}^{\otimes}}((i, C_1, \ldots, C_n), (i'', E_1, \ldots, E_{n''}))$ spanned by those morphisms which cover the map $\beta \circ \alpha : \langle n \rangle \to \langle n'' \rangle$ in $\mathcal{F}\mathrm{in}_*$. If $i = i' = 0$, this follows immediately from the definitions. Otherwise, we have $i' = i'' = 1$; it will therefore suffice to prove that for $j'' \in \langle n'' \rangle^{\circ}$, the induced map

$$\mathrm{Map}_{\mathbf{A}}(\bigotimes_{\beta(j')=j''} D_{j'}, E_{j''}) \to \mathrm{Map}_{\mathbf{A}}(\bigotimes_{(\beta \circ \alpha)(j)=j''} C_j, E_{j''})$$

is a homotopy equivalence of Kan complexes. Since each $E_{j''}$ is a fibrant object of $\mathbf{A}$, it will suffice to show that the map

$$\eta : \bigotimes_{(\beta \circ \alpha)(j)=j''} C_i \to \bigotimes_{\beta(j')=j''} D_j'$$

is a weak equivalence of cofibrant in $\mathbf{A}$. This follows from the observation that η can be identified with the tensor product of the maps $\eta_{j'}$, each of which is a weak equivalence between cofibrant objects of $\mathbf{A}$. $\square$

Variant 4.1.3.17. Suppose that $\mathbf{A}$ is a simplicial monoidal model category. Let $\mathbf{A}^{\circ}$ denote the full subcategory of $\mathbf{A}$ spanned by the fibrant-cofibrant objects. We regard $\mathbf{A}^{\circ}$ as a simplicial colored operad as follows: given a finite set I of cardinality n, we set

$$\mathrm{Mul}_{\mathbf{A}^{\circ}}^{\mathcal{A}\mathrm{ss}}(\{X_i\}_{i \in I}, Y) = \coprod_{\alpha} \mathrm{Map}_{\mathbf{A}}(\otimes_{1 \le j \le n} X_{\alpha(j)}, Y),$$

where the coproduct is taken over all bijective maps $\alpha : \{1, \ldots, n\} \to I$ (equivalently, over all linear orderings of I). We let $\mathrm{N}_{\mathcal{A}\mathrm{ss}}^{\otimes}(\mathbf{A}^{\circ})$ denote the operadic nerve of this simplicial colored operad. The proof of Proposition 4.1.3.15 yields the following:

(1) The map $\mathrm{N}_{\mathcal{A}\mathrm{ss}}^{\otimes}(\mathbf{A}^{\circ}) \to \mathcal{A}\mathrm{ss}^{\otimes}$ exhibits $\mathrm{N}_{\mathcal{A}\mathrm{ss}}^{\otimes}(\mathbf{A}^{\circ})$ as a monoidal ∞-category.

(2) Let $\mathrm{N}_{\mathcal{A}\mathrm{ss}}^{\otimes}(\mathbf{A}^c)$ denote the monoidal ∞-category associated to the discrete monoidal category $\mathbf{A}^c$. Then there exists a monoidal functor $\theta : \mathrm{N}_{\mathcal{A}\mathrm{ss}}^{\otimes}(\mathbf{A}^c) \to \mathrm{N}_{\mathcal{A}\mathrm{ss}}^{\otimes}(\mathbf{A}^{\circ})$ which induces an equivalence of ∞-categories $\mathrm{N}(\mathbf{A}^c)[W^{-1}] \simeq \mathrm{N}(\mathbf{A}^{\circ})$, where W is the collection of weak equivalences in $\mathbf{A}^c$ (in other words, the monoidal functor θ exhibits $\mathrm{N}(\mathbf{A}^{\circ})$ as the underlying ∞-category of $\mathbf{A}$).

We now describe an application of Variant 4.1.3.17:

Example 4.1.3.18. Let $\mathcal{P}\mathrm{Op}_{\infty}$ denote the category of ∞-preoperads, endowed with the monoidal model structure of Proposition 2.2.5.7. It is easy to see that this monoidal structure is compatible with the simplicial structure on $\mathcal{P}\mathrm{Op}_{\infty}$ (in the sense of Definition 4.1.3.8), so that $\mathcal{P}\mathrm{Op}_{\infty}$ is a simplicial monoidal model category. Applying Variant 4.1.3.17, we obtain a coCartesian fibration of ∞-operads $\mathrm{N}_{\mathcal{A}\mathrm{ss}}^{\otimes}(\mathcal{P}\mathrm{Op}_{\infty}{}^{\circ}) \to \mathcal{A}\mathrm{ss}^{\otimes}$. There is an obvious map from the simplicial colored operad $\mathcal{P}\mathrm{Op}_{\infty}{}^{\circ}$ of Construction 2.2.5.12 to the simplicial colored operad $\mathrm{Op}_{\infty}^{\Delta}$ of Construction 2.2.5.12, which determines a commutative diagram

$$\begin{array}{ccc} \mathrm{N}_{\mathcal{A}\mathrm{ss}}^{\otimes}(\mathcal{P}\mathrm{Op}_{\infty}^{\circ}) & \longrightarrow & \mathcal{A}\mathrm{ss}^{\otimes} \\ \downarrow & & \downarrow \\ \mathrm{Op}(\infty)^{\otimes} & \longrightarrow & \mathrm{N}(\mathcal{F}\mathrm{in}_*). \end{array}$$

In fact, this is a homotopy pullback diagram: that is, it induces an equivalence of ∞-operads $\mathrm{N}^{\otimes}_{\mathcal{A}\mathrm{ss}}(\mathcal{P}\mathrm{Op}_{\infty}{}^{\mathrm{o}}) \to \mathcal{A}\mathrm{ss}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{Op}(\infty)^{\otimes}$. To see this, it suffices to observe that for any sequence of ∞-operads $\{\mathcal{O}^{\otimes}_i\}_{1 \leq i \leq n}$ and any other ∞-operad $\mathcal{O}'^{\otimes}$, the canonical map

$$\mathrm{Mul}^{F}_{\mathrm{Op}^{\Delta}_{\infty}}(\{\mathcal{O}^{\otimes}_i\}, \mathcal{O}'^{\otimes}) \to \mathrm{Mul}_{\mathrm{Op}^{\Delta}_{\infty}}(\{\mathcal{O}^{\otimes}_i\}, \mathcal{O}'^{\otimes})$$

is a homotopy equivalence. In fact, this map is the inclusion of a fiber of the Kan fibration

$$\mathrm{Mul}_{\mathrm{Op}^{\Delta}_{\infty}}(\{\mathcal{O}_i\}, \mathcal{O}'^{\otimes}) \to \mathrm{N}(\mathfrak{S}(\{1, \ldots, n\})),$$

and the base $\mathrm{N}(\mathfrak{S}(\{1, \ldots, n\}))$ is a contractible Kan complex.

Remark 4.1.3.19. The underlying ∞-category of the ∞-operad $\mathrm{N}^{\otimes}_{\mathcal{A}\mathrm{ss}}(\mathcal{P}\mathrm{Op}_{\infty}{}^{\mathrm{o}})$ can be identified with $\mathrm{N}(\mathcal{P}\mathrm{Op}_{\infty}{}^{\mathrm{o}}) \simeq \mathrm{Op}_{\infty}$.

4.1.4 Rectification of Associative Algebras

Let $\mathbf{A}$ be a monoidal model category. In §4.1.3, we saw that the underlying ∞-category $\mathrm{N}(\mathbf{A}^c)[W^{-1}]$ of $\mathbf{A}$ admits a monoidal structure, so it makes sense to consider algebra objects $A \in \mathrm{Alg}(\mathrm{N}(\mathbf{A}^c)[W^{-1}])$. In this case, we can think of A as an object of $\mathbf{A}$ equipped with a multiplication $m : A \otimes A \to A$ which is unital and associative up to coherent homotopy.

Our goal in this section is to prove a rectification result which asserts (under some mild hypotheses on $\mathbf{A}$) that A is equivalent to a strictly unital and associative algebra in the ordinary category $\mathbf{A}$. More precisely, we will show that $\mathrm{Alg}(\mathrm{N}(\mathbf{A}^c)[W-1])$ is equivalent to the underlying ∞-category of a simplicial model category $\mathrm{Alg}(\mathbf{A})$ whose objects are associative algebras in $\mathbf{A}$. Our first step is to describe a model structure on the ordinary category $\mathrm{Alg}(\mathbf{A})$. Here we adopt the convention that if $\mathcal{C}$ is any monoidal category, then $\mathrm{Alg}(\mathcal{C})$ denotes the category of associative algebra objects of $\mathcal{C}$. Note that there is a canonical equivalence $\mathrm{N}(\mathrm{Alg}(\mathcal{C})) \simeq \mathrm{Alg}(\mathrm{N}(\mathcal{C}))$.

In the arguments which follow, we will need to invoke the following hypothesis (formulated originally by Schwede and Shipley; see [127]):

Definition 4.1.4.1 (Monoid Axiom)**.** Let $\mathbf{A}$ be a combinatorial symmetric monoidal model category. Let U be the collection of all morphisms of $\mathbf{A}$ having the form

$$X \otimes Y \overset{\mathrm{id}_X \otimes f}{\longrightarrow} X \otimes Y',$$

where f is a trivial cofibration, and let $\overline{U}$ denote the weakly saturated class of morphisms generated by U (Definition T.A.1.2.2). We will say that $\mathbf{A}$ *satsifies the monoid axiom* if every morphism of $\overline{U}$ is a weak equivalence in $\mathbf{A}$.

Remark 4.1.4.2. Let $\mathbf{A}$ be a combinatorial symmetric monoidal model category in which every object is cofibrant, and let U and $\overline{U}$ be as in Definition 4.1.4.1. Then every morphism belonging to U is a trivial cofibration. Since the collection of trivial cofibrations in $\mathbf{A}$ is weakly saturated, we conclude that $\mathbf{A}$ satisfies the monoid axiom.

We will need the following result of [127]:

Proposition 4.1.4.3. *[Schwede-Shipley] Let $\mathbf{A}$ be a combinatorial monoidal model category. Assume that either every object of $\mathbf{A}$ is cofibrant, or that $\mathbf{A}$ is a symmetric monoidal model category which satisfies the monoid axiom. Then:*

(1) *The category $\mathrm{Alg}(\mathbf{A})$ admits a combinatorial model structure, where:*

    (W) *A morphism $f : A \to B$ of algebra objects of $\mathbf{A}$ is a weak equivalence if it is a weak equivalence when regarded as a morphism in $\mathbf{A}$.*

(F) *A morphism $f : A \to B$ of algebra objects of $\mathbf{A}$ is a fibration if it is a fibration when regarded as a morphism in $\mathbf{A}$.*

(2) *The forgetful functor $\theta : \mathrm{Alg}(\mathbf{A}) \to \mathbf{A}$ is a right Quillen functor.*

(3) *If $\mathbf{A}$ is equipped with a compatible simplicial structure, then $\mathrm{Alg}(\mathbf{A})$ inherits the structure of a simplicial model category.*

(4) *Suppose that every object of $\mathbf{A}$ is cofibrant and that the collection of weak equivalences in $\mathbf{A}$ is stable under filtered colimits. Then $\mathrm{Alg}(\mathbf{A})$ is left proper.*

Assuming Proposition 4.1.4.3 for the moment, we can now state our main result:

Theorem 4.1.4.4. *Let $\mathbf{A}$ be a combinatorial monoidal model category. Assume either:*

(A) *Every object of $\mathbf{A}$ is cofibrant.*

(B) *The model category $\mathbf{A}$ is left proper, the class of cofibrations in $\mathbf{A}$ is generated by cofibrations between cofibrant objects, the monoidal structure on $\mathbf{A}$ is symmetric, and $\mathbf{A}$ satisfies the monoid axiom.*

Let W be the collection of weak equivalences in $\mathbf{A}^c$, and W' the collection of weak equivalences in $\mathrm{Alg}(\mathbf{A})^c$. Then the canonical map

$$\mathrm{N}(\mathrm{Alg}(\mathbf{A})^c)[W'^{-1}] \to \mathrm{Alg}(\mathrm{N}(\mathbf{A}^c)[W^{-1}])$$

is an equivalence of ∞-categories.

Remark 4.1.4.5. In the situation of Proposition 4.1.4.3, assume that $\mathbf{A}$ is a simplicial monoidal model category (in the sense of Definition 4.1.3.8). Then $\mathrm{Alg}(\mathbf{A})$ inherits the structure of a simplicial model category. The simplicial structure on $\mathrm{Alg}(\mathbf{A})$ can be described as follows: for any pair of algebras $A, B \in \mathrm{Alg}(\mathbf{A})$, the mapping space $\mathrm{Map}_{\mathrm{Alg}(\mathbf{A})}(A, B)$ is the simplicial subset of $\mathrm{Map}_{\mathbf{A}}(A, B)$ characterized by the following property: a map of simplicial sets $K \to \mathrm{Map}_{\mathbf{A}}(A, B)$ factors through $\mathrm{Map}_{\mathrm{Alg}(\mathbf{A})}(A, B)$ if and only if the diagrams

$$
\begin{array}{ccc}
\Delta^0 \times K & \longrightarrow & \mathrm{Map}_{\mathbf{A}}(1, A) \times \mathrm{Map}_{\mathcal{C}}(A, B) \\
\downarrow & & \downarrow \\
\Delta^0 & \longrightarrow & \mathrm{Map}_{\mathbf{A}}(1, B)
\end{array}
$$

$$
\begin{array}{ccccc}
K \longrightarrow K \times K & \longrightarrow & \mathrm{Map}_{\mathbf{A}}(A, B) \times \mathrm{Map}_{\mathbf{A}}(A, B) & \longrightarrow & \mathrm{Map}_{\mathbf{A}}(A \otimes A, B \otimes B) \\
\searrow & & & & \downarrow \\
\Delta^0 \times K & \longrightarrow & \mathrm{Map}_{\mathbf{A}}(A \otimes A, A) \times \mathrm{Map}_{\mathbf{A}}(A, B) & \longrightarrow & \mathrm{Map}_{\mathbf{A}}(A \otimes A, B)
\end{array}
$$

commute.

Example 4.1.4.6. Let $\mathbf{A}$ be the category of *symmetric spectra*, as defined in [73]. Then $\mathbf{A}$ admits several model structures which satisfy assumption (B) of Theorem 4.1.4.4. It follows that $\mathrm{N}(\mathbf{A}^\circ)$ is equivalent, as a monoidal ∞-category, to the ∞-category Sp of spectra (endowed with the smash product monoidal structure; see §6.3.2). Using Theorem 4.1.4.4, we deduce that the ∞-category $\mathrm{Alg}(\mathrm{Sp})$ of associative algebra objects of Sp is equivalent to the underlying ∞-category of $\mathrm{Alg}(\mathbf{A})$.

Example 4.1.4.7. [∞-Categorical MacLane Coherence Theorem] Let $\mathbf{A}$ be the category of marked simplicial sets (see §T.3.1). Then $\mathbf{A}$ is a simplicial model category, which satisfies the hypotheses of Example 2.4.1.10. Using Theorem 1.3.4.20, we can identify the underlying ∞-category of $\mathbf{A}$ with $\mathrm{N}(\mathbf{A}^\circ) \simeq \mathcal{C}\mathrm{at}_\infty$, the ∞-category of ∞-categories. Proposition 2.4.2.5 implies that composition with the evident Cartesian structure $\mathrm{N}^\otimes(\mathbf{A}^\circ) \to \mathrm{N}(\mathbf{A}^\circ)$ induces an equivalence of ∞-categories $\mathrm{Alg}(\mathrm{N}(\mathbf{A}^\circ)) \to \mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_\infty)$. Combining this

observation with Theorem 4.1.4.4, we conclude that the ∞-category of monoid objects of Cat_∞ is equivalent to the ∞-category underlying the category of *strictly* associative monoids in $\mathbf{A}$. In other words, every monoidal ∞-category $\mathcal{C}$ is equivalent (as a monoidal ∞-category) to an ∞-category $\mathcal{C}'$ equipped with a strictly associative multiplication $\mathcal{C}' \times \mathcal{C}' \to \mathcal{C}'$ (so that we can regard $\mathcal{C}'$ as a simplicial monoid). We regard this assertion as an ∞-categorical analogue of *MacLane's coherence theorem*, which asserts that every monoidal category is equivalent to a strict monoidal category (that is, a monoidal category in which the tensor product operation $\otimes$ is associative up to equality, and the associativity isomorphisms are simply the identity maps).

We now turn to the proof of Proposition 4.1.4.3.

Lemma 4.1.4.8. *Let $\mathbf{A}$ be a combinatorial monoidal model category, and let $\mathrm{N}(\mathbf{A}^c)[W^{-1}]$ be its underlying ∞-category. Then the induced tensor product on $\mathrm{N}(\mathbf{A}^c)[W^{-1}]$ preserves small colimits separately in each variable.*

Proof. For each object $A \in \mathbf{A}$, the operation of tensor product with A (either on the left or on the right) determines a left Quillen functor from $\mathbf{A}$ to itself. The desired result now follows from Corollary 1.3.4.26. $\square$

Remark 4.1.4.9. Lemma 4.1.4.8 admits a converse. Suppose that $\mathcal{C}$ is a presentable ∞-category endowed with a monoidal structure, and that the associated bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. Then $\mathcal{C}$ is equivalent (as a monoidal ∞-category) to the underlying ∞-category of a combinatorial simplicial monoidal model category $\mathbf{A}$. Since we will not need this fact, we will only give a sketch of proof.

First, we apply Example 4.1.4.7 to reduce to the case where $\mathcal{C}$ is a *strict* monoidal ∞-category; that is, $\mathcal{C}$ is a simplicial monoid. Now choose a regular cardinal κ such that $\mathcal{C}$ is κ-accessible. Enlarging κ if necessary, we may suppose that the full subcateogry $\mathcal{C}^\kappa \subseteq \mathcal{C}$ spanned by the κ-compact objects contains the unit object of $\mathcal{C}$ and is stable under tensor products.

The ∞-category $\mathcal{C}^\kappa$ is essentially small. We define a sequence of simplicial subsets

$$\mathcal{D}(0) \subseteq \mathcal{D}(1) \subseteq \ldots \subseteq \mathcal{C}^\kappa$$

as follows. Let $\mathcal{D}(0) = \emptyset$, and for $i \geq 0$ let $\mathcal{D}(i+1)$ be a small simplicial subset of $\mathcal{C}^\kappa$ which is categorically equivalent to $\mathcal{C}^\kappa$ and contains the submonoid of $\mathcal{C}^\kappa$ generated by $\mathcal{D}(i)$. Let $\mathcal{D} = \bigcup \mathcal{D}(i)$, so that $\mathcal{D}$ is a small simplicial submonoid of $\mathcal{C}^\kappa$ such that the inclusion $\mathcal{D} \subseteq \mathcal{C}^\kappa$ is a categorical equivalence.

The proof of Theorem T.5.5.1.1 shows that $\mathcal{C}$ can be identified with an accessible localization of $\mathcal{P}(\mathcal{D})$. According to Proposition T.5.1.1.1, we can identify $\mathcal{P}(\mathcal{D})$ with $\mathrm{N}(\mathbf{A}^o)$, where $\mathbf{A}$ denotes the category $(\mathrm{Set}_\Delta)_{/\mathcal{D}}$ endowed with the contravariant model structure (see §T.2.1.4). Let $L : \mathcal{P}(\mathcal{D}) \to \mathcal{C}$ be a localization functor, and let $\mathbf{B}$ be the category $(\mathrm{Set}_\Delta)_{/\mathcal{D}}$ endowed with the following *localized* model structure:

(C) A morphism $\alpha : X \to Y$ in $(\mathrm{Set}_\Delta)_{/\mathcal{D}}$ is a cofibration in $\mathbf{B}$ if and only if α is a monomorphism of simplicial sets.

(W) A morphism $\alpha : X \to Y$ in $(\mathrm{Set}_\Delta)_{/\mathcal{D}}$ is a weak equivalence in $\mathbf{B}$ if and only if the $L(\beta)$ is an isomorphism in the homotopy category $\mathrm{h}\mathcal{C}$, where β denotes the corresponding morphism in $\mathrm{h}\mathcal{P}(\mathcal{D}) \simeq \mathrm{h}\mathbf{A}$.

(F) A morphism $\alpha : X \to Y$ in $(\mathrm{Set}_\Delta)_{/\mathcal{D}}$ is a fibration in $\mathbf{B}$ if and only if it has the right lifting property with respect to every morphism which is simultaneously a cofibration and a weak equivalence in $\mathbf{B}$.

Proposition T.A.3.7.3 implies that $\mathbf{B}$ is a (combinatorial) simplicial model category, and that the underlying ∞-category $\mathrm{N}(\mathbf{B}^o)$ is equivalent to $\mathcal{C}$.

The category $(\mathrm{Set}_\Delta)_{/\mathcal{D}}$ is endowed with a monoidal structure, which may be described as follows: given a finite collection of objects $X_1, \ldots, X_n \in \mathcal{S}_{/\mathcal{D}}$, we let $X_1 \otimes \ldots \otimes X_n$ denote the product $X_1 \times \ldots \times X_n$ of the underlying simplicial sets, mapping to $\mathcal{D}$ via the composition $X_1 \times \ldots \times X_n \to \mathcal{D}^n \to \mathcal{D}$, where the second map is given by the monoid structure on $\mathcal{D}$. It is not difficult to verify that this monoidal structure is

compatible with the model structure on $\mathbf{B}$. Applying Variant 4.1.3.17, we deduce that $\mathrm{N}(\mathbf{B}^{\mathrm{o},\otimes})$ determines a monoidal structure on $\mathrm{N}(\mathbf{B}^{\mathrm{o}}) \simeq \mathcal{C}$. One can show that this monoidal structure coincides (up to equivalence) with the structure determined by the associative multiplication on $\mathcal{C}$.

We now turn to the proof of Proposition 4.1.4.3.

Notation 4.1.4.10. Let $f : X \to X'$ and $g : Y \to Y'$ be morphisms in a monoidal category $\mathbf{A}$ which admits pushouts. We define the *pushout product* of i and j to be the induced map

$$f \wedge g : (X \otimes Y') \coprod_{X \otimes Y} (X' \otimes Y) \to X' \otimes Y'.$$

The operation $\wedge$ endows the category $\mathrm{Fun}([1], \mathbf{A})$ with a monoidal structure, which is symmetric if the monoidal structure on $\mathbf{A}$ is symmetric.

Lemma 4.1.4.11. *Let $\mathbf{A}$ be a combinatorial symmetric monoidal model category which satisfies the monoid axiom, and let $\overline{U}$ be as in Definition 4.1.4.1. Then:*

(1) *If $f : X \to X'$ belongs to $\overline{U}$ and Y is an object of $\mathbf{A}$, then $f \otimes \mathrm{id}_Y : X \otimes Y \to X' \otimes Y$ belongs to $\overline{U}$.*

(2) *If $f, g \in \overline{U}$, then $f \wedge g \in \overline{U}$.*

Proof. To prove (1), let S denote the collection of all morphisms f in $\mathbf{A}$ such that $f \otimes \mathrm{id}_Y$ belongs to $\overline{U}$. It is easy to see that S is weakly saturated. It will therefore suffice to show that $U \subseteq S$, which is obvious.

To prove (2), we use the same argument. Fix g, and let S' be the set of all morphisms $f \in \mathbf{A}$ such that $f \wedge g$ belongs to $\overline{U}$. We wish to prove that $\overline{U} \subseteq S'$. Since S' is weakly saturated, it will suffice to show that $U \subseteq S'$. In other words, we may assume that f is of the form $f_0 \otimes \mathrm{id}_A$, where f_0 is a trivial cofibration in $\mathbf{A}$. Similarly, we may assume that $g = g_0 \otimes \mathrm{id}_B$. Then $f \wedge g = (f_0 \wedge g_0) \otimes (\mathrm{id}_{A \otimes B})$, which belongs to U since $f_0 \wedge g_0$ is a trivial cofibration in $\mathbf{A}$. $\square$

Lemma 4.1.4.12. *Let $\mathbf{A}$ be a monoidal model category. Suppose given commutative diagrams*

$$
\begin{array}{ccc}
A \longrightarrow B & \qquad & X \longrightarrow Y \\
\downarrow \qquad \downarrow & \qquad & \downarrow \qquad \downarrow \\
A' \longrightarrow B' & \qquad & X' \longrightarrow Y'
\end{array}
$$

where every object is cofibrant, the horizontal arrows are cofibrations, and the vertical arrows are weak equivalences. Then in the induced diagram

$$
\begin{array}{ccc}
(A \otimes Y) \coprod_{A \otimes X} (B \otimes X) & \longrightarrow & B \otimes Y \\
\downarrow & & \downarrow \\
(A' \otimes Y') \coprod_{A' \otimes X'} (B' \otimes X') & \longrightarrow & B' \otimes Y'
\end{array}
$$

has the same properties.

Proof. The assertion that the horizontal arrows are cofibrations follows immediately from the definition of a monoidal model category. Since every object appearing in either of the original diagrams, we deduce that each of the morphisms

$$A \otimes X \to A' \otimes X' \qquad A \otimes Y \to A' \otimes Y'$$

$$B \otimes X \to B' \otimes X' \qquad B \otimes Y \to B' \otimes Y'$$

is a weak equivalence between cofibrant objects. We obtain a weak equivalence of cofibrant diagrams

$$
\begin{array}{ccc}
A \otimes Y & \longleftarrow A \otimes X \longrightarrow & B \otimes X \\
\downarrow & \downarrow & \downarrow \\
A' \otimes Y' & \longleftarrow A' \otimes X' \longrightarrow & B' \otimes X',
\end{array}
$$

from which we obtain (by passing to the colimit) a weak equivalence of cofibrant objects $(A \otimes Y) \coprod_{A \otimes X} (B \otimes X) \to (A' \otimes Y') \coprod_{A' \otimes X'} (B' \otimes X')$ as desired. $\square$

Proof of Proposition 4.1.4.3. We first observe that the category $\mathrm{Alg}(\mathbf{A})$ is presentable (this follows, for example, from Corollary 3.2.3.5). Recall that a collection S of morphisms in a presentable category $\mathcal{C}$ is *weakly saturated* if it is stable under pushouts, retracts, and transfinite composition (see Definition T.A.1.2.2); we will say that S is *generated by* a subset $S_0 \subset S$ if S is the smallest weakly saturated collection of morphisms containing S_0.

Since $\mathcal{C}$ is combinatorial, there exists a (small) collection of morphisms $I = \{i_\alpha : C \to C'\}$ which generates the class of cofibrations in $\mathbf{A}$, and a (small) collection of morphisms $J = \{j_\alpha : D \to D'\}$ which generates the class of trivial cofibrations in $\mathbf{A}$.

Let $F : \mathbf{A} \to \mathrm{Alg}(\mathbf{A})$ be a left adjoint to the forgetful functor. Let $\overline{F(I)}$ be the weakly saturated class of morphisms in $\mathrm{Alg}(\mathbf{A})$ generated by $\{F(i) : i \in I\}$, and let $\overline{F(J)}$ be defined similarly. Unwinding the definitions, we see that a morphism in $\mathrm{Alg}(\mathbf{A})$ is a trivial fibration if and only if it has the right lifting property with respect to $F(i)$, for every $i \in I$. Invoking the small object argument, we deduce that every morphism $f : A \to C$ in $\mathrm{Alg}(\mathbf{A})$ admits a factorization $A \xrightarrow{f'} B \xrightarrow{f''} C$ where $f' \in \overline{F(I)}$ and f'' is a trivial fibration. Similarly, we can find an analogous factorization where $f' \in \overline{F(J)}$ and f'' is a fibration.

Using a standard argument, we may reduce the proof of (1) to the problem of showing that every morphism belonging to $\overline{F(J)}$ is a weak equivalence in $\mathrm{Alg}(\mathbf{A})$. If $\mathbf{A}$ is symmetric and satisfies the monoid axiom, then let $\overline{U}$ be as in Definition 4.1.4.1; otherwise let $\overline{U}$ be the collection of all trivial cofibrations in $\mathbf{A}$. Let S be the collection of all morphisms in $\mathrm{Alg}(\mathbf{A})$ such that the induced map in $\mathbf{A}$ belongs to $\overline{U}$. Then S is weakly saturated, and every element of S is a weak equivalence. To complete the proof, it will suffice to show that $F(J) \subseteq S$. In other words, we must prove:

(∗) Let

$$
\begin{array}{ccc}
F(C) & \xrightarrow{F(i)} & F(C') \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & A'
\end{array}
$$

be a pushout diagram in $\mathrm{Alg}(\mathbf{A})$. If i is a trivial cofibration in $\mathbf{A}$, then $f \in S$.

Let $\emptyset$ be an initial object of $\mathbf{A}$, and let $j : \emptyset \to A$ be the unique morphism. We now observe that A' can be obtained as the direct limit of a sequence

$$
A = A^{(0)} \xrightarrow{f_1} A^{(1)} \xrightarrow{f_2} \dots
$$

of objects of $\mathbf{A}$, where each f_n is a pushout of $j \wedge i \wedge j \wedge \dots \wedge i \wedge j$; here the factor i appears n times. If every object of $\mathbf{A}$ is cofibrant, then we conclude that f_n is a trivial cofibration using the definition of a monoidal model category. If the monoidal structure on $\mathbf{A}$ is symmetric and satisfies the monoid axiom, then repeated application of Lemma 4.1.4.11 shows that $f_n \in \overline{U}$. Since $\overline{U}$ is weakly saturated, it follows that $f \in S$ as desired. This completes the proof of (1).

Assertion (2) is obvious. To prove (3), we observe both $\mathbf{A}$ and $\mathrm{Alg}(\mathbf{A})$ are cotensored over simplicial sets, and that we have canonical isomorphisms $\theta(A^K) \simeq \theta(A)^K$ for $A \in \mathrm{Alg}(\mathbf{A})$, $K \in \mathcal{S}\mathrm{et}_\Delta$. To prove that

$\mathrm{Alg}(\mathbf{A})$ is a simplicial model category, it will suffice to show that $\mathrm{Alg}(\mathbf{A})$ is tensored over simplicial sets, and that given a fibration $i : A \to A'$ in $\mathrm{Alg}(\mathbf{A})$ and a cofibration $j : K \to K'$ in $\mathcal{S}\mathrm{et}_{\Delta}$, the induced map $A^{K'} \to A^K \times_{A'^K} A'^{K'}$ is a fibration, trivial if either i or j is a fibration. The second claim follows from the fact that θ detects fibrations and trivial fibrations. For the first, it suffices to prove that for $K \in \mathcal{S}\mathrm{et}_{\Delta}$, the functor $A \mapsto A^K$ has a left adjoint; this follows from the adjoint functor theorem.

We now prove (4). Let T be the collection of all morphisms $\alpha : X \to Y$ in $\mathrm{Alg}(\mathbf{A})$ with the following property: for every pushout diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \alpha\ } & Y \\
\downarrow & & \downarrow \\
A & \longrightarrow & A' \\
\ \downarrow{\scriptstyle \beta} & & \ \downarrow{\scriptstyle \beta'} \\
B & \longrightarrow & B',
\end{array}
$$

if β is a weak equivalence, then β' is a weak equivalence. To prove that $\mathbf{A}$ is left proper, it will suffice to show that every cofibration belongs to T. Using the assumption that the collection of weak equivalences in $\mathbf{A}$ is stable under filtered colimits, we deduce that T is a weakly saturated class of morphisms. It will therefore suffice to show that T contains every generating cofibration of the form $F(i) : F(x) \to F(y)$, where $i : x \to y$ is a cofibration in $\mathbf{A}$. The algebra objects A' and B' can be defined as the direct limit of sequences

$$
A = A^{(0)} \xrightarrow{f_1} A^{(1)} \xrightarrow{f_2} \ldots
$$

$$
B = B^{(0)} \xrightarrow{g_1} B^{(1)} \xrightarrow{g_2} \ldots
$$

as in the proof of $(*)$. Since the collection of weak equivalences in $\mathbf{A}$ is stable under filtered colimits, it will suffice to show that each of the maps $\beta_k : A^{(k)} \to B^{(k)}$ is a weak equivalence. For $k = 0$ there is nothing to prove; in the general case we work by induction on k. We have a map between the homotopy pushout diagrams

$$
\begin{array}{ccc}
K & \xrightarrow{\ f'_k\ } & A \otimes y \otimes \ldots \otimes y \otimes A \\
\downarrow & & \downarrow \\
A^{(k-1)} & \xrightarrow{\ f_k\ } & A^{(k)}
\end{array}
\qquad
\begin{array}{ccc}
L & \xrightarrow{\ g'_k\ } & B \otimes y \otimes \ldots \otimes y \otimes B \\
\downarrow & & \downarrow \\
B^{(k-1)} & \xrightarrow{\ g_k\ } & B^{(k)}.
\end{array}
$$

Consequently, to prove that β_k is a weak equivalence, it will suffice to show that β_{k-1} is a weak equivalence (which follows from the inductive hypothesis) and that the maps

$$
K \to L \qquad A \otimes y \otimes \ldots \otimes y \otimes A \to B \otimes y \otimes \ldots \otimes y \otimes B
$$

are weak equivalences, which follows from Lemma 4.1.4.12. $\qquad\square$

We now turn to the proof of Theorem 4.1.4.4. The main point is to establish the following:

Lemma 4.1.4.13. *Let $\mathbf{A}$ be a combinatorial monoidal model category and let $\mathcal{C}$ be a small category such that $\mathrm{N}(\mathcal{C})$ is sifted (Definition T.5.5.8.1). Assume either that every object of $\mathbf{A}$ is cofibrant, or that $\mathbf{A}$ satisfies the following pair of conditions:*

(A) *The monoidal structure on $\mathbf{A}$ is symmetric, and $\mathbf{A}$ satisfies the monoid axiom.*

(B) *The model category $\mathbf{A}$ is left proper and the class of cofibrations in $\mathbf{A}$ is generated by cofibrations between cofibrant objects (this is automatic if every object of $\mathbf{A}$ is cofibrant).*

Let W be the collection of weak equivalences in $\mathbf{A}^c$ and W' the collection of weak equivalences in $\mathrm{Alg}(\mathbf{A})^c$. Then the forgetful functor $\mathrm{N}(\mathrm{Alg}(\mathbf{A})^c)[W'^{-1}] \to \mathrm{N}(\mathbf{A}^c)[W^{-1}]$ preserves $\mathrm{N}(\mathcal{C})$-indexed colimits.

Proof. In view of Propositions 1.3.4.24 and 1.3.4.25, it will suffice to show that the forgetful functor θ : $\mathrm{Alg}(\mathbf{A}) \to \mathbf{A}$ preserves homotopy colimits indexed by $\mathcal{C}$. Let us regard $\mathrm{Alg}(\mathbf{A})^{\mathcal{C}}$ and $\mathbf{A}^{\mathcal{C}}$ as endowed with the projective model structure (see §T.A.3.3). Let $F : \mathbf{A}^{\mathcal{C}} \to \mathbf{A}$ and $F_{\mathrm{Alg}} : \mathrm{Alg}(\mathbf{A})^{\mathcal{C}} \to \mathrm{Alg}(\mathbf{A})$ be colimit functors, and let $\theta^{\mathcal{C}} : \mathrm{Alg}(\mathbf{A})^{\mathcal{C}} \to \mathbf{A}^{\mathcal{C}}$ be given by composition with θ. Since $\mathrm{N}(\mathcal{C})$ is sifted, there is a canonical isomorphism of functors $\alpha : F \circ \theta^{\mathcal{C}} \simeq \theta \circ F_{\mathrm{Alg}}$. We wish to prove that this isomorphism persists after deriving all of the relevant functors. Since θ and $\theta^{\mathcal{C}}$ preserve weak equivalences, they can be identified with their right derived functors. Let LF and LF_{Alg} be the left derived functors of F and F_{Alg}, respectively. Then α induces a natural transformation $\overline{\alpha} : LF \circ \theta^{\mathcal{C}} \to \theta \circ LF_{\mathrm{Alg}}$; we wish to show that $\overline{\alpha}$ is an isomorphism. Let $A : \mathcal{C} \to \mathrm{Alg}(\mathbf{A})$ be a strongly cofibrant object of $\mathrm{Alg}(\mathbf{A})^{\mathcal{C}}$; we must show that the natural map

$$LF(\theta^{\mathcal{C}}(A)) \to \theta(LF_{\mathrm{Alg}}(A)) \simeq \theta(F_{\mathrm{Alg}}(A)) \simeq F(\theta^{\mathcal{C}}(A))$$

is a weak equivalence in $\mathbf{A}$.

Let us say that an object $X \in \mathbf{A}^{\mathcal{C}}$ is *good* if each of the objects $X(C) \in \mathbf{A}$ is cofibrant, the object $F(X) \in \mathbf{A}$ is cofibrant, and the natural map $LF(X) \to F(X)$ is a weak equivalence in $\mathbf{A}$: in other words, if the colimit of X is also a homotopy colimit of X. To complete the proof, it will suffice to show that $\theta^{\mathcal{C}}(A)$ is good, whenever A is a strongly cofibrant object of $\mathrm{Alg}(\mathbf{A})^{\mathcal{C}}$. This is not obvious, since $\theta^{\mathcal{C}}$ is a right Quillen functor and does not preserve strongly cofibrant objects in general (note that we have not yet used the full strength of our assumption that $\mathrm{N}(\mathcal{C})$ is sifted). To continue the proof, we will need a relative version of the preceding condition. We will say that a morphism $f : X \to Y$ in $\mathbf{A}^{\mathcal{C}}$ is *good* if the following conditions are satisfied:

(i) The objects $X, Y \in \mathbf{A}^{\mathcal{C}}$ are good.

(ii) For each $C \in \mathcal{C}$, the induced map $X(C) \to Y(C)$ is a cofibration in $\mathbf{A}$.

(iii) The map $F(X) \to F(Y)$ is a cofibration in $\mathbf{A}$.

We now make the following observations:

(1) The collection of good morphisms is stable under transfinite composition. More precisely, suppose given an ordinal α and a direct system of objects $\{X^{\beta}\}_{\beta < \alpha}$ of $\mathbf{A}^{\mathcal{C}}$. Suppose further that for every $0 < \beta < \alpha$, the map $\varinjlim\{X^{\gamma}\}_{\gamma < \beta} \to X^{\beta}$ is good. Then the induced map $X^0 \to \varinjlim\{X^{\beta}\}_{\beta < \alpha}$ is good. The only nontrivial point is to verify that the object $X = \varinjlim\{X^{\beta}\}_{\beta < \alpha}$ is good. For this, we observe X is a homotopy colimit of the system $\{X^{\beta}\}$ (in virtue of (ii)), while $F(X)$ is a homotopy colimit of the system $\{F(X^{\beta})\}$ (in virtue of (iii)), and that the collection of homotopy colimit diagrams is stable under homotopy colimits.

(2) Suppose given a pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\ f'\ } & Y' \end{array}$$

in $\mathbf{A}^{\mathcal{C}}$. If f is good and X' is good, then f' is good. Once again, the only nontrivial point is to show that Y' is good. To see this, we observe that our hypotheses imply that Y' is homotopy pushout of Y with X' over X. Similarly, $F(Y')$ is a homotopy pushout of $F(Y)$ with $F(X')$ over $F(X)$. We now invoke once again the fact that the class of homotopy colimit diagrams is stable under homotopy colimits.

(3) Let $F : \mathcal{C} \to \mathbf{A}$ be a constant functor whose value is a cofibrant object of $\mathbf{A}$. Then F is good. This follows from the fact that $\mathrm{N}(\mathcal{C})$ is weakly contractible (use Propositions 1.3.4.24 and T.5.5.8.7).

(4) Every projectively cofibrant object of $\mathbf{A}^{\mathcal{C}}$ is good. Every projective cofibration between projectively cofibrant objects of $\mathbf{A}^{\mathcal{C}}$ is good.

(5) If X and Y are good objects of $\mathbf{A}^{\mathcal{C}}$, then $X \otimes Y$ is good. To prove this, we first observe that the collection of cofibrant objects of $\mathbf{A}$ is stable under tensor products. Because $N(\mathcal{C})$ is sifted, Proposition 1.3.4.24 supplies a chain of isomorphisms in $h\mathbf{A}$:

$$LF(X \otimes Y) \simeq LF(X) \otimes LF(Y) \simeq F(X) \otimes F(Y) \simeq F(X \otimes Y).$$

(6) Let $f : X \to X'$ be a good morphism in $\mathbf{A}^{\mathcal{C}}$, and let Y be a good object of $\mathbf{A}^{\mathcal{C}}$. Then the morphism $f \otimes \mathrm{id}_Y$ is good. Condition (i) follows from (5), condition (ii) follows from the fact that tensoring with each $Y(C)$ preserves cofibrations (since $Y(C)$ is cofibrant), and condition (iii) follows by applying the same argument to $F(Y)$ (and invoking the fact that F commutes with tensor products).

(7) Let $f : X \to X'$ and $g : Y \to Y'$ be good morphisms in $\mathbf{A}^{\mathcal{C}}$. Then

$$f \wedge g : (X \otimes Y') \coprod_{X \otimes Y} (X' \otimes Y) \to X' \otimes Y'$$

is good. Condition (ii) follows immediately from the fact that $\mathbf{A}$ is a monoidal model category. Condition (iii) follows from the same argument, together with the observation that F commutes with pushouts and tensor products. Condition (i) follows by combining (5), (6), and (2).

We observe that our assumption (B) implies an analogous result for $\mathbf{A}^{\mathcal{C}}$:

(B') The collection of all strong cofibrations in $\mathbf{A}^{\mathcal{C}}$ is generated by strong cofibrations between strongly cofibrant objects.

Let $T : \mathbf{A}^{\mathcal{C}} \to \mathrm{Alg}(\mathbf{A})^{\mathcal{C}}$ be a left adjoint to $\theta^{\mathcal{C}}$. Using the small object argument and (B'), we conclude that for every strongly cofibrant object $A \in \mathrm{Alg}(\mathbf{A})^{\mathcal{C}}$ there exists a transfinite sequence $\{A^\beta\}_{\beta \leq \alpha}$ in $\mathrm{Alg}(\mathbf{A})^{\mathcal{C}}$ with the following properties:

(a) The object A^0 is initial in $\mathrm{Alg}(\mathbf{A})^{\mathcal{C}}$.

(b) The object A is a retract of A^α.

(c) If $\lambda \leq \alpha$ is a limit ordinal, then $A^\lambda \simeq \mathrm{colim}\{A^\beta\}_{\beta < \lambda}$.

(d) For each $\beta < \alpha$, there is a pushout diagram

$$
\begin{array}{ccc}
T(X') & \xrightarrow{T(f)} & T(X) \\
\downarrow & & \downarrow \\
A^\beta & \longrightarrow & A^{\beta+1}
\end{array}
$$

where f is a strong cofibration between strongly cofibrant objects of $\mathbf{A}^{\mathcal{C}}$.

We wish to prove that $\theta^{\mathcal{C}}(A)$ is good. In view of (b), it will suffice to show that $\theta^{\mathcal{C}}(A^\alpha)$ is good. We will prove a more general assertion: for every $\gamma \leq \beta \leq \alpha$, the induced morphism $u_{\gamma,\beta} : \theta^{\mathcal{C}}(A^\gamma) \to \theta^{\mathcal{C}}(A^\beta)$ is good. The proof is by induction on β. If $\beta = 0$, then we are reduced to proving that $\theta^{\mathcal{C}}(A^0)$ is good. This follows from (a) and (3). If β is a nonzero limit ordinal, then the desired result follows from (c) and (1). It therefore suffices to treat the case where $\beta = \beta' + 1$ is a successor ordinal. Moreover, we may suppose that $\gamma = \beta'$: if $\gamma < \beta'$, then we observe that $u_{\gamma,\beta} = u_{\beta',\beta} \circ u_{\gamma,\beta'}$ and invoke (1), while if $\gamma > \beta'$, then $\gamma = \beta$ and we are reduced to proving that $\theta^{\mathcal{C}}(A^\beta)$ is good, which follows from the assertion that $u_{\beta',\beta}$ is good. We are now reduced to proving the following:

$(*)$ Let

$$
\begin{array}{ccc}
T(X') & \xrightarrow{T(f)} & T(X) \\
\downarrow & & \downarrow \\
B' & \xrightarrow{\quad v \quad} & B
\end{array}
$$

be a pushout diagram in $\mathrm{Alg}(\mathbf{A})^{\mathcal{C}}$, where $f : X' \to X$ is a projective cofibration between projectively cofibrant objects of $\mathbf{A}^{\mathcal{C}}$. If $\theta^{\mathcal{C}}(B')$ is good, then $\theta^{\mathcal{C}}(v)$ is good.

To prove $(*)$, we set $Y = \theta^{\mathcal{C}}(B) \in \mathbf{A}^{\mathcal{C}}$, $Y' = \theta^{\mathcal{C}}(B') \in \mathbf{A}^{\mathcal{C}}$. Let $g : \emptyset \to Y'$ the unique morphism, where $\emptyset$ denotes an initial object of $\mathbf{A}^{\mathcal{C}}$. As in the proof of Proposition 4.1.4.3, Y can be identified with the colimit of a sequence

$$
Y^{(0)} \xrightarrow{w_1} Y^{(1)} \xrightarrow{w_2} \ldots
$$

where $Y^{(0)} = Y'$, and w_k is a pushout of the morphism $f^{(k)} = g \wedge f \wedge g \wedge \ldots \wedge f \wedge g$, where the factor f appears k times. In view of (1) and (2), it will suffice to prove that each $f^{(k)}$ is a good morphism. Since Y' is good, we conclude immediately that g is good. It follows from (4) that f is good. Repeated application of (7) allows us to deduce that $f^{(k)}$ is good, and to conclude the proof. □

We are now ready to prove our main result.

Proof of Theorem 4.1.4.4. Consider the diagram

$$
\begin{array}{ccc}
\mathrm{N}(\mathrm{Alg}(\mathbf{A})^c)[W'^{-1}] & \xrightarrow{\hspace{5cm}} & \mathrm{Alg}(\mathrm{N}(\mathbf{A})[W^{-1}]) \\
& {\scriptstyle G} \searrow \qquad \swarrow {\scriptstyle G'} & \\
& \mathrm{N}(\mathbf{A})[W^{-1}]. &
\end{array}
$$

It will suffice to show that this diagram satisfies the hypotheses of Corollary 6.2.2.14:

(a) The ∞-categories $\mathrm{N}(\mathrm{Alg}(\mathbf{A})^c)[W'^{-1}]$ and $\mathrm{Alg}(\mathrm{N}(\mathbf{A}^c)[W^{-1}])$ admit geometric realizations of simplicial objects. In fact, both of these ∞-categories are presentable. For $\mathrm{N}(\mathrm{Alg}(\mathbf{A})^c)[W'^{-1}]$, this follows from Proposition 1.3.4.22. For $\mathrm{Alg}(\mathrm{N}(\mathbf{A})^{\mathrm{o}})$, we first observe that $\mathrm{N}(\mathbf{A}^c)[W^{-1}]$ is presentable (Proposition 1.3.4.22) and that the tensor product preserves colimits separately in each variable (Lemma 4.1.4.8), and apply Corollary 3.2.3.5.

(b) The functors G and G' admit left adjoints F and F'. The existence of a left adjoint to G is clear, and the existence of a left adjoint to G' follows from Corollary 3.1.3.5.

(c) The functor G' is conservative and preserves geometric realizations of simplicial objects. This follows from Proposition 3.2.3.1 and Lemma 3.2.2.6.

(d) The functor G is conservative and preserves geometric realizations of simplicial objects. The first assertion is immediate from the definition of the weak equivalences in $\mathrm{Alg}(\mathbf{A})$, and the second follows from Lemma 4.1.4.13.

(e) The canonical map $G' \circ F' \to G \circ F$ is an equivalence of functors. This follows from the observation that both sides induce, on the level of homotopy categories, the free algebra functor $C \mapsto \coprod_{n \geq 0} C^{\otimes n}$ (Proposition 4.1.1.14).

□

Corollary 4.1.4.14. *Let $\mathbf{A}$ be combinatorial model category. Assume that:*

(i) *The Cartesian product functor endows $\mathbf{A}$ with the structure of a monoidal model category.*

(*ii*) *The model category* **A** *is left proper and the class of cofibrations is generated by cofibrations between cofibrant objects.*

(*iii*) *The Cartesian product functor on* **A** *satisfies the monoid axiom.*

Let $G : \mathrm{Alg}(\mathbf{A}) \to \mathbf{A}^{\boldsymbol{\Delta}^{op}}$ *be the functor which assigns to each algebra object of* **A** *the corresponding monoid object. Then:*

(1) *The functor G admits a left adjoint F.*

(2) *The functors F and G determine a Quillen adjunction between $\mathrm{Alg}(\mathbf{A})$ and $\mathbf{A}^{\boldsymbol{\Delta}^{op}}$, where the latter category is endowed with the projective model structure.*

(3) *The right derived functor RG is fully faithful, and its essential image in the homotopy category of $\mathbf{A}^{\boldsymbol{\Delta}^{op}}$ consists of those simplicial objects $A_\bullet$ of* **A** *which determine monoid objects of the homotopy category* h**A***.*

Remark 4.1.4.15. Conditions (*ii*) and (*iii*) in the statement of Corollary 4.1.4.14 are automatic if we assume that every object of **A** is cofibrant.

Proof. Assertion (1) follows from the adjoint functor theorem, since G is an accessible functor which commutes with small limits. Assertion (2) is clear, since G preserves weak equivalences and fibrations. To prove (3), it will suffice to show that the induced map of ∞-categories $F' : \mathrm{N}((\mathbf{A}^{\boldsymbol{\Delta}^{op}})^c)[W^{-1}] \to \mathrm{N}(\mathrm{Alg}(\mathbf{A})^c)[W'^{-1}]$ admits a fully faithful right adjoint G', whose essential image is the collection of diagrams $\boldsymbol{\Delta}^{op} \to \mathbf{A}$ which determine monoid objects in the homotopy category h**A**. Here W and W' denote the collections of weak equivalences between cofibrant objects in $\mathbf{A}^{\boldsymbol{\Delta}^{op}}$ and $\mathrm{Alg}(\mathbf{A})$, respectively.

We have a homotopy commutative diagram

$$
\begin{array}{ccc}
\mathrm{N}(\mathrm{Alg}(\mathbf{A})^c)[W'^{-1}] & \longrightarrow & \mathrm{N}((\mathbf{A}^{\boldsymbol{\Delta}^{op}})^c)[W^{-1}] \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\
\mathrm{Alg}(\mathrm{N}(\mathbf{A}^c)[V^{-1}]) & \xrightarrow{\ G''\ } & \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}^{op}), \mathrm{N}(\mathbf{A}^c)[V^{-1}]),
\end{array}
$$

where V is the collection of weak equivalences in $\mathbf{A}^c$, the map p is the categorical equivalence of Theorem 4.1.4.4, and q is the categorical equivalence supplied by Proposition 1.3.4.25. It will therefore suffice to show that G'' is fully faithful, and that its essential image is the class of monoid objects of $\mathrm{N}(\mathbf{A}^c)[W^{-1}]$. This follows immediately Propositions 4.1.2.6 and 2.4.2.5. $\qquad\square$

4.2 Left and Right Modules

Let $\mathcal{C}$ be a monoidal category with unit object **1** and let A be an associative algebra object of $\mathcal{C}$. A *left A-module* in $\mathcal{C}$ is an object $M \in \mathcal{C}$ equipped with an action map $a : A \otimes M \to M$ such that the following diagrams commute

$$
\begin{array}{ccc}
A \otimes A \otimes M & \xrightarrow{m \otimes \mathrm{id}} & A \otimes M \\
\downarrow{\scriptstyle \mathrm{id} \otimes a} & & \downarrow{\scriptstyle a} \\
A \otimes M & \xrightarrow{\ a\ } & M
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbf{1} \otimes M & \xrightarrow{u \otimes \mathrm{id}} & A \otimes M \\
& \searrow \quad \swarrow{\scriptstyle a} & \\
& M,
\end{array}
$$

where $m : A \otimes A \to A$ and $u : \mathbf{1} \to A$ denote the multiplication and unit of A, respectively. The collection of all left A-modules can be organized into a category $\mathrm{LMod}_A(\mathcal{C})$.

Our goal in this section is to generalize the theory of left modules to the ∞-categorical setting. Recall that if $\mathcal{C}^\otimes$ is a symmetric monoidal ∞-category (with underlying ∞-category $\mathcal{C}$), then an associative algebra

object of $\mathcal{C}$ is defined to be a map of ∞-operads $\mathcal{A}ss^{\otimes} \to \mathcal{C}^{\otimes}$ (Definition 4.1.1.10). In §4.2.1, we will introduce a larger ∞-operad $\mathcal{LM}^{\otimes}$, which contains $\mathcal{A}ss^{\otimes}$ as a full subcategory. If $A : \mathcal{A}ss^{\otimes} \to \mathcal{C}^{\otimes}$ is an associative algebra object of $\mathcal{C}$, then we can define a *left A-module* to be a map of ∞-operads $M : \mathcal{LM}^{\otimes} \to \mathcal{C}^{\otimes}$ such that $M|\,\mathcal{A}ss^{\otimes} = A$.

If A is an associative algebra of a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, then the collection of all left A-module objects of $\mathcal{C}$ can be organized into an ∞-category, which we will denote by $\mathrm{LMod}_A(\mathcal{C})$. In practice, it is often useful to know that A-module structures survive as we perform various categorical constructions in $\mathcal{C}$. In §4.2.3, we will describe several instances of this phenomenon, by showing that (under some mild hypotheses) the ∞-category $\mathrm{LMod}_A(\mathcal{C})$ admits limits and colimits which are preserved by the forgetful functor $\mathrm{LMod}_A(\mathcal{C}) \to \mathcal{C}$. To prove assertions of this type, it will be useful to adopt an alternative definition of left module (Definition 4.2.2.10) which we describe in §4.2.2, building on the ideas of §4.1.2.

The theory of left modules developed in this section can be regarded as a generalization of the classical theory of left modules in the following sense: if $\mathcal{C}$ is a symmetric monoidal category and A is an associative algebra object of $\mathcal{C}$, then we have an equivalence of ∞-categories

$$\theta : \mathrm{N}(\mathrm{LMod}_A(\mathcal{C})) \to \mathrm{LMod}_A(\mathrm{N}(\mathcal{C})).$$

where the ∞-category $\mathrm{N}(\mathcal{C})$ inherits a symmetric monoidal structure from $\mathcal{C}$ (see Example 2.1.2.21) and we identify A with the corresponding associative algebra object of $\mathrm{N}(\mathcal{C})$. More generally, if $\mathcal{C}$ is a fibrant simplicial category, then we can define a functor θ as indicated above (where $\mathrm{N}(\mathcal{C})$ now denotes the homotopy coherent nerve of $\mathcal{C}$), but it need not be an equivalence: objects of $\mathrm{LMod}_A(\mathrm{N}(\mathcal{C}))$ can be thought of as objects $M \in \mathcal{C}$ together with a coherently associative left action of A on M, which cannot always be rectified to a strictly associative action of A on M. In §4.3.3, we will see that this rectification can often be achieved when $\mathcal{C}$ is the underlying ∞-category of a monoidal model category; in these cases, θ is an equivalence when restricted to the subcategory of cofibrant objects of $\mathrm{LMod}_A(\mathcal{C})$ (see Theorem 4.3.3.17 for a precise statement). The key to the proof is a structure theorem for free modules (Proposition 4.2.4.2) which we will prove in §4.2.4, using the general machinery developed in Chapter 3.

We close this section by giving a brief discussion of the theory of duality in a monoidal ∞-category (§4.2.5).

4.2.1 The ∞-Operad $\mathcal{LM}^{\otimes}$

Our goal in this section is to lay the foundations for an ∞-categorical theory of left modules over associative algebras. As a first step, we recast the classical theory of left modules using the language of colored operads.

Definition 4.2.1.1. We define a colored operad **LM** as follows:

(i) The set of objects of **LM** has two elements, which we will denote by $\mathfrak{a}$ and $\mathfrak{m}$.

(ii) Let $\{X_i\}_{i \in I}$ be a finite collection of objects of **LM** and let Y be another object of **LM**. If $Y = \mathfrak{a}$, then $\mathrm{Mul}_{\mathbf{LM}}(\{X_i\}, Y)$ is the collection of all linear orderings of I provided that each $X_i = \mathfrak{a}$, and is empty otherwise. If $Y = \mathfrak{m}$, then $\mathrm{Mul}_{\mathbf{LM}}(\{X_i\}, Y)$ is the collection of all linear orderings $\{i_1 < \cdots < i_n\}$ on the set I such that $X_{i_n} = \mathfrak{m}$ and $X_{i_j} = \mathfrak{a}$ for $j < n$ (by convention, we agree that this set is empty if I is empty).

(iii) The composition law on **LM** is determined by the composition of linear orderings, as described in Definition 4.1.1.1.

Remark 4.2.1.2. Restricting our attention to the object $\mathfrak{a} \in \mathbf{LM}$, we obtain a sub-colored operad of **LM**, which is isomorphic to the colored operad **Ass** of Definition 4.1.1.1. We will often abuse notation by identifying **Ass** with this sub-colored operad of **LM**.

Remark 4.2.1.3. If $\mathcal{C}$ is a symmetric monoidal category and $F : \mathbf{LM} \to \mathcal{C}$ is a map of colored operads, then $F|\,\mathbf{Ass}$ is a map of colored operads $\mathbf{Ass} \to \mathcal{C}$, which we can identify with an associative algebra object

$F(\mathfrak{a}) = A \in \mathcal{C}$ (see Remark 4.1.1.2). Let $M = F(\mathfrak{m}) \in \mathcal{C}$. The unique operation $\phi \in \mathrm{Mul}_{\mathbf{LM}}(\{\mathfrak{a}, \mathfrak{m}\}, \mathfrak{m})$ determines a map $F(\phi) : A \otimes M \to M$. It is not difficult to see that $F(\phi)$ exhibit M as a left A-module.

Conversely, suppose we are given a map of colored operads $F_0 : \mathbf{Ass} \to \mathcal{C}$ corresponding to an associative algebra object A of $\mathcal{C}$, and $a : A \otimes M \to M$ be a map which exhibits M as a left A-module. To every finite linearly ordered set I, we can associate a map

$$A^{\otimes I} \otimes M \to A \otimes M \xrightarrow{a} M,$$

where the first map is given by the associative algebra structure on A. This construction determines a map of colored operads $F : \mathbf{LM} \to \mathcal{C}$ extending F_0, such that $F(\mathfrak{m}) = M$.

We can summarize the above discussion as follows: if $F_0 : \mathbf{Ass} \to \mathcal{C}$ classifies an associative algebra object $A \in \mathcal{C}$, then giving a left A-module is equivalent to giving a map of colored operads $F : \mathbf{LM} \to \mathcal{C}$ which extends F_0.

Remark 4.2.1.4. Our notation for the objects of $\mathbf{LM}$ is motivated by Remark 4.2.1.3: a map of colored operads from $\mathbf{LM}$ to a symmetric monoidal category $\mathcal{C}$ carries $\mathfrak{a}$ to an associative algebra in $\mathcal{C}$, and $\mathfrak{m}$ to a left module over that algebra.

Remark 4.2.1.5. Every operation $\phi \in \mathrm{Mul}_{\mathbf{LM}}(\{X_i\}_{i \in I}, Y)$ determines a linear ordering on the set I. Passage from ϕ to this linear ordering determines a map of colored operads $\mathbf{LM} \to \mathbf{Ass}$. This map can be understood as follows: for every symmetric monoidal category $\mathcal{C}$ and every map of colored operads $F : \mathbf{Ass} \to \mathcal{C}$, the composite map $\mathbf{LM} \to \mathbf{Ass} \to \mathcal{C}$ corresponds to the pair (A, M), where A is the associative algebra object of $\mathcal{C}$ determined by F and $M = A$, regarded as a left module over itself.

Notation 4.2.1.6. We let $\mathbf{LM}^{\otimes}$ denote the category obtained by applying Construction 2.1.1.7 to the colored operad $\mathbf{LM}$. We can describe this category more concretely as follows (see Remark 4.1.1.4):

(1) The objects of $\mathbf{LM}^{\otimes}$ are pairs $(\langle n \rangle, S)$, where $\langle n \rangle$ is an object of $\mathcal{F}\mathrm{in}_*$ and S is a subset of $\langle n \rangle^{\circ}$.

(2) A morphism from $(\langle n \rangle, S)$ to $(\langle n' \rangle, S')$ in $\mathbf{LM}^{\otimes}$ consists of a morphism $\alpha : \langle n \rangle \to \langle n' \rangle$ in $\mathbf{Ass}^{\otimes}$ satisfying the following conditions:

   (i) The map α carries $S \cup \{*\}$ into $S' \cup \{*\}$.

   (ii) If $m' \in S'$, then $\alpha^{-1}\{s'\}$ contains exactly one element of S, and that element is maximal with respect to the linear ordering of $\alpha^{-1}\{s'\}$.

In terms of this description, the object $\mathfrak{a} \in \mathbf{LM}$ corresponds to the object $(\langle 1 \rangle, \emptyset) \in \mathbf{LM}^{\otimes}$, while the object $\mathfrak{m} \in \mathbf{LM}$ corresponds to $(\langle 1 \rangle, \langle 1 \rangle^{\circ})$.

We now introduce the ∞-categorical analogue of Definition 4.2.1.1.

Definition 4.2.1.7. We let $\mathcal{LM}^{\otimes}$ denote the nerve of the category $\mathbf{LM}^{\otimes}$. We regard $\mathcal{LM}^{\otimes}$ as an ∞-operad via the forgetful functor $\mathcal{LM}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ (see Example 2.1.1.21).

Remark 4.2.1.8. The underlying ∞-category $\mathcal{LM}$ of $\mathcal{LM}^{\otimes}$ is isomorphic to the discrete simplicial set $\Delta^0 \coprod \Delta^0$ with two vertices, corresponding to the objects $\mathfrak{a}, \mathfrak{m} \in \mathbf{LM}$.

Remark 4.2.1.9. The assertion that the forgetful functor $\mathcal{LM}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ exhibits $\mathcal{LM}^{\otimes}$ as an ∞-operad admits the following refinement: the map of colored operads $\mathbf{LM} \to \mathbf{Ass}$ appearing in Remark 4.2.1.5 induces a fibration of ∞-operads $\mathcal{LM}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$.

Remark 4.2.1.10. The inclusion of colored operads $\mathbf{Ass} \hookrightarrow \mathbf{LM}$ of Remark 4.2.1.2 determines a map $\mathcal{A}\mathrm{ss}^{\otimes} \hookrightarrow \mathcal{LM}^{\otimes}$, which is an isomorphism from $\mathcal{A}\mathrm{ss}^{\otimes}$ onto the full subcategory of $\mathcal{LM}^{\otimes}$ spanned by objects of the form $(\langle n \rangle, \emptyset)$. We will generally abuse notation and identify $\mathcal{A}\mathrm{ss}^{\otimes}$ with its image in $\mathcal{LM}^{\otimes}$.

Notation 4.2.1.11. Let $\mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a fibration of ∞-operads. We let $\mathcal{C}_{\mathfrak{a}}^{\otimes}$ denote the fiber product $\mathcal{C}^{\otimes} \times_{\mathcal{LM}^{\otimes}} \mathcal{Ass}^{\otimes}$ (so that $\mathcal{C}_{\mathfrak{a}}^{\otimes}$ is a planar ∞-operad, in the sense of Definition 4.1.1.6). We will denote the underlying ∞-category of $\mathcal{C}_{\mathfrak{a}}^{\otimes}$ by $\mathcal{C}_{\mathfrak{a}} = \mathcal{C}^{\otimes} \times_{\mathcal{LM}^{\otimes}} \{\mathfrak{a}\}$. We let $\mathcal{C}_{\mathfrak{m}}$ denote the fiber product $\mathcal{C}^{\otimes} \times_{\mathcal{LM}^{\otimes}} \{\mathfrak{m}\}$.

Definition 4.2.1.12. Let $\mathcal{C}^{\otimes} \to \mathcal{Ass}^{\otimes}$ be a planar ∞-operad and let $\mathcal{M}$ be an ∞-category. A *weak enrichment of $\mathcal{M}$ over $\mathcal{C}^{\otimes}$* is a fibration of ∞-operads $q : \mathcal{O}^{\otimes} \to \mathcal{LM}^{\otimes}$ together with isomorphisms $\mathcal{O}_{\mathfrak{a}}^{\otimes} \simeq \mathcal{C}^{\otimes}$ and $\mathcal{O}_{\mathfrak{m}}^{\otimes} \simeq \mathcal{M}$. In this situation, we will say that q *exhibits $\mathcal{M}$ as weakly enriched over $\mathcal{C}^{\otimes}$*.

Definition 4.2.1.13. Let $\mathcal{C}^{\otimes} \to \mathcal{Ass}^{\otimes}$ be a planar ∞-operad, $\mathcal{M}$ an ∞-category, and let $q : \mathcal{O}^{\otimes} \to \mathcal{LM}^{\otimes}$ exhibit $\mathcal{M}$ as weakly enriched over $\mathcal{C}^{\otimes}$. We let $\mathrm{LMod}(\mathcal{M})$ denote the ∞-category $\mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O})$. We will refer to $\mathrm{LMod}(\mathcal{M})$ as *the ∞-category of left module objects of $\mathcal{M}$*. Composition with the inclusion $\mathcal{Ass}^{\otimes} \hookrightarrow \mathcal{LM}^{\otimes}$ determines a categorical fibration

$$\mathrm{LMod}(\mathcal{M}) = \mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O}) \to \mathrm{Alg}_{/\mathcal{Ass}^{\otimes}}(\mathcal{C}) = \mathrm{Alg}(\mathcal{C}).$$

If A is an algebra object of $\mathcal{C}$, we let $\mathrm{LMod}_A(\mathcal{M})$ denote the fiber $\mathrm{LMod}(\mathcal{M}) \times_{\mathrm{Alg}(\mathcal{C})} \{A\}$; we will refer to $\mathrm{LMod}_A(\mathcal{M})$ as the *∞-category of left A-module objects of $\mathcal{M}$*.

Remark 4.2.1.14. The notation of Definition 4.2.1.13 is somewhat abusive: the ∞-category $\mathrm{LMod}(\mathcal{M})$ depends not only on the ∞-category $\mathcal{M}$, but also on the planar ∞-operad $\mathcal{C}^{\otimes}$ and on the weak enrichment of $\mathcal{M}$ over $\mathcal{C}^{\otimes}$.

Remark 4.2.1.15. Let $\mathcal{M}$ be an ∞-category which is weakly enriched over a planar ∞-operad $\mathcal{C}^{\otimes}$. We can think of the objects of $\mathrm{LMod}(\mathcal{M})$ as given by pairs (A, M), where A is an associative algebra object of $\mathcal{C}$ and M is a left A-module in $\mathcal{M}$.

Example 4.2.1.16. Let $\mathcal{C}^{\otimes} \to \mathcal{Ass}^{\otimes}$ be a planar ∞-operad. Form the fiber product $\mathcal{O}^{\otimes} = \mathcal{C}^{\otimes} \times_{\mathcal{Ass}^{\otimes}} \mathcal{LM}^{\otimes}$ using the fibration of ∞-operads $\mathcal{LM}^{\otimes} \to \mathcal{Ass}^{\otimes}$ of Remark 4.2.1.9. Then $\mathcal{O}^{\otimes}$ exhibits the ∞-category $\mathcal{C}$ as weakly enriched over $\mathcal{C}^{\otimes}$. We can therefore consider the ∞-category $\mathrm{LMod}(\mathcal{C}) = \mathrm{Alg}_{\mathcal{LM}/\mathcal{Ass}}(\mathcal{C})$.

Example 4.2.1.17. Let $\mathcal{C}^{\otimes} \to \mathcal{Ass}^{\otimes}$ be a planar ∞-operad. Composition with the forgetful functor $\mathcal{LM}^{\otimes} \to \mathcal{Ass}^{\otimes}$ of Remark 4.2.1.9 determines a map $s : \mathrm{Alg}(\mathcal{C}) \to \mathrm{LMod}(\mathcal{C})$, which is a section of the projection map $\mathrm{LMod}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$. This section can be interpreted as follows: for every algebra object $A \in \mathrm{Alg}(\mathcal{C})$, the object $s(A) \in \mathrm{LMod}_A(\mathcal{C})$ can be identified with A, regarded as a left module over itself. For this reason, we will often not distinguish in notation between A and $s(A)$.

Example 4.2.1.18. Let $\mathcal{C}^{\otimes} \to \mathcal{Ass}^{\otimes}$ be a monoidal ∞-category. The objects of $\mathrm{LMod}(\mathcal{C}) = \mathrm{Alg}_{\mathcal{LM}/\mathcal{Ass}}(\mathcal{C})$ are given by functors $F : \mathcal{LM}^{\otimes} \to \mathcal{C}^{\otimes}$. Then $F|\mathcal{Ass}^{\otimes}$ is an algebra object of $\mathcal{C}$, which we will identify with its underlying object $F(\mathfrak{a}) = A \in \mathcal{C}$. We also have an object $F(\mathfrak{m}) = M \in \mathcal{C}$. As in Remark 4.2.1.3, we see that the unique operation $\phi \in \mathrm{Mul}_{\mathbf{LM}}(\{\mathfrak{a}, \mathfrak{m}\}, \mathfrak{m})$ determines a map $a : A \otimes M \to M$ in $\mathcal{C}$, which is well-defined up to homotopy. The requirement that F be defined on the entirety of the ∞-operad $\mathcal{LM}^{\otimes}$ guarantees that the action map a is compatible with the associative multiplication on A, up to coherent homotopy. In particular, if $m : A \otimes A \to A$ and $u : \mathbf{1} \to A$ denote the multiplication and unit map on A, then the diagrams

$$
\begin{array}{ccc}
A \otimes A \otimes M & \xrightarrow{m \otimes \mathrm{id}} & A \otimes M \\
\downarrow{\scriptstyle \mathrm{id} \otimes a} & & \downarrow{\scriptstyle a} \\
A \otimes M & \xrightarrow{\quad a \quad} & M
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbf{1} \otimes M & \xrightarrow{u \otimes \mathrm{id}} & A \otimes M \\
& \searrow \qquad \swarrow{\scriptstyle a} & \\
& M, &
\end{array}
$$

commute up to homotopy.

Definition 4.2.1.19. Let $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a fibration of ∞-operads. We will say that q *exhibits $\mathcal{C}_{\mathfrak{m}}$ as left-tensored over $\mathcal{C}_{\mathfrak{a}}$* if q is a coCartesian fibration of ∞-operads.

Remark 4.2.1.20. Let $\mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a coCartesian fibration of ∞-operads. Then the induced map $\mathcal{C}_{\mathfrak{a}}^{\otimes} \to \mathcal{A}ss^{\otimes}$ is also a coCartesian fibration of ∞-operads, so that $\mathcal{C}_{\mathfrak{a}}^{\otimes}$ is a monoidal ∞-category.

Remark 4.2.1.21. Let $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a coCartesian fibration of ∞-operads. Then q is classified by a map $\chi : \mathcal{LM}^{\otimes} \to \mathcal{C}at_{\infty}$ which is an $\mathcal{LM}$-monoid object of $\mathcal{C}at_{\infty}$ (Example 2.4.2.4). We can identify χ with an object of $\mathrm{Mon}_{\mathcal{LM}}(\mathcal{C}at_{\infty}) \simeq \mathrm{Alg}_{\mathcal{LM}}(\mathcal{C}at_{\infty}) = \mathrm{LMod}(\mathcal{C}at_{\infty})$ (Proposition 2.4.2.5). More informally: q can be thought of as giving an associative algebra $\mathcal{C}_{\mathfrak{a}}$ in $\mathcal{C}at_{\infty}$, together with a left module $\mathcal{C}_{\mathfrak{m}}$ over $\mathcal{C}_{\mathfrak{a}}$. In particular, q determines an action map

$$\otimes : \mathcal{C}_{\mathfrak{a}} \times \mathcal{C}_{\mathfrak{m}} \to \mathcal{C}_{\mathfrak{m}}$$

which is well-defined up to homotopy (and compatible with the monoidal structure on $\mathcal{C}_{\mathfrak{a}}$ up to homotopy).

Notation 4.2.1.22. Let $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a fibration of ∞-operads, so that q exhibits $\mathcal{C}_{\mathfrak{m}}$ as weakly enriched over the planar operad $\mathcal{C}_{\mathfrak{a}}^{\otimes}$. Given an ordered sequence of objects $C_1, \ldots, C_n \in \mathcal{C}_{\mathfrak{a}}$ and a pair of objects $M, N \in \mathcal{C}_{\mathfrak{m}}$, we let $\mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C_1, \ldots, C_n\} \otimes M, N)$ denote the summand of the mapping space $\mathrm{Mul}_{\mathcal{C}}(\{C_1, \ldots, C_n, M\}, \{N\})$ corresponding to the linear ordering $\{1 < 2 < \cdots < n\}$ of the set $\langle n \rangle$.

Remark 4.2.1.23. In the special case $n = 0$, the mapping space $\mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C_1, \ldots, C_n\} \times M, N)$ can be identified with the usual mapping space $\mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(M, N)$ in the ∞-category $\mathcal{C}_{\mathfrak{m}}$.

Example 4.2.1.24. Suppose that $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ is a coCartesian fibration of ∞-operads, so that q determines tensor product functors

$$\otimes : \mathcal{C}_{\mathfrak{a}} \times \mathcal{C}_{\mathfrak{a}} \to \mathcal{C}_{\mathfrak{a}} \qquad \otimes : \mathcal{C}_{\mathfrak{a}} \times \mathcal{C}_{\mathfrak{m}} \to \mathcal{C}_{\mathfrak{m}} \,.$$

Unwinding the definitions, we obtain a homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C_1, \ldots, C_n\} \otimes M, N) \simeq \mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(C_1 \otimes \cdots \otimes C_n \otimes M, N),$$

generalizing the homotopy equivalence of Remark 4.2.1.23 to the case $n > 0$.

Our next goal is to give a criterion for testing if a fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ is a coCartesian fibration.

Definition 4.2.1.25. Let $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a fibration of ∞-operads. We will say that $\mathcal{C}_{\mathfrak{m}}$ is *pseudo-enriched* over $\mathcal{C}_{\mathfrak{a}}^{\otimes}$ if the following conditions are satisfied:

(1) The planar ∞-operad $\mathcal{C}_{\mathfrak{a}}^{\otimes}$ is a monoidal ∞-category.

(2) For every sequence of objects $C_1, \ldots, C_n \in \mathcal{C}_{\mathfrak{a}}$ and every pair of objects $M, N \in \mathcal{C}_{\mathfrak{m}}$, the canonical map

$$\mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C_1 \otimes \cdots \otimes C_n\} \otimes M, N) \to \mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C_1, \ldots, C_n\} \otimes M, N)$$

is a homotopy equivalence.

Proposition 4.2.1.26. *Let* $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ *be a fibration of ∞-operads. Then q is a coCartesian fibration if and only if the following conditions are satisfied:*

(1) *The map q exhibits $\mathcal{C}_{\mathfrak{m}}$ as pseudo-enriched over $\mathcal{C}_{\mathfrak{a}}^{\otimes}$.*

(2) *For every object $M \in \mathcal{C}_{\mathfrak{m}}$ and every object $C \in \mathcal{C}_{\mathfrak{a}}$, there exists a morphism $\phi \in \mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C\} \otimes M, M')$ with the following universal property: for every pair of objects $C' \in \mathcal{C}_{\mathfrak{a}}$, $N \in \mathcal{C}_{\mathfrak{m}}$, composition with ϕ induces a homotopy equivalence*

$$\mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C'\} \otimes M', N) \to \mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C', C\} \otimes M, N).$$

Remark 4.2.1.27. If the hypotheses of Proposition 4.2.1.26 are satisfied, then the object M' appearing in hypothesis (3) can be identified with $C \otimes M$, where $\otimes : \mathcal{C}_{\mathfrak{a}} \times \mathcal{C}_{\mathfrak{m}} \to \mathcal{C}_{\mathfrak{m}}$ is the tensor product functor of Remark 4.2.1.21.

Proof. The necessity of conditions (1) and (2) is obvious. We will prove the sufficiency. Assume that conditions (1) and (2) are satisfied. Fix an object $X \in \mathcal{C}^{\otimes}$ lying over an object $(\langle n \rangle, S)$ in $\mathcal{LM}^{\otimes}$, and let $\alpha : (\langle n \rangle, S) \to (\langle n'' \rangle, S'')$ be a morphism in $\mathcal{LM}^{\otimes}$; we wish to show that α can be lifted to a q-coCartesian morphism $X \to X''$ in $\mathcal{C}^{\otimes}$. We may assume without loss of generality that α is active. We observe that α admits a canonical factorization

$$(\langle n \rangle, S) \xrightarrow{\alpha'} (\langle n' \rangle, S') \xrightarrow{\alpha''} (\langle n'' \rangle, S'')$$

with the following properties:

(*i*) The map α' induces a bijection $S \simeq S'$.

(*ii*) For each $i \in \langle n'' \rangle^{\circ}$, the inverse image $\alpha''^{-1}\{i\}$ contains exactly only element of $\langle n' \rangle^{\circ} - S'$.

According to Proposition T.2.4.1.7, it will suffice to show that α' can be lifted to a q-coCartesian morphism $\overline{\alpha}' : X \to X'$ and that α'' can be lifted to a q-coCartesian morphism $\overline{\alpha}'' : X' \to X''$. We first use assumption (1) to choose a locally q-coCartesian morphism $\overline{\alpha}' : X \to X'$. We claim that $\overline{\alpha}'$ is q-coCartesian. Using Proposition T.2.4.4.3, we are reduced to proving that a product of maps of the form

$$\mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C_1 \otimes \cdots \otimes C_{k_1}, C_{k_1+1} \otimes \cdots \otimes C_{k_2}, \ldots, C_{k_{n-1}+1} \otimes \cdots \otimes C_{k_n}\} \otimes M, N) \to \mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C_1, \ldots, C_{k_n}\} \otimes M, N)$$

is a homotopy equivalence. This follows from (1), which implies that both sides are homotopy equivalent to $\mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C_1 \otimes \cdots \otimes C_{k_n}\} \otimes M, N)$.

We next use (2) to deduce the existence of a locally q-coCartesian morphism $\overline{\alpha}''$ lifting α''. Once again, we prove that $\overline{\alpha}''$ is q-coCartesian using the criterion of Proposition T.2.4.4.3. Unwinding the definitions, we must show that if $\phi \in \mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C\} \otimes M, M')$ is as in (3) and we are given a finite sequence of objects $C_1, \ldots, C_n \in \mathcal{C}_{\mathfrak{a}}$, then for each $N \in \mathcal{C}_{\mathfrak{m}}$ the induced map

$$\psi : \mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C_1, \ldots, C_n\} \otimes M', N) \to \mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C_1, \ldots, C_n, C\} \otimes M, N)$$

is a homotopy equivalence. This follows from our assumption on M', since condition (1) allows us to identify ψ with the homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C'\} \otimes M', N) \to \mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C', C\} \otimes M, N) \simeq \mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C' \otimes C\} \otimes M, N),$$

where $C' = C_1 \otimes \cdots \otimes C_n$. $\qquad\square$

Motivated by the second hypothesis of Proposition 4.2.1.26, we introduce two variants of Definition 4.2.1.19.

Definition 4.2.1.28. Let $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a fibration of ∞-operads which exhibits $\mathcal{C}_{\mathfrak{m}}$ as pseudo-enriched over $\mathcal{C}_{\mathfrak{a}}^{\otimes}$.

Let M and N be objects of $\mathcal{M} = \mathcal{C}_{\mathfrak{m}}$. A *morphism object* for M and N is an object $\mathrm{Mor}_{\mathcal{M}}(M, N) \in \mathcal{C}_{\mathfrak{a}}$ equipped with a map $\alpha \in \mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{\mathrm{Mor}_{\mathcal{M}}(M, N)\} \otimes M, N)$ with the following universal property: for every object $C \in \mathcal{C}_{\mathfrak{a}}$, composition with α induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}_{\mathfrak{a}}}(C, \mathrm{Mor}_{\mathcal{M}}(M, N)) \to \mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C\} \otimes M, N).$$

Let N be an object of $\mathcal{C}_{\mathfrak{m}}$ and C an object of $\mathcal{C}_{\mathfrak{a}}$. An *exponential object* is an object ${}^{C}N \in \mathcal{M}$ together with a map $\beta \in \mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C\} \otimes {}^{C}N, N)$ with the following universal property: for every pair of objects $C' \in \mathcal{C}_{\mathfrak{a}}$ and $M \in \mathcal{C}_{\mathfrak{m}}$, composition with β induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C'\} \otimes M, {}^{C}N) \to \mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C, C'\} \otimes M, N)$.

We will say that q *exhibits $\mathcal{C}_{\mathfrak{m}}$ as enriched over* $\mathcal{C}_{\mathfrak{a}}^{\otimes}$ if, for every pair of objects $M, N \in \mathcal{C}_{\mathfrak{m}}$, there exists a morphism object $\mathrm{Mor}_{\mathcal{M}}(M, N) \in \mathcal{C}_{\mathfrak{a}}$. We will say that q *exhibits $\mathcal{C}_{\mathfrak{m}}$ as cotensored over* $\mathcal{C}_{\mathfrak{a}}^{\otimes}$ if, for every pair of objects $C \in \mathcal{C}_{\mathfrak{a}}$, $N \in \mathcal{C}_{\mathfrak{m}}$, there exists an exponential object ${}^{C}N \in \mathcal{C}_{\mathfrak{m}}$.

Remark 4.2.1.29. In the situation of Definition 4.2.1.28, a morphism object $\mathrm{Mor}_{\mathcal{M}}(M, N) \in \mathcal{C}_{\mathfrak{a}}$ is determined up to canonical equivalence (provided that it exists). Similarly, an exponential object $^C N$ is determined up to equivalence.

Remark 4.2.1.30. Suppose that $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ is a coCartesian fibration of ∞-operads. Then a map $\beta \in \mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C\} \otimes {}^C N, N) \simeq \mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(C \otimes {}^C N, N)$ exhibits $^C N$ as an exponential of N by C if and only if, for every object $M \in \mathcal{C}_{\mathfrak{m}}$, composition with β induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(M, {}^C N) \to \mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(C \otimes M, N).$$

Remark 4.2.1.31. Let $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a fibration of ∞-operads which exhibits $\mathcal{M} = \mathcal{C}_{\mathfrak{m}}$ as enriched over $\mathcal{C}_{\mathfrak{a}}^{\otimes}$. Then the morphism objects $\mathrm{Mor}_{\mathcal{M}}(M, N) \in \mathcal{C}_{\mathfrak{a}}$ depend functorially on $M, N \in \mathcal{C}_{\mathfrak{m}}$. To see this, we observe that we have a trifunctor $(\mathcal{C}_{\mathfrak{a}} \times \mathcal{C}_{\mathfrak{m}})^{op} \times \mathcal{C}_{\mathfrak{m}} \to \mathcal{S}$, given informally by the formula $(C, M, N) \mapsto \mathrm{Map}_{\mathcal{C}_{\mathfrak{m}}}(\{C\} \otimes M, N)$. We may identify this map with a bifunctor $e : \mathcal{C}_{\mathfrak{m}}^{op} \times \mathcal{C}_{\mathfrak{m}} \to \mathrm{Fun}(\mathcal{C}_{\mathfrak{a}}^{op}, \mathcal{S})$. Since $\mathcal{C}_{\mathfrak{m}}$ is enriched over $\mathcal{C}_{\mathfrak{a}}$, then the image of e is contained in full subcategory $\mathrm{Fun}'(\mathcal{C}_{\mathfrak{a}}^{op}, \mathcal{S}) \subseteq \mathrm{Fun}(\mathcal{C}_{\mathfrak{a}}^{op}, \mathcal{S})$ spanned by the essential image of the Yoneda embedding $j : \mathcal{C}_{\mathfrak{a}} \to \mathrm{Fun}'(\mathcal{C}_{\mathfrak{a}}^{op}, \mathcal{S})$. Composing e with a homotopy inverse to j, we obtain the desired functor $\mathrm{Mor} : \mathcal{C}_{\mathfrak{m}}^{op} \times \mathcal{C}_{\mathfrak{m}} \to \mathcal{C}_{\mathfrak{a}}$.

Similarly, if $\mathcal{C}_{\mathfrak{m}}$ is cotensored over $\mathcal{C}_{\mathfrak{a}}$, then formation of exponential objects $(C, N) \mapsto {}^C N$ can be regarded as a functor $\mathcal{C}_{\mathfrak{a}}^{op} \times \mathcal{C}_{\mathfrak{m}} \to \mathcal{C}_{\mathfrak{m}}$.

Example 4.2.1.32. Let $\mathcal{C}$ be a monoidal ∞-category, and regard $\mathcal{C}$ as left-tensored over itself (Example 4.2.1.16). Then $\mathcal{C}$ is enriched over itself if and only if it is *right closed*, and cotensored over itself if and only if it is *left closed* (Definition 4.1.1.17).

The following result provides a large supply of examples of enriched and cotensored ∞-categories:

Proposition 4.2.1.33. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a coCartesian fibration of ∞-operads. Assume that $\mathcal{C}_{\mathfrak{m}}$ and $\mathcal{C}_{\mathfrak{a}}$ are presentable ∞-categories.*

(1) *If, for every $C \in \mathcal{C}_{\mathfrak{a}}$, the functor $C \otimes \bullet : \mathcal{C}_{\mathfrak{m}} \to \mathcal{C}_{\mathfrak{m}}$ preserves small colimits, then $\mathcal{C}_{\mathfrak{m}}$ is cotensored over $\mathcal{C}_{\mathfrak{a}}$.*

(2) *If, for every $M \in \mathcal{C}_{\mathfrak{m}}$, the functor $\bullet \otimes M : \mathcal{C}_{\mathfrak{a}} \to \mathcal{C}_{\mathfrak{m}}$ preserves small colimits, then $\mathcal{C}_{\mathfrak{m}}$ is enriched over $\mathcal{C}_{\mathfrak{a}}$.*

Proof. This follows immediately from the representability criterion of Proposition T.5.5.2.2 (together with Remark 4.2.1.30, in the case of assertion (1)). □

Remark 4.2.1.34. A much larger class of examples of enriched ∞-categories can be obtained by combining Proposition 4.2.1.33 with the following observation: if $\mathcal{C}_{\mathfrak{m}}$ is an ∞-category which is enriched over a monoidal ∞-category $\mathcal{C}_{\mathfrak{a}}^{\otimes}$, then every full subcategory of $\mathcal{C}_{\mathfrak{m}}$ is also enriched over $\mathcal{C}_{\mathfrak{a}}^{\otimes}$.

Remark 4.2.1.35. Let $\mathcal{C}_{\mathfrak{a}}^{\otimes}$ be a monoidal ∞-category. The theory of $\mathcal{C}_{\mathfrak{a}}$-enriched ∞-categories is important for many applications of higher category theory. For example, it can be used as the basis for an inductive definition of the notion of (∞, n)-category for all integers $n \geq 0$: an (∞, n)-category is an ∞-category which is enriched over $\mathcal{C}\mathrm{at}_{(\infty, n-1)}$, where $\mathcal{C}\mathrm{at}_{(\infty, n-1)}$ is the ∞-category of $(\infty, n-1)$-categories (endowed with the Cartesian symmetric monoidal structure).

Variant 4.2.1.36. By slightly modifying the definitions presented in this section, one can develop an entirely parallel theory of right modules in the ∞-categorical setting. We define a category $\mathbf{RM}^{\otimes}$ as follows (compare with Notation 4.2.1.6):

(1) The objects of $\mathbf{RM}^{\otimes}$ are pairs $(\langle n \rangle, S)$, where $\langle n \rangle$ is an object of $\mathfrak{Fin}_*$ and S is a subset of $\langle n \rangle^{\circ}$.

(2) A morphism from $(\langle n \rangle, S)$ to $(\langle n' \rangle, S')$ in $\mathbf{RM}$ consists of a morphism $\alpha : \langle n \rangle \to \langle n' \rangle$ in $\mathbf{Ass}^{\otimes}$ satisfying the following conditions:

(*i*) The map α carries $S \cup \{*\}$ into $S' \cup \{*\}$.

(*ii*) If $m' \in S'$, then $\alpha^{-1}\{s'\}$ contains exactly one element of S, and that element is minimal with respect to the linear ordering of $\alpha^{-1}\{s'\}$.

We let $\mathcal{RM}^{\otimes}$ denote the ∞-operad $\mathrm{N}(\mathbf{RM}^{\otimes})$. We let $\mathcal{RM}$ denote the underlying ∞-category of $\mathcal{RM}^{\otimes}$, which is isomorphic to the discrete simplicial set $\{\mathfrak{a}, \mathfrak{m}\}$. Given a fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{RM}^{\otimes}$, we let $\mathrm{RMod}(\mathcal{C}_{\mathfrak{m}})$ denote the ∞-category $\mathrm{Alg}_{/\mathcal{RM}}(\mathcal{C})$; we will refer to $\mathrm{RMod}(\mathcal{C}_{\mathfrak{m}})$ as the ∞-*category of right module objects of* $\mathcal{C}_{\mathfrak{m}}$. There is an evident forgetful functor $\mathrm{RMod}(\mathcal{C}_{\mathfrak{m}}) \to \mathrm{Alg}(\mathcal{C}_{\mathfrak{a}})$. For every algebra object $A \in \mathcal{C}_{\mathfrak{a}}$, we let $\mathrm{RMod}_A(\mathcal{C}_{\mathfrak{m}})$ denote the fiber $\mathrm{RMod}(\mathcal{C}_{\mathfrak{m}}) \times_{\mathrm{Alg}(\mathcal{C}_{\mathfrak{a}})} \{A\}$; we refer to $\mathrm{RMod}_A(\mathcal{C}_{\mathfrak{m}})$ as the ∞-*category of right A-module objects of* $\mathcal{C}_{\mathfrak{m}}$. All of our discussion concerning left modules can be adapted to the case of right modules without any essential changes.

Remark 4.2.1.37. If $\mathcal{O}^{\otimes}$ is a planar ∞-operad, then the theory of left modules in $\mathcal{O}^{\otimes}$ is equivalent to the theory of right $\mathcal{O}^{\otimes}_{\mathrm{rev}}$ modules, where $\mathcal{O}^{\otimes}_{\mathrm{rev}}$ is the reverse of $\mathcal{O}^{\otimes}$; see Remark 4.1.1.8.

4.2.2 Simplicial Models for Algebras and Modules

The ∞-operad $\mathcal{LM}^{\otimes}$ of Definition 4.2.1.7 gives rise to a good theory of left modules over associative algebras in the ∞-categorical setting. However, for many purposes it is inconvenient to work with the entire simplicial set $\mathcal{LM}^{\otimes}$. Our goal in this section is to describe an alternative approach to the theory of left modules, which based on the ideas introduced in §4.1.2. In §4.2.3, we will apply these ideas to the study of limits and colimits in ∞-categories of left modules.

We begin by studying left modules in the category of sets. Let M be a monoid with multiplication map $m : M \times M \to M$ and unit element 1, and let X be a set equipped with a left action of M: that is, there is a map of sets $M \times X \to X$ such that the diagrams

$$
\begin{array}{ccc}
M \times M \times X & \xrightarrow{m \otimes \mathrm{id}} & M \otimes X \\
\downarrow{\scriptstyle \mathrm{id} \otimes a} & & \downarrow{\scriptstyle a} \\
M \otimes X & \xrightarrow{\quad a \quad} & X
\end{array}
\qquad\qquad
\begin{array}{ccc}
\{1\} \times X & \xrightarrow{u \otimes \mathrm{id}} & M \otimes X \\
& \searrow & \downarrow{\scriptstyle a} \\
& & X
\end{array}
$$

commute. As explained in Example 4.1.2.1, we can associate to the monoid M a category $\mathcal{D}_M$ with a single object E, such that $\mathrm{Hom}_{\mathcal{D}_M}(E, E) = M$ and composition of morphisms in $\mathcal{D}_M$ is given by the multiplication m. Unwinding the definitions, we see that giving a left action of M on a set X is equivalent to giving a functor $F : \mathcal{D}_M \to \mathcal{S}\mathrm{et}$ such that $F(E) = X$; the action of M on X is then encoded by the induced map $M = \mathrm{Hom}_{\mathcal{D}_M}(E, E) \to \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(X, X)$. Applying the Grothendieck construction to the functor F (see §T.2.1.1), we can obtain a new category $\overline{\mathcal{D}}_M$, whose objects are pairs (e, x) where e is an object of $\mathcal{D}_M$ (so that $e = E$, since $\mathcal{D}_M$ has a unique object) and x is an element of $F(e) = F(E) = X$. Finally, we can take the nerve of the category $\overline{\mathcal{D}}_M$ to obtain a simplicial set, which we will denote by $B_X M$. The forgetful functor $\overline{\mathcal{D}}_M \to \mathcal{D}_M$ induces a map of simplicial sets $B_X M \to BM$, where BM is the simplicial set described in Example 4.1.2.1.

Remark 4.2.2.1. Unwinding the definitions, we can describe the simplicial set $B_X M$ concretely as follows. The collection of n-simplices $\mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^n, B_X M)$ is isomorphic to $M^n \times X$. Under these isomorphisms, the face and degeneracy maps are given by the formulae

$$
d_i(m_1, \ldots, m_n, x) = \begin{cases} (m_2, \ldots, m_n, x) & \text{if } i = 0 \\ (m_1, \ldots, m_i m_{i+1}, m_{i+2}, \ldots, m_n, x) & \text{if } 0 < i < n \\ (m_1, \ldots, m_{n-1}, m_n x) & \text{if } i = n. \end{cases}
$$

$$
s_i(m_1, \ldots, m_n, x) = (m_1, \ldots, m_i, 1, m_{i+1}, \ldots, m_n, x).
$$

Each step in the construction of $B_X M$ is reversible: the category $\overline{\mathcal{D}}_M$ can be recovered as the homotopy category of $B_X M \simeq \mathrm{N}(\overline{\mathcal{D}}_M)$, the functor $F : \mathcal{D}_M \to \mathcal{S}\mathrm{et}$ can be recovered from the forgetful functor $\overline{\mathcal{D}}_M \to \mathcal{D}_M$, and F encodes the left action of M on X. We can summarize the situation as follows: the construction $X \mapsto B_X M$ determines a fully faithful embedding from the category of left M-modules to the category of simplicial sets mapping to BM. Moreover, it is easy to describe the essential image of this functor: given a map of simplicial sets $K \to BM$, there exists a left action of M on a set X and an isomorphism $K \simeq B_X M$ if and only if, for every integer $n \geq 0$, the inclusion $\{n\} \hookrightarrow [n]$ induces a bijection

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^n, K) \to \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^n, BM) \times \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^0, K);$$

in this case, X can be recovered as the set of vertices of K.

Motivated by the above discussion, we introduce the definition of a left action of a monoid in an arbitrary ∞-category.

Definition 4.2.2.2. Let $\mathcal{C}$ be an ∞-category. A *left action object* of $\mathcal{C}$ is a natural transformation $\alpha : M' \to M$ in $\mathrm{Fun}(\mathrm{N}(\Delta)^{op}, \mathcal{C})$ with the following properties:

(*i*) The simplicial object $M \in \mathrm{Fun}(\mathrm{N}(\Delta)^{op}, \mathcal{C})$ is a monoid object of $\mathcal{C}$, in the sense of Definition 4.1.2.2.

(*ii*) For each integer $n \geq 0$, the map $M'([n]) \to M([n])$ and the map $M'([n]) \to M'(\{n\}) \simeq M'([0])$ exhibit $M'([n])$ as a product of $M'([0])$ with $M([n])$ in $\mathcal{C}$.

We let $\mathrm{LMon}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}(\Delta)^{op} \times \Delta^1, \mathcal{C})$ spanned by the left action objects of $\mathcal{C}$.

Remark 4.2.2.3. Let $\alpha : M' \to M$ be a left action object of an ∞-category $\mathcal{C}$. Following the abuse of Remark 4.1.2.3, we will sometimes say that $M([1]) \in \mathcal{C}$ is the underlying monoid object of α and that $M'([0]) \in \mathcal{C}$ is equipped with a left action of $M([1])$. This left action is encoded by the map

$$M([1]) \times M'([0]) = M([1]) \times M'(\{1\}) \xleftarrow{\sim} M'([1]) \to M'(\{0\}) = M'([0]).$$

Example 4.2.2.4. Let $T : \mathrm{N}(\Delta)^{op} \to \mathrm{N}(\Delta)^{op}$ be the translation functor given by $[n] \mapsto [n] \star [0] \simeq [n+1]$. The inclusion maps $[n] \to [n] \star [0]$ determine a natural transformation $T \to \mathrm{id}$. This natural transformation gives a map $\mathrm{Fun}(\mathrm{N}(\Delta)^{op}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}(\Delta)^{op} \times \Delta^1, \mathcal{C})$, which carries $\mathrm{Mon}(\mathcal{C})$ into $\mathrm{LMon}(\mathcal{C})$. We can summarize the situation informally as follows: if M is any monoid object of $\mathcal{C}$, then there is a natural left action of M on itself.

Notation 4.2.2.5. Let $\mathcal{C}$ be an ∞-category. By construction, there is a forgetful functor $\mathrm{LMon}(\mathcal{C}) \to \mathrm{Mon}(\mathcal{C})$, which carries a natural transformation $\alpha : M' \to M$ to the simplicial object $M \in \mathrm{Fun}(\mathrm{N}(\Delta)^{op}, \mathcal{C})$. If $M \in \mathrm{Mon}(\mathcal{C})$ is an object, then we will denote the fiber product $\mathrm{LMon}(\mathcal{C}) \times_{\mathrm{Mon}(\mathcal{C})} \{M\}$ by $\mathrm{LMon}_M(\mathcal{C})$.

Our first goal in this section is to prove an analogue of Proposition 4.1.2.6, which compares left action objects of $\mathcal{C}$ with $\mathcal{L}\mathcal{M}^\otimes$-monoid objects of $\mathcal{C}$ (Proposition 4.2.2.9 below). To formulate this result, we need a slightly more elaborate version of Construction 4.1.2.5.

Construction 4.2.2.6. For every finite linearly ordered set $[n]$, let $\mathrm{LCut}_0([n])$ denote the collection of all downward-closed subsets $S \subseteq [n]$. There is a canonical bijection $\langle n + 1 \rangle \simeq \mathrm{LCut}_0([n])$, which carries the base point $* \in \langle n+1 \rangle$ to the empty subset $\emptyset \subset [n]$ and an integer $i \in \langle n+1 \rangle^\circ$ to the subset $\{j \in [n] : j < i\}$. In what follows, we will identify $\mathrm{LCut}_0([n])$ with $\langle n+1 \rangle$ under this bijection.

The construction $[n] \mapsto (\mathrm{LCut}_0([n]), \{[n]\}) \simeq (\langle n+1 \rangle, \{n+1\})$ determines a functor $\mathrm{LCut} : \Delta^{op} \to \mathbf{LM}^\otimes$. Namely, if we are given a map $\alpha : [m] \to [n]$ in Δ, there is an induced map $\alpha' : \mathrm{LCut}_0([n]) \to \mathrm{LCut}_0([m])$, given by the formula $\alpha'(S) = \alpha^{-1} S$. The map α' preserves base points (since the inverse image of an empty set is empty). Moreover, for each $\emptyset \neq S \in \mathrm{LCut}_0([m])$, the inverse image $\alpha'^{-1}\{S\} \subseteq \mathrm{LCut}_0([n])$ is linearly ordered by inclusion (under the bijection $\mathrm{LCut}_0([n]) \simeq \langle n+1 \rangle$, this linear ordering is induced by the usual ordering of $\langle n+1 \rangle^\circ = \{1 < \cdots < n+1\}$).

More explicitly, the functor LCut can be described as follows:

(1) For each $n \geq 0$, we have $\mathrm{Cut}([n]) = (\langle n+1 \rangle, \{n+1\})$.

(2) Given a morphism $\alpha : [n] \to [m]$ in $\mathbf{\Delta}$, the associated morphism $\mathrm{Cut}(\alpha) : \langle m+1 \rangle \to \langle n+1 \rangle$ is given by the formula

$$\mathrm{Cut}(\alpha)(i) = \begin{cases} j & \text{if } (\exists j)[1 \leq j \leq n \wedge \alpha(j-1) < i \leq \alpha(j)] \\ n+1 & \text{if } \alpha(n) < i \\ * & \text{otherwise.} \end{cases}$$

for $i \in \langle m+1 \rangle^{\circ}$, where we endow each $\mathrm{Cut}(\alpha)^{-1}\{j\}$ with the linear ordering induced by its inclusion into $\langle n \rangle^{\circ}$.

Passing to nerves, the functor LCut determines a map of simplicial sets $\mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{LM}^{\otimes}$, which we will also denote by LCut.

Remark 4.2.2.7. The functor LCut_0 of Construction 4.2.2.6 is *almost* identical to the functor Cut of Construction 4.1.2.5. Both constructions assign to an object $[n] \in \mathbf{\Delta}$ a set which parametrizes decompositions $[n] \simeq S \coprod T$, where S is a downward closed subset of $[n]$ and T its complement. The only difference is that the decompositions $[n] \simeq [n] \coprod \emptyset$ and $[n] \simeq \emptyset \coprod [n]$ are considered to represent the same element of $\mathrm{Cut}([n]) \simeq \langle n \rangle$ (namely, the base point of $\langle n \rangle$), but different elements of $\mathrm{LCut}_0([n]) \simeq \langle n+1 \rangle$ (the decomposition $[n] \simeq [n] \coprod \emptyset$ corresponds to the element $n+1 \in \langle n+1 \rangle$, while $[n] \simeq \emptyset \coprod [n]$ corresponds to the base point $* \in \langle n+1 \rangle$).

Remark 4.2.2.8. Let us identify $\mathcal{A}ss^{\otimes}$ with the full subcategory of $\mathcal{LM}^{\otimes}$ spanned by objects of the form $(\langle n \rangle, T)$ where T is empty, as in Remark 4.2.1.10. We can therefore think of the functor Cut of Construction 4.1.2.5 as a map from $\mathrm{N}(\mathbf{\Delta})^{op}$ into $\mathcal{LM}^{\otimes}$. For each $[n] \in \mathbf{\Delta}$, there is an evident map of sets $\theta : \mathrm{LCut}_0([n]) \to \mathrm{LCut}([n])$, which carries a subset $S \subseteq [n]$ to the equivalence relation $\sim$, where $i \sim j$ if either $i, j \in S$ or $i, j \notin S$. More concretely, $\theta : \langle n+1 \rangle \to \langle n \rangle$ is given by the formula

$$\theta(k) = \begin{cases} * & \text{if } k = n+1 \\ k & \text{otherwise.} \end{cases}$$

This construction determines a morphism $\gamma : \mathrm{LCut} \to \mathrm{Cut}$ in the ∞-category $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{LM}^{\otimes})$. When convenient, we will identify γ with a map $\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1 \to \mathcal{LM}^{\otimes}$. It is not difficult to see that this map is an approximation to the ∞-operad $\mathcal{LM}^{\otimes}$ (see the proof of Proposition 4.1.2.10).

Combining Remark 4.2.2.8 with Proposition 4.1.2.11, we obtain the following result:

Proposition 4.2.2.9. *Let $\mathcal{C}$ be an ∞-category finite admits finite products. Composition with the functor $\gamma : \mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1 \to \mathcal{LM}^{\otimes}$ induces an equivalence of ∞-categories*

$$\mathrm{Mon}_{\mathcal{LM}}(\mathcal{C}) \to \mathrm{LMon}(\mathcal{C}).$$

Combining Proposition 4.2.2.9 with Proposition 2.4.2.5, we obtain an efficient description of left module objects in an arbitrary ∞-category $\mathcal{C}$ which is equipped with a Cartesian symmetric monoidal structure. Our goal for the remainder of this section is to obtain a generalization of this description which applies to a general fibration of generalized ∞-operads $\mathcal{O}^{\otimes} \to \mathcal{LM}^{\otimes}$. First, we need to introduce the analogue of Definition 4.2.2.2.

Definition 4.2.2.10. Let $q : \mathcal{O}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a fibration of ∞-operads, so that q exhibits the ∞-category $\mathcal{M} = \mathcal{O}_{\mathfrak{m}}$ as weakly enriched over the planar ∞-operad $\mathcal{O}_{\mathfrak{a}}^{\otimes}$. Let $\gamma : \mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1 \to \mathcal{LM}^{\otimes}$ be as in Remark 4.2.2.8. We let $^{\Delta}\mathrm{LMod}(\mathcal{M})$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{LM}^{\otimes}}(\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1, \mathcal{O}^{\otimes})$ spanned by those maps $f : \mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1 \to \mathcal{O}^{\otimes}$ which satisfy the following conditions:

(1) The restriction $f|(\mathrm{N}(\mathbf{\Delta})^{op} \times \{1\})$ belongs to $^{\Delta}\mathrm{Alg}(\mathcal{C}) \subseteq \mathrm{Fun}_{\mathcal{A}ss^{\otimes}}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{C}^{\otimes})$ (see Definition 4.1.2.14).

(2) If $\alpha : [m] \to [n]$ is an inert morphism in $\mathbf{\Delta}$ such that $\alpha(m) = n$, then the induced map $f([n], 0) \to f([m], 0)$ is an inert morphism in $\mathcal{O}^\otimes$.

(3) For each $[n] \in \mathbf{\Delta}$, the induced map $f([n], 0) \to f([n], 1)$ is an inert morphism in $\mathcal{O}^\otimes$.

Combining Remark 4.2.2.8 with Theorem 2.3.3.23, we obtain the following counterpart of Proposition 4.2.2.9:

Proposition 4.2.2.11. *Let $p : \mathcal{O}^\otimes \to \mathcal{LM}^\otimes$ be a fibration of ∞-operads and let $\mathcal{M} = \mathcal{O}_\mathfrak{m}$. Then composition with the functor $\gamma : \mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1 \to \mathcal{LM}^\otimes$ of Remark 4.2.2.8 induces an equivalence of ∞-categories $\mathrm{Alg}_{/\,\mathcal{LM}}(\mathcal{O}) \to {}^\Delta\mathrm{LMod}(\mathcal{M})$.*

Let M be as in Remark 4.2.2.25, so that the functor γ of Remark 4.2.2.8 defines a map of ∞-preoperads $\overline{\gamma} : (\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1, M) \to \mathcal{LM}^{\otimes, \natural}$. Note that Proposition 4.2.2.11 is equivalent to the statement that $\overline{\gamma}$ is weak equivalence with respect to the ∞-operadic model structure of Proposition 2.1.4.6. For later use, we establish a stronger version of Proposition 4.2.2.11:

Proposition 4.2.2.12. *Let $(\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1, M)$ be the marked simplicial set described in Remark 4.2.2.25. Then the map $\overline{\gamma} : (\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1, M) \to \mathcal{LM}^{\otimes, \natural}$ is a weak equivalence with respect to the generalized ∞-operadic model structure described in Remark 2.3.2.4.*

The proof is a mild variation on the proof of Theorem 2.3.3.23, and will be given at the end of this section.

Remark 4.2.2.13. Let $p : \mathcal{O}^\otimes \to \mathcal{LM}^\otimes$ be a fibration of ∞-operads, which exhibits the ∞-category $\mathcal{M} = \mathcal{O}_\mathfrak{m}$ as weakly enriched over the planar ∞-operad $\mathcal{O}_\mathfrak{a}^\otimes$. We will generally abuse terminology by referring to ${}^\Delta\mathrm{LMod}(\mathcal{M})$ as the ∞-*category of left module objects of* $\mathcal{M}$. Though this terminology is in conflict with that of Definition 4.2.1.13, our abuse is justified by Proposition 4.2.2.11.

Notation 4.2.2.14. Let $\mathcal{M}$ be an ∞-category which is weakly enriched over a planar ∞-operad $\mathcal{C}^\otimes$. By construction, the inclusion $\mathrm{N}(\mathbf{\Delta})^{op} \times \{1\} \hookrightarrow \mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1$ induces a forgetful functor ${}^\Delta\mathrm{LMod}(\mathcal{M}) \to {}^\Delta\mathrm{Alg}(\mathcal{C})$. If $A \in {}^\Delta\mathrm{Alg}(\mathcal{C})$, we let ${}^\Delta\mathrm{LMod}_A(\mathcal{M})$ denote the fiber ${}^\Delta\mathrm{LMod}(\mathcal{M}) \times_{{}^\Delta\mathrm{Alg}(\mathcal{C})} \{A\}$. Following the abuse of Remark 4.2.2.13, we will refer to ${}^\Delta\mathrm{LMod}_A(\mathcal{M})$ as the ∞-*category of left A-module objects of* $\mathcal{M}$.

Let $\mathcal{M}$ be an ∞-category which is weakly enriched over a planar ∞-operad $\mathcal{C}^\otimes$. The functor $\mathrm{LMod}(\mathcal{M}) \to {}^\Delta\mathrm{LMod}(\mathcal{M})$ fits into a commutative diagram

$$
\begin{array}{ccc}
\mathrm{LMod}(\mathcal{M}) & \longrightarrow & {}^\Delta\mathrm{LMod}(\mathcal{M}) \\
\downarrow & & \downarrow \\
\mathrm{Alg}(\mathcal{C}) & \longrightarrow & {}^\Delta\mathrm{Alg}(\mathcal{C}).
\end{array}
$$

Combining Propositions 4.2.2.11 and 4.1.2.15, we obtain:

Corollary 4.2.2.15. *Let $\mathcal{M}$ be an ∞-category which is weakly enriched over a planar ∞-operad $\mathcal{C}^\otimes$. Let $A \in \mathrm{Alg}(\mathcal{C})$ be an algebra object of $\mathcal{C}$ and let A' be its image in ${}^\Delta\mathrm{Alg}(\mathcal{C})$. Then the functor γ of Remark 4.2.2.8 induces an equivalence of ∞-categories ${}^\Delta\mathrm{LMod}_A(\mathcal{M}) \to \mathrm{LMod}_A(\mathcal{M})$.*

The remainder of this section is devoted to introducing some notation which will facilitate the application of Proposition 4.2.2.11.

Notation 4.2.2.16. Let $q : \mathcal{O}^\otimes \to \mathcal{LM}^\otimes$ be a fibration of ∞-operads, so that q exhibits the ∞-category $\mathcal{M} = \mathcal{O}_\mathfrak{m}$ as weakly enriched over the planar operad $\mathcal{C}^\otimes = \mathcal{O}_\mathfrak{a}^\otimes$. We let $\mathrm{Cut} : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{A}\mathrm{ss}^\otimes$ be the functor of Construction 4.1.2.5, and $\mathcal{C}^\circledast$ denote the fiber product $\mathcal{C}^\otimes \times_{\mathcal{A}\mathrm{ss}^\otimes} \mathrm{N}(\mathbf{\Delta})^{op}$.

We define a simplicial set $\overline{\mathcal{M}}^{\circledast}$ together with a map $\overline{\mathcal{M}}^{\circledast} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$ so that the following universal property is satisfied: for every simplicial set K equipped with a map $K \to \mathrm{N}(\boldsymbol{\Delta})^{op}$, there is a canonical bijection

$$\mathrm{Hom}_{(\mathrm{Set}_\Delta)_{/\,\mathrm{N}(\boldsymbol{\Delta})^{op}}}(K, \overline{\mathcal{M}}^{\circledast}) \simeq \mathrm{Hom}_{(\mathrm{Set}_\Delta)_{/\,\mathcal{L}\mathcal{M}^{\otimes}}}(K \times \Delta^1, \mathcal{O}^{\otimes}).$$

Here we regard $K \times \Delta^1$ as an object of $(\mathrm{Set}_\Delta)_{/\,\mathcal{L}\mathcal{M}^{\otimes}}$ via the map

$$K \times \Delta^1 \to \mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1 \xrightarrow{\gamma} \mathcal{L}\mathcal{M}^{\otimes},$$

where γ is the functor described in Remark 4.2.2.8.

Unwinding the definitions, we see that a vertex of $\overline{\mathcal{M}}^{\circledast}$ lying over an object $[n] \in \boldsymbol{\Delta}^{op}$ corresponds to a morphism α in $\mathcal{O}^{\otimes}$ such that $q(\alpha)$ is the map $\theta : (\langle n+1\rangle, \{n+1\}) \to (\langle n\rangle, \emptyset)$ in $\mathcal{L}\mathcal{M}^{\otimes}$ which appears in Remark 4.2.2.8. We let $\mathcal{M}^{\circledast}$ denote the full simplicial subset of $\overline{\mathcal{M}}^{\circledast}$ spanned by those vertices for which α is inert.

Remark 4.2.2.17. Let $q : \mathcal{O}^{\otimes} \to \mathcal{L}\mathcal{M}^{\otimes}$ be a fibration of ∞-operads, so that q exhibits the ∞-category $\mathcal{M} = \mathcal{O}_{\mathfrak{m}}$ as weakly enriched over the planar operad $\mathcal{C}^{\otimes} = \mathcal{O}_{\mathfrak{a}}^{\otimes}$. Using Proposition T.4.3.2.15, we deduce that composition with the inclusion $\{0\} \hookrightarrow \Delta^1$ induces a trivial Kan fibration $\mathcal{M}^{\circledast} \to \mathcal{O}^{\otimes} \times_{\mathcal{L}\mathcal{M}^{\otimes}} \mathrm{N}(\boldsymbol{\Delta})^{op}$, where $\mathrm{N}(\boldsymbol{\Delta})^{op}$ maps to $\mathcal{L}\mathcal{M}^{\otimes}$ via the functor LCut of Construction 4.2.2.6. In particular, the fiber of $\mathcal{M}^{\circledast}$ over an object $[n] \in \boldsymbol{\Delta}^{op}$ is canonically equivalent to $\mathcal{M} \times \mathcal{C}^n$.

Remark 4.2.2.18. Let $\mathcal{M}$ be an ∞-category which is weakly enriched over a planar ∞-operad $\mathcal{C}^{\otimes}$. We have categorical fibrations

$$\mathcal{M}^{\circledast} \xrightarrow{q} \mathcal{C}^{\circledast} \xrightarrow{p} \mathrm{N}(\boldsymbol{\Delta})^{op};$$

in particular, both $\mathcal{M}^{\circledast}$ and $\mathcal{C}^{\circledast}$ are ∞-categories.

Let us identify $^\Delta\mathrm{Alg}(\mathcal{C})$ with a full subcategory of the ∞-category $\mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{C}^{\circledast})$ of sections of p, and $^\Delta\mathrm{LMod}(\mathcal{M})$ with a full subcategory of the ∞-category $\mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{M}^{\circledast})$ of sections of $p \circ q$. If $A \in {}^\Delta\mathrm{Alg}(\mathcal{C})$, then A determines a functor $\mathrm{N}(\boldsymbol{\Delta})^{op} \times \mathcal{C}^{\circledast}$, and we can identify $\mathrm{LMod}_A(\mathcal{M})$ with a full subcategory of the ∞-category $\mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{M}^{\circledast} \times_{\mathcal{C}^{\circledast}} \mathrm{N}(\boldsymbol{\Delta})^{op}))$.

Note that an object $F \in \mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{M}^{\circledast})$ belongs to $^\Delta\mathrm{LMod}(\mathcal{M})$ if and only if the following conditions are satisfied:

(1′) The image of $F \in \mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{C}^{\circledast})$ belongs to $^\Delta\mathrm{Alg}(\mathcal{C})$.

(2′) If $\alpha : [m] \to [n]$ is an inert morphism in $\boldsymbol{\Delta}$ such that $\alpha(m) = n$, then the induced map $f([n]) \to f([m])$ is a $(p \circ q)$-coCartesian morphism in $\mathcal{M}^{\circledast}$.

We now record a few simple properties of the forgetful map $\mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast}$ which will be useful later.

Lemma 4.2.2.19. *Suppose we are given a coCartesian fibration of ∞-operads $\mathcal{O}^{\otimes} \to \mathcal{L}\mathcal{M}^{\otimes}$, which exhibits $\mathcal{M} = \mathcal{O}_{\mathfrak{m}}$ as left-tensored over the monoidal ∞-category $\mathcal{C}^{\otimes} = \mathcal{O}_{\mathfrak{a}}^{\otimes}$. Then the associated functor $\mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast}$ is a locally coCartesian fibration.*

Proof. For each $n \geq 0$, the map $p_{[n]} : \mathcal{M}_{[n]}^{\circledast} \to \mathcal{C}_{[n]}^{\circledast}$ is equivalent to the projection $\mathcal{C}_{[n]}^{\circledast} \times \mathcal{M} \to \mathcal{C}_{[n]}^{\circledast}$ (see Remark 4.2.2.17). The desired result now follows from Proposition T.2.4.2.11. $\qquad\square$

Remark 4.2.2.20. In the situation of Lemma 4.2.2.19, suppose we are given a morphism $\overline{\alpha} : C \to D$ in $\mathcal{C}^{\circledast}$, covering a map $\alpha : [m] \to [n]$ in $\boldsymbol{\Delta}$. We can identify C with an n-tuple of objects $(C_1, \ldots, C_n) \in \mathcal{C}^n$, D with an m-tuple $(D_1, \ldots, D_m) \in \mathcal{C}^m$, and $\overline{\alpha}$ with a collection of morphisms $C_{\alpha(i-1)+1} \otimes \ldots \otimes C_{\alpha(i)} \to D_i$. The fibers of the map $p : \mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast}$ over the objects C and D are both canonically equivalent to $\mathcal{M}$, and the induced functor $\overline{\alpha}_! : \mathcal{M} \to \mathcal{M}$ is given (up to equivalence) by the formula $M \mapsto C_{\alpha(m)+1} \otimes \ldots \otimes C_n \otimes M$.

Remark 4.2.2.21. Let $p : \mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast}$ be as in Lemma 4.2.2.19, and suppose given a commutative triangle

$$
\begin{array}{ccc}
 & M' & \\
{}^{f}\nearrow & & \searrow^{g} \\
M & \xrightarrow{\ \ h\ \ } & M''
\end{array}
$$

in $\mathcal{M}^{\circledast}$, covering a triangle

$$
\begin{array}{ccc}
 & [m] & \\
{}^{\alpha}\swarrow & & \nwarrow^{\beta} \\
[n] & \xleftarrow{\ \ \gamma\ \ } & [k]
\end{array}
$$

in the category $\boldsymbol{\Delta}$. Suppose furthermore that f and g are locally p-coCartesian, and that α induces a bijection

$$\{i \in [m] : \beta(k) < i\} \simeq \{j \in [n] : \gamma(k) < j \leq \alpha(m)\}$$

(so that, in particular, $\beta(k) < i \leq m$ implies $\gamma(k) < \alpha(i)$). Then the description given in Remark 4.2.2.20 implies that h is also locally p-coCartesian.

Remark 4.2.2.22. Let $p : \mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast}$ be as in Lemma 4.2.2.19, and let $\overline{\alpha}$ be a morphism in $\mathcal{M}^{\circledast}$ which covers a map $\alpha : [m] \to [n]$ in $\boldsymbol{\Delta}$. Then:

(1) Suppose that $\alpha(m) = n$ and $\overline{\alpha}$ is locally p-Cartesian. Then $\overline{\alpha}$ is p-Cartesian.

(2) Suppose that $m \leq n$, that $\alpha : [m] \to [n]$ is the canonical inclusion, and that $\overline{\alpha}$ is locally p-coCartesian. Then $\overline{\alpha}$ is p-coCartesian.

Assertion (1) follows from Remark 4.2.2.21 and Lemma T.5.2.2.3, while (2) follows from Remark 4.2.2.21 and Lemma T.2.4.2.7.

Proposition 4.2.2.23. *Suppose we are given a coCartesian fibration of ∞-operads $\mathcal{O}^{\otimes} \to \mathcal{LM}^{\otimes}$, which exhibits $\mathcal{M} = \mathcal{O}_{\mathfrak{m}}$ as left-tensored over the monoidal ∞-category $\mathcal{C}^{\otimes} = \mathcal{O}_{\mathfrak{a}}^{\otimes}$. Let $\overline{\alpha}$ be a morphism in $\mathcal{M}^{\circledast}$ which covers a map $\alpha : [m] \to [n]$ in $\boldsymbol{\Delta}$. Then:*

(1) *Suppose that $\alpha(m) = n$ and $\overline{\alpha}$ is locally p-Cartesian. Then $\overline{\alpha}$ is p-Cartesian.*

(2) *Suppose that $m \leq n$, that $\alpha : [m] \to [n]$ is the canonical inclusion, and that $\overline{\alpha}$ is locally p-coCartesian. Then $\overline{\alpha}$ is p-coCartesian.*

Proof. Assertion (1) follows from Remark 4.2.2.21 and Lemma T.5.2.2.3, while (2) follows from Remark 4.2.2.21 and Lemma T.2.4.2.7. $\qquad \square$

Remark 4.2.2.24. Let $q : \mathcal{O}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a coCartesian fibration of ∞-operads. Then q is classified by a functor $\mathcal{LM}^{\otimes} \to \mathcal{C}\mathrm{at}_{\infty}$ which is an $\mathcal{LM}$-monoid object of $\mathcal{C}\mathrm{at}_{\infty}$ (Example 2.4.2.4). According to Proposition 4.2.2.9, this $\mathcal{LM}$-monoid object is determined up to equivalence by the induced map $\mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1 \to \mathcal{C}\mathrm{at}_{\infty}$, which we can view as a morphism in $\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{C}\mathrm{at}_{\infty})$. This morphism classifies the map $\mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast}$ of Notation 4.2.2.16. In other words, the data contained in the coCartesian fibration of ∞-operads $q : \mathcal{O}^{\otimes} \to \mathcal{LM}^{\otimes}$ is *equivalent* to the data contained in the diagram

$$\mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast} \to \mathrm{N}(\boldsymbol{\Delta})^{op};$$

either can be reconstructed from the other, up to equivalence.

We now turn to the proof of Proposition 4.2.2.12.

Remark 4.2.2.25. Let $\gamma : N(\mathbf{\Delta})^{op} \times \Delta^1 \to \mathcal{LM}^{\otimes}$ be as in Remark 4.2.2.8, and let M denote the collection of all morphisms f in $N(\mathbf{\Delta})^{op} \times \Delta^1$ such that $\gamma(f)$ is an inert morphism in $\mathcal{LM}^{\otimes}$. We note that if $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ is a fibration of ∞-operads, then a functor $F \in \mathrm{Fun}_{\mathcal{LM}^{\otimes}}(N(\mathbf{\Delta})^{op} \times \Delta^1, \mathcal{C}^{\otimes})$ belongs to $^{\Delta}\mathrm{LMod}(\mathcal{C}_{\mathfrak{m}})$ if and only if F carries each morphism in M to an inert morphism in $\mathcal{C}^{\otimes}$. The "if" direction is obvious. For the converse, we note that $F \in {}^{\Delta}\mathrm{LMod}(\mathcal{C}_{\mathfrak{m}})$ and let $\alpha : ([m], i) \to ([n], j)$ be a morphism in $N(\mathbf{\Delta})^{op} \times \Delta^1$ be such that $\gamma(\alpha)$ is inert. If $i = j$, then it follows immediately from the definition that $F(\alpha)$ is inert. If $i \neq j$, then α factors as a composition

$$([m], 0) \xrightarrow{\alpha'} ([n], 0) \xrightarrow{\alpha''} ([n], 1),$$

where $F(\alpha')$ and $F(\alpha'')$ are inert, so that $F(\alpha)$ is also inert.

Proof of Proposition 4.2.2.12. Let $q : \mathcal{O}^{\otimes} \to N(\mathcal{F}\mathrm{in}_*)$ be a generalized ∞-operad and let $\mathrm{Fun}'_{N(\mathcal{F}\mathrm{in}_*)}(N(\mathbf{\Delta})^{op} \times \Delta^1, \mathcal{O}^{\otimes})$ be the full subcategory of $\mathrm{Fun}_{N(\mathcal{F}\mathrm{in}_*)}(N(\mathbf{\Delta})^{op} \times \Delta^1, \mathcal{O}^{\otimes})$ spanned by those functors which carry every morphism in M to an inert morphism in $\mathcal{O}^{\otimes}$. We wish to prove that composition with the map $\gamma : N(\mathbf{\Delta})^{op} \times \Delta^1 \to \mathcal{LM}^{\otimes}$ induces an equivalence of ∞-categories $\theta : \mathrm{Alg}_{\mathcal{LM}}(\mathcal{O}) \to \mathrm{Fun}'_{N(\mathcal{F}\mathrm{in}_*)}(N(\mathbf{\Delta})^{op} \times \Delta^1, \mathcal{O}^{\otimes})$. Using Using Propositions 2.3.2.9 and 2.3.2.11, we may assume that q factors as a composition

$$\mathcal{O}^{\otimes} \xrightarrow{q'} \mathcal{C} \times N(\mathcal{F}\mathrm{in}_*) \to N(\mathcal{F}\mathrm{in}_*),$$

where q' exhibits $\mathcal{O}^{\otimes}$ as a $\mathcal{C}$-family of ∞-operads. Let $q'_0 : \mathcal{O}^{\otimes} \to \mathcal{C}$ be the composition of q' with the projection to $\mathcal{C}$.

Let $\mathcal{J}$ denote the categorical mapping cylinder of the functor $\mathbf{\Delta}^{op} \times [1] \to \mathbf{LM}^{\otimes}$ determined by the map γ appearing in Remark 4.2.2.8. More precisely, we can describe $\mathcal{J}$ as follows:

(1) An object of $\mathcal{J}$ is either an object of $\mathbf{\Delta}^{op} \times [1]$ or an object of $\mathbf{LM}$.

(2) Morphisms in $\mathcal{J}$ are give by the formulas

$$\mathrm{Hom}_{\mathcal{J}}(([m], i), ([n], j)) = \mathrm{Hom}_{\mathbf{\Delta}^{op} \times [1]}(([m], i), ([n], j))$$

$$\mathrm{Hom}_{\mathcal{J}}((\langle m \rangle, S), (\langle n \rangle, T)) = \mathrm{Hom}_{\mathbf{LM}^{\otimes}}((\langle m \rangle, S), (\langle n \rangle, T))$$

$$\mathrm{Hom}_{\mathcal{J}}((\langle m \rangle, S), ([n], 0)) = \mathrm{Hom}_{\mathbf{LM}}((\langle m \rangle, S), \mathrm{LCut}([n]))$$

$$\mathrm{Hom}_{\mathcal{J}}((\langle m \rangle, S), ([n], 1)) = \mathrm{Hom}_{\mathbf{LM}}((\langle m \rangle, S), \mathrm{Cut}([n]))$$

$$\mathrm{Hom}_{\mathcal{J}}(([n], i), (\langle m \rangle, T)) = \emptyset.$$

There is a canonical retraction r from $\mathcal{J}$ onto $\mathbf{LM}^{\otimes}$, given on objects of $\mathbf{\Delta}^{op} \times [1]$ by the formula $r([n], 0) = \mathrm{LCut}([n])$, $r([n], 1) = \mathrm{Cut}([n])$.

We let $\mathfrak{X}$ denote the full subcategory of $\mathrm{Fun}_{N(\mathcal{F}\mathrm{in}_*)}(N(\mathcal{J}), \mathcal{O}^{\otimes})$ spanned by those functors $f : N(\mathcal{J}) \to \mathcal{O}^{\otimes}$ which satisfy the following codnitions:

(i) For every object $([n], i) \in \mathbf{\Delta}^{op} \times [1]$, the functor f carries the canonical map $r([n], i) \to ([n], i)$ in $\mathcal{J}$ to an equivalence in $\mathcal{O}^{\otimes}$.

(ii) The restriction $f| \mathcal{LM}^{\otimes}$ is carries inert morphisms in $\mathcal{LM}^{\otimes}$ to inert morphisms in $\mathcal{O}^{\otimes}$.

Note that conditions (i) and (ii) immediately imply the following:

(iii) The restriction $f|(N(\mathbf{\Delta})^{op} \times \Delta^1)$ belongs to $\mathrm{Fun}'_{N(\mathcal{F}\mathrm{in}_*)}(N(\mathbf{\Delta})^{op} \times \Delta^1, \mathcal{O}^{\otimes})$.

Condition (i) is equivalent to the requirement that f is a q-left Kan extension of $f| \mathcal{LM}^{\otimes}$. Since every functor $f_0 \in \mathrm{Fun}_{N(\mathcal{F}\mathrm{in}_*)}(\mathcal{LM}^{\otimes}, \mathcal{O}^{\otimes})$ admits a q-left Kan extension $f \in \mathrm{Fun}_{N(\mathcal{F}\mathrm{in}_*)}(N(\mathcal{J}), \mathcal{O}^{\otimes})$ (given, for example, by $f_0 \circ r$), Proposition T.4.3.2.15 implies that the restriction map $p : \mathfrak{X} \to \mathrm{Alg}_{\mathcal{LM}}(\mathcal{O})$ is a trivial Kan fibration. The map θ is the composition of a section to p (given by composition with r) with the restriction map $p' : \mathfrak{X} \to \mathrm{Fun}'_{N(\mathcal{F}\mathrm{in}_*)}(N(\mathbf{\Delta})^{op} \times \Delta^1, \mathcal{O}^{\otimes})$. It will therefore suffice to show that p' is a trivialKan fibration. In view of Proposition T.4.3.2.15, this will follow from the following pair of assertions:

(a) Every $f_0 \in \mathrm{Fun}'_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1, \mathcal{O}^{\otimes})$ admits a q-right Kan extension $f \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N}(\mathcal{I}), \mathcal{O}^{\otimes})$.

(b) Given $f \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N}(\mathcal{I}), \mathcal{O}^{\otimes})$ satisfying (iii), the functor f is a q-right Kan extension of $f_0 = f|(\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1)$ if and only if it also satisfies conditions (i) and (ii).

We first prove (a). Fix $f_0 \in \mathrm{Fun}'_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1, \mathcal{O}^{\otimes})$. Then $q'_0 \circ f_0$ carries every morphism of M to an equivalence in $\mathcal{C}$. Using the two-out-of-three property, we deduce that $q'_0 \circ f_0$ carries every morphism in $\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1$ to an equivalence in $\mathcal{C}$. Since $\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1$ is a weakly contractible simplicial set, we may assume without loss of generality that $q'_0 \circ f_0$ factors as a composition

$$\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1 \to \{C\} \hookrightarrow \mathcal{C}$$

for some object $C \in \mathcal{C}$; that is, we may regard f_0 as a functor $\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1 \to \mathcal{O}^{\otimes}_C$, where $\mathcal{O}^{\otimes}_C$ denotes the ∞-operad $\mathcal{O}^{\otimes} \times_{\mathcal{C}} \{C\}$. Let $q_C : \mathcal{O}^{\otimes}_C \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ denote the restriction of q to $\mathcal{O}^{\otimes}_C$.

Fix an object $(\langle n \rangle, S) \in \mathcal{L}\mathcal{M}^{\otimes}$. Let $\mathcal{J}$ denote the category $(\mathbf{\Delta}^{op} \times [1]) \times_{\mathcal{I}} (\mathcal{I})_{(\langle n \rangle, S)/}$, and let g denote the composition $\mathrm{N}(\mathcal{J}) \to \mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1 \xrightarrow{f_0} \mathcal{O}^{\otimes}$. According to Lemma T.4.3.2.13, it will suffice to show that g admits a q-limit in $\mathcal{O}^{\otimes}$ (compatible with the evident map $\mathrm{N}(\mathcal{J}^{\triangleleft}) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*))$. The objects of $\mathcal{J}$ can be identified with morphisms $\alpha : (\langle n \rangle, S) \to r([m], i)$ in $\mathcal{L}\mathcal{M}^{\otimes}$. Let $\mathcal{J}_0 \subseteq \mathcal{J}$ denote the full subcategory spanned by those objects for which α is inert. The inclusion $\mathcal{J}_0 \subseteq \mathcal{J}$ has a right adjoint and therefore induces a right cofinal map $\mathrm{N}(\mathcal{J}_0) \to \mathrm{N}(\mathcal{J})$. Consequently, it will suffice to show that $g_0 = g|\mathrm{N}(\mathcal{J}_0)$ admits a q-limit in $\mathcal{O}^{\otimes}$ (compatible with the evident map $\mathrm{N}(\mathcal{J}_0)^{\triangleleft} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*))$.

Let $\mathcal{J}_1$ denote the full subcategory of $\mathcal{J}_0$ spanned by the morphism which are either of the form $\rho^j : (\langle n \rangle, S) \to r([0], 0)$ where $j \in S$, $\rho^j : (\langle n \rangle, S) \to r([1], 1)$ when $j \notin S$, or $(\langle n \rangle, S) \to r([0], 1) \simeq (\langle 0 \rangle, \emptyset)$. Let $g_1 = g_0|\mathrm{N}(\mathcal{J}_1)$. We note that $\mathrm{N}(\mathcal{J}_1)$ is weakly contractible. Since $f_0 \in \mathrm{Fun}'_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1, \mathcal{O}^{\otimes})$, g_1 carries every morphism in $\mathrm{N}(\mathcal{J}_1)$ to an inert morphism in $\mathcal{O}^{\otimes}$. We first show that g_1 can be extended to a q_C-limit diagram in $\mathcal{O}^{\otimes}_C$ (compatible with the map $\mathrm{N}(\mathcal{J}_1)^{\triangleleft} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

Let $\mathcal{J}_2$ be the full subcategory of $\mathcal{J}_1$ obtained by removing objects of the form $(\langle n \rangle, S) \to r([0], 1)$. Note that $\mathcal{J}_2$ is isomorphic to the discrete category with object set $\{1, 2, \ldots, n\}$. Since $\mathcal{O}^{\otimes}_C$ is an ∞-operad, the functor $g_2 = g_1|\mathrm{N}(\mathcal{J}_2)$ can be extended to a q_C-limit diagram $\bar{g}_2$ (lying over the evident map $\mathrm{N}(\mathcal{J}_2)^{\triangleleft} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*))$. Since q' exhibits $\mathcal{O}^{\otimes}$ as a $\mathcal{C}$-family of ∞-operads, $\bar{g}_2$ is also a q'-limit diagram. Note that g_1 is a q_C-right Kan extension of g_2, so (by Lemma T.4.3.2.7), we can extend g_2 to a q_C-limit diagram $\bar{g}_2 : \mathrm{N}(\mathcal{J}_2)^{\triangleleft} \to \mathcal{O}^{\otimes}_C$. Since g_1 is also a q'-right Kan extension of g_2, $\bar{g}_2$ is also a q'-limit diagram. We claim that $\bar{g}_2$ is also a q-limit diagram. In view of Proposition T.4.3.1.5, it will suffice to show that $q' \circ \bar{g}_2$ is a q''-limit diagram in $\mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Equivalently, we must show that the constant functor $q'_0 \circ \bar{g}_2$ is a limit diagram in $\mathcal{C}$, which follows from the fact that the simplicial set $\mathrm{N}(\mathcal{J}_1)$ is weakly contractible (Corollary T.4.4.4.10). This completes the proof of (a). Moreover, the proof shows that an extension $f : \mathrm{N}(\mathcal{I}) \to \mathcal{O}^{\otimes}$ is a q-right Kan extension of f_0 if and only if it satisfies the following condition:

(i') For every object $(\langle n \rangle, S) \in \mathcal{L}\mathcal{M}^{\otimes}$ as above and every morphism $\alpha : (\langle n \rangle, S) \to ([m], i)$ in $\mathcal{I}$ belonging to $\mathcal{J}_1$, the morphism $f(\alpha)$ is inert in $\mathcal{O}^{\otimes}$.

To prove (b), it will suffice to show that if $f \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N}(\mathcal{I}), \mathcal{O}^{\otimes})$ satisfies condition (iii), then it satisfies conditions (i) and (ii) if and only if it satisfies condition (i'). We first prove the "only if" direction. Assume that $f \in \mathcal{X}$, and let $\alpha : (\langle n \rangle, S) \to ([m], i)$ be as in (i'). Then α factors as a composition

$$(\langle n \rangle, S) \xrightarrow{\alpha'} r([m], i) \xrightarrow{\alpha''} ([m], i),$$

where α' is inert (so that $f(\alpha')$ is inert by (ii)) and $f(\alpha'')$ is an equivalence by virtue of (i).

Suppose now that f satisfies (i') and (iii). We first show that f satisfies (i). Fix an object $([n], i)$ in $\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1$; we wish to show that f carries the canonical map $\alpha : r([n], i) \to ([n], i)$ to an equivalence in $\mathcal{O}^{\otimes}$. For $1 \leq j \leq n$, let $\beta j : ([n], i) \to ([1], 1)$ denote the be the map carrying $[1]$ to the interval $\{j-1, j\} \subseteq [n]$, and if $i = 0$ let $\beta_0 : ([n], i) \to ([0], 0)$ be the map induced by the inclusion $[0] \simeq \{n\} \hookrightarrow [n]$. Condition (iii)

guarantees that the maps $\{f(\beta_j)\}_{i \leq j \leq n}$ determine a q'-product diagram in $\mathcal{O}^{\otimes}$, and condition (i') guarantees that the composite maps $\{f(\beta_j \circ \alpha)\}_{i \leq j \leq n}$ also determine a q'-product diagram in $\mathcal{O}^{\otimes}$. It follows from the uniqueness of relative limits that $f(\alpha)$ is an equivalence in $\mathcal{O}^{\otimes}$, as desired.

It remains to show that f satisfies (ii). For every object $(\langle n \rangle, S) \in \mathcal{LM}^{\otimes}$, we have a commutative diagram

$$
\begin{array}{ccc}
 & (\langle n \rangle, S) & \\
 {}^{\alpha}\swarrow & & \searrow{}^{\alpha''} \\
(\langle 0 \rangle, \emptyset) & \xrightarrow{\ \ \alpha' \ \ } & ([0], 1).
\end{array}
$$

Condition (i') implies that $(q'_0 \circ f)(\alpha'')$ and $(q' \circ f)(\alpha')$ are equivalences in $\mathcal{C}$, so that $(q'_0 \circ f)(\alpha)$ is an equivalence in $\mathcal{C}$. It follows that $q'_0 \circ f$ carries every morphism in $\mathcal{LM}^{\otimes}$ to an equivalence in $\mathcal{C}$. The simplicial set $\mathcal{LM}^{\otimes}$ is weakly contractible (it has a final object, given by $(\langle 0 \rangle, \emptyset)$); we may therefore assume without loss of generality that $f | \mathcal{LM}^{\otimes}$ is the constant map $\mathcal{LM}^{\otimes} \to \{C\} \hookrightarrow \mathcal{C}$, for some $C \in \mathcal{C}$. We wish to show that $f | \mathcal{LM}^{\otimes}$ defines a map of ∞-operads $\mathcal{LM}^{\otimes} \to \mathcal{O}_C^{\otimes}$. By virtue of Remark 2.1.2.9, it will suffice to show that $f(\alpha)$ is inert whenever α is an inert morphism of $\mathcal{LM}^{\otimes}$ of the form $(\langle n \rangle, S) \to (\langle 1 \rangle, T)$. Write $(\langle 1 \rangle, T) = r([i], i)$ for $i \in \{0, 1\}$. Since the map $f(\langle 1 \rangle, T) \to f([i], i)$ is an equivalence by (i), it will suffice to show that the composite map $f(\langle n \rangle, S) \to f(\langle 1 \rangle, T) \to f([i], i)$ is inert, which follows immediately from (i'). $\qquad\square$

4.2.3 Limits and Colimits of Modules

Let A be a ring and suppose we are given a collection of A-modules $\{M_i\}_{i \in I}$. Then the abelian groups

$$
\prod_{i \in I} M_i \qquad \bigoplus_{i \in I} M_i
$$

inherit A-module structures. Moreover, the resulting A-modules can be identified with the product and coproduct of the family $\{M_i\}_{i \in I}$ in the category of A-modules. We can summarize the situation as follows: the category of A-modules admits products and coproducts, and the forgetful functor U from the category of A-modules to the category of abelian groups preserves products and coproducts. In fact, U preserves all limits and colimits.

Our goal in this section is to show that this is a general phenomenon. Let A be an algebra object of an arbitrary monoidal ∞-category $\mathcal{C}$, and let $\mathcal{M}$ be an ∞-category which is left-tensored over $\mathcal{C}$. We can summarize our results as follows:

(1) Let K be a simplicial set such that $\mathcal{M}$ admits K-indexed limits. Then $\mathrm{LMod}_A(\mathcal{M})$ admits K-indexed limits, and the forgetful functor $\mathrm{LMod}_A(\mathcal{M}) \to \mathcal{M}$ preserves K-indexed limits (Corollary 4.2.3.3).

(2) Let K be a simplicial set such that $\mathcal{M}$ admits K-indexed colimits and the tensor product functor $M \mapsto A \otimes M$ preserves K-indexed colimits. Then $\mathrm{LMod}_A(\mathcal{M})$ admits K-indexed colimits, and the forgetful functor $\mathrm{LMod}_A(\mathcal{M}) \to \mathcal{M}$ preserves K-indexed colimits (Corollary 4.2.3.5).

We begin with the study of limits in $\mathrm{LMod}_A(\mathcal{M})$. In fact, we will work a little bit more generally, and consider *relative* limits with respect to the forgetful functor $\mathrm{LMod}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{C})$ (we refer the reader to §T.4.3 for a discussion of relative limits in general). We have the following general result:

Proposition 4.2.3.1. *Let $\mathcal{C}$ be a monoidal ∞-category and let $\mathcal{M}$ be an ∞-category which is left-tensored over $\mathcal{C}$. Let K be a simplicial set such that $\mathcal{M}$ admits K-indexed limits, and let $\theta : \mathrm{LMod}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{C})$ be the forgetful functor. Then:*

(1) *For every diagram*

$$
\begin{array}{ccc}
K & \longrightarrow & \mathrm{LMod}(\mathcal{M}) \\
\downarrow & \nearrow & \downarrow{\scriptstyle\theta} \\
K^{\triangleleft} & \longrightarrow & \mathrm{Alg}(\mathcal{C})
\end{array}
$$

  there exists a dotted arrow as indicated, which is a θ-limit diagram.

(2) *An arbitrary map $\overline{g} : K^{\triangleleft} \to \mathrm{LMod}(\mathcal{M})$ is a θ-limit diagram if and only if the induced map $K^{\triangleleft} \to \mathcal{M}$ is a limit diagram.*

Proof. Let $\theta' : {}^{\triangle}\mathrm{LMod}(\mathcal{M}) \to {}^{\triangle}\mathrm{Alg}(\mathcal{C})$ be the forgetful functor. By virtue of Propositions 4.1.2.15 and 4.2.2.11, it will suffice to prove the following analogues of (1) and (2):

(1′) For every diagram

$$
\begin{array}{ccc}
K & \longrightarrow & {}^{\triangle}\mathrm{LMod}(\mathcal{M}) \\
\downarrow & \nearrow & \downarrow{\scriptstyle\theta'} \\
K^{\triangleleft} & \longrightarrow & {}^{\triangle}\mathrm{Alg}(\mathcal{C})
\end{array}
$$

  there exists a dotted arrow as indicated, which is a θ'-limit diagram.

(2′) An arbitrary map $\overline{g} : K^{\triangleleft} \to {}^{\triangle}\mathrm{LMod}(\mathcal{M})$ is a θ'-limit diagram if and only if the induced map $K^{\triangleleft} \to \mathcal{M}$ is a limit diagram.

Let $\mathcal{M}^{\circledast} \xrightarrow{q} \mathcal{C}^{\circledast} \xrightarrow{p} \mathrm{N}(\boldsymbol{\Delta})^{op}$ be as in Notation 4.2.2.16. For each $n \geq 0$, the equivalence $\mathcal{M}^{\circledast}_{[n]} \simeq \mathcal{C}^{\circledast}_{[n]} \times \mathcal{M}$ implies the following:

($1'_n$) For every diagram

$$
\begin{array}{ccc}
K & \longrightarrow & \mathcal{M}^{\circledast}_{[n]} \\
\downarrow & \nearrow & \downarrow{\scriptstyle q_{[n]}} \\
K^{\triangleleft} & \longrightarrow & \mathcal{C}^{\circledast}_{[n]}
\end{array}
$$

  there exists a dotted arrow as indicated, which is a $q_{[n]}$-limit diagram.

($2'_n$) An arbitrary diagram $K^{\triangleleft} \to \mathcal{M}^{\circledast}_{[n]}$ is a $q_{[n]}$-limit diagram if and only the composite map $K^{\triangleleft} \to \mathcal{M}^{\otimes}_{[n]} \to \mathcal{M}$ is a limit diagram.

Combining this observation with Corollary T.4.3.1.15, we deduce:

($1''_n$) For every diagram

$$
\begin{array}{ccc}
K & \longrightarrow & \mathcal{M}^{\circledast}_{[n]} \\
\downarrow & \nearrow & \downarrow{\scriptstyle q_{[n]}} \\
K^{\triangleleft} & \longrightarrow & \mathcal{C}^{\circledast}_{[n]}
\end{array}
$$

  there exists a dotted arrow as indicated, which is a q-limit diagram.

($2''_n$) An arbitrary diagram $K^{\triangleleft} \to \mathcal{M}^{\circledast}_{[n]}$ is a q-limit diagram if and only the composition

$$
K^{\triangleleft} \to \mathcal{M}^{\circledast}_{[n]} \to \mathcal{M}
$$

  is a limit diagram.

Let $^{\Delta}\mathrm{LMod}'(\mathcal{M})$ be the full subcategory of $\mathrm{Map}_{\mathrm{N}(\mathbf{\Delta})^{op}}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{M}^{\circledast})$ spanned by those objects whose image in $\mathrm{Fun}_{\mathrm{N}(\mathbf{\Delta})^{op}}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{C}^{\circledast})$ belongs to $^{\Delta}\mathrm{Alg}(\mathcal{C})$. Combining $(1''_n)$, $(2''_n)$, and Lemma 3.2.2.9, we deduce:

$(1'')$ For every diagram

$$
\begin{array}{ccc}
K & \longrightarrow & ^{\Delta}\mathrm{LMod}'(\mathcal{M}) \\
\downarrow & \nearrow & \downarrow{\scriptstyle\theta''} \\
K^{\triangleleft} & \longrightarrow & ^{\Delta}\mathrm{Alg}(\mathcal{C})
\end{array}
$$

there exists a dotted arrow as indicated, which is a θ''-limit diagram.

$(2'')$ An arbitrary map $\bar{p} : K^{\triangleleft} \to {}^{\Delta}\mathrm{LMod}'(\mathcal{M})$ is a θ'-limit diagram if and only if, for every $n \geq 0$, the composite map $K^{\triangleleft} \to \mathcal{M}_{[n]}^{\otimes} \to \mathcal{M}$ is a limit diagram.

To deduce $(1')$ from these assertions, it will suffice to show that if if $\bar{g} : K^{\triangleright} \to {}^{\Delta}\mathrm{LMod}'(\mathcal{M})$ satisfies the hypothesis of $(2'')$ and $g = \bar{g}|K$ factors through $^{\Delta}\mathrm{LMod}(\mathcal{M})$, then $\bar{g}$ factors through $^{\Delta}\mathrm{LMod}(\mathcal{M})$. Let $f : [m] \to [n]$ be an inert morphism in $\mathbf{\Delta}$ such that $f(m) = n$. Then f induces a natural transformation $\bar{g}_{[n]} \to \bar{g}_{[m]}$ of functors $K^{\triangleleft} \to \mathcal{M}^{\circledast}$. We wish to show that this natural transformation is $(p \circ q)$-coCartesian. Since $f(m) = n$, we have a homotopy commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}_{[n]}^{\otimes} & \longrightarrow & \mathcal{M}_{[m]}^{\otimes} \\
{\scriptstyle\alpha}\searrow & & \swarrow{\scriptstyle\beta} \\
& \mathcal{M}, &
\end{array}
$$

and it suffices to show that the associated transformation $\bar{t} : \alpha \circ \bar{g}_{[n]} \to \beta \circ \bar{g}_{[m]}$ is an equivalence. Our hypothesis implies that $\bar{t}$ restricts to an equivalence $t : \alpha \circ g_{[n]} \to \beta \circ g_{[m]}$. Since $\bar{g}$ satisfies $(2'')$, the maps $\alpha \circ \bar{g}_{[n]}$ and $\beta \circ \bar{g}_{[m]}$ are both limit diagrams in $\mathcal{M}$. It follows that $\bar{t}$ is an equivalence as well, as desired.

We now complete the proof by observing that if $\bar{g} : K^{\triangleleft} \to {}^{\Delta}\mathrm{LMod}'(\mathcal{M})$ factors through $^{\Delta}\mathrm{LMod}(\mathcal{M})$, then the criteria of $(2')$ and $(2'')$ are equivalent. $\square$

Corollary 4.2.3.2. *Let $\mathcal{C}$ be a monoidal ∞-category, $\mathcal{M}$ an ∞-category which is left-tensored over $\mathcal{C}$, and $\theta : \mathrm{LMod}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{C})$ the forgetful functor. Then θ is a Cartesian fibration. Moreover, a morphism f in $\mathrm{LMod}(\mathcal{M})$ is θ-Cartesian if and only if the image of f in $\mathcal{M}$ is an equivalence.*

Proof. Apply Proposition 4.2.3.1 in the case $K = \Delta^0$. $\square$

Corollary 4.2.3.3. *Let $\mathcal{C}$ be a monoidal ∞-category, $\mathcal{M}$ an ∞-category which is left-tensored over $\mathcal{C}$, and $\theta : \mathrm{LMod}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{C})$ the forgetful functor. Let A be an algebra object of $\mathcal{C}$. Let K be a simplicial set such that $\mathcal{M}$ admits K-indexed limits. Then:*

(1) *The ∞-category $\mathrm{LMod}_A(\mathcal{M})$ admits K-indexed limits.*

(2) *A map $\bar{p} : K^{\triangleleft} \to \mathrm{LMod}_A(\mathcal{M})$ is a limit diagram if and only if the induced map $K^{\triangleleft} \to \mathcal{M}$ is a limit diagram.*

(3) *Given a morphism $\phi : B \to A$ of algebra objects of $\mathcal{A}$, the induced functor $\mathrm{LMod}_A(\mathcal{M}) \to \mathrm{LMod}_B(\mathcal{M})$ preserves K-indexed limits.*

We now turn to the problem of constructing colimits in ∞-categories of modules. We begin with the following very general principle:

Proposition 4.2.3.4. *Let $\mathcal{A} \subseteq \widehat{\mathrm{Cat}}_\infty$ be a subcategory of the ∞-category of (not necessarily small) ∞-categories. Assume that $\mathcal{A}$ has the following properties:*

(a) *The ∞-category $\mathcal{A}$ admits small limits, and the inclusion $\mathcal{A} \subseteq \widehat{\mathrm{Cat}}_\infty$ preserves small limits.*

(b) *If X belongs to $\mathcal{A}$, then $\mathrm{Fun}(\Delta^1, X)$ belongs to $\mathcal{A}$.*

(c) *If X and Y belong to $\mathcal{C}$, then a functor $X \to \mathrm{Fun}(\Delta^1, Y)$ is a morphism in $\mathcal{A}$ if and only if, for every vertex v of Δ^1, the composite functor $X \to \mathrm{Fun}(\Delta^1, Y) \to \mathrm{Fun}(\{v\}, Y) \simeq Y$ is a morphism of $\mathcal{A}$.*

Let $\mathcal{C}$ be a monoidal ∞-category, A an algebra object of $\mathcal{C}$, $\mathcal{M}$ an ∞-category which is left-tensored over $\mathcal{C}$. Suppose $\mathcal{M}$ is an object of $\mathcal{A}$, and that the functor $A \otimes \bullet : \mathcal{M} \to \mathcal{M}$ is a morphism of $\mathcal{A}$. Then:

(1) *The ∞-category $\mathrm{Mod}_A(\mathcal{M})$ is an object of $\mathcal{A}$.*

(2) *For every ∞-category $\mathcal{N}$ belonging to $\mathcal{A}$, a functor $\mathcal{N} \to \mathrm{Mod}_A(\mathcal{M})$ is a morphism in $\mathcal{A}$ if and only if the composite functor $\mathcal{N} \to \mathrm{Mod}_A(\mathcal{M}) \to \mathcal{M}$ is a morphism in $\mathcal{A}$.*

In particular, the forgetful functor $\mathrm{Mod}_A(\mathcal{M}) \to \mathcal{M}$ is a morphism in $\mathcal{A}$.

Proof. Let $p : \mathcal{M}^\otimes \to \mathcal{C}^\otimes$ exhibit $\mathcal{M}$ as left-tensored over $\mathcal{C}$. Form a pullback diagram

$$
\begin{array}{ccc}
X & \longrightarrow & \mathcal{M}^\otimes \\
\Big\downarrow{\scriptstyle p'} & & \Big\downarrow{\scriptstyle p} \\
\mathrm{N}(\mathbf{\Delta})^{op} & \xrightarrow{\ A\ } & \mathcal{C}^\otimes .
\end{array}
$$

We observe that p' is a locally coCartesian fibration (Lemma 4.2.2.19), each fiber of p' is equivalent to $\mathcal{M}$, and each of the associated functors can be identified with an iterate of the functor $A \otimes \bullet : \mathcal{M} \to \mathcal{M}$. It follows from Proposition T.5.4.7.11 that $^\Delta \mathrm{LMod}_A(\mathcal{M}) \in \mathcal{A}$, so that $\mathrm{LMod}_A(\mathcal{M}) \in \mathcal{A}$ by Corollary 4.2.2.15.

Now suppose that $f : \mathcal{N} \to \mathrm{Mod}_A(\mathcal{M})$ is as in (2). Using Proposition T.5.4.7.11 and Corollary 4.2.2.15, we conclude that f is a morphism of $\mathcal{A}$ if and only if, for every $n \geq 0$, the composite map $\mathcal{N} \to \mathrm{Mod}_A(\mathcal{M}) \to X_{[n]}$ belongs to $\mathcal{A}$. We complete the proof by observing that each of the functors $\mathrm{Mod}_A(\mathcal{M}) \to X_{[n]}$ is equivalent to the forgetful functor $\mathrm{Mod}_A(\mathcal{M}) \to \mathcal{M}$. $\qquad\square$

Corollary 4.2.3.5. *Let $\mathcal{C}$ be a monoidal ∞-category, let $\mathcal{M}$ be an ∞-category which is left-tensored over $\mathcal{C}$. Let K be a simplicial set such that $\mathcal{M}$ admits K-indexed colimits. Let $A \in \mathrm{Alg}(\mathcal{C})$, and suppose that the functor $A \otimes \bullet : \mathcal{M} \to \mathcal{M}$ preserves K-indexed colimits. Then:*

(1) *Every diagram $f : K \to \mathrm{LMod}_A(\mathcal{M})$ has a colimit.*

(2) *An arbitrary diagram $\overline{f} : K^\triangleright \to \mathrm{LMod}_A(\mathcal{M})$ is a colimit diagram if and only if it induces a colimit diagram $K^\triangleright \to \mathcal{M}$.*

Proof. We will abuse notation by identifying A with its image in $^\Delta\mathrm{Alg}(\mathcal{C})$. In view of Corollary 4.2.2.15, it will suffice to prove the following analogues of (1) and (2):

(1′) *Every diagram $f : K \to {}^\Delta\mathrm{LMod}_A(\mathcal{M})$ has a colimit.*

(2′) *An arbitrary diagram $\overline{f} : K^\triangleright \to {}^\Delta\mathrm{LMod}_A(\mathcal{M})$ is a colimit diagram if and only if it induces a colimit diagram $K^\triangleright \to \mathcal{M}$.*

Let $q : \mathcal{M}^\circledast \to \mathcal{C}^\circledast$ be as in Notation 4.2.2.16, regard A as a map $\mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{C}^\circledast$, and let $\mathcal{X}$ denote the fiber product $\mathcal{M}^\circledast \times_{\mathcal{C}^\circledast} \mathrm{N}(\mathbf{\Delta})^{op}$, so that $^\Delta\mathrm{LMod}_A(\mathcal{M})$ can be identified with a full subcategory of the ∞-category $\mathrm{Fun}_{\mathrm{N}(\mathbf{\Delta})^{op}}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{X})$ of sections of the induced map $q' : \mathcal{X} \to \mathrm{N}(\mathbf{\Delta})^{op}$. Note that q' is a locally coCartesian fibration whose fibers are equivalent to $\mathcal{M}$ (see Lemma 4.2.2.19 and Remark 4.2.2.20).

Let $\mathcal{A} \subseteq \widehat{\mathrm{Cat}}_\infty$ be the subcategory whose objects are ∞-categories which admit K-indexed colimits, and whose morphisms are functors which preserve K-indexed colimits. Applying Proposition 4.2.3.4, we conclude that $^\Delta\mathrm{LMod}_A(\mathcal{M})$ belongs to $\mathcal{A}$, and that the forgetful functor $^\Delta\mathrm{LMod}_A(\mathcal{M}) \to \mathcal{M}$ is a morphism of $\mathcal{A}$. This proves (1), and the "only if" direction of (2). The "if" direction then follows formally, since the forgetful functor $^\Delta\mathrm{LMod}_A(\mathcal{M}) \to \mathcal{M}$ is conservative (Corollary 4.2.3.2). $\qquad\square$

Remark 4.2.3.6. In the situation of Corollary 4.2.3.5, any colimit diagram $\overline{f} : K^\triangleright \to \mathrm{LMod}_A(\mathcal{M})$ is also a q-colimit diagram, where q denotes the projection $\mathrm{LMod}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{C})$. This follows immediately from Corollary T.4.3.1.16, since q is a Cartesian fibration (Corollay 4.2.3.2).

Corollary 4.2.3.7. *Let $\mathcal{C}$ be an ∞-category equipped with a monoidal structure and let $\mathcal{M}$ be an ∞-category which is left-tensored over $\mathcal{C}$. Suppose that $\mathcal{M}$ is presentable and that, for each $C \in \mathcal{C}$, the functor $C \otimes \bullet :$ $\mathcal{M} \to \mathcal{M}$ preserves small colimits. Then:*

(1) *For every $A \in \mathrm{Alg}(\mathcal{C})$, the ∞-category $\mathrm{LMod}_A(\mathcal{M})$ is presentable.*

(2) *For every morphism $A \to B$ of algebra objects of $\mathcal{C}$, the associated functor $\mathrm{LMod}_B(\mathcal{M}) \to \mathrm{LMod}_A(\mathcal{M})$ preserves small limits and colimits.*

(3) *The forgetful functor $\theta : \mathrm{LMod}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{C})$ is a presentable fibration (Definition T.5.5.3.2).*

Proof. Assertion (1) follows from Proposition 4.2.3.4 as in the proof of Corollary 4.2.3.5. To prove (2), we observe that Corollary 4.2.3.2 implies that the diagram

$$
\begin{array}{ccc}
\mathrm{LMod}_B(\mathcal{M}) & \longrightarrow & \mathrm{LMod}_A(\mathcal{M}) \\
& \searrow \quad \swarrow & \\
& \mathcal{M} &
\end{array}
$$

commutes up to homotopy. Assertion (2) then follows immediately from Corollaries 4.2.3.3 and 4.2.3.5. Assertion (3) follows from (1) and (2), by virtue of Proposition T.5.5.3.3. $\qquad\square$

Remark 4.2.3.8. Under the hypotheses of Corollary 4.2.3.7, if $A \to B$ is a morphism of algebra objects of $\mathcal{C}$. then the forgetful functor $\psi : \mathrm{LMod}_B(\mathcal{M}) \to \mathrm{LMod}_A(\mathcal{M})$ admits both left and right adjoints (Corollary T.5.5.2.9). In §4.3.6 we will prove the existence of a left adjoint to ψ under much weaker assumptions (Proposition 4.3.7.14).

4.2.4 Free Modules

Let A be an associative ring and M_0 an abelian group. The tensor product $M = A \otimes M_0$ has the structure of a left A-module, given by the map

$$A \otimes M \simeq A \otimes (A \otimes M_0) \simeq (A \otimes A) \otimes M_0 \to A \otimes M_0 \simeq M.$$

Moreover, M is characterized up to isomorphism by the condition that it is the *free* left A-module generated by M_0. More precisely, there exists a map of abelian groups $\phi : M_0 \to M$ with the following universal property: for every left A-module M, composition with ϕ induces a bijection $\mathrm{Hom}_A(M, N) \to \mathrm{Hom}(M_0, N)$, where $\mathrm{Hom}_A(M, N)$ denotes the set of A-module homomorphisms from M to N and $\mathrm{Hom}(M_0, N)$ the set of abelian group homomorphisms from M_0 to N.

Our first goal in this section is to generalize the above discussion to the ∞-categorical setting. We begin with a point of notation: if $\mathcal{C}^\otimes$ is a planar ∞-operad and $\mathcal{M}$ is an ∞-category weakly enriched over $\mathcal{C}^\otimes$, then evaluation on the object $\mathfrak{m} \in \mathcal{LM}$ determines a forgetful functor $\theta : \mathrm{LMod}_A(\mathcal{M}) \to \mathcal{M}$, for each $A \in \mathrm{Alg}(\mathcal{C})$. We will generally abuse notation by not distinguishing between an object $M \in \mathrm{LMod}_A(\mathcal{M})$ and its image $\theta(M) \in \mathcal{M}$.

Definition 4.2.4.1. Let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category and let $\mathcal{M}$ be an ∞-category left-tensored over $\mathcal{C}^{\otimes}$. Suppose we are given an algebra object $A \in \mathrm{Alg}(\mathcal{C})$, a left A-module $M \in \mathrm{LMod}_A(\mathcal{M})$, and a morphism $\lambda : M_0 \to M$ in $\mathcal{M}$. We will say that λ *exhibits M as a free left A-module generated by M_0* if the induced map

$$A \otimes M_0 \xrightarrow{\mathrm{id} \otimes \lambda} A \otimes M \to M$$

is an equivalence in $\mathcal{M}$.

The main result of this section can be stated as follows:

Proposition 4.2.4.2. *Let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category and $\mathcal{M}$ an ∞-category which is left-tensored over $\mathcal{C}^{\otimes}$. Suppose further that we are given objects $A \in \mathrm{Alg}(\mathcal{C})$, $M_0 \in \mathcal{M}$. Then:*

(1) *There exists an object $M \in \mathrm{LMod}_A(\mathcal{M})$ and a morphism $\lambda : M_0 \to M$ in $\mathcal{M}$ which exhibits M as a free left A-module generated by M_0.*

(2) *Let $M \in \mathrm{LMod}_A(\mathcal{M})$ and let $\lambda : M_0 \to M$ be a morphism which exhibits M as a free left A-module generated by M_0. For every pair of objects $B \in \mathrm{Alg}(\mathcal{C})$, $N \in \mathrm{LMod}_B(\mathcal{M})$, composition with λ induces a homotopy equivalence*

$$\mathrm{Map}_{\mathrm{LMod}(\mathcal{M})}((A, M), (B, N)) \to \mathrm{Map}_{\mathrm{Alg}(\mathcal{C})}(A, B) \times \mathrm{Map}_{\mathcal{M}}(M_0, N).$$

Proof. The action of $\mathcal{C}^{\otimes}$ on $\mathcal{M}$ is encoded by a coCartesian fibration of ∞-operads $\mathcal{O}^{\otimes} \to \mathcal{LM}^{\otimes}$ such that $\mathcal{O}_{\mathfrak{a}}^{\otimes} \simeq \mathcal{C}^{\otimes}$ and $\mathcal{O}_{\mathfrak{m}} \simeq \mathcal{M}$. Let $\mathcal{LM}_0^{\otimes}$ be the subcategory of $\mathcal{LM}^{\otimes}$ spanned by all objects, together with those morphisms $\alpha : (\langle n \rangle, S) \to (\langle n' \rangle, S')$ such that $\alpha^{-1}(S \cup \{*\}) = S' \cup \{*\}$. The inclusion $\mathcal{Ass}^{\otimes} \hookrightarrow \mathcal{LM}^{\otimes}$ of Remark 4.2.1.10 extends to an isomorphism $\mathcal{Ass}^{\otimes} \boxplus \mathcal{Triv}^{\otimes} \simeq \mathcal{LM}_0^{\otimes}$, which carries the unique object of $\mathcal{Triv}$ to $\mathfrak{m}$. Using Theorem 2.2.3.6 and Example 2.1.3.5, we deduce that the forgetful functor $\mathrm{Alg}_{\mathcal{LM}_0 / \mathcal{LM}}(\mathcal{O}) \to \mathrm{Alg}(\mathcal{C}) \times \mathcal{M}$ is an equivalence of ∞-categories.

The ∞-category $\mathcal{LM}_{0,\mathrm{act}}^{\otimes} \times_{\mathcal{LM}_{\mathrm{act}}^{\otimes}} (\mathcal{LM}_{\mathrm{act}}^{\otimes})_{/\mathfrak{a}}$ contains $\mathrm{id} : \mathfrak{a} \to \mathfrak{a}$ as a final object, and the ∞-category $\mathcal{LM}_{0,\mathrm{act}}^{\otimes} \times_{\mathcal{LM}_{\mathrm{act}}^{\otimes}} (\mathcal{LM}_{\mathrm{act}}^{\otimes})_{/\mathfrak{m}}$ contains the unique active map $(\langle 2 \rangle, \{1\}) \to \mathfrak{m}$ as a final object. Note that if $X \in \mathrm{Alg}_{\mathcal{LM}_0 / \mathcal{LM}}(\mathcal{O})$ corresponds to a pair $(A, M_0) \in \mathrm{Alg}(\mathcal{C}) \times \mathcal{M}$ and $X' \in \mathrm{Alg}_{/ \mathcal{LM}}(\mathcal{O})$ to a pair (A', M), then a natural transformation $\gamma : X \to X' | \mathcal{LM}_0^{\otimes}$ exhibits X' as the free $\mathcal{LM}$-algebra object generated by X (in the sense of Definition 3.1.3.1) if and only if γ induces an equivalence $A \simeq A'$ in $\mathrm{Alg}(\mathcal{C})$ and a map $M_0 \to M$ in $\mathcal{M}$ which exhibits M as the free left A'-module generated by M_0 (in the sense of Definition 4.2.4.1). Assertion (1) now follows immediately from Proposition 3.1.3.3 and assertion (2) from Proposition 3.1.3.2. $\qquad \square$

From Proposition 4.2.4.2, we can immediately deduce the following uniqueness result for free modules:

Corollary 4.2.4.3. *Let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category containing an algebra object A, and let $\mathcal{M}$ be an ∞-category left-tensored over $\mathcal{C}^{\otimes}$ containing an object M_0. Suppose we are given objects $M, M' \in \mathrm{LMod}_A(\mathcal{M})$, together with morphisms $\lambda : M_0 \to M$, $\lambda' : M_0 \to M'$ in $\mathcal{M}$ which exhibit M and M' as free left A-modules generated by M_0. Then there exists a morphism $\phi : M \to M'$ in $\mathrm{LMod}_A(\mathcal{M})$ which extends to a commutative diagram*

$$
\begin{array}{ccc}
 & M_0 & \\
{\lambda} \swarrow & & \searrow {\lambda'} \\
M & \xrightarrow{\ \ \phi\ \ } & M'
\end{array}
$$

in $\mathcal{M}$. The morphism ϕ is uniquely determined up to homotopy and is an equivalence in $\mathrm{LMod}_A(\mathcal{M})$.

Corollary 4.2.4.4. *Let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category and $\mathcal{M}$ an ∞-category which is left-tensored over $\mathcal{C}^{\otimes}$. Then the forgetful functor $G : \mathrm{LMod}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{C}) \times \mathcal{M}$ admits a left adjoint F, given informally by $F(A, M_0) = (A, M) \in \mathrm{LMod}(\mathcal{M})$, where M is a free left A-module generated by M_0.*

Remark 4.2.4.5. Corollary 4.2.4.4 shows that that a free left A-module generated by $M_0 \in \mathcal{M}$ depends functorially on the pair (A, M_0).

If we regard the algebra $A \in \mathrm{Alg}(\mathcal{C})$ as fixed, the above results take a slightly simpler form:

Corollary 4.2.4.6. *Let $\mathcal{C}^\otimes$ be a monoidal ∞-category and $\mathcal{M}$ an ∞-category which is left-tensored over $\mathcal{C}^\otimes$. Let $A \in \mathrm{Alg}(\mathcal{C})$, $M \in \mathrm{LMod}_A(\mathcal{M})$, and suppose there exists a map $\lambda : M_0 \to M$ in $\mathcal{M}$ which exhibits M as the free A-module generated by M_0. Then for any $N \in \mathrm{LMod}_A(\mathcal{M})$, composition with λ induces a homotopy equivalence $\mathrm{Map}_{\mathrm{LMod}_A(\mathcal{M})}(M, N) \to \mathrm{Map}_{\mathcal{M}}(M_0, N)$.*

Corollary 4.2.4.7. *Let $\mathcal{C}^\otimes$ be a monoidal ∞-category containing an algebra object $A \in \mathrm{Alg}(\mathcal{C})$. Regard $\mathcal{C}$ as left-tensored over itself and A as a left module over itself (as in Example 4.2.1.17). Let $1_\mathcal{C}$ be the unit object of $\mathcal{C}$ and M an object of $\mathrm{LMod}_A(\mathcal{C})$. Then composition with the unit map of A induces a homotopy equivalence $\mathrm{Map}_{\mathrm{LMod}_A(\mathcal{C})}(A, M) \to \mathrm{Map}_\mathcal{C}(1_\mathcal{C}, M)$.*

Proof. Apply Corollary 4.2.4.6 to the unit map $\mathbf{1} \to A$. $\qquad\qquad\square$

Corollary 4.2.4.8. *Let $\mathcal{C}$ be a monoidal ∞-category and let $\mathcal{M}$ be an ∞-category which is left-tensored over $\mathcal{C}$. Let $A \in \mathrm{Alg}(\mathcal{C})$. Then the forgetful functor $G : \mathrm{LMod}_A(\mathcal{M}) \to \mathcal{M}$ admits a left adjoint F, which carries each $M_0 \in \mathcal{M}$ to a free left A-module generated by M_0. The composition $G \circ F : \mathcal{M} \to \mathcal{M}$ is the functor given by tensor product with A.*

Corollary 4.2.4.8 guarantees that the theory of modules over a trivial algebra is very simple.

Proposition 4.2.4.9. *Let $\mathcal{C}^\otimes$ be a monoidal ∞-category and $\mathcal{M}$ an ∞-category which is left-tensored over $\mathcal{C}$. Let A be an algebra object of $\mathrm{Alg}(\mathcal{C})$ such that the unit map $1_\mathcal{C} \to A$ is an equivalence in $\mathcal{C}$. Then the forgetful functor $G : \mathrm{LMod}_A(\mathcal{M}) \to \mathcal{M}$ is an equivalence of ∞-categories.*

Proof. Let F be the left adjoint to $\mathbf{G}$ supplied by Corollary 4.2.4.8, and let

$$u : \mathrm{id}_\mathcal{M} \to G \circ F, \quad v : F \circ G \to \mathrm{id}_\mathcal{M}$$

be a compatible unit and counit for the adjunction. We wish to prove that u and v are equivalences of functors.

We first consider the functor u. Corollary 4.2.4.8 implies that the composition $G \circ F$ can be identified with the functor $M \mapsto A \otimes M$. The unit map u is given by tensor product with the unit map $u_0 : 1_\mathcal{C} \to A$ of the algebra A. By hypothesis, u_0 is an equivalence in $\mathcal{C}$, so that u is an equivalence in $\mathrm{Fun}(\mathcal{M}, \mathcal{M})$.

Note that a morphism $(A_0, M_0) \to (A_1, M_1)$ in $\mathrm{LMod}(\mathcal{M})$ is an equivalence if and only if the induced map $A_0 \to A_1$ is an equivalence in $\mathcal{C}$ and the induced map $M_0 \to M_1$ is an equivalence in $\mathcal{M}$. It follows that the functor G is conservative (see Corollary 4.2.3.2 for a stronger version of this statement). Consequently, to show that v is an equivalence it suffices to show that the induced transformation $\alpha : G \circ F \circ G \to G$ is an equivalence of functors. We now observe that u provides a right inverse to this α. Since u is an equivalence, we conclude also that α is an equivalence. $\qquad\square$

4.2.5 Duality in Monoidal ∞-Categories

Let k be a field, and let V be a vector space over k. The *dual space* $V^\vee$ is defined to be the set $\mathrm{Hom}_k(V, k)$ of k-linear maps from V into k. It can be characterized by the following universal property: for every k-vector space W, giving a k-linear map $W \to V^\vee$ is equivalent to giving a k-linear map $W \otimes_k V \to k$ (or, equivalently, a bilinear map $W \times V \to k$). In particular, the identity $\mathrm{id} : V^\vee \to V^\vee$ classifies a linear map $e : V^\vee \otimes_k V \to k$, which corresponds to the bilinear map $(\lambda, v) \mapsto \lambda(v)$.

If V is a finite-dimensional vector space over k, then the dual $V^\vee$ admits a second description: namely, for every k-vector space W, the tensor product $W \otimes_k V^\vee$ can be identified with the set $\mathrm{Hom}_k(V, W)$ of k-linear maps from V into W. In particular, $V \otimes_k V^\vee$ can be identified with the set of k-linear endomorphisms of V,

and there is a canonical map $c : k \to V \otimes_k V^\vee$ which carries the element $1 \in k$ to the identity endomorphism of V. It is not difficult to verify that the composite maps

$$V \simeq k \otimes_k V \xrightarrow{c \otimes \mathrm{id}} V \otimes_k V^\vee \otimes_k V \xrightarrow{\mathrm{id} \otimes e} V \otimes_k k \simeq V$$

$$V^\vee \simeq V^\vee \otimes_k k \xrightarrow{\mathrm{id} \otimes c} V^\vee \otimes_k V \otimes_k V^\vee \xrightarrow{e \otimes \mathrm{id}} k \otimes_k V^\vee \simeq V^\vee$$

correpond to the identity on V and $V^\vee$, respectively.

In this section, we will generalize the notion of duality, replacing the ordinary category of vector spaces over a field k with an arbitrary monoidal ∞-category. As a first step, we recall how the above ideas generalize to the setting of monoidal categories.

Definition 4.2.5.1. Let $\mathcal{C}$ be a monoidal category. A *duality datum* in $\mathcal{C}$ consists of the following data:

(*i*) A pair of objects $X, X^\vee \in \mathcal{C}$.

(*ii*) A pair of morphisms
$$c : \mathbf{1} \to X \otimes X^\vee \qquad e : X^\vee \otimes X \to \mathbf{1},$$
where $\mathbf{1}$ denotes the unit object of $\mathcal{C}$.

These morphisms are required to satisfy the following conditions:

(*iii*) The composite maps
$$X \xrightarrow{c \otimes \mathrm{id}} X \otimes X^\vee \otimes X \xrightarrow{\mathrm{id} \otimes e} X$$
$$X^\vee \xrightarrow{\mathrm{id} \otimes c} X^\vee \otimes X \otimes X^\vee \xrightarrow{e \otimes \mathrm{id}} X^\vee$$
are the identity on X and $X^\vee$, respectively.

In this case, we will say that e and c exhibit $X^\vee$ as a *right dual* of X, or X as a *left dual* of $X^\vee$.

Let $\mathcal{C}$ be a monoidal ∞-category, and let $\mathcal{M}$ be an ∞-category right-tensored over $\mathcal{C}$. If $M, N \in \mathcal{M}$ and $C \in \mathcal{C}$, we will say that a morphism $q : N \otimes C \to M$ in $\mathcal{C}$ *exhibits N as an exponential of M by C* if, for every object $N' \in \mathcal{M}$, the composite map

$$\mathrm{Map}_{\mathcal{M}}(N', N) \to \mathrm{Map}_{\mathcal{M}}(N' \otimes C, N \otimes C) \xrightarrow{q \circ} \mathrm{Map}_{\mathcal{M}}(N' \otimes C, M)$$

is a homotopy equivalence.

Remark 4.2.5.2. If $\mathcal{C}$ and $\mathcal{M}$ are presentable ∞-categories and the tensor product $\mathcal{M} \times \mathcal{C} \to \mathcal{M}$ preserves colimits separately in each variable, then for every object $(M, C) \in \mathcal{M} \times \mathcal{C}^{op}$, the construction $N \mapsto \mathrm{Map}_{\mathcal{C}}(N \otimes C, M)$ determines a limit-preserving functor $\mathcal{M}^{op} \to \mathcal{S}$, which is representable by an object $M^C \in \mathcal{M}$ (Proposition T.5.5.2.2). By definition, this object is equipped with a canonical map $M^C \otimes C \to M$ which exhibits M^C as an exponential of M by C.

Remark 4.2.5.3. Let $\mathcal{C}$ be a monoidal ∞-category and $\mathcal{M}$ an ∞-category right-tensored over $\mathcal{C}$. The construction $(M, C) \mapsto \mathrm{Map}_{\mathcal{C}}(\bullet \otimes C, M)$ determines a functor $\chi : \mathcal{M} \times \mathcal{C}^{op} \to \mathrm{Fun}(\mathcal{M}^{op}, \mathcal{S})$. Suppose that, for every object $(M, C) \in \mathcal{M} \times \mathcal{C}^{op}$, there exists an exponential $M^C \in \mathcal{M}$. Then χ factors through the essential image of the Yoneda embedding $j : \mathcal{M} \to \mathrm{Fun}(\mathcal{M}^{op}, \mathcal{S})$. Since j is fully faithful, we can write $\chi \simeq j \circ \chi_0$ for some functor $\chi_0 : \mathcal{M} \times \mathcal{C}^{op} \to \mathcal{M}$. We will denote this functor by $(M, C) \mapsto M^C$.

Remark 4.2.5.4. Let $\mathcal{C}$ be as in Remark 4.2.5.3 and let $C \in \mathcal{C}$. Then the functor $M \mapsto M^C$ preserves all limits which exist in $\mathcal{C}$.

Lemma 4.2.5.5. *Let $\mathcal{C}$ be a monoidal ∞-category and let $\mathcal{M}$ be an ∞-category right-tensored over $\mathcal{C}$. Suppose that we are given a duality datum $(X, X^\vee, e, c)$ in the homotopy category $h\mathcal{C}$. Then, for every object $M \in \mathcal{M}$, the map*

$$(M \otimes X^\vee) \otimes X \simeq M \otimes (X^\vee \otimes X) \xrightarrow{e} M \otimes \mathbf{1} \simeq M$$

exhibits $M \otimes X^\vee$ as an exponential of M by X.

Proof. Fix an object $N \in \mathcal{M}$. We wish to show that the composite map

$$\phi : \mathrm{Map}_{\mathcal{M}}(N, M \otimes X^\vee) \to \mathrm{Map}_{\mathcal{M}}(N \otimes X, M \otimes X^\vee \otimes X) \xrightarrow{e} \mathrm{Map}_{\mathcal{M}}(N \otimes X, M)$$

is a homotopy equivalence. Let ψ denote the composition

$$\mathrm{Map}_{\mathcal{M}}(N \otimes X, M) \to \mathrm{Map}_{\mathcal{M}}(N \otimes X \otimes X^\vee, M \otimes X^\vee) \xrightarrow{c} \mathrm{Map}_{\mathcal{M}}(N, M \otimes X^\vee).$$

Using the compatibility of e and c, we deduce that ϕ and ψ are homotopy inverse to one another. $\square$

Lemma 4.2.5.6. *Let $\mathcal{C}$ be a monoidal ∞-category and let $e : B \otimes C \to \mathbf{1}$ be a morphism in $\mathcal{C}$. The following conditions are equivalent:*

(1) *The triple (B, C, e) can be extended to a duality datum in the homotopy category $h\mathcal{C}$. That is, there exists a map $c : \mathbf{1} \to C \otimes B$ such that the compositions*

$$C \to C \otimes B \otimes C \to C \qquad C \to C \otimes B \otimes C \to C$$

are homotopic to the identity.

(2) *For every object $A \in \mathcal{C}$, the map*

$$A \otimes B \otimes C \xrightarrow{\mathrm{id}\, \otimes e} A$$

exhibits $A \otimes B$ as an exponential of A by C. In other words, for every object $D \in \mathcal{C}$, e induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}}(D, A \otimes B) \to \mathrm{Map}_{\mathcal{C}}(D \otimes C, A).$$

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemma 4.2.5.5 (applied to the right action of $\mathcal{C}$ on itself). Conversely, suppose that (2) is satisfied. Then the map

$$\theta : \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, C \otimes B) \xrightarrow{\otimes C} \mathrm{Map}_{\mathcal{C}}(C, C \otimes B \otimes C) \xrightarrow{e} \mathrm{Map}_{\mathcal{C}}(C, C)$$

is a homotopy equivalence. In particular, there exists a map $c : \mathbf{1} \to C \otimes B$ whose image under θ is an equivalence. It follows that the composition

$$\phi : C \xrightarrow{c \otimes \mathrm{id}} C \otimes B \otimes C \xrightarrow{\mathrm{id}\, \otimes e} C$$

is homotopic to the identity. To complete the proof, it suffices to show that the composition

$$\psi : B \xrightarrow{\mathrm{id}\, \otimes c} B \otimes C \otimes B \xrightarrow{e \otimes \mathrm{id}} B$$

is homotopic to the identity. Assumption (2) (applied in the case $A = \mathbf{1}$) guarantees that e exhibits B as an exponential $\mathbf{1}^C$, so that ψ' is classified by up to homotopy by some map $e' : B \otimes C \to \mathbf{1}$; we wish to prove that $e = e'$. Unwinding the definitions, we see that e' is given by the composition

$$B \otimes C \xrightarrow{\mathrm{id}\, \otimes c \otimes \mathrm{id}} B \otimes C \otimes B \otimes C \tag{4.1}$$

$$\xrightarrow{e \otimes e} \mathbf{1} \otimes \mathbf{1} \tag{4.2}$$

$$\simeq \mathbf{1}. \tag{4.3}$$

This map can be written as the composition

$$B \otimes C \xrightarrow{\mathrm{id}\, \otimes \phi} B \otimes C \xrightarrow{e} \mathbf{1},$$

which is homotopic to e since ϕ is homotopic to id_Y. $\square$

Definition 4.2.5.7. Let $\mathcal{C}$ be a monoidal ∞-category. We say that an object $C \in \mathcal{C}$ is *right dualizable* (*left dualizable*) if there exists an object $B \in \mathcal{C}$ and a map $e : B \otimes C \to \mathbf{1}$ ($e : C \otimes B \to \mathbf{1}$) satisfying the equivalent conditions of Lemma 4.2.5.6.

Notation 4.2.5.8. Let $\mathcal{C}$ be a monoidal ∞-category. We let $\mathrm{DDat}(\mathcal{C})$ denote the full subcategory of

$$(\mathcal{C} \times \mathcal{C}) \times_{\mathrm{Fun}(\{0\},\mathcal{C})} \mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathrm{Fun}(\{1\},\mathcal{C})} \{\mathbf{1}\}$$

spanned by those triples $(B, C, e : B \otimes C \to \mathbf{1})$ which satisfy the equivalent conditions of Lemma 4.2.5.6. We will refer to $\mathrm{DDat}(\mathcal{C})$ as the ∞-*groupoid of duality data in* $\mathcal{C}$.

The construction $(B, C, e) \mapsto B$ defines a forgetful functor $\pi : \mathrm{DDat}(\mathcal{C}) \to \mathcal{C}$.

Remark 4.2.5.9. We observe that the image of π is contained in the subcategory $\mathcal{C}^{\mathrm{rd}} \subseteq \mathcal{C}$ whose objects are right dualizable objects of $\mathcal{C}$ and whose morphisms are equivalences in $\mathcal{C}$. The condition on objects follows from Lemma 4.2.5.6. The condition on morphisms follows from the following observation: if $\alpha : (B, C, e) \to (B', C', e')$ is a morphism in $\mathrm{DDat}(\mathcal{C})$, then dual of the underlying map $C \to C'$ is a homotopy inverse to the underlying map $B \to B'$.

Lemma 4.2.5.10. *Let $\mathcal{C}$ be a monoidal ∞-category and let $\mathcal{C}^{\mathrm{rd}}$ be defined as in Remark 4.2.5.9. Then the map $\pi : \mathrm{DDat}(\mathcal{C}) \to \mathcal{C}^{\mathrm{rd}}$ is a trivial Kan fibration.*

Proof. The map π is obviously an essentially surjective categorical fibration. To show that it is an equivalence, it suffices to prove that π is fully faithful. In other words, we must show that for every pair of objects $(B, C, e), (B', C', e') \in \mathcal{D}$, the map π induces a homotopy equivalence $\mathrm{Map}_{\mathrm{DDat}(\mathcal{C})}((B, C, e), (B', C', e')) \to \mathrm{Iso}_{\mathcal{C}}(B, B')$; here $\mathrm{Iso}_{\mathcal{C}}(B, B')$ denotes the summand of the mapping space $\mathrm{Map}_{\mathcal{C}}(X, X')$ consisting of those connected components which correspond to equivalences from B to B'.

Unwinding the definitions, we see that $\mathrm{Map}_{\mathrm{DDat}(\mathcal{C})}((B, C, e), (B', C', e'))$ can be identified with the homotopy fiber of a map $\phi : \mathrm{Iso}_{\mathcal{C}}(B, B') \times \mathrm{Iso}_{\mathcal{C}}(C, C') \to \mathrm{Map}_{\mathcal{C}}(B \otimes C, \mathbf{1})$ over the point e. The pairings e and e' determine homotopy equivalences $\mathrm{Iso}_{\mathcal{C}}(C, C') \simeq \mathrm{Iso}_{\mathcal{C}}(B', B)$ and $\mathrm{Map}_{\mathcal{C}}(B \otimes C, \mathbf{1}) \simeq \mathrm{Map}_{\mathcal{C}}(B, B)$. Under these identifications, ϕ corresponds to the obvious composition map $\mathrm{Iso}_{\mathcal{C}}(B, B') \times \mathrm{Iso}_{\mathcal{C}}(B', B) \to \mathrm{Map}_{\mathcal{C}}(B, B)$, and $\mathrm{Map}_{\mathrm{DDat}(\mathcal{C})}((B, B', e), (B', C', e'))$ corresponds to the homotopy fiber lying over the identity. Note that ϕ factors through the summand $\mathrm{Iso}_{\mathcal{C}}(B, B')$. We have a homotopy pullback diagram

$$
\begin{array}{ccc}
\mathrm{Map}_{\mathrm{DDat}(\mathcal{C})}((B, C, e), (B', C', e')) & \longrightarrow & \mathrm{Iso}_{\mathcal{C}}(B, B') \times \mathrm{Iso}_{\mathcal{C}}(B', B) \\
\downarrow & & \downarrow{\scriptstyle \psi} \\
\mathrm{Iso}_{\mathcal{C}}(B, B') & \longrightarrow & \mathrm{Iso}_{\mathcal{C}}(B, B') \times \mathrm{Iso}_{\mathcal{C}}(B, B),
\end{array}
$$

where ψ is given by the formula $(f, g) \mapsto (f, gf)$. It therefore suffices to show that ψ is a homotopy equivalence, which is clear. $\qquad\square$

Proposition 4.2.5.11. *Let $\{\mathcal{C}_\alpha\}$ be a diagram of monoidal ∞-categories having a limit $\mathcal{C}$. Then an object of $\mathcal{C}$ is right dualizable if and only if its image in each $\mathcal{C}_\alpha$ is right dualizable.*

Proof. The "only if" direction is obvious. For the converse, we observe that $\mathrm{DDat}(\mathcal{C}) \simeq \varprojlim \mathrm{DDat}(\mathcal{C}_\alpha)$ and apply Lemma 4.2.5.10. $\qquad\square$

Remark 4.2.5.12. If $\mathcal{C}$ is a symmetric monoidal ∞-category, then an object $C \in \mathcal{C}$ is left dualizable if and only if it is right dualizable; in this case, we will simply say that C is *dualizable*.

4.3 Bimodules

Let $\mathcal{C}$ be a monoidal category. Given a pair of associative algebra objects $A, B \in \mathrm{Alg}(\mathcal{C})$, one can define a category $_A\mathrm{BMod}_B(\mathcal{C})$ of *A-B bimodule objects* of $\mathcal{C}$. By definition, an *A-B* bimodule is an object $M \in \mathcal{C}$ equipped with multiplication maps $m : A \otimes M \to M$ and $m' : M \otimes B \to M$ with the following properties:

(i) The multiplication m determines a left action of A on M. That is, if $m_A : A \otimes A \to A$ and $e : \mathbf{1} \to A$ denote the multiplication and unit maps of A, then the

$$
\begin{array}{ccc}
A \otimes A \otimes M & \xrightarrow{\ m_A \otimes \mathrm{id}\ } & A \otimes M \\
\downarrow & & \downarrow{\scriptstyle m} \\
A \otimes M & \xrightarrow{\quad m \quad} & M
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbf{1} \otimes M & \xrightarrow{\quad e \quad} & A \otimes M \\
 & \searrow & \downarrow{\scriptstyle m} \\
 & & M
\end{array}
$$

commute.

(ii) The multiplication m' determines a right action of B on M. That is, if $m_B : B \otimes B \to B$ and $e' : \mathbf{1} \to B$ denote the multiplication and unit maps of B, then the diagrams

$$
\begin{array}{ccc}
M \otimes B \otimes B & \xrightarrow{\ \mathrm{id} \otimes m_B\ } & M \otimes B \\
\downarrow & & \downarrow{\scriptstyle m'} \\
M \otimes B & \xrightarrow{\quad m' \quad} & M
\end{array}
\qquad\qquad
\begin{array}{ccc}
M \otimes \mathbf{1} & \xrightarrow{\quad e' \quad} & M \otimes B \\
 & \searrow & \downarrow{\scriptstyle m'} \\
 & & M
\end{array}
$$

commute.

(iii) The left action of A on M commutes with the right action of B on M: that is, the diagram

$$
\begin{array}{ccc}
A \otimes M \otimes B & \xrightarrow{\ m \otimes \mathrm{id}\ } & M \otimes B \\
\downarrow{\scriptstyle \mathrm{id} \otimes m'} & & \downarrow{\scriptstyle m'} \\
A \otimes M & \xrightarrow{\quad m \quad} & M
\end{array}
$$

is commutative.

Our goal in this section is to develop an analogous theory of bimodules in the ∞-categorical setting. We will follow the basic pattern of our approach to left modules in §4.2. We begin in §4.3.1 by introducing an ∞-operad $\mathcal{BM}^{\otimes}$ equipped with a pair of inclusions $\mathcal{A}ss^{\otimes} \hookrightarrow \mathcal{BM}^{\otimes} \hookleftarrow \mathcal{A}ss^{\otimes}$. If $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category and $A, B \in \mathrm{Alg}(\mathcal{C})$, then we will denote the fiber product $\{A\} \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{Alg}_{\mathcal{BM}}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \{B\}$ by $_A\mathrm{BMod}_B(\mathcal{C})$; we will refer to $_A\mathrm{BMod}_B(\mathcal{C})$ as the *∞-category of A-B-bimodule objects of* $\mathcal{C}$.

As in the classical case, it is possible to understand view an *A-B*-bimodule object of a symmetric monoidal ∞-category $\mathcal{C}$ as an object of $\mathcal{C}$ which is equipped with a left action of A and a right action of B which commute up to coherent homotopy. In §4.3.2, we will make this precise by showing that the ∞-category of right B-module objects of $\mathcal{C}$ is canonically left-tensored over $\mathcal{C}$, and there is an equivalence of ∞-categories $_A\mathrm{BMod}_B(\mathcal{C}) \simeq \mathrm{LMod}_A(\mathrm{RMod}_B(\mathcal{C}))$. Using this equivalence, we will show that many of the results proven in §4.2 for left modules (such as the existence of limits and colimits and the structure of free modules) admit straightforward generalizations to the setting of bimodules.

Let A, B, and C be associative rings, let M be an *A-B*-bimodule, N a *B-C*-bimodule, and X an *A-C*-bimodule. In §4.3.4, we discussed the notion of a *bilinear pairing* from M and N to X: this can be formulated either as a map of sets $M \times N \to X$ or as a map of abelian groups $M \otimes N \to X$, which in either case must be required to satisfy certain axioms. However, there is another approach which is often more convenient. Namely, one can define the *relative tensor product* $M \otimes_B N$ to be the coequalizer of the diagram

$$
M \otimes B \otimes N \underset{g}{\overset{f}{\rightrightarrows}} M \otimes N,
$$

where f is given by the right action of B on M and g by the left action of B on N. This coequalizer inherits a left action of A and a right action of C, and may therefore be viewed as an A-C-bimodule. Moreover, it is not difficult to see that giving a bilinear pairing $M \times N \to X$ is equivalent to giving a map of A-C-bimodules $M \otimes_B N \to X$.

In §4.3.4, we will generalize the notion of a bilinear pairing to the setting of an arbitrary monoidal ∞-category $\mathcal{C}$. Under some mild hypotheses, one can show this notion determines a corepresentable functor: that is, we can associate to every pair of bimodules $M \in {}_A\mathrm{BMod}_B(\mathcal{C})$ and $N \in {}_B\mathrm{BMod}_C(\mathcal{C})$ a new bimodule $M \otimes_B N \in {}_A\mathrm{BMod}_C(\mathcal{C})$, which is universal among bimodule objects which receive a B-bilinear map from the pair (M, N). We will study the relative tensor product functor in §4.3.5. Note that the construction of $M \otimes_B N$ is somewhat more complicated than in the classical case: it is generally not given by the coequalizer of the diagram $M \otimes B \otimes N \rightrightarrows M \otimes N$. Instead it is computed by the classical *two-sided bar construction*: that is, as the geometric realization of the simplicial bimodule $\mathrm{Bar}_B(M, N)_\bullet$ given informally by the formula

$$\mathrm{Bar}_B(M, N)_n = M \otimes B^{\otimes n} \otimes N.$$

We will use this description in §4.3.6 to study associativity properties of the relative tensor product.

The relative tensor product construction plays an important role in the study of "base change" for module objects. Suppose we are given algebra objects $A, B, B' \in \mathrm{Alg}(\mathcal{C})$ of a symmetric monoidal ∞-category $\mathcal{C}$ and a morphism of algebra objects $\phi : B \to B'$. Then ϕ induces a forgetful functor ${}_A\mathrm{BMod}_{B'}(\mathcal{C}) \to {}_A\mathrm{BMod}_B(\mathcal{C})$. Under some mild hypotheses, one can construct a left adjoint to this forgetful functor. In §4.3.7, we will show that this left adjoint is described by the formula $M \mapsto M \otimes_B B'$. The proof will make use of some general results concerning the duality of bimodule objects of $\mathcal{C}$, building on the the ideas introduced in §4.2.5

4.3.1   The ∞-Operad $\mathcal{BM}^{\otimes}$

Our goal in this section is to lay the foundations for an ∞-categorical theory of bimodules. We begin by formulating the classical theory of bimodules in terms of colored operads.

Definition 4.3.1.1. We define a colored operad **BM** as follows:

(*i*) The set of objects of **LM** has three elements, which we will denote by $\mathfrak{a}_-$, $\mathfrak{a}_+$, and $\mathfrak{m}$.

(*ii*) Let $\{X_i\}_{i \in I}$ be a finite collection of objects of **BM** and let Y be another object of **LM**. If $Y = \mathfrak{a}_-$, then $\mathrm{Mul}_{\mathbf{BM}}(\{X_i\}, Y)$ is the collection of all linear orderings of I provided that each $X_i = \mathfrak{a}_-$, and is empty otherwise. If $Y = \mathfrak{a}_+$, then $\mathrm{Mul}_{\mathbf{BM}}(\{X_i\}, Y)$ is the collection of all linear orderings of I provided that each $X_i = \mathfrak{a}_+$, and is empty otherwise. If $Y = \mathfrak{m}$, then $\mathrm{Mul}_{\mathbf{LM}}(\{X_i\}, Y)$ is the collection of all linear orderings on $\{i_1 < \cdots < i_n\}$ on the set I with the following property: there is exactly one index $i_k \in I$ such that $X_{i_k} = \mathfrak{m}$. We have $X_{i_j} = \mathfrak{a}_-$ for $j < k$ and $X_{i_j} = \mathfrak{a}_+$ for $j > k$.

(*iii*) The composition law on **BM** is determined by the composition of linear orderings, as described in Definition 4.1.1.1.

Remark 4.3.1.2. Restricting our attention to the pair of objects $\mathfrak{a}_-, \mathfrak{m} \in \mathbf{BM}$, we obtain a colored suboperad isomorphic to the colored operad **LM** of Definition 4.2.1.1. Similarly, the pair of objects $\mathfrak{a}_+, \mathfrak{m} \in \mathbf{BM}$ determine a colored suboperad $\mathbf{RM} \subseteq \mathbf{BM}$, which underlies the ∞-operad $\mathcal{RM}^{\otimes}$ of Variant 4.2.1.36.

Remark 4.3.1.3. If $\mathcal{C}$ is a symmetric monoidal category and $F : \mathbf{BM} \to \mathcal{C}$ is a map of colored operads, then $F|\mathbf{LM}$ determines an associative algebra $A_- = F(\mathfrak{a}_-)$ in $\mathcal{C}$ and a left A-module $M = F(\mathfrak{m}) \in \mathcal{C}$ (see Remark 4.2.1.3). Similarly, the restriction $F|\mathbf{RM}$ determines an associative algebra $A_+ = F(\mathfrak{a}_+)$ such that M has the structure of a right module over A_+. The left action of A_- commutes with the right action of A_+, in the sense that that the diagram σ :

$$
\begin{array}{ccc}
A_- \otimes M \otimes A_+ & \longrightarrow & A_- \otimes M \\
\downarrow & & \downarrow \\
M \otimes A_+ & \longrightarrow & M
\end{array}
$$

is commutative. To see this, it suffices to observe that both of the composite maps $A_- \otimes M \otimes A_+ \to M$ are given by the unique operation $\phi \in \mathrm{Mul}_{\mathbf{BM}}(\{\mathfrak{a}_-, \mathfrak{m}, \mathfrak{a}_+\}, \mathfrak{m})$. We conclude that M has the structure of an A_--A_+-bimodule object of $\mathcal{C}$.

Conversely, suppose we are given an object $M \in \mathcal{C}$ equipped with a left action of an associative algebra A_- and a right action of another associative algebra A_+. According to Remark 4.2.1.3, the pair (A_-, M) determines a map of colored operads $F_- : \mathbf{LM} \to \mathcal{C}$. Similarly, the pair (A_+, M) determines a map of colored operads $F_+ : \mathbf{RM} \to \mathcal{C}$. If the diagram σ commutes, then F_- and F_+ admit a unique amalgamation to a map of colored operads $F : \mathbf{BM} \to \mathcal{C}$, which assigns to each operation

$$\psi \in \mathrm{Mul}_{\mathbf{BM}}(\{\mathfrak{a}_-, \ldots, \mathfrak{a}_-, \mathfrak{m}, \mathfrak{a}_+, \ldots, \mathfrak{a}_+\}, \mathfrak{m})$$

the map

$$A_- \otimes \cdots \otimes A_- \otimes M \otimes A_+ \otimes \cdots \otimes A_+ \overset{m}{\to} A_- \otimes M \otimes A_+ \overset{a}{\to} M,$$

where m is given by the multiplication in the algebras A_- and A_+ and a is the map appearing in the diagram σ.

We can summarize the above discussion as follows: giving a map of colored operads $\mathbf{BM} \to \mathcal{C}$ is equivalent to giving a pair of associative algebras A_-, A_+ in $\mathcal{C}$, together with an A_--A_+-bimodule $M \in \mathcal{C}$.

Remark 4.3.1.4. Every operation $\phi \in \mathrm{Mul}_{\mathbf{RM}}(\{X_i\}_{i \in I}, Y)$ determines a linear ordering on the set I. Passage from ϕ to this linear ordering determines a map of colored operads $\mathbf{BM} \to \mathbf{Ass}$. This map can be understood as follows: for every symmetric monoidal category $\mathcal{C}$ and every map of colored operads $F : \mathbf{Ass} \to \mathcal{C}$ corresponding to an associative algebra $A \in \mathcal{C}$, the composite map $\mathbf{BM} \to \mathbf{Ass} \to \mathcal{C}$ corresponds to A, regarded as a bimodule over itself.

Notation 4.3.1.5. We let $\mathbf{BM}^\otimes$ denote the category obtained by applying Construction 2.1.1.7 to the colored operad $\mathbf{BM}$. We can describe this category concretely as follows:

(1) The objects of $\mathbf{BM}^\otimes$ are pairs $(\langle n \rangle, c_-, c_+)$, where $\langle n \rangle$ is an object of $\mathcal{F}\mathrm{in}_*$ and $c_-, c_+ : \langle n \rangle^\circ \to [1]$ are maps satsifying $c_-(i) \le c_+(i)$ for $1 \le i \le n$.

(2) A morphism from $(\langle n \rangle, c_-, c_+)$ to $(\langle n' \rangle, c'_-, c'_+)$ in $\mathbf{BM}^\otimes$ consists of a morphism $\alpha : \langle n \rangle \to \langle n' \rangle$ in $\mathbf{Ass}^\otimes$ satisfying the following condition, for each $j \in \langle n \rangle^\circ$ with $\alpha^{-1}\{j\} = \{i_1 \prec \cdots \prec i_m\}$:

$$c'_-(j) = c_-(i_1) \le c_+(i_1) = c_-(i_2) \le c_+(i_2) = c_-(i_3) \le \cdots \le c_+(i_{m-1}) = c_-(i_m) \le c_+(i_m) = c'_+(j).$$

In terms of this description, the object $\mathfrak{a}_- \in \mathbf{BM}^\otimes$ to the triple $(\langle 1 \rangle, c_-, c_+)$ where $c_-(1) = c_+(1) = 0$. The object $\mathfrak{a}_+$ corresponds to the triple $(\langle 1 \rangle, c_-, c_+)$ with $c_-(1) = c_+(1) = 1$, and the object $\mathfrak{m}$ corresponds to the triple $(\langle 1 \rangle, c_-, c_+)$ with $c_-(1) = 0$ and $c_+(1) = 1$.

We now introduce the ∞-categorical analogue of Definition 4.3.1.1.

Definition 4.3.1.6. We let $\mathcal{BM}^\otimes$ denote the nerve of the category $\mathbf{BM}^\otimes$. We regard $\mathcal{BM}^\otimes$ as an ∞-operad via the forgetful functor $\mathcal{BM}^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ (see Example 2.1.1.21).

Remark 4.3.1.7. The underlying ∞-category $\mathcal{BM}$ of $\mathcal{BM}^\otimes$ is isomorphic to the discrete simplicial set $\Delta^0 \coprod \Delta^0 \coprod \Delta^0$ with three vertices, corresponding to the objects $\mathfrak{a}_-, \mathfrak{m}, \mathfrak{a}_+ \in \mathbf{BM}$.

Remark 4.3.1.8. The map of colored operads $\mathbf{BM} \to \mathbf{Ass}$ appearing in Remark 4.3.1.4 induces a fibration of ∞-operads $\mathcal{BM}^\otimes \to \mathcal{A}\mathrm{ss}^\otimes$.

Remark 4.3.1.9. The inclusions of colored operads $\mathbf{LM} \hookrightarrow \mathbf{BM} \hookleftarrow \mathbf{RM}$ of Remark 4.3.1.2 determine isomorphisms of $\mathcal{LM}^\otimes$ and $\mathcal{RM}^\otimes$ onto full subcategories of $\mathcal{BM}^\otimes$. We will generally abuse notation and identify $\mathcal{LM}^\otimes$ and $\mathcal{RM}^\otimes$ with their images in $\mathcal{BM}^\otimes$.

Remark 4.3.1.10. The inclusions $\mathcal{LM}^{\otimes}, \mathcal{RM}^{\otimes} \hookrightarrow \mathcal{BM}^{\otimes}$ determine two *different* embeddings of $\mathcal{A}ss^{\otimes}$ into $\mathcal{BM}^{\otimes}$. We will denote the images of these embeddings by $\mathcal{A}ss_{-}^{\otimes}$ and $\mathcal{A}ss_{+}^{\otimes}$, respectively. Note that the inclusions $\mathcal{A}ss_{-}^{\otimes}, \mathcal{A}ss_{+}^{\otimes} \subseteq \mathcal{BM}^{\otimes}$ extend to an isomorphism from $\mathcal{A}ss^{\otimes} \boxplus \mathcal{A}ss^{\otimes}$ to the full subcategory of $\mathcal{BM}^{\otimes}$ spanned by objects of the form $(\langle n \rangle, c_{-}, c_{+})$ where $c_{-} = c_{+}$.

Notation 4.3.1.11. Let $\mathcal{C}^{\otimes} \to \mathcal{BM}^{\otimes}$ be a fibration of ∞-operads. We let $\mathcal{C}_{-}^{\otimes}$ denote the fiber product $\mathcal{C}^{\otimes} \times_{\mathcal{BM}^{\otimes}} \mathcal{A}ss_{-}^{\otimes}$ (so that $\mathcal{C}_{-}^{\otimes}$ is a planar ∞-operad, in the sense of Definition 4.1.1.6). We will denote the underlying ∞-category of $\mathcal{C}_{-}^{\otimes}$ by $\mathcal{C}_{-} = \mathcal{C}^{\otimes} \times_{\mathcal{BM}^{\otimes}} \{\mathfrak{a}_{-}\}$. Similarly, we define $\mathcal{C}_{+}^{\otimes} = \mathcal{C}^{\otimes} \times_{\mathcal{BM}^{\otimes}} \mathcal{A}ss_{+}^{\otimes}$, $\mathcal{C}_{+} = \mathcal{C}^{\otimes} \times_{\mathcal{BM}^{\otimes}} \{\mathfrak{a}_{+}\}$, and $\mathcal{C}_{\mathfrak{m}} = \mathcal{C}^{\otimes} \times_{\mathcal{BM}^{\otimes}} \{\mathfrak{m}\}$. Note that $\mathcal{C}_{\mathfrak{m}}$ is an ∞-category which is weakly enriched over the planar ∞-operad $\mathcal{C}_{-}^{\otimes}$ (in the sense of Definition 4.2.1.12). Similarly, $\mathcal{C}_{\mathfrak{m}}$ is weakly enriched over the *reverse* of the planar ∞-operad $\mathcal{C}_{+}^{\otimes}$ (see Remark 4.1.1.8).

Definition 4.3.1.12. Let $q : \mathcal{C}^{\otimes} \to \mathcal{BM}^{\otimes}$ be a fibration of ∞-operads and let $\mathcal{M}$ denote the fiber $\mathcal{C}_{\mathfrak{m}}$. We let $\mathrm{BMod}(\mathcal{M})$ denote the ∞-category $\mathrm{Alg}_{/\mathcal{BM}}(\mathcal{C})$. We will refer to $\mathrm{BMod}(\mathcal{M})$ as the ∞-*category of bimodule objects of* $\mathcal{M}$.

Composition with the inclusions $\mathcal{A}ss_{-}^{\otimes}, \mathcal{A}ss_{+}^{\otimes} \subseteq \mathcal{BM}^{\otimes}$ determines a categorical fibration $\mathrm{BMod}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{C}_{-}) \times \mathrm{Alg}(\mathcal{C}_{+})$. Given algebra objects $A \in \mathrm{Alg}(\mathcal{C}_{-})$ and $B \in \mathrm{Alg}(\mathcal{C}_{+})$, we let $_A\mathrm{BMod}_B(\mathcal{M})$ denote the fiber

$$\{A\} \times_{\mathrm{Alg}(\mathcal{C}_{-})} \mathrm{BMod}(\mathcal{M}) \times_{\mathrm{Alg}(\mathcal{C}_{+})} \{B\}.$$

We will refer to $_A\mathrm{BMod}_B(\mathcal{M})$ as the ∞-*category of A-B-bimodule objects of* $\mathcal{M}$.

Remark 4.3.1.13. The notation of Definition 4.3.1.12 is somewhat abusive: the ∞-category $\mathrm{BMod}(\mathcal{M})$ depends not only on the ∞-category $\mathcal{M}$, but also on the fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{BM}^{\otimes}$ having $\mathcal{M}$ as a fiber.

Remark 4.3.1.14. Let $q : \mathcal{C}^{\otimes} \to \mathcal{BM}^{\otimes}$ be a fibration of ∞-operads and $\mathcal{M} = \mathcal{C}_{\mathfrak{m}}$. We can think of the objects of $\mathrm{BMod}(\mathcal{M})$ as given by triples (A, B, M), where $A \in \mathrm{Alg}(\mathcal{C}_{-})$, $B \in \mathrm{Alg}(\mathcal{C}_{+})$, and M is an A-B-bimodule object of $\mathcal{M}$.

Example 4.3.1.15. Let $\mathcal{C}^{\otimes} \to \mathcal{A}ss^{\otimes}$ be a planar ∞-operad. Form the fiber product $\mathcal{O}^{\otimes} = \mathcal{C}^{\otimes} \times_{\mathcal{A}ss^{\otimes}} \mathcal{BM}^{\otimes}$ using the fibration of ∞-operads $\mathcal{LM}^{\otimes} \to \mathcal{A}ss^{\otimes}$ of Remark 4.3.1.8. The fiber $\mathcal{O}_{\mathfrak{m}}$ is isomorphic to $\mathcal{C}$, and the planar ∞-operads $\mathcal{O}_{-}^{\otimes}$ and $\mathcal{O}_{+}^{\otimes}$ are isomorphic to $\mathcal{C}^{\otimes}$. We can therefore consider the ∞-category $\mathrm{BMod}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{BM}/\mathcal{A}ss}(\mathcal{C})$. of bimodule objects of $\mathcal{C}$.

Example 4.3.1.16. Let $\mathcal{C}^{\otimes} \to \mathcal{A}ss^{\otimes}$ be a planar ∞-operad. Composition with the forgetful functor $\mathcal{BM}^{\otimes} \to \mathcal{A}ss^{\otimes}$ of Remark 4.3.1.8 determines a map $s : \mathrm{Alg}(\mathcal{C}) \to \mathrm{BMod}(\mathcal{C})$. This map carries each $A \in \mathrm{Alg}(\mathcal{C})$ to an object $s(A) \in {_A\mathrm{BMod}_A}(\mathcal{C})$. We will can think of $s(A)$ as A, regarded as a bimodule over itself. For this reason, we will often not distinguish in notation between A and $s(A)$.

Definition 4.3.1.17. Let $q : \mathcal{C}^{\otimes} \to \mathcal{BM}^{\otimes}$ be a fibration of ∞-operads. We will say that q *exhibits* $\mathcal{C}_{\mathfrak{m}}$ *as bitensored over* $\mathcal{C}_{-}$ *and* $\mathcal{C}_{+}$ if the map q is a coCartesian fibration.

Remark 4.3.1.18. Let $\mathcal{C}^{\otimes} \to \mathcal{BM}^{\otimes}$ be a coCartesian fibration of ∞-operads. Then $\mathcal{C}_{-}^{\otimes}$ and $\mathcal{C}_{+}^{\otimes}$ are monoidal ∞-categories. The ∞-category $\mathcal{C}_{\mathfrak{m}}$ is left-tensored over $\mathcal{C}_{-}^{\otimes}$ (in the sense of Definition 4.2.1.19). Similarly, we can regard $\mathcal{C}_{\mathfrak{m}}$ as right-tensored over $\mathcal{C}_{+}^{\otimes}$ (see Variant 4.2.1.36). Moreover, the left and right actions of $\mathcal{C}_{-}$ and $\mathcal{C}_{+}$ on $\mathcal{C}_{\mathfrak{m}}$ commute up to coherent homotopy; in particular, the diagram of ∞-categories

$$\begin{array}{ccc}
\mathcal{C}_{-} \times \mathcal{C}_{\mathfrak{m}} \times \mathcal{C}_{+} & \longrightarrow & \mathcal{C}_{-} \times \mathcal{C}_{\mathfrak{m}} \\
\downarrow & & \downarrow \\
\mathcal{C}_{\mathfrak{m}} \times \mathcal{C}_{+} & \longrightarrow & \mathcal{C}_{\mathfrak{m}}
\end{array}$$

commutes up to canonical equivalence.

Remark 4.3.1.19. Suppose that $q : \mathcal{C}^{\otimes} \to \mathcal{BM}^{\otimes}$ is a coCartesian fibration of ∞-operads, so that q exhibits $\mathcal{M} = \mathcal{C}_{\mathfrak{m}}$ as bitensored over the monoidal ∞-categories $\mathcal{C}_{-}^{\otimes}$ and $\mathcal{C}_{+}^{\otimes}$. Let us use the symbol $\otimes$ to indicate all of the induced functors

$$\mathcal{C}_{-} \times \mathcal{C}_{-} \to \mathcal{C}_{-} \qquad \mathcal{C}_{+} \times \mathcal{C}_{+} \to \mathcal{C}_{+}$$
$$\mathcal{C}_{-} \times \mathcal{M} \to \mathcal{M} \qquad \mathcal{M} \times \mathcal{C}_{+} \to \mathcal{M}.$$

Let $F : \mathcal{BM}^{\otimes} \to \mathcal{C}^{\otimes}$ be a bimodule object of $\mathcal{M}$. Then $A_{-} = F(\mathfrak{a}_{-})$ and $A_{+} \in F(\mathfrak{a}_{+})$ are algebra objects of the monoidal ∞-categories $\mathcal{C}_{-}^{\otimes}$ and $\mathcal{C}_{+}^{\otimes}$, respectively. The object $M = F(\mathfrak{m}) \in \mathcal{M}$ is equipped with left and right actions

$$A_{-} \otimes M \xrightarrow{a} M \xleftarrow{a'} M \otimes A_{+}$$

which exhibit M as an A_{-}-A_{+}-bimodule object of the homotopy category $h\mathcal{M}$. In other words, the following diagrams commute up to homotopy:

$$
\begin{array}{ccc}
A_{-} \otimes A_{-} \otimes M & \xrightarrow{m \otimes \mathrm{id}_M} & A_{-} \otimes M \\
{\scriptstyle \mathrm{id}_{A_{-}} \otimes a} \downarrow & & \downarrow {\scriptstyle a} \\
A_{-} \otimes M & \xrightarrow{a} & M
\end{array}
\qquad
\begin{array}{ccc}
M \otimes A_{+} \otimes A_{+} & \xrightarrow{\mathrm{id}_M \otimes m} & M \otimes A_{+} \\
{\scriptstyle a' \otimes \mathrm{id}_{A_{+}}} \downarrow & & \downarrow {\scriptstyle a'} \\
M \otimes A_{+} & \xrightarrow{a'} & M
\end{array}
$$

$$
\begin{array}{ccc}
A_{-} \otimes M \otimes A_{+} & \xrightarrow{\mathrm{id}_{A_{-}} \otimes a'} & M \otimes A_{-} \otimes M \\
{\scriptstyle a \otimes \mathrm{id}_{A_{+}}} \downarrow & & \downarrow {\scriptstyle a} \\
M \otimes A_{+} & \xrightarrow{a'} & M.
\end{array}
$$

4.3.2 Bimodules, Left Modules, and Right Modules

Let $\mathcal{C}$ be a monoidal category containing an object M. If A and B are algebra objects of $\mathcal{C}$, then an A-B-bimodule structure on M is determined by the data of a left action of A on M and a right action of B on M. These data are merely required to satisfy a condition: the commutativity of the diagram σ :

$$
\begin{array}{ccc}
A \otimes M \otimes B & \longrightarrow & A \otimes M \\
\downarrow & & \downarrow \\
M \otimes B & \longrightarrow & M.
\end{array}
$$

In the ∞-categorical setting, the situation is more subtle. The diagram σ is required to commute up to homotopy, and that homotopy is taken as part of the data defining a bimodule structure on M. Consequently, to describe an A-B-bimodule structure on M, it is not sufficient to specify the action of A on M and the right action of B on M individually. However, we *can* recover the bimodule structure on M if we view M as a right B-module in the ∞-category of left A-modules, rather than the ∞-category $\mathcal{C}$ itself. Our goal in this section is to give a precise formulation and proof of this assertion.

We begin by considering a more general situation. Let $q : \mathcal{C}^{\otimes} \to \mathcal{BM}^{\otimes}$ be a coCartesian fibration of ∞-operads, so that q exhibits the ∞-category $\mathcal{C}_{\mathfrak{m}}$ as bitensored over the monoidal ∞-categories $\mathcal{C}_{-}^{\otimes}$ and $\mathcal{C}_{+}^{\otimes}$. In particular, $\mathcal{C}_{\mathfrak{m}}$ is left-tensored over $\mathcal{C}_{-}^{\otimes}$, so that we can consider the ∞-category $\mathrm{LMod}(\mathcal{C}_{\mathfrak{m}})$ whose objects are pairs (A, M), where B is an associative algebra object of $\mathcal{C}_{-}^{\otimes}$ and M is a left A-module. Our first goal is to show that $\mathrm{LMod}(\mathcal{C}_{\mathfrak{m}})$ is right-tensored over $\mathcal{C}_{+}^{\otimes}$. The left action of $\mathcal{C}_{+}^{\otimes}$ can be described informally by the formula $(A, M) \otimes B = (A, M \otimes B)$: in other words, we claim that if M has the structure of a left A-module, then the tensor product $M \otimes B$ inherits a left action of A.

Our first step is to construct a functor $\mathcal{LM}^{\otimes} \times \mathcal{RM}^{\otimes} \to \mathcal{BM}^{\otimes}$. In what follows, we adopt the following notational convention: if we are given a subset $K \subseteq \langle m \rangle^{\circ} \times \langle n \rangle^{\circ}$, then we implicitly choose an order-preserving isomorphism $\langle k \rangle^{\circ} \to K$ (where the ordering on K is inherited from the lexicographical ordering

on $\langle m\rangle^\circ \times \langle n\rangle^\circ$). This gives an isomorphism of finite pointed sets $K_* = K \cup \{*\} \simeq \langle k\rangle$; we will abuse notation and identify K_* with the corresponding object of Fin_*.

Construction 4.3.2.1. We define a functor $\mathbf{Pr} : \mathbf{LM}^\otimes \times \mathbf{RM}^\otimes \to \mathbf{BM}^\otimes$ is defined as follows:

(1) Let $(\langle m\rangle, S)$ be an object of $\mathbf{LM}^\otimes$ and $(\langle n\rangle, T)$ an object of $\mathbf{RM}^\otimes$. We let $\mathbf{Pr}((\langle m\rangle, S), (\langle n\rangle, T)) = (X_*, c_-, c_+)$, where X is the finite set

$$(\langle m\rangle^\circ \times T) \coprod_{S \times T} (S \times \langle n\rangle^\circ) \subseteq \langle m\rangle^\circ \times \langle n\rangle^\circ \simeq \langle mn\rangle^\circ$$

and the functions $c_-, c_+ : X \to [1]$ are given by the formulas

$$c_-(i,j) = \begin{cases} 0 & \text{if } j \in T \\ 1 & \text{if } j \notin T \end{cases} \qquad c_+(i,j) = \begin{cases} 0 \text{ if } i \notin S \\ 1 \qquad\qquad \text{if } i \in S. \end{cases}$$

(2) Let $\alpha : (\langle m\rangle, S) \to (\langle m'\rangle, S')$ be a morphism in $\mathbf{LM}^\otimes$ and $\beta : (\langle n\rangle, T) \to (\langle n'\rangle, T')$ a morphism in $\mathbf{RM}^\otimes$. Let $(X_*, c_-, c_+) = \mathbf{Pr}((\langle m\rangle, S), (\langle n\rangle, T))$ and $(X'_*, c'_-, c'_+) = \mathbf{Pr}((\langle m'\rangle, S'), (\langle n'\rangle, T'))$ be defined as above. Then $\mathbf{Pr}(\alpha, \beta) : (X_*, c_-, c_+) \to (X'_*, c'_-, c'_+)$ is the unique morphism in $\mathbf{BM}^\otimes$ lying over the map $\gamma : X_* \to X'_*$ which is described as follows:

(i) If $(i, j) \in X \subseteq \langle m\rangle^\circ \times \langle n\rangle^\circ$, then

$$\gamma(i,j) = \begin{cases} (\alpha(i), \beta(j)) & \text{if } \alpha(i) \in \langle m'\rangle^\circ, \beta(j) \in \langle n'\rangle^\circ \\ * & \text{otherwise.} \end{cases}$$

(ii) Let $i' \in \langle m'\rangle^\circ - S$ and $j' \in T'$, so that $j' = \beta(j)$ for a unique element $j \in T$. Then the linear ordering on $\gamma^{-1}\{(i', j')\} = \alpha^{-1}\{i'\} \times \{j\}$ is determined by the map α in $\mathbf{Ass}^\otimes$.

(iii) Let $i' \in S$ and $j' \in \langle n'\rangle^\circ - T'$, so that $i' = \alpha(i)$ for a unique element $i \in S$. Then the linear ordering on $\gamma^{-1}\{(i', j')\} = \{i\} \times \beta^{-1}\{j'\}$ is determined by the map β in $\mathbf{Ass}^\otimes$.

(iv) Let $i' \in S'$ and $j' \in T'$, so that $i' = \alpha(i)$ and $j' = \beta(j)$ for unique elements $i \in S$, $j \in T$. Then $\gamma^{-1}\{(i', j')\} = (\alpha^{-1}\{i'\} \times \{j\}) \coprod_{\{(i,j)\}} (\{i\} \times \beta^{-1}\{j'\})$. We endow $\gamma^{-1}\{(i', j')\}$ with the unique linear ordering which is compatible with the linear orders on $\alpha^{-1}\{i'\}$ and $\beta^{-1}\{j'\}$ determined by α and β (so that $x \leq y$ for $x \in \alpha^{-1}\{i'\} \times \{j\}$) and $y \in \{i\} \times \beta^{-1}\{j'\}$, with equality if and only if $x = y = (i, j)$).

We will also use $\mathbf{Pr}$ to denote the induced map of ∞-categories $\mathcal{LM}^\otimes \times \mathcal{RM}^\otimes \to \mathcal{BM}^\otimes$.

Construction 4.3.2.2. Let $q : \mathcal{C}^\otimes \to \mathcal{BM}^\otimes$ be a fibration of ∞-operads. We define a map of simplicial sets $\overline{\mathrm{LMod}}(\mathcal{C}_\mathfrak{m})^\otimes \to \mathcal{RM}^\otimes$ so that the following universal property is satisfied: for every map of simplicial sets $K \to \mathcal{RM}^\otimes$, there is a canonical bijection

$$\mathrm{Hom}_{(\mathrm{Set}_\Delta)_{/\mathcal{RM}^\otimes}}(K, \overline{\mathrm{LMod}}(\mathcal{C}_\mathfrak{m})^\otimes) \simeq \mathrm{Hom}_{(\mathrm{Set}_\Delta)_{/\mathcal{BM}^\otimes}}(\mathcal{LM}^\otimes \times K, \mathcal{C}^\otimes).$$

Let $\mathrm{LMod}(\mathcal{C}_\mathfrak{m})^\otimes$ denote the full simplicial subset of $\overline{\mathrm{LMod}}(\mathcal{C}_\mathfrak{m})^\otimes$ spanned by those vertices which correspond to a vertex $X \in \mathcal{RM}^\otimes$ together with a functor $F : \mathcal{LM}^\otimes \times \{X\} \to \mathcal{C}^\otimes$ which carries inert morphisms in $\mathcal{LM}^\otimes$ to inert morphisms in $\mathcal{C}^\otimes$.

Remark 4.3.2.3. As usual, our notation is somewhat abusive: the ∞-category $\mathrm{LMod}(\mathcal{C}_\mathfrak{m})^\otimes$ depends on the fibration of ∞-operads $q : \mathcal{C}^\otimes \to \mathcal{BM}^\otimes$, and not just the fiber $\mathcal{C}_\mathfrak{m}$.

Remark 4.3.2.4. The composite map

$$\mathcal{LM}^\otimes \times \{\mathfrak{m}\} \hookrightarrow \mathcal{LM}^\otimes \times \mathcal{RM}^\otimes \xrightarrow{\mathbf{Pr}} \mathcal{BM}^\otimes$$

coincides with the inclusion $\mathcal{LM}^\otimes \hookrightarrow \mathcal{BM}^\otimes$ of Remark 4.3.1.9. If $\mathcal{C}^\otimes \to \mathcal{BM}^\otimes$ is a fibration of ∞-operads, we obtain a canonical isomorphism of simplicial sets

$$\mathrm{LMod}(\mathcal{C}_\mathfrak{m})^\otimes \times_{\mathcal{RM}^\otimes} \{\mathfrak{m}\} \simeq \mathrm{LMod}(\mathcal{C}_\mathfrak{m}),$$

where $\mathrm{LMod}(\mathcal{C}_\mathfrak{m})$ denotes the ∞-category of left modules associated to the fibration of ∞-operads

$$\mathcal{C}^\otimes \times_{\mathcal{BM}^\otimes} \mathcal{LM}^\otimes \to \mathcal{LM}^\otimes.$$

We can now state our first main result:

Proposition 4.3.2.5. *Let $q : \mathcal{C}^\otimes \to \mathcal{BM}^\otimes$ be a fibration of ∞-operads. Then:*

(1) *The induced map map $p : \mathrm{LMod}(\mathcal{C}_\mathfrak{m})^\otimes \to \mathcal{RM}^\otimes$ is also a fibration of ∞-operads.*

(2) *A morphism α in $\mathrm{LMod}(\mathcal{C}_\mathfrak{m})^\otimes$ is inert if and only if $p(\alpha)$ is an inert morphism in $\mathcal{RM}^\otimes$ and, for every object $X \in \mathcal{LM}$, the image $\alpha(X)$ is an inert morphism in $\mathcal{C}^\otimes$.*

(3) *Suppose that q is a coCartesian fibration of ∞-operads. Then p is a coCartesian fibration of ∞-operads.*

(4) *Assume that q is a coCartesian fibration of ∞-operads. Then a morphism $\alpha \in \mathrm{LMod}(\mathcal{C}_\mathfrak{m})^\otimes$ is p-coCartesian if and only if, for every $X \in \mathcal{LM}$, the image $\alpha(X)$ is a q-coCartesian morphism in $\mathcal{C}^\otimes$.*

Proof. We will prove (1) and (2); the proofs of (3) and (4) are similar. The proof is essentially the same as that of Proposition 3.2.4.3, despite the fact that $\mathbf{Pr} : \mathcal{LM}^\otimes \times \mathcal{RM}^\otimes \to \mathcal{BM}^\otimes$ is not a bifunctor of ∞-operads. We note that the condition on a morphism $\alpha : M \to M' \in \mathrm{LMod}(\mathcal{C}_\mathfrak{m})^\otimes$ appearing in (2) is equivalent to the following apparently stronger condition:

(∗) For every inert morphism $\beta : X \to Y$ in $\mathcal{LM}^\otimes$, the induced map $M(X) \to M'(Y)$ is an inert morphism in $\mathcal{C}^\otimes$.

For every ∞-operad $\mathcal{O}^\otimes$, let $\mathfrak{P}_\mathcal{O}$ be the categorical pattern appearing in the proof of Proposition 3.2.4.3. Using the functor $\mathbf{Pr}$, we see that the construction $\overline{X} \mapsto \mathcal{LM}^{\otimes,\natural} \times \overline{X}$ determines a functor $F : (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}_{\mathcal{RM}}} \to (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}_{\mathcal{BM}}}$. Let G denote a right adjoint to F. Assertions (1) and (2) are equivalent to the requirement that G preserves fibrant objects. We will complete the proof by showing that F is a left Quillen functor. Since $\mathcal{LM}^{\otimes,\natural} \in (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}_{\mathcal{LM}}}$ is cofibrant, it will suffice to show that the product functor

$$(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}_{\mathcal{LM}}} \times (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}_{\mathcal{RM}}} \to (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}_{\mathcal{BM}}}$$

determined by $\mathbf{Pr}$ is a left Quillen bifunctor. This follows immediately from Remark B.2.5 and Proposition B.2.9. $\square$

Let $q : \mathcal{C}^\otimes \to \mathcal{BM}^\otimes$ be a fibration of ∞-operads. We note that the inclusion $\{\mathfrak{m}\} \hookrightarrow \mathcal{LM}^\otimes$ determines a forgetful functor $\theta : \mathrm{LMod}(\mathcal{C}_\mathfrak{m})^\otimes \to \mathcal{C}^\otimes \times_{\mathcal{BM}^\otimes} \mathcal{RM}^\otimes$. Using Proposition 4.3.2.5, we see that θ is a fibration of ∞-operads. Moreover, if q is a coCartesian fibration of ∞-operads, then θ is an $\mathcal{RM}$-monoidal functor.

Proposition 4.3.2.6. *Let $q : \mathcal{C}^\otimes \to \mathcal{BM}^\otimes$ be a fibration of ∞-operads. Then the forgetful functor*

$$\mathrm{LMod}(\mathcal{C}_\mathfrak{m})^\otimes \to \mathcal{C}^\otimes \times_{\mathcal{BM}^\otimes} \mathcal{RM}^\otimes$$

induces a trivial Kan fibration of planar ∞-operads

$$\mathrm{LMod}(\mathcal{C}_\mathfrak{m})^\otimes_\mathfrak{a} \to \mathcal{C}^\otimes_+.$$

Proof. Let $\mathfrak{X}$ denote the full subcategory of $\mathrm{Fun}(\mathcal{LM}^{\otimes}, \mathcal{BM}^{\otimes})$ spanned by those functors of the form $X \mapsto \mathbf{Pr}(X, Y)$, where $Y \in \mathcal{Ass}_{+}^{\otimes} \subseteq \mathcal{RM}^{\otimes}$. Let $\mathfrak{X}'$ denote the full subcategory of $\mathrm{Fun}(\mathcal{LM}^{\otimes}, \mathcal{C}^{\otimes})$ spanned by those functors $F : \mathcal{LM}^{\otimes} \to \mathcal{C}^{\otimes}$ such that $q \circ F \in \mathfrak{X}$ and F carries inert morphisms in $\mathcal{LM}^{\otimes}$ to inert morphisms in $\mathcal{C}^{\otimes}$. There is an evident pullback diagram

$$
\begin{array}{ccc}
\mathrm{LMod}(\mathcal{C}_{\mathfrak{m}})_{\mathfrak{a}}^{\otimes} & \longrightarrow & \mathfrak{X}' \\
\downarrow & & \downarrow{\scriptstyle \theta} \\
\mathcal{C}_{+}^{\otimes} & \longrightarrow & \mathfrak{X} \times_{\mathrm{Fun}(\{\mathfrak{m}\}, \mathcal{BM}^{\otimes})} \mathrm{Fun}(\{\mathfrak{m}\}, \mathcal{C}^{\otimes}).
\end{array}
$$

It will therefore suffice to show that the map θ is a trivial Kan fibration.

Let $\mathcal{LM}_0^{\otimes}$ denote the full subcategory of $\mathcal{LM}^{\otimes}$ spanned by those objects of the form $(\langle n \rangle, S)$, where $S = \langle n \rangle^{\circ}$. Then $\mathcal{LM}_0^{\otimes}$ is isomorphic to the trivial ∞-operad $\mathcal{Triv}^{\otimes}$ of Example 2.1.1.20. Let $\mathcal{Z}'$ denote the full subcategory of $\mathrm{Fun}(\mathcal{LM}_0^{\otimes}, \mathcal{C}^{\otimes})$ spanned by those functors which carry each morphism in $\mathcal{LM}_0^{\otimes}$ to an inert morphism in $\mathcal{C}^{\otimes}$. The map θ factors as a composition

$$
\mathfrak{X}' \xrightarrow{\theta'} \mathfrak{X} \times_{\mathrm{Fun}(\mathcal{LM}_0^{\otimes}, \mathcal{BM}^{\otimes})} \mathcal{Z}' \xrightarrow{\theta''} \mathfrak{X} \times_{\mathrm{Fun}(\{\mathfrak{m}\}, \mathcal{BM}^{\otimes})} \mathrm{Fun}(\{\mathfrak{m}\}, \mathcal{C}^{\otimes}).
$$

We will complete the proof by showing that θ' and θ'' are trivial Kan fibrations.

We first show that θ' is a trivial Kan fibration. In view of Proposition T.4.3.2.15, it will suffice to show the following:

(a) Let $Y \in \mathcal{Ass}_{+}^{\otimes}$ and let $f : \mathcal{LM}^{\otimes} \to \mathcal{BM}^{\otimes}$ be given by the formula $f(X) = \mathbf{Pr}(X, Y)$. Suppose that $f | \mathcal{LM}_0^{\otimes}$ lifts to an object $\overline{f} \in \mathcal{Z}'$. Then there exists a dotted arrow as indicated in the diagram

$$
\begin{array}{ccc}
\mathcal{LM}_0^{\otimes} & \xrightarrow{\ \overline{f}\ } & \mathcal{C}^{\otimes} \\
\downarrow & {\scriptstyle F}\ \nearrow & \downarrow{\scriptstyle q} \\
\mathcal{LM}^{\otimes} & \xrightarrow{\ f\ } & \mathcal{BM}^{\otimes},
\end{array}
$$

such that F is a q-left Kan extension of $\overline{f}$.

(b) Given a commutative diagram as above, F is a q-left Kan extension of $\overline{f}$ if and only if F carries inert morphisms in $\mathcal{LM}^{\otimes}$ to inert morphisms in $\mathcal{C}^{\otimes}$.

Note that the inclusion $\mathcal{LM}_0^{\otimes} \hookrightarrow \mathcal{LM}^{\otimes}$ admits a right adjoint r, which carries an object $(\langle n \rangle, S) \in \mathcal{LM}^{\otimes}$ to the pair (S_*, S) (where we abuse notation by identify the finite pointed set $S_* = S \cup \{*\}$ with an object of $\mathcal{Fin}_*$). For each object $X \in \mathcal{LM}^{\otimes}$, our assumption that $Y \in \mathcal{Ass}_{+}^{\otimes}$ guarantees that the functor f carries the counit map $v_X : r(X) \to X$ to degenerate edge of $\mathcal{BM}^{\otimes}$. To prove (a), we take $F = \overline{f} \circ r$. Moreover, we obtain the following version of (b):

(b') Given a commutative diagram as in (a), the functor F is a q-left Kan extension of $\overline{f}$ if and only if, for each $X \in \mathcal{LM}^{\otimes}$, the induced map $F(r(X)) \to F(X)$ is an equivalence in $\mathcal{C}^{\otimes}$.

We now prove (b). Assume we are given a commutative diagram as in (a). Suppose first that F carries inert morphisms in $\mathcal{LM}^{\otimes}$ to inert morphisms in $\mathcal{C}^{\otimes}$. Let $X \in \mathcal{LM}^{\otimes}$, and note that there exists a commutative diagram

$$
\begin{array}{ccc}
 & X & \\
{\scriptstyle v_X}\nearrow & & \searrow{\scriptstyle \beta} \\
r(X) & \xrightarrow{\ \ \mathrm{id}\ \ } & r(X).
\end{array}
$$

Then $F(\beta)$ is an inert morphism in the fiber $\mathcal{C}^{\otimes}_{f(X)}$ and therefore an equivalence. Since $F(v_X)$ is a right inverse to $F(\beta)$, we conclude that $F(v_X)$ is an equivalence. Using (b'), we conclude that F is a q-left Kan extension of $\overline{f}$. Conversely, suppose that F is a q-left Kan extension of $\overline{f}$,and let $\alpha : X \to Y$ be an inert morphism in $\mathcal{LM}^{\otimes}$. We have a commutative diagram

$$
\begin{array}{ccc}
r(X) & \xrightarrow{v_X} & X \\
\downarrow{\scriptstyle r(\alpha)} & & \downarrow{\scriptstyle \alpha} \\
r(Y) & \xrightarrow{v_Y} & Y
\end{array}
$$

in $\mathcal{LM}^{\otimes}$. Using (b'), we see that $F(v_X)$ and $F(v_Y)$ are equivalences in $\mathcal{C}^{\otimes}$. Our assumption that $\overline{f} \in \mathcal{Z}'$ guarantees that $F(r(\alpha))$ is inert, so that $F(\alpha)$ is inert as desired. This completes the proof that θ' is a trivial Kan fibration.

Let $\mathcal{Z}$ denote the full subcategory of $\mathrm{Fun}(\mathcal{LM}^{\otimes}_0, \mathcal{BM}^{\otimes})$ spanned by those functors of the form $X \mapsto \mathbf{Pr}(X, Y)$ for some $Y \in \mathcal{A}\mathrm{ss}^{\otimes}_+$. We observe that θ'' is a pullback of the forgetful functor

$$
\psi : \mathcal{Z}' \to \mathcal{Z} \times_{\mathrm{Fun}(\{\mathfrak{m}\}, \mathcal{BM}^{\otimes})} \mathrm{Fun}(\{\mathfrak{m}\}, \mathcal{RM}^{\otimes}).
$$

It will therefore suffice to show that ψ is a trivial Kan fibration. We proceed as in Example 2.1.3.5. Using Proposition T.4.3.2.15, we are reduced to proving the following:

(c) Let $Y \in \mathcal{A}\mathrm{ss}^{\otimes}_+$ and let $g : \mathcal{LM}^{\otimes}_0 \to \mathcal{BM}^{\otimes}$ be given by the formula $g(X) = \mathbf{Pr}(X, Y)$. Let $\overline{Y}$ be an object of the fiber $\mathcal{C}^{\otimes}_Y$. Then there exists a dotted arrow as indicated in the diagram

$$
\begin{array}{ccc}
\{\mathfrak{m}\} & \xrightarrow{\overline{Y}} & \mathcal{C}^{\otimes} \\
\downarrow & {\scriptstyle G}\;\nearrow & \downarrow{\scriptstyle q} \\
\mathcal{LM}^{\otimes}_0 & \xrightarrow{g} & \mathcal{BM}^{\otimes},
\end{array}
$$

such that G is a q-right Kan extension of $G|\{\mathfrak{m}\}$.

(d) Given a commutative diagram as above, G is a q-right Kan extension of $G|\{\mathfrak{m}\}$ if and only if carries inert morphisms in $\mathcal{LM}^{\otimes}_0$ to inert morphisms in $\mathcal{C}^{\otimes}$.

Assertion (c) follows immediately from Remark 2.1.2.11, since q is a fibration of ∞-operads. The characterization of assertion (d) follows from Remark 2.1.2.9 (applied to G, regarded as a section of the ∞-operad fibration $\mathcal{C}^{\otimes} \times_{\mathcal{BM}^{\otimes}} \mathcal{LM}^{\otimes}_0$). $\qquad\square$

The main result of this section can be stated as follows:

Theorem 4.3.2.7. *Let* $q : \mathcal{C}^{\otimes} \to \mathcal{BM}^{\otimes}$ *be a fibration of* ∞*-operads. Then composition with the functor* $\mathbf{Pr} : \mathcal{LM}^{\otimes} \times \mathcal{RM}^{\otimes} \to \mathcal{BM}^{\otimes}$ *induces an equivalence of* ∞*-categories*

$$
\theta : \mathrm{BMod}(\mathcal{C}_{\mathfrak{m}}) \to \mathrm{RMod}(\mathrm{LMod}(\mathcal{C}_{\mathfrak{m}})).
$$

Proof. Let $\mathcal{X}$ denote the fiber product $\mathcal{C}^{\otimes} \times_{\mathcal{BM}^{\otimes}} \mathcal{LM}^{\otimes} \times \mathcal{RM}^{\otimes}$. The canonical map

$$
\mathcal{X} \to \mathcal{LM}^{\otimes} \times \mathcal{RM}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \mathcal{RM}^{\otimes}
$$

exhibits $\mathcal{X}$ as a $\mathcal{RM}^{\otimes}$-family of ∞-operads. Let $\gamma : \mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1 \to \mathcal{LM}^{\otimes}$ be as in Remark 4.2.2.8 and let M be the collection of morphisms f in $\mathrm{N}(\boldsymbol{\Delta}\,0^{op} \times \Delta^1$ such that $\gamma(f)$ is an inert morphism in $\mathcal{LM}^{\otimes}$. Let $\mathcal{Y}^{\otimes}$ denote the full subcategory of

$$
\mathrm{Fun}_{\mathcal{LM}^{\otimes}}(\mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1, \mathcal{X}) \times_{\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1, \mathcal{RM}^{\otimes})} \mathcal{RM}^{\otimes}
$$

spanned by those pairs (F, Y), where $Y \in \mathcal{RM}^\otimes$ and $F : \mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1 \to \mathfrak{X}_Y$ is a functor which carries every morphism in M to an inert morphism in the ∞-operad $\mathfrak{X}_Y$. It follows from Proposition 4.2.2.12 that composition with γ induces an equivalence of ∞-categories $\mathrm{LMod}(\mathcal{C}_{\mathfrak{m}})^\otimes \to \mathcal{Y}$. In particular, the categorical fibration $p : \mathcal{Y}^\otimes \to \mathcal{RM}^\otimes$ is a fibration of ∞-operads. Note also that a morphism in $\mathcal{Y}^\otimes$ is inert if and only if its image in $\mathcal{RM}^\otimes$ is inert and the corresponding functor $\mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1 \times \Delta^1 \to \mathfrak{X}$ carries every morphism in

$$M \times \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^1, \Delta^1) \subseteq \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^1, \mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1 \times \Delta^1)$$

to an inert morphism in $\mathfrak{X}$.

Reversing the order of linearly ordered sets determines isomorphisms

$$\mathrm{rev} : \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathrm{N}(\boldsymbol{\Delta})^{op} \qquad \mathrm{rev} : \mathcal{LM}^\otimes \to \mathcal{RM}^\otimes .$$

Let $\gamma' : \mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1 \to \mathcal{RM}^\otimes$ be defined so that the diagram

$$
\begin{array}{ccc}
\mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1 & \xrightarrow{\ \gamma\ } & \mathcal{LM}^\otimes \\
{\scriptstyle \mathrm{rev} \times \mathrm{id}}\downarrow & & \downarrow {\scriptstyle \mathrm{rev}} \\
\mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1 & \xrightarrow{\ \gamma'\ } & \mathcal{RM}^\otimes
\end{array}
$$

commutes, and let M' be the collection of all morphisms f in $\mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1$ such that $\gamma'(f)$ is an inert morphism in $\mathcal{RM}^\otimes$. Let $\mathcal{Z}$ be the full subcategory of $\mathrm{Fun}_{\mathcal{RM}^\otimes}(\mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1, \mathcal{Y}^\otimes)$ spanned by those functors which carry every morphism in M' to an inert morphism in $\mathcal{Y}$. The analogue of Proposition 4.2.2.12 for right modules implies that γ' induces an equivalence of ∞-categories $\theta'' : \mathrm{RMod}(\mathcal{Y}_{\mathfrak{m}}) \to \mathcal{Z}$.

Let $\phi : \mathrm{BMod}(\mathcal{C}_{\mathfrak{m}}) \to \mathcal{Z}$ be the composite functor

$$\mathrm{BMod}(\mathcal{C}_{\mathfrak{m}}) \xrightarrow{\theta} \mathrm{RMod}(\mathrm{LMod}(\mathcal{C})) \xrightarrow{\theta'} \mathrm{RMod}(\mathcal{Y}_{\mathfrak{m}}) \xrightarrow{\theta''} \mathcal{Z} .$$

The above argument shows that θ' is an equivalence of ∞-categories, and the analogue of Proposition 4.2.2.11 for right modules implies that θ'' is an equivalence of ∞-categories. To complete the proof, it will suffice to show that ϕ is an equivalence of ∞-categories.

Let $\mathcal{I}_1$ denote the category $\boldsymbol{\Delta}^{op} \times \boldsymbol{\Delta}^{op} \times [1] \times [1]$. Let $\psi : \mathcal{I}_1 \to \mathbf{BM}^\otimes$ be the composite functor

$$\mathcal{I}_1 \xrightarrow{\gamma \times \gamma'} \mathbf{LM}^\otimes \times \mathbf{RM}^\otimes \xrightarrow{\ \mathbf{Pr}\ } \mathbf{BM}^\otimes .$$

Unwinding the definitions, we see that $\mathcal{Z}$ can be identified with the full subcategory of $\mathrm{Fun}_{\mathcal{BM}^\otimes}(\mathrm{N}(\mathcal{I}_1), \mathcal{C}^\otimes)$ spanned by those functors which carry every morphism belonging to $M \times M'$ to an inert morphism in $\mathcal{C}^\otimes$.

Let $\mathcal{I}$ denote the categorical mapping cylinder of the functor $\psi : \mathcal{I}_1 \to \mathbf{BM}^\otimes$. More precisely, we define $\mathcal{I}$ as follows:

(1) An object of $\mathcal{I}$ is either an object of $\mathbf{BM}^\otimes$ or an object of $\mathcal{I}_1$.

(2) Morphisms in $\mathcal{I}$ are given by the formulas

$$
\mathrm{Hom}_{\mathcal{I}}(X, Y) = \begin{cases}
\mathrm{Hom}_{\mathbf{BM}^\otimes}(X, Y) & \text{if } X, Y \in \mathbf{BM}^\otimes \\
\mathrm{Hom}_{\mathcal{I}_1}(X, Y) & \text{if } X, Y \in \mathcal{I}_1 \\
\mathrm{Hom}_{\mathbf{BM}^\otimes}(X, \psi(Y)) & \text{if } X \in \mathbf{BM}^\otimes, Y \in \mathcal{I}_1 \\
\emptyset & \text{if } X \in \mathcal{I}_1, Y \in \mathbf{BM}^\otimes .
\end{cases}
$$

The functor ψ extends to a retraction of $\mathcal{I}$ onto the subcategory $\mathbf{BM}^\otimes \subseteq \mathcal{I}$. We let $\mathcal{Z}'$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{BM}^\otimes}(\mathrm{N}(\mathcal{I}), \mathcal{C}^\otimes)$ spanned by those functors $f : \mathrm{N}(\mathcal{I}) \to \mathcal{C}^\otimes$ which satisfy the following codnitions:

(i) For every object $X \in \mathcal{I}_0$, the canonical map $f(\psi(X)) \to f(X)$ is an equivalence in $\mathcal{C}^{\otimes}$.

(ii) The restriction $f|\,\mathcal{BM}^{\otimes}$ is carries inert morphisms in $\mathcal{BM}^{\otimes}$ to inert morphisms in $\mathcal{C}^{\otimes}$.

Note that conditions (i) and (ii) immediately imply the following:

(iii) The restriction $f|\,\mathrm{N}(\mathcal{I}_1)$ carries morphisms in $M \times M'$ to inert morphisms in $\mathcal{C}^{\otimes}$.

Condition (i) is equivalent to the requirement that f is a q-left Kan extension of $f|\,\mathcal{BM}^{\otimes}$. Since every functor $f_0 \in \mathrm{Fun}_{\mathcal{BM}^{\otimes}}(\mathcal{BM}^{\otimes}, \mathcal{C}^{\otimes})$ admits a q-left Kan extension $f \in \mathrm{Fun}_{\mathcal{BM}^{\otimes}}(\mathrm{N}(\mathcal{I}), \mathcal{C}^{\otimes})$ (given, for example, by $f_0 \circ r$), Proposition T.4.3.2.15 implies that the restriction map $p : \mathcal{Z}' \to \mathrm{BMod}(\mathcal{C}_\mathfrak{m})$ is a trivial Kan fibration. The map ϕ is the composition of a section to p (given by composition with r) with the restriction map $p' : \mathcal{Z}' \to \mathcal{Z}$ given by $f \mapsto f|\,\mathcal{I}_1$. It will therefore suffice to show that p' is a trivial Kan fibration. In view of Proposition T.4.3.2.15, this can be deduced from the following pair of assertions:

(a) Every $f_0 \in \mathcal{Z} \subseteq \mathrm{Fun}_{\mathcal{BM}^{\otimes}}(\mathrm{N}(\mathcal{I}_1), \mathcal{C}^{\otimes})$ admits a q-right Kan extension $f \in \mathrm{Fun}_{\mathcal{BM}^{\otimes}}(\mathrm{N}(\mathcal{I}), \mathcal{C}^{\otimes})$.

(b) Given $f \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N}(\mathcal{I}), \mathcal{C}^{\otimes})$ satisfying (iii), the functor f is a q-right Kan extension of $f|\,\mathrm{N}(\mathcal{I}_1)$ if and only if it satisfies conditions (i) and (ii).

We first prove (a). Fix $f_0 \in \mathcal{Z}$ and let $(\langle n \rangle, c_-, c_+)$ be an object of $\mathcal{BM}^{\otimes}$. Let $\mathcal{J}$ denote the category $\mathcal{I}_1 \times_{\mathcal{I}} (\mathcal{I})_{(\langle n \rangle, c_-, c_+)/}$, and let g denote the composition $\mathrm{N}(\mathcal{J}) \to \mathrm{N}(\mathcal{I}_1) \xrightarrow{f_0} \mathcal{O}^{\otimes}$, so that $q \circ g$ extends canonically to a map $G : \mathrm{N}(\mathcal{J})^{\triangleleft} \to \mathrm{N}(\mathcal{I}) \xrightarrow{r} \mathcal{BM}^{\otimes}$. According to Lemma T.4.3.2.13, it will suffice to show that g can be extended to a q-limit diagram in $\mathcal{C}^{\otimes}$ lying over G.

The objects of $\mathcal{J}$ can be identified with morphisms $\alpha : (\langle n \rangle, c_-, c_+) \to \psi(X)$ in $\mathcal{BM}^{\otimes}$, where $X \in \mathcal{I}_1$. Let $\mathcal{J}_0 \subseteq \mathcal{J}$ denote the full subcategory spanned by those objects for which α is inert. The inclusion $\mathcal{J}_0 \subseteq \mathcal{J}$ has a right adjoint, so that $\mathrm{N}(\mathcal{J}_0) \to \mathrm{N}(\mathcal{J})$ is right cofinal. Consequently, it will suffice to show that $g_0 = g|\,\mathrm{N}(\mathcal{J}_0)$ admits a q-limit in $\mathcal{C}^{\otimes}$ (compatible with G).

Let $\mathcal{J}_1$ denote the full subcategory of $\mathcal{J}_0$ spanned by the morphism which are either of the form $\rho^j : (\langle n \rangle, c_-, c_+) \to r([0], [0], 0, 0) \simeq \mathfrak{m}$ where $c_-(j) = 0 < 1 = c_+(j)$, $\rho^j : (\langle n \rangle, c_-, c_+) \to r([1], [n], 0, 1) = \mathfrak{a}_-$ where $c_-(j) = c_+(j) = 0$, or $\rho^j : (\langle n \rangle, c_-, c_+) \to r([m], [1], 1, 0) = \mathfrak{a}_+$ where $c_-(j) = c_+(j) = 1$. Note that $\mathcal{J}_1$ decomposes as a disjoint union of full subcategories $\mathcal{J}_1 \simeq \coprod_{1 \le j \le n} \mathcal{J}_1(j)$.

Let $g_1 = g_0|\,\mathrm{N}(\mathcal{J}_1)$. We claim that g_0 is a q-right Kan extension of g_1. To prove this, consider an object of $\overline{X} \in \mathcal{J}_1$, corresponding an morphism $\alpha : (\langle n \rangle, c_-, c_+) \to \psi(X)$ in $\mathcal{BM}^{\otimes}$. Let $S = \{j \in \langle n \rangle^{\circ} : \alpha(j) \ne *\}$, so that $(\mathcal{J}_1)_{X/}$ decomposes as a disjoint union $\coprod_{j \in S} (\mathcal{J}_1(j))_{X/}$, where $\mathcal{J}_1(j)_{X/}$ denotes the subcategory $\mathcal{J}_1(j) \times_{\mathcal{J}_1} (\mathcal{J}_1)_{X/} \subseteq (\mathcal{J}_1)_{X/}$. Each of the subcategories $\mathcal{J}_1(j)_{X/}$ has an initial object, given by a morphism $\beta_j : X \to X_j$ in $\mathcal{I}_1$. If $X = ([m], [n], i, i')$, then

$$X_j = \begin{cases} ([0], [0], 0, 0) & \text{if } c_-(j) = 0 < 1 = c_+(j) \\ ([1], [n], 0, 1) & \text{if } c_-(j) = c_+(j) = 0 \\ ([m], [1], 1, 0) & \text{if } c_-(j) = c_+(j) = 1. \end{cases}$$

In each of these cases, the morphism β_j belongs to $M \times M'$, so that $f_0(\beta_j)$ is inert (since $f_0 \in \mathcal{Z}$). Since q is a fibration of ∞-operads, we conclude that f_0 exhibits $f_0(X)$ is a q-product of the objects $f_0(X_j)$, and therefore a q-limit of $f_0|(\mathcal{J}_1)_{X/}$.

Since g_0 is a q-right Kan extension of g_0, it will suffice to show that g_1 can be extended to a q-limit diagram in $\mathcal{C}^{\otimes}$ which is compatible with G (Lemma T.4.3.2.7). Each of the ∞-categories $\mathcal{J}_1(j)$ is isomorphic either to Δ^0 to $\mathrm{N}(\boldsymbol{\Delta})^{op}$, and is in particular weakly contractible. That each restriction $g_1|\,\mathrm{N}(\mathcal{J}_1(j))$ takes values in the ∞-category $\mathcal{C}_-$ (if $c_-(j) = c_+(j) = 0$), $\mathcal{C}_\mathfrak{m}$ (if $c_-(j) = 0 < 1 = c_+(j)$), or $\mathcal{C}_+$ (if $c_-(j) = c_+(j) = 1$). The assumption $f_0 \in \mathcal{Z}$ guarantees that $g_1|\,\mathrm{N}(\mathcal{J}_1(j))$ carries each morphism in $\mathcal{J}_1(j)$ to an equivalence. Since $\mathrm{N}(\mathcal{J}_1(j))$ is weakly contractible, we conclude that $g_1|\,\mathrm{N}(\mathcal{J}_1(j))$ is equivalent to a constant diagram and admits a q-limit $Y_i \in \mathcal{C}$ (Corollary T.4.4.4.10). Since q is a fibration of ∞-operads, the objects $Y_i \in \mathcal{C}$ admit a q-product in $\mathcal{C}^{\otimes}_{(\langle n \rangle, c_-, c_+)}$ which is a q-limit of g_1 compatible with G. This completes the proof of (a). Moreover,

the proof shows that a functor $f : \mathrm{N}(\mathcal{I}) \to \mathcal{C}^{\otimes}$ is a q-right Kan extension of f_0 if and only if it satisfies the following condition:

(i') For every object $(\langle n \rangle, c_-, c_+) \in \mathcal{BM}^{\otimes}$ as above and every object $\alpha : (\langle n \rangle, S) \to X$ belonging to the category $\mathcal{J}_1$ defined above, the image $f(\alpha)$ is an inert morphism in $\mathcal{C}^{\otimes}$.

To prove (b), it will suffice to show that if $f \in \mathrm{Fun}_{\mathcal{BM}^{\otimes}}(\mathrm{N}(\mathcal{I}), \mathcal{O}^{\otimes})$ satisfies condition (iii), then it satisfies conditions (i) and (ii) if and only if it satisfies condition (i'). We first prove the "only if" direction. Assume that $f \in \mathcal{Z}'$, and let $\alpha : (\langle n \rangle, c_-, c_+) \to X$ be as in (i'). Then α factors as a composition

$$(\langle n \rangle, c_-, c_+) \xrightarrow{\alpha'} \psi(X) \xrightarrow{\alpha''} X,$$

where α' is inert (so that $f(\alpha')$ is inert by (ii)) and $f(\alpha'')$ is an equivalence by virtue of (i).

Suppose now that f satisfies (i') and (iii). We first show that f satisfies (i). Fix an object $X = ([m], [n], i, j)$ in $\mathcal{I}$; we wish to show that f carries the canonical map $\alpha : \psi(X) \to X$ to an equivalence in $\mathcal{C}^{\otimes}$. Let $\psi(X) = (\langle n \rangle, c_-, c_+)$, and let $\mathcal{J}_1$ be the category defined above, so that $(\mathcal{J}_1)_{X/} \simeq \coprod_{1 \leq j \leq n} \mathcal{J}_1(j)_{X/}$ where each $\mathcal{J}_1(j)$ contains an initial object given by a morphism $\beta_j : X \to X_j$ in $\mathcal{I}_1$. Assumption (i') shows that each $f(\beta_j \circ \alpha)$ is inert, and assumption (iii) guarantees that each $f(\beta_j)$ is inert (since $\beta_j \in M \times M'$). It follows that the image of $f(\alpha)$ under the functor $u : \mathcal{C}^{\otimes}_{\psi(X)} \to \prod_{1 \leq j \leq n} \mathcal{C}^{\otimes}_{\psi(X_j)}$ is an equivalence. Since q is a fibration of ∞-operads, the functor u is an equivalence so that $f(\alpha)$ is an equivalence in the ∞-category $\mathcal{C}^{\otimes}_{\psi(X)}$.

It remains to show that f satisfies (ii). For this, it will suffice to show that $f(\alpha)$ is inert whenever α is an inert morphism of $\mathcal{LM}^{\otimes}$ of the form $\alpha : (\langle n \rangle, c_-, c_+) \to (\langle 1 \rangle, c'_-, c'_+)$. We observe that in this case we can write $(\langle 1 \rangle, c'_-, c'_+) \simeq \psi(X)$ so that α determines an object of the category $\mathcal{J}_1$ defined above. Let $\beta : \psi(X) \to X$ be the evident morphism in $\mathcal{I}$. Condition (i') guarantees that $f(\beta \circ \alpha)$ is inert, and condition (i) guarantees that $f(\beta)$ is an equivalence. It follows that $f(\alpha)$ is inert, as desired. $\square$

Corollary 4.3.2.8. *Let $\mathcal{M}$ be an ∞-category which is bitensored over a pair of monoidal ∞-categories $\mathcal{C}^{\otimes}_-$ and $\mathcal{C}^{\otimes}_+$. Let $B \in \mathrm{Alg}(\mathcal{C}_+)$, and assume that the unit map $\mathbf{1} \to B$ is an equivalence in $\mathcal{C}_+$. Then the forgetful functor*

$$\mathrm{BMod}(\mathcal{M}) \times_{\mathrm{Alg}(\mathcal{C}_+)} \{B\} \to \mathrm{LMod}(\mathcal{M})$$

is an equivalence of ∞-categories. In particular, if $A \in \mathrm{Alg}(\mathcal{C}_-)$, then the forgetful functor

$$_A\mathrm{BMod}_B(\mathcal{M}) \to \mathrm{LMod}_A(\mathcal{M})$$

is an equivalence of ∞-categories.

Proof. Combine Theorem 4.3.2.7, Proposition 4.3.2.6, and Proposition 4.2.4.9. $\square$

4.3.3   Limits, Colimits, and Free Bimodules

Let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category and suppose we are given a pair of algebra objects $A, B \in \mathrm{Alg}(\mathcal{C})$. In this section, we study the relationship between the ∞-category $_A\mathrm{BMod}_B(\mathcal{C})$ of A-B-bimodule objects of $\mathcal{C}$ and the underlying ∞-category $\mathcal{C}$. We can summarize our main results as follows:

(a) Let K be a simplicial set such that $\mathcal{C}$ admits K-indexed limits. Then $_A\mathrm{BMod}_B(() \mathcal{C}$ admits K-indexed limits, and the forgetful functor $\mathrm{LMod}_A(\mathcal{M}) \to \mathcal{M}$ preserves K-indexed limits (Corollary 4.3.3.3).

(b) Let K be a simplicial set such that $\mathcal{C}$ admits K-indexed colimits and the tensor product functors $M \mapsto A \otimes M$ and $M \mapsto M \otimes B$ preserve K-indexed colimits. Then $_A\mathrm{BMod}_B(\mathcal{C})$ admits K-indexed colimits, and the forgetful functor $_A\mathrm{BMod}_B(\mathcal{C}) \to \mathcal{C}$ preserves K-indexed colimits (Proposition 4.3.3.9).

(c) The forgetful functor $_A\mathrm{BMod}_B(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint, given on objects by $M \mapsto A \otimes M \otimes B$ (Corollary 4.3.3.14).

At the end of this section, we describe an application of (c), which gives a simple description of bimodule objects in the underlying ∞-category of a monoidal model category (Theorem 4.3.3.17).

The analogues of assertions (a), (b), and (c) for left modules were proven in §4.2. It is possible to obtain the results of this section by generalizing the methods of §4.2. We will adopt a different approach, which uses Theorem 4.3.2.7 to reduce questions about bimodule objects to simpler questions about left and right modules. For example, Theorem 4.3.2.7, Proposition 4.2.3.1, and Proposition T.4.3.1.5 immediately imply the following:

Proposition 4.3.3.1. *Let $\mathcal{M}$ be an ∞-category which is bitensored over a pair of monoidal ∞-categories $\mathcal{C}_-^\otimes$ and $\mathcal{C}_+^\otimes$. Let K be a simplicial set such that $\mathcal{M}$ admits K-indexed limits, and let $\theta : \mathrm{BMod}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{C}_-) \times \mathrm{Alg}(\mathcal{C}_+)$ be the forgetful functor. Then:*

(1) *For every diagram*

$$
\begin{array}{ccc}
K & \longrightarrow & \mathrm{BMod}(\mathcal{M}) \\
\downarrow & \nearrow & \downarrow{\scriptstyle \theta} \\
K^\triangleleft & \longrightarrow & \mathrm{Alg}(\mathcal{C}_-) \times \mathrm{Alg}(\mathcal{C}_+)
\end{array}
$$
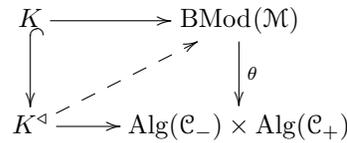
there exists a dotted arrow as indicated, which is a θ-limit diagram.

(2) *An arbitrary map $\overline{g} : K^\triangleleft \to \mathrm{BMod}(\mathcal{M})$ is a θ-limit diagram if and only if the induced map $K^\triangleleft \to \mathcal{M}$ is a limit diagram.*

Corollary 4.3.3.2. *Let $\mathcal{M}$ be an ∞-category which is bitensored over a pair of monoidal ∞-categories $\mathcal{C}_-^\otimes$ and $\mathcal{C}_+^\otimes$. Then the forgetful functor $\theta : \mathrm{BMod}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{C}_-) \times \mathrm{Alg}(\mathcal{C}_+)$ is a Cartesian fibration. A morphism f in $\mathrm{BMod}(\mathcal{M})$ is θ-Cartesian if and only if the image of f in $\mathcal{M}$ is an equivalence.*

Proof. Apply Proposition 4.3.3.1 in the case $K = \Delta^0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Corollary 4.3.3.3. *Let $\mathcal{M}$ be an ∞-category which is bitensored over a pair of monoidal ∞-categories $\mathcal{C}_-^\otimes$ and $\mathcal{C}_+^\otimes$. Let $A \in \mathrm{Alg}(\mathcal{C}_-)$ and $B \in \mathrm{Alg}(\mathcal{C}_+)$ be algebra objects, and let K be a simplicial set such that $\mathcal{M}$ admits K-indexed limits. Then:*

(1) *The ∞-category ${}_A\mathrm{BMod}_B(\mathcal{M})$ admits K-indexed limits.*

(2) *A map $\overline{p} : K^\triangleleft \to {}_A\mathrm{BMod}_B(\mathcal{M})$ is a limit diagram if and only if the induced map $K^\triangleleft \to \mathcal{M}$ is a limit diagram.*

(3) *Given maps of algebra objects $A \to A'$, $B \to B'$, the induced functor ${}_{A'}\mathrm{BMod}_{B'}(\mathcal{M}) \to {}_A\mathrm{BMod}_B(\mathcal{M})$ preserves K-indexed limits.*

We would next like to discuss an analogue of Proposition 4.3.3.1 for colimits of bimodules. This will require a bit of additional notation.

Notation 4.3.3.4. Let $\tau : \mathcal{RM}^\otimes \to \mathcal{Triv}^\otimes$ be the map of ∞-operads given by the formula $(\langle n \rangle, S) \mapsto S_*$; here we abuse notation by identifying the pointed set $S_* = S \cup \{*\}$ with an object of $\mathcal{F}\mathrm{in}_*$. The functor $\wedge : \mathcal{F}\mathrm{in}_* \times \mathcal{F}\mathrm{in}_* \to \mathcal{F}\mathrm{in}_*$ of Notation 2.2.5.1 refines to a bifunctor of ∞-operads

$$\wedge' : \mathcal{A}\mathrm{ss}^\otimes \times \mathcal{Triv}^\otimes \to \mathcal{A}\mathrm{ss}^\otimes \simeq \mathcal{A}\mathrm{ss}_-^\otimes \subseteq \mathcal{BM}^\otimes .$$

Given a fibration of ∞-operads $p : \mathcal{C}^\otimes \to \mathcal{BM}$, we let $\mathrm{Alg}(\mathcal{C}_-)^\otimes$ denote the fiber product

$$\mathcal{RM}^\otimes \times_{\mathcal{Triv}^\otimes} \mathrm{Alg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C})^\otimes,$$

where $\mathrm{Alg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C})^{\otimes} \to \mathcal{T}\mathrm{riv}^{\otimes}$ is the fibration of ∞-operads obtained by applying Construction 3.2.4.1 to the map $\wedge'$. Note that the composition

$$\mathcal{A}\mathrm{ss}^{\otimes} \times \mathcal{RM}^{\otimes} \xrightarrow{\mathrm{id} \times \tau} \mathcal{A}\mathrm{ss}^{\otimes} \times \mathcal{T}\mathrm{riv}^{\otimes} \xrightarrow{\wedge'} \mathcal{BM}^{\otimes}$$

coincides with the restriction $\mathbf{Pr}\,|(\mathcal{A}\mathrm{ss}^{\otimes} \times \mathcal{RM}^{\otimes})$. Consequently, composition with the inclusion functor $\mathcal{A}\mathrm{ss}^{\otimes} \times \mathcal{RM}^{\otimes} \hookrightarrow \mathcal{LM}^{\otimes} \times \mathcal{RM}^{\otimes}$ induces a fibration of ∞-operads $\mathrm{LMod}(\mathcal{C}_{\mathfrak{m}})^{\otimes} \to \mathrm{Alg}(\mathcal{C}_{-})^{\otimes}$.

Remark 4.3.3.5. Let $\mathcal{C}^{\otimes} \to \mathcal{BM}^{\otimes}$ be as in Notation 4.3.3.4. Then $\mathrm{Alg}(\mathcal{C}_{-})_{\mathfrak{m}}^{\otimes} \simeq \mathrm{Alg}(\mathcal{C}_{-})$, and the forgetful functor $\mathrm{Alg}(\mathcal{C}_{-})_{\mathfrak{a}}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ is a trivial Kan fibration. Example 2.1.3.5 guarantees that evaluation at $\mathfrak{m}$ induces an equivalence of ∞-categories $\mathrm{RMod}(\mathrm{Alg}(\mathcal{C}_{-})) \simeq \mathrm{Alg}(\mathcal{C}_{-})$.

Notation 4.3.3.6. Let $\mathcal{C}^{\otimes} \to \mathcal{BM}^{\otimes}$ be a fibration of ∞-operads and let $A \in \mathrm{Alg}(\mathcal{C}_{-})$. The composite functor

$$\mathcal{A}\mathrm{ss}^{\otimes} \times \mathcal{RM}^{\otimes} \xrightarrow{\mathrm{id} \times \tau} \mathcal{A}\mathrm{ss}^{\otimes} \times \mathcal{T}\mathrm{riv}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes} \xrightarrow{A} \mathcal{C}_{-}^{\otimes}$$

can be identified with a map $A' : \mathcal{RM}^{\otimes} \to \mathrm{Alg}(\mathcal{C}_{-})^{\otimes}$, which is a preimage of A under the trivial Kan fibration $\mathrm{RMod}(\mathrm{Alg}(\mathcal{C}_{-})) \to \mathrm{Alg}(\mathcal{C}_{-})$ of Remark 4.3.3.5. We let $\mathrm{LMod}_{A}(\mathcal{C}_{\mathfrak{m}})^{\otimes}$ denote the fiber product

$$\mathrm{LMod}(\mathcal{C}_{\mathfrak{m}})^{\otimes} \times_{\mathrm{Alg}(\mathcal{C}_{-})^{\otimes}} \mathcal{RM}^{\otimes}\,.$$

Remark 4.3.3.7. Let $p : \mathcal{C}^{\otimes} \to \mathcal{BM}^{\otimes}$ be a fibration of ∞-operads and let $A \in \mathrm{Alg}(\mathcal{C}_{-})$. Then the map $\mathrm{LMod}_{A}(\mathcal{C}_{\mathfrak{m}})^{\otimes} \to \mathcal{RM}^{\otimes}$ is a fibration of ∞-operads, which exhibits the ∞-category $\mathrm{LMod}_{A}(\mathcal{C}_{\mathfrak{m}})_{\mathfrak{m}} \simeq \mathrm{LMod}_{A}(\mathcal{C}_{\mathfrak{m}})$ as weakly enriched over the reverse of the planar ∞-operad $q : \mathrm{LMod}_{A}(\mathcal{C}_{\mathfrak{m}})_{\mathfrak{a}}^{\otimes} \simeq \mathrm{LMod}(\mathcal{C}_{\mathfrak{m}})_{\mathfrak{a}}^{\otimes}$ (which is equivalent to $\mathcal{C}_{+}^{\otimes}$; see Proposition 4.3.2.6). If p is a coCartesian fibration of ∞-operads, then q is also a coCartesian fibration of ∞-operads. It follows from Remark 4.3.3.5 that $\mathrm{RMod}(\mathrm{LMod}_{A}(\mathcal{C}_{\mathfrak{m}}))$ can be identified with the homotopy fiber of the map

$$\mathrm{RMod}(\mathrm{LMod}(\mathcal{C}_{\mathfrak{m}})) \to \mathrm{LMod}(\mathcal{C}_{\mathfrak{m}}) \to \mathrm{Alg}(\mathcal{C}_{-})$$

over the object $A \in \mathrm{Alg}(\mathcal{C}_{-})$.

Remark 4.3.3.8. Let $q : \mathcal{C}^{\otimes} \to \mathcal{BM}^{\otimes}$ be a fibration of ∞-operads and let $A \in \mathrm{Alg}(\mathcal{C}_{-})$, $B \in \mathrm{Alg}(\mathcal{C}_{+})$. We let $\mathrm{RMod}_{B}(\mathrm{LMod}_{A}(\mathcal{C}_{\mathfrak{m}})$ denote the fiber

$$\mathrm{RMod}(\mathrm{LMod}_{A}(\mathcal{C}_{\mathfrak{m}}) \times_{\mathrm{Alg}(\mathcal{C}_{+})} \{B\},$$

where the map $\mathrm{RMod}(\mathrm{LMod}_{A}(\mathcal{C}_{\mathfrak{m}})) \to \mathrm{Alg}(\mathcal{C}_{+})$ is the categorical fibration given by composition with the inclusion

$$\{\mathfrak{m}\} \times \mathcal{A}\mathrm{ss}^{\otimes} \hookrightarrow \mathcal{LM}^{\otimes} \times \mathcal{RM}^{\otimes}\,.$$

Since $\mathrm{Alg}(\mathcal{C}_{-})_{\mathfrak{a}}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ is a trivial Kan fibration, Proposition 4.3.2.6 guarantees that the categorical fibration $\theta : \mathrm{Alg}(\mathrm{LMod}_{A}(\mathcal{C}_{\mathfrak{m}})_{\mathfrak{a}}) \to \mathrm{Alg}(\mathcal{C}_{+})$ is an equivalence of ∞-categories, and therefore a trivial Kan fibration. We may therefore assume that $B = \theta(\overline{B})$ for some $\overline{B} \in \mathrm{Alg}(\mathrm{LMod}_{A}(\mathcal{C}_{\mathfrak{m}})_{\mathfrak{a}})$, and the inclusion

$$\mathrm{RMod}_{\overline{B}}(\mathrm{LMod}_{A}(\mathcal{C}_{\mathfrak{m}}) \to \mathrm{RMod}_{B}(\mathrm{LMod}_{A}(\mathcal{C}_{\mathfrak{m}})$$

is a categorical equivalence. Combining Remark 4.3.3.7 with Theorem 4.3.2.7, we obtain an equivalence of ∞-categories

$$_{A}\mathrm{BMod}_{B}(\mathcal{C}_{\mathfrak{m}}) \to \mathrm{RMod}_{B}(\mathrm{LMod}_{A}(\mathcal{C}_{\mathfrak{m}}))\,.$$

We are now ready to discuss colimits in ∞-categories of bimodules.

Proposition 4.3.3.9. *Let $\mathcal{M}$ be an ∞-category which is bitensored over a pair of monoidal ∞-categories $\mathcal{C}_-^{\otimes}$ and $\mathcal{C}_+^{\otimes}$. Let $A \in \mathrm{Alg}(\mathcal{C}_-)$ and $B \in \mathrm{Alg}(\mathcal{C}_+)$ be algebra objects. Let K be a simplicial set such that $\mathcal{M}$ admits K-indexed colimits, and assume that the functors*

$$\mathcal{M} \simeq \{A\} \times \mathcal{M} \hookrightarrow \mathcal{C}_- \times \mathcal{M} \to \mathcal{M}$$

$$\mathcal{M} \simeq \mathcal{M} \times \{B\} \hookrightarrow \mathcal{M} \times \mathcal{C}_+ \to \mathcal{M}$$

preserve K-indexed colimits. Then:

(1) *Every diagram $f : K \to {}_A\mathrm{BMod}_B(\mathcal{M})$ has a colimit.*

(2) *An arbitrary diagram $\overline{f} : K^{\triangleright} \to {}_A\mathrm{BMod}_B(\mathcal{M})$ is a colimit diagram if and only if it induces a colimit diagram $K^{\triangleright} \to \mathcal{M}$.*

Proof. Applying Corollary 4.2.3.5, we deduce that the ∞-category $\mathrm{LMod}_A(\mathcal{M})$ admits K-indexed colimits; moreover, a map $K^{\triangleright} \to \mathrm{LMod}_A(\mathcal{M})$ is a colimit diagram if and only if the induced map $K^{\triangleright} \to \mathcal{M}$ is a colimit diagram. Let $\overline{B} \in \mathrm{Alg}(\mathrm{LMod}_A(\mathcal{M})_{\mathfrak{a}})$ be a preimage of B under the trivial Kan fibration $\mathrm{Alg}(\mathrm{LMod}_A(\mathcal{C}_{\mathfrak{m}})_{\mathfrak{a}}) \to \mathrm{Alg}(\mathcal{C}_+)$ of Remark 4.3.3.8. Since the right action of B on $\mathcal{M}$ preserves K-indexed colimits, the action of $\overline{B}$ on $\mathrm{LMod}_A(\mathcal{M})$ also preserves K-indexed colimits. Applying Corollary 4.2.3.5 again, we deduce that $\mathrm{RMod}_{\overline{B}}(\mathrm{LMod}_A(\mathcal{M}))$ admits K-indexed colimits; moreover, a map $\overline{f} : K^{\triangleright} \to \mathrm{RMod}_{\overline{B}}(\mathrm{LMod}_A(\mathcal{M}))$ is a colimit diagram if and only if the composite map $K^{\triangleright} \to \mathrm{LMod}_A(\mathcal{M})$ is a colimit diagram (which is equivalent to the requirement that the underlying map $K^{\triangleright} \to \mathcal{M}$ is a colimit diagram). Assertions (1) and (2) now follow immediately from the equivalences

$$_A\mathrm{BMod}_B(\mathcal{M}) \to \mathrm{RMod}_B(\mathrm{LMod}_A(\mathcal{M})) \leftarrow \mathrm{RMod}_{\overline{B}}(\mathrm{LMod}_A(\mathcal{M}))$$

of Remark 4.3.3.8. $\square$

Corollary 4.3.3.10. *Let $\mathcal{M}$ be an ∞-category which is bitensored over a pair of monoidal ∞-categories $\mathcal{C}_-^{\otimes}$ and $\mathcal{C}_+^{\otimes}$. Assume that $\mathcal{M}$ is presentable and that for every pair of objects $X \in \mathcal{C}_-^{\otimes}$, $Y \in \mathcal{C}_+^{\otimes}$, the functors*

$$\mathcal{M} \simeq \{X\} \times \mathcal{M} \hookrightarrow \mathcal{C}_- \times \mathcal{M} \to \mathcal{M}$$

$$\mathcal{M} \simeq \mathcal{M} \times \{Y\} \hookrightarrow \mathcal{M} \times \mathcal{C}_+ \to \mathcal{M}$$

preserve small colimits. Then:

(1) *For every $A \in \mathrm{Alg}(\mathcal{C}_-)$, $B \in \mathrm{Alg}(\mathcal{C}_+)$, the ∞-category $_A\mathrm{BMod}_B(\mathcal{M})$ is presentable.*

(2) *For every $f : A \to A'$ in $\mathrm{Alg}(\mathcal{C}_-)$ and every $g : B \to B'$ in $\mathrm{Alg}(\mathcal{C}_+)$, the associated functor $_{A'}\mathrm{BMod}_{B'}(\mathcal{M}) \to {}_A\mathrm{BMod}_B(\mathcal{M})$ preserves small limits and colimits.*

(3) *The forgetful functor $\theta : \mathrm{BMod}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{C}_-) \times \mathrm{Alg}(\mathcal{C})_+$ is a presentable fibration (Definition T.5.5.3.2).*

Proof. Let $\overline{B}$ be as in Remark 4.3.3.8. Assertion (1) follows from iterated application of Corollary 4.2.3.7, using the equivalences

$$_A\mathrm{BMod}_B(\mathcal{M}) \to \mathrm{RMod}_B(\mathrm{LMod}_A(\mathcal{M})) \leftarrow \mathrm{RMod}_{\overline{B}}(\mathrm{LMod}_A(\mathcal{M})).$$

Assertion (2) follows from Proposition 4.3.3.9 and Corollary 4.3.3.3. Assertion (3) follows from (1) and (2), by virtue of Proposition T.5.5.3.3. $\square$

We next use Theorem 4.3.2.7 and the results of §4.2.4 to develop a good theory of free bimodules.

Definition 4.3.3.11. Let $\mathcal{M}$ be an ∞-category which is bitensored over a pair of monoidal ∞-categories $\mathcal{C}_-^\otimes$ and $\mathcal{C}_+^\otimes$. Let $A \in \mathrm{Alg}(\mathcal{C}_-)$, let $B \in \mathrm{Alg}(\mathcal{C}_+)$, and let $M \in {}_A\mathrm{BMod}_B(\mathcal{M})$ be an A-B-bimodule. We will abuse notation by identifying M with its image in $\mathcal{M}$. We will say that a morphism $M_0 \to M$ in $\mathcal{M}$ *exhibits* M *as the free A-B-bimodule generated by* M_0 if the composite map

$$A \otimes M_0 \otimes B \to A \otimes M \otimes B \to M$$

is an equivalence in $\mathcal{M}$.

Using Remark 4.3.3.8 and Proposition 4.2.4.2, we immediately deduce the following:

Proposition 4.3.3.12. *Let $\mathcal{M}$ be an ∞-category which is bitensored over a pair of monoidal ∞-categories $\mathcal{C}_-^\otimes$ and $\mathcal{C}_+^\otimes$. Suppose that we are given objects $A \in \mathrm{Alg}(\mathcal{C}_-)$, $B \in \mathrm{Alg}(\mathcal{C}_+)$, and $M_0 \in \mathcal{M}$. Then:*

(1) *There exists an object $M \in {}_A\mathrm{BMod}_B(\mathcal{M})$ and a morphism $\lambda : M_0 \to M$ in $\mathcal{M}$ which exhibits M as a free A-B-bimodule generated by M_0.*

(2) *Let $M \in {}_A\mathrm{BMod}_B(\mathcal{M})$ and let $\lambda : M_0 \to M$ be a morphism which exhibits M as a free left A-module generated by M_0. For every $N \in {}_A\mathrm{BMod}_B(\mathcal{M})$, composition with λ induces a homotopy equivalence*

$$\mathrm{Map}_{{}_A\mathrm{BMod}_B(\mathcal{M})}(M, N) \to \mathrm{Map}_\mathcal{M}(M_0, N).$$

Corollary 4.3.3.13. *Let $\mathcal{M}$ be an ∞-category which is bitensored over a pair of monoidal ∞-categories $\mathcal{C}_-^\otimes$ and $\mathcal{C}_+^\otimes$, and let $A \in \mathrm{Alg}(\mathcal{C}_-)$ and $B \in \mathrm{Alg}(\mathcal{C}_+)$. Then the forgetful functor ${}_A\mathrm{BMod}_B(\mathcal{M}) \to \mathcal{M}$ admits a left adjoint, which carries an object $M_0 \in \mathcal{M}$ to a free A-B-bimodule generated by M_0.*

Corollary 4.3.3.14. *Let $\mathcal{M}$ be an ∞-category which is bitensored over a pair of monoidal ∞-categories $\mathcal{C}_-^\otimes$ and $\mathcal{C}_+^\otimes$. Then the forgetful functor $\mathrm{BMod}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{C}_-) \times \mathcal{M} \times \mathrm{Alg}(\mathcal{C}_+)$ admits a left adjoint, which carries each triple (A, M_0, B) to a free A-B-bimodule $A \otimes M_0 \otimes B \in {}_A\mathrm{BMod}_B(\mathcal{M})$.*

Proof. Combine Proposition 4.3.3.12 with Corollary 4.3.3.2. $\square$

We conclude this section with a rectification result for bimodule objects in a monoidal model category. Let $\mathbf{A}$ be a simplicial monoidal model category, and $\mathcal{C} = \mathrm{N}(\mathbf{A}^\circ)$ its underlying ∞-category. In §4.1.4, we saw that there is often a close relationship between associative algebra objects $A \in \mathrm{Alg}(\mathcal{C})$ and (strictly) associative algebra objects of $\mathbf{A}$ itself (Theorem 4.1.4.4). We will conclude this section by proving an analogous result for bimodules (Theorem 4.3.3.17). The key point is that we have a similar understanding of free bimodules in both $\mathbf{A}$ and $\mathcal{C}$, thanks to Proposition 4.3.3.12.

Let $\mathbf{A}$ be a monoidal model category and let A and B be algebra objects of $\mathbf{A}$. We let ${}_A\mathrm{BMod}_B(\mathbf{A})$ denote the ordinary category of A-B-bimodule objects in $\mathbf{A}$.

Proposition 4.3.3.15. *Let $\mathbf{A}$ be a combinatorial monoidal model category and suppose that A and B are associative algebra objects of $\mathbf{A}$ which are cofibrant as objects of $\mathbf{A}$. Then ${}_A\mathrm{BMod}_B(\mathbf{A})$ has the structure of a combinatorial model category, where:*

(W) *A morphism $f : M \to N$ is a weak equivalence in ${}_A\mathrm{BMod}_B(\mathbf{A})$ if and only if it is a weak equivalence in $\mathbf{A}$.*

(F) *A morphism $f : M \to N$ is a fibration in ${}_A\mathrm{BMod}_B(\mathbf{A})$ if and only if it is a fibration in $\mathbf{A}$.*

The forgetful functor ${}_A\mathrm{BMod}_B(\mathbf{A}) \to \mathbf{A}$ is both a left Quillen functor and a right Quillen functor. Moreover, if $\mathbf{A}$ is equipped with a compatible simplicial structure, then the induced simplicial structure on ${}_A\mathrm{BMod}_B(\mathbf{A})$ endows ${}_A\mathrm{BMod}_B(\mathbf{A})$ with the structure of a simplicial model category.

Proof. The proof is similar to that of Proposition 4.1.4.3. We first observe that $_A\mathrm{BMod}_B(\mathbf{A})$ is presentable (Corollary 4.3.3.10). Let $T : {_A}\mathrm{BMod}_B(\mathbf{A}) \to \mathbf{A}$ be the forgetful functor. Then T admits a left adjoint given by the formula $F(X) = A \otimes X \otimes B$, and a right adjoint given by the formula $G(X) = {^A}X^B$. Since $\mathbf{A}$ is combinatorial, there exists a (small) collection of morphisms $I = \{i_\alpha : C \to C'\}$ which generates the class of cofibrations in $\mathbf{A}$, and a (small) collection of morphisms $J = \{j_\alpha : D \to D'\}$ which generates the class of trivial cofibrations in $\mathbf{A}$. Let $\overline{F(I)}$ be the weakly saturated class of morphisms in $_A\mathrm{BMod}_B(\mathbf{A})$ generated by $\{F(i) : i \in I\}$, and let $\overline{F(J)}$ be defined similarly. Unwinding the definitions, we see that a morphism in $_A\mathrm{BMod}_B(\mathbf{A})$ is a trivial fibration if and only if it has the right lifting property with respect to $F(i)$, for every $i \in I$. Invoking the small object argument, we deduce that every morphism $f : M \to N$ in $_A\mathrm{BMod}_B(\mathbf{A})$ admits a factorization $M \xrightarrow{f'} N' \xrightarrow{f''} N$ where $f' \in \overline{F(I)}$ and f'' is a trivial fibration. Similarly, we can find an analogous factorization where $f' \in \overline{F(J)}$ and f'' is a fibration. Using standard arguments, we are reduced to the problem of showing that each morphism belonging to $\overline{F(J)}$ is a weak equivalence in $_A\mathrm{BMod}_B(\mathbf{A})$. Let S be the collection of all morphisms $f : M \to N$ in $_A\mathrm{BMod}_B(\mathbf{A})$ such that $T(f)$ is a trivial cofibration in $\mathbf{A}$. We wish to prove that $\overline{F(J)} \subseteq S$. Since T preserves colimits, we conclude that S is weakly saturated; it will therefore suffice to show that for each $j \in J$, $F(j) \in S$. In other words, we must show that if $j : X \to Y$ is a trivial cofibration in $\mathbf{A}$, then the induced map $A \otimes X \otimes B \to A \otimes Y \otimes B$ is again a trivial cofibration in $\mathbf{A}$. This follows immediately from the definition of a monoidal model category, in view of our assumption that A and B are cofibrant objects of $\mathbf{A}$. This completes the proof that $\mathrm{LMod}_A(\mathbf{A})$ is a model category.

The forgetful functor $T : {_A}\mathrm{BMod}_B(\mathbf{A}) \to \mathbf{A}$ is a right Quillen functor by construction. To see that T is also a left Quillen functor, it suffices to show that the right adjoint $G : \mathbf{A} \to {_A}\mathrm{BMod}_B(\mathbf{A})$ preserves fibrations and trivial fibrations. In view of the definition of fibrations and trivial fibrations in $_A\mathrm{BMod}_B(\mathbf{A})$, this is equivalent to the assertion that the composition $T \circ G : \mathbf{A} \to \mathbf{A}$ preserves fibrations and trivial fibrations. This follows immediately from the definition of a monoidal model category, since A and B are assumed to be cofibrant in $\mathbf{A}$.

Now suppose that $\mathbf{A}$ is equipped with a compatible simplicial structure. We claim that $_A\mathrm{BMod}_B(\mathbf{A})$ inherits the structure of a simplicial model category. For this, we suppose that $f : M \to N$ is a fibration in $_A\mathrm{BMod}_B(\mathbf{A})$ and that $g : X \to Y$ is a cofibration of simplicial sets. We wish to show that the induced map $M^Y \to M^X \times_{N^X} N^Y$ is a fibration in $\mathrm{LMod}_A(\mathbf{A})$, which is trivial if either f or g is trivial. This follows immediately from the analogous statement in the simplicial model category $\mathbf{A}$. $\square$

Remark 4.3.3.16. Proposition 4.3.3.15 admits the following generalization: suppose that $\mathbf{A}$ is a combinatorial model category, and that $F : \mathbf{A} \to \mathbf{A}$ is a left Quillen functor which is equipped with the structure of a monad (that is, an associative algebra in the monoidal category $\mathrm{Fun}(\mathbf{A}, \mathbf{A})$ of endofunctors of $\mathbf{A}$). Then the category of algebras over F inherits a model structure, where the fibrations and weak equivalences are defined at the level of the underlying objects of $\mathbf{A}$.

In the situation of Proposition 4.3.3.15, let $\mathbf{A}^c \subseteq \mathbf{A}$ and $(_A\mathrm{BMod}_B(\mathbf{A}))^c \subseteq {_A}\mathrm{BMod}_B(\mathbf{A})$ denote the full subcategories of $\mathbf{A}$ and $_A\mathrm{BMod}_B(\mathbf{A})^c$ spanned by the cofibrant objects. Let W be the collection of weak equivalences in $\mathbf{A}^c$ and W' the collection of weak equivalences in $_A\mathrm{BMod}_B(\mathbf{A})^c$. The monoidal functor $\mathrm{N}(\mathbf{A}^c) \to \mathrm{N}(\mathbf{A}^c)[W^{-1}]$ determines a map

$$\theta : \mathrm{N}(_A\mathrm{BMod}_B(\mathbf{A})^c)[W'^{-1}] \to {_A}\mathrm{BMod}_B(\mathrm{N}(\mathbf{A}^c)[W^{-1}])$$

(where we abuse notation by identifying A and B with their images in $\mathrm{Alg}(\mathrm{N}(\mathbf{A}^c)[W^{-1}])$).

Theorem 4.3.3.17. *Let $\mathbf{A}$ be a combinatorial monoidal model category and let $A, B \in \mathrm{Alg}(\mathbf{A})$ be associative algebras in $\mathbf{A}$ which are cofibrant as objects of $\mathbf{A}$. Then the functor $\theta : \mathrm{N}(_A\mathrm{BMod}_B(\mathbf{A})^c)[W'^{-1}] \to {_A}\mathrm{BMod}_B(\mathrm{N}(\mathbf{A}^c)[W^{-1}])$ described above is an equivalence of ∞-categories.*

Proof. Consider the diagram

$$N(_A\mathrm{BMod}_B(\mathbf{A})^c)[W'^{-1}] \xrightarrow{\quad\quad\theta\quad\quad} {}_A\mathrm{BMod}_B(N(\mathbf{A}^c)[W^{-1}])$$

$$\begin{array}{ccc} & G \searrow & \swarrow G' \\ & N(\mathbf{A}^c)[W^{-1}]. & \end{array}$$

We will show that this diagram satisfies the hypotheses of Corollary 6.2.2.14:

(a) The ∞-categories $N(_A\mathrm{BMod}_B(\mathbf{A})^c)[W'^{-1}]$ and $_A\mathrm{BMod}_B(N(\mathbf{A}^c)[W^{-1}])$ admit geometric realizations of simplicial objects. In fact, both of these ∞-categories are presentable. For $N(_A\mathrm{BMod}_B(\mathbf{A})^c)[W'^{-1}]$, this follows from Propositions 1.3.4.22 and 4.3.3.15. For $_A\mathrm{BMod}_B(N(\mathbf{A}^c)[W^{-1}])$, we first observe that $N(\mathbf{A}^c)[W^{-1}]$ is presentable (Proposition 1.3.4.22) and that left tensor product by A and right tensor product by B preserve small colimits (Corollary 4.1.4.8), and then apply Corollary 4.3.3.10.

(b) The functors G and G' admit left adjoints F and F'. The existence of a left adjoint to G follows from the fact that G is determined by a right Quillen functor. The existence of a left adjoint to G' follows from Corollary 4.3.3.14.

(c) The functor G' is conservative and preserves geometric realizations of simplicial objects. This follows from Corollaries 4.3.3.2 and 4.3.3.9.

(d) The functor G is conservative and preserves geometric realizations of simplicial objects. The first assertion is immediate from the definition of the weak equivalences in $_A\mathrm{BMod}_B(\mathbf{A})$, and the second follows from the fact that G is also a left Quillen functor.

(e) The natural map $G' \circ F' \to G \circ F$ is an equivalence. Unwinding the definitions, we are reduced to proving that if N is a cofibrant object of $\mathbf{A}$, then the natural map $N \to A \otimes N \otimes B$ induces an equivalence $F'(N) \simeq A \otimes N \otimes B$. This follows from the explicit description of F' given in Corollary 4.3.3.13.

$\square$

4.3.4   Multilinear Maps

Let A, B, and C be associative rings. Suppose that M is an A-B-bimodule, N is a B-C-bimodule, and X is an A-C-bimodule. A *bilinear pairing* $f : M \times N \to X$ is a map f satisfying the equations

$$f(m + m', n) = f(m, n) + f(m', n) \qquad f(m, n + n') = f(m, n) + f(m, n')$$

$$f(am, n) = af(m, n) \qquad f(mb, n) = f(m, bn) \qquad f(m, nc) = f(m, n)c.$$

We can reformulate this definition categorically as follows: a bilinear map is a map $F : M \otimes N \to X$ such that the diagrams

$$\begin{array}{ccc} A \otimes M \otimes N & \xrightarrow{F} & A \otimes X \\ {\scriptstyle m_{A,M} \otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle m_{A,X}} \\ A \otimes X & \xrightarrow{m_{A,X}} & X \end{array} \qquad \begin{array}{ccc} M \otimes B \otimes N & \xrightarrow{m_{B,M}} & M \otimes N \\ {\scriptstyle \mathrm{id} \otimes m_{B,N}}\downarrow & & \downarrow{\scriptstyle F} \\ M \otimes N & \xrightarrow{F} & X \end{array} \qquad \begin{array}{ccc} M \otimes N \otimes C & \xrightarrow{m_{C,N}} & M \otimes N \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ X \otimes C & \xrightarrow{m_{C,X}} & X \end{array}$$

commute; here $m_{A,M} : A \otimes M \to M$ denotes the left action of A on M, and the maps $m_{A,X}$, $m_{B,M}$, $m_{B,N}$, $m_{C,N}$, and $m_{C,X}$ are defined similarly. The advantage of this reformulation is that it makes sense more generally: if $\mathcal{C}$ is an arbitrary monoidal category containing algebra objects A, B, and C and bimodules

$M \in {}_A\mathrm{BMod}_B(\mathcal{C})$, $N \in {}_B\mathrm{BMod}_C(\mathcal{C})$, and $X \in {}_A\mathrm{BMod}_C(\mathcal{C})$, then we can define a *bilinear map* from $M \otimes N$ to X to be a map $F : M \otimes N \to X$ such that the above diagrams commute.

Our goal in this section is to generalize the theory of bilinear maps to the ∞-categorical setting. That is, if $\mathcal{C}$ is a symmetric monoidal ∞-category (or, more generally, a planar ∞-operad) and we are given $A, B, C \in \mathrm{Alg}(\mathcal{C})$, $M \in {}_A\mathrm{BMod}_B(\mathcal{C})$, $N \in {}_B\mathrm{BMod}_C(\mathcal{C})$, and $X \in {}_A\mathrm{BMod}_C(\mathcal{C})$, then we will introduce the notion of a *bilinear map* from the pair $\{M, N\}$ to X. The definition is naturally given in the language of ∞-operads. Namely, we will introduce a correspondence of ∞-operads $\mathrm{Tens}^{\otimes}_{\succ} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^1$ with the following properties:

(a) There is a homotopy pushout diagram of ∞-operads

$$
\begin{array}{ccc}
\mathcal{A}\mathrm{ss}^{\otimes} & \xrightarrow{\;\;i\;\;} & \mathcal{B}\mathcal{M}^{\otimes} \\
\downarrow{\scriptstyle j} & & \downarrow \\
\mathcal{B}\mathcal{M}^{\otimes} & \longrightarrow & \mathrm{Tens}^{\otimes}_{\succ} \times_{\Delta^1} \{0\},
\end{array}
$$

where i is the inclusion $\mathcal{A}\mathrm{ss}^{\otimes} \simeq \mathcal{A}\mathrm{ss}^{\otimes}_{-} \hookrightarrow \mathcal{B}\mathcal{M}^{\otimes}$ and j the inclusion $\mathcal{A}\mathrm{ss}^{\otimes} \simeq \mathcal{A}\mathrm{ss}^{\otimes}_{+} \hookrightarrow \mathcal{B}\mathcal{M}^{\otimes}$. In other words, giving a $\mathcal{M}_0$-algebra object of $\mathcal{C}$ is equivalent to giving a pair of bimodule objects $M \in {}_A\mathrm{BMod}_B(\mathcal{C})$, $N \in {}_{B'}\mathrm{BMod}_C(\mathcal{C})$ such that $B' = B$.

(b) The fiber $\mathrm{Tens}^{\otimes}_{\succ} \times_{\Delta^1} \{1\}$ is isomorphic to $\mathcal{B}\mathcal{M}^{\otimes}$, so a $\mathcal{M}_1$-algebra object of $\mathcal{C}$ is an object $X \in {}_{A'}\mathrm{BMod}_{C'}(\mathcal{X})$.

(c) Suppose we are given algebra objects $\gamma_0 : \mathrm{Tens}^{\otimes}_{\succ} \times_{\Delta^1} \{0\} \to \mathcal{C}^{\otimes}$ and $\gamma_1 : \mathrm{Tens}^{\otimes}_{\succ} \times_{\Delta^1} \{1\} \to \mathcal{C}^{\otimes}$, corresponding to a triple $M \in {}_A\mathrm{BMod}_B(\mathcal{C})$, $N \in {}_B\mathrm{BMod}_C(\mathcal{C})$, and $X \in {}_{A'}\mathrm{BMod}_{C'}(\mathcal{C})$. Then extending γ_0 and γ_1 to a map of generalized ∞-operads $\mathrm{Tens}^{\otimes}_{\succ} \to \mathcal{C}^{\otimes}$ is equivalent (by definition) to giving a pair of associative algebra maps $A \to A'$, $C \to C'$, together with a map $M \otimes N \to X$ which is bilinear in the sense described above.

For later applications, it will be convenient to develop a slightly more general theory of *multilinear maps*. If $\mathcal{C}$ is a symmetric monoidal ∞-category and we are given a sequence of bimodules $M_{0,1} \in {}_{A_0}\mathrm{BMod}_{A_1}(\mathcal{C})$, $M_{1,2} \in {}_{A_1}\mathrm{BMod}_{A_2}(\mathcal{C}), \ldots, M_{n-1,n} \in {}_{A_{n-1}}\mathrm{BMod}_{A_n}(\mathcal{C})$, $X \in {}_{A_0}\mathrm{BMod}_{A_n}(\mathcal{C})$, then we can define the notion of a *multilinear map* from $\{M_{i-1,i}\}_{1 \le i \le n}$ to X: namely, a map $F : M_{0,1} \otimes \cdots \otimes M_{n-1,n} \to X$ which is compatible with the left action of A_0, the right action of A_n, and which *coequalizes* the right action of A_i on $M_{i-1,i}$ and the left action of A_i on $M_{i,i+1}$ for $0 < i < n$, all up to coherent homotopy. Moreover, we should be able to compose multilinear maps in a natural way. To encode all of this structure, we introduce a $\mathrm{N}(\mathbf{\Delta})^{op}$ family of ∞-operads $\mathrm{Tens}^{\otimes}$ such $\mathrm{Tens}^{\otimes}_{\succ} \simeq \mathrm{Tens}^{\otimes} \times_{\mathrm{N}(\mathbf{\Delta})^{op}} \Delta^1$, where Δ^1 maps to $\mathrm{N}(\mathbf{\Delta})^{op}$ via the morphism $[1] \simeq \{0, 2\} \hookrightarrow [2]$ in $\mathbf{\Delta}$.

Definition 4.3.4.1. Let $\mathbf{Ass}^{\otimes}$ be the category of Definition 4.1.1.3. We define a new category $\mathbf{Tens}^{\otimes}$ as follows:

(1) An object of $\mathbf{Tens}^{\otimes}$ consists of an object $\langle n \rangle \in \mathbf{Ass}^{\otimes}$, an object $[k] \in \mathbf{\Delta}^{op}$, and a pair of maps $c_-, c_+ : \langle n \rangle^{\circ} \to [k]$ satisfying $c_-(i) \le c_+(i) \le c_-(i) + 1$ for $i \in \langle n \rangle^{\circ}$.

(2) Let $(\langle n \rangle, [k], c_-, c_+)$ and $(\langle n' \rangle, [k'], c'_-, c'_+)$ be objects of $\mathbf{Tens}^{\otimes}$. A morphism from $(\langle n \rangle, [k], c_-, c_+)$ to $(\langle n' \rangle, [k'], c'_-, c'_+)$ consists of a morphism $\alpha : \langle n \rangle \to \langle n' \rangle$ in $\mathbf{Ass}^{\otimes}$ together with a morphism $\lambda : [k'] \to [k]$ in $\mathbf{\Delta}$ such that, for every $j \in \langle n' \rangle^{\circ}$ with $\alpha^{-1}\{j\} = \{\{i_0 \prec i_1 \prec \cdots \prec i_m\}$, we have $\lambda(c'_-(j)) = c_-(i_0)$, $\lambda(c'_+(j)) = c_+(i_m)$, and

$$
c_-(i_0) \le c_+(i_0) = c_-(i_1) \le c_+(i_1) = c_-(i_2) \le \cdots \le c_+(i_{m-1}) = c_-(i_m) \le c_+(i_m).
$$

We let $\mathrm{Tens}^{\otimes}$ denote the nerve of the category $\mathbf{Tens}^{\otimes}$.

By construction, the ∞-category $\mathrm{Tens}^{\otimes}$ is equipped with forgetful functors $\mathrm{N}(\mathbf{\Delta})^{op} \leftarrow \mathrm{Tens}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$. The following statement follows easily from the definitions:

Proposition 4.3.4.2. *The forgetful functor* $\mathrm{Tens}^{\otimes} \to \mathrm{N}(\mathbf{\Delta})^{op} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *exhibits* $\mathrm{Tens}^{\otimes}$ *as a* $\mathrm{N}(\mathbf{\Delta})^{op}$-*family of* ∞-*operads (in the sense of Definition 2.3.2.10). In particular,* $\mathrm{Tens}^{\otimes}$ *is a generalized* ∞-*operad.*

Remark 4.3.4.3. In fact, something slightly stronger is true: the forgetful functor $\mathrm{Tens}^{\otimes} \to \mathrm{N}(\mathbf{\Delta})^{op} \times \mathcal{A}\mathrm{ss}^{\otimes}$ is a fibration of generalized ∞-operads.

Remark 4.3.4.4. A morphism $\alpha : (\langle n \rangle, [k], c_-, c_+) \to (\langle n' \rangle, [k'], c'_-, c'_+)$ in **Tens**$^{\otimes}$ is inert if and only if it induces an inert morphism $\langle n \rangle \to \langle n' \rangle$ in **Ass**$^{\otimes}$ and an isomorphism $[k'] \to [k]$ in $\mathbf{\Delta}$.

Notation 4.3.4.5. If $[n]$ is an object of $\mathbf{\Delta}^{op}$, we let $\mathrm{Tens}^{\otimes}_{[k]}$ denote the fiber product $\mathrm{Tens}^{\otimes} \times_{\mathrm{N}(\mathbf{\Delta})^{op}} \{[k]\}$. Each $\mathrm{Tens}^{\otimes}_{[k]}$ is equipped with a fibration of ∞-operads $\mathrm{Tens}^{\otimes}_{[k]} \to \mathcal{A}\mathrm{ss}^{\otimes}$.

Example 4.3.4.6. The map $\mathrm{Tens}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ restricts to an isomorphism $\mathrm{Tens}^{\otimes}_{[0]} \simeq \mathcal{A}\mathrm{ss}^{\otimes}$.

Example 4.3.4.7. The ∞-operad $\mathrm{Tens}^{\otimes}_{[1]}$ is isomorphic to the ∞-operad $\mathcal{BM}^{\otimes}$ of §4.3.1; see Notation 4.3.1.5.

Remark 4.3.4.8. Fix integer $0 \leq i \leq k$. Let $\mathcal{O}(i)^{\otimes}$ denote the full subcategory of $\mathrm{Tens}^{\otimes}_{[k]}$ spanned by objects of the form $(\langle n \rangle, [k], c_-, c_+)$ where $c_-(m) = c_+(m) = i$ for $1 \leq m \leq n$. Then the composite map $\mathcal{O}(i)^{\otimes} \subseteq \mathrm{Tens}^{\otimes}_{[k]} \to \mathcal{A}\mathrm{ss}^{\otimes}$ is an isomorphism. Choosing a section, we obtain a map of ∞-operads $\epsilon_i : \mathcal{A}\mathrm{ss}^{\otimes} \to \mathrm{Tens}^{\otimes}_{[k]}$.

Notation 4.3.4.9. For $0 \leq i \leq k$, we let $\mathfrak{a}_i \in \mathrm{Tens}_{[k]}$ denote the image of $\langle 1 \rangle \in \mathcal{A}\mathrm{ss}$ under the embedding $\epsilon_i : \mathcal{A}\mathrm{ss}^{\otimes} \to \mathrm{Tens}^{\otimes}_{[k]}$ of Remark 4.3.4.8. If $0 \leq i, j \leq k$ with $j = i + 1$, we let $\mathfrak{m}_{i,j}$ denote the object of $\mathrm{Tens}_{[k]}$ given by $(\langle 1 \rangle, [k], c_-, c_+)$ where $c_-(1) = i$ and $c_+(1) = j$. Note that every object of $\mathrm{Tens}_{[k]}$ has the form $\mathfrak{a}_i$ (for $0 \leq i \leq k$) or $\mathfrak{m}_{i,j}$ (for $0 \leq i, j \leq k$ with $j = i + 1$).

Let $\theta : \mathrm{Tens}^{\otimes}_{[k]} \to \mathcal{C}^{\otimes}$ be a map of ∞-operads. For $0 \leq i \leq k$, $A_i = \theta(\mathfrak{a}_i) \in \mathcal{C}$ has the structure of an associative algebra object of $\mathcal{C}$. The intuition behind Definition 4.3.4.1 is the following: the functor θ is determined by the sequence of associative algebras $(A_0, \ldots, A_k)$, together with an A_i-A_j-bimodule $M_{i,j} = \theta(\mathfrak{m}_{i,j})$ when $j = i + 1$. Our first main result of this section (Proposition 4.3.4.11) makes this idea more precise. To state it, we will need a bit of terminology.

Notation 4.3.4.10. Every morphism $\alpha : [k] \to [k']$ in $\mathbf{\Delta}$ determines an edge $\Delta^1 \to \mathrm{N}(\mathbf{\Delta})^{op}$. We let $\mathrm{Tens}^{\otimes}_{\alpha}$ denote the fiber product $\mathrm{Tens}^{\otimes} \times_{\mathrm{N}(\mathbf{\Delta})^{op}} \Delta^1$, so that $\mathrm{Tens}^{\otimes}_{\alpha}$ is a correspondence of ∞-operads from $\mathrm{Tens}^{\otimes}_{\alpha} \times_{\Delta^1} \{0\} \simeq \mathrm{Tens}^{\otimes}_{[k']}$ to $\mathrm{Tens}^{\otimes}_{\alpha} \times_{\Delta^1} \{1\} \simeq \mathrm{Tens}^{\otimes}_{[k]}$. If the image of α is convex (that is, $\alpha(i+1) \leq \alpha(i) + 1$ for $0 \leq i < k$), then this correspondence is associated with a map of ∞-operads $v_\alpha : \mathrm{Tens}^{\otimes}_{[k]} \to \mathrm{Tens}^{\otimes}_{[k']}$, given by composition with α.

Given a nonempty linearly ordered set I, we let $\mathrm{Tens}^{\otimes}_I$ denote the ∞-operad $\mathrm{Tens}^{\otimes}_{[m]}$ where $m \geq 0$ is chosen so that there exists an isomorphism of linearly ordered sets $I \simeq [m]$. We will apply this notation in particular when I is a convex subset of $[k]$ for some $k \geq 0$. In this case, the inclusion $I \to [k]$ induces a map of ∞-operads $\mathrm{Tens}^{\otimes}_I \to \mathrm{Tens}^{\otimes}_{[k]}$, which defines an isomorphism of $\mathrm{Tens}^{\otimes}_I$ onto the full subcategory of $\mathrm{Tens}^{\otimes}_{[k]}$ spanned by those objects $(\langle n \rangle, [k], c_-, c_+)$ where $c_-, c_+ : \langle n \rangle^\circ \to [k]$ take values in I.

Proposition 4.3.4.11 (Segal Condition). *Fix $k \geq 0$. Then* $\mathrm{Tens}^{\otimes}_{[k]}$ *can be identified with the colimit (in the* ∞-*category* Op_∞ *of* ∞-*operads) of the diagram*
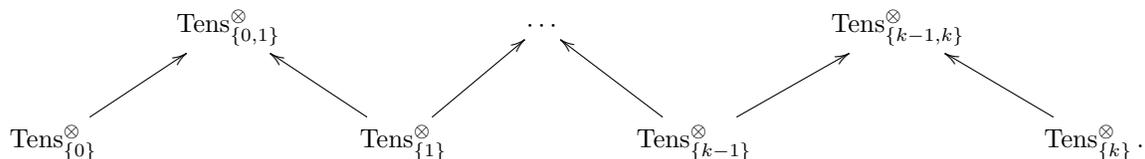
For the purpose of proving Proposition 4.3.4.11 it will be useful to package the structure of the generalized ∞-operad $\mathbf{Tens}^{\otimes}$ in a more compact form.

Definition 4.3.4.12. Let Step denote the full subcategory of $\mathrm{Fun}([1], \boldsymbol{\Delta})^{op}$ spanned by those morphisms $f : [n] \to [k]$ in $\boldsymbol{\Delta}$ with the following property: for $1 \le i \le n$, we have $f(i) \le f(i-1) + 1$.

Let $\phi : \boldsymbol{\Delta}^{op} \to \mathbf{Ass}^{\otimes}$ be the functor of Construction 4.1.2.5. We define a functor $\Phi : \mathrm{Step} \to \mathbf{Tens}^{\otimes}$ such that the diagram

$$
\begin{array}{ccc}
\mathrm{Step} & \xrightarrow{\ \Phi\ } & \mathbf{Tens}^{\otimes} \\
\downarrow & & \downarrow \\
\boldsymbol{\Delta}^{op} \times \boldsymbol{\Delta}^{op} & \xrightarrow{\ \phi \times \mathrm{id}\ } & \mathbf{Ass}^{\otimes} \times \boldsymbol{\Delta}^{op} \,.
\end{array}
$$

- Let $f : [n] \to [k]$ be a morphism in $\boldsymbol{\Delta}$, viewed as an object of Step. Then $\Phi(f) = (\phi([n]), [k], c_-, c_+)$, where $c_-, c_+ : \langle n \rangle^{\circ} \to [k]$ are given by

$$
c_-(i) = f(i-1) \qquad c_+(i) = f(i).
$$

- Let α be a morphism in Step, corresponding to a commutative diagram

$$
\begin{array}{ccc}
[n] & \xrightarrow{\ f\ } & [k] \\
{\scriptstyle \alpha_0} \downarrow & & \downarrow {\scriptstyle \alpha_1} \\
[n'] & \xrightarrow{\ f'\ } & [k'].
\end{array}
$$

Then $\Phi(\alpha)$ is the morphism $(\phi(\alpha_0), \alpha_1)$ of $\mathbf{Tens}^{\otimes}$.

Definition 4.3.4.13. We will say that a morphism α in $\mathrm{N}(\mathrm{Step})$ is *inert* if $\Phi(\alpha)$ is an inert morphism in the generalized ∞-operad $\mathbf{Tens}^{\otimes}$.

Remark 4.3.4.14. We can identify a morphism α in $\mathrm{N}(\mathrm{Step})$ with a commutative diagram

$$
\begin{array}{ccc}
[n] & \xrightarrow{\ f\ } & [k] \\
{\scriptstyle \alpha_0} \downarrow & & \downarrow {\scriptstyle \alpha_1} \\
[n'] & \xrightarrow{\ f'\ } & [k'].
\end{array}
$$

in $\boldsymbol{\Delta}$; using Remark 4.3.4.4, we conclude that α is inert if and only if α_1 is an isomorphism and α_0 satisfies $\alpha_0(i) = \alpha_0(i-1) + 1$ for $1 \le i \le n$.

Notation 4.3.4.15. Let S be an ∞-category equipped with a map $S \to \mathrm{N}(\boldsymbol{\Delta})^{op}$. We let $\mathrm{Tens}_S^{\otimes}$ denote the S-family of ∞-operads $\mathrm{Tens}^{\otimes} \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} S$, and $\mathrm{N}(\mathrm{Step})_S$ the fiber product $\mathrm{N}(\mathrm{Step}) \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} S$. We will say that a morphism in $\mathrm{N}(\mathrm{Step})_S$ is *inert* if its image in S is an equivalence and its image in $\mathrm{N}(\mathrm{Step})$ is inert.

Suppose we are given a categorical fibration $\mathcal{C}^{\otimes} \to \mathrm{Tens}_S^{\otimes}$ which exhibits $\mathcal{C}^{\otimes}$ as an S-family of ∞-operads. We let $\mathrm{Alg}_S(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}_{\mathrm{Tens}_S^{\otimes}}(\mathrm{N}(\mathrm{Step})_S, \mathcal{C}^{\otimes})$ spanned by those functors $F : \mathrm{N}(\mathrm{Step}_S) \to \mathcal{C}^{\otimes}$ which carry every inert morphism in $\mathrm{N}(\mathrm{Step})_S$ to an inert morphism in $\mathcal{C}^{\otimes}$.

Variant 4.3.4.16. Suppose we are given a fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$. For every ∞-category S equipped with a map $S \to \mathrm{N}(\boldsymbol{\Delta})^{op}$, let $\mathcal{C}_S^{\otimes}$ denote the fiber product

$$
S \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} \mathrm{Tens}^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathcal{C}^{\otimes} \,.
$$

Then the induced map $\mathcal{C}_S^{\otimes} \to \mathrm{Tens}_S^{\otimes}$ exhibits $\mathcal{C}^{\otimes}$ as an S-family of ∞-operads, so that the ∞-category $\mathrm{Alg}_S(\mathcal{C}_S)$ is defined as in Notation 4.3.4.15. We will abuse notation and denote this ∞-category simply by $\mathrm{Alg}_S(\mathcal{C})$; note that it can be identified with a full subcategory of $\mathrm{Fun}_{\mathcal{A}\mathrm{ss}^{\otimes}}(\mathrm{N}(\mathrm{Step})_S, \mathcal{C}^{\otimes})$.

We can now formulate a precise sense in which the ∞-category N(Step) is a "model" for the generalized ∞-operad $\mathrm{Tens}^{\otimes}$.

Proposition 4.3.4.17. *Let S be an ∞-category equipped with a map $S \to \mathrm{N}(\boldsymbol{\Delta})^{op}$, and suppose we are given a categorical fibration $q : \mathcal{C}^{\otimes} \to \mathrm{Tens}_S^{\otimes}$ which exhibits $\mathcal{C}^{\otimes}$ as an S-family of ∞-operads. Then composition with the functor $\Phi : \mathrm{N}(\mathrm{Step}) \to \mathrm{Tens}^{\otimes}$ of Definition 4.3.4.12 induces an equivalence of ∞-categories*

$$\mathrm{Alg}_{/\mathrm{Tens}_S}(\mathcal{C}) \to \mathrm{Alg}_S(\mathcal{C}).$$

We defer the proof of Propositions 4.3.4.11 and 4.3.4.17 until the end of this section.

Remark 4.3.4.18. Proposition 4.1.2.15 is the special case of Proposition 4.3.4.17 where we assume that $S = \{[0]\}$.

Let $\mathcal{C}^{\otimes}$ be a planar ∞-operad. Since the ∞-operad $\mathcal{A}\mathrm{ss}^{\otimes}$ is coherent (Proposition 4.1.1.16), we can associate to each associative algebra $A \in \mathrm{Alg}(\mathcal{C})$ a new planar ∞-operad $\mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C})^{\otimes}$ (see §3.3.3). Our next goal is to show that the underlying ∞-category $\mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C})$ is equivalent to the ∞-category $_A\mathrm{BMod}_A(\mathcal{C})$ of A-A-bimodule objects of $\mathcal{C}$. Moreover, we show that under this equivalence, the planar ∞-operad structure on $\mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C})$ can be described in terms of the theory of multilinear maps developed in this section (see Theorem 4.3.4.28 for a precise statement). We begin by introducing some notation.

Definition 4.3.4.19. Let $p : \mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ be a fibration of ∞-operads. We define simplicial sets $\overline{\mathrm{Bim}}(\mathcal{C})$ and $\overline{\mathrm{Bim}}'(\mathcal{C})$ equipped with maps $\overline{\mathrm{Bim}}(\mathcal{C}) \to \mathrm{N}(\boldsymbol{\Delta})^{op} \leftarrow \overline{\mathrm{Bim}}'(\mathcal{C})$ so that the following universal properties are satisfied: for every simplicial set K, we have isomorphisms

$$\mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(K, \mathrm{Bim}(\mathcal{C})) \simeq \mathrm{Fun}_{\mathcal{A}\mathrm{ss}^{\otimes}}(K \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} \mathrm{Tens}^{\otimes}, \mathcal{C}^{\otimes})$$

$$\mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(K, \mathrm{Bim}'(\mathcal{C})) \simeq \mathrm{Fun}_{\mathcal{A}\mathrm{ss}^{\otimes}}(K \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} \mathrm{N}(\mathrm{Step}), \mathcal{C}^{\otimes}).$$

Let $\mathrm{Bim}(\mathcal{C})$ be the full simplicial subset of $\overline{\mathrm{Bim}}(\mathcal{C})$ spanned by those vertices which determine ∞-operad maps $\mathrm{Tens}_{[n]}^{\otimes} \to \mathcal{C}^{\otimes}$ (for some $n \geq 0$), and let $\mathrm{Bim}'(\mathcal{C})$ be the full simplicial subset of $\overline{\mathrm{Bim}}'(\mathcal{C})$ spanned by those vertices which correspond to maps $\mathrm{N}(\mathrm{Step})_{[n]} \to \mathcal{C}^{\otimes}$ which preserve inert morphisms (for some $n \geq 0$).

Proposition 4.3.4.20. *Let $\mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ be a fibration of ∞-operads. Then:*

(1) *The maps $\overline{\mathrm{Bim}}(\mathcal{C}) \to \mathrm{N}(\boldsymbol{\Delta})^{op} \leftarrow \overline{\mathrm{Bim}}'(\mathcal{C})$ are categorical fibrations of simplicial sets. In particular, both $\overline{\mathrm{Bim}}(\mathcal{C})$ and $\overline{\mathrm{Bim}}'(\mathcal{C})$ are ∞-categories.*

(2) *The maps $\mathrm{Bim}(\mathcal{C}) \to \mathrm{N}(\boldsymbol{\Delta})^{op} \leftarrow \mathrm{Bim}'(\mathcal{C})$ are categorical fibrations of simplicial sets. In particular, both $\mathrm{Bim}(\mathcal{C})$ and $\mathrm{Bim}'(\mathcal{C})$ are ∞-categories.*

(3) *Composition with the functor $\Phi : \mathrm{N}(\mathrm{Step}) \to \mathrm{Tens}^{\otimes}$ of Definition 4.3.4.12 induces an equivalence of ∞-categories $\mathrm{Bim}(\mathcal{C}) \to \mathrm{Bim}'(\mathcal{C})$.*

Proof. Since $\boldsymbol{\Delta}$ is a category in which the only morphisms are identity maps, assertion (1) is equivalent to the statement that the maps $\overline{\mathrm{Bim}}(\mathcal{C}) \to \mathrm{N}(\boldsymbol{\Delta})^{op} \leftarrow \overline{\mathrm{Bim}}'(\mathcal{C})$ are inner fibrations. This follows from Proposition B.3.14, since $\mathrm{Tens}^{\otimes}$ and $\mathrm{N}(\mathrm{Step})$ are flat over $\mathrm{N}(\boldsymbol{\Delta})^{op}$ (Propositions 4.3.6.21 and 4.3.6.1). Assertion (2) is an immediate consequence of (1), since the inclusions $\mathrm{Bim}(\mathcal{C}) \subseteq \overline{\mathrm{Bim}}(\mathcal{C})$ and $\mathrm{Bim}'(\mathcal{C}) \subseteq \overline{\mathrm{Bim}}'(\mathcal{C})$ are categorical fibrations. To prove (3), it suffices to show that for every simplicial set S equipped with a map $S \to \mathrm{N}(\boldsymbol{\Delta})^{op}$, the induced map

$$\mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(S, \mathrm{Bim}(\mathcal{C})) \to \mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(S, \mathrm{Bim}'(\mathcal{C}))$$

is an equivalence of ∞-categories, which is a special case of Proposition 4.3.4.17. $\square$

Notation 4.3.4.21. Let $\mathcal{C}^{\otimes} \to \mathcal{A}ss^{\otimes}$ be a fibration of ∞-operads. For $n \geq 0$, we let $\mathrm{Bim}(\mathcal{C})_{[n]}$ and $\mathrm{Bim}'(\mathcal{C})_{[n]}$ denote the fiber products $\mathrm{Bim}(\mathcal{C}) \times_{\mathrm{N}(\mathbf{\Delta})^{op}} \{[n]\}$ and $\mathrm{Bim}'(\mathcal{C}) \times_{\mathrm{N}(\mathbf{\Delta})^{op}} \{[n]\}$. Unwinding the definitions, we see that $\mathrm{Bim}(\mathcal{C})_{[n]}$ can be identified with the ∞-category $\mathrm{Alg}_{\mathrm{Tens}_{[n]}/\mathcal{A}ss}(\mathcal{C})$. In particular, we have canonical isomorphisms

$$\mathrm{Bim}(\mathcal{C})_{[0]} \simeq \mathrm{Alg}(\mathcal{C}) \qquad \mathrm{Bim}(\mathcal{C})_{[1]} \simeq \mathrm{BMod}(\mathcal{C}).$$

Similarly, we have a canonical isomorphism $\mathrm{Bim}'(\mathcal{C})_{[0]} \simeq {}^{\Delta}\mathrm{Alg}(\mathcal{C})$.

Notation 4.3.4.22. Let Step_0 denote the full subcategory of Step spanned by those objects which correspond to constant maps $[n] \to [k]$ in $\mathbf{\Delta}$.

Remark 4.3.4.23. The forgetful functor $\mathrm{Step} \to \mathbf{\Delta}^{op}$ induces a Cartesian fibration $p : \mathrm{N}(\mathrm{Step}_0) \to \mathrm{N}(\mathbf{\Delta})^{op}$. In particular, p is a flat categorical fibration (Example B.3.11). The fiber of p over an object $[k] \in \mathbf{\Delta}^{op}$ is isomorphic to a disjoint union of $k+1$ copies of $\mathrm{N}(\mathbf{\Delta})^{op}$.

Construction 4.3.4.24. Let $\mathcal{C}^{\otimes} \to \mathcal{A}ss^{\otimes}$ be a fibration of ∞-operads. We define a simplicial set $\overline{\mathrm{Bim}'}_0(\mathcal{C})$ equipped with a map $\overline{\mathrm{Bim}'}_0(\mathcal{C}) \to \mathrm{N}(\mathbf{\Delta})^{op}$ so that the following universal property is satisfied: for every simplicial set K equipped with a map $K \to \mathrm{N}(\mathbf{\Delta})^{op}$, there is a canonical isomorphism

$$\mathrm{Fun}_{\mathrm{N}(\mathbf{\Delta})^{op}}(K, \overline{\mathrm{Bim}'}_0(\mathcal{C})) \simeq \mathrm{Fun}_{\mathcal{A}ss^{\otimes}}(K \times_{\mathrm{N}(\mathbf{\Delta})^{op}} \mathrm{N}(\mathrm{Step}_0), \mathcal{C}^{\otimes}).$$

We say that a morphism in Step_0 is inert if its image in Step is inert; let $\mathrm{Bim}'_0(\mathcal{C})$ be the full simplicial subset of $\overline{\mathrm{Bim}'}(\mathcal{C})$ spanned by those vertices which correspond to maps $\mathrm{N}(\mathrm{Step}_0)_{[k]} \to \mathcal{C}^{\otimes}$ which carry inert morphisms to inert morphisms (for some $k \geq 0$).

Remark 4.3.4.25. Let $\mathcal{C}^{\otimes} \to \mathcal{A}ss^{\otimes}$ be a fibration of ∞-operads. It follows from Theorem B.4.2 that the forgetful functor $\mathrm{Bim}'_0(\mathcal{C}) \to \mathrm{N}(\mathbf{\Delta})^{op}$ is a coCartesian fibration. Moreover, for each $k \geq 0$ the fiber $\mathrm{Bim}'_0(\mathcal{C})_{[k]}$ is canonically isomorphic to the product $\mathrm{Alg}(\mathcal{C})^{k+1}$.

An object of $\mathrm{Bim}'(\mathcal{C})_{[k]}$ can be thought of as a sequence of algebra objects $(A_0, A_1, \ldots, A_k)$ of $\mathcal{C}$, together with a sequence $\{M_i\}_{1 \leq i \leq n}$ where each M_i is an A_{i-1}-A_i-bimodule. The inclusion $\mathrm{Step}_0 \hookrightarrow \mathrm{Step}$ induces a forgetful functor $q : \mathrm{Bim}'(\mathcal{C}) \to \mathrm{Bim}'_0(\mathcal{C})$. Informally, this functor forgets the data of the bimodules M_i.

Remark 4.3.4.26. Let $\mathcal{C}^{\otimes} \to \mathcal{A}ss^{\otimes}$ be a fibration of ∞-operads. There is an evident functor $\mathrm{Step}_0 \to \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}$, given by the formula $(f : [n] \to [k]) \mapsto ([n], [k])$. Composition with this functor determines a diagonal map $\delta : \mathrm{N}(\mathbf{\Delta})^{op} \times {}^{\Delta}\mathrm{Alg}(\mathcal{C}) \to \mathrm{Bim}'_0(\mathcal{C})$. For each $k \geq 0$, the induced map of fibers $\delta_{[k]} \to \{[k]\} \times {}^{\Delta}\mathrm{Alg}(\mathcal{C}) \to \mathrm{Bim}'_0(\mathcal{C})_{[k]}$ can be identified with the diagonal map ${}^{\Delta}\mathrm{Alg}(\mathcal{C}) \to {}^{\Delta}\mathrm{Alg}(\mathcal{C})^{k+1}$.

Construction 4.3.4.27. Let $\mathcal{C}^{\otimes} \to \mathcal{A}ss^{\otimes}$ be a fibration of ∞-operads, and let $\phi : \mathbf{\Delta}^{op} \to \mathbf{Ass}^{\otimes}$ be the functor described in Construction 4.1.2.5. Composition with ϕ induces a map $\mathrm{N}(\mathrm{Fun}([1], \mathbf{\Delta}))^{op} \to \mathrm{Fun}(\Delta^1, \mathcal{A}ss^{\otimes})$, which carries $\mathrm{N}(\mathrm{Step})$ into $\mathcal{K}_{\mathcal{A}ss}$ (see Notation 3.3.2.1). Composition with this functor induces a map $\mathrm{Mod}^{\mathcal{A}ss}(\mathcal{C})^{\otimes} \times_{\mathcal{A}ss^{\otimes}} \mathrm{N}(\mathbf{\Delta})^{op} \to \mathrm{Bim}'(\mathcal{C})$. Unwinding the definitions, we see that the composition of this functor with the restriction map $\mathrm{Bim}'(\mathcal{C}) \to \mathrm{Bim}'_0(\mathcal{C})$ can be identified with the composition

$$\mathrm{Mod}^{\mathcal{A}ss}(\mathcal{C})^{\otimes} \times_{\mathcal{A}ss^{\otimes}} \mathrm{N}(\mathbf{\Delta})^{op} \to (\mathrm{Alg}(\mathcal{C}) \times \mathcal{A}ss^{\otimes}) \times_{\mathcal{A}ss^{\otimes}} \mathrm{N}(\mathbf{\Delta})^{op} \to {}^{\Delta}\mathrm{Alg}(\mathcal{C}) \times \mathrm{N}(\mathbf{\Delta})^{op} \xrightarrow{\delta} \mathrm{Bim}'_0(\mathcal{C}).$$

Theorem 4.3.4.28. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{A}ss^{\otimes}$ be a fibration of ∞-operads. Then the resulting diagram*

$$
\begin{array}{ccc}
\mathrm{Mod}^{\mathcal{A}ss}(\mathcal{C})^{\otimes} \times_{\mathcal{A}ss^{\otimes}} \mathrm{N}(\mathbf{\Delta})^{op} & \longrightarrow & \mathrm{Bim}'(\mathcal{C}) \\
\downarrow & & \downarrow \\
\mathrm{Alg}(\mathcal{C}) \times \mathrm{N}(\mathbf{\Delta})^{op} & \longrightarrow & \mathrm{Bim}'_0(\mathcal{C})
\end{array}
$$

is a homotopy pullback square of ∞-categories.

Proof. Let $^{\mathrm{p}}\mathrm{Alg}_{/\,\mathcal{A}\mathrm{ss}}(\mathcal{C})$ and $\overline{\mathcal{M}\mathrm{od}}^{\mathcal{A}\mathrm{ss}}(\mathcal{C})^{\otimes}$ be defined as in §3.3.3. We have a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Mod}^{\mathcal{A}\mathrm{ss}}(\mathcal{C}) \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathrm{N}(\boldsymbol{\Delta})^{op} & \longrightarrow & \overline{\mathcal{M}\mathrm{od}}^{\mathcal{A}\mathrm{ss}}(\mathcal{C})^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathrm{N}(\boldsymbol{\Delta})^{op} & \longrightarrow & \mathrm{Bim}'(\mathcal{C}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Alg}(\mathcal{C}) \times \mathrm{N}(\boldsymbol{\Delta})^{op} & \longrightarrow & {}^{\mathrm{p}}\mathrm{Alg}_{/\,\mathcal{A}\mathrm{ss}}(\mathcal{C}) \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathrm{N}(\boldsymbol{\Delta})^{op} & \longrightarrow & \mathrm{Bim}'_0(\mathcal{C}).
\end{array}
$$

In view of Remark 3.3.3.16, the horizontal maps on the left are categorical equivalences; it will therefore suffice to show that the square on the right is a homotopy pullback diagram. Fix an ∞-category S equipped with a map $S \to \mathrm{N}(\boldsymbol{\Delta})^{op}$; we will show that the induced diagram

$$
\begin{array}{ccc}
\mathrm{Fun}_{\mathcal{A}\mathrm{ss}^{\otimes}}(S, \overline{\mathcal{M}\mathrm{od}}^{\mathcal{A}\mathrm{ss}}(\mathcal{C})) & \longrightarrow & \mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(S, \mathrm{Bim}'(\mathcal{C})) \\
\downarrow & & \downarrow \\
\mathrm{Fun}_{\mathcal{A}\mathrm{ss}^{\otimes}}(S, {}^{\mathrm{p}}\mathrm{Alg}_{/\,\mathcal{A}\mathrm{ss}}(\mathcal{C})) & \longrightarrow & \mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(S, \mathrm{Bim}'_0(\mathcal{C}))
\end{array}
$$

is a homotopy pullback square of ∞-categories.

We observe that the ∞-category $\mathcal{K}_{\mathcal{A}\mathrm{ss}} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathrm{N}(\boldsymbol{\Delta})^{op}$ is isomorphic to the nerve of a category $\mathcal{D}$: an object of $\mathcal{D}$ is given by a triple $(\langle n \rangle, [k], \alpha)$ where $[k] \in \boldsymbol{\Delta}^{op}$, $\langle n \rangle \in \mathbf{Ass}^{\otimes}$, and $\alpha : \phi([k]) \to \langle n \rangle$ is a semi-inert morphism (where $\phi : \boldsymbol{\Delta}^{op} \to \mathbf{Ass}^{\otimes}$ is the functor described in Construction 4.1.2.5). Let $\psi : \mathrm{Step} \to \mathcal{D}$ be the functor described in Construction 4.3.4.27, given on objects by $\psi(f : [n] \to [k]) = ([k], \phi([n]), \phi(f))$. Let $\mathcal{E}$ denote the categorical mapping cylinder of ψ, which may be described more explicitly as follows:

- An object of $\mathcal{E}$ is either an object of $\mathcal{D}$ or an object of Step.

- Morphisms in $\mathcal{E}$ are given by the formula

$$
\mathrm{Hom}_{\mathcal{E}}(X, Y) = \begin{cases}
\mathrm{Hom}_{\mathcal{D}}(X, Y) & \text{if } X, Y \in \mathcal{D} \\
\mathrm{Hom}_{\mathrm{Step}}(X, Y) & \text{if } X, Y \in \mathrm{Step} \\
\mathrm{Hom}_{\mathcal{D}}(X, \psi Y) & \text{if } X \in \mathcal{D}, Y \in \mathrm{Step} \\
\emptyset & \text{otherwise.}
\end{cases}
$$

Let $\mathrm{N}(\mathcal{E})^0$ denote the full subcategory of $\mathrm{N}(\mathcal{E})$ spanned by those objects which belong either to Step_0 or to $\mathcal{K}^0_{\mathcal{A}\mathrm{ss}} \subseteq \mathcal{K}_{\mathcal{A}\mathrm{ss}}$ Let $\mathrm{N}(\mathcal{E})_S$ denote the fiber product $\mathrm{N}(\mathcal{D}) \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} S$, and define $\mathrm{N}(\mathcal{E})^0_S$. Let $\mathrm{Fun}'_{\mathcal{A}\mathrm{ss}^{\otimes}}(\mathrm{N}(\mathcal{E})_S, \mathcal{C}^{\otimes})$ be the full subcategory of $\mathrm{Fun}_{\mathcal{A}\mathrm{ss}^{\otimes}}(\mathrm{N}(\mathcal{E})_S, \mathcal{C}^{\otimes})$ spanned by those functors F with the following properties:

(*i*) The functor F is a p-left Kan extension of $F_0 = F|(\mathcal{K}_{\mathcal{A}\mathrm{ss}} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} S)$.

(*ii*) The functor F_0 determines a map $S \to \overline{\mathcal{M}\mathrm{od}}^{\mathcal{A}\mathrm{ss}}(\mathcal{C})$.

Similarly, we let $\mathrm{Fun}'_{\mathcal{A}\mathrm{ss}^{\otimes}}(\mathrm{N}(\mathcal{E})^0_S, \mathcal{C}^{\otimes})$ be the full subcategory of $\mathrm{Fun}_{\mathcal{A}\mathrm{ss}^{\otimes}}(\mathrm{N}(\mathcal{E})^0_S, \mathcal{C}^{\otimes})$ spanned by those functors F which satisfy the following conditions:

(*i'*) The functor F is a p-left Kan extension of $F_0 = F|(\mathcal{K}^0_{\mathcal{A}\mathrm{ss}} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} S)$.

(*ii'*) The functor F_0 determines a map $S \to {}^{\mathrm{p}}\mathrm{Alg}_{/\,\mathcal{A}\mathrm{ss}}(\mathcal{C})$.

Note that the inclusion $\mathrm{N}(\mathrm{Step}) \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} S$ determines restriction maps

$$
\mathrm{Fun}'_{\mathcal{A}\mathrm{ss}^{\otimes}}(\mathrm{N}(\mathcal{E})_S, \mathcal{C}^{\otimes}) \to \mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(S, \mathrm{Bim}'(\mathcal{C})) \qquad \mathrm{Fun}'_{\mathcal{A}\mathrm{ss}^{\otimes}}(\mathrm{N}(\mathcal{E}^0_S, \mathcal{C}^{\otimes}) \to \mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(S, \mathrm{Bim}'_0(\mathcal{C})).
$$

We have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Fun}_{\mathcal{A}\mathrm{ss}^\otimes}(S, \overline{\mathrm{Mod}}^{\mathcal{A}\mathrm{ss}}(\mathcal{C})) & \longrightarrow & \mathrm{Fun}'_{\mathcal{A}\mathrm{ss}^\otimes}(\mathrm{N}(\mathcal{E})_S, \mathcal{C}^\otimes) & \longrightarrow & \mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(S, \mathrm{Bim}'(\mathcal{C})) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Fun}_{\mathcal{A}\mathrm{ss}^\otimes}(S, {}^{\mathrm{p}}\mathrm{Alg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C})) & \longrightarrow & \mathrm{Fun}'_{\mathcal{A}\mathrm{ss}^\otimes}(\mathrm{N}(\mathcal{E})_S^0, \mathcal{C}^\otimes) & \longrightarrow & \mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(S, \mathrm{Bim}'_0(\mathcal{C}))
\end{array}$$

where the horizontal maps on the left are induced by the evident retraction map $r : \mathcal{E} \to \mathcal{D}$. Note that these left horizontal maps are sections of the restriction maps

$$\mathrm{Fun}'_{\mathcal{A}\mathrm{ss}^\otimes}(\mathrm{N}(\mathcal{E})_S, \mathcal{C}^\otimes) \to \mathrm{Fun}_{\mathcal{A}\mathrm{ss}^\otimes}(S, \overline{\mathrm{Mod}}^{\mathcal{A}\mathrm{ss}}(\mathcal{C}))\, \mathrm{Fun}'_{\mathcal{A}\mathrm{ss}^\otimes}(\mathrm{N}(\mathcal{E})_S^0, \mathcal{C}^\otimes) \to \mathrm{Fun}_{\mathcal{A}\mathrm{ss}^\otimes}(S, {}^{\mathrm{p}}\mathrm{Alg}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C})),$$

which are trivial Kan fibrations (Proposition T.4.3.2.15). To complete the proof, it will therefore suffice to show that the square appearing on the right of the above diagram is a homotopy pullback. In fact, we will show that the map

$$\mathrm{Fun}'_{\mathcal{A}\mathrm{ss}^\otimes}(\mathrm{N}(\mathcal{E})_S, \mathcal{C}^\otimes) \to \mathrm{Fun}'_{\mathcal{A}\mathrm{ss}^\otimes}(\mathrm{N}(\mathcal{E})_S^0, \mathcal{C}^\otimes) \times_{\mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(S, \mathrm{Bim}'_0(\mathcal{C}))} \mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(S, \mathrm{Bim}'(\mathcal{C}))$$

is a trivial Kan fibration.

Let $\mathrm{N}(\mathcal{E})_S^1$ be the full subcategory of $\mathrm{N}(\mathcal{E})_S$ spanned by the objects of $\mathrm{N}(\mathcal{E})_S^0$ together with the objects of $\mathrm{N}(\mathrm{Step}) \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} S$. Let $\mathrm{Fun}'_{\mathcal{A}\mathrm{ss}^\otimes}(\mathrm{N}(\mathcal{E})_S^1, \mathcal{C}^\otimes)$ be the full subcategory of $\mathrm{Fun}_{\mathcal{A}\mathrm{ss}^\otimes}(\mathrm{N}(\mathcal{E})_S^1, \mathcal{C}^\otimes)$ spanned by those functors F such that $F \,|\, \mathrm{N}(\mathcal{E})_S^0 \in \mathrm{Fun}'_{\mathcal{A}\mathrm{ss}^\otimes}(\mathrm{N}(\mathcal{E})_S^0, \mathcal{C}^\otimes)$ and $F \,|\, (\mathrm{N}(\mathrm{Step}) \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} S)$ determines a map $S \to \mathrm{Bim}'(\mathcal{C})$. In view of Proposition T.4.3.2.15, it will suffice to prove the following:

(a) Every functor $F_0 \in \mathrm{Fun}'_{\mathcal{A}\mathrm{ss}^\otimes}(\mathrm{N}(\mathcal{E})_S^1, \mathcal{C}^\otimes)$ admits a p-right Kan extension $F \in \mathrm{Fun}_{\mathcal{A}\mathrm{ss}^\otimes}(\mathrm{N}(\mathcal{E})_S, \mathcal{C}^\otimes)$.

(b) Let $F \in \mathrm{Fun}_{\mathcal{A}\mathrm{ss}^\otimes}(\mathrm{N}(\mathcal{E})_S, \mathcal{C}^\otimes)$ be an arbitrary functor such that $F_0 = F \,|\, \mathrm{N}(\mathcal{E})_S^1$ belongs to the ∞-category $\mathrm{Fun}'_{\mathcal{A}\mathrm{ss}^\otimes}(\mathrm{N}(\mathcal{E})_S^1, \mathcal{C}^\otimes)$. Then F is a p-right Kan extension of F_0 if and only if F belongs to $\mathrm{Fun}'_{\mathcal{A}\mathrm{ss}^\otimes}(\mathrm{N}(\mathcal{E})_S, \mathcal{C}^\otimes)$.

We first prove (a). Fix $F_0 \in \mathrm{Fun}'_{\mathcal{A}\mathrm{ss}^\otimes}(\mathrm{N}(\mathcal{E})_S^1, \mathcal{C}^\otimes)$. According to Lemma T.4.3.2.13, it will suffice to show that for every object $X \in \mathrm{N}(\mathcal{E})_S$, if we set $\mathcal{X} = \mathrm{N}(\mathcal{E})_S^1 \times_{\mathrm{N}(\mathcal{E})_S} (\mathrm{N}(\mathcal{E})_S)_{X/}$, then the map

$$\theta : \mathcal{X} \to \mathrm{N}(\mathcal{E})_S^1 \overset{F_0}{\to} \mathcal{C}^\otimes$$

can be extended to a p-limit diagram covering the evident map

$$\gamma : \mathcal{X}^\triangleright \to \mathrm{N}(\mathcal{E})_S \to \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathcal{A}\mathrm{ss}^\otimes .$$

Without loss of generality, we may assume that $X \notin \mathrm{N}(\mathcal{E})_S^1$. Let $s \in S$ denote the image of X and let $[k] \in \boldsymbol{\Delta}^{op}$ be the image of s, so that we can identify X with a semi-inert morphism $\phi([k]) \to \langle n \rangle$ in $\mathcal{A}\mathrm{ss}^\otimes$. Let $\mathcal{E}_{[k]}$ denote the fiber of the map $\mathcal{E} \to \boldsymbol{\Delta}^{op}$ over the object $[k]$, and let X' denote the image of X in $\mathcal{E}_{[k]}$. Let $\mathcal{X}_0$ be the full subcategory of $(\mathcal{E}_{[k]})_{X/}$ spanned by morphisms of the form $\alpha : X \to Y$ which satisfy one of the following two conditions:

(1) The object Y belongs to $\mathcal{D}_{[k]}$ and corresponds to a null morphism $\phi([k]) \to \langle m \rangle$, and α induces an inert map $\langle n \rangle \to \langle m \rangle$.

(2) The object Y belongs to $\mathrm{Step}_{[k]}$ and corresponds to a map $f : [m] \to [k]$. Moreover, the map α induces an inert map $\langle n \rangle \to \phi([m])$.

It is not difficult to show that the inclusion $\mathfrak{X}_0 \hookrightarrow \mathfrak{X}$ admits a right adjoint and is therefore right cofinal. Consequently, it will suffice to show that the restriction $\theta_0 = \theta | \mathfrak{X}_0$ can be extended to a p-limit diagram compatible with γ.

Let $\mathfrak{X}_1 \subseteq \mathfrak{X}_0$ be the full subcategory spanned by those objects which satisfy conditions (1) or (2) above, with $m = 1$. Using our assumption that p is a fibration of ∞-operads, we deduce that θ_0 is a p-right Kan extension of $\theta_1 = \theta | \mathfrak{X}_1$. Using Lemma T.4.3.2.7, we are reduced to proving that θ_1 can be extended to a p-limit diagram (compatible with γ).

Let $\mathfrak{X}_2 \subseteq \mathfrak{X}_1$ be the full subcategory spanned by those objects which either satisfy condition (1) with $m = 1$, or satisfy the following stronger version of condition (2):

(2′) The object Y belongs to $\mathrm{Step}_{[k]}$ and corresponds to the $f : [1] \simeq \{i - 1, i\} \hookrightarrow [k]$ for some $1 \leq i \leq k$. Moreover, the map α induces an inert map $\langle n \rangle \to \phi([m])$.

It is not hard to see that the inclusion $\mathfrak{X}_2 \subseteq \mathfrak{X}_1$ admits a right adjoint and is therefore right cofinal. We are therefore reduced to proving that $\theta_2 = \theta_1 | \mathfrak{X}_2$ can be extended to a p-limit diagram (compatible with γ). We note that $\mathfrak{X}_2$ is equivalent to a discrete simplicial set, whose objects are morphisms $\alpha_i : X \to X_i$ for $1 \leq i \leq n$. The existence of a p-limit of θ_2 now follows from our assumption that p is a fibration of ∞-operads. This proves (a). Moreover, the proof gives the following version of (b):

(b′) Let $F \in \mathrm{Fun}_{\mathcal{A}\mathrm{ss}^{\otimes}}(\mathrm{N}(\mathcal{E})_S, \mathcal{C}^{\otimes})$ be an arbitrary functor such that $F_0 = F | \mathrm{N}(\mathcal{E})_S^1$ belongs to the ∞-category $\mathrm{Fun}'_{\mathcal{A}\mathrm{ss}^{\otimes}}(\mathrm{N}(\mathcal{E})_S^1, \mathcal{C}^{\otimes})$. Then F is a p-right Kan extension of F_0 if and only if, for every object $X \in \mathrm{N}(\mathcal{E})_S$ as above, each of the maps $F(\alpha_i)$ is an inert morphism in $\mathcal{C}^{\otimes}$.

To complete the proof, it will suffice show that if $F \in \mathrm{Fun}_{\mathcal{A}\mathrm{ss}^{\otimes}}(\mathrm{N}(\mathcal{E})_S, \mathcal{C}^{\otimes})$, then F satisfies conditions (i) and (ii) if and only if it satisfies the criterion of (b'). Assume first that F satisfies (i) and (ii), and let $X \in \mathrm{N}(\mathcal{E})_S$ be as above. Fix $1 \leq i \leq n$; we wish to show that $F(\alpha_i)$ is an inert morphism in $\mathcal{C}^{\otimes}$. If i does not lie in the image of the semi-inert morphism $\beta : \phi([k]) \to \langle n \rangle$, then α_i is an inert morphism in $\mathcal{K}_{\mathcal{A}\mathrm{ss}} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \{s\}$ and the desired result follows from (ii). If $i = \beta(j)$ for $1 \leq j \leq k$, then α_i factors as a composition

$$X \overset{\alpha_i'}{\to} (s, \rho^i : \langle n \rangle \to \langle 1 \rangle) \overset{\alpha_i''}{\to} X_i',$$

where $F(\alpha_i')$ is inert by virtue of (ii) and $F(\alpha_i'')$ is an equivalence by (i), from which it follows that $F(\alpha_i)$ is inert.

Now suppose that F satisfies the criterion described in (b'); we wish to show that conditions (i) and (ii) are satisfied. We first prove (i). Fix an object $Y \in \mathrm{N}(\mathrm{Step}) \times_{\mathrm{N}(\mathbf{\Delta})^{op}} S$, corresponding to a vertex $s \in S$ lying over $[k] \in \mathbf{\Delta}^{op}$ and a map $f : [n] \to [k]$ in $\mathbf{\Delta}$. Let $X \in \mathrm{N}(\mathcal{E})_S$ be the image of Y under the retraction r. We wish to prove that F carries the natural map $X \to Y$ to an equivalence in $\mathcal{C}^{\otimes}$. For $1 \leq i \leq n$, let $\epsilon_i : Y \to Y_i$ be the morphism in $\mathrm{N}(\mathrm{Step}) \times_{\mathrm{N}(\mathbf{\Delta})^{op}} \{s\}$ corresponding to the map $[1] \simeq \{i - 1, i\} \hookrightarrow [n]$. Since $F_0 \in \mathrm{Fun}'_{\mathcal{A}\mathrm{ss}^{\otimes}}(\mathrm{N}(\mathcal{E})_S^1, \mathcal{C}^{\otimes})$, we conclude that each $F(\epsilon_i)$ is an inert morphism in $\mathcal{C}^{\otimes}$. Since $\mathcal{C}^{\otimes}$ is an ∞-operad, it will suffice to show that each of the composite maps $F(X) \to F(Y) \to F(Y_i)$ is inert. If i lies in the image of the map $\phi([k]) \to \phi([n])$, then this composition can be identified with $F(\alpha_i)$ and the desired result follows from the criterion of (b'). If i does not belong to the image of $\phi([k])$, then the composite map factors as a composition

$$F(X) \overset{F(\alpha_i)}{\longrightarrow} F(X_i) \to F(Y_i)$$

where $F(\alpha_i)$ is inert and the second map is an equivalence, since F_0 satisfies condition (i').

We now complete the proof by showing that F satisfies (ii). In view of Remark 2.1.2.9, it will suffice to show that for every object $X \in \mathrm{N}(\mathcal{E})_S$ corresponding to an object $s \in S$ lying over $[k] \in \mathbf{\Delta}^{op}$ and a semi-inert morphism $\beta : \phi([k]) \to \langle n \rangle$ as above, if we choose inert morphisms $\alpha_i' : X \to X_i' = (s, \phi([k]) \to \langle 1 \rangle)$ lying over $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \leq i \leq n$, then each $F(\alpha_i')$ is an inert morphism in $\mathcal{C}^{\otimes}$. If i does not lie in the image of β, then $\alpha_i \simeq \alpha_i'$ and the desired result is an immediate consequence of the criterion of (b'). If i does lie in the image of β, then α_i factors as a composition

$$X \overset{\alpha_i'}{\to} X_i' \overset{\alpha_i''}{\to} X_i.$$

Since $F(\alpha_i'')$ is an equivalence (by assumption (i')) and $F(\alpha_i)$ is inert (by the criterion of (b')), we conclude that $F(\alpha_i')$ is inert as desired. $\square$

We now turn to the proofs of Propositions 4.3.4.17 and 4.3.4.11.

Proof of Proposition 4.3.4.17. We use the same idea as in the proof of Theorem 2.3.3.23, though the details are slightly more complicated. First, let $\mathcal{J}$ denote the categorical mapping cylinder of the functor Φ. That is, an object of $\mathcal{J}$ consists either of an object of Step or an object of $\mathbf{Tens}^{\otimes}$, with morphisms in $\mathcal{J}$ defined by the formula

$$\mathrm{Hom}_{\mathcal{J}}(x,y) = \begin{cases} \mathrm{Hom}_{\mathrm{Step}}(x,y) & \text{if } x,y \in \mathrm{Step} \\ \mathrm{Hom}_{\mathbf{Tens}^{\otimes}}(x,y) & \text{if } x,y \in \mathbf{Tens}^{\otimes} \\ \mathrm{Hom}_{\mathbf{Tens}^{\otimes}}(x,\Phi(y)) & \text{if } x \in \mathbf{Tens}^{\otimes}, y \in \mathrm{Step} \\ \emptyset & \text{if } x \in \mathrm{Step}, y \in \mathbf{Tens}^{\otimes}. \end{cases}$$

We regard $\mathbf{Tens}^{\otimes}$ and Step as full subcategories of $\mathcal{J}$. Note that there is a retraction of $\mathcal{J}$ onto $\mathbf{Tens}^{\otimes}$, whose restriction to Step coincides with the functor Φ. In particular, we can use the composite map $\mathcal{J} \to \mathbf{Tens}^{\otimes} \to \mathbf{\Delta}^{op}$ to define an ∞-category $\mathrm{N}(\mathcal{J})_S = \mathrm{N}(\mathcal{J}) \times_{\mathrm{N}(\mathbf{\Delta})^{op}} S$. There is an evident functor $\mathrm{N}(\mathcal{J})_S \to \Delta^1$, whose fibers are given by

$$\mathrm{N}(\mathcal{J})_S \times_{\Delta^1} \{0\} \simeq \mathrm{Tens}^{\otimes}_S \qquad \mathrm{N}(\mathcal{J})_S \times_{\Delta^1} \{1\} \simeq \mathrm{N}(\mathrm{Step})_S.$$

We will use these isomorphisms to identify $\mathrm{N}(\mathrm{Step})_S$ and $\mathrm{Tens}^{\otimes}_S$ with full subcategories of $\mathrm{N}(\mathcal{J})_S$. Moreover, the retraction of $\mathcal{J}$ onto $\mathbf{Tens}^{\otimes}$ determines a retraction $r : \mathrm{N}(\mathcal{J})_S \to \mathrm{Tens}^{\otimes}_S$, whose restriction to $\mathrm{N}(\mathrm{Step})_S$ is the functor $\Phi_S : \mathrm{N}(\mathrm{Step})_S \to \mathrm{Tens}^{\otimes}_S$ induced by Φ.

Let $\mathcal{A}$ be the full subcategory of $\mathrm{Fun}_{\mathrm{Tens}^{\otimes}_S}(\mathrm{N}(\mathcal{J})_S, \mathcal{C}^{\otimes})$ spanned by those functors $F : \mathrm{N}(\mathcal{J})_S \to \mathcal{C}^{\otimes}$ with the following properties:

(i) Let $x \in \mathrm{N}(\mathrm{Step})_S$ be an object and $\alpha : \Phi_S(x) \to x$ the evident map. Then $F(\alpha)$ is an equivalence in $\mathcal{C}^{\otimes}$.

(ii) The restriction $F| \mathrm{Tens}^{\otimes}_S$ belongs to $\mathrm{Alg}_{/\mathrm{Tens}_S}(\mathcal{C})$.

Conditions (i) and (ii) together imply that $F| \mathrm{N}(\mathrm{Step})_S$ is equivalent to the composition $(F| \mathrm{Tens}^{\otimes}_S) \circ \Phi_S$, and therefore F also satisfies:

(iii) The restriction $F| \mathrm{N}(\mathrm{Step})_S$ belongs to $\mathrm{Alg}_S(\mathcal{C})$.

Note that the functor $\mathrm{Alg}_{/\mathrm{Tens}_S}(\mathcal{C}) \to \mathrm{Alg}_S(\mathcal{C})$ factors as a composition

$$\mathrm{Alg}_{/\mathrm{Tens}_S}(\mathcal{C}) \to \mathcal{A} \to \mathrm{Alg}_S(\mathcal{C})$$

where the first map is given by composition with r. We note that r has a left inverse, given by the restriction map $\mathcal{A} \to \mathrm{Alg}_{/\mathrm{Tens}_S}(\mathcal{C})$. Note that a functor F satisfies condition (i) if and only if F is a q-left Kan extension of $F| \mathrm{Tens}^{\otimes}_S$. Using Proposition T.4.3.2.15, we deduce that the $\mathcal{A} \to \mathrm{Alg}_{/\mathrm{Tens}_S}(\mathcal{C})$ is a trivial Kan fibration, and therefore r is an equivalence of ∞-categories. To complete the proof, it will suffice to show that the restriction map $\mathcal{A} \to \mathrm{Alg}_S(\mathcal{C})$ is also a trivial Kan fibration. In view of Proposition T.4.3.2.15, it will suffice to prove the following:

(a) Every $F_0 \in \mathrm{Alg}_S(\mathcal{C})$ admits a q-right Kan extension $F \in \mathrm{Fun}_{\mathrm{Tens}^{\otimes}_S}(\mathrm{N}(\mathcal{J})_S, \mathcal{C}^{\otimes})$.

(b) A functor $F \in \mathrm{Fun}_{\mathrm{Tens}^{\otimes}_S}(\mathrm{N}(\mathcal{J})_S, \mathcal{C}^{\otimes})$ belongs to $\mathcal{A}$ if and only if $F| \mathrm{N}(\mathrm{Step})_S \in \mathrm{Alg}_S(\mathcal{C})$ and F is a q-right Kan extension of $F| \mathrm{N}(\mathrm{Step})_S$.

We first prove (b). Choose a functor $F_0 \in \mathrm{N}(\mathrm{Step})_S$; we wish to prove that F_0 admits a q-right Kan extension $F \in \mathrm{Fun}_{\mathrm{Tens}_S^\otimes}(\mathrm{N}(\mathcal{J})_S, \mathcal{C}^\otimes)$. Fix an object of $\bar{s} \in \mathrm{Tens}_S^\otimes$, corresponding to an object $s \in S$ and an object $(\langle n \rangle, [k], c_-, c_+)$ having the same image in $\mathbf{\Delta}^{op}$. Let $\mathcal{J}$ denote the ∞-category $\mathrm{N}(\mathrm{Step})_S \times_{\mathcal{A}} \mathcal{A}_{\bar{s}/}$. According to Lemma T.4.3.2.13, it will suffice to show that the functor

$$f : \mathcal{J} \to \mathcal{A}_{\bar{s}/} \to \mathcal{A} \xrightarrow{F} \mathcal{C}^\otimes$$

can be extended to a q-limit diagram lying over the map

$$g : \mathcal{J}^\triangleleft \to \mathcal{A}_{\bar{s}/}^\triangleleft \to \mathcal{A} \xrightarrow{r} \mathrm{Tens}_S^\otimes .$$

Let $\mathcal{J}_0$ be the full subcategory of $\mathcal{J}$ spanned by those morphisms $\alpha : \bar{s} \to \Phi_S(\bar{t})$, where $\bar{t}$ is an object of $\mathrm{N}(\mathrm{Step})_S$ lying over $t \in S$ and α induces an equivalence $s \to t$ in the ∞-category S. We note that the inclusion $\mathcal{J}_0 \subseteq \mathcal{J}$ admits a right adjoint and is therefore right cofinal. Consequently, we are reduced to proving that $f|\mathcal{J}_0$ can be extended to a q-limit diagram lying over $g|\mathcal{J}_0^\triangleleft$.

Let $\mathcal{A}_s$ denote the fiber product $\mathcal{A} \times_S \{s\}$, let $\mathrm{N}(\mathrm{Step})_s$ be defined similarly, and let $\mathcal{J}_1$ denote the fiber product $\mathrm{N}(\mathrm{Step})_S \times_{\mathcal{A}_s} (\mathcal{A}_s)_{\bar{s}/}$. Note that the canonical map $\mathcal{J}_1 \to \mathcal{J}$ restricts to an equivalence $\mathcal{J}_1 \simeq \mathcal{J}_0$. It therefore suffices to prove that $f|\mathcal{J}_1$ can be extended to a q-limit diagram lying over $g|\mathcal{J}_1^\triangleleft$.

Unwinding the definitions, we can identify objects of $\mathcal{J}_1$ with pairs consisting of a map $\alpha : [m] \to [k]$ in $\mathbf{\Delta}$ and a map $\beta : \langle n \rangle \to \phi([m])$ in $\mathcal{A}\mathrm{ss}^\otimes$ which satisfy the following condition:

- For $1 \leq j \leq m$, if $\beta^{-1}\{j\}$ is the linearly ordered set $\{i_0 \prec \cdots \prec i_k\}$, then we have

$$\alpha(j-1) = c_-(i_0) \leq c_+(i_0) = c_-(i_1) \leq c_+(i_1) = \cdots = c_-(i_k) \leq c_+(i_k) = \alpha(j) \leq \alpha(j-1) + 1.$$

Let $\mathcal{J}_2$ denote the full subcategory of $\mathcal{J}_1$ spanned by those pairs (α, β) such that β is inert. We note that the inclusion $\mathcal{J}_2 \subseteq \mathcal{J}_1$ admits a right adjoint and is therefore right cofinal. We are therefore reduced to proving that $f|\mathcal{J}_2$ can be extended to a q-limit diagram lying over $g|\mathcal{J}_2^\triangleleft$.

Let $\mathcal{J}_3$ be the full subcategory of $\mathcal{J}_2$ spanned by those pairs $(\alpha : [m] \to [k], \beta : \langle n \rangle \to \phi([m]))$ where $m = 1$. Using our assumption that $F_0 \in \mathrm{Alg}_S(\mathcal{C})$, we deduce that $f|\mathcal{J}_2$ is a q-right Kan extension of $f|\mathcal{J}_3$. Using Lemma T.4.3.2.7, we are reduced to proving that $f|\mathcal{J}_3$ admits a q-limit diagram lying over $g|\mathcal{J}_3^\triangleleft$. We note that $\mathcal{J}_3$ is a discrete category whose objects correspond bijectively to the elements of $\langle n \rangle^\circ$. The existence of the desired extension now follows immediately from our assumption that $\mathcal{C}^\otimes \to \mathrm{Tens}_S^\otimes$ is a fibration of S-families of ∞-operads. This proves (a). Let $\mathcal{A}'$ be the full subcategory of $F \in \mathrm{Fun}_{\mathrm{Tens}_S^\otimes}(\mathrm{N}(\mathcal{J})_S, \mathcal{C}^\otimes)$ spanned by those functors such that $F|\mathrm{N}(\mathrm{Step})_S \in \mathrm{Alg}_S(\mathcal{C})$ and F is a q-right Kan extension of $F|\mathrm{N}(\mathrm{Step})_S$. We obtain the following version of (b):

(b') Let $F \in \mathrm{Fun}_{\mathrm{Tens}_S^\otimes}(\mathrm{N}(\mathcal{J})_S, \mathcal{C}^\otimes)$ be a functor such that $F|\mathrm{N}(\mathrm{Step})_S$ belongs to $\mathrm{Alg}_S(\mathcal{C})$. Then $F \in \mathcal{A}'$ if and only if, for every object $\bar{s} \in \mathrm{Tens}_S^\otimes$ as above, the maps $\{\rho^i : \langle n \rangle \to \phi([1])\}_{1 \leq i \leq n}$ exhibit $F(\bar{s})$ as a q-product of the objects $F(s, [1] \xrightarrow{c_-(i), c_+(i)} [k])$ in $\mathcal{C}^\otimes$.

To prove (b), we wish to show that $\mathcal{A} = \mathcal{A}'$. If $F \in \mathcal{A}$, then we may assume without loss of generality that F factors as a composition

$$\mathcal{A} \xrightarrow{r} \mathrm{Tens}_S^\otimes \xrightarrow{F'} \mathcal{C}^\otimes$$

where $F' \in \mathrm{Alg}_{/\mathrm{Tens}_S}(\mathcal{C})$. In this case, the criterion of (b') is satisfied so that $F \in \mathcal{A}'$. Conversely, suppose that $F \in \mathcal{A}'$; we must show that F satisfies conditions (i) and (ii). To verify (i), choose an object $x = (s, \gamma : [n] \to [k]) \in \mathrm{N}(\mathrm{Step})_S$ and let $\alpha : \Phi_S(x) \to x$ be the canonical map in $\mathcal{A}$. Let $f : \mathcal{J}_3 \to \mathcal{C}^\otimes$ be defined as above with $\bar{s} = \Phi_S(x)$. Criterion (b') insures that F exhibits $F(\Phi_S(x))$ as a q-limit of f and the assumption that $F|\mathrm{N}(\mathrm{Step})_S$ belongs to $\mathrm{Alg}_S(\mathcal{C})$ implies that F exhibits $F(x)$ as a q-limit of f; the essential uniqueness of q-limit diagrams guarantees that $F(\alpha)$ is an equivalence.

We now verify condition (ii). According to Remark 2.1.2.9, it will suffice to verify that for every object $\bar{s} \in \mathrm{Tens}_S^{\otimes}$ as above, if we let $\alpha_i : \bar{s} \to \bar{s}_i$ in $\mathrm{Tens}_{\{s\}}^{\otimes}$ be inert morphisms covering $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \le i \le n$ (in the ∞-operad $\mathrm{Tens}_{\{s\}}^{\otimes}$), then each $F(\alpha_i)$ is an inert morphism in $\mathcal{C}^{\otimes}$. Note that $\bar{s}_i = \Phi_S(t_i)$ for some object $t_i = (s, [1] \to [k]) \in \mathrm{N(Step)}_S$. In view of condition (i), it suffices to show that each of the composite maps

$$\beta_i : \bar{s} \xrightarrow{\alpha_i} \bar{s}_i \to t_i$$

is inert, which follows immediately from criterion (b'). $\qquad\square$

Proof of Proposition 4.3.4.11. For every finite nonempty linearly ordered set I, let $\mathrm{N(Step)}_I$ denote the fiber product $\mathrm{N(Step)} \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} \{[m]\}$ where $I \simeq [m]$. Let $\overline{\mathrm{N(Step)}}_I = (\mathrm{N(Step)}_I, M_I)$, where M_I is the collection of inert morphisms in $\mathrm{N(Step)}_I$; we regard $\overline{\mathrm{N(Step)}}_I$ as an ∞-preoperad. The functor Φ of Definition 4.3.4.12 defines a map of ∞-preoperads $\overline{\mathrm{N(Step)}}_I \to \mathrm{Tens}_I^{\otimes,\natural}$ (see Notation 2.1.4.5), and Proposition 4.3.4.17 implies that this map is a weak equivalence of ∞-preoperads. In view of Theorem T.4.2.4.1, it will suffice to show that $\overline{\mathrm{N(Step)}}_{[k]}$ is a homotopy colimit of the diagram

$$
\begin{array}{ccccccc}
& \overline{\mathrm{N(Step)}}_{\{0,1\}} & & \cdots & & \overline{\mathrm{N(Step)}}_{\{k-1,k\}} & \\
& \nearrow \quad \nwarrow & & \nwarrow \quad \nearrow & & \nearrow \quad \nwarrow & \\
\overline{\mathrm{N(Step)}}_{\{0\}} & & \overline{\mathrm{N(Step)}}_{\{1\}} & & \overline{\mathrm{N(Step)}}_{\{k-1\}} & & \overline{\mathrm{N(Step)}}_{\{k\}}
\end{array}
$$
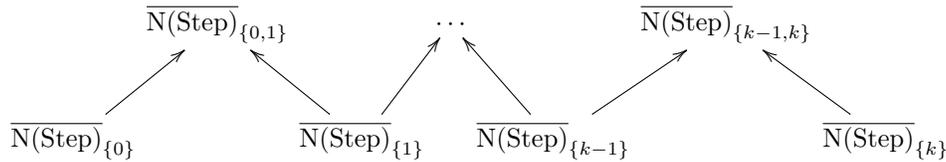
in $\mathcal{P}\mathrm{Op}_\infty$. We observe that this diagram is cofibrant, and its colimit is isomorphic to the pair $(\mathfrak{X}, M)$, where $\mathfrak{X}$ is the full subcategory of $\mathrm{N(Step)}_{[k]}$ spanned by those objects $f : [n] \to [k]$ whose image in $[k]$ has cardinality ≤ 2 and M is the collection of inert morphisms in $\mathfrak{X}$. It will therefore suffice to show that the inclusion $(\mathfrak{X}, M) \subseteq \overline{\mathrm{N(Step)}}_{[k]}$ is a weak equivalence of ∞-preoperads. Unwinding the definitions, we must prove the following:

$(*)$ Let $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad. Then the restriction functor $\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N(Step)}_{[k]}, \mathcal{C}^{\otimes}) \to \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathfrak{X}, \mathcal{C}^{\otimes})$ induces a trivial Kan fibration $\mathrm{Fun}^0_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N(Step)}_{[k]}, \mathcal{C}^{\otimes}) \to \mathrm{Fun}^0_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathfrak{X}, \mathcal{C}^{\otimes})$, where $\mathrm{Fun}^0_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N(Step)}_{[k]}, \mathcal{C}^{\otimes})$ is the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N(Step)}_{[k]}, \mathcal{C}^{\otimes})$ spanned by those functors which carry inert morphisms in $\mathrm{N(Step)}_{[k]}$ to inert morphisms in $\mathcal{C}^{\otimes}$, and $\mathrm{Fun}^0_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathfrak{X}, \mathcal{C}^{\otimes})$ is defined similarly.

In view of Proposition T.4.3.2.15, $(*)$ will follow from the following pair of assertions:

(a) Let $F_0 \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathfrak{X}, \mathcal{C}^{\otimes})$ be a functor which carries every inert morphism in $\mathfrak{X}$ to an inert morphism in $\mathcal{C}^{\otimes}$. Then F_0 admits a p-right Kan extension $F \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N(Step)}_{[k]}, \mathcal{C}^{\otimes})$.

(b) Let $F \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N(Step)}_{[k]}, \mathcal{C}^{\otimes})$ be a functor such that $F_0 = F | \mathfrak{X}$ carries inert morphisms in $\mathfrak{X}$ to inert morphisms in $\mathcal{C}^{\otimes}$. Then F is a p-right Kan extension of F_0 if and only if F carries inert morphisms in $\mathrm{N(Step)}_{[k]}$ to inert morphisms in $\mathcal{C}^{\otimes}$.

We begin by proving (a). Fix a functor $F_0 \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathfrak{X}, \mathcal{C}^{\otimes})$ which carries inert morphisms in $\mathfrak{X}$ to inert morphisms in $\mathcal{C}^{\otimes}$. Let X be an object of $\mathrm{N(Step)}_{[k]}$, corresponding to a map of linearly ordered sets $f : [n] \to [k]$. Let $\mathfrak{X}_{X/} = \mathfrak{X} \times_{\mathrm{N(Step)}_{[k]}} (\mathrm{N(Step)}_{[k]})_{X/}$; the objects of $\mathfrak{X}_{/X}$ can be identified with maps of linearly ordered sets $g : [m] \to [n]$ such that $f \circ g$ carries $[m]$ to a convex subset of $[k]$ having cardinality at most 2. According to Lemma T.4.3.2.13, it will suffice to show that the map $F_0 | \mathfrak{X}_{X/}$ can be extended to a p-limit diagram covering the evident map

$$\lambda : \mathfrak{X}_{X/}^{\triangleright} \hookrightarrow (\mathrm{N(Step)}_{[k]})_{X/}^{\triangleright} \to \mathrm{N(Step)}_{[k]} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*).$$

Let $\mathfrak{X}^0_{X/}$ be the full subcategory of $\mathfrak{X}_{X/}$ spanned by those objects for which $g : [m] \to [n]$ is an injective map whose image is a convex subset of $[n]$. The inclusion $\mathfrak{X}^0_{X/} \subseteq \mathfrak{X}_{X/}$ admits a right adjoint and so is right cofinal. It will therefore suffice to show that $F_0|\,\mathfrak{X}^0_{X/}$ can be extended to a p-limit diagram covering $\lambda|(\mathfrak{X}^0_{X/})^{\triangleright}$.

Let $\mathfrak{X}^1_{X/}$ be the full subcategory of $\mathfrak{X}^0_{X/}$ spanned by those objects $g : [m] \to [n]$ such that $m = 1$. Since F_0 preserves inert morphisms and $\mathcal{C}^{\otimes}$ is an ∞-operad, we conclude that $F_0|\,\mathfrak{X}^0_{X/}$ is a p-right Kan extension of $F_0|\,\mathfrak{X}^1_{X/}$. Using Lemma T.4.3.2.7, we are reduced to proving that $F_0|\,\mathfrak{X}^1_{X/}$ can be extended to a p-limit diagram covering the map $\lambda|(\mathfrak{X}^1_{X/})^{\triangleright}$. We now observe that $\mathfrak{X}^1_{X/}$ is isomorphic to a discrete simplicial set, whose objects correspond to the maps

$$\alpha_i : [1] \simeq \{i - 1, i\} \subseteq [n]$$

for $1 \le i \le n$, each of which determines determines an object $X_i \in \mathfrak{X}$. We are therefore reduced to proving that there exists an object in $\mathcal{C}^{\otimes}_{\langle n \rangle}$ which is a p-product of the objects $\{F_0(X_i) \in \mathcal{C}\}_{1 \le i \le n}$, which follows from our assumption that $\mathcal{C}^{\otimes}$ is an ∞-operad. This proves (a). Moreover, the proof yields the following version of (b):

(b') Let $F \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N}(\mathrm{Step})_{[k]}, \mathcal{C}^{\otimes})$ be a functor such that $F_0 = F|\,\mathfrak{X}$ carries inert morphisms in $\mathfrak{X}$ to inert morphisms in $\mathcal{C}^{\otimes}$. Then F is a p-right Kan extension of F_0 if and only if, for every object $X = (f : [n] \to [k]) \in \mathrm{N}(\mathrm{Step})_{[k]}$ as above, the induced maps $F(X) \to F(X_i)$ are inert for $1 \le i \le n$.

Assertion (b') immediately implies the "if" direction of (b). For the converse, suppose that $F \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N}(\mathrm{Step})_{[k]}, \mathcal{C}^{\otimes})$ is a functor which satisfies the conditions described in (b'); we wish to prove that F preserves inert morphisms. Let $\beta : X \to X'$ be an inert morphism in $\mathrm{N}(\mathrm{Step})_{[k]}$, corresponding to a commutative diagram

$$
\begin{array}{ccc}
[n'] & \xrightarrow{\ \ h\ \ } & [n] \\
 & \searrow{\scriptstyle f'} \quad \swarrow{\scriptstyle f} & \\
 & [k] &
\end{array}
$$

such that h carries $[n']$ isomorphically to a convex subset of $[n]$. We wish to prove that $F(\beta)$ is an inert morphism in $\mathcal{C}^{\otimes}$. Since $\mathcal{C}^{\otimes}$ is an ∞-operad, it will suffice to show that for every inert morphism $\gamma : F(X') \to C$ in $\mathcal{C}^{\otimes}$ such that $C \in \mathcal{C}$, the composite map $\gamma \circ F(\beta)$ is inert. Since F satisfies the criterion of (b'), we may assume without loss of generality that γ is the map $F(X') \to F(X'')$ induced by the inclusion $[1] \simeq \{i - 1, i\} \hookrightarrow [n']$ for some $1 \le i \le n'$. We may therefore replace $[n']$ by $[1]$, and thereby reduce to the case described in (b').   □

4.3.5   Tensor Products and the Bar Construction

Let $\mathcal{C}$ be a symmetric monoidal ∞-category, and suppose we are given algebra objects $A, B, C \in \mathrm{Alg}(\mathcal{C})$ and bimodules $M \in {}_A\mathrm{BMod}_B(\mathcal{C})$ and $N \in {}_B\mathrm{BMod}_C(\mathcal{C})$. Our goal in this section is to define the relative tensor product $M \otimes_B N$ as an object of ${}_A\mathrm{BMod}_C(\mathcal{C})$. The definition is based on the theory of operadic left Kan extensions developed in §3.1.2, applied to diagrams indexed by the ∞-operad family $\mathrm{Tens}^{\otimes}$ of Definition 4.3.4.1.

Notation 4.3.5.1. The morphism $[1] \simeq \{0, 2\} \hookrightarrow [2]$ in $\mathbf{\Delta}$ determines a map of simplicial sets $\Delta^1 \to \mathrm{N}(\mathbf{\Delta})^{op}$. We let $\mathrm{Tens}^{\otimes}_{\succ}$ denote the fiber product

$$\mathrm{Tens}^{\otimes} \times_{\mathrm{N}(\mathbf{\Delta})^{op}} \Delta^1.$$

Remark 4.3.5.2. The map $\mathrm{Tens}^{\otimes}_{\succ} \to \Delta^1$ is a correspondence of ∞-operads from $\mathrm{Tens}^{\otimes}_{[2]} \simeq \mathrm{Tens}^{\otimes}_{\succ} \times_{\Delta^1} \{0\}$ to $\mathrm{Tens}^{\otimes}_{[1]} \simeq \mathrm{Tens}^{\otimes}_{\succ} \times_{\Delta^1} \{1\} \simeq \mathcal{BM}^{\otimes}$.

Let $\mathcal{C}^{\otimes}$ be an ∞-operad. Proposition 4.3.4.11 guarantees that the canonical map

$$\mathrm{Alg}_{\mathrm{Tens}_{[2]}^{\otimes}}(\mathcal{C}) \to \mathrm{BMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{BMod}(\mathcal{C})$$

is an equivalence of ∞-categories. In other words, we can identify objects of $\mathrm{Alg}_{\mathrm{Tens}_{[2]}^{\otimes}}(\mathcal{C})$ with pairs $M, N \in$ $\mathrm{BMod}(\mathcal{C})$ having the same image in $\mathrm{Alg}(\mathcal{C})$ (that is, $M \in {}_A\mathrm{BMod}_B(\mathcal{C})$ and $N \in {}_B\mathrm{BMod}_C(\mathcal{C})$ for some $A, B, C \in \mathrm{Alg}(\mathcal{C})$). Similarly, we can identify objects of $\mathrm{Alg}_{\mathrm{Tens}_{[1]}^{\otimes}}(\mathcal{C})$ with objects $X \in \mathrm{BMod}(\mathcal{C})$.

Definition 4.3.5.3. Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads and suppose we are given a map of generalized ∞-operads $F : \mathrm{Tens}_{\succ}^{\otimes} \to \mathcal{C}^{\otimes}$, such that $F | \mathrm{Tens}_{[2]}^{\otimes}$ determines a pair of bimodules object $M, N \in \mathrm{BMod}(\mathcal{C})$ and $F | \mathrm{Tens}_{[1]}^{\otimes}$ is a bimodule object $X \in \mathrm{BMod}(\mathcal{C})$. We will say that F *exhibits X as a relative tensor product of M and N* if F is an operadic q-colimit diagram.

Our main goal in this section is to give a criterion for the existence of relative tensor products.

Notation 4.3.5.4. Let $\mathrm{Step} \subseteq \mathrm{Fun}([1], \boldsymbol{\Delta})^{op}$ be the category of appearing in Definition 4.3.4.12. We let $u : \boldsymbol{\Delta}^{op} \to \mathrm{Step}$ be the functor given by the formula

$$[n] \mapsto ([0] \star [n] \star [0] \simeq [n+2] \xrightarrow{f} [2]),$$

where f is given by the formula

$$f(i) = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } 0 < i < n+2 \\ 2 & \text{if } i = n+2. \end{cases}$$

We can extend u to a functor $u_+ : \boldsymbol{\Delta}_+^{op} \to \mathrm{Step}$, where u_+ carries the object $[-1] \in \boldsymbol{\Delta}_+^{op}$ to the object $(\mathrm{id} : [1] \to [1]) \in \mathrm{Step}$. On morphisms, the extension u_+ is defined by assigning to each map $[n] \to [-1]$ in $\boldsymbol{\Delta}^{op}$ the morphism in Step corresponding to the commutative diagram

$$\begin{array}{ccccc} [1] & \xrightarrow{\sim} & \{0, n+2\} & \longrightarrow & [n+2] \\ \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle f} \\ [1] & \xrightarrow{\sim} & \{0, 2\} & \longrightarrow & [2]. \end{array}$$

in $\boldsymbol{\Delta}$.

Composing u and u_+ with the functor $\Phi : \mathrm{Step} \to \mathbf{Tens}^{\otimes}$ (and passing to nerves), we obtain functors $U : \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathrm{Tens}_{[2]}^{\otimes}$ and $U_+ : \mathrm{N}(\boldsymbol{\Delta}_+)^{op} \to \mathrm{Tens}_{\succ}^{\otimes}$.

Proposition 4.3.5.5. *Suppose we are given a commutative diagram of generalized ∞-operads*

$$\begin{array}{ccc} \mathrm{Tens}_{[2]}^{\otimes} & \xrightarrow{F_0} & \mathcal{C}^{\otimes} \\ \downarrow & {\scriptstyle F}\nearrow & \downarrow{\scriptstyle q} \\ \mathrm{Tens}_{\succ}^{\otimes} & \xrightarrow{f} & \mathcal{O}^{\otimes}, \end{array}$$

where q is a fibration of ∞-operads and F_0 corresponds to a pair of bimodule objects $M \in {}_A\mathrm{BMod}_B(\mathcal{C}), N \in {}_B\mathrm{BMod}_C(\mathcal{C})$. Then there exists a dotted arrow F as indicated in the diagram, which exhibits $X = F | \mathrm{Tens}_{[1]}^{\otimes}$ as a relative tensor product of M and N, if and only if the following conditions are satisfied:

(i) *Let $\mathfrak{a}_0, \mathfrak{a}_2 \in \mathrm{Tens}_{[2]}$ and $\mathfrak{a}_-, \mathfrak{a}_+ \in \mathrm{Tens}_{[1]} \simeq \mathcal{BM}$ be as in Notation 4.3.4.9 and Definition 4.3.1.1. Then the evident maps $f(\mathfrak{a}_0) \to f(\mathfrak{a}_-)$ and $f(\mathfrak{a}_2) \to f(\mathfrak{a}_+)$ in $\mathcal{O}$ can be lifted to morphisms $A \to A'$, $C \to C'$ in $\mathcal{C}$, which are given by operadic q-colimit diagrams $\Delta^1 \to \mathcal{C}^{\otimes}$.*

(ii) *Let U and U_+ be as in Notation 4.3.5.4. Then the composition $f \circ U_+$ can be lifted to an operadic q-colimit diagram extending the functor $F_0 \circ U : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{C}^{\otimes}$.*

Moreover, if $F : \mathrm{Tens}_{\succ}^{\otimes} \to \mathcal{C}^{\otimes}$ is any map making the above diagram commute and $X = F | \mathrm{Tens}_{[1]}^{\otimes} \in {}_{A'}\mathrm{BMod}_{C'}(\mathcal{C})$, then F is an operadic q-left Kan extension of F_0 if and only if the following conditions are satisfied:

(i') *The functor F induces maps $A \to A'$ and $C \to C'$ which are given by operadic q-colimit diagrams $\Delta^1 \to \mathcal{C}^{\otimes}$.*

(ii') *The composition $F \circ U_+ : \mathrm{N}(\mathbf{\Delta}_+)^{op} \to \mathcal{C}^{\otimes}$ is an operadic q-colimit diagram.*

Proof. The ∞-category $\mathrm{Tens}_{[1]} \simeq \mathcal{BM}$ has three objects, given by $\mathfrak{a}_-$, $\mathfrak{m}$, and $\mathfrak{a}_+$. Let $\mathcal{D}(\mathfrak{a}_-)$ denote the ∞-category $(\mathrm{Tens}_{\succ,\mathrm{act}}^{\otimes})_{/\mathfrak{a}_-} \times_{\Delta^1} \{0\}$, and define $\mathcal{D}(\mathfrak{m})$ and $\mathcal{D}(\mathfrak{a}_+)$ similarly. Assertions (i') and (ii') are immediate consequences of the following:

(i'') *The inclusions $\{\mathfrak{a}_0\} \hookrightarrow \mathcal{D}(\mathfrak{a}_-)$ and $\{\mathfrak{a}_2\} \hookrightarrow \mathcal{D}(\mathfrak{a}_+)$ are left cofinal.*

(ii'') *The functor U_+ induces a left cofinal map $\theta : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{D}(\mathfrak{m})$.*

Furthermore, conditions (i) and (ii) follow from (i'') and (ii'') together with Theorem 3.1.2.3. Assertion (i'') is obvious (the ∞-categories $\mathcal{D}(\mathfrak{a}_-)$ and $\mathcal{D}(\mathfrak{a}_+)$ contain $\mathfrak{a}_0$ and $\mathfrak{a}_2$ as final objects). To prove (ii''), we observe that θ admits a left adjoint. $\qquad\square$

Remark 4.3.5.6. In the situation of Proposition 4.3.5.5, suppose that the maps

$$f(\mathfrak{a}_0) \to f(\mathfrak{a}_-) \qquad f(\mathfrak{a}_2) \to f(\mathfrak{a}_+)$$

are equivalences in $\mathcal{O}$. Then condition (i) is automatically satisfied, and condition (i') is equivalent to the requirement that the functor F induces equivalences $A \to A'$ and $C \to C'$.

Our next goal is to make the construction of relative tensor products more explicit.

Construction 4.3.5.7. Let $U : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathrm{Tens}_{[2]}^{\otimes}$ be as in Notation 4.3.5.4. The extension $U_+ : \mathrm{N}(\mathbf{\Delta}_+)^{op} \to \mathrm{Tens}_{\succ}^{\otimes}$ determines a morphism $\beta : U \to U'$ in $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathrm{Tens}_{\succ}^{\otimes})$, where $U' : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathrm{Tens}_{[1]}^{\otimes} \subseteq \mathrm{Tens}_{\succ}^{\otimes}$ is the constant functor taking the value $\mathfrak{m} \in \mathrm{Tens}_{[1]} \simeq \mathcal{BM}$.

Suppose we are given a commutative diagram of generalized ∞-operads

$$
\begin{array}{ccc}
\mathrm{Tens}_{[2]}^{\otimes} & \xrightarrow{\;F_0\;} & \mathcal{C}^{\otimes} \\
\downarrow & & \downarrow{\scriptstyle q} \\
\mathrm{Tens}_{\succ}^{\otimes} & \xrightarrow{\;f\;} & \mathcal{O}^{\otimes}
\end{array}
$$

where q is a coCartesian fibration of ∞-operads. The functor F_0 corresponds to a pair of bimodules $M \in {}_A\mathrm{BMod}_B(\mathcal{C})$, $N \in {}_B\mathrm{BMod}_C(\mathcal{C})$. Let $\mathcal{C}_{\mathfrak{m}}$ denote the fiber product $\mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \{\mathfrak{m}\}$, where

$$\{\mathfrak{m}\} \subseteq \mathrm{Tens}_{[1]}^{\otimes} \subset \mathrm{Tens}_{\succ}^{\otimes}$$

maps to $\mathcal{O}^{\otimes}$ via f.

Since q is a coCartesian fibration, the natural transformation $f \circ U \to f \circ U'$ determined by β can be lifted to a q-coCartesian natural transformation $F_0 \circ U \to X_\bullet$ in $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{C}^{\otimes})$. Here $X_\bullet$ is a simplicial object of the ∞-category $\mathcal{C}_{\mathfrak{m}}$, which is well-defined up to a contractible space of choices. We will denote this simplicial object by $\mathrm{Bar}_B(M, N)_\bullet$ and refer to it as the (*two-sided*) *bar construction* on M and N.

Theorem 4.3.5.8. *Let $q : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a coCartesian fibration of ∞-operads which is compatible with $N(\mathbf{\Delta})^{op}$-indexed colimits, in the sense of Definition 3.1.1.18.*

Suppose we are given a commutative diagram of generalized ∞-operads

$$
\begin{array}{ccc}
\mathrm{Tens}_{[2]}^\otimes & \xrightarrow{\ F_0\ } & \mathcal{C}^\otimes \\
\downarrow & \overset{F}{\nearrow} & \downarrow{\scriptstyle q} \\
\mathrm{Tens}_{\succ}^\otimes & \xrightarrow{\ f\ } & \mathcal{O}^\otimes,
\end{array}
$$

where F_0 corresponds to a pair of bimodule objects $M \in {}_A\mathrm{BMod}_B(\mathcal{C}), N \in {}_B\mathrm{BMod}_C(\mathcal{C})$. Then there exists an extension F of F_0 as indicated in the diagram, which exhibits $X = F|\mathrm{Tens}_{[1]}^\otimes$ as a relative tensor product of M and N. Moreover, if F is an arbitrary extension of F_0 making the above diagram commute, then F exhibits $X \in {}_{A'}\mathrm{BMod}_{B'}(\mathcal{C})$ as a relative tensor product of M and N if and only if the following conditions are satisfied:

(*i*) *The functor F induces q-coCartesian morphisms $A \to A'$, $B \to B'$.*

(*ii*) *The functor F induces an equivalence*

$$|\mathrm{Bar}_B(M,N)_\bullet| \to F(\mathfrak{m}).$$

Proof. Since $N(\mathbf{\Delta})^{op}$ is weakly contractible, the projection map $N(\mathbf{\Delta})^{op} \to \Delta^0$ is left cofinal; it follows that q is also compatible with Δ^0-indexed colimits. The desired result now follows from Propositions 3.1.1.15, 3.1.1.16, and 4.3.5.5. $\qquad\square$

Remark 4.3.5.9. In the situation of Construction 4.3.5.7, suppose that B is a trivial algebra object of the monoidal ∞-category $\mathcal{C}_{\mathfrak{a}_1}^\otimes$ (see §3.2.1). Then the simplicial object $\mathrm{Bar}_B(M,N)_\bullet)$ is essentially constant, so that $|\mathrm{Bar}_B(M,N)_\bullet|$ is equivalent to $\mathrm{Bar}_B(M,N)_0$. It follows that the canonical map $M \otimes N \to M \otimes_B N$ is an equivalence.

Definition 4.3.5.10. Let $q : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a coCartesian fibration of ∞-operads which is compatible with $N(\mathbf{\Delta})^{op}$-indexed colimits. Suppose we are given a map of generalized ∞-operads $\mathrm{Tens}_{\succ}^\otimes \to \mathcal{O}^\otimes$.

The formation of operadic q-left Kan extensions induces a functor

$$T : \mathrm{Alg}_{\mathrm{Tens}_{[2]}/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Tens}_{[1]}/\mathcal{O}}(\mathcal{C}).$$

We will refer to this functor as the *relative tensor product functor*. Given an object of $\mathrm{Alg}_{\mathrm{Tens}_{[2]}/\mathcal{O}}(\mathcal{C})$ corresponding to a pair of bimodules $M \in {}_A\mathrm{BMod}_B(\mathcal{C})$, $N \in {}_B\mathrm{BMod}_C(\mathcal{C})$, we will denote the image of of pair (M,N) under the functor T by $M \otimes_B N$.

Example 4.3.5.11. Let $q : \mathcal{C}^\otimes \to \mathcal{A}\mathrm{ss}^\otimes$ be a monoidal ∞-category. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and that the tensor product functors $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects separately in each variable (which is equivalent to the assumption that $\otimes$ preserves geometric realizations of simplicial objects, since the simplicial set $N(\mathbf{\Delta})^{op}$ is sifted). Consider the map of generalized ∞-operads given by the composition $\mathrm{Tens}_{\succ}^\otimes \hookrightarrow \mathrm{Tens}^\otimes \to \mathcal{A}\mathrm{ss}^\otimes$.

The relative tensor product determines a functor

$$T : \mathrm{BMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{BMod}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathrm{Tens}_{[2]}/\mathcal{A}\mathrm{ss}}(\mathcal{C}) \to \mathrm{BMod}(\mathcal{C}).$$

Criterion (*i*) of Theorem 4.3.5.8 guarantees that the diagram

$$
\begin{array}{ccc}
\mathrm{BMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{BMod}(\mathcal{C}) & \xrightarrow{\quad\quad T \quad\quad} & \mathrm{BMod}(\mathcal{C}) \\
& \searrow \qquad \swarrow{\scriptstyle \theta} & \\
& \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{C}) &
\end{array}
$$

commutes up to canonical homotopy; we may therefore assume without loss of generality that this diagram is commutative (since the map θ is a categorical fibration). It follows that for every triple of algebra objects $A, B, C \in \mathrm{Alg}(\mathcal{C})$, the functor T restricts to a map

$$_A\mathrm{BMod}_B(\mathcal{C}) \times {}_B\mathrm{BMod}_C(\mathcal{C}) \to {}_A\mathrm{BMod}_C(\mathcal{C}),$$

which we will also refer to as the *relative tensor product functor* and denote by $(M, N) \mapsto M \otimes_B N \in {}_A\mathrm{BMod}_C(\mathcal{C})$.

Example 4.3.5.12. There is a retraction r of $\mathrm{Tens}_{\succeq}^{\otimes}$ onto the full subcategory $\mathrm{Tens}_{[1]}^{\otimes} \subseteq \mathrm{Tens}_{\succeq}^{\otimes}$, which is given on $\mathrm{Tens}_{[2]}^{\otimes}$ by composition with the map $u : [2] \to [1]$ in $\mathbf{\Delta}$ defined by $u(0) = u(1) = 0 < 1 = u(2)$.

Let $q : \mathcal{C}^{\otimes} \to \mathcal{BM}^{\otimes}$ be a coCartesian fibration of ∞-operads, corresponding to an ∞-category $\mathcal{N} = \mathcal{C}_{\mathfrak{m}}$ which is bitensored over a pair of monoidal ∞-categories $\mathcal{C}_{-}^{\otimes}$ and $\mathcal{C}_{+}^{\otimes}$. We note that $\mathrm{Alg}_{\mathrm{Tens}_{[2]} / \mathcal{BM}}(\mathcal{C})$ can be identified with the fiber product

$$\mathrm{BMod}(\mathcal{C}_{-}) \times_{\mathrm{Alg}(\mathcal{C}_{-})} \mathrm{BMod}(\mathcal{N}),$$

whose objects are pairs (M, N) where $M \in {}_A\mathrm{BMod}_B(\mathcal{C}_{-})$, $N \in {}_B\mathrm{BMod}_C(\mathcal{N})$ for some $A, B \in \mathrm{Alg}(\mathcal{C}_{-})$, $C \in \mathrm{Alg}(\mathcal{C}_{+})$.

Assume that q is compatible with $\mathrm{N}(\mathbf{\Delta})^{op}$-indexed colimits: that is, the ∞-categories $\mathcal{C}_{-}$, $\mathcal{N}$, and $\mathcal{C}_{+}$ admit geometric realizations of simplicial objects, and the tensor product functors

$$\mathcal{C}_{-} \times \mathcal{C}_{-} \to \mathcal{C}_{-} \qquad \mathcal{C}_{+} \times \mathcal{C}_{+} \to \mathcal{C}_{+}$$

$$\mathcal{C}_{-} \times \mathcal{N} \to \mathcal{N} \qquad \mathcal{N} \otimes \mathcal{C}_{+} \to \mathcal{N}$$

preserve geometric realizations separately in each variable. Then Theorem 4.3.5.8 defines a relative tensor product functor

$$\mathrm{BMod}(\mathcal{C}_{-}) \times_{\mathrm{Alg}(\mathcal{C}_{-})} \mathrm{BMod}(\mathcal{N}) \simeq \mathrm{Alg}_{\mathrm{Tens}_{[2]} / \mathcal{BM}}(\mathcal{C}) \to \mathrm{BMod}(\mathcal{N}).$$

As in Example 4.3.5.11, we also obtain for every triple $A, B \in \mathrm{Alg}(\mathcal{C}_{-})$ $C \in \mathrm{Alg}(\mathcal{C}_{+})$ an induced functor

$$_A\mathrm{BMod}_B(\mathcal{C}_{-}) \times {}_B\mathrm{BMod}_C(\mathcal{N}) \to {}_A\mathrm{BMod}_C(\mathcal{N}).$$

Remark 4.3.5.13. In the situation of Example 4.3.5.12, we obtain another retraction of $\mathrm{Tens}_{\succeq}^{\otimes}$ onto $\mathrm{Tens}_{[1]}^{\otimes}$ using the map $u' : [2] \to [1]$ given by $u(0) = 0 < 1 = u(1) = u(2)$. This gives a relative tensor product functor

$$_A\mathrm{BMod}_B(\mathcal{N}) \times {}_B\mathrm{BMod}_C(\mathcal{C}_{+}) \to {}_A\mathrm{BMod}_C(\mathcal{N})$$

defined for algebra objects $A \in \mathrm{Alg}(\mathcal{C}_{-})$ and $B, C \in \mathrm{Alg}(\mathcal{C}_{+})$.

We conclude this section by studying the behavior of the relative tensor product with respect to colimits. First, we need to recall a bit of notation. For $i \in \{0, 1, 2\}$, we let $\mathrm{Tens}_{\{i\}}^{\otimes}$ denote the full subcategory of $\mathrm{Tens}_{[2]}^{\otimes}$ defined in Notation 4.3.4.10 (so that $\mathrm{Tens}_{\{i\}}^{\otimes}$ is isomorphic to $\mathcal{A}ss^{\otimes}$). We let $\mathcal{A}ss_{-}^{\otimes}$ and $\mathcal{A}ss_{+}^{\otimes}$ be the full subcategories of $\mathcal{BM}^{\otimes} \simeq \mathrm{Tens}_{[1]}^{\otimes}$ defined in Remark 4.3.1.10. We regard $\mathcal{A}ss_{\{i\}}^{\otimes}$, $\mathcal{A}ss_{-}^{\otimes}$, and $\mathcal{A}ss_{+}^{\otimes}$ as full subcategories of $\mathrm{Tens}_{\succeq}^{\otimes}$. Similarly, we let $\mathfrak{m}_{0,1}$ and $\mathfrak{m}_{1,2}$ denote the objects of $\mathrm{Tens}_{[2]}^{\otimes} \subseteq \mathrm{Tens}_{\succeq}^{\otimes}$ defined in Notation 4.3.4.9, and $\mathfrak{m} \in \mathcal{BM}^{\otimes} \simeq \mathrm{Tens}_{[1]}^{\otimes} \subseteq \mathrm{Tens}_{\succeq}^{\otimes}$ the object defined in Remark 4.3.1.3.

Proposition 4.3.5.14. *Let K be a simplicial set and let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads which is compatible with $\mathrm{N}(\mathbf{\Delta})^{op}$-indexed colimits and with K-indexed colimits.*

Suppose we are given a map of generalized ∞-operads $\mathrm{Tens}_{\succeq}^{\otimes} \to \mathcal{O}^{\otimes}$ and algebra objects $A \in \mathrm{Alg}_{\mathcal{A}ss_0 / \mathcal{O}}(\mathcal{C})$, $B \in \mathrm{Alg}_{\mathcal{A}ss_1 / \mathcal{O}}(\mathcal{C})$, and $C \in \mathrm{Alg}_{\mathcal{A}ss_2 / \mathcal{O}}(\mathcal{C})$. Using Theorem 3.1.2.3, we can choose q-coCartesian natural transformations $A \to A'$ and $C \to C'$, where $A' \in \mathrm{Alg}_{\mathcal{A}ss_{-} / \mathcal{O}}(\mathcal{C})$ and $C' \in \mathrm{Alg}_{\mathcal{A}ss_{+} / \mathcal{O}}(\mathcal{C})$. Then the relative tensor product defines a functor

$$T : {}_A\mathrm{BMod}_B(\mathcal{C}_{\mathfrak{m}_{0,1}}) \times {}_B\mathrm{BMod}_C(\mathcal{C}_{\mathfrak{m}_{1,2}}) \to {}_{A'}\mathrm{BMod}_{B'}(\mathcal{C}_{\mathfrak{m}})$$

which preserves K-indexed colimits separately in each variable.

Corollary 4.3.5.15. *Let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category and K a simplicial set. Assume that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves $\mathrm{N}(\boldsymbol{\Delta})^{op}$-indexed colimits and K-indexed colimits separately in each variable, and let $A, B, C \in \mathrm{Alg}(\mathcal{C})$. Then the relative tensor product functor*

$$_{A}\mathrm{BMod}_{B}(\mathcal{C}) \times {}_{B}\mathrm{BMod}_{C}(\mathcal{C}) \to {}_{A}\mathrm{BMod}_{C}(\mathcal{C})$$

preserves K-indexed colimits separately in each variable.

Corollary 4.3.5.16. *Let $\mathcal{N}$ be an ∞-category which is bitensored over a pair of monoidal ∞-categories $\mathcal{C}_{-}^{\otimes}$ and $\mathcal{C}_{+}^{\otimes}$. Let K be a simplicial set. Assume that the tensor product functors*

$$\mathcal{C}_{-} \times \mathcal{C}_{-} \to \mathcal{C}_{-} \qquad \mathcal{C}_{+} \times \mathcal{C}_{+} \to \mathcal{C}_{+}$$

$$\mathcal{C}_{-} \times \mathcal{N} \to \mathcal{N} \qquad \mathcal{N} \otimes \mathcal{C}_{+} \to \mathcal{N}$$

preserve $\mathrm{N}(\boldsymbol{\Delta})^{op}$-indexed colimits and K-indexed colimits separately in each variable. Given a triple of algebra objects $A, B \in \mathrm{Alg}(\mathcal{C}_{-})$, $C \in \mathrm{Alg}(\mathcal{C}_{+})$, the relative tensor product functor

$$_{A}\mathrm{BMod}_{B}(\mathcal{C}_{-}) \times {}_{B}\mathrm{BMod}_{C}(\mathcal{N}) \to {}_{A}\mathrm{BMod}_{C}(\mathcal{N})$$

of Example 4.3.5.12 preserves K-indexed colimits separately in each variable.

Proof of Proposition 4.3.5.14. In view of Proposition 4.3.3.9, it will suffice to show that the composite functor

$$_{A}\mathrm{BMod}_{B}(\mathcal{C}_{\mathfrak{m}_{0,1}}) \times {}_{B}\mathrm{BMod}_{C}(\mathcal{C}_{\mathfrak{m}_{1,2}}) \xrightarrow{T} {}_{A'}\mathrm{BMod}_{B'}(\mathcal{C}_{\mathfrak{m}}) \to \mathcal{C}_{\mathfrak{m}}$$

preserves K-indexed colimits separately in each variable. Theorem 4.3.5.8 shows that this composite functor is given by $(M, N) \mapsto |\mathrm{Bar}_{B}(M, N)_{\bullet}|$. In view of Lemma T.5.5.2.3, it will suffice to show that for each $k \geq 0$, the functor

$$\theta : {}_{A}\mathrm{BMod}_{B}(\mathcal{C}_{\mathfrak{m}_{0,1}}) \times {}_{B}\mathrm{BMod}_{C}(\mathcal{C}_{\mathfrak{m}_{1,2}}) \to \mathcal{C}_{\mathfrak{m}}$$

given by $(M, N) \to \mathrm{Bar}_{B}(M, N)_{k}$ preserves K-indexed colimits separately in each variable. We note that θ factors as a composition

$$_{A}\mathrm{BMod}_{B}(\mathcal{C}_{\mathfrak{m}_{0,1}}) \times {}_{B}\mathrm{BMod}_{C}(\mathcal{C}_{\mathfrak{m}_{1,2}}) \xrightarrow{\phi \times \psi} \mathcal{C}_{\mathfrak{m}_{0,1}} \times \mathcal{C}_{\mathfrak{m}_{1,2}} \xrightarrow{\theta'} \mathcal{C}_{\mathfrak{m}} .$$

Here the functors ϕ and ψ preserve K-indexed colimits by Proposition 4.3.3.9, and the functor θ' preserves K-indexed colimits separately in each variable by virtue of our assumption that $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is compatible with K-indexed colimits. $\square$

4.3.6 Associativity of the Tensor Product

Let $\mathcal{A}b$ denote the category of abelian groups. If A and B are rings, we let $_{A}\mathrm{BMod}_{B}(\mathcal{A}b)$ denote the category of A-B-bimodules. Suppose we are given associative rings A, B, C, and D, together with bimodules $M \in {}_{A}\mathrm{BMod}_{B}(\mathcal{A}b)$, $N \in {}_{B}\mathrm{BMod}_{C}(\mathcal{A}b)$, and $P \in {}_{C}\mathrm{BMod}_{D}(\mathcal{A}b)$. Then there is a canonical isomorphism of A-D-bimodules

$$M \otimes_{B} (N \otimes_{C} P) \simeq (M \otimes_{B} N) \otimes_{C} P.$$

To see this, it suffices to show that $M \otimes_{B} (N \otimes_{C} P)$ and $(M \otimes_{B} N) \otimes_{C} P$ corepresent the same functor on the category $_{A}\mathrm{BMod}_{D}()$. In fact, for any A-D-bimodule X, both $\mathrm{Hom}_{_{A}\mathrm{BMod}_{D}()}((M \otimes_{B} N) \otimes_{C} P, X)$ and $\mathrm{Hom}_{_{A}\mathrm{BMod}_{D}()}(M \otimes_{B} (N \otimes_{C} P), X)$ can be identified with the set of maps $f : M \times N \times P \to X$ satisfying the conditions

$$f(m + m', n, p) = f(m, n, p) + f(m', n, p) \qquad f(m, n + n', p) = f(m, n, p) + f(m, n', p)$$

$$f(m, n, p + p') = f(m, n, p) + f(m, n, p')$$

$$f(am, n, p) = af(m, n, p) \qquad f(mb, n, p) = f(m, bn, p)$$

$$f(m, nc, p) = f(m, n, cp) \qquad f(m, n, pd) = f(m, n, p)d.$$

Our goal in this section is to prove an analogous associativity property for the tensor product of bimodule objects in an arbitrary monoidal ∞-category. The crucial ingredient is the following technical result, whose proof will be given at the end of this section:

Theorem 4.3.6.1. *The forgetful functor* $\mathrm{Tens}^{\otimes} \to \mathrm{N}(\mathbf{\Delta})^{op}$ *is a flat categorical fibration (see Definition B.3.8).*

Corollary 4.3.6.2. *Let S be a simplicial set equipped with a map $S \to \mathrm{N}(\mathbf{\Delta})^{op}$, and let $\mathrm{Tens}_S^{\otimes}$ be the S-family of ∞-operads defined in Notation 4.3.4.15. For every simplex $\sigma : \Delta^n \to S$, we let $\mathrm{Tens}_\sigma^{\otimes}$ denote the fiber product $\mathrm{Tens}_S^{\otimes} \times_S \Delta^n$.*

Suppose we are given a fibration of ∞-operads $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and a map of generalized ∞-operads $\mathrm{Tens}_S^{\otimes} \to \mathcal{O}^{\otimes}$. Suppose further that the following condition is satisfied:

(∗) *For every edge $\alpha : s \to s'$ in S and every $F_0 \in \mathrm{Alg}_{\mathrm{Tens}_s / \mathcal{O}}(\mathcal{C})$, there exists an operadic q-left Kan extension $F \in \mathrm{Alg}_{\mathrm{Tens}_\alpha / \mathcal{O}}(\mathcal{C})$ of F_0.*

Define a map of simplicial sets $p : X \to S$ so that the following universal property is satisfied: for every map of simplicial sets $K \to S$, there is a canonical bijection $\mathrm{Hom}_{(\mathrm{Set}_\Delta)_{/S}}(K, X) \simeq \mathrm{Alg}_{\mathrm{Tens}_K / \mathcal{O}}(\mathcal{C})$. Then the projection map $p : X \to S$ is a coCartesian fibration. Moreover, if $\overline{\alpha}$ is an edge of X with $p(\overline{\alpha}) = \alpha : s \to s'$, then $\overline{\alpha}$ is p-coCartesian if and only if it corresponds to an object $F \in \mathrm{Alg}_{\mathrm{Tens}_\alpha / \mathcal{O}}(\mathcal{C})$ which is an operadic q-left Kan extension of $F_0 = F | \mathrm{Tens}_s^{\otimes}$.

Proof. It follows from Theorem 4.3.6.1 that p is an inner fibration. Let us call an edge $\overline{\alpha}$ of X *special* if $p(\overline{\alpha}) = \alpha : s \to s'$ and $\overline{\alpha}$ corresponds to an object $F \in \mathrm{Alg}_{\mathrm{Tens}_\alpha / \mathcal{O}}(\mathcal{C})$ which is an operadic q-left Kan extension of $F_0 = F | \mathrm{Tens}_s^{\otimes}$. Theorem 3.1.2.3 implies that every special edge of X is p-coCartesian. It follows from (∗) that p is a locally coCartesian fibration. Using Theorem 4.3.6.1 and Theorem 3.1.4.1, we conclude that the class of special edges is closed under composition, so that p is a coCartesian fibration (Proposition T.2.4.2.8). $\qquad\square$

To apply Corollary 4.3.6.2 in practice, we need to describe the functors induced by operadic left Kan extension along correspondences of the form $\mathrm{Tens}_\alpha^{\otimes} = \mathrm{Tens}^{\otimes} \times_{\mathrm{N}(\mathbf{\Delta})^{op}} \Delta^1$, where $\Delta^1 \to \mathrm{N}(\mathbf{\Delta})^{op}$ is induced by a morphism $\alpha : [n] \to [m]$ in $\mathbf{\Delta}$.

Notation 4.3.6.3. Let $\alpha : [n] \to [m]$ be a morphism in $\mathbf{\Delta}$, which we identify with a map $\Delta^1 \to \mathrm{N}(\mathbf{\Delta})^{op}$. We let $\mathrm{Tens}_\alpha^{\otimes}$ denote the fiber product $\mathrm{Tens}^{\otimes} \times_{\mathrm{N}(\mathbf{\Delta})^{op}} \Delta^1$. We observe that $\mathrm{Tens}_\alpha^{\otimes}$ can be identified with a correspondence of ∞-operads from $\mathrm{Tens}_{[m]}^{\otimes}$ to $\mathrm{Tens}_{[n]}^{\otimes}$. For $0 \leq i \leq m$, we let $\mathfrak{a}_i \in \mathrm{Tens}_{[m]} \subseteq \mathrm{Tens}_\alpha$ be defined as in Notation 4.3.4.9, and for $0 \leq i < m$ we let $\mathfrak{m}_{i,i+1} \in \mathrm{Tens}_{[m]} \subseteq \mathrm{Tens}_\alpha$ defined similarly. To avoid confusion, we will denote the corresponding objects of $\mathrm{Tens}_{[n]} \subseteq \mathrm{Tens}_\alpha$ by $\{\mathfrak{b}_j\}_{0 \leq j \leq n}$ and $\{\mathfrak{n}_{j,j+1}\}_{0 \leq j < n}$.

We would like to analyze the behavior of operadic left Kan extension along a correspondence of the form $\mathrm{Tens}_\alpha^{\otimes}$. We begin by considering an easy special case.

Remark 4.3.6.4. Let $\alpha : [n] \to [m]$ be a map of linearly ordered sets. Assume that the image of α is convex (that is, we have $\alpha(i + 1) \leq \alpha(i) + 1$ for $0 \leq i < n$). We let $v_\alpha : \mathrm{Tens}_{[n]}^{\otimes} \to \mathrm{Tens}_{[m]}^{\otimes}$ be the map given by composition with α (Notation 4.3.4.10).

The projection map $\mathrm{Tens}_\alpha^{\otimes} \to \Delta^1$ is the Cartesian fibration associated to the induced functor $\mathrm{Tens}_{[n]}^{\otimes} \to \mathrm{Tens}_{[m]}^{\otimes}$. It follows that for every object $X \in \mathrm{Tens}_{[n]}$, the ∞-category $\mathrm{Tens}_{[m]}^{\otimes} \times_{\mathrm{Tens}_\alpha^{\otimes}} (\mathrm{Tens}_\alpha^{\otimes})_{/X}^{\mathrm{act}}$ contains $v_\alpha(X)$ as a final object.

Combining Remark 4.3.6.4 and Theorem 3.1.2.3, we deduce:

Lemma 4.3.6.5. *Let $\alpha : [n] \to [m]$ be a morphism in $\mathbf{\Delta}$ with convex image and let $q : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a fibration of ∞-operads. Suppose we are given a diagram of generalized ∞-operads*

$$
\begin{array}{ccc}
\mathrm{Tens}^\otimes_{[m]} & \xrightarrow{\ F_0\ } & \mathcal{C}^\otimes \\
\downarrow & {}^{F} \nearrow & \downarrow q \\
\mathrm{Tens}^\otimes_\alpha & \xrightarrow{\ f\ } & \mathcal{O}^\otimes
\end{array}
$$

Then:

(1) *There exists a functor F as indicated in the diagram, which is an operadic q-left Kan extension of F_0, if and only if for every object $X \in \mathrm{Tens}_{[n]}$, there exists a map $F_0(v_\alpha X) \to C$ in $\mathcal{C}$ lying over the map $f(v_\alpha X) \to f(X)$, given by an operadic q-colimit diagram $\Delta^1 \to \mathcal{C}^\otimes$.*

(2) *Let $F : \mathrm{Tens}^\otimes_\alpha \to \mathcal{C}^\otimes$ be any map of generalized ∞-operads making the above diagram commute. Then F is an operadic q-left Kan extension of F_0 if and only if, for each $X \in \mathrm{Tens}_{[n]}$, the induced map $F(v_\alpha X) \to F(X)$ is given by an operadic q-colimit diagram $\Delta^1 \to \mathcal{C}^\otimes$.*

Example 4.3.6.6. In the situation of Lemma 4.3.6.5, suppose that f can be written as a composition

$$
\mathrm{Tens}^\otimes_\alpha \xrightarrow{v_\alpha} \mathrm{Tens}^\otimes_{[m]} \to \mathcal{O}^\otimes .
$$

Then the condition stated in (1) is always satisfied. Moreover, we can construct an operadic q-left Kan extension of F_0 by setting $F = F_0 \circ v_\alpha$.

We now discuss the process of operadic left Kan extension along a correspondence $\mathrm{Tens}^\otimes_\alpha$, where α is a face map in $\mathbf{\Delta}$.

Notation 4.3.6.7. For $0 \le i \le m$, we let $\alpha_i : [m-1] \to [m]$ be the ith face map (given by $\alpha(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j > i \end{cases}$, so that α is an isomorphism from $[m-1]$ onto $\{0, \ldots, i-1, i+1, \ldots, m\} \subseteq [m]$.

If $0 < i < m$, then the commutative diagram

$$
\begin{array}{ccccc}
[1] & \xrightarrow{\ \sim\ } & \{i-1, i\} & \longrightarrow & [m-1] \\
\downarrow & & \downarrow & & \downarrow \alpha_i \\
[2] & \xrightarrow{\ \sim\ } & \{i-1, i, i+1\} & \longrightarrow & [m]
\end{array}
$$

determines a map of generalized ∞-operads $\xi : \mathrm{Tens}^\otimes_\succ \to \mathrm{Tens}^\otimes_{\alpha_i}$.

Lemma 4.3.6.8. *Let $0 \le i \le m$ and let $\alpha_i : [m-1] \to [m]$ be as in Notation 4.3.6.7. Let $q : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a fibration of ∞-operads, and suppose we are given a commutative diagram of generalized ∞-operads*

$$
\begin{array}{ccc}
\mathrm{Tens}^\otimes_{[m]} & \xrightarrow{\ F_0\ } & \mathcal{C}^\otimes \\
\downarrow & {}^{F} \nearrow & \downarrow q \\
\mathrm{Tens}^\otimes_\alpha & \xrightarrow{\ f\ } & \mathcal{O}^\otimes .
\end{array}
$$

There exists a map F as indicated in the diagram, which is an operadic q-left Kan extension of F_0, if and only if the following conditions are satisfied:

(a) *For $j \in [m-1]$, there exists a morphism $F_0(\mathfrak{a}_{\alpha(j)}) \to B$ in $\mathcal{C}$ lying over the morphism $f(\mathfrak{a}_{\alpha(j)}) \to f(\mathfrak{b}_j)$ in $\mathcal{O}$, given by an operadic q-colimit diagram $\Delta^1 \to \mathcal{C}^{\otimes}$.*

(b) *For $0 < j < i$, there exists a morphism $F_0(\mathfrak{m}_{j-1,j}) \to N$ in $\mathcal{C}$ lying over the morphism $f(\mathfrak{m}_{j-1,j}) \to f(\mathfrak{n}_{j-1,j})$ in $\mathcal{O}$, given by an operadic q-colimit diagram $\Delta^1 \to \mathcal{C}^{\otimes}$.*

(c) *For $i < j < m$, there exists a morphism $F_0(\mathfrak{m}_{j,j+1}) \to N$ in $\mathcal{C}$ lying over the the morphism $f(\mathfrak{m}_{j,j+1}) \to f(\mathfrak{n}_{j,j+1})$, given by an operadic q-colimit diagram $\Delta^1 \to \mathcal{C}^{\otimes}$.*

(d) *If $0 < i < m$ and $\xi : \mathrm{Tens}^{\otimes}_{\succ} \to \mathrm{Tens}^{\otimes}_{\alpha_i}$ is defined as in Notation 4.3.6.7, then the diagram*

$$
\begin{array}{ccc}
\mathrm{Tens}^{\otimes}_{[2]} & \xrightarrow{\ F_0 \circ \xi\ } & \mathcal{C}^{\otimes} \\
\downarrow & \nearrow & \downarrow{\scriptstyle q} \\
\mathrm{Tens}^{\otimes}_{\succ} & \xrightarrow{\ f \circ \xi\ } & \mathcal{O}^{\otimes}
\end{array}
$$

admits an extension as indicated, which is an operadic q-left Kan extension of $F_0 \circ \xi$.

Moreover, a map of generalized ∞-operads $F : \mathrm{Tens}^{\otimes}_{\alpha_i}$ making the above diagrams commute is an operadic q-left Kan extension if and only if the following conditions are satisfied:

(a') *For $j \in [m-1]$, the map $F(\mathfrak{a}_{\alpha(j)}) \to F(\mathfrak{b}_j)$ in $\mathcal{C}$ is given by an operadic q-colimit diagram $\Delta^1 \to \mathcal{C}^{\otimes}$.*

(b') *For $0 < j < i$, the map $F(\mathfrak{m}_{j-1,j}) \to F(\mathfrak{n}_{j-1,j})$ is given by an operadic q-colimit diagram $\Delta^1 \to \mathcal{C}^{\otimes}$.*

(c') *For $i < j < m$, the map $F(\mathfrak{m}_{j,j+1}) \to F(\mathfrak{n}_{j-1,j})$ is given by an operadic q-colimit diagram $\Delta^1 \to \mathcal{C}^{\otimes}$.*

(d') *If $0 < i < m$, then $F \circ \xi$ is an operadic q-left Kan extension of $F \circ \xi|\mathrm{Tens}^{\otimes}_{[2]}$.*

Proof. For every object $X \in \mathrm{Tens}_{[m-1]}$, we let $\mathcal{D}(X)$ denote the fiber product

$$
\mathrm{Tens}^{\otimes}_{[m]} \times_{\mathrm{Tens}^{\otimes}_{\alpha_i}} (\mathrm{Tens}^{\otimes}_{\alpha_i})^{\mathrm{act}}_{/X}.
$$

The objects of $\mathrm{Tens}_{[m-1]}$ have the form $\{\mathfrak{b}_j\}_{0 \leq j < m}$ and $\{\mathfrak{n}_{j-1,j}\}_{0 < j < m}$. The desired result follows from Theorem 3.1.2.3 together with the following observations:

(a'') *If $X = \mathfrak{b}_j$, then $\mathcal{D}(X)$ contains $\mathfrak{a}_{\alpha(j)}$ as a final object.*

(b'') *If $X = \mathfrak{n}_{j-1,j}$ for $0 < j < i$, then $\mathcal{D}(X)$ contains $\mathfrak{m}_{j-1,j}$ as a final object.*

(c'') *If $X = \mathfrak{n}_{j-1,j}$ for $i < j < m$, then $\mathcal{D}(X)$ contains $\mathfrak{m}_{j,j+1}$ as a final object.*

(d'') *If $0 < i < m$, then ξ induces an isomorphism from $\mathrm{Tens}^{\otimes}_{[2]} \times_{\mathrm{Tens}^{\otimes}_{\succ}} (\mathrm{Tens}^{\otimes}_{\succ})^{\mathrm{act}}_{/\mathfrak{m}}$ to $\mathcal{D}(\mathfrak{m}_{i-1,i})$*

$\square$

Lemma 4.3.6.9. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{A}ss^{\otimes}$ be a monoidal ∞-category. Assume that $\mathcal{C}^{\otimes}$ admits geometric realizations of simplicial objects and that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects separately in each variable. Let $p : X \to \mathrm{N}(\boldsymbol{\Delta})^{op}$ be defined as in Corollary 4.3.6.2 (applied to identity map $\mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$). Then:*

(1) *The map p is a coCartesian fibration.*

(2) *If $\alpha : [n] \to [m]$ is a morphism in $\boldsymbol{\Delta}$ with convex image, then the associated functor*

$$
\alpha_! : \mathrm{Alg}_{\mathrm{Tens}_{[m]} / \mathcal{A}ss}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Tens}_{[n]} / \mathcal{A}ss}(\mathcal{C})
$$

is given by composition with the map v_α of Remark 4.3.6.4.

(3) *If $\alpha_i : [m-1] \to [m]$ is the morphism in $\mathbf{\Delta}$ described in Notation 4.3.6.7 for $0 < i < m$, then the associated functor $(\alpha_i)_! : X_{[m]} \to X_{[m-1]}$ is given by the composition*

$$
\begin{aligned}
X_{[m]} &\simeq \mathrm{Alg}_{\mathrm{Tens}_{[m]} / \mathcal{A}\mathrm{ss}}(\mathcal{C}) \\
&\simeq \mathrm{BMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \cdots \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{BMod}(\mathcal{C}) \\
&\xrightarrow{T} \mathrm{BMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \cdots \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{BMod}(\mathcal{C}) \\
&\simeq \mathrm{Alg}_{\mathrm{Tens}_{[m-1]} / \mathcal{A}\mathrm{ss}}(\mathcal{C}) \\
&\simeq X_{[m-1]}.
\end{aligned}
$$

Here the functor T is given by applying the relative tensor product functor

$$
\mathrm{BMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{BMod}(\mathcal{C}) \to \mathrm{BMod}(\mathcal{C})
$$

to the ith and $(i+1)$st factors.

Proof. To prove (1), it will suffice to show that for every map $\alpha : [n] \to [m]$ in $\mathbf{\Delta}$ and every $F_0 \in \mathrm{Alg}_{\mathrm{Tens}_{[m]} / \mathcal{A}\mathrm{ss}}(\mathcal{C})$, there exists an operadic q-left Kan extension $F \in \mathrm{Alg}_{\mathrm{Tens}_\alpha / \mathcal{A}\mathrm{ss}}(\mathcal{C})$ of F (Corollary 4.3.6.2). We can factor α as a composition

$$
[n] \xrightarrow{\beta} [k] \to [k+1] \to \cdots \to [m-1] \to [m],
$$

where β has convex image and remaining maps are of the form $\alpha_i : [p-1] \to [p]$, where $0 < i < p$. Using Theorems 4.3.6.1 and 3.1.4.1, we can assume either that $\alpha = \beta$ or that α has the form $\alpha_i : [p-1] \to [p]$ for $0 < i < p$. In these cases, the desired result (and the more explicit description of the associated functor supplied by (2) and (3)) follow from Lemmas 4.3.6.5 and 4.3.6.8, respectively. $\square$

Definition 4.3.6.10. If $\mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ is as in Lemma 4.3.6.9, we will denote the simplicial set X by $\mathrm{BMod}(\mathcal{C})^{\circledast}$.

Remark 4.3.6.11. Let $\mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ be as in Lemma 4.3.6.9, and let $p : \mathrm{BMod}(\mathcal{C})^{\circledast} \to \mathrm{N}(\mathbf{\Delta})^{op}$ be the associated coCartesian fibration. We have canonical isomorphisms

$$
\mathrm{BMod}(\mathcal{C})^{\circledast}_{[0]} \simeq \mathrm{Alg}(\mathcal{C}) \qquad \mathrm{BMod}(\mathcal{C})^{\circledast}_{[1]} \simeq \mathrm{BMod}(\mathcal{C}).
$$

Using Proposition 4.3.4.11, we see that for each $n \geq 0$, the inclusions $[1] \simeq \{i-1, i\} \hookrightarrow [n]$ induce an equivalence of ∞-categories

$$
\begin{aligned}
\mathrm{BMod}(\mathcal{C})^{\circledast}_{[n]} &\to \mathrm{BMod}(\mathcal{C})^{\otimes}_{[1]} \times_{\mathrm{BMod}(\mathcal{C})^{\circledast}_{[0]}} \cdots \times_{\mathrm{BMod}(\mathcal{C})^{\circledast}_{[0]}} \mathrm{BMod}(\mathcal{C})^{\circledast}_{[1]} \\
&\simeq \mathrm{BMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \cdots \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{BMod}(\mathcal{C}).
\end{aligned}
$$

One can regard the coCartesian fibration p as encoding the structure of an $(\infty, 2)$-category in which the objects are associative algebra objects of $\mathcal{C}$ and the morphisms from $A \in \mathrm{Alg}(\mathcal{C})$ to $B \in \mathrm{Alg}(\mathcal{C})$ are given by the $(\infty, 1)$-category ${}_A\mathrm{BMod}_B(\mathcal{C})$.

We now describe some applications of Lemma 4.3.6.9.

Proposition 4.3.6.12. *Let $\mathcal{C}$ be a monoidal ∞-category and let A be an associative algebra object of $\mathcal{C}$. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and that the tensor product on $\mathcal{C}$ preserves geometric realizations of simplicial objects separately in each variable. Then the planar ∞-operad $\mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C})^{\otimes}$ is a monoidal ∞-category. Moreover, the tensor product on $\mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C})$ corresponds (under the equivalence $\mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C}) \simeq {}_A\mathrm{BMod}_A(\mathcal{C})$ provided by Theorem 4.3.4.28) to the relative tensor product functor $\otimes_A : {}_A\mathrm{BMod}_A(\mathcal{C}) \times {}_A\mathrm{BMod}_A(\mathcal{C}) \to {}_A\mathrm{BMod}_A(\mathcal{C})$ described in §4.3.5.*

Remark 4.3.6.13. In the situation of Proposition 4.3.6.12, the tensor product on $\mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C})$ again preserves geometric realizations of simplicial objects separately in each variable (Corollary 4.3.5.15).

Proof. We wish to show that the forgetful functor $\mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C})^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ is a coCartesian fibration. According to Corollary T.2.4.2.10, it will suffice to show that for every map $\sigma : \Delta^n \to \mathcal{A}\mathrm{ss}^{\otimes}$, the induced map $\mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C})^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \Delta^n \to \Delta^n$ is a coCartesian fibration (in fact, we may assume that $n \leq 2$. Let Cut : $\mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathcal{A}\mathrm{ss}^{\otimes}$ be the functor of Construction 4.1.2.5. We observe that σ factors as a composition $\Delta^n \to \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathcal{A}\mathrm{ss}^{\otimes}$. It will therefore suffice to show that the pullback $\mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C})^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$ is a coCartesian fibration. This follows from Theorem 4.3.4.28 and Lemma 4.3.6.9. $\qquad\square$

Proposition 4.3.6.14 (Associativity of the Tensor Product). *Let $q : \mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ be a monoidal ∞-category. Assume that $\mathcal{C}^{\otimes}$ admits geometric realizations of simplicial objects and that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects separately in each variable. Let*

$$T : \mathrm{BMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{BMod}(\mathcal{C}) \to \mathrm{BMod}(\mathcal{C})$$

be the relative tensor product functor. Then the diagram

$$
\begin{array}{ccc}
& \mathrm{BMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{BMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{BMod}(\mathcal{C}) & \\
\end{array}
$$

$T \times \mathrm{id}$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\mathrm{id} \times T$

$$\mathrm{BMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{BMod}(\mathcal{C}) \qquad\qquad\qquad\qquad \mathrm{BMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{BMod}(\mathcal{C})$$

$T \qquad\qquad\qquad\qquad\qquad\qquad T$

$$\mathrm{BMod}(\mathcal{C})$$

commutes up to canonical homotopy.

Proof. Using Lemma 4.3.6.9, we see that both functors are induced by operadic q-left Kan extension along the correspondence $\mathrm{Tens}_{\beta}^{\otimes}$, where β is the morphism $[1] \simeq \{0, 3\} \hookrightarrow [3]$ in $\boldsymbol{\Delta}$. $\qquad\square$

Remark 4.3.6.15. In the situation of Proposition 4.3.6.14, suppose we are given a quintuple of algebra objects $A, B, C, D, E \in \mathrm{Alg}(\mathcal{C})$ and bimodules $M \in {}_A\mathrm{BMod}_B(\mathcal{C})$, $N \in {}_B\mathrm{BMod}_C(\mathcal{C})$, $P \in {}_C\mathrm{BMod}_D(\mathcal{C})$, and $Q \in {}_D\mathrm{BMod}_E(\mathcal{C})$. Proposition 4.3.6.14 supplies a diagram of equivalences

$$
\begin{array}{ccc}
& (M \otimes_B N) \otimes_C (P \otimes_D Q) & \\
& & \\
M \otimes_B (N \otimes_C (P \otimes_D Q)) & & ((M \otimes_B N) \otimes_C P) \otimes_D Q \\
\downarrow & & \downarrow \\
M \otimes_B ((N \otimes_C P) \otimes_D Q) & \longrightarrow & (M \otimes_B (N \otimes_C P)) \otimes_D Q
\end{array}
$$

in the ∞-category ${}_A\mathrm{BMod}_E(\mathcal{C})$. This diagram commutes up to (canonical) homotopy: indeed, each of its terms can be canonically identified with the image of $(M, N, P, Q) \in \mathrm{Alg}_{\mathrm{Tens}_{[4]} / \mathcal{A}\mathrm{ss}}(\mathcal{C})$ under the functor $\mathrm{Alg}_{\mathrm{Tens}_{[4]} / \mathcal{A}\mathrm{ss}}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Tens}_{[1]} / \mathcal{A}\mathrm{ss}}(\mathcal{C}) \simeq \mathrm{BMod}(\mathcal{C})$ given by operadic q-left Kan extension along $\mathrm{Tens}_{\beta}^{\otimes}$, where β is the morphism $[1] \simeq \{0, 4\} \hookrightarrow [4]$ in $\boldsymbol{\Delta}$. In fact, the coCartesian fibration $q : X \to \mathrm{N}(\boldsymbol{\Delta}_s)^{op}$ can be regarded as witnessing the fact that the relative tensor product operation on bimodules is associative up to coherent homotopy.

Proposition 4.3.6.16 (Unitality of the Tensor Product). *Let $q : \mathcal{C}^{\otimes} \to \mathcal{A}ss^{\otimes}$ be a monoidal ∞-category. Assume that $\mathcal{C}^{\otimes}$ admits geometric realizations of simplicial objects and that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects separately in each variable. Let $u : \mathrm{Alg}(\mathcal{C}) \to \mathrm{BMod}(\mathcal{C})$ be given by composition with the forgetful functor $\mathcal{BM}^{\otimes} \to \mathcal{A}ss^{\otimes}$ (so that u carries an algebra object $A \in \mathrm{Alg}(\mathcal{C})$ to the underlying object of $\mathcal{C}$, regarded as an A-A-bimodule), and let $T : \mathrm{BMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{BMod}(\mathcal{C}) \to \mathrm{BMod}(\mathcal{C})$ be the relative tensor product functor. Then the composite functors*

$$\mathrm{BMod}(\mathcal{C}) \simeq \mathrm{Alg}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{BMod}(\mathcal{C}) \xrightarrow{u \times \mathrm{id}} \mathrm{BMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{BMod}(\mathcal{C}) \xrightarrow{T} \mathrm{BMod}(\mathcal{C})$$

$$\mathrm{BMod}(\mathcal{C}) \simeq \mathrm{BMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{Alg}(\mathcal{C}) \xrightarrow{\mathrm{id} \times u} \mathrm{BMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{BMod}(\mathcal{C}) \xrightarrow{T} \mathrm{BMod}(\mathcal{C})$$

are canonically homotopic to the identity.

We can state Proposition 4.3.6.16 more informally as follows: if M is an A-B-bimodule object of $\mathcal{C}$, then we have canonical equivalences

$$A \otimes_A M \simeq M \simeq M \otimes_B B.$$

Proof. Apply Lemma 4.3.6.9 to the commutative diagrams

$$
\begin{array}{ccc}
 & [2] & \\
{}^{\alpha}\nearrow & & \searrow^{\beta} \\
[1] \xrightarrow{\quad \mathrm{id} \quad} & & [1]
\end{array}
$$

in $\mathbf{\Delta}$, where α is the inclusion $[1] \simeq \{0, 2\} \hookrightarrow [2]$ and β is a left inverse to α. $\qquad\square$

Remark 4.3.6.17. In the situation of Proposition 4.3.6.16, if we are given a pair of bimodule objects $M \in {}_A\mathrm{BMod}_B(\mathcal{C})$ and $N \in {}_B\mathrm{BMod}_C(\mathcal{C})$, then the diagrams

$$
\begin{array}{ccc}
A \otimes_A (M \otimes_B N) & \longrightarrow & (A \otimes_A M) \otimes_B C \\
& \searrow \quad \swarrow & \\
& M \otimes_B N &
\end{array}
$$

$$
\begin{array}{ccc}
(M \otimes_B N) \otimes_C C & \longrightarrow & M \otimes_B (N \otimes_C C) \\
& \searrow \quad \swarrow & \\
& M \otimes_B N &
\end{array}
$$

$$
\begin{array}{ccc}
(M \otimes_B B) \otimes_B N & \longrightarrow & M \otimes_B (B \otimes_B N) \\
& \searrow \quad \swarrow & \\
& M \otimes_B N &
\end{array}
$$

in the ∞-category ${}_A\mathrm{BMod}_C(\mathcal{C})$ commute up to canonical homotopy.

Remark 4.3.6.18. In the situation of Proposition 4.3.6.16, let A be an associative algebra object of $\mathcal{C}$, and regard A as a bimodule over itself. For any $M \in {}_A\mathrm{BMod}_A(\mathcal{C})$, Proposition 4.3.6.16 supplies equivalences

$$\phi : A \otimes_A M \simeq M \qquad \psi : M \otimes_A A \simeq M.$$

When $M = A$, the maps ϕ and ψ are canonically homotopic. Indeed, both are determined the object of $\mathrm{Alg}_{\mathrm{Tens}_{\succ} / \mathcal{A}ss}(\mathcal{C})$ given by the composition

$$\mathrm{Tens}_{\succ}^{\otimes} \to \mathcal{A}ss^{\otimes} \xrightarrow{A} \mathcal{C}^{\otimes}.$$

We conclude this section by giving the proof of Theorem 4.3.6.1. First, we need some preliminaries.

Lemma 4.3.6.19. *Let S and T be finite linearly ordered sets, where T is nonempty. Let $\mathcal{C}$ be the category whose objects are linearly ordered sets $\widetilde{S}$ equipped with monotone maps*

$$S \xleftarrow{\alpha} \widetilde{S} \xrightarrow{\beta} T$$

such that α is surjective. Then $\mathrm{N}(\mathcal{C})$ is weakly contractible.

Proof. We work by induction on the number of elements of S. If S is empty, then the result is obvious. Otherwise, S has a smallest element s. Let $\mathcal{D}$ be the category whose objects are nonempty linearly ordered sets $\widetilde{T}$ equipped with a map $\widetilde{T} \to T$, and let $f : \mathcal{C} \to \mathcal{D}$ be the functor given by $\widetilde{S} \mapsto \alpha^{-1}\{s\}$. We observe that f induces a Cartesian fibration $\mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{D})$. Moreover, if $\gamma : \widetilde{T} \to T$ is an object of $\mathcal{D}$ and $t_0 \in T$ is the largest element in $\gamma(\widetilde{T})$, then the fiber of f over γ can be identified with the category $\mathcal{C}'$ whose objects are diagrams

$$S - \{s\} \xleftarrow{\alpha'} \widetilde{S}' \xrightarrow{\beta'} \{t \in T : t \geq t_0\}$$

such that α' is surjective. The inductive hypothesis guarantees that $\mathcal{C}'$ has a weakly contractible nerve, so that Lemma T.4.1.3.2 guarantees that the map $\mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{D})$ is left cofinal (and therefore a weak homotopy equivalence). It will therefore suffice to show that $\mathrm{N}(\mathcal{D})$ is weakly contractible. This is clear, since $\mathcal{D}$ has a final object (given by the identity map $T \to T$). $\qquad\square$

Lemma 4.3.6.20. *Fix maps of finite linearly ordered sets $\mu : S \to T$, $\nu : S \to S'$, where μ is injective. Let $\mathcal{C}_{\mu,\nu}$ be the category whose objects are commutative diagrams of finite linearly ordered sets*

$$
\begin{array}{ccc}
S & \xrightarrow{\mu} & T \\
\downarrow{\scriptstyle\nu} & & \downarrow \\
S' & \xrightarrow{\mu'} & T',
\end{array}
$$

where μ' is injective. Then the nerve $\mathrm{N}(\mathcal{C}_{\mu,\nu})$ is weakly contractible.

Proof. Let $\mathcal{C}^0_{\mu,\nu}$ be the full subcategory of $\mathcal{C}_{\mu,\nu}$ spanned by those diagrams which induce a surjection $T \coprod S' \to T'$. It is easy to see that the inclusion $\mathcal{C}^0_{\mu,\nu} \subseteq \mathcal{C}_{\mu,\nu}$ admits a right adjoint. It will therefore suffice to show that the simplicial set $\mathrm{N}(\mathcal{C}^0_{\mu,\nu})$ is weakly contractible.

We work by induction on the number of elements of the set $T - \mu(S)$. If $T = \mu(S)$, then $\mathcal{C}^0_{\mu,\nu}$ has an initial object (characterized by the requirement that μ' be a bijection) and the result is obvious. If $T - \mu(S)$ is nonempty, then we can factor μ as a composition of injective maps

$$S \xrightarrow{\mu_0} U \xrightarrow{\mu_1} T.$$

where $U - \mu_0(S)$ consists of a single element u, and μ_1 is a monomorphism. Let $\mathcal{D}$ be the category whose objects are commutative diagrams

$$
\begin{array}{ccccc}
S & \xrightarrow{\mu_0} & U & \xrightarrow{\mu_1} & T \\
\downarrow{\scriptstyle\nu} & & \downarrow & & \downarrow \\
S' & \xrightarrow{\mu'_0} & U' & \xrightarrow{\mu'_1} & T'
\end{array}
$$

where μ'_0 and μ'_1 are injective and the maps $U \coprod S' \to U'$ and $T \coprod U' \to T'$ are surjective. There is an evident forgetful functor $\mathcal{D} \to \mathcal{C}^0_{\mu,\nu}$. This functor has a right adjoint and therefore induces a weak homotopy equivalence $\mathrm{N}(\mathcal{D}) \to \mathrm{N}(\mathcal{C}^0_{\mu,\nu})$; it will therefore suffice to show that $\mathcal{D}$ is weakly contractible. We have another forgetful functor $\mathcal{D} \to \mathcal{C}^0_{\mu_0,\nu}$, which induces a Cartesian fibration $\phi : \mathrm{N}(\mathcal{D})^{op} \to \mathrm{N}(\mathcal{C}^0_{\mu_0,\nu})^{op}$. Each fiber of ϕ

is equivalent to an ∞-category of the form $N(\mathcal{C}^0_{\mu_1,\nu'})^{op}$ for some map $\nu' : U \to U'$, and is therefore weakly contractible by the inductive hypothesis. Lemma T.4.1.3.2 implies that ϕ is left cofinal and therefore a weak homotopy equivalence. We are therefore reduced to proving that the simplicial set $N(\mathcal{C}^0_{\mu_0,\nu})$ is weakly contractible. We may therefore replace μ by μ_0 and thereby reduce to the case where $T - \mu(S)$ consists of a single element, which we will again denote by u.

Let $S_- = \{s \in S : \mu(s) < u\}$ and $S_+ = \{s \in S : \mu(s) > u\}$. We will assume for simplicity that S_- and S_+ are nonempty (the cases where either S_- or S_+ require slight modifications). Then S_- has a largest element s_-, and S_+ has a largest element s_+. Let $S'_0 = \{s' \in S' : \nu(s_-) \leq s' \leq \nu(s_+)\}$. As an ordered set, we can write $S'_0 = \{\nu_{s_-} = s'_0 < s'_1 < \cdots < s'_n = \nu(s_+)\}$. For $0 \leq i \leq n$, let X_i denote the object of $\mathcal{C}^0_{\mu,\nu}$ corresponding to the diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\mu} & T \\
\downarrow{\nu} & & \downarrow{\nu'} \\
S' & \xrightarrow{\mu'} & S'
\end{array}
$$

where $\nu'(u) = i$. For $1 \leq i \leq n$, let Y_i denote the object of $\mathcal{C}^0_{\mu,\nu}$ corresponding to the diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\mu} & T \\
\downarrow{\nu} & & \downarrow{\nu'} \\
S' & \xrightarrow{\mu'} & S' \cup \{u'\},
\end{array}
$$

where $\nu'(u) = u'$ and the ordering on $S' \cup \{u'\}$ is such that $u \leq s'$ if and only if $s'_i \leq s'$. It is easy to see that every object of $\mathcal{C}^0_{\mu,\nu}$ is uniquely isomorphic either to X_i (for some $0 \leq i \leq n$) or Y_i (for some $1 \leq i \leq n$); moreover, $\mathcal{C}^0_{\mu,\nu}$ is equivalent to the category depicted by the diagram

$$X_0 \leftarrow Y_1 \to X_1 \leftarrow Y_2 \to \cdots \leftarrow Y_n \to X_n.$$

It follows that the geometric realization $|N(\mathcal{C}^0_{\mu,\nu})|$ is homeomorphic to a closed interval and is therefore contractible, as desired. $\square$

Proposition 4.3.6.21. *The forgetful functor* $N(\text{Step}) \to N(\mathbf{\Delta})^{op}$ *is a flat categorical fibration (see Definition B.3.8).*

Proof. We will prove that the map $N(\text{Step})^{op} \to N(\mathbf{\Delta})$ is flat. Fix a 2-simplex σ of $N(\mathbf{\Delta})$, corresponding to a composable pair of morphisms $[k] \xrightarrow{\alpha} [k'] \xrightarrow{\beta} [k'']$ in $\mathbf{\Delta}$. Let $\mathcal{C} = N(\text{Step})^{op} \times_{N(\mathcal{F}\text{in}_*)} \Delta^2$; for $i \in \{0,1,2\}$ we let $\mathcal{C}_i$ denote the fiber product $\mathcal{C} \times_{\Delta^2} \{i\}$. Fix a morphism $f : X \to Z$ in $\mathcal{C}$, where $X \in \mathcal{C}_0$ and $Z \in \mathcal{C}_2$, and let $\mathcal{D} = \mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C}_{X//Z}$; we wish to show that $\mathcal{D}$ is weakly contractible. We can identify X with a morphism $\gamma : [n] \to [k]$ and Z with a morphism $\gamma'' : [n''] \to [k'']$, so that an object of $\mathcal{D}$ corresponds to a commutative diagram

$$
\begin{array}{ccc}
[n] & \xrightarrow{\overline{\alpha}} [n'] & \xrightarrow{\overline{\beta}} [n''] \\
\downarrow{\gamma} & \downarrow{\gamma'} & \downarrow{\gamma''} \\
[k] & \xrightarrow{\alpha} [k'] & \xrightarrow{\beta} [k'']
\end{array}
$$

in $\mathbf{\Delta}$, where $\gamma'([n'])$ is a convex subset of $[k']$.

Let $\overline{S} \subseteq [k']$ be the smallest convex subset of $[k']$ which contains $(\beta \circ \gamma)([n])$, and let $S = \overline{S} - \overline{S} \cap \beta([n])$. Let $\mathcal{D}'$ be the full subcategory of $\mathcal{D}$ spanned by diagrams with the following property that $[n'] = \alpha([n]) \cup \gamma'^{-1}S$. It is not difficult to see that the inclusion $\mathcal{D}' \subseteq \mathcal{D}$ admits a right adjoint, and is therefore a weak homotopy equivalence. It will therefore suffice to show that $\mathcal{D}'$ is weakly contractible.

Let $T = \beta(S) \subseteq [k'']$, and let $\widetilde{T} = \gamma''^{-1}T \subseteq [n'']$. Note that T is contained in the smallest convex subset of $[k'']$ which contains the image of the map $[n] \to [k] \to [k'] \to [k'']$. Since the image of γ'' is convex, we conclude that γ'' induces a surjection $\widetilde{T} \to T$. Unwinding the definitions, we see that the construction $[n'] \mapsto \gamma'^{-1}S$ determines an equivalence from $\mathcal{D}'$ to the nerve of the category whose objects are linearly ordered sets $\widetilde{S}$ which fit into a commutative diagram

$$\begin{array}{ccc} \widetilde{S} & \longrightarrow & \widetilde{T} \\ \downarrow{\scriptstyle \delta} & & \downarrow \\ S & \longrightarrow & T, \end{array}$$

where δ is surjective. We now conclude by observing that this category is equivalent to a product, over the elements of T, of categories of the type described in Lemma 4.3.6.19 (and therefore has a weakly contractible nerve). $\qquad\square$

Remark 4.3.6.22. Suppose we are given a morphism $(\langle n' \rangle, [k'], c'_-, c'_+) \to (\langle n \rangle, [k], c_-, c_+)$ in $\mathbf{Tens}^{\otimes}$, corresponding to a nondecreasing map of linearly ordered sets $\lambda : [k] \to [k']$ and a morphism $\alpha : \langle n' \rangle \to \langle n \rangle$ in $\mathbf{Ass}^{\otimes}$. For $1 \leq i \leq n$, let $L_i = \{j_1 \prec j_2 \prec \vdots \prec j_p\}$ denote the inverse image $\alpha^{-1}\{i\}$, regarded as a linearly ordered set. For $\lambda c_-(i) < m \leq \lambda c_+(i)$, there is a unique element $\mu_i(m) \in L_i$ such that $c'_-(\mu_i(m)) = m - 1$ and $c'_+(\mu_i(m)) = m$. The function μ_i determines an injective map of linearly ordered sets $\{\lambda c_-(i) + 1, \ldots, \lambda c_+(i)\} \to L_i$.

Conversely, suppose we are given an object $(\langle n \rangle, [k], c_-, c_+) \in \mathbf{Tens}^{\otimes}$ and a morphism $\lambda : [k] \to [k']$ in $\mathbf{\Delta}$. Given any collection of injective maps of finite linearly ordered sets

$$\mu_i : \{\lambda c_-(i) + 1, \ldots, \lambda c_+(i)\} \to L_i$$

for $1 \leq i \leq n$, we can reconstruct a morphism $(\langle n' \rangle, [k'], c'_-, c'_+) \to (\langle n \rangle, [k], c_-, c_+)$ covering λ, which is unique up to canonical isomorphism.

Proof of Theorem 4.3.6.1. Fix a 2-simplex σ of $\mathrm{N}(\mathbf{\Delta})^{op}$ corresponding to a composable pair of morphisms $[k] \overset{\lambda}{\to} [k'] \overset{\lambda'}{\to} [k'']$ in $\mathbf{\Delta}$. Let $\mathcal{C} = \mathrm{Tens}^{\otimes} \times_{\mathrm{N}(\mathbf{\Delta})^{op}} \Delta^2$, and let $\mathcal{C}_i = \mathcal{C} \times_{\Delta^2} \{i\}$ for $i \in \{0, 1, 2\}$. Fix a map $f : X \to Z$ in $\mathcal{C}$ where $X = (\langle n'' \rangle, [k''], c''_-, c''_+) \in \mathcal{C}_0$ and $Z = (\langle n \rangle, [k], c_-, c_+) \in \mathcal{C}_2$. We wish to prove that the simplicial set $\mathcal{D} = \mathcal{C}_{X//Z} \times_{\Delta^2} \{1\}$ is weakly contractible. According to Remark 4.3.6.22, we can identify f with a collection of finite linearly ordered sets L_i equipped with injective monotone maps $\mu_i : \{\lambda'\lambda c_-(i) + 1, \cdots, \lambda'\lambda c_+(i)\} \to L_i$ for $1 \leq i \leq n$. Similarly, we can identify an object of $\mathcal{D}$ with a collection of commutative diagrams of finite linearly ordered sets

$$\begin{array}{ccc} \{\lambda'\lambda c_-(i) + 1, \ldots, \lambda'\lambda c_+(i)\} & \overset{\mu_i}{\longrightarrow} & L_i \\ \downarrow{\scriptstyle \nu_i} & & \downarrow \\ \{\lambda c_-(i) + 1, \cdots, \lambda c_+(i)\} & \overset{\mu'_i}{\longrightarrow} & L'_i \end{array}$$

defined for $1 \leq i \leq n$, where μ'_i is injective and the map ν_i is characterized by the inequalities

$$\lambda'(\nu_i(m) - 1) \leq m - 1 < m \leq \lambda'(\nu_i(m)).$$

It follows that $\mathcal{D}$ equivalent to a finite product of (the nerves) of categories of the type described in Lemma 4.3.6.20, and is therefore weakly contractible. $\qquad\square$

4.3.7 Duality of Bimodules

Let $\mathcal{C}$ be a monoidal ∞-category. In §4.2.5, we introduced the notion of the *right dual* of an object $X \in \mathcal{C}$: that is, an object $X^\vee \in \mathcal{C}$ equipped with evaluation and coevaluation maps

$$e : X^\vee \otimes X \to \mathbf{1} \qquad c : X \otimes X^\vee \to \mathbf{1}$$

for which the composite maps

$$X \xrightarrow{c \otimes \mathrm{id}} X \otimes X^\vee \otimes X \xrightarrow{\mathrm{id} \otimes e} X$$
$$X^\vee \xrightarrow{\mathrm{id} \otimes c} X^\vee \otimes X \otimes X^\vee \xrightarrow{e \otimes \mathrm{id}} X^\vee$$

are the identity on X and $X^\vee$, respectively. In this section, we will discuss a generalization where we suppose that X is equipped with the structure of an A-B-bimodule, for some algebra objects $A, B \in \mathrm{Alg}(\mathcal{C})$.

We begin by proving a counterpart to Lemma 4.2.5.6.

Proposition 4.3.7.1. *Let $\mathcal{C}$ be a monoidal ∞-category. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects. Let $A, B \in \mathrm{Alg}(\mathcal{C})$ and suppose we are given bimodule objects $X \in {}_A\mathrm{BMod}_B(\mathcal{C})$, $Y \in {}_B\mathrm{BMod}_A(\mathcal{C})$, and a morphism $c : B \to Y \otimes_A X$ in ${}_B\mathrm{BMod}_B(\mathcal{C})$. The following conditions are equivalent:*

(1) *There exists a morphism $e : X \otimes_B Y \to A$ in ${}_A\mathrm{BMod}_A(\mathcal{C})$ such that the composite maps*

$$X \simeq X \otimes_B B \xrightarrow{\mathrm{id} \otimes c} X \otimes_B Y \otimes_A X \xrightarrow{e \otimes \mathrm{id}} A \otimes_A X \simeq X$$

$$Y \simeq B \otimes_B Y \xrightarrow{c \otimes \mathrm{id}} Y \otimes_A X \otimes_B Y \xrightarrow{\mathrm{id} \otimes e} Y \otimes_A A \simeq Y$$

are homotopic to id_X and id_Y, respectively.

(2) *Let $\mathcal{M}$ be an ∞-category left-tensored over $\mathcal{C}$ for that $\mathcal{M}$ admits geometric realizations and the action map $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$ preserves geometric realizations. Let $F : \mathrm{LMod}_B(\mathcal{M}) \to \mathrm{LMod}_A(\mathcal{M})$ be the functor given by $M \mapsto X \otimes_B M$, and let $G : \mathrm{LMod}_A(\mathcal{M}) \to \mathrm{LMod}_B(\mathcal{M})$ be the functor given by $M \mapsto Y \otimes_A M$. Then the map c induces a natural transformation $u : \mathrm{id} \to G \circ F$ which exhibits F as left adjoint to G.*

(3) *Let $\mathcal{N}$ be an ∞-category right-tensored over $\mathcal{C}$ such that $\mathcal{N}$ admits geometric realizations and the action map $\mathcal{N} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations. Let $G : \mathrm{RMod}_A(\mathcal{N}) \to \mathrm{RMod}_B(\mathcal{N})$ be the functor given by $N \mapsto N \otimes_A X$, and let $F : \mathrm{RMod}_B(\mathcal{N}) \to \mathrm{RMod}_A(\mathcal{N})$ be the functor given by $N \mapsto N \otimes_B Y$. Then the map c induces a natural transformation $\mathrm{id} \to G \circ F$ which exhibits F as left adjoint to G.*

Proof. We will prove the equivalence $(1) \Leftrightarrow (2)$; the equivalence $(1) \Leftrightarrow (3)$ will follow by symmetry. The implication $(1) \Rightarrow (2)$ is obvious: if there exists a map e as in (1), then e induces a natural transformation $v : F \circ G \to \mathrm{id}$ which is a unit map compatible with the counit v. Conversely, suppose that condition (2) is satisfied. Let $\mathcal{M} = \mathrm{RMod}_A(\mathcal{C})$, which we regard as an ∞-category left-tensored over $\mathcal{C}$ via the construction of §6.3.3. Regard A as an object of $\mathrm{LMod}_A(\mathcal{M})$, let $u : \mathrm{id} \to G \circ F$ be as in (2), and choose a counit map $v : F \circ G \to \mathrm{id}$ compatible with u. Then v induces a map $v(A) : X \otimes_B Y \to A$. The compatibility between u and v implies that the composite map $G \to (G \circ F) \circ G = G \circ (F \circ G) \to G$ is homotopic to the identity. Evaluating on the object $A \in \mathrm{LMod}_A(\mathcal{M})$, we conclude that the composite map

$$\alpha : Y \xrightarrow{c \otimes \mathrm{id}} Y \otimes_A X \otimes_B Y \xrightarrow{\mathrm{id} \otimes v(A)} Y$$

is homotopic to id_Y. To complete the proof, it will suffice to show that the map

$$\beta : X \xrightarrow{\mathrm{id} \otimes c} X \otimes_B Y \otimes_A X \xrightarrow{v(A) \otimes \mathrm{id}} X$$

is homotopic to id_X. Let $\mathcal{M}' = \mathrm{RMod}_B(\mathcal{C})$, let $F' : \mathrm{LMod}_B(\mathcal{M}') \to \mathrm{LMod}_A(\mathcal{M}')$ $G' : \mathrm{LMod}_A(\mathcal{M}') \to \mathrm{LMod}_B(\mathcal{M}')$ be given by tensor product with X and Y, respectively, let $u' : \mathrm{id}_{\mathrm{LMod}_A(\mathcal{M}')} \to G' \circ F'$ be the

map described in (2). We regard B as an object of $\mathrm{LMod}_B(\mathcal{M}')$, so that $Y \simeq F'(B)$. Since u' is the unit of an adjunction, to show that β is homotopic to id_X it will suffice to show that the composite map

$$B \xrightarrow{e} (G' \circ F')(B) \simeq G'(X) \xrightarrow{G'(\beta)} G'(X) \simeq Y \otimes_A X$$

is homotopic to c. Unwinding the definitions, we see that this composition is given by

$$B \simeq B \otimes_B B \xrightarrow{c \otimes c} Y \otimes_A X \otimes_B Y \otimes_A X \xrightarrow{\mathrm{id} \otimes v(A) \otimes \mathrm{id}} Y \otimes_A X.$$

This map is homotopic to the composition

$$B \xrightarrow{c} Y \otimes_A X \xrightarrow{\mathrm{id} \otimes \alpha} Y \otimes_A X$$

and therefore also to c, since α is homotopic to id_Y. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.3.7.2. In the setting of Proposition 4.3.7.1, it suffices to verify condition (2) in the special case where $\mathcal{M}$ has the form $\mathrm{RMod}_R(\mathcal{C})$ for some $R \in \mathrm{Alg}(\mathcal{C})$: in fact, the proof requires only the special cases $R = A$ and $R = B$.

Definition 4.3.7.3. Let $\mathcal{C}$ be a monoidal ∞-category which admits geometric realizations and for which the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations. Let $A, B \in \mathrm{Alg}(\mathcal{C})$, let $X \in {}_A\mathrm{BMod}_B(\mathcal{C})$, $Y \in {}_B\mathrm{BMod}_A(\mathcal{C})$, and let $c : B \to Y \otimes_A X$ be a map. We will say that *c exhibits X as the right dual of Y*, or *c exhibits Y as the left dual of X*, if the equivalent conditions of Proposition 4.3.7.1 are satisfied.

Given $X \in {}_A\mathrm{BMod}_B(\mathcal{C})$, we will say that X is *left dualizable* if there exists an object $Y \in {}_B\mathrm{BMod}_A(\mathcal{C})$ and a morphism $c : B \to Y \otimes_A X$ ${}_B\mathrm{BMod}_B(\mathcal{C})$ which exhibits Y as a left dual of X. Similarly, we say that an object $Y \in {}_B\mathrm{BMod}_A(\mathcal{C})$ *right dualizable* if there exists an object $X \in {}_A\mathrm{BMod}_B(\mathcal{C})$ and a morphism $c : B \to Y \otimes_A X$ in ${}_A\mathrm{BMod}_A(\mathcal{C})$ which exhibits X as a right dual of Y.

Example 4.3.7.4. Let $\mathcal{C}$ be a monoidal ∞-category and let $\mathbf{1}$ denote the unit object of $\mathcal{C}$, regarded as a trivial algebra so that the forgetful functor ${}_{\mathbf{1}}\mathrm{BMod}_{\mathbf{1}}(\mathcal{C}) \to \mathcal{C}$ is an equivalence of (monoidal) ∞-categories. An object $X \in {}_{\mathbf{1}}\mathrm{BMod}_{\mathbf{1}}(\mathcal{C})$ is left dualizable (right dualizable) in the sense of Definition 4.3.7.3 if and only if its image in $\mathcal{C}$ is left dualizable (right dualizable) in the sense of Definition 4.2.5.7.

Example 4.3.7.5. Let $\mathcal{C}$ be a monoidal ∞-category with unit object $\mathbf{1}$ (which we regard as a trivial algebra object of $\mathcal{C}$) and let $A \in \mathrm{Alg}(\mathcal{C})$ be an algebra object of $\mathcal{C}$. Suppose that $c : \mathbf{1} \to X \otimes X^\vee$ is a morphism in $\mathcal{C}$ which exhibits $X^\vee$ as a left dual of X. Let us regard $X \otimes A$ as an object of $\mathrm{RMod}_A(\mathcal{C}) \simeq {}_{\mathbf{1}}\mathrm{BMod}_A(\mathcal{C})$ and $A \otimes X^\vee$ as an object of $\mathrm{LMod}_A(\mathcal{C}) \simeq {}_A\mathrm{BMod}_{\mathbf{1}}(\mathcal{C})$. Then the map

$$c' : \mathbf{1} \to X \otimes X^\vee \simeq X \otimes \mathbf{1} \otimes X^\vee \to X \otimes A \otimes X^\vee \simeq (X \otimes A) \otimes_A (A \otimes X^\vee)$$

in ${}_A\mathrm{BMod}_A(\mathcal{C})$ exhibits $A \otimes X^\vee$ as a left dual of $X \otimes A$. To see this, we choose a evaluation map $e : X^\vee \otimes X \to \mathbf{1}$ compatible with c, and note that the map

$$e' : (A \otimes X^\vee) \otimes (X \otimes A) \simeq A \otimes (X^\vee \otimes X) \otimes A \xrightarrow{\mathrm{id} \otimes e \otimes \mathrm{id}} A \otimes \mathbf{1} \otimes A \simeq A \otimes A \to A$$

satisfies condition (1) of Proposition 4.3.7.1.

Definition 4.3.7.6. Let $p : \mathcal{M}^\otimes \to \mathcal{LM}^\otimes$ and $q : \mathcal{N}^\otimes \to \mathcal{LM}^\otimes$ be coCartesian fibrations of ∞-operads, which exhibit $\mathcal{M} = \mathcal{M}^\otimes_{\mathfrak{m}}$ and $\mathcal{N} = \mathcal{N}^\otimes_{\mathfrak{m}}$ as left-tensored over the same monoidal ∞-category

$$\mathcal{M}^\otimes \times_{\mathcal{LM}^\otimes} \mathcal{A}\mathrm{ss}^\otimes \xrightarrow[\simeq]{\alpha} \mathcal{C}^\otimes \xleftarrow[\simeq]{\beta} \mathcal{N}^\otimes \times_{\mathcal{LM}^\otimes} \mathcal{A}\mathrm{ss}^\otimes .$$

A *$\mathcal{C}$-linear functor* from $\mathcal{M}$ to $\mathcal{N}$ is an $\mathcal{LM}$-monoidal functor from $\mathcal{M}^\otimes$ to $\mathcal{N}^\otimes$ which is the identity on $\mathcal{C}^\otimes$. In other words, a $\mathcal{C}$-linear functor from $\mathcal{M}$ to $\mathcal{N}$ is a functor $F : \mathcal{M}^\otimes \to \mathcal{N}^\otimes$ satisfying the following conditions:

(*i*) The diagram

$$\begin{array}{ccc} \mathcal{M}^{\otimes} & \xrightarrow{\ F\ } & \mathcal{N}^{\otimes} \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle q} \\ & \mathcal{LM}^{\otimes} & \end{array}$$

is commutative.

(*ii*) The functor F carries p-coCartesian morphisms of $\mathcal{M}^{\otimes}$ to q-coCartesian morphisms of $\mathcal{N}^{\otimes}$.

(*iii*) The restriction $F|(\mathcal{M}^{\otimes} \times_{\mathcal{LM}^{\otimes}} \mathcal{Ass}^{\otimes})$ is given by $\beta \circ \alpha$.

We let $\mathrm{LinFun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{LM}^{\otimes}}(\mathcal{M}^{\otimes}, \mathcal{N}^{\otimes}) \times_{\mathrm{Fun}_{\mathcal{Ass}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{C}^{\otimes})} \{\mathrm{id}\}$ spanned by the $\mathcal{C}$-linear functors from $\mathcal{M}$ to $\mathcal{N}$.

Remark 4.3.7.7. In the situation of Definition 4.3.7.6, evaluation at $\mathfrak{m} \in \mathcal{LM}$ determines a forgetful functor $\theta : \mathrm{LinFun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) \to \mathrm{Fun}(\mathcal{M}, \mathcal{N})$. We will often abuse terminology by identifying F with the underlying functor $\theta(F) : \mathcal{M} \to \mathcal{N}$.

Suppose that $\mathcal{K}$ is a collection of simplicial sets such that both $\mathcal{M}$ and $\mathcal{N}$ admit $\mathcal{K}$-indexed colimits. We let $\mathrm{LinFun}_{\mathcal{C}}^{\mathcal{K}}(\mathcal{M}, \mathcal{N})$ denote the full subcategory of $\mathrm{LinFun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ spanned by those functors F such that $\theta(F) : \mathcal{M} \to \mathcal{N}$ preserves $\mathcal{K}$-indexed colimits.

Remark 4.3.7.8. Let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category, and let $F : \mathcal{M}^{\otimes} \to \mathcal{N}^{\otimes}$ be a $\mathcal{C}$-linear functor from between ∞-categories left-tensored over $\mathcal{C}^{\otimes}$. Then composition with F determines a commutative diagram

$$\begin{array}{ccc} \mathrm{LMod}(\mathcal{M}) & \xrightarrow{\hspace{3cm}} & \mathrm{LMod}(\mathcal{N}) \\ & \searrow \qquad \swarrow & \\ & \mathrm{Alg}(\mathcal{C}). & \end{array}$$

In particular, for every algebra object $A \in \mathrm{Alg}(\mathcal{C})$, we have an induced functor $\mathrm{LMod}_A(\mathcal{M}) \to \mathrm{LMod}_A(\mathcal{N})$. This construction depends functorially on F in an obvious sense, so we get a functor

$$\mathrm{LinFun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) \to \mathrm{Fun}(\mathrm{LMod}_A(\mathcal{M}), \mathrm{LMod}_A(\mathcal{N})).$$

The following result gives a useful criterion for left dualizability:

Proposition 4.3.7.9. *Let $\mathcal{C}$ be a monoidal ∞-category which admits geometric realizations and for which the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations. Let $A, B \in \mathrm{Alg}(\mathcal{C})$ and let $X \in {}_A\mathrm{BMod}_B(\mathcal{C})$. Then X is left dualizable if and only if the following conditions are satisfied:*

(1) *Let $\mathcal{M}$ be an ∞-category right-tensored over $\mathcal{M}$. Assume that $\mathcal{M}$ admits geometric realizations and that the action $\mathcal{M} \times \mathcal{C} \to \mathcal{M}$ preserves geometric realizations. Then the functor $G_{\mathcal{M}} : \mathrm{RMod}_A(\mathcal{M}) \to \mathrm{RMod}_B(\mathcal{M})$, given by $M \mapsto M \otimes_A X$, admits a left adjoint.*

(2) *Let $U : \mathcal{M} \to \mathcal{M}'$ be a $\mathcal{C}$-linear functor between ∞-categories $\mathcal{M}$ and $\mathcal{M}'$ which are right-tensored over $\mathcal{C}$ (see Definition 4.3.7.6). Assume that $\mathcal{M}$ and $\mathcal{M}'$ satisfy the hypotheses of (1) and that U preserves geometric realizations of simplicial objects. Then the diagram of ∞-categories*

$$\begin{array}{ccc} \mathrm{RMod}_A(\mathcal{M}) & \xrightarrow{\ G_{\mathcal{M}}\ } & \mathrm{RMod}_B(\mathcal{M}) \\ \downarrow & & \downarrow \\ \mathrm{RMod}_A(\mathcal{M}') & \xrightarrow{\ G_{\mathcal{M}'}\ } & \mathrm{RMod}_B(\mathcal{M}') \end{array}$$

is left adjointable (Definition 6.2.3.13).

Proof. Assume first that X admits a left dual $Y \in {}_B\mathrm{BMod}_A(\mathcal{C})$, and let $\mathcal{M}$ be as in (1). Then $G_{\mathcal{M}}$ admits a left adjoint $F_{\mathcal{M}}$, given informally by $M \mapsto M \otimes_B Y$ (Proposition 4.3.7.1). This immediately implies (1), and assertion (2) follows from the formula for $F_{\mathcal{M}}$. Conversely, suppose that (1) and (2) are satisfied. Set $\mathcal{M} = \mathrm{LMod}_B(\mathcal{C})$ and let $F_{\mathcal{M}}$ be the left adjoint to $G_{\mathcal{M}}$ whose existence is guaranteed by (1). We regard B as an object of $\mathrm{RMod}_B(\mathcal{M})$ and let $Y = F_{\mathcal{M}}(B)$. Let $u_{\mathcal{M}} : \mathrm{id} \to G_{\mathcal{M}} \circ F_{\mathcal{M}}$ be a unit for the adjunction between $F_{\mathcal{M}}$ and $G_{\mathcal{M}}$, so that $u_{\mathcal{M}}(A)$ can be regarded as a morphism $c : B \to Y \otimes_A X$ in $\mathrm{RMod}_B(\mathcal{M}) \simeq {}_B\mathrm{BMod}_B(\mathcal{C})$. We will show that c exhibits Y as the left dual of X by verifying the analogue of condition (3) of Proposition 4.3.7.1. Let $\mathcal{M}'$ be an ∞-category which is right-tensored over $\mathcal{C}$, such that $\mathcal{M}'$ admits geometric realizations and the action $\mathcal{M}' \times \mathcal{C} \to \mathcal{M}'$ preserves geometric realizations of simplicial objects. Let $T : \mathrm{RMod}_B(\mathcal{M}') \to \mathrm{RMod}_A(\mathcal{M}')$ be the functor given by $M \mapsto M \otimes_B Y$, so that c determines a natural transformation $u : \mathrm{id}_{\mathrm{RMod}_B(\mathcal{M}')} \to G_{\mathcal{M}'} \circ T$. Condition (1) guarantees that $G_{\mathcal{M}'}$ admits a right adjoint $F_{\mathcal{M}'}$, so that u determines a natural transformation $\alpha : F_{\mathcal{M}'} \to T$. We wish to show that α is an equivalence. To this end, choose an object $M \in \mathrm{RMod}_B(\mathcal{M}')$; we wish to show that $\alpha(M) : F_{\mathcal{M}'}(M) \to M \otimes_B Y$ is an equivalence. The construction $N \mapsto M \otimes_B N$ determines a $\mathcal{C}$-linear functor $U : \mathcal{M} = \mathrm{LMod}_B(\mathcal{C}) \to \mathcal{M}'$ which carries $B \in \mathrm{RMod}_B(\mathcal{M})$ to $M \in \mathrm{RMod}_B(\mathcal{M}')$ and commutes with geometric realizations (see Theorem 6.3.4.1 for a more detailed construction). The assertion that $\alpha(M)$ is an equivalence now follows immediately from the left adjointability of the diagram

$$
\begin{array}{ccc}
\mathrm{RMod}_A(\mathcal{M}) & \xrightarrow{G_{\mathcal{M}}} & \mathrm{RMod}_B(\mathcal{M}) \\
\downarrow & & \downarrow \\
\mathrm{RMod}_A(\mathcal{M}') & \xrightarrow{G_{\mathcal{M}'}} & \mathrm{RMod}_B(\mathcal{M}).
\end{array}
$$

$\square$

Remark 4.3.7.10. Let $\mathcal{C}$ be as in Proposition 4.3.7.9 and suppose that $X \in {}_A\mathrm{BMod}_B(\mathcal{C})$ satisfies conditions (1) and (2). Let $G : {}_A\mathrm{BMod}_B(\mathcal{C}) \to {}_B\mathrm{BMod}_B(\mathcal{C})$ be given by $M \mapsto M \otimes_A X$, and let F be a left adjoint to G. Suppose we are given a bimodule $Y \in {}_B\mathrm{BMod}_A(\mathcal{C})$ and a morphism $c : B \to Y \otimes_A X$ in ${}_B\mathrm{BMod}_B(\mathcal{C})$. Then c exhibits Y as a left dual of X if and only if it is adjoint to an equivalence $F(B) \to Y$ in ${}_B\mathrm{BMod}_A(\mathcal{C})$. In particular, the left dual Y and the map c are determined (up to a contractible space of choices) provided that they exist.

Remark 4.3.7.11. Let $\mathcal{C}$ be a monoidal ∞-category which admits geometric realizations and for which the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations. Suppose that $X \in {}_A\mathrm{BMod}_B(\mathcal{C})$, $Y \in {}_B\mathrm{BMod}_A(\mathcal{C})$, and $e : X \otimes_B Y \to A$ is a map which exhibits Y as a left dual of X. Let $B' \to B$ be a morphism in $\mathrm{Alg}(\mathcal{C})$, let X' denote the image of X in ${}_A\mathrm{BMod}_{B'}(\mathcal{C})$, and let $Y' \in {}_{B'}\mathrm{BMod}_A(\mathcal{C})$ be defined similarly. Then the composite map

$$
e' : X' \otimes_{B'} Y' \to X \otimes_B Y \xrightarrow{e} A
$$

exhibits Y' as a left dual of X'. To see this, choose a map $c : B \to Y \otimes_A X$ as in Proposition 4.3.7.1. Note that $Y' \otimes_A X'$ can be identified with the image of $Y \otimes_A X$ under the forgetful functor ${}_B\mathrm{BMod}_B(\mathcal{C}) \to {}_{B'}\mathrm{BMod}_{B'}(\mathcal{C})$, so that c determines a morphism $c' : B' \to B \to Y' \otimes_A X'$ in ${}_{B'}\mathrm{BMod}_{B'}(\mathcal{C})$. It is easy to see that c' satisfies the requirements of condition (1) of Proposition 4.3.7.1.

Using Proposition 4.3.7.9, we can prove a converse to Remark 4.3.7.11:

Proposition 4.3.7.12. *Let $\mathcal{C}$ be a monoidal ∞-category which admits geometric realizations and for which the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations. Let $A, B \in \mathrm{Alg}(\mathcal{C})$ and let $X \in {}_A\mathrm{BMod}_B(\mathcal{C})$. The following conditions are equivalent:*

(a) *The object X is left dualizable.*

(b) *For every map $B' \to B$ in $\mathrm{Alg}(\mathcal{C})$, the image of X in ${}_A\mathrm{BMod}_{B'}(\mathcal{C})$ is left dualizable.*

(*c*) *There exists a map $B' \to B$ in* $\mathrm{Alg}(\mathcal{C})$ *such that the image of X in* $_A\mathrm{BMod}_{B'}(\mathcal{C})$ *is left dualizable.*

(*d*) *Let* $\mathbf{1}$ *denote the unit object of* $\mathcal{C}$, *regarded as an initial object of* $\mathrm{Alg}(\mathcal{C})$. *Then the image of X in* $\mathrm{LMod}_A(\mathcal{C}) \simeq {}_A\mathrm{BMod}_{\mathbf{1}}(\mathcal{C})$ *is left dualizable.*

Proof. The implication $(b) \Rightarrow (c)$ is obvious, and the implications $(a) \Rightarrow (b)$ and $(c) \Rightarrow (d)$ follow from Remark 4.3.7.11. We will show that (d) implies (a). Assume that the image of X in $\mathrm{LMod}_A(\mathcal{C})$ is left dualizable; we will show that X satisfies conditions (1) and (2) of Proposition 4.3.7.9.

We begin by verifying condition (1). Let $\mathcal{M}$ be an ∞-category right-tensored over $\mathcal{C}$, and assume that the action $\mathcal{M} \times \mathcal{C} \to \mathcal{M}$ preserves geometric realizations. Let $G_{\mathcal{M}} : \mathrm{RMod}_A(\mathcal{M}) \to \mathrm{RMod}_B(\mathcal{M})$ be the functor given by tensor product with X; we wish to show that $G_{\mathcal{M}}$ admits a left adjoint. Let $\mathfrak{X} \subseteq \mathrm{RMod}_B(\mathcal{M})$ be the full subcategory spanned by those objects M for which the functor $N \mapsto \mathrm{Map}_{\mathrm{RMod}_B(\mathcal{M})}(M, G_{\mathcal{M}}(N))$ is corepresentable by an object of $\mathrm{RMod}_A(\mathcal{M})$; we wish to show that $\mathfrak{X} = \mathrm{RMod}_B(\mathcal{M})$. Since $\mathrm{RMod}_A(\mathcal{M})$ admits geometric realizations of simplicial objects (Corollary 4.2.3.5), the ∞-category $\mathfrak{X}$ is closed under the formation of geometric realizations. It will therefore suffice to show that $\mathfrak{X}$ contains every free right module $M_0 \otimes B$ (Corollary 6.2.2.12). In this case, the relevant functor is given by $\mathrm{Map}_{\mathcal{M}}(M_0, G(N))$, where $G : \mathrm{RMod}_A(\mathcal{M}) \to \mathcal{M}$ is the forgetful functor. The desired result now follows from the observation that G admits a left adjoint (by virtue of (d) and Proposition 4.3.7.9).

The verification of condition (2) is similar. Suppose we are given a $\mathcal{C}$-linear functor $U : \mathcal{M} \to \mathcal{M}'$ between ∞-categories right-tensored over $\mathcal{C}$, which preserves geometric realizations. Let $G_{\mathcal{M}}$ and $G_{\mathcal{M}'}$ be defined as above; we wish to show that the diagram

$$
\begin{array}{ccc}
\mathrm{RMod}_A(\mathcal{M}) & \xrightarrow{\ G_{\mathcal{M}}\ } & \mathrm{RMod}_B(\mathcal{M}) \\
\downarrow & & \downarrow \\
\mathrm{RMod}_A(\mathcal{M}') & \xrightarrow{\ G_{\mathcal{M}'}\ } & \mathrm{RMod}_B(\mathcal{M}')
\end{array}
$$

is left adjointable. Let $F_{\mathcal{M}}$ and $F_{\mathcal{M}'}$ be the left adjoints to $G_{\mathcal{M}}$ and $G_{\mathcal{M}'}$, respectively, and let $\mathfrak{X}$ be the full subcategory of $\mathrm{RMod}_B(\mathcal{M})$ spanned by those objects M for which the map $F_{\mathcal{M}'}(UM) \to UF_{\mathcal{M}}(M)$ is an equivalence. We wish to show that $\mathfrak{X} = \mathrm{RMod}_B(\mathcal{M})$. It is easy to see that $\mathfrak{X}$ is closed under the formation of geometric realizations of simplicial objects; it will therefore suffice to show that $\mathfrak{X}$ contains every free right module of the form $M_0 \otimes B$. This follows from the left adjointability of the outer square and rightmost square in the diagram

$$
\begin{array}{ccccc}
\mathrm{RMod}_A(\mathcal{M}) & \xrightarrow{\ G_{\mathcal{M}}\ } & \mathrm{RMod}_B(\mathcal{M}) & \longrightarrow & \mathcal{M} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{RMod}_A(\mathcal{M}') & \xrightarrow{\ G_{\mathcal{M}'}\ } & \mathrm{RMod}_B(\mathcal{M}') & \longrightarrow & \mathcal{M}',
\end{array}
$$

which follow from (d) and Proposition 4.3.7.9. $\qquad\square$

Notation 4.3.7.13. Let $\mathcal{C}$ be a monoidal ∞-category which admits geometric realizations and for which the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations. If A is an associative algebra object of $\mathcal{C}$, we let $_AA_A$ denote its image under the map $\mathrm{Alg}(\mathcal{C}) \to \mathrm{BMod}(\mathcal{C})$ given by composition with the forgetful functor $\mathcal{BM}^{\otimes} \to \mathcal{A}ss^{\otimes}$. Then $_AA_A$ is both left and right dual to itself, via the homotopy inverse equivalences

$$
_AA_A \to {}_AA_A \otimes_A {}_AA_A \qquad _AA_A \otimes_A {}_AA_A \to {}_AA_A.
$$

Given maps of algebra objects $f : B \to A$, $f' : C \to A$, we let $_BA_C$ denote the image of $_AA_A$ under the forgetful functor $_A\mathrm{BMod}_A(\mathcal{C}) \to {}_B\mathrm{BMod}_C(\mathcal{C})$ determined by f and f' (see Corollary 4.3.3.2) The bimodule object $_BA_C$ is determined up to canonical equivalence by f and f'. It follows from Remark 4.3.7.11 that the canonical map

$$
e : {}_AA_B \otimes_B {}_BA_A \to {}_AA_A \otimes_A {}_AA_A \simeq A
$$

exhibits ${}_A A_B$ as a right dual of ${}_B A_A$. Combining this observation with Proposition 4.3.7.1, we obtain the following:

Proposition 4.3.7.14. *Let $\mathcal{C}$ be a monoidal ∞-category and let $\mathcal{M}$ be an ∞-category left-tensored over $\mathcal{C}$. Assume that $\mathcal{C}$ and $\mathcal{M}$ admit geometric realizations of simplicial objects and that the tensor product functors*

$$\mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \mathcal{C} \times \mathcal{M} \to \mathcal{M}$$

preserve geometric realizations of simplicial objects. Then, for every map $f : B \to A$ in $\mathrm{Alg}(\mathcal{C})$, the forgetful functor $\mathrm{LMod}_A(\mathcal{M}) \to \mathrm{LMod}_B(\mathcal{M})$ admits a left adjoint, given by the relative tensor product construction $M \mapsto {}_A B_A \otimes_B M$.

Let $\mathcal{C}$ be a monoidal ∞-category which admits geometric realizations and for which the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations, and let $X \in {}_A\mathrm{BMod}_B(\mathcal{C})$. According to Proposition 4.3.7.12, X is left dualizable if and only if its image in ${}_A\mathrm{BMod}_{\mathbf{1}}(\mathcal{C}) \simeq \mathrm{LMod}_A(\mathcal{C})$ is left dualizable. There is therefore no loss of generality in restricting our attention to case where $B = \mathbf{1}$. In this case, we can simplify the criterion of Proposition 4.3.7.1:

Proposition 4.3.7.15. *Let $\mathcal{C}$ be a monoidal ∞-category. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects. Let $A \in \mathrm{Alg}(\mathcal{C})$, let $X \in \mathrm{LMod}_A(\mathcal{C})$ and let $Y \in \mathrm{RMod}_A(\mathcal{C})$. A morphism $c : \mathbf{1} \to Y \otimes_A X$ exhibits Y as a left dual of X if and only if the following condition is satisfied:*

($\ast$) *For each $C \in \mathcal{C}$ and each $M \in \mathrm{RMod}_A(\mathcal{C})$, the composite map*

$$\mathrm{Map}_{\mathrm{RMod}_A(\mathcal{C})}(C \otimes Y, M) \to \mathrm{Map}_{\mathcal{C}}(C \otimes Y \otimes_A X, M \otimes_A X) \overset{\circ c}{\to} \mathrm{Map}_{\mathcal{C}}(C, M \otimes_A X)$$

  is a homotopy equivalence.

Proof. Let $\mathcal{M}$ be an ∞-category right-tensored over $\mathcal{C}$ such that $\mathcal{M}$ admits geometric realizations and the action map $\mathcal{M} \times \mathcal{C} \to \mathcal{M}$ preserves geometric realizations. Let $F : \mathcal{M} \to \mathrm{RMod}_A(\mathcal{M})$ be given by $M \mapsto N \otimes Y$. and let $G : \mathrm{RMod}_A(\mathcal{M}) \to \mathcal{M}$ be given by tensor product with X. Then c induces a natural transformation $u : \mathrm{id} \to G \circ F$. According to Proposition 4.3.7.1, it will suffice to show that u is the unit of an adjunction between F and G. In view of Remark 4.3.7.2, it suffices to prove this in the special cases $\mathcal{M} = \mathcal{C}$ and $\mathcal{M} = \mathrm{LMod}_A(\mathcal{C})$. When $\mathcal{M} = \mathcal{C}$, this is a reformulation of ($\ast$). When $\mathcal{M} = \mathrm{LMod}_A(\mathcal{C})$, it is equivalent to the following variant of($\ast$):

($\ast'$) *For each $C \in \mathrm{LMod}_A(\mathcal{C})$ and each $M \in {}_A\mathrm{BMod}_A(\mathcal{C})$, the composite map*

$$\theta_C : \mathrm{Map}_{{}_A\mathrm{BMod}_A(\mathcal{C})}(C \otimes Y, M) \to \mathrm{Map}_{\mathrm{LMod}_A(\mathcal{C})}(C \otimes Y \otimes_A X, M \otimes_A X) \overset{\circ c}{\to} \mathrm{Map}_{\mathrm{LMod}_A(\mathcal{C})}(C, M \otimes_A X)$$

  is a homotopy equivalence.

We complete the proof by showing that ($\ast$) $\Rightarrow$ ($\ast'$). Fix the bimodule $M \in {}_A\mathrm{BMod}_A(\mathcal{C})$, and let $\mathcal{X} \subseteq \mathrm{LMod}_A(\mathcal{C})$ be the full subcategory of $\mathrm{LMod}_A(\mathcal{C})$ spanned by those objects C for which θ_C is a homotopy equivalence. We wish to prove that $\mathcal{X} = \mathrm{LMod}_A(\mathcal{C})$. Since $\mathcal{X}$ is closed under geometric realizations in $\mathrm{LMod}_A(\mathcal{C})$, it suffices to show that $\mathcal{X}$ contains every free left module $C = A \otimes C_0$. In this case, we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Map}_{{}_A\mathrm{BMod}_A(\mathcal{C})}(C \otimes Y, M) & \overset{\theta_C}{\longrightarrow} & \mathrm{Map}_{\mathrm{LMod}_A(\mathcal{C})}(C, M \otimes_A X) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathrm{RMod}_A(\mathcal{C})}(C_0 \otimes Y, M) & \overset{\theta'}{\longrightarrow} & \mathrm{Map}_{\mathcal{C}}(C_0, M \otimes_A X),
\end{array}
$$

where the vertical maps are homotopy equivalences and θ' is a homotopy equivalence by ($\ast$). $\qquad\square$

4.4 Modules over Commutative Algebras

This section is devoted to the study of algebras and modules over the commutative ∞-operad $\mathrm{Comm}^{\otimes} = \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ of Example 2.1.1.18. Our principal results can be summarized as follows:

(1) Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $\mathrm{CAlg}(\mathcal{C})$ be the ∞-category of commutative algebra objects of $\mathcal{C}$. Since $\mathrm{Comm}^{\otimes}$ is a coherent ∞-operad (Example 3.3.1.12), we can associate to each commutative algebra object $A \in \mathrm{CAlg}(\mathcal{C})$ an ∞-operad $\mathrm{Mod}_A^{\mathrm{Comm}}(\mathcal{C})^{\otimes}$. In §4.4.1, we will show that the underlying ∞-category $\mathrm{Mod}_A^{\mathrm{Comm}}(\mathcal{C})$ is equivalent to the ∞-category $\mathrm{LMod}_A(\mathcal{C})$ of left A-module objects of $\mathcal{C}$ studied in §4.2 (Proposition 4.4.1.4); here we abuse notation by identifying A with the underlying associative algebra object of $\mathcal{C}$ (that is, the image of A under the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$).

(2) If $A \in \mathrm{CAlg}(\mathcal{C})$, Proposition 4.4.1.4 also yields an equivalence $\mathrm{Mod}_A^{\mathrm{Comm}}(\mathcal{C}) \simeq \mathrm{RMod}_A(\mathcal{C})$. In other words, any object $M \in \mathrm{Mod}_A^{\mathrm{Comm}}(\mathcal{C})$ can be viewed as both a left A-module and a right A-module. In fact, these structures are compatible: M can be viewed as an A-A-bimodule. This construction determines a forgetful functor $\theta : \mathrm{Mod}_A^{\mathrm{Comm}}(\mathcal{C}) \to {}_A\mathrm{BMod}_A(\mathcal{C})$. In §4.4.2, we will see that θ can often be promoted to a monoidal functor. That is, under some mild hypotheses, we will show that the ∞-operad $\mathrm{Mod}_A^{\mathrm{Comm}}(\mathcal{C})^{\otimes}$ is a symmetric monoidal ∞-category, and that the tensor product on $\mathrm{Mod}_A^{\mathrm{Comm}}(\mathcal{C})^{\otimes}$ can be identified with the relative tensor product over A studied in §4.3.5 (Theorem 4.4.2.1).

(3) Suppose we are given a map $f : A \to B$ in $\mathrm{CAlg}(\mathcal{C})$. In §3.4.3, we saw that f determines a map of ∞-operads $G : \mathrm{Mod}_B^{\mathrm{Comm}}(\mathcal{C}) \to \mathrm{Mod}_A^{\mathrm{Comm}}(\mathcal{C})$. Under some mild hypotheses, Proposition 4.3.7.14 implies that G admits a left adjoint, given by the relative tensor product $M \mapsto B \otimes_A M$. In §4.4.3, we will show that this left adjoint can be promoted to a symmetric monoidal functor from $\mathrm{Mod}_A^{\mathrm{Comm}}(\mathcal{C})^{\otimes}$ to $\mathrm{Mod}_B^{\mathrm{Comm}}(\mathcal{C})^{\otimes}$ (Theorem 4.4.3.1).

(4) Suppose that $\mathcal{C}$ is given as the underlying ∞-category of a model category $\mathbf{A}$, and that the symmetric monoidal structure on $\mathcal{C}$ is determined by a symmetric monoidal structure on $\mathbf{A}$ (see Example 4.1.3.6). There is often a close relationship between commutative algebra objects of $\mathcal{C}$ and (strictly) commutative algebra objects of $\mathbf{A}$. In §4.4.4, we will formulate conditions on $\mathbf{A}$ which guarantee an equivalence of $\mathrm{CAlg}(\mathcal{C})$ with the underlying ∞-category for a suitable model structure on the category $\mathrm{CAlg}(\mathbf{A})$ of commutative algebras in $\mathbf{A}$ (Theorem 4.4.4.7).

4.4.1 Left and Right Modules over Commutative Algebras

Let R be an associative ring. We can associate to R a category LMod_R of left R-modules and a category RMod_R of right R-modules. If R is commutative, then these categories are equivalent to one another: every left action of R on an abelian group M can also be viewed as a right action of R on M. In this section, we will describe an ∞-categorical analogue of this phenomenon. Suppose that A is a commutative algebra object of a symmetric monoidal ∞-category $\mathcal{C}$, and let us abuse notation by identifying A with its image in $\mathrm{Alg}(\mathcal{C})$. We will show that there there is canonical equivalence $\mathrm{LMod}_A(\mathcal{C}) \simeq \mathrm{RMod}_A(\mathcal{C})$. In fact, both of these ∞-categories are equivalent to the (symmetrically defined) ∞-category $\mathrm{Mod}_A^{\mathrm{Comm}}(\mathcal{C})$ introduced in §3.3.3.

In what follows, it will be convenient to introduce a slight simplification in notation:

Definition 4.4.1.1. Let $\mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a generalized ∞-operad. We let $\mathrm{Mod}(\mathcal{C})^{\otimes}$ denote the generalized ∞-operad $\mathrm{Mod}^{\mathrm{Comm}}(\mathcal{C})^{\otimes}$ of Definition 3.3.3.8. If $A \in \mathrm{CAlg}(\mathcal{C})$, we let $\mathrm{Mod}_A(\mathcal{C})^{\otimes}$ denote the ∞-operad $\mathrm{Mod}(\mathcal{C})^{\otimes} \times_{\mathrm{CAlg}(\mathcal{C})} \{A\}$, and $\mathrm{Mod}_A(\mathcal{C})$ its underlying ∞-category.

Remark 4.4.1.2. Let $f : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a map of coherent ∞-operads, and let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of generalized ∞-operads. We let $\mathrm{Mod}^{\mathcal{O}'}(\mathcal{C})^{\otimes}$ denote the ∞-category $\mathrm{Mod}^{\mathcal{O}'}(\mathcal{C}')^{\otimes}$, where $\mathcal{C}'^{\otimes} = \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{O}'^{\otimes}$.

The map f induces a map $F : \mathcal{K}_{\mathcal{O}'} \to \mathcal{K}_{\mathcal{O}} \times_{\mathcal{O}^\otimes} \mathcal{O}'^\otimes$. Composition with F determines a functor

$$\mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^\otimes \times_{\mathcal{O}^\otimes} \mathcal{O}'^\otimes \to \mathrm{Mod}^{\mathcal{O}'}(\mathcal{C})^\otimes.$$

In particular, for any generalized ∞-operad $\mathcal{C}^\otimes$ and any coherent ∞-operad $\mathcal{O}$, we have a canonical map

$$\mathrm{Mod}(\mathcal{C})^\otimes \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathcal{O}^\otimes \to \mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^\otimes.$$

Construction 4.4.1.3. Let $\mathcal{C}^\otimes$ be an ∞-operad. Applying Remark 4.4.1.2 to the ∞-operad $\mathcal{A}\mathrm{ss}^\otimes$, we obtain a forgetful functor $\theta : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}^{\mathcal{A}\mathrm{ss}}(\mathcal{C})$. Let $\mathrm{Bim}'(\mathcal{C})$ be defined as in Construction 4.3.4.24 (applied to the planar ∞-operad $\mathcal{C}^\otimes \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathcal{A}\mathrm{ss}^\otimes$) and let $\mathrm{Bim}'(\mathcal{C})_{[1]}$ denote the fiber $\mathrm{Bim}'(\mathcal{C}) \times_{\mathrm{N}(\boldsymbol{\Delta})} \{[1]\}$, so that Theorem 4.3.4.28 supplies an equivalence $\mathrm{Mod}^{\mathcal{A}\mathrm{ss}}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C}) \times_{\Delta \mathrm{Alg}(\mathcal{C}) \times \Delta \mathrm{Alg}(\mathcal{C})} \mathrm{Bim}'(\mathcal{C})_{[1]}$. Let $^\Delta \mathrm{LMod}(\mathcal{C})$ be defined as in Definition 4.2.2.10 (where we view the ∞-category $\mathcal{C}$ as weakly enriched over itself), so there is an evident forgetful functor $\mathrm{Bim}'(\mathcal{C})_{[1]} \to {}^\Delta \mathrm{LMod}(\mathcal{C})$. Composing these functors, we obtain a map $\Theta : \mathrm{Mod}(\mathcal{C}) \to {}^\Delta \mathrm{LMod}(\mathcal{C})$. We observe that Θ fits into a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Mod}(\mathcal{C}) & \longrightarrow & {}^\Delta \mathrm{LMod}(\mathcal{C}) \\
\downarrow & & \downarrow \\
\mathrm{CAlg}(\mathcal{C}) & \longrightarrow & {}^\Delta \mathrm{Alg}(\mathcal{C}).
\end{array}
$$

The main result of this section can now be stated as follows:

Proposition 4.4.1.4. *Let $p : \mathcal{C}^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad. Then Construction 4.4.1.3 determines a homotopy pullback diagram of ∞-categories*

$$
\begin{array}{ccc}
\mathrm{Mod}(\mathcal{C}) & \longrightarrow & {}^\Delta \mathrm{LMod}(\mathcal{C}) \\
\downarrow & & \downarrow \\
\mathrm{CAlg}(\mathcal{C}) & \longrightarrow & {}^\Delta \mathrm{Alg}(\mathcal{C}).
\end{array}
$$

Corollary 4.4.1.5. *Let $\mathcal{C}^\otimes$ be an ∞-operad, let $A \in \mathrm{CAlg}(\mathcal{C})$, and let us abuse notation by identifying A with its image under the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C}) \to {}^\Delta \mathrm{Alg}(\mathcal{C})$. Then the functor Θ of Construction 4.4.1.3 induces an equivalence of ∞-categories $\mathrm{Mod}_A(\mathcal{C}) \to {}^\Delta \mathrm{LMod}_A(\mathcal{C})$.*

Corollary 4.4.1.6. *Let $\mathcal{C}^\otimes$ be an ∞-operad. Then $\mathrm{Mod}(\mathcal{C})$ can be identified with the homotopy fiber product of ∞-categories $\mathrm{LMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{CAlg}(\mathcal{C})$. If $A \in \mathrm{CAlg}(\mathcal{C})$, there is a canonical equivalence of ∞-categories $\mathrm{Mod}_A(\mathcal{C}) \simeq \mathrm{LMod}_A(\mathcal{C})$ (here we abuse notation by identifying A with its image in $\mathrm{Alg}(\mathcal{C})$).*

Proof. Combine Propositions 4.4.1.4 and Proposition 4.2.2.11. ∎

Proof of Proposition 4.4.1.4. Consider the functor $\Phi : \boldsymbol{\Delta}^{op} \times [1] \to (\mathcal{F}\mathrm{in}_*)_{\langle 1 \rangle /}$ described as follows:

- For $n \geq 0$, we have $\Phi([n], 0) = (\alpha : \langle 1 \rangle \to \psi([n] \star [0]))$, where ψ denotes the composition of the functor $\phi : \boldsymbol{\Delta}^{op} \to \mathbf{Ass}^\otimes$ with the forgetful functor $\mathbf{Ass}^\otimes \to \mathcal{F}\mathrm{in}_*$ and $\alpha : \langle 1 \rangle \to \psi([n] \star [0])$ carries $1 \in \langle 1 \rangle$ to the point $[n] \in \psi([n] \star [0])$.

- For $n \geq 0$, we have $\Phi([n], 1) = (\alpha : \langle 1 \rangle \to \psi([n]))$, where α is the null morphism given by the composition $\langle 1 \rangle \to \langle 0 \rangle \to \psi([n]) \simeq \langle n \rangle$.

Let $\mathcal{I}_0$ denote the full subcategory of $(\mathcal{F}\mathrm{in}_*)_{\langle 1 \rangle /}$ spanned by the semi-inert morphisms $\langle 1 \rangle \to \langle n \rangle$. We observe that Φ defines a functor $\boldsymbol{\Delta}^{op} \times [1] \to \mathcal{I}_0$. Let $\mathcal{I}$ denote the categorical mapping cylinder of the functor Φ. More precisely, the category $\mathcal{I}$ is defined as follows:

- An object of $\mathcal{I}$ is either an object $\alpha : \langle 1 \rangle \to \langle n \rangle$ of $\mathcal{I}_0$ or an object $([n], i)$ of $\mathbf{\Delta}^{op} \times [1]$.

- Morphisms in $\mathcal{I}$ are defined as follows:

$$\mathrm{Hom}_{\mathcal{I}}(\alpha, \alpha') = \mathrm{Hom}_{\mathcal{I}_0}(\alpha, \alpha') \qquad \mathrm{Hom}_{\mathcal{I}}(([n], i), ([n'], i')) = \mathrm{Hom}_{\mathbf{\Delta}^{op} \times [1]}(([n], i), ([n'], i'))$$

$$\mathrm{Hom}_{\mathcal{I}}(\alpha, ([n], i)) = \mathrm{Hom}_{\mathcal{I}_0}(\alpha, \Phi([n], i)) \qquad \mathrm{Hom}_{\mathcal{I}}(([n], i), \alpha) = \emptyset.$$

The full subcategory of $\mathcal{K}_{\mathrm{Comm}} \times_{\mathrm{Comm}^{\otimes}} \langle 1 \rangle$ spanned by the null morphisms $\langle 1 \rangle \to \langle n \rangle$ is actually isomorphic (rather than merely equivalent) to $\mathrm{Comm}^{\otimes}$. Consequently, we have an isomorphism $\mathrm{Mod}^{\mathrm{Comm}}(\mathcal{C}) \simeq \overline{\mathrm{Mod}}^{\mathrm{Comm}}(\mathcal{C})$, where the latter can be identified with a full subcategory of $\mathrm{Fun}_{\mathrm{Comm}^{\otimes}}(\mathrm{N}(\mathcal{I}_0), \mathcal{C}^{\otimes})$. We regard $\mathrm{N}(\mathcal{I})$ as equipped with a forgetful functor to $\mathrm{Comm}^{\otimes}$, given by composing the retraction $r : \mathcal{I} \to \mathcal{I}_0$ with the forgetful functor $\mathrm{N}(\mathcal{I}_0) \to \mathrm{Comm}^{\otimes}$. Let $\mathcal{D}$ denote the full subcategory of $\mathrm{Fun}_{\mathrm{Comm}^{\otimes}}(\mathrm{N}(\mathcal{I}), \mathcal{C}^{\otimes})$ spanned by those functors F which satisfy the following conditions:

(i) The restriction of F to $\mathrm{N}(\mathcal{I}_0)$ belongs to $\mathrm{Mod}^{\mathrm{Comm}}(\mathcal{C})$.

(ii) For every object $([n], i) \in \mathbf{\Delta}^{op} \times [1]$, the canonical map $F(\Phi([n], i)) \to F([n], i)$ is an equivalence in $\mathcal{C}^{\otimes}$ (equivalently: F is a p-left Kan extension of $F| \mathrm{N}(\mathcal{I}_0)$).

Let $\mathcal{I}'$ denote the full subcategory of $\mathcal{I}$ spanned by those objects corresponding to null morphisms $\langle 1 \rangle \to \langle n \rangle$ in $\mathcal{F}in_*$ or having the form $\alpha : \langle 1 \rangle \to \langle n \rangle$ where α is null, or pairs $([n], i) \in \mathbf{\Delta}^{op} \times [1]$ where $i = 1$. Let $\mathcal{D}'$ denote the full subcategory of $\mathrm{Fun}_{\mathrm{Comm}^{\otimes}}(\mathrm{N}(\mathcal{I}'), \mathcal{C}^{\otimes})$ spanned by those functors F satisfying the following conditions:

(i') The restriction of F to $\mathrm{N}(\mathcal{I}_0 \cap \mathcal{I}')$ belongs to $\mathrm{CAlg}(\mathcal{C})$.

(ii') For $n \geq 0$, the canonical map $F(\Phi([n], 1)) \to F([n], 1)$ is an equivalence in $\mathcal{C}^{\otimes}$ (equivalently: F is a p-left Kan extension of $F| \mathrm{N}(\mathcal{I}_0 \cap \mathcal{I}')$).

Using Proposition T.4.3.2.15, we deduce that inclusion $\mathcal{I}_0 \hookrightarrow \mathcal{I}$ induces a trivial Kan fibrations

$$\mathcal{D} \to \mathrm{Mod}^{\mathrm{Comm}}(\mathcal{C}) \qquad \mathcal{D}_0 \to \mathrm{CAlg}(\mathcal{C}).$$

We have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Mod}^{\mathrm{Comm}}(\mathcal{C}) & \longrightarrow & \mathcal{D} & \longrightarrow & {}^{\Delta}\mathrm{LMod}(\mathcal{C}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{CAlg}(\mathcal{C}) & \longrightarrow & \mathcal{D}' & \longrightarrow & {}^{\Delta}\mathrm{Alg}(\mathcal{C})
\end{array}$$

where the horizontal maps on the left are given by composition with the retraction r (since these are sections of the trivial Kan fibrations above, they are categorical equivalences) and the horizontal maps on the right are given by composition with the inclusion $\mathbf{\Delta}^{op} \times [1] \hookrightarrow \mathcal{I}$. Consequently, it will suffice to prove that the square on the right is a homotopy pullback diagram. Since the vertical maps in this square are categorical fibrations between ∞-categories, it will suffice to show that the map $\mathcal{D} \to {}^{\Delta}\mathrm{LMod}(\mathcal{C}) \times_{{}_{\Delta}\mathrm{Alg}(\mathcal{C})} \mathcal{D}'$ is a trivial Kan fibration. Let $\mathcal{I}''$ denote the full subcategory of $\mathcal{I}$ spanned by the objects of $\mathcal{I}'$ and $\mathbf{\Delta}^{op} \times [1]$. We observe that ${}^{\Delta}\mathrm{LMod}(\mathcal{C}) \times_{{}_{\Delta}\mathrm{Alg}(\mathcal{C})} \mathcal{D}'$ can be identified with the full subcategory of $\mathcal{D}'' \subseteq \mathrm{Fun}_{\mathrm{Comm}^{\otimes}}(\mathrm{N}(\mathcal{I}''), \mathcal{C}^{\otimes})$ spanned by those functors F which satisfy conditions (i'), (ii'), and the following additional condition:

(iii') The restriction of F to $\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1$ belongs to ${}^{\Delta}\mathrm{LMod}(\mathcal{C})$.

We wish to prove that $\mathcal{D} \to \mathcal{D}''$ is a trivial Kan fibration. In view of Proposition T.4.3.2.15, it will suffice to prove the following:

(a) Every functor $F_0 \in \mathcal{D}''$ admits a p-right Kan extension $F \in \mathrm{Fun}_{\mathrm{Comm}^{\otimes}}(\mathrm{N}(\mathcal{I}), \mathcal{C}^{\otimes})$.

(b) Let $F \in \mathrm{Fun}_{\mathrm{Comm}^{\otimes}}(\mathrm{N}(\mathcal{I}), \mathcal{C}^{\otimes})$ be a functor such that $F_0 = F|\,\mathrm{N}(\mathcal{I}'') \in \mathcal{D}''$. Then $F \in \mathcal{D}$ if and only if F is a p-right Kan extension of F_0.

We first prove (a). Let $F_0 \in \mathcal{D}''$, and consider an object $\alpha : \langle 1 \rangle \to \langle n \rangle$ of $\mathcal{I}$ which does not belong to $\mathcal{I}''$, and let $i = \alpha(1) \in \langle n \rangle^{\circ}$. Let $\mathcal{J} = \mathcal{I}'' \times_{\mathcal{I}} \mathcal{I}_{\alpha/}$. We wish to prove that the diagram $\mathrm{N}(\mathcal{J}) \to \mathcal{C}^{\otimes}$ determined by F_0 can be extended to a p-limit diagram covering the map

$$\mathrm{N}(\mathcal{J})^{\triangleleft} \to \mathrm{N}(\mathcal{I}_{\alpha/})^{\triangleleft} \to \mathrm{N}(\mathcal{I}) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*).$$

Let $\mathcal{J}_0$ denote the full subcategory of $\mathcal{J}$ spanned by the objects of $\mathcal{I}' \times_{\mathcal{I}} \mathcal{I}_{\alpha/}$ together with those maps $\alpha \to ([m], 0)$ in $\mathcal{I}$ for which the underlying map $u : \langle n \rangle \to \psi([m] \star [0])$ satisfies the following condition: if $u(j) = u(i)$ for $j \in \langle n \rangle^{\circ}$, then $i = j$. It is not difficult to see that the inclusion $\mathcal{J}_0 \subseteq \mathcal{J}$ admits a right adjoint, so that $\mathrm{N}(\mathcal{J}_0) \subseteq \mathrm{N}(\mathcal{J})$ is right cofinal. Consequently, it will suffice to show that the induced map $G : \mathrm{N}(\mathcal{J}_0) \to \mathcal{C}^{\otimes}$ can be extended to a p-limit diagram (compatible with the underlying map $\mathrm{N}(\mathcal{J}_0)^{\triangleleft} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$).

Let $\mathcal{J}_1$ denote the full subcategory of $\mathcal{J}_0$ spanned by the objects of $\mathcal{I}' \times_{\mathcal{I}} \mathcal{I}_{\alpha/}$ together with the morphism $s : \alpha \to ([0], 0)$ in $\mathcal{I}$ given by the map $\rho^i : \langle n \rangle \to \psi([0] \star [0]) \simeq \langle 1 \rangle$. We claim that G is a p-right Kan extension of $G|\,\mathrm{N}(\mathcal{J}_1)$. To prove this, consider an arbitrary object $J \in \mathcal{J}_0$ which does not belong to $\mathcal{J}_1$, which we identify with a map $t : \alpha \to ([m], 0)$ in $\mathcal{I}$. We have a commutative diagram

$$
\begin{array}{ccc}
(t : \alpha \to ([m], 0)) & \longrightarrow & (s : \alpha \to ([0], 0)) \\
\downarrow & & \downarrow \\
(t' : \alpha \to ([m], 1)) & \longrightarrow & (s'\alpha \to ([0], 1))
\end{array}
$$

in $\mathcal{J}_0$, which we can identify with a diagram $s \to s' \leftarrow t'$ in $(\mathcal{J}_0)_{J/} \times_{\mathcal{J}_0} \mathcal{J}_1$. Using Theorem T.4.1.3.1, we deduce that this diagram determines a left cofinal map $\Lambda_0^2 \to \mathrm{N}((\mathcal{J}_0)_{J/} \times_{\mathcal{J}_0} \mathcal{J}_1)^{op}$. Consequently, to prove that G is a p-right Kan extension of $G|\,\mathrm{N}(\mathcal{J}_1)$, it suffices to verify that the diagram

$$
\begin{array}{ccc}
F_0([m], 0) & \longrightarrow & F_0([0], 0) \\
\downarrow & & \downarrow \\
F_0([m], 1) & \longrightarrow & F_0([m], 1)
\end{array}
$$

is a p-limit diagram, which follows from (iii'). Using Lemma T.4.3.2.7, we are are reduced to proving that $G_1 = G|\,\mathrm{N}(\mathcal{J}_1)$ can be extended to a p-limit diagram lifting the evident map $\mathrm{N}(\mathcal{J}_1)^{\triangleleft} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

Let $\beta : \langle n \rangle \to \langle n - 1 \rangle$ be an inert morphism such that $\beta(i) = *$, and let $\alpha' : \langle 1 \rangle \to \langle n - 1 \rangle$ be given by the composition $\beta \circ \alpha$. Let $\mathcal{J}_2$ denote the full subcategory of $\mathcal{J}_1$ spanned by the map $s : \alpha \to ([0], 0)$, the induced map $s' : \alpha \to ([0], 1)$, and the natural transformation $\alpha \to \alpha'$ induced by β. Using Theorem T.4.1.3.1, we deduce that the inclusion $\mathrm{N}(\mathcal{J}_2) \subseteq \mathrm{N}(\mathcal{J}_1)$ is right cofinal. Consequently, we are reduced to proving that $G_2 = G|\,\mathrm{N}(\mathcal{J}_2)$ can be extended to a p-limit diagram lifting the evident map $\mathrm{N}(\mathcal{J}_2)^{\triangleleft} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. In other words, we must find a p-limit diagram

$$
\begin{array}{ccc}
F(\alpha) & \longrightarrow & F_0([0], 0) \\
\downarrow & & \downarrow \\
F_0(\alpha') & \longrightarrow & F_0([0], 1)
\end{array}
$$

covering the diagram

$$
\begin{array}{ccc}
\langle n \rangle & \xrightarrow{\rho^i} & \langle 1 \rangle \\
\downarrow{\scriptstyle \beta} & & \downarrow \\
\langle n - 1 \rangle & \longrightarrow & \langle 0 \rangle
\end{array}
$$

in $\mathfrak{Fin}_*$. The existence of such a diagram follows immediately from our assumption that $p : \mathcal{C}^\otimes \to \mathrm{N}(\mathfrak{Fin}_*)$ is an ∞-operad (see Proposition 2.3.2.5). This completes the proof of (a). Moreover, the proof also gives the following analogue of (b):

(b') Let $F \in \mathrm{Fun}_{\mathrm{Comm}^\otimes}(\mathrm{N}(\mathcal{I}), \mathcal{C}^\otimes)$ be a functor such that $F_0 = F| \,\mathrm{N}(\mathcal{I}'') \in \mathcal{D}''$. Then F is a p-right Kan extension of F_0 if and only if it satisfies the following condition:

$(*)$ For every object $\alpha : \langle 1 \rangle \to \langle n \rangle$ of $\mathcal{I}$ which does not belong to $\mathcal{I}''$, the induced diagram

$$
\begin{array}{ccc}
F(\alpha) & \xrightarrow{\;u\;} & F_0([0],0) \\
\downarrow{\scriptstyle v} & & \downarrow \\
F_0(\alpha') & \longrightarrow & F_0([0],1)
\end{array}
$$

is a p-limit diagram.

We observe that $(*)$ can be reformulated as follows:

$(*')$ For every object $\alpha : \langle 1 \rangle \to \langle n \rangle$ of $\mathcal{I}$ which does not belong to $\mathcal{I}''$, the induced morphisms $u : F(\alpha) \to F_0([0],0)$ and $v : F(\alpha) \to F_0(\alpha')$ are inert.

To complete the proof, it will suffice to show that if $F \in \mathrm{Fun}_{\mathrm{Comm}^\otimes}(\mathrm{N}(\mathcal{I}), \mathcal{C}^\otimes)$ satisfies (i'), (ii'), and (iii'), then F satisfies condition $(*')$ if and only if it satisfies conditions (i) and (ii). We first prove the "if" direction. The assertion that v is inert follows immediately from (i). To see that u is inert, choose a map $u_0 : \alpha \to ([n-1],0)$ in $\mathcal{I}$ corresponding to an isomorphism $\langle n \rangle \simeq \psi([n-1] \star [0])$, so that u factors as a composition

$$F(\alpha) \xrightarrow{F(u_0)} F([n-1],0) \xrightarrow{F(u_1)} F([0],0).$$

The map $F(u_0)$ is inert by virtue of (ii), and the map $F(u_1)$ is inert by virtue of (iii').

We now prove the "only if" direction. Suppose that F satisfies (i'), (ii'), (iii'), and $(*')$. We first claim that F satisfies (i). Let $f : \alpha \to \beta$ be a morphism in $\mathcal{I}_0$ whose image in $\mathfrak{Fin}_*$ is inert; we wish to prove that $F(f)$ is an inert morphism in $\mathcal{C}^\otimes$. If $\alpha \in \mathcal{I}'$, then this follows from assumption (i'). If $\beta \in \mathcal{I}'$ and $\alpha \notin \mathcal{I}'$, then we can factor f as a composition

$$\alpha \xrightarrow{f'} \alpha' \xrightarrow{f''} \beta$$

where $\alpha' \in \mathcal{I}'$ is defined as above. Then $F(f'')$ is inert by virtue of (i'), while $F(f')$ is inert by $(*')$, so that $F(f)$ is inert as desired. Finally, suppose that $\beta : \langle 1 \rangle \to \langle m \rangle$ does not belong to $\mathcal{I}'$, so that $\beta(1) = i \in \langle m \rangle^\circ$. To prove that $F(f)$ is inert, it will suffice to show that for each $j \in \langle m \rangle^\circ$, there exists an inert morphism $\gamma_j : F(\beta) \to C_{j'}$ in $\mathcal{C}^\otimes$ covering $\rho^j : \langle m \rangle \to \langle 1 \rangle$ such that $\gamma_j \circ F(f)$ is inert. If $j \neq i$, we can take $\gamma_j = F(g)$, where $\beta_j = \rho^j \circ \beta$ and $g : \beta \to \beta_j$ is the induced map; then $\beta_j \in \mathcal{I}'$ so that $F(g)$ and $F(g \circ f)$ are both inert by the arguments presented above. If $i = j$, we take $\gamma_j = F(g)$ where g is the map $\beta \to ([0],0)$ in $\mathcal{I}$ determined by ρ^i. Then $F(g)$ and $F(g \circ f)$ are both inert by virtue of $(*')$. This completes the proof of (i).

We now prove (ii). Let $([n],i) \in \mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1$, let $\alpha = \Psi([n],i) \in \mathcal{I}_0$, and let $f : \alpha \to ([n],i)$ be the canonical morphism in $\mathcal{I}$. We wish to prove that $F(f)$ is an equivalence in $\mathcal{C}^\otimes$. If $i = 1$, this follows from (ii'). Assume therefore that $i = 0$. Observe that α can be identified with the morphism $\langle 1 \rangle \to \langle n+1 \rangle$ carrying $1 \in \langle 1 \rangle$ to $n + 1 \in \langle n + 1 \rangle$. To prove that $F(f)$ is an equivalence, it will suffice to show that for each $j \in \langle n + 1 \rangle^\circ$, there exists an inert morphism $\gamma_j : F([n],0) \to C_i$ in $\mathcal{C}^\otimes$ covering $\rho^j : \langle n+1 \rangle \to \langle 1 \rangle$ such that $\gamma_j \circ F(f)$ is also inert. If $j \neq n + 1$, we take $\gamma_j = F(g)$, where $g : ([n],0) \to ([1],1)$ is a morphism in $\boldsymbol{\Delta}^{op} \times[1]$ covering the map ρ^i. Then γ_j is inert by virtue of (iii'), while $\gamma_j \circ F(f) \simeq F(g \circ f)$ can be written as a composition

$$F(\alpha) \xrightarrow{F(h')} F(\alpha') \xrightarrow{F(h'')} F(r([1],1)) \xrightarrow{F(h''')} F([1],1).$$

Assumption $(*)$ guarantees that $F(h')$ is inert, assumption (i') guarantees that $F(h'')$ is inert, and assumption (ii') guarantees that $F(h''')$ is inert. It follows that $F(h''' \circ h'' \circ h') \simeq F(g \circ f)$ is inert as desired. In the

case $j = n + 1$, we instead take $\gamma_j = F(g)$ where $g : ([n], 0) \to ([0], 0)$ is the morphism in $\mathfrak{I}$ determined by the map $[0] \to [n]$ in $\mathbf{\Delta}$ carrying $0 \in [0]$ to $n \in [n]$. Then $\gamma_j = F(g)$ is inert by virtue of (iii'), while $\gamma_j \circ F(f) \simeq F(g \circ f)$ is inert by virtue of $(*')$. $\hfill\square$

4.4.2 Tensor Products over Commutative Algebras

Let R be a commutative ring. Then the category of R-modules is endowed with a symmetric monoidal structure, whose tensor product operation is given by $(M, N) \mapsto M \otimes_R N$. Our goal in this section is to establish a generalization of this statement, where we replace the category of abelian groups by an arbitrary symmetric monoidal ∞-category $\mathcal{C}$. Let A be a commutative algebra object of $\mathcal{C}$, and let us abuse notation by identifying A with the underlying associative algebra object of $\mathcal{C}$. We have seen that the ∞-category $\mathrm{Mod}_A^{Ass}(\mathcal{C})$ can be identified with the ∞-category of A-A-bimodules (Theorem 4.3.4.28). Moreover, under some mild hypotheses, the relative tensor product functor $\otimes_A$ endows $\mathrm{Mod}_A^{Ass \otimes}(\mathcal{C})$ with the structure of a monoidal ∞-category (Proposition 4.3.6.12). Our main result can be stated as follows:

Theorem 4.4.2.1. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects, and that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects separately in each variable. Let A be a commutative algebra object of $\mathcal{C}$. Then:*

(1) *The ∞-operad $\mathrm{Mod}_A(\mathcal{C})^{\otimes}$ is a symmetric monoidal ∞-category.*

(2) *The forgetful functor $\psi : \mathrm{Mod}_A(\mathcal{C})^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} Ass^{\otimes} \to \mathrm{Mod}_A^{Ass}(\mathcal{C})^{\otimes}$ of Remark 4.4.1.2 is monoidal. In particular, we can identify the tensor product operation on $\mathrm{Mod}_A(\mathcal{C})$ with the composition*

$$\mathrm{Mod}_A(\mathcal{C}) \times \mathrm{Mod}_A(\mathcal{C}) \to {}_A\mathrm{BMod}_A(\mathcal{C}) \times {}_A\mathrm{BMod}_A(\mathcal{C}) \overset{\otimes_A}{\to} {}_A\mathrm{BMod}_A(\mathcal{C}) \to \mathrm{LMod}_A(\mathcal{C}) \simeq \mathrm{Mod}_A(\mathcal{C})$$

(see Proposition 4.3.6.12).

The proof of Theorem 4.4.2.1 will make use of the following criterion for detecting symmetric monoidal structures.

Proposition 4.4.2.2. *Let $q : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad. Then q is a coCartesian fibration if and only if the following conditions are satisfied:*

(1) *The fibration q has units (in the sense of Definition 3.2.1.1).*

(2) *Let $\alpha : \langle 2 \rangle \to \langle 1 \rangle$ be an active morphism in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, and let $X \in \mathcal{C}_{\langle 2 \rangle}^{\otimes}$. Then there exists a map $\overline{\alpha} : X \to X'$ with $q(\overline{\alpha}) = \alpha$, which determines an operadic q-colimit diagram $\Delta^1 \to \mathcal{C}^{\otimes}$.*

Proof. If q is a coCartesian fibration, then conditions (1) and (2) follow immediately from Proposition 3.1.1.13. Conversely, suppose that (1) and (2) are satisfied; we wish to prove that q is a coCartesian fibration. In view of Proposition 3.1.1.13, it will suffice to prove that the following stronger version of (2) is satisfied:

(2′) *Let $\alpha : \langle n \rangle \to \langle 1 \rangle$ be an active morphism in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, and let $X \in \mathcal{C}_{\langle n \rangle}^{\otimes}$. Then there a map $\overline{\alpha} : X \to X'$ lifting α which is given by an operadic q-colimit diagram.*

We verify (2′) using induction on n. If $n = 0$, the desired result follows from (1). If $n = 1$, then α is the identity and we can choose $\overline{\alpha}$ to be a degenerate edge. Assume that $n \geq 2$. Choose any partition of this set into nonempty disjoint subsets $\langle n \rangle^{\circ}_-$ and $\langle n \rangle^{\circ}_+$. This decomposition determines a factorization of α as a composition

$$\langle n \rangle \overset{\alpha'}{\to} \langle 2 \rangle \overset{\alpha''}{\to} \langle 1 \rangle.$$

Let n_- and n_+ be the cardinalities of $\langle n \rangle^{\circ}_-$ and $\langle n \rangle^{\circ}_+$, respectively, so that our decomposition induces inert morphisms $\langle n \rangle \to \langle n_+ \rangle$ and $\langle n \rangle \to \langle n_- \rangle$. Since q is a fibration of ∞-operads, we can lift these maps to

inert morphisms $X \to X_-$ and $X \to X_+$ in $\mathbb{C}^\otimes$. Using the inductive hypothesis, we can choose morphisms $X_- \to X'_-$ and $X_+ \to X'_-$ lying over the maps $\langle n_- \rangle \to \langle 1 \rangle \leftarrow \langle n_+ \rangle$ in $\mathcal{A}ss$, which are classified by operadic q-colimit diagrams $\Delta^1 \to \mathbb{C}^\otimes$. Let $\overline{\alpha}'$ denote the induced morphism $X \simeq X_- \oplus X_+ \to X'_- \oplus X'_+$. Using (2), we can choose a morphism $\overline{\alpha}'' : X'_- \oplus X'_+ \to X'$ lifting α'', which is classified by an operadic q-colimit diagram. To complete the proof, it will suffice to show that $\overline{\alpha}'' \circ \overline{\alpha}'$ is classified by an operadic q-colimit diagram. Choose an object $Y \in \mathbb{C}^\otimes$; we must show that the composite map

$$X \oplus Y \simeq X_- \oplus X_+ \oplus Y \xrightarrow{\beta} X'_- \oplus X'_+ \oplus Y \xrightarrow{\gamma} X' \oplus Y$$

is classified by a weak operadic q-colimit diagram. To prove this, it suffices to show that $\gamma \circ \beta$ is q-coCartesian. In view of Proposition T.2.4.1.7, it suffices to show that γ and β are q-coCartesian, which follows from Proposition 3.1.1.10. $\qquad\square$

Proof of Theorem 4.4.2.1. We will prove (1) using Proposition 4.4.2.2. Note that $\mathrm{Mod}_A(\mathbb{C})^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ automatically has units (Example 3.4.4.5). It will therefore suffice to prove the following:

($*$) For every object $X \in \mathrm{Mod}_A(\mathbb{C})^\otimes_{\langle 2 \rangle}$, there exists a morphism $f : X \to Y$ in $\mathrm{Mod}_A(\mathbb{C})^\otimes$ covering the active morphism $\langle 2 \rangle \to \langle 1 \rangle$ in $\mathcal{F}\mathrm{in}_*$ and classified by an operadic p-colimit diagram, where $p : \mathrm{Mod}_A(\mathbb{C})^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ denotes the projection.

Let $\mathcal{D}$ denote the subcategory of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{\langle 2 \rangle/}$ whose objects are injective maps $\langle 2 \rangle \to \langle n \rangle$ and whose morphisms are diagrams
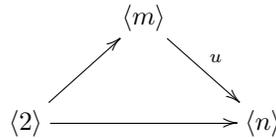
where u is active, so that $\mathcal{D}^\triangleright$ maps to $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ by a map carrying the cone point of $\mathcal{D}^\triangleright$ to $\langle 1 \rangle$. The category $\mathcal{D}$ has an evident forgetful functor $\mathcal{D} \to \mathcal{K}_{\mathrm{Comm}} \times_{\mathrm{Comm}^\otimes} \{\langle 2 \rangle\}$, so that X determines a functor $F_0 \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{D}, \mathbb{C}^\otimes)$. In view of Theorem 3.4.4.3, to prove ($*$) it will suffice to show that F_0 can be extended to an operadic p-colimit diagram $F \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{D}^\triangleright, \mathbb{C}^\otimes)$.

In view of our assumption on $\mathbb{C}^\otimes$ and Proposition 3.1.1.20, the existence of F will follow provided that we can exhibit a left cofinal map $\phi : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{D}$. We define ϕ by a variation of Construction 4.1.2.5. Let $\mathcal{J}$ denote the category whose objects are finite sets S containing a pair of distinct elements $x, y \in S$, so we have a canonical equivalence $\mathcal{D} \simeq \mathrm{N}(\mathcal{J})$. We will obtain ϕ as the nerve of a functor $\phi_0 : \mathbf{\Delta}^{op} \to \mathcal{J}$, where $\phi_0([n])$ is the set of all downward-closed subsets of $[n]$ (with distinguished points given by $\emptyset, [n] \subseteq [n]$). To prove that ϕ is left cofinal, it will suffice to show that for each $\overline{S} = (S, x, y) \in \mathcal{J}$, the category $\mathcal{I} = \mathbf{\Delta}^{op} \times_{\mathcal{J}} \mathcal{J}_{\overline{S}/}$ has weakly contractible nerve. Writing $S = S_0 \coprod \{x, y\}$, we can identify $\mathcal{I}^{op}$ with the category of simplices of the simplicial set $(\Delta^1)^{S_0}$. Since $(\Delta^1)^{S_0}$ is weakly contractible, it follows that $\mathrm{N}(\mathcal{I})$ is weakly contractible. This proves (1).

We now prove (2). Since the forgetful functor ψ clearly preserves units (Corollary 3.4.4.4), it will suffice to verify the following:

($*'$) Let $X \in \mathrm{Mod}_A(\mathbb{C})^\otimes_{\langle 2 \rangle}$, let $\alpha : \langle 2 \rangle \to \langle 1 \rangle$ be an active morphism in $\mathcal{A}ss^\otimes$, let α_0 be its image in $\mathrm{Comm}^\otimes$, and let $\overline{\alpha}_0 : X \to Y$ be a q-coCartesian morphism in $\mathrm{Mod}_A(\mathbb{C})^\otimes$ lifting α_0. Then the induced morphism $\overline{\alpha}$ in $\mathrm{Mod}_A^{\mathcal{A}ss}(\mathbb{C})^\otimes$ is p'-coCartesian (where $p' : \mathrm{Mod}_A^{\mathcal{A}ss}(\mathbb{C})^\otimes \to \mathcal{A}ss^\otimes$ denotes the projection).

Using Proposition 3.1.1.15, we deduce that $\overline{\alpha}_0$ is classified by an operadic q-colimit diagram $\Delta^1 \to \mathrm{Mod}_A(\mathbb{C})^\otimes$. Applying Theorem 3.4.4.3, we deduce that the underlying diagram $\mathcal{D}^\triangleright \to \mathbb{C}^\otimes$ is an operadic p-colimit diagram. By the cofinality argument above, we conclude that $\overline{\alpha}_0$ induces an operadic p-colimit diagram $\mathrm{N}(\mathbf{\Delta}^{op})^\triangleright \to \mathbb{C}^\otimes$. The desired result now follows from the description of the tensor product on $\mathrm{Mod}_A^{\mathcal{A}ss}(\mathbb{C})$ supplied by Proposition 4.3.6.12 (together with Theorem 4.3.5.8). $\qquad\square$

4.4.3   Change of Algebra

Let $f : A \to B$ be a commutative ring. Via the map f, every B-module can be regarded as an A-module. This construction determines a forgetful functor from the (ordinary) category Mod_B of B-modules to the (ordinary) category Mod_A of A-modules. This forgetful functor has a left adjoint, given by the construction $M \mapsto B \otimes_A M$. This left adjoint is symmetric monoidal: that is, for every pair of A-modules M and N, there is a canonical isomorphism

$$B \otimes_A (M \otimes_A N) \simeq (B \otimes_A M) \otimes_B (B \otimes_A N).$$

Our goal in this section is to prove an analogous result, where the commutative rings A and B are replaced by commutative algebra objects in an arbitrary symmetric monoidal ∞-category. Our main result can be stated as follows:

Theorem 4.4.3.1. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Assume that the $\mathcal{C}$ admits geometric realizations of simplicial objects, and that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations separately in each variable. Then the map $p : \mathrm{Mod}(\mathcal{C})^{\otimes} \to \mathrm{CAlg}(\mathcal{C}) \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a coCartesian fibration.*

Remark 4.4.3.2. Let $\mathcal{C}^{\otimes}$ be as in Theorem 4.4.3.1. The coCartesian fibration $\mathrm{Mod}(\mathcal{C})^{\otimes} \to \mathrm{CAlg}(\mathcal{C}) \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is classified by functor $\mathrm{CAlg}(\mathcal{C}) \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathcal{C}\mathrm{at}_\infty$, which we can interpret as a functor $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{Mon}_{\mathrm{Comm}}(\mathcal{C}\mathrm{at}_\infty)$. In particular, every morphism $f : A \to B$ of commutative algebra objects of $\mathcal{C}$ determines a morphism in $\mathrm{Mon}_{\mathrm{Comm}}(\mathcal{C}\mathrm{at}_\infty) \subseteq \mathrm{Fun}(\mathrm{N}(\mathcal{F}\mathrm{in}_*), \mathcal{C}\mathrm{at}_\infty)$, which classifies a symmetric monoidal functor $\mathrm{Mod}_A(\mathcal{C})^{\otimes} \to \mathrm{Mod}_B(\mathcal{C})^{\otimes}$.

Remark 4.4.3.3. Theorem 4.4.3.1 singles out a special property enjoyed by the commutative ∞-operad. If $f : A \to B$ is a map of associative algebra objects of $\mathcal{C}$, then the forgetful functor

$$_B\mathrm{BMod}_B(\mathcal{C}) \simeq \mathrm{Mod}_B^{\mathcal{A}\mathrm{ss}}(\mathcal{C}) \to \mathrm{Mod}_A^{\mathcal{A}\mathrm{ss}}(\mathcal{C}) \simeq {}_A\mathrm{BMod}_A(\mathcal{C})$$

also has a left adjoint, given by the construction $M \mapsto B \otimes_A M \otimes_A B$. However, this left adjoint is not a monoidal functor: if M and N are objects of $_A\mathrm{BMod}_A(\mathcal{C})$, then there is an induced map

$$B \otimes_A M \otimes_A N \otimes_A B \to B \otimes_A M \otimes_A B \otimes_A N \otimes_A N \simeq (B \otimes_A M \otimes_A B) \otimes_B (B \otimes_A N \otimes_A B)$$

which is in general not an equivalence.

The proof of Theorem 4.4.3.1 will require some preliminaries.

Lemma 4.4.3.4. *Let S be an ∞-category and let $p : \mathcal{C}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an S-family of ∞-operads. Assume that:*

(a) *The composite map $p' : \mathcal{C}^{\otimes} \to S$ is a Cartesian fibration; moreover, the image in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ of any p'-Cartesian morphism of $\mathcal{C}^{\otimes}$ is an equivalence.*

(b) *For each $s \in S$, the induced map $p_s : \mathcal{C}_s^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a coCartesian fibration.*

(c) *The underlying map $p_0 : \mathcal{C} \to S$ is a coCartesian fibration.*

Then:

(1) *The map p is a locally coCartesian fibration.*

(2) *A morphism f in $\mathcal{C}^{\otimes}$ is locally p-coCartesian if and only if it factors as a composition $f'' \circ f'$, where f' is a p_s-coCartesian morphism in $\mathcal{C}_s^{\otimes}$ for some $s \in S$ (here $p_s : \mathcal{C}_s^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ denotes the restriction*

of p) and $f'' : Y \to Z$ is a morphism in $\mathcal{C}^{\otimes}_{\langle n \rangle}$ with the following property: for $1 \le i \le n$, there exists a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{f''} & Z \\
\downarrow & & \downarrow \\
Y_i & \xrightarrow{f''_i} & Z_i
\end{array}
$$

where the vertical maps are inert morphisms of $\mathcal{C}^{\otimes}$ lying over $\rho^i : \langle n \rangle \to \langle 1 \rangle$ and the map f''_i is a locally p_0-coCartesian morphism in $\mathcal{C}$.

Proof. Let $f \simeq f' \circ f''$ be as in (2), and let $y, z \in S$ denote the images of $Y, Z \in \mathcal{C}^{\otimes}$. Using the equivalences

$$
\mathcal{C}^{\otimes} \times_{S \times \mathrm{N}(\mathfrak{Fin}_*)} (y, \langle n \rangle) \simeq \mathcal{C}^n_y \qquad \mathcal{C}^{\otimes} \times_{S \times \mathrm{N}(\mathfrak{Fin}_*)} (z, \langle n \rangle) \simeq \mathcal{C}^n_z,
$$

we deduce that if f'' satisfies the stated condition, then f'' is locally p-coCartesian. If f' is a p_s-coCartesian morphism in $\mathcal{C}^{\otimes}_s$, then condition (a) and Corollary T.4.3.1.15 guarantee that f' is p-coCartesian. Replacing p by the induced fibration $\mathcal{C}^{\otimes} \times_{S \times \mathrm{N}(\mathfrak{Fin}_*)} \Delta^2 \to \Delta^2$ and applying Proposition T.2.4.1.7, we deduce that f is locally p-coCartesian. This proves the "if" direction of (2).

To prove (1), consider an object $X \in \mathcal{C}^{\otimes}$ lying over $(s, \langle n \rangle) \in S \times \mathrm{N}(\mathfrak{Fin}_*)$, and let $f_0 : (s, \langle n \rangle) \to (s', \langle n' \rangle)$ be a morphism in $S \times \mathrm{N}(\mathfrak{Fin}_*)$. Then f_0 factors canonically as a composition

$$
(s, \langle n \rangle) \xrightarrow{f'_0} (s, \langle n' \rangle) \xrightarrow{f''_0} (s', \langle n' \rangle).
$$

Using assumption (b), we can lift f'_0 to a p_s-coCartesian morphism $f' : X \to Y$ in $\mathcal{C}^{\otimes}_s$, and using (c) we can lift f''_0 to a morphism $f'' : Y \to Z$ in $\mathcal{C}^{\otimes}_{\langle n \rangle}$ satisfying the condition given in (2). It follows from the above argument that $f = f'' \circ f'$ is a locally p-coCartesian morphism of $\mathcal{C}^{\otimes}$ lifting f_0. This proves (1).

The "only if" direction of (2) now follows from the above arguments together with the uniqueness properties of locally p-coCartesian morphisms. $\square$

For the statement of the next lemma, we need to introduce a bit of terminology. Let $p : \mathcal{C}^{\otimes} \to S \times \mathrm{N}(\mathfrak{Fin}_*)$ be as in Lemma 4.4.3.4, and let $f : X \to Y$ be a locally p-coCartesian morphism in $\mathcal{C}^{\otimes}_{\langle n \rangle}$ for some $n \ge 0$. If $\alpha : \langle n \rangle \to \langle n' \rangle$ is a morphism in $\mathfrak{Fin}_*$, we will say that f is α-good if the following condition is satsified:

(∗) Let $x, y \in S$ denote the images of $X, Y \in \mathcal{C}^{\otimes}$, and choose morphisms $\overline{\alpha}_x : X \to X'$ in $\mathcal{C}^{\otimes}_x$ and $\overline{\alpha}_y : Y \to Y'$ in $\mathcal{C}^{\otimes}_y$ which are p_x and p_y-coCartesian lifts of α, respectively, so that we have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\overline{\alpha}_x} & & \downarrow{\overline{\alpha}_y} \\
X' & \xrightarrow{f'} & Y'.
\end{array}
$$

Then f' is locally p-coCartesian.

Lemma 4.4.3.5. *Let $p : \mathcal{C}^{\otimes} \to S \times \mathrm{N}(\mathfrak{Fin}_*)$ be as in Lemma 4.4.3.4. The following conditions are equivalent:*

(1) *The map p is a coCartesian fibration.*

(2) *For every morphism $f : X \to Y$ in $\mathcal{C}^{\otimes}_{\langle n \rangle}$ and every morphism $\alpha : \langle n \rangle \to \langle n' \rangle$, the morphism f is α-good.*

(3) *For every morphism $f : X \to Y$ in $\mathcal{C}^{\otimes}_{\langle n \rangle}$ where $n \in \{0, 2\}$ and every active morphism $\alpha : \langle n \rangle \to \langle 1 \rangle$, the morphism f is α-good.*

Proof. The implication $(1) \Rightarrow (2)$ follows from Proposition T.2.4.1.7 and the implication $(2) \Rightarrow (3)$ is obvious. We next prove that $(2) \Rightarrow (1)$. Since p is a locally coCartesian fibration (Lemma 4.4.3.4), it will suffice to show that if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are locally p-coCartesian morphsims in $\mathcal{C}^{\otimes}$, then $g \circ f$ is locally p-coCartesian (Proposition T.2.4.2.8). Using Lemma 4.4.3.4, we can assume that $f = f'' \circ f'$, where f' is p-coCartesian and f'' is a locally p-coCartesian morphism in $\mathcal{C}^{\otimes}_{\langle n \rangle}$ for some $n \geq 0$. To prove that $g \circ f \simeq (g \circ f'') \circ f'$ is locally p-coCartesian, it will suffice to show that $g \circ f''$ is locally p-coCartesian. We may therefore replace f by f'' and thereby assume that f has degenerate image in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Applying Lemma 4.4.3.4 again, we can write $g = g'' \circ g'$ where $g' : Y \rightarrow Y'$ is a p-coCartesian morphism in $\mathcal{C}^{\otimes}_y$ covering some map $\alpha : \langle n \rangle \rightarrow \langle n' \rangle$, and g'' is a locally p-coCartesian morphism in $\mathcal{C}^{\otimes}_{\langle n'' \rangle}$. Let x denote the image of X in S, and let $k : X \rightarrow X'$ be a p-coCartesian morphism in $\mathcal{C}^{\otimes}_x$ lying over α. We have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ k\ } & Y \\
\downarrow & & \downarrow{\scriptstyle g'} \ \searrow{\scriptstyle g} \\
X' & \xrightarrow[\ h\ ]{} & Y' \xrightarrow[\ g''\ ]{} Z.
\end{array}
$$

Since f is α-good (by virtue of assumption (2)), we deduce that h is locally p-coCartesian. To prove that $g \circ f \simeq (g'' \circ h) \circ k$ is locally p-coCartesian, it will suffice (by virtue of Lemma 4.4.3.4) to show that $g'' \circ h$ is locally p-coCartesian. We claim more generally that the collection of locally p-coCartesian edges in $\mathcal{C}^{\otimes}_{\langle n \rangle}$ is closed under composition. This follows from the observation that $\mathcal{C}^{\otimes}_{\langle n \rangle} \rightarrow S$ is a coCartesian fibration (being equivalent to a fiber power of the coCartesian fibration $p_0 : \mathcal{C} \rightarrow S$). This completes the proof that $(2) \Rightarrow (3)$.

We now prove that $(3) \Rightarrow (2)$. Let us say that a morphism $\alpha : \langle n \rangle \rightarrow \langle m \rangle$ is *perfect* if every locally p-coCartesian morphism f in $\mathcal{C}^{\otimes}_{\langle n \rangle}$ is α-good. Assumption (3) guarantees that the active morphisms $\langle 2 \rangle \rightarrow \langle 1 \rangle$ and $\langle 0 \rangle \rightarrow \langle 1 \rangle$ are perfect; we wish to prove that every morphism in $\mathcal{F}\mathrm{in}_*$ is perfect. This follows immediately from the following three claims:

(i) If $\alpha : \langle n \rangle \rightarrow \langle m \rangle$ is perfect, then for each $k \geq 0$ the induced map $\langle n + k \rangle \rightarrow \langle m + k \rangle$ is perfect.

(ii) The unique (inert) morphism $\langle 1 \rangle \rightarrow \langle 0 \rangle$ is perfect.

(iii) The collection of perfect morphisms in $\mathcal{F}\mathrm{in}_*$ is closed under composition.

Assertions (i) and (ii) are obvious. To prove (iii), suppose that we are given perfect morphisms $\alpha : \langle n \rangle \rightarrow \langle m \rangle$ and $\beta : \langle m \rangle \rightarrow \langle k \rangle$. Let $f : X \rightarrow Y$ be a locally p-coCartesian morphism in $\mathcal{C}^{\otimes}_{\langle n \rangle}$, and form a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\downarrow & & \downarrow \\
X' & \xrightarrow{\ f'\ } & Y' \\
\downarrow & & \downarrow \\
X'' & \xrightarrow{\ f''\ } & Y''
\end{array}
$$

where the upper vertical maps are p-coCartesian lifts of α and the lower vertical maps are p-coCartesian lifts of β. Since α is perfect, the map f' is locally p-coCartesian. Since β is perfect, the map f'' is locally p-coCartesian, from which it follows that f is $\beta \circ \alpha$-good as desired. $\qquad\square$

Lemma 4.4.3.6. *Let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ preserves geometric realizations separately in each variable. Then the forgetful functor $\theta : \mathrm{LMod}(\mathcal{C}) \rightarrow \mathrm{Alg}(\mathcal{C})$ is a coCartesian fibration.*

Proof. Corollary 4.2.3.2 guarantees that θ is a Cartesian fibration. In view of Proposition T.5.2.2.5, it will suffice to show that for each morphism $f : A \to B$ in $\mathrm{Alg}(\mathcal{C})$, the forgetful functor $\mathrm{LMod}_B(\mathcal{C}) \to \mathrm{LMod}_A(\mathcal{C})$ has a left adjoint. This follows from Proposition 4.3.7.14. $\square$

Proof of Theorem 4.4.3.1. We first show that p is a locally coCartesian fibration by verifying the hypotheses of Lemma 4.4.3.4:

(a) The map $p' : \mathrm{Mod}(\mathcal{C})^{\otimes} \to \mathrm{CAlg}(\mathcal{C})$ is a Cartesian fibration by virtue of Corollary 3.4.3.4; moreover, a morphism f in $\mathrm{Mod}(\mathcal{C})^{\otimes}$ is p'-Cartesian if and only if its image in $\mathcal{C}^{\otimes}$ is an equivalence (which implies that its image in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is an equivalence).

(b) For each $A \in \mathrm{CAlg}(\mathcal{C})$, the fiber $\mathrm{Mod}_A(\mathcal{C})^{\otimes}$ is a symmetric monoidal ∞-category (Theorem 4.4.2.1).

(c) The map $p_0 : \mathrm{Mod}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C})$ is a coCartesian fibration. This follows from Corollary 4.4.1.6, because the forgetful functor $\mathrm{LMod}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$ is a coCartesian fibration (Lemma 4.4.3.6).

To complete the proof, it will suffice to show that p satisfies condition (3) of Lemma 4.4.3.5. Fix a map $f_0 : A \to B$ in $\mathrm{CAlg}(\mathcal{C})$ and a locally p-coCartesian morphism $f : M_A \to M_B$ in $\mathrm{Mod}(\mathcal{C})^{\otimes}_{\langle n \rangle}$ covering f; we wish to prove that f is α-good, where $\alpha : \langle n \rangle \to \langle 1 \rangle$ is the unique active morphism (here $n = 0$ or $n = 2$). Let $\mathcal{D}^{\otimes} = \mathrm{Mod}_A(\mathcal{C})^{\otimes}$. Theorem 4.4.2.1 implies that $\mathcal{D}^{\otimes}$ is a symmetric monoidal ∞-category, Corollaries 4.4.1.6 and 4.2.3.5 imply that $\mathcal{D}$ admits geometric realizations of simplicial objects, and Corollary 4.3.5.15 implies that the tensor product on $\mathcal{D}$ preserves geometric realizations separately in each variable. Corollary 3.4.1.8 implies that the forgetful functor $\mathrm{Mod}(\mathcal{D})^{\otimes} \to \mathrm{Mod}(\mathcal{C})^{\otimes} \times_{\mathrm{CAlg}(\mathcal{C})} \mathrm{CAlg}(\mathcal{C})^{A/}$ is an equivalence of ∞-categories. Replacing $\mathcal{C}^{\otimes}$ by $\mathcal{D}^{\otimes}$, we may assume without loss of generality that A is a unit algebra in $\mathcal{C}^{\otimes}$.

Suppose first that $n = 0$. Unwinding the definitions, we are required to show that if $M_A \to M'_A$ and $M_B \to M'_B$ are morphisms $\mathrm{Mod}(\mathcal{C})^{\otimes}$ which exhibit $M'_A \in \mathrm{Mod}_A(\mathcal{C})$ as a p_A-unit (see Definition 3.2.1.1; here $p_A : \mathrm{Mod}_A(\mathcal{C})^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ denotes the restriction of p) and $M'_B \in \mathrm{Mod}_B(\mathcal{C})$ as a p_B-unit (where p_B is defined similarly), then the induced map $f' : M'_A \to M'_B$ is locally p-coCartesian. Using Corollary 3.4.3.4, it suffices to show that $\phi(f')$ is an equivalence, where $\phi : \mathrm{Mod}(\mathcal{C}) \to \mathcal{C}$ is the forgetful functor. Using Corollary 3.4.4.4 and Proposition 4.4.1.4, this translates into the following assertion: the map $A \to B$ exhibits B as the free left B-module generated by A. This follows from Corollary 4.2.4.8.

We now treat the case $n = 2$. Since A is a unit algebra, Proposition 3.4.2.1 allows us to identify M_A with a pair of objects $P, Q \in \mathcal{C}$. We can identify M_B with a pair of objects $P_B, Q_B \in \mathrm{Mod}_B(\mathcal{C})$. Let us abuse notation by identifying P_B and Q_B with their images in $\mathrm{Mod}_B^{\mathrm{Ass}}(\mathcal{C}) \simeq {}_B\mathrm{BMod}_B(\mathcal{C})$. Unwinding the definitions (and using Theorem 4.4.2.1), it suffices to show that the canonical map $f' : P \otimes Q \to (P_B \otimes_B Q_B)$ exhibits the relative tensor product $P_B \otimes_B Q_B$ as the free left B-module generated by $P \otimes Q$. This follows from the calculation

$$B \otimes (P \otimes Q) \simeq (B \otimes P) \otimes Q \simeq (B \otimes P) \otimes_{\mathbf{1}} Q \simeq (B \otimes P) \otimes_B (B \otimes Q).$$

(see Remark 4.3.5.9). $\square$

4.4.4 Rectification of Commutative Algebras

Let $\mathbf{A}$ be a symmetric monoidal model category (Definition 4.1.3.8). In §4.1.3, we saw that the underlying ∞-category $\mathrm{N}(\mathbf{A}^c)[W^{-1}]$ of $\mathbf{A}$ inherits the structure of a symmetric monoidal ∞-category, for which the natural map $\mathrm{N}(\mathbf{A}^c) \to \mathrm{N}(\mathbf{A}^c)[W^{-1}]$ can be promoted to a symmetric monoidal functor. In particular, every associative algebra object of $\mathbf{A}$ (which is cofibrant as an object of $\mathbf{A}$) determines a associative algebra object of the underlying ∞-category $\mathrm{N}(\mathbf{A}^c)[W^{-1}]$. Under some mild hypotheses, Proposition 4.1.4.3 guarantees that the category of associative algebra objects $\mathrm{Alg}(\mathbf{A})$ is equipped with a model structure, and Theorem 4.1.4.4 implies that the underlying ∞-category of $\mathrm{Alg}(\mathbf{A})$ is equivalent to $\mathrm{Alg}(\mathrm{N}(\mathbf{A}^c)[W^{-1}])$. Our goal in this section is to prove analogous results concerning the relationship between commutative algebra objects of

A and $\mathrm{N}(\mathbf{A}^c)[W^{-1}]$. These results will require some rather strong assumptions relating the model structure on **A** and the symmetric monoidal structure on **A**, which we now formulate.

Notation 4.4.4.1. Let **A** be a symmetric monoidal category which admits colimits. Given a pair of morphisms $f : A \to A'$, $g : B \to B'$, we let $f \wedge g$ denote the induced map

$$(A \otimes B') \coprod_{A \otimes B} (A' \otimes B) \to A' \otimes B'.$$

We observe that the operation $\wedge$ determines a symmetric monoidal structure on the category of morphisms in **A**. In particular, for every morphism $f : X \to Y$, we iterate the above construction to obtain a map

$$\wedge^n(f) : \square^n(f) \to Y^{\otimes n}.$$

Here the source and target of $\wedge^n(f)$ carry actions of the symmetric group Σ_n, and $\wedge^n(f)$ is a Σ_n-equivariant map. Passing to Σ_n-coinvariants, we obtain a new map, which we will denote by $\sigma^n(f) : \mathrm{Sym}^n(Y; X) \to \mathrm{Sym}^n(Y)$.

   Before giving the next definition, we need to review a bit of terminology. Recall that a collection S of morphisms in a presentable category **A** is *weakly saturated* if it is stable under pushouts, retracts, and transfinite composition (see Definition T.A.1.2.2). For every collection S of morphisms in **A**, there is a smallest weakly saturated collection of morphisms $\overline{S}$ containing S. In this case, we will say that $\overline{S}$ is generated by S.

Definition 4.4.4.2. Let **A** be a combinatorial symmetric monoidal model category. We will say that a morphism $f : X \to Y$ is a *power cofibration* if the following condition is satisfied:

($\star$) For every $n \geq 0$, the induced map $\wedge^n(f) : \square^n(f) \to Y^{\otimes n}$ is a cofibration in $\mathbf{A}^{\Sigma_n}$. Here $\mathbf{A}^{\Sigma_n}$ denotes the category of objects of **A** equipped with an action of the symmetric group Σ_n, endowed with the *projective* model structure (see §T.A.3.3).

We will say that an object $X \in \mathbf{A}$ is *power cofibrant* if the map $\emptyset \to X$ is a power cofibration, where $\emptyset$ is an initial object of **A**.
   Let V be a collection of morphisms in a combinatorial symmetric monoidal model category **A**. We will say that **A** is *freely powered* if the following conditions are satisfied:

($F1$) The model category **A** satisfies the monoid axiom of [128] (see Definition 4.1.4.1): if $\overline{U}$ denotes the weakly saturated class generated by morphisms of the form $\mathrm{id}_X \otimes f$, where f is a trivial cofibration in **A**, then every morphism in $\overline{U}$ is a weak equivalence in **A**.

($F2$) The model category **A** is left proper, and the collection of cofibrations in $\mathbf{A}_V$ is generated (as a weakly saturated class) by cofibrations between cofibrant objects.

($F3$) Every cofibration in **A** is a power cofibration.

Remark 4.4.4.3. If $f : X \to Y$ is a cofibration in **A**, then the definition of a monoidal model category guarantees that $\wedge^n(f)$ is a cofibration *in* **A**, which is trivial if f is trivial. Condition ($\star$) is much stronger: roughly speaking, it guarantees that the symmetric group Σ_n acts *freely* on the object $Y^{\otimes n}$ (see Lemma 4.4.4.11 below).

Remark 4.4.4.4. In the situation of Definition 4.4.4.2, if every object of **A** is cofibrant, then conditions ($F1$) and ($F2$) are automatic.

Remark 4.4.4.5. Let **A** be a combinatorial symmetric monoidal model category. Then every power cofibration in **A** is a cofibration (take $n = 1$).

We now turn to the study of commutative algebras in a symmetric monoidal model category $\mathbf{A}$. Let $\mathrm{CAlg}(\mathbf{A})$ denote the category whose objects are commutative algebras in $\mathbf{A}$. The main results of this section can be stated as follows:

Proposition 4.4.4.6. *Let $\mathbf{A}$ be a combinatorial symmetric monoidal model category which is freely powered. Then:*

(1) *The category $\mathrm{CAlg}(\mathbf{A})$ admits a combinatorial model structure, where:*

   (W) *A morphism $f : A \to B$ of commutative algebra objects of $\mathbf{A}$ is a weak equivalence if it is a weak equivalence when regarded as a morphism in $\mathbf{A}$.*

   (F) *A morphism $f : A \to B$ of commutative algebra objects of $\mathbf{A}$ is a fibration if it is a fibration when regarded as a morphism in $\mathbf{A}$.*

(2) *The forgetful functor $\theta : \mathrm{CAlg}(\mathbf{A}) \to \mathbf{A}$ is a right Quillen functor.*

Theorem 4.4.4.7. *Let $\mathbf{A}$ be a combinatorial symmetric monoidal model category which is freely powered. Assume that the forgetful functor $\theta : \mathrm{CAlg}(\mathbf{A}) \to \mathbf{A}$ preserves fibrant-cofibrant objects. Let $\mathbf{A}^c$ and $\mathrm{CAlg}(\mathbf{A})^c$ denote the full subcategories of $\mathbf{A}$ and $\mathrm{CAlg}(\mathbf{A})$ spanned by the cofibrant objects, let W be the collection of weak equivalences in $\mathbf{A}^c$, and let W' be the collection of weak equivalences in $\mathrm{CAlg}(\mathbf{A})^c$. Then the canonical map*

$$\mathrm{N}(\mathrm{CAlg}(\mathbf{A})^c)[W'^{-1}] \to \mathrm{CAlg}(\mathrm{N}(\mathbf{A}^c)[W^{-1}])$$

is an equivalence of ∞-categories.

We will prove Proposition 4.4.4.6 and Theorem 4.4.4.7 at the end of this section.

Remark 4.4.4.8. Suppose that $\mathbf{A}$ is a combinatorial simplicial symmetric monoidal model category, in the sense of Definition 4.1.3.8. The category $\mathrm{CAlg}(\mathbf{A})$ inherits the structure of a simplicial category from $\mathbf{A}$: namely, we regard $\mathrm{CAlg}(\mathbf{A})$ as cotensored over simplicial sets using the observation that any commutative algebra structure on an object $A \in \mathbf{A}$ induces a commutative algebra structure on A^K for every simplicial set K. It is not difficult to see that the model structure of Proposition 4.4.4.6 is compatible with this simplicial structure on $\mathrm{CAlg}(\mathbf{A})$.

Remark 4.4.4.9. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. In order for $\mathcal{C}$ to arise from the situation described in Proposition 4.1.3.10, it is necessary for $\mathcal{C}$ to be presentable and for the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ to preserve small colimits separately in each variable. We do not know if these conditions are also sufficient.

We now turn to the proof of Proposition 4.4.4.6. First, we need a few preliminaries.

Lemma 4.4.4.10. *Let $\mathbf{A}$ be a combinatorial symmetric monoidal model category. For every category $\mathcal{C}$, let $\mathbf{A}^{\mathcal{C}}$ denote the associated diagram category, endowed with the projective model structure (see §T.A.3.3). Let f be a cofibration in $\mathbf{A}^{\mathcal{C}}$, and g a cofibration in $\mathbf{A}^{\mathcal{C}'}$. Then the smash product $f \wedge g$ (see Notation 4.4.4.1) is a cofibration in $\mathbf{A}^{\mathcal{C} \times \mathcal{C}'}$. In particular, if $X \in \mathbf{A}^{\mathcal{C}}$ and $Y \in \mathbf{A}^{\mathcal{C}'}$ are cofibrant, then $X \otimes Y$ is a cofibrant object of $\mathbf{A}^{\mathcal{C} \times \mathcal{C}'}$.*

Proof. Let S denote the collection of all morphisms f in $\mathbf{A}^{\mathcal{C}}$ for which the conclusion of the Lemma holds. It is not difficult to see that S is weakly saturated, in the sense of Definition T.A.1.2.2. Consequently, it will suffice to prove that S contains a set of generating cofibrations for $\mathbf{A}^{\mathcal{C}}$. Let $i^C : \{*\} \to \mathcal{C}$ be the inclusion of an object $C \in \mathcal{C}$, and let $i_!^C : \mathbf{A} \to \mathbf{A}^{\mathcal{C}}$ be the corresponding left Kan extension functor (a left adjoint to the evaluation at C). Then the collection of cofibrations in $\mathbf{A}^{\mathcal{C}}$ is generated by morphisms of the form $i_!^C(f_0)$, where f_0 is a cofibration in $\mathbf{A}$. We may therefore assume that $f = i_!^C(f_0)$. Using the same argument, we may assume that $g = i_!^{C'}(g_0)$, where $C' \in \mathcal{C}$ and g_0 is a cofibration in $\mathbf{A}$. We now observe that $f \wedge g$ is isomorphic to $i_!^{(C,C')}(f_0 \wedge g_0)$. Since $i_!^{(C,C')} : \mathbf{A} \to \mathbf{A}^{\mathcal{C} \times \mathcal{C}'}$ is a left Quillen functor, it will suffice to show that $f_0 \wedge g_0$ is a cofibration in $\mathbf{A}$, which follows from our assumption that $\mathbf{A}$ is a (symmetric) monoidal model category. $\square$

Lemma 4.4.4.11. *Let* $\mathbf{A}$ *be a combinatorial symmetric monoidal model category. Then:*

(1) *Let* $f : X \to Y$ *be a power cofibration in* $\mathbf{A}$*. Then the induced map*

$$\sigma^n(f) : \operatorname{Sym}^n(Y; X) \to \operatorname{Sym}^n(Y)$$

   is a cofibration, which is weak equivalence if f *is a weak equivalence (see Notation 4.4.4.1) and* $n > 0$*.*

(2) *Let* Y *be a power cofibrant object of* $\mathbf{A}$*. Then* $\operatorname{Sym}^n(Y)$ *is a homotopy colimit for the action of* Σ_n *on* $Y^{\otimes n}$*.*

(3) *Let* $f : X \to Y$ *be a power cofibration between power-cofibrant objects of* $\mathbf{A}$*. Then the object* $\square^n(f) \in \mathbf{A}^{\Sigma_n}$ *is cofibrant (with respect to the projective model structure).*

Proof. Let $F : \mathbf{A}^{\Sigma_n} \to \mathbf{A}$ be a left adjoint to the diagonal functor, so that F carries an object $X \in \mathbf{A}^{\Sigma_n}$ to the object of coinvariants X_{Σ_n}. We observe that F is a left Quillen functor, and that $\sigma^n(f) = F(\wedge^n(f))$ for every morphism in $\mathbf{A}$. Assertion (1) now follows immediately from the definitions (and the observation that $\wedge^n(f)$ is a weak equivalence if $n > 0$ and f is a trivial cofibration). Assertion (2) follows immediately from the definition of a homotopy colimit.

We now prove (3). Let $f : X \to Y$ be a power cofibration between power cofibrant objects of $\mathbf{A}$. We observe that $Y^{\otimes n}$ admits a Σ_n-equivariant filtration

$$X^{\otimes n} = Z_0 \xrightarrow{\gamma_1} Z_1 \xrightarrow{\gamma_2} \dots \xrightarrow{\gamma_{n-1}} Z_{n-1} = \square^n(f) \longrightarrow Z_n = Y^{\otimes n}.$$

It will therefore suffice to prove that each of the maps $\{\gamma_i\}_{1 \leq i \leq n-1}$ is a cofibration in $\mathbf{A}^{\Sigma_n}$. For this, we observe that there is a pushout diagram

$$
\begin{array}{ccc}
\pi_!(\square^i(f) \times X^{\otimes(n-i)}) & \xrightarrow{\pi_!(\wedge^i(f) \otimes \mathrm{id})} & \pi_!(Y^{\otimes i} \otimes X^{\otimes n-i}) \\
\downarrow & & \downarrow \\
Z_{i-1} & \xrightarrow{\quad\gamma_i\quad} & Z_i,
\end{array}
$$

where $\pi_!$ denotes the left adjoint to the forgetful functor $\mathbf{A}^{\Sigma_n} \to \mathbf{A}^{\Sigma_i \times \Sigma_{n-i}}$. Since $\pi_!$ is a left Quillen functor, it suffices to show that each $\wedge^i(f) \otimes \mathrm{id}$ is a cofibration in $\mathbf{A}^{\Sigma_i \times \Sigma_{n-i}}$. This follows from our assumption that X is power cofibrant, our assumption that f is a power cofibration, and Lemma 4.4.4.10. $\square$

Proof of Proposition 4.4.4.6. We first observe that the category $\operatorname{CAlg}(\mathbf{A})$ is presentable (this is a special case of Corollary 3.2.3.5). Since $\mathbf{A}$ is combinatorial, there exists a (small) collection of morphisms $I = \{i_\alpha : C \to C'\}$ which generates the class of cofibrations in $\mathbf{A}$, and a (small) collection of morphisms $J = \{j_\alpha : D \to D'\}$ which generates the class of trivial cofibrations in $\mathbf{A}$.

Let $F : \mathbf{A} \to \operatorname{CAlg}(\mathbf{A})$ be a left adjoint to the forgetful functor. Let $\overline{F(I)}$ be the weakly saturated class of morphisms in $\operatorname{CAlg}(\mathbf{A})$ generated by $\{F(i) : i \in I\}$, and let $\overline{F(J)}$ be defined similarly. Unwinding the definitions, we see that a morphism in $\operatorname{CAlg}(\mathbf{A})$ is a trivial fibration if and only if it has the right lifting property with respect to $F(i)$, for every $i \in I$. Invoking the small object argument, we deduce that every morphism $f : A \to C$ in $\operatorname{CAlg}(\mathbf{A})$ admits a factorization $A \xrightarrow{f'} B \xrightarrow{f''} C$ where $f' \in \overline{F(I)}$ and f'' is a trivial fibration. Similarly, we can find an analogous factorization where $f' \in \overline{F(J)}$ and f'' is a fibration.

Using a standard argument, we may reduce the proof of (1) to the problem of showing that every morphism belonging to $\overline{F(J)}$ is a weak equivalence in $\operatorname{CAlg}(\mathbf{A})$. Let $\overline{U}$ be as in Definition 4.4.4.2, and let S be the collection of morphisms in $\operatorname{CAlg}(\mathbf{A})$ such that the underlying morphism in $\mathbf{A}$ belongs to $\overline{U}$. Since $\mathbf{A}$ satisfies condition $(F1)$, S consists of weak equivalences in $\mathbf{A}$. It will therefore suffice to show that $\overline{F(J)} \subseteq S$. Because S is weakly saturated, it will suffice to show that $F(J) \subseteq S$. Unwinding the definitions, we are reduced to proving the following:

($*$) Let

$$F(C) \xrightarrow{F(i)} F(C')$$

be a pushout diagram in $\mathrm{CAlg}(\mathbf{A})$. If i is a trivial cofibration in $\mathbf{A}$, then $f \in S$.

To prove ($*$), we observe that $F(C')$ admits a filtration by $F(C)$-modules

$$F(C) \simeq B_0 \to B_1 \to B_2 \to \dots,$$

where $F(C') \simeq \mathrm{colim}\{B_i\}$ and for each $n > 0$ there is a pushout diagram

$$F(C) \otimes \mathrm{Sym}^n(C'; C) \xrightarrow{\mathrm{id}_{F(C)} \otimes \sigma^n(i)} F(C) \otimes \mathrm{Sym}^n(C')$$
$$B_{n-1} \longrightarrow B_n.$$

(See Notation 4.4.4.1.) It follows that A' admits a filtration by A-modules

$$A \simeq B_0' \to B_1' \to B_2' \to \dots,$$

where $A' \simeq \mathrm{colim}\{B_i'\}$ and for each $n > 0$ there is a pushout diagram

$$A \otimes \mathrm{Sym}^n(C'; C) \xrightarrow{\mathrm{id}_A \otimes \sigma^n(i)} A \otimes \mathrm{Sym}^n(C')$$
$$B_{n-1}' \longrightarrow B_n'.$$

Since i is a trivial power cofibration, Lemma 4.4.4.11 implies that $\sigma^n(i)$ is a trivial cofibration in $\mathbf{A}$. It follows that $\mathrm{id}_A \otimes \sigma^n(i)$ belongs to $\overline{U}$. Since $\overline{U}$ is stable under transfinite composition, we conclude that f belongs to S. This completes the proof of (1). Assertion (2) is obvious. $\qquad\square$

The proof of Theorem 4.4.4.7 rests on the following analogue of Lemma 4.1.4.13:

Lemma 4.4.4.12. *Let $\mathbf{A}$ be as in Theorem 4.4.4.7, and let $\mathcal{C}$ be a small category such that $\mathrm{N}(\mathcal{C})$ is sifted (Definition T.5.5.8.1). Then the forgetful functor $\mathrm{N}(\mathrm{CAlg}(\mathbf{A}))^c[W'^{-1}] \to \mathrm{N}(\mathbf{A}^c)[W^{-1}]$ preserves $\mathrm{N}(\mathcal{C})$-indexed colimits.*

Proof. In view of Propositions 1.3.4.24 and 1.3.4.25, it will suffice to prove that the forgetful functor $\theta : \mathrm{CAlg}(\mathbf{A}) \to \mathbf{A}$ preserves homotopy colimits indexed by $\mathcal{C}$. Let us regard $\mathrm{CAlg}(\mathbf{A})^{\mathcal{C}}$, $\mathbf{A}^{\mathcal{C}}$, and $\mathbf{A}^{\mathcal{C}}$ as endowed with the projective model structure (see §T.A.3.3). Let $F : \mathbf{A}^{\mathcal{C}} \to \mathbf{A}$ and $F_{\mathrm{CAlg}} : \mathrm{CAlg}(\mathbf{A})^{\mathcal{C}} \to \mathrm{CAlg}(\mathbf{A})$ be colimit functors, and let $\theta^{\mathcal{C}} : \mathrm{CAlg}(\mathbf{A})^{\mathcal{C}} \to \mathbf{A}^{\mathcal{C}}$ be given by composition with θ. Since $\mathrm{N}(\mathcal{C})$ is sifted, there is a canonical isomorphism of functors $\alpha : F \circ \theta^{\mathcal{C}} \simeq \theta \circ F_{\mathrm{CAlg}}$. We wish to prove that this isomorphism persists after deriving all of the relevant functors. Since θ and $\theta^{\mathcal{C}}$ preserve weak equivalences, they can be identified with their right derived functors. Let LF and LF_{CAlg} be the left derived functors of F and F_{CAlg}, respectively. Then α induces a natural transformation $\overline{\alpha} : LF \circ \theta^{\mathcal{C}} \to \theta \circ LF_{\mathrm{CAlg}}$; we wish to show that $\overline{\alpha}$ is an isomorphism. Let $A : \mathcal{C} \to \mathrm{CAlg}(\mathbf{A})$ be a projectively cofibrant object of $\mathrm{CAlg}(\mathbf{A})^{\mathcal{C}}$; we must show that the natural map

$$LF(\theta^{\mathcal{C}}(A)) \to \theta(LF_{\mathrm{CAlg}}(A)) \simeq \theta(F_{\mathrm{CAlg}}(A)) \simeq F(\theta^{\mathcal{C}}(A))$$

is a weak equivalence in $\mathbf{A}$.

Let us say that an object $X \in \mathbf{A}^{\mathcal{C}}$ is *good* if each $X(C) \in \mathbf{A}$ is cofibrant, the colimit $F(X) \in \mathbf{A}$ is cofibrant, and the canonical map the natural map $LF(X) \to F(X)$ is an isomorphism in the homotopy category h$\mathbf{A}$ (in other words, the colimit of X is also a homotopy colimit of X). To complete the proof, it will suffice to show that $\theta^{\mathcal{C}}(A)$ is good, whenever A is a projectively cofibrant object of $\mathrm{CAlg}(\mathbf{A})^{\mathcal{C}}$. This is not obvious, since $\theta^{\mathcal{C}}$ is a right Quillen functor and does not preserve projectively cofibrant objects in general (note that we have not yet used the full strength of our assumption that $\mathrm{N}(\mathcal{C})$ is sifted). To continue the proof, we will need a relative version of the preceding condition. We will say that a morphism $f : X \to Y$ in $\mathbf{A}^{\mathcal{C}}$ is *good* if the following conditions are satisfied:

(i) The objects $X, Y \in \mathbf{A}^{\mathcal{C}}$ are good.

(ii) For each $C \in \mathcal{C}$, the induced map $X(C) \to Y(C)$ is a cofibration in $\mathbf{A}$.

(iii) The map $F(X) \to F(Y)$ is a cofibration in $\mathbf{A}$.

As in the proof of Lemma 4.1.4.13, we have the following:

(1) The collection of good morphisms is stable under transfinite composition.

(2) Suppose given a pushout diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\downarrow & & \downarrow \\
X' & \xrightarrow{\ f'\ } & Y'
\end{array}
$$

in $\mathbf{A}^{\mathcal{C}}$. If f is good and X' is good, then f' is good.

(3) Let $F : \mathcal{C} \to \mathbf{A}$ be a constant functor whose value is a cofibrant object of $\mathbf{A}$. Then F is good.

(4) Every projectively cofibrant object of $\mathbf{A}^{\mathcal{C}}$ is good. Every strong cofibration between projectively cofibrant objects of $\mathbf{A}^{\mathcal{C}}$ is good.

(5) If X and Y are good objects of $\mathbf{A}^{\mathcal{C}}$, then $X \otimes Y$ is good.

(6) Let $f : X \to X'$ be a good morphism in $\mathbf{A}^{\mathcal{C}}$, and let Y be a good object of $\mathbf{A}^{\mathcal{C}}$. Then the morphism $f \otimes \mathrm{id}_Y$ is good.

(7) Let $f : X \to X'$ and $g : Y \to Y'$ be good morphisms in $\mathbf{A}^{\mathcal{C}}$. Then

$$
f \wedge g : (X \otimes Y') \coprod_{X \otimes Y} (X' \otimes Y) \to X' \otimes Y'
$$

is good.

Moreover, our assumption that $\mathbf{A}$ is freely powered ensures that the class of good morphisms has the following additional property:

(8) Let $f : X \to Y$ be a good morphism in $\mathbf{A}^{\mathcal{C}}$. Then the induced map $\sigma^n(f) : \mathrm{Sym}^n(Y; X) \to \mathrm{Sym}^n(Y)$ (see Notation 4.4.4.1) is good. Condition (ii) follows immediately from Lemma 4.4.4.11, and condition (iii) follows from Lemma 4.4.4.11 and the observation that $F(\sigma^n(f)) = \sigma^n(F(f))$ (since the functor F commutes with colimits and tensor products). It will therefore suffice to show that the objects

$\operatorname{Sym}^n(Y; X)$ and $\operatorname{Sym}^n(Y)$ are good. Let $D : \mathbf{A}^{\Sigma_n} \to \mathbf{A}$ be the coinvariants functor, and consider the following diagram of left Quillen functors (which commutes up to canonical isomorphism):

$$
\begin{array}{ccc}
\mathbf{A}^{\mathcal{C} \times \Sigma_n} & \xrightarrow{\ D^{\mathcal{C}}\ } & \mathbf{A}^{\mathcal{C}} \\
\Big\downarrow{\scriptstyle F} & & \Big\downarrow{\scriptstyle F} \\
\mathbf{A}^{\Sigma_n} & \xrightarrow{\ D\ } & \mathbf{A}.
\end{array}
$$

Let us regard $\wedge^n(f)$ as an object in the category of arrows of $\mathrm{h}\mathbf{A}^{\mathcal{C} \times \Sigma_n}$. We wish to show that the canonical map $LF(\sigma^n(f)) \to F(\sigma^n(f))$ is an isomorphism (in the category of morphisms in $\mathrm{h}\mathbf{A}$). We now observe that $\sigma^n(f) = D^{\mathcal{C}}(\wedge^n(f))$. Lemma 4.4.4.11 implies that the canonical map $LD^{\mathcal{C}} \wedge^n (f) \to D^{\mathcal{C}} \wedge^n (f)$ is a weak equivalence. It will therefore suffice to show that the transformation

$$
\alpha : L(F \circ D^{\mathcal{C}})(\wedge^n(f)) \to (F \circ D^{\mathcal{C}})(\wedge^n(f))
$$

is an isomorphism (in the category of morphisms of $\mathrm{h}\mathbf{A}$). Using the commutativity of the above diagram, we can identify α with the map

$$
(LD \circ LF) \wedge^n (f) \to (D \circ F)(\wedge^n(f)) = D \wedge^n (F(f)).
$$

Using Lemma 4.4.4.11 again, we can identify the right hand side with $LD \wedge^n (F(f))$. It will therefore suffice to show that the map $LF \wedge^n (f) \to F \wedge^n (f)$ is an isomorphism in the category of morphisms of $\mathrm{h}\mathbf{A}^{\Sigma_n}$. Since the forgetful functor $\mathbf{A}^{\Sigma_n} \to \mathbf{A}$ preserves homotopy colimits (it is also a left Quillen functor) and detects equivalences, we are reduced to proving that the morphism $\wedge^n(f)$ is good. This follows from (7) using induction on n.

We observe that axiom $(F2)$ of Definition 4.4.4.2 has the following consequence:

$(F2')$ The collection of all projective cofibrations in $\mathbf{A}^{\mathcal{C}}$ is generated by projective cofibrations between projectively cofibrant objects.

Let $T : \mathbf{A}^{\mathcal{C}} \to \operatorname{CAlg}(\mathbf{A})^{\mathcal{C}}$ be a left adjoint to $\theta^{\mathcal{C}}$. Using the small object argument and (B'), we conclude that for every projectively cofibrant object $A \in \operatorname{CAlg}(\mathbf{A})^{\mathcal{C}}$ there exists a transfinite sequence $\{A^{\beta}\}_{\beta \leq \alpha}$ in $\operatorname{CAlg}(\mathbf{A})^{\mathcal{C}}$ with the following properties:

(a) The object A^0 is initial in $\operatorname{CAlg}(\mathbf{A})^{\mathcal{C}}$.

(b) The object A is a retract of A^{α}.

(c) If $\lambda \leq \alpha$ is a limit ordinal, then $A^{\lambda} \simeq \operatorname{colim}\{A^{\beta}\}_{\beta < \lambda}$.

(d) For each $\beta < \alpha$, there is a pushout diagram

$$
\begin{array}{ccc}
T(X') & \xrightarrow{\ T(f)\ } & T(X) \\
\Big\downarrow & & \Big\downarrow \\
A^{\beta} & \xrightarrow{\hspace{1cm}} & A^{\beta+1}
\end{array}
$$

where f is a projective cofibration between projectively cofibrant objects of $\mathbf{A}^{\mathcal{C}}$.

We wish to prove that $\theta^{\mathcal{C}}(A)$ is good. In view of (b), it will suffice to show that $\theta^{\mathcal{C}}(A^{\alpha})$ is good. We will prove a more general assertion: for every $\gamma \leq \beta \leq \alpha$, the induced morphism $u_{\gamma,\beta} : \theta^{\mathcal{C}}(A^{\gamma}) \to \theta^{\mathcal{C}}(A^{\beta})$ is good. The proof is by induction on β. If $\beta = 0$, then we are reduced to proving that $\theta^{\mathcal{C}}(A^0)$ is good. This follows from (a) and (3). If β is a nonzero limit ordinal, then the desired result follows from (c) and (1). It

therefore suffices to treat the case where $\beta = \beta' + 1$ is a successor ordinal. Moreover, we may suppose that $\gamma = \beta'$: if $\gamma < \beta'$, then we observe that $u_{\gamma,\beta} = u_{\beta',\beta} \circ u_{\gamma,\beta'}$ and invoke (1), while if $\gamma > \beta'$, then $\gamma = \beta$ and we are reduced to proving that $\theta^{\mathcal{C}}(A^\beta)$ is good, which follows from the assertion that $u_{\beta',\beta}$ is good. We are now reduced to proving the following:

($*$) Let

$$
\begin{array}{ccc}
T(X') & \xrightarrow{\ T(f)\ } & T(X) \\
\downarrow & & \downarrow \\
B' & \xrightarrow{\quad v \quad} & B
\end{array}
$$

be a pushout diagram in $\mathrm{CAlg}(\mathbf{A})^{\mathcal{C}}$, where $f : X' \to X$ is a projective cofibration between projectively cofibrant objects of $\mathbf{A}^{\mathcal{C}}$. If $\theta^{\mathcal{C}}(B')$ is good, then $\theta^{\mathcal{C}}(v)$ is good.

To prove ($*$), we set $Y = \theta^{\mathcal{C}}(B) \in \mathbf{A}^{\mathcal{C}}$, $Y' = \theta^{\mathcal{C}}(B') \in \mathbf{A}^{\mathcal{C}}$. Let $g : \emptyset \to Y'$ the unique morphism, where $\emptyset$ denotes an initial object of $\mathbf{A}^{\mathcal{C}}$. As in the proof of Proposition 4.4.4.6, Y can be identified with the colimit of a sequence

$$
Y' = Y^{(0)} \xrightarrow{w_1} Y^{(1)} \xrightarrow{w_2} \cdots
$$

where $Y^{(0)} = Y'$, and w_k is a pushout of the morphism $f^{(k)} = B' \otimes \sigma^k(f)$. The desired result now follows immediately from (4), (6) and (8). □

We are now ready to prove our main result:

Proof of Theorem 4.4.4.7. Consider the diagram

$$
\begin{array}{ccc}
\mathrm{N}(\mathrm{CAlg}(\mathbf{A})^c)[W'^{-1}] & \longrightarrow & \mathrm{CAlg}(\mathrm{N}(\mathbf{A}^c)[W^{-1}]) \\
& \searrow{\scriptstyle G} \qquad \swarrow{\scriptstyle G'} & \\
& \mathrm{N}(\mathbf{A}^c)[W^{-1}]. &
\end{array}
$$

It will suffice to show that this diagram satisfies the hypotheses of Corollary 6.2.2.14:

(a) The ∞-categories $\mathrm{N}(\mathrm{CAlg}(\mathbf{A})^c)[W'^{-1}]$ and $\mathrm{CAlg}(\mathrm{N}(\mathbf{A}^c)[W^{-1}])$ admit geometric realizations of simplicial objects. In fact, both of these ∞-categories are presentable. For $\mathrm{N}(\mathrm{CAlg}(\mathbf{A})^c)[W'^{-1}]$, this follows from Propositions 1.3.4.22 and 4.4.4.6. For $\mathrm{CAlg}(\mathrm{N}(\mathbf{A}^c)[W^{-1}])$, we first observe that $\mathrm{N}(\mathbf{A}^c)[W^{-1}]$ is presentable (Proposition 1.3.4.22) and that the tensor product preserves colimits separately in each variable (Corollary 4.1.4.8), and then apply Corollary 3.2.3.5.

(b) The functors G and G' admit left adjoints F and F'. The existence of a left adjoint to G' follows from Corollary 3.1.3.5, and a left adjoint to G is induced by the left Quillen functor $\mathbf{A} \to \mathrm{CAlg}(\mathbf{A})$.

(c) The functor G' is conservative and preserves geometric realizations of simplicial objects. This follows from Corollary 3.2.3.2 and Lemma 3.2.2.6.

(d) The functor G is conservative and preserves geometric realizations of simplicial objects. The first assertion is immediate from the definition of the weak equivalences in $\mathrm{CAlg}(\mathbf{A})$, and the second follows from Lemma 4.4.4.12.

(e) The canonical map $G' \circ F' \to G \circ F$ is an equivalence of functors. Unwinding the definitions, we must show that for every cofibrant object $C \in \mathbf{A}$, the free strictly commutative algebra $\coprod_n \mathrm{Sym}^n(C) \in \mathrm{CAlg}(\mathbf{A})$ is a free algebra generated by C, in the sense of Definition 3.1.3.1. In view of Proposition 3.1.3.11, it suffices to show that the colimit defining the total symmetric power $\coprod_n \mathrm{Sym}^n(C)$ in $\mathbf{A}$ is also a homotopy colimit. This follows immediately from part (3) of Lemma 4.4.4.11.

□

Chapter 5

Little Cubes and Factorizable Sheaves

Let X be a topological space equipped with a base point $*$. We let ΩX denote the loop space of X, which we will identify with the space of continuous map $p : [-1, 1] \to X$ such that $f(-1) = * = f(1)$. Given a pair of loops $p, q \in \Omega X$, we can define a composite loop $p \circ q$ by concatenating p with q: that is, we define $p \circ q : [-1, 1] \to X$ by the formula

$$(p \circ q)(t) = \begin{cases} q(2t + 1) & \text{if } -1 \le t \le 0 \\ p(2t - 1) & \text{if } 0 \le t \le 1. \end{cases}$$

This composition operation is associative up to homotopy, and endows the set of path components $\pi_0 \Omega X$ with the structure of a group: namely, the fundamental group $\pi_1(X, *)$. However, composition of paths is not strictly associative: given a triple of paths $p, q, r \in \Omega X$, we have

$$(p \circ (q \circ r))(t) = \begin{cases} r(4t + 3) & \text{if } -1 \le t \le \frac{-1}{2} \\ q(4t + 1) & \text{if } \frac{-1}{2} \le t \le 0 \\ p(2t - 1) & \text{if } 0 \le t \le 1. \end{cases} \qquad ((p \circ q) \circ r)(t) = \begin{cases} r(2t + 1) & \text{if } -1 \le t \le 0 \\ q(4t - 1) & \text{if } 0 \le t \le \frac{1}{2} \\ p(4t - 3) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

The paths $p \circ (q \circ r)$ and $(p \circ q) \circ r$ follow the same trajectories but are parametrized differently; they are homotopic but not identical.

One way to compensate for the failure of strict associativity is to consider not one composition operation but several. For every finite set S, let $\mathrm{Rect}((-1, 1) \times S, (-1, 1))$ denote the collection of finite sequences of maps $\{f_S : (-1, 1) \to (-1, 1)\}_{s \in S}$ with the following properties:

(a) For $s \ne t$, the maps f_s and f_t have disjoint images.

(b) For each $s \in S$, the map f_s is given by a formula $f_s(t) = at + b$ where $a > 0$.

If X is any pointed topological space, then there is an evident map

$$\theta : (\Omega X)^S \times \mathrm{Rect}((-1, 1) \times S, (-1, 1)) \to \Omega X,$$

given by the formula

$$\theta(\{p_s\}_{s \in S}, \{f_s\}_{s \in S})(t) = \begin{cases} p_s(t') & \text{if } t = f_s(t') \\ * & \text{otherwise.} \end{cases}$$

Each of the spaces $\mathrm{Rect}((-1, 1) \times S, (-1, 1))$ is equipped with a natural topology (with respect to which the map θ is continuous), and the collection of spaces $\{\mathrm{Rect}((-1, 1) \times S, (-1, 1))\}_I$ can be organized into a topological operad, which we will denote by $^t\mathbb{E}_1$. We can summarize the situation as follows:

($*$) For every pointed topological space X, the loop space ΩX carries an action of the topological operad $^t\mathbb{E}_1$.

Every point of $\mathrm{Rect}((-1,1) \times S, (-1,1))$ determines a linear ordering of the finite set S. Conversely, if we fix a linear ordering of S, then the corresponding subspace of $\mathrm{Rect}((-1,1) \times S, (-1,1))$ is contractible. In other words, there is a canonical homotopy equivalence of $\mathrm{Rect}((-1,1) \times S, (-1,1))$ with the (discrete) set of linear orderings of S. Together, these homotopy equivalences determine a weak equivalence of the topological operad $^t\mathbb{E}_1$ with the associative operad (Definition 4.1.1.1). Consequently, an action of the operad $^t\mathbb{E}_1$ can be regarded as a homotopy-theoretic substitute for an associative algebra structure. In other words, assertion ($*$) articulates the idea that the loop space ΩX is equipped with a multiplication which associative up to coherent homotopy.

If X is a pointed space, then we can consider also the k-fold loop space $\Omega^k X$, which we will identify with the space of all maps $f : [-1,1]^k \to X$ which carry the boundary of the cube $[-1,1]^k$ to the base point of X. If $k > 0$, then we can identify $\Omega^k X$ with $\Omega(\Omega^{k-1} X)$, so that $\Omega^k X$ is equipped with a coherently associative multiplication given by concatenation of loops. However, if $k > 1$, then the structure of $\Omega^k X$ is much richer. To investigate this structure, it is convenient to introduce a higher-dimensional version of the topological $^t\mathbb{E}_1$, the *little k-cubes operad* introduced by Boardman and Vogt ([19]). In §5.1, we will review the definition of this topological operad and study the properties of its operadic nerve, which we denote by $\mathbb{E}_k^\otimes$ and refer to as the ∞-*operad of little k-cubes*.

The structure of the ∞-operad $\mathbb{E}_k^\otimes$ is closely related to topology of manifolds. If M is a k-dimensional topological manifold, then we can associate to M an ∞-operad $\mathbb{E}_M^\otimes$, which is equivalent to $\mathbb{E}_k^\otimes$ in the special case where $M \simeq \mathbf{R}^k$ is a Euclidean space. If $\mathcal{C}$ is a symmetric monoidal ∞-category, then the (nonunital) $\mathbb{E}_M$-algebra objects of $\mathcal{C}$ can be identified with *factorizable cosheaves* on M with values in $\mathcal{C}$ (Theorem 5.3.4.10). In §5.3, we will discuss the notion of factorizable cosheaf together with the closely related theory of *topological chiral homology*. Our discussion makes use of a general construction which produces variants of the ∞-operad $\mathbb{E}_k^\otimes$, which we discuss in §5.2.

5.1   Definitions and Basic Properties

Our goal in this section is to study the ∞-categorical avatar of the little k-cubes operads introduced by Boardman and Vogt. We begin with some definitions.

Definition 5.1.0.1. Fix an integer $k \geq 0$. We let $\square^k = (-1,1)^k$ denote an open cube of dimension k. We will say that a map $f : \square^k \to \square^k$ is a *rectilinear embedding* if it is given by the formula

$$f(x_1, \ldots, x_k) = (a_1 x_1 + b_1, \ldots, a_k x_k + b_k)$$

for some real constants a_i and b_i, with $a_i > 0$. More generally, if S is a finite set, then we will say that a map $\square^k \times S \to \square^k$ is a *rectilinear embedding* if it is an open embedding whose restriction to each connected component of $\square^k \times S$ is rectilinear. Let $\mathrm{Rect}(\square^k \times S, \square^k)$ denote the collection of all rectitlinear embeddings from $\square^k \times S$ into $\square^k$. We will regard $\mathrm{Rect}(\square^k \times S, \square^k)$ as a topological space (it can be identified with an open subset of $(\mathbf{R}^{2k})^S$).

The spaces $\mathrm{Rect}(\square^k \times \{1, \ldots, n\}, \square^k)$ constitute the n-ary operations of a topological operad, which we will denote by $^t\mathbb{E}_k$ and refer to as the *little k-cubes operad*.

Definition 5.1.0.2. We define a topological category $^t\mathbb{E}_k^\otimes$ as follows:

(1) The objects of $^t\mathbb{E}_k^\otimes$ are the objects $\langle n \rangle \in \mathcal{F}\mathrm{in}_*$.

(2) Given a pair of objects $\langle m \rangle, \langle n \rangle \in^t \mathbb{E}_k^\otimes$, a morphism from $\langle m \rangle$ to $\langle n \rangle$ in $^t\mathbb{E}_k^\otimes$ consists of the following data:

– A morphism $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$.

– For each $j \in \langle n \rangle^{\circ}$ a rectilinear embedding $\square^k \times \alpha^{-1}\{j\} \to \square^k$.

(3) For every pair of objects $\langle m \rangle, \langle n \rangle \in {}^t \mathbb{E}_k^{\otimes}$, we regard $\mathrm{Hom}_{{}^t \mathbb{E}_k^{\otimes}}(\langle m \rangle, \langle n \rangle)$ as endowed with the topology induced by the presentation

$$\mathrm{Hom}_{{}^t \mathbb{E}_k^{\otimes}}(\langle m \rangle, \langle n \rangle) = \coprod_{f : \langle m \rangle \to \langle n \rangle} \prod_{1 \leq j \leq n} \mathrm{Rect}(\square^k \times f^{-1}\{j\}, \square^k).$$

(4) Composition of morphisms in ${}^t \mathbb{E}_k^{\otimes}$ is defined in the obvious way.

We let $\mathbb{E}_k^{\otimes}$ denote the nerve of the topological category ${}^t \mathbb{E}_k^{\otimes}$.

Corollary T.1.1.5.12 implies that $\mathbb{E}_k^{\otimes}$ is an ∞-category. There is an evident forgetful functor from ${}^t \mathbb{E}_k^{\otimes}$ to the (discrete) category $\mathcal{F}\mathrm{in}_*$, which induces a functor $\mathbb{E}_k^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

Proposition 5.1.0.3. *The functor* $\mathbb{E}_k^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *exhibits* $\mathbb{E}_k^{\otimes}$ *as an* ∞-*operad.*

Proof. We have a canonical isomorphism $\mathbb{E}_k^{\otimes} \simeq \mathrm{N}^{\otimes}(\mathcal{O})$, where $\mathcal{O}$ denotes the simplicial colored operad having a single object $\square^k$ with $\mathrm{Mul}_{\mathcal{O}}(\{\square^k\}_{i \in I}, \square^k) = \mathrm{Sing}\,\mathrm{Rect}(\square^k \times I, \square^k)$. Since $\mathcal{O}$ is a fibrant simplicial colored operad, $\mathbb{E}_k^{\otimes}$ is an ∞-operad by virtue of Proposition 2.1.1.27. $\square$

Definition 5.1.0.4. We will refer to the ∞-operad $\mathbb{E}_k^{\otimes}$ as the ∞-*operad of little k-cubes.*

Remark 5.1.0.5. Let $\mathrm{Env}(\mathbb{E}_k)^{\otimes}$ be the symmetric monoidal envelope of $\mathbb{E}_k^{\otimes}$, as defined in §2.2.4. We can describe the underlying ∞-category $\mathrm{Env}(\mathbb{E}_k)$ informally as follows: its objects are topological spaces of the form $\coprod_{i \in I} \square^k$, and its morphisms are given by embeddings which are rectilinear on each component. The symmetric monoidal structure on $\mathrm{Env}(\mathbb{E}_k)$ is given by disjoint union.

Example 5.1.0.6. Suppose that $k = 0$. Then $\square^k$ consists of a single point, and the only rectilinear embedding from $\square^k$ to itself is the identity map. A finite collection $\{f_i : \square^k \to \square^k\}_{i \in I}$ of rectlinear embeddings have disjoint images if and only if the index set I has at most one element. It follows that ${}^t \mathbb{E}_k$ is isomorphic (as a topological category) to the subcategory of $\mathcal{F}\mathrm{in}_*$ spanned by the injective morphisms in $\mathcal{F}\mathrm{in}_*$. We conclude that $\mathbb{E}_k^{\otimes}$ is isomorphic to the ∞-operad $\mathbb{E}_0^{\otimes} \subseteq \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ introduced in Example 2.1.1.19.

Example 5.1.0.7. Suppose that $k = 1$, so that we can identify the cube $\square^k$ with the interval $(-1, 1)$. Every rectangular embedding $(-1, 1) \times I \to (-1, 1)$ determines a linear ordering of the set I, where $i < j$ if and only if $f(t, i) < f(t', j)$ for all $t, t' \in (-1, 1)$. This construction determines a composition of the space $\mathrm{Rect}((-1, 1) \times I, (-1, 1))$ into components $\mathrm{Rect}_{<}((-1, 1) \times I, (-1, 1))$, where $<$ ranges over all linear orderings on I. Each of the spaces $\mathrm{Rect}_{<}((-1, 1) \times I, (-1, 1))$ can be realized as a nonempty convex subset of $(\mathbf{R}^2)^I$ and is therefore contractible. It follows that $\mathrm{Rect}((-1, 1) \times I, (-1, 1))$ is homotopy equivalent to the discrete set of all linear orderings on I.

Using these homotopy equivalences, we obtain a weak equivalence of topological categories ${}^t \mathbb{E}_1^{\otimes} \to \mathbf{Ass}^{\otimes}$, where $\mathbf{Ass}^{\otimes}$ is the category of Definition 4.1.1.3. Passing to the homotopy coherent nerves, we obtain an equivalence of ∞-operads $\mathbb{E}_1^{\otimes} \simeq \mathcal{A}\mathrm{ss}^{\otimes}$.

Example 5.1.0.7 implies that, when $k = 1$, the theory of $\mathbb{E}_k$-algebras reduces to the theory of associative algebras studied in Chapter 4. The ∞-operads $\mathbb{E}_k^{\otimes}$ have a related interpretation when $k > 1$: namely, giving an $\mathbb{E}_k$-algebra object of a symmetric monoidal ∞-category $\mathcal{C}$ is equivalent to giving an object $A \in \mathcal{C}$ which is equipped with k different associative multiplications, which are compatible with one another in a suitable sense. This is a consequence of Theorem 5.1.2.2, which asserts that for $k, k' \geq 0$ we can identify $\mathbb{E}_{k+k'}^{\otimes}$ with the tensor product of the ∞-operads $\mathbb{E}_k^{\otimes}$ and $\mathbb{E}_{k'}^{\otimes}$. We will prove this result in §5.1.2.

If X is a pointed topological space, then the k-fold loop space $\Omega^k X$ carries an action of the topological operad ${}^t \mathbb{E}_k$. This observation admits the following converse, which highlights the importance of the operads ${}^t \mathbb{E}_k$ in algebraic topology:

Theorem 5.1.0.8 (May)**.** *Let Y be a topological space equipped with an action of the little cubes operad ${}^t\mathbb{E}_k$. If $k > 0$, assume that $\pi_0 Y$ is a group under the induced monoid structure (see Definition 5.1.3.2). Then Y is weakly homotopy equivalent to $\Omega^k X$, for some pointed topological space X.*

In §5.1.3, we will prove a variant of this result, which describes $\mathbb{E}_k$-algebra objects of the ∞-category $\mathcal{S}$ of spaces (see Theorem 5.1.3.6). Theorem 5.1.0.8 can be interpreted as saying that, in some sense, the topological operad ${}^t\mathbb{E}_k$ encodes precisely the structure that a k-fold loop space should be expected to possess. In the case $k = 1$, we recover a familiar notion: namely, a (coherently) associative multiplication. This makes sense in a variety of contexts outside of algebraic topology. For example, one can consider associative algebra objects in the category of abelian groups (that is, associative rings), or more generally associative algebras in any monoidal ∞-category (as in Chapter 4). A similar phenomenon occurs for larger values of k: that is, it is interesting to study algebras over the topological operads ${}^t\mathbb{E}_k$ in categories other than the category of topological spaces. For example, the theory of *structured ring spectra* (which is the subject of Chapter 8) is obtained by considering $\mathbb{E}_k$-algebras in the setting of spectra.

For algebraic applications, it is important to consider not only $\mathbb{E}_k$-algebras but also modules over them. Here the essential observation is that the ∞-operads $\mathbb{E}_k^{\otimes}$ are coherent, so that the general theory of Chapter 3 gives a robust theory of modules. We will prove this coherence result in §5.1.1, using some general observations relating the spaces of operations in $\mathbb{E}_k^{\otimes}$ to configuration spaces of points on manifolds. It follows that for every $\mathbb{E}_k$-algebra object A of a sufficiently nice symmetric monoidal ∞-category $\mathcal{C}$, we can define an ∞-category $\mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})$ which is equipped with an $\mathbb{E}_k$-monoidal structure. In §5.1.4, we will show that the tensor product in $\mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})$ can be described in terms of the relative tensor product over A which was studied in §4.3.

In §5.1.5, we will study the tensor product operation on $\mathbb{E}_k$-algebras themselves. If $\mathcal{C}$ is a symmetric monoidal ∞-category, then the ∞-category $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$ inherits a symmetric monoidal structure (given by pointwise tensor product). We will see that, at least when restricted to free algebras, this tensor product is closely related to the coproduct on $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$ (Theorem 5.1.5.1).

5.1.1   Little Cubes and Configuration Spaces

Our main goal in this section is to prove the following result:

Let k be a nonnegative integer and let $\mathbb{E}_k^{\otimes}$ be the ∞-operad of little k-cubes introduced in Definition 5.1.0.2. If $k = 1$, then $\mathbb{E}_k^{\otimes}$ is equivalent to the associative ∞-operad $\mathcal{A}ss^{\otimes}$ (Example 5.1.0.7), which is coherent by Proposition 4.1.1.16. Our ultimate goal in this section is to prove that this is general phenomenon:

Theorem 5.1.1.1. *Let $k \geq 0$ be a nonnegative integer. Then the little cubes ∞-operad $\mathbb{E}_k^{\otimes}$ is coherent.*

In order to prove Theorem 5.1.1.1, we will need a good understanding of the spaces of operations in the ∞-operad $\mathbb{E}_k^{\otimes}$. By construction, these can be realized as spaces of rectilinear embeddings between cubes. However, for many purposes it is convenient to replace these embedding spaces by configuration spaces.

Definition 5.1.1.2. Let I be a finite set and let M be a topological manifold. We let $\mathrm{Conf}(I; M)$ denote the space of all injective maps from I into M, which we identify with a subspace of M^I. We refer to $\mathrm{Conf}(I; M)$ as the *configuration space* of maps $I \to M$.

Lemma 5.1.1.3. *Let I and J be finite sets, and suppose we are given an injective map $p : J \to \square^k$. Let $U \subseteq \mathrm{Rect}(\square^k \times (I \coprod J), \square^k)$ be the open subset consisting of those rectilinear embeddings $f : \square^k \times (I \coprod J) \to \square^k$ such that, for every $j \in J$, we have $p(j) \in f(\square^k \times \{j\})$. Then evaluation at the origin $0 \in \square^k$ determines a homotopy equivalence $U \to \mathrm{Conf}(I, \square^k - p(J))$.*

Proof. Let $\overline{U}$ denote the collection of all maps $\square^k \times (I \coprod J) \to \square^k$ which which either belong to U, or factor as a composition

$$\square^k \times (I \coprod J) \to I \coprod J \xrightarrow{\overline{p}} \square^k$$

where $\bar{p}$ is an injection with $\bar{p}|J = p$. Then θ factors as a composition

$$U \xrightarrow{\theta'} \overline{U} \xrightarrow{\theta''} \mathrm{Conf}(I; \square^k - p(J))$$

where θ' is the open inclusion and θ'' is given by evaluation at the origin $0 \in \square^k$. We claim that both θ' and θ'' are homotopy equivalences:

(i) For every map $f \in \overline{U}$, let $\epsilon(f)$ denote the infimum over $a, b \in I \coprod J$ of the distance from $f(\square^k \times \{a\})$ to $f(\square^k \times \{a\})$ and the distance from $f(\square^k \times \{a\})$ to the boundary of $\square^k$. We then define a family of maps $\{f_t\}_{t \in [0,1]}$ by the formula

$$f_t(x_1, \ldots, x_k, i) = f(x_1, \ldots, x_k, i) + \frac{t\epsilon(f)}{2k}(x_1, \ldots, x_k).$$

This construction determines a map $H : \overline{U} \times [0,1] \to \overline{U}$ such that $H|(\overline{U} \times \{0\})$ is the identity map and H carries $(\overline{U} \times (0,1])$ into U. It follows that $H|(\overline{U} \times \{1\})$ is a homotopy inverse to θ'.

(ii) There is an evident homeomorphism $\mathrm{Conf}(I; \square^k - p(J)) \simeq \overline{U} - U$, which determines a map $j : \mathrm{Conf}(I, \square^k - p(J)) \to \overline{U}$. We claim that j is a homotopy equivalence. Indeed, there is a deformation retraction of $\overline{U}$ onto $\overline{U} - U$, which carries a map $f : \square^k \times (I \coprod J) \to \square^k$ to the family of maps $\{f_t : \square^k \times (I \coprod J) \to \square^k\}_{t \in [0,1]}$ given by the formula

$$f_t(x_1, \ldots, x_k, a) = \begin{cases} f(tx_1, \ldots, tx_k, a) & \text{if } a \in I \\ tf(x_1, \ldots, x_k, b) + (1-t)p(b) & \text{if } b \in J. \end{cases}$$

Since θ'' is a left inverse to j, it follows that θ'' is a homotopy equivalence.

$\square$

We can use the relationship between rectilinear embedding spaces and configuration spaces to establish some basic connectivity properties of the ∞-operads $\mathbb{E}_k^\otimes$:

Proposition 5.1.1.4. *Let $k \geq 0$. For every pair of integers $m, n \geq 0$, the map of topological spaces $\mathrm{Map}_{{}^t\mathbb{E}_k^\otimes}(\langle m \rangle, \langle n \rangle) \to \mathrm{Hom}_{\mathcal{F}\mathrm{in}_*}(\langle m \rangle, \langle n \rangle)$ is $(k-1)$-connective.*

Proof. Unwinding the definitions, this is equivalent to the requirement that for every finite set I, the space of rectinilinear embeddings $\mathrm{Rect}(\square^k \times I, \square^k)$ is $(k-1)$-connective. This space is homotopy equivalent (via evaluation at the origin) to the configuration space $\mathrm{Conf}(I; \square^k)$ of injective maps $I \hookrightarrow \square^k$ (Lemma 5.1.1.3). We will prove more generally that $\mathrm{Conf}(J, \square^k - F)$ is $(k-1)$-connective, where J is any finite set and F is any finite subset of $\square^k$. The proof proceeds by induction on the number of elements of J. If $J = \emptyset$, then $\mathrm{Conf}(J, \square^k - F)$ consists of a single point and there is nothing to prove. Otherwise, choose an element $j \in J$. Evaluation at j determines a Serre fibration $\mathrm{Conf}(J, \square^k - F) \to \square^k - F$, whose fiber over a point x is the space $\mathrm{Conf}(J - \{j\}, \square^k - (F \cup \{x\}))$. The inductive hypothesis guarantees that these fibers are $(k-1)$-connective. Consequently, to show that $\mathrm{Conf}(J, \square^k - F)$ is $(k-1)$-connective, it suffices to show that $\square^k - F$ is $(k-1)$-connective. In other words, we must show that for $m < k$, every map $g_0 : S^{m-1} \to \square^k - F$ can be extended to a map $g : D^m \to \square^k - F$, where D^m denotes the unit disk of dimension m and S^{m-1} its boundary sphere. Without loss of generality, we may assume that g_0 is smooth. Since $\square^k$ is contractible, we can extend g_0 to a map $g : D^m \to \square^k$, which we may also assume to be smooth and transverse to the submanifold $F \subseteq \square^k$. Since F has codimension k in $\square^k$, $g^{-1}F$ has codimension k in D^m, so that $g^{-1}F = \emptyset$ (since $m < k$) and g factors through $\square^k - F$, as desired. $\square$

For each $k \geq 0$, there is a stabilization functor ${}^t\mathbb{E}_k \to {}^t\mathbb{E}_{k+1}$ which is the identity on objects and is given on morphisms by taking the product with the interval $(-1, 1)$. This functor induces a map of ∞-operads $\mathbb{E}_k^\otimes \to \mathbb{E}_{k+1}^\otimes$. Proposition 5.1.1.4 immediately implies the following:

Corollary 5.1.1.5. *The colimit of the sequence of ∞-operads*

$$\mathbb{E}_0^{\otimes} \to \mathbb{E}_1^{\otimes} \to \mathbb{E}_2^{\otimes} \to \dots$$

is equivalent to the commutative ∞-operad $\mathrm{Comm}^{\otimes} = \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

Notation 5.1.1.6. Motivated by Corollary 5.1.1.5, we adopt the following convention: when $k = \infty$, we let $\mathbb{E}_k^{\otimes}$ denote the commutative ∞-operad $\mathrm{Comm}^{\otimes}$ (so that $\mathbb{E}_\infty^{\otimes} \simeq \varinjlim_{j \geq 0} \mathbb{E}_j^{\otimes}$).

Consequently, if $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category, then the ∞-category $\mathrm{CAlg}(\mathcal{C})$ of commutative algebra objects of $\mathcal{C}$ can be identified with the homotopy limit of the tower of ∞-categories $\{\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})\}_{k \geq 0}$. In many situations, this tower actually stabilizes at some finite stage:

Corollary 5.1.1.7. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Let $n \geq 1$, and assume that the underlying ∞-category $\mathcal{C}$ is equivalent to an n-category (that is, the mapping spaces $\mathrm{Map}_{\mathcal{C}}(X, Y)$ are $(n-1)$-truncated for every pair of objects $X, Y \in \mathcal{C}$; see §T.2.3.4). Then the map $\mathbb{E}_k^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ induces an equivalence of ∞-categories $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$ for $k > n$.*

Proof. Let C and D be objects of $\mathcal{C}^{\otimes}$, corresponding to finite sequences of objects $(X_1, \dots, X_m)$ and $(Y_1, \dots, Y_{m'})$ of objects of $\mathcal{C}$. Then $\mathrm{Map}_{\mathcal{C}^{\otimes}}(C, D)$ can be identified with the space

$$\coprod_{\alpha : \langle m \rangle \to \langle n \rangle} \prod_{1 \leq j \leq m'} \mathrm{Map}_{\mathcal{C}}(\otimes_{\alpha(i) = j} X_i, Y_j),$$

and is therefore also $(n-1)$-truncated. Consequently, $\mathcal{C}^{\otimes}$ is equivalent to an n-category. Proposition 5.1.1.4 implies that the forgetful functor $\mathbb{E}_k^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ induces an equivalence of the underlying homotopy n-categories, and therefore induces an equivalence $\theta : \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N}(\mathcal{F}\mathrm{in}_*), \mathcal{C}^{\otimes}) \to \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathbb{E}_k^{\otimes}, \mathcal{C}^{\otimes})$. The desired result now follows from the observation that a map $A \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N}(\mathcal{F}\mathrm{in}_*), \mathcal{C}^{\otimes})$ is a commutative algebra object of $\mathcal{C}$ if and only if $\theta(A)$ is an $\mathbb{E}_k$-algebra object of $\mathcal{C}$. $\square$

We now turn to the proof of Theorem 5.1.1.1. We first formulate a general coherence criterion for ∞-operads that are obtained as the operadic nerves of simplicial colored operads (Proposition 5.1.1.11). We will then use our analysis of configuration spaces to show that this criterion applies in the case of the ∞-operad $\mathbb{E}_k^{\otimes}$.

Notation 5.1.1.8. Let $\mathcal{O}$ be a simplicial operad (that is, a simplicial colored operad having a single distinguished object), and let $\mathcal{O}^{\otimes}$ be the simplicial category described in Notation 2.1.1.22: the objects of $\mathcal{O}^{\otimes}$ are objects $\langle n \rangle \in \mathcal{F}\mathrm{in}_*$, and the morphisms spaces $\mathcal{O}^{\otimes}$ are given by the formula

$$\mathrm{Map}_{\mathcal{O}^{\otimes}}(\langle m \rangle, \langle n \rangle) = \coprod_{\alpha : \langle m \rangle \to \langle n \rangle} \prod_{1 \leq i \leq n} \mathrm{Mul}_{\mathcal{O}}(\alpha^{-1}\{i\}, \{i\})$$

where α ranges over all maps $\langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$. We will say that a morphism in $\mathcal{O}^{\otimes}$ is *active* if its image in $\mathcal{F}\mathrm{in}_*$ is active, and we let $\mathrm{Map}_{\mathcal{O}^{\otimes}}^{\mathrm{act}}(\langle m \rangle, \langle n \rangle)$ denote the summand of $\mathrm{Map}_{\mathcal{O}^{\otimes}}(\langle m \rangle, \langle n \rangle)$ spanned by the active morphisms.

We will say that $\mathcal{O}$ is *unital* if $\mathrm{Mul}_{\mathcal{O}}(\emptyset, \{0\})$ is isomorphic to Δ^0; in this case, every semi-inert morphism $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$ can be lifted *uniquely* to a morphism $\overline{\alpha}$ in $\mathcal{O}^{\otimes}$. In particular, the canonical inclusion $i : \langle m \rangle \to \langle m + 1 \rangle$ admits a unique lift $\overline{i} : \langle m \rangle \to \langle m + 1 \rangle$ in $\mathcal{O}^{\otimes}$. Composition with $\overline{i}$ induces a map of simplicial sets

$$\theta : \mathrm{Map}_{\mathcal{O}^{\otimes}}^{\mathrm{act}}(\langle m + 1 \rangle, \langle n \rangle) \to \mathrm{Map}_{\mathcal{O}^{\otimes}}^{\mathrm{act}}(\langle m \rangle, \langle n \rangle).$$

For every active morphism $f : \langle m \rangle \to \langle n \rangle$ in $\mathcal{O}^{\otimes}$, we will denote the simplicial set $\theta^{-1}\{f\}$ by $\mathrm{Ext}_\Delta(f)$; we will refer to $\mathrm{Ext}_\Delta(f)$ as the *space of strict extensions of f*.

Construction 5.1.1.9. Let $\mathcal{O}$ be a fibrant simplicial operad, and let $N^{\otimes}(\mathcal{O})$ be the underlying ∞-operad (Definition 2.1.1.23). Suppose we are given a sequence of active morphisms

$$\langle m_0 \rangle \xrightarrow{f_1} \langle m_1 \rangle \xrightarrow{f_2} \ldots \xrightarrow{f_n} \langle m_n \rangle$$

in the simplicial category $\mathcal{O}^{\otimes}$. This sequence determines an n-simplex σ of $N(\mathcal{O})^{\otimes}$. Let $S \subseteq [n]$ be a proper nonempty subset having maximal element $j-1$. We define a map of simplicial sets $\theta : \mathrm{Ext}_{\Delta}(f_j) \to \mathrm{Ext}(\sigma, S)$ as follows: for every k-simplex $\tau : \Delta^k \to \mathrm{Ext}_{\Delta}(f_j)$, $\theta(\tau)$ is a k-simplex of $\mathrm{Ext}(\sigma, S)$ corresponding to a map of simplicial categories $\psi : \mathfrak{C}[\Delta^n \times \Delta^{k+1}] \to \mathcal{O}^{\otimes}$, which may be described as follows:

(*i*) On objects, the functor ψ is given by the formula

$$\psi(n', k') = \begin{cases} \langle m_{n'} \rangle & \text{if } k' = 0 \text{ or } n' \notin S \\ \langle m_{n'} + 1 \rangle & \text{otherwise.} \end{cases}$$

(*ii*) Fix a pair of vertices $(n', k'), (n'', k'') \in \Delta^n \times \Delta^{k+1}$. Then ψ induces a map of simplicial sets $\phi : \mathrm{Map}_{\mathfrak{C}[\Delta^n \times \Delta^{k+1}]}((n', k'), (n'', k'')) \to \mathrm{Map}_{\mathcal{O}^{\otimes}}(\psi(n', k'), \psi(n'', k''))$. The left hand side can be identified with the nerve of the partially ordered set P of chains

$$(n', k') = (n_0, k_0) \leq (n_1, k_1) \leq \ldots \leq (n_p, k_p) = (n'', k'')$$

in $[n] \times [k+1]$. If $\psi(n', k') = \langle m_{n'} \rangle$ or $\psi(n'', k'') = \langle m_{n''} + 1 \rangle$, then ϕ is given by the constant map determined by $f_{n''} \circ \cdots \circ f_{n'+1}$. Otherwise, ϕ is given by composing the morphisms $f_{j-1} \circ \cdots \circ f_{n'+1}$ and $f_{n''} \circ \cdots \circ f_{j+1}$ with the map

$$N(P) \xrightarrow{\phi_0} \Delta^k \xrightarrow{\tau} \mathrm{Ext}_{\Delta}(f_j) \to \mathrm{Map}_{\mathcal{O}^{\otimes}}(\langle m_{j-1} + 1 \rangle, \langle m_j \rangle),$$

where ϕ_0 is induced by the map of partially ordered sets $P \to [k]$ which carries a chain $(n', k') = (n_0, k_0) \leq (n_1, k_1) \leq \ldots \leq (n_p, k_p) = (n'', k'')$ to the supremum of the set $\{k_i - 1 : n_i \in S\} \subseteq [k]$.

Remark 5.1.1.10. In the situation of Construction 5.1.1.9, the simplicial set $\mathrm{Ext}(\sigma, S)$ can be identified with the homotopy fiber of the map

$$\beta : \mathrm{Map}^{\mathrm{act}}_{\mathcal{O}^{\otimes}}(\langle m_{j-1} + 1 \rangle, \langle m_j \rangle) \to \mathrm{Map}^{\mathrm{act}}_{\mathcal{O}^{\otimes}}(\langle m_{j-1} \rangle, \langle m_j \rangle),$$

while $\mathrm{Ext}_{\Delta}(f_j)$ can be identified with the actual fiber of β. The map θ of Construction 5.1.1.9 can be identified with the canonical map from the actual fiber to the homotopy fiber.

Proposition 5.1.1.11. *Let $\mathcal{O}$ be a fibrant simplicial operad, and assume that every morphism in the underlying simplicial category $\mathcal{O} = \mathcal{O}^{\otimes}_{\langle 1 \rangle}$ admits a homotopy inverse. Suppose that, for every pair active morphisms $f_0 : \langle m \rangle \to \langle n \rangle$ and $g_0 : \langle n \rangle \to \langle 1 \rangle$ in $\mathcal{O}^{\otimes}$, there exist morphisms $f : \langle m \rangle \to \langle n \rangle$, $h : \langle n \rangle \to \langle n \rangle$, and $g : \langle n \rangle \to \langle 1 \rangle$ satisfying the following conditions:*

(*i*) *The map f is homotopic to f_0, the map g is homotopic to g_0, and the map h is homotopic to $\mathrm{id}_{\langle n \rangle}$.*

(*ii*) *Each of the sequences*

$$\mathrm{Ext}_{\Delta}(h) \to \mathrm{Map}^{\mathrm{act}}_{\mathcal{O}^{\otimes}}(\langle n+1 \rangle, \langle n \rangle) \to \mathrm{Map}^{\mathrm{act}}_{\mathcal{O}^{\otimes}}(\langle n \rangle, \langle n \rangle)$$

$$\mathrm{Ext}_{\Delta}(g \circ h) \to \mathrm{Map}^{\mathrm{act}}_{\mathcal{O}^{\otimes}}(\langle n+1 \rangle, \langle 1 \rangle) \to \mathrm{Map}^{\mathrm{act}}_{\mathcal{O}^{\otimes}}(\langle n \rangle, \langle 1 \rangle)$$

$$\mathrm{Ext}_{\Delta}(h \circ f) \to \mathrm{Map}^{\mathrm{act}}_{\mathcal{O}^{\otimes}}(\langle m+1 \rangle, \langle n \rangle) \to \mathrm{Map}^{\mathrm{act}}_{\mathcal{O}^{\otimes}}(\langle m \rangle, \langle n \rangle)$$

$$\mathrm{Ext}_{\Delta}(g \circ h \circ f) \to \mathrm{Map}^{\mathrm{act}}_{\mathcal{O}^{\otimes}}(\langle m+1 \rangle, \langle 1 \rangle) \to \mathrm{Map}^{\mathrm{act}}_{\mathcal{O}^{\otimes}}(\langle m \rangle, \langle 1 \rangle)$$

is a homotopy fiber sequence.

(iii) *The diagram*

$$\begin{array}{ccc} \mathrm{Ext}_\Delta(h) & \longrightarrow & \mathrm{Ext}_\Delta(g \circ h) \\ \downarrow & & \downarrow \\ \mathrm{Ext}_\Delta(h \circ f) & \longrightarrow & \mathrm{Ext}_\Delta(g \circ h \circ f) \end{array}$$

is a homotopy pushout square of simplicial sets.

Then the ∞-operad $\mathrm{N}^{\otimes}(\mathcal{O})$ is coherent.

Proof. It is clear that $\mathrm{N}^{\otimes}(\mathcal{O})$ satisfies conditions (1) and (2) of Definition 3.3.1.9. We will show that condition (3) is also satisfied. Suppose we are given a degenerate 3-simplex σ :

$$\begin{array}{ccc} & \langle n \rangle \xrightarrow{\;\;g_0\;\;} \langle 1 \rangle & \\ {\scriptstyle f_0}\nearrow & {\scriptstyle \mathrm{id}} \quad {\scriptstyle g_0}\nearrow & \\ \langle m \rangle \xrightarrow{\;\;f_0\;\;} \langle n \rangle & & \end{array}$$

in $\mathrm{N}(\mathcal{O})^{\otimes}$, where f and g are active. We wish to show that the diagram

$$\begin{array}{ccc} \mathrm{Ext}(\sigma, \{0,1\}) & \longrightarrow & \mathrm{Ext}(\sigma|\Delta^{\{0,1,3\}}, \{0,1\}) \\ \downarrow & & \downarrow \\ \mathrm{Ext}(\sigma|\Delta^{\{0,2,3\}}, \{0\}) & \longrightarrow & \mathrm{Ext}(\sigma|\Delta^{\{0,3\}}, \{0\}) \end{array}$$

is a homotopy pushout square of Kan complexes. In proving this, we are free to replace σ by any equivalent diagram $\sigma' : \Delta^3 \to \mathrm{N}^{\otimes}(\mathcal{O})$. We may therefore assume that σ' is determined by a triple of morphisms $f : \langle m \rangle \to \langle n \rangle$, $h : \langle n \rangle \to \langle n \rangle$, and $g : \langle n \rangle \to \langle 1 \rangle$ satisfying conditions (ii) and (iii) above. Using Remark 5.1.1.10, we see that Construction 5.1.1.9 determines a weak homotopy equivalence between the diagrams

$$\begin{array}{ccc} \mathrm{Ext}_\Delta(h) & \longrightarrow & \mathrm{Ext}_\Delta(g \circ h) \\ \downarrow & & \downarrow \\ \mathrm{Ext}_\Delta(h \circ f) & \longrightarrow & \mathrm{Ext}_\Delta(g \circ h \circ f) \end{array} \qquad \begin{array}{ccc} \mathrm{Ext}(\sigma, \{0,1\}) & \longrightarrow & \mathrm{Ext}(\sigma|\Delta^{\{0,1,3\}}, \{0,1\}) \\ \downarrow & & \downarrow \\ \mathrm{Ext}(\sigma|\Delta^{\{0,2,3\}}, \{0\}) & \longrightarrow & \mathrm{Ext}(\sigma|\Delta^{\{0,3\}}, \{0\}). \end{array}$$

Since the diagram on the left is a homotopy pushout square by virtue of (iii), the diagram on the right is also a homotopy pushout square. $\square$

Proof of Theorem 5.1.1.1. Let $\mathcal{O} = \mathrm{Sing}^t \, \mathbb{E}_k$ denote the simplicial operad associated to the topological operad $^t\mathbb{E}_k$. We will say that a rectilinear embedding $f \in \mathrm{Rect}(\square^k \times \langle n \rangle^{\circ}, \square^k)$ is *generic* if f can be extended to an embedding $\overline{f} : \overline{\square^k} \times \langle n \rangle^{\circ} \to \square^k$, where $\overline{\square^k} = [-1,1]^k$ is a closed cube of dimension k. We will say that an active morphism $f : \langle n \rangle \to \langle m \rangle$ in $\mathcal{O}^{\otimes}$ is *generic* if it corresponds to a sequence of m rectilinear embeddings which are generic.

    We observe the following:

(a) If f is generic, then the difference $\square^k - \overline{f}(\overline{\square^k} \times \langle n \rangle^{\circ})$ is homotopy equivalent to $\square^k - f(\{0\} \times \langle n \rangle^{\circ})$. It follows that the sequence $\mathrm{Ext}_\Delta(f) \to \mathrm{Map}_{\mathcal{O}^{\otimes}}(\langle n+1 \rangle, \langle 1 \rangle) \to \mathrm{Map}_{\mathcal{O}^{\otimes}}(\langle n \rangle, \langle 1 \rangle)$ is homotopy equivalent to the fiber sequence of configuration spaces (see Lemma 5.1.3)

$$\square^k - f(\{0\} \times \langle n \rangle^{\circ}) \to \mathrm{Conf}(\langle n+1 \rangle^{\circ}, \square^k) \to \mathrm{Conf}(\langle n \rangle^{\circ}, \square^k),$$

   hence also a homotopy fiber sequence. More generally, if $f : \langle n \rangle \to \langle m \rangle$ is generic, then

$$\mathrm{Ext}_\Delta(f) \to \mathrm{Map}^{\mathrm{act}}_{\mathcal{O}^{\otimes}}(\langle n+1 \rangle, \langle m \rangle) \to \mathrm{Map}^{\mathrm{act}}_{\mathcal{O}^{\otimes}}(\langle n \rangle, \langle m \rangle),$$

   is a fiber sequence.

(b) Every rectlinear embedding $f_0 \in \text{Rect}(\square^k \times \langle n \rangle^\circ, \square^k)$ is homotopic to a generic rectilinear embedding f (for example, we can take f to be the composition of f_0 with the "contracting" map $\square^k \times \langle n \rangle^\circ \simeq (\frac{-1}{2}, \frac{1}{2})^k \times \langle n \rangle^\circ \hookrightarrow \square^k \times \langle n \rangle^\circ)$. Similarly, every active morphism in $\mathcal{O}^\otimes$ is homotopic to a generic morphism.

(c) The collection of generic morphisms in $\mathcal{O}^\otimes$ is stable under composition.

To prove that $\mathbb{E}_k^\otimes$ is coherent, it will suffice to show that the simplicial operad $\mathcal{O}$ satisfies the criteria of Proposition 5.1.1.11. It is clear that every map in $\mathcal{O}$ admits a homotopy inverse (in fact, every rectilinear embedding from $\square^k$ to itself is homotopic to the identity). In view of (a), (b), and (c) above, it will suffice to show that the diagram

$$
\begin{array}{ccc}
\text{Ext}_\Delta(h) & \longrightarrow & \text{Ext}_\Delta(g \circ h) \\
\downarrow & & \downarrow \\
\text{Ext}_\Delta(h \circ f) & \longrightarrow & \text{Ext}_\Delta(g \circ h \circ f)
\end{array}
$$

is a homotopy pushout square for every triple of active morphisms

$$
\langle m \rangle \xrightarrow{f} \langle n \rangle \xrightarrow{h} \langle n \rangle \xrightarrow{g} \langle 1 \rangle
$$

in $\mathcal{O}^\otimes$, provided that each of the underlying rectilinear embeddings is generic.

Let $U_0 \subseteq U_1 \subseteq U_2$ be the images of $g \circ h \circ f$, $g \circ h$, and g, respectively. Let $\overline{U}_i$ denote the closure of U_i. We now set

$$
V = \square^k - \overline{U}_1 \qquad W = U_2 - \overline{U}_0.
$$

Note that $V \cup W = \square^k - \overline{U}_0$ and $V \cap W = U_2 - \overline{U}_1$. The argument of Lemma 5.1.1.3 shows that evaluation at the origin of $\square^k$ determines weak homotopy equivalences

$$
\text{Ext}_\Delta(h) \to \text{Sing}(V \cap W) \qquad \text{Ext}_\Delta(g \circ h) \to \text{Sing}(V)
$$

$$
\text{Ext}_\Delta(h \circ f) \to \text{Sing}(W) \qquad \text{Ext}_\Delta(g \circ h \circ f) \to \text{Sing}(W \cup V).
$$

It will therefore suffice to show that the diagram

$$
\begin{array}{ccc}
\text{Sing}(V \cap W) & \longrightarrow & \text{Sing}(V) \\
\downarrow & & \downarrow \\
\text{Sing}(W) & \longrightarrow & \text{Sing}(W \cup V)
\end{array}
$$

is a homotopy pushout square of Kan complexes, which follows from Theorem A.3.1. $\qquad \square$

5.1.2 The Additivity Theorem

If K is a pointed topological space, then the k-fold loop space $\Omega^k(K)$ carries an action of the (topological) little cubes operad ${}^t\mathbb{E}_k$. Passing to singular complexes, we deduce that if $X \in \mathcal{S}_*$, then the k-fold loop space $\Omega^k(X)$ can be promoted to an $\mathbb{E}_k^\otimes$-algebra object of the ∞-category $\mathcal{S}$ of spaces. The work of May provides a converse to this observation: if Z is a *grouplike* $\mathbb{E}_k$-algebra object of $\mathcal{S}$ (see Definition 5.1.3.2), then Z is equivalent to $\Omega^k(Y)$ for some pointed space $Y \in \mathcal{S}_*$ (see Theorem 5.1.3.6 for a precise statement). The delooping process $Z \mapsto Y$ is compatible with products in Z. Consequently, if Z is equipped with a *second* action of the ∞-operad $\mathbb{E}_{k'}^\otimes$, which is suitable compatible with the $\mathbb{E}_k^\otimes$-action on Z, then we should expect that the space Y again carries an action of $\mathbb{E}_{k'}^\otimes$, and is therefore *itself* homotopy equivalent to $\Omega^{k'}(X)$ for some pointed space $X \in \mathcal{S}_*$. Then $Z \simeq \Omega^{k+k'}(X)$ carries an action of the ∞-operad $\mathbb{E}_{k+k'}^\otimes$. Our goal in this section is to show that this phenomenon is quite general, and applies to algebra objects of an arbitrary

symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$: namely, giving an $\mathbb{E}_{k+k'}$-algebra object of $\mathcal{C}$ is equivalent to giving an object $A \in \mathcal{C}$ which is equipped with *commuting* actions of the ∞-operads $\mathbb{E}_k^{\otimes}$ and $\mathbb{E}_{k'}^{\otimes}$. More precisely, we have a canonical equivalence $\mathrm{Alg}_{\mathbb{E}_{k+k'}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Alg}_{\mathbb{E}_{k'}}(\mathcal{C}))$ (Theorem 5.1.2.2). Equivalently, we can identify $\mathbb{E}_{k+k'}^{\otimes}$ with the *tensor product* of the ∞-operads $\mathbb{E}_k^{\otimes}$ and $\mathbb{E}_{k'}^{\otimes}$ (see Definition 2.2.5.3). We first describe the bifunctor $\mathbb{E}_k^{\otimes} \times \mathbb{E}_{k'}^{\otimes} \to \mathbb{E}_{k+k'}^{\otimes}$ which gives rise to this identification.

Construction 5.1.2.1. Choose nonnegative integers k, k'. We define a topological functor $\times :^t \mathbb{E}_k^{\otimes} \times^t \mathbb{E}_{k'}^{\otimes} \to^t \mathbb{E}_{k+k'}^{\otimes}$ as follows:

(1) The diagram of functors

$$
\begin{array}{ccc}
{}^t\mathbb{E}_k^{\otimes} \times^t \mathbb{E}_{k'}^{\otimes} & \xrightarrow{\ \times\ } & {}^t\mathbb{E}_{k+k'}^{\otimes} \\
\downarrow & & \downarrow \\
\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) & \xrightarrow{\ \wedge\ } & \mathrm{N}(\mathcal{F}\mathrm{in}_*)
\end{array}
$$

commutes, where $\wedge$ denotes the smash product functor on pointed finite sets (Notation 2.2.5.1). In particular, the functor $\times$ is given on objects by the formula $\langle m \rangle \wedge \langle n \rangle = \langle mn \rangle$.

(2) Suppose we are given a pair of morphisms $\overline{\alpha} : \langle m \rangle \to \langle n \rangle$ in ${}^t\mathbb{E}_k^{\otimes}$ and $\overline{\beta} : \langle m' \rangle \to \langle n' \rangle$ in ${}^t\mathbb{E}_{k'}^{\otimes}$. Write $\overline{\alpha} = (\alpha, \{f_j : \square^k \times \alpha^{-1}\{j\}, \square^k\}_{j \in \langle n \rangle^{\circ}})$ and $\overline{\beta} = (\beta, \{f'_j : \square^k \times \beta^{-1}\{j\}, \square^k\}_{j' \in \langle n' \rangle^{\circ}})$. We then define $\overline{\alpha} \times \overline{\beta} : \langle mm' \rangle \to \langle nn' \rangle$ to be given by the pair

$$(\alpha \wedge \beta, \{f_j \times f_{j'} : \square^{k+k'} \times \alpha^{-1}\{j\} \times \beta^{-1}\{j'\}, \square^{k+k'}\}_{j \in \langle n \rangle^{\circ}, j' \in \langle n' \rangle^{\circ}}).$$

Passing to homotopy coherent nerves, we obtain a bifunctor of ∞-operads (see Definition 2.2.5.3) $\mathbb{E}_k^{\otimes} \times \mathbb{E}_{k'}^{\otimes} \to \mathbb{E}_{k+k'}^{\otimes}$.

A version of the following fundamental result was proven by Dunn (see [41]):

Theorem 5.1.2.2 (Dunn Additivity Theorem). *Let $k, k' \geq 0$ be nonnegative integers. Then the bifunctor $\mathbb{E}_k^{\otimes} \times \mathbb{E}_{k'}^{\otimes} \to \mathbb{E}_{k+k'}^{\otimes}$ of Construction 5.1.2.1 exhibits the ∞-operad $\mathbb{E}_{k+k'}^{\otimes}$ as a tensor product of the ∞-operads $\mathbb{E}_k^{\otimes}$ and $\mathbb{E}_{k'}^{\otimes}$ (see Definition 2.2.5.3).*

Example 5.1.2.3 (Baez-Dolan Stabilization Hypothesis). Theorem 5.1.2.2 implies that supplying an $\mathbb{E}_k$-monoidal structure on an ∞-category $\mathcal{C}$ is equivalent to supplying k compatible monoidal structures on $\mathcal{C}$. Fix an integer $n \geq 1$, and let $\mathcal{C}\mathrm{at}_{\infty}^{\leq n}$ denote the full subcategory of $\mathcal{C}\mathrm{at}_{\infty}$ spanned by those ∞-categories which are equivalent to n-categories. For $\mathcal{C}, \mathcal{D} \in \mathcal{C}\mathrm{at}_{\infty}^{\leq n}$, the mapping space $\mathrm{Map}_{\mathcal{C}\mathrm{at}_{\infty}}(\mathcal{C}, \mathcal{D})$ is the underlying Kan complex of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, which is equivalent to an n-category (Corollary T.2.3.4.8). It follows that $\mathcal{C}\mathrm{at}_{\infty}^{\leq n}$ is equivalent to an $(n+1)$-category. Let us regard $\mathcal{C}\mathrm{at}_{\infty}^{\leq n}$ as endowed with the Cartesian monoidal structure. Corollary 5.1.1.7 implies that $\mathrm{CAlg}(\mathcal{C}\mathrm{at}_{\infty}^{\leq n}) \simeq \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}\mathrm{at}_{\infty}^{\leq n})$ for $k \geq n+2$. It follows that if $\mathcal{C}$ is an n-category, then supplying an $\mathbb{E}_k$-monoidal structure on $\mathcal{C}$ is equivalent to supplying a symmetric monoidal structure on $\mathcal{C}$ (when $k \geq n+2$). This can be regarded as a version of the "stabilization hypothesis" proposed in [6] (the formulation above applies to n-categories where all m-morphisms are invertible for $m > 1$, but the same argument can be applied more generally.)

Example 5.1.2.4 (Braided Monoidal Categories). Let $\mathcal{C}$ be an ordinary category. According to Example 5.1.2.3, supplying an $\mathbb{E}_k$-monoidal structure on $\mathrm{N}(\mathcal{C})$ is equivalent to supplying a symmetric monoidal structure on $\mathcal{C}$ if $k \geq 3$. If $k = 1$, then supplying an $\mathbb{E}_k$-monoidal structure on $\mathrm{N}(\mathcal{C})$ is equivalent to supplying a monoidal structure on $\mathcal{C}$ (combine Example 2.4.2.4 and Proposition 2.4.2.5) Let us therefore focus our attention on the case $k = 2$. Giving an $\mathbb{E}_2$-monoidal structure on $\mathrm{N}(\mathcal{C})$ is equivalent to exhibiting $\mathrm{N}(\mathcal{C})$ as an $\mathbb{E}_2$-algebra object of $\mathcal{C}\mathrm{at}_{\infty}$. Theorem 5.1.2.2 provides an equivalence $\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{C}\mathrm{at}_{\infty}) \simeq \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}\mathrm{at}_{\infty}))$. Combining this with Example 2.4.2.4 and Proposition 2.4.2.5, we can view $\mathrm{N}(\mathcal{C})$ as an (associative) monoid

object in the ∞-category $\mathrm{Mon}(\mathcal{C}\mathrm{at}_\infty)$ of monoidal ∞-categories. This structure allows us to view $\mathcal{C}$ as a monoidal category with respect to some tensor product $\otimes$, together with a *second* multiplication given by a monoidal functor

$$\boxtimes : (\mathcal{C}, \otimes) \times (\mathcal{C}, \otimes) \to (\mathcal{C}, \otimes).$$

This second multiplication also has a unit, which is a functor from the one-object category $[0]$ into $\mathcal{C}$. Since this functor is required to be monoidal, it carries the unique object of $[0]$ to the unit object $\mathbf{1} \in \mathcal{C}$, up to canonical isomorphism. It follows that $\mathbf{1}$ can be regarded as a unit with respect to both tensor product operations $\otimes$ and $\boxtimes$.

We can now exploit the classical Eckmann-Hilton argument to show that the tensor product functors $\otimes, \boxtimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ are isomorphic. Namely, our assumption that $\boxtimes$ is a monoidal functor gives a chain of isomorphisms

$$
\begin{align}
X \boxtimes Y \quad &\simeq \quad (X \otimes \mathbf{1}) \boxtimes (\mathbf{1} \otimes Y) \tag{5.1} \\
&\simeq \quad (X \boxtimes \mathbf{1}) \otimes (\mathbf{1} \boxtimes Y) \tag{5.2} \\
&\simeq \quad X \otimes Y \tag{5.3}
\end{align}
$$

depending naturally on X and Y. Consequently, $\boxtimes$ is determined by $\otimes$ as a functor from $\mathcal{C} \times \mathcal{C}$ into $\mathcal{C}$. However, it gives rise to additional data when viewed as a *monoidal* functor: a monoidal structure on the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ supplies a canonical isomorphism

$$(W \otimes X) \otimes (Y \otimes Z) \simeq (W \otimes Y) \otimes (X \otimes Z).$$

Taking W and Z to be the unit object, we get a canonical isomorphism $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$. Conversely, if we are given a collection of isomorphisms $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$, we can try to endow $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with the structure of a monoidal functor by supplying the isomorphisms

$$(W \otimes X) \otimes (Y \otimes Z) \simeq W \otimes (X \otimes Y) \otimes Z \overset{\sigma_{X,Y}}{\simeq} W \otimes (Y \otimes X) \otimes Z \simeq (W \otimes Y) \otimes (X \otimes Z)$$

together with the evident isomorphism $\mathbf{1} \otimes \mathbf{1} \simeq \mathbf{1}$. One can show that construction supplies $\otimes$ with the structure of a monoidal functor if and only if the following conditions are satisfied:

(1) For every triple of objects $X, Y, Z \in \mathcal{C}$, the isomorphism $\sigma_{X, Y \otimes Z}$ is given by the composition

$$X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z \overset{\sigma_{X,Y}}{\simeq} (Y \otimes X) \otimes Z \simeq Y \otimes (X \otimes Z) \overset{\sigma_{X,Z}}{\simeq} Y \otimes (Z \otimes X) \simeq (Y \otimes Z) \otimes X.$$

(2) For every triple of objects $X, Y, Z \in \mathcal{C}$, the isomorphism $\sigma_{X \otimes Y, Z}$ is given by the composition

$$(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z) \overset{\sigma_{Y,Z}}{\simeq} X \otimes (Z \otimes Y) \simeq (X \otimes Z) \otimes Y \overset{\sigma_{X,Z}}{\simeq} (Z \otimes X) \otimes Y \simeq Z \otimes (X \otimes Y).$$

(Equivalently, the inverse maps $\sigma_{X,Y}^{-1} : Y \otimes X \simeq X \otimes Y$ satisfy condition (1).)

A natural isomorphism $\sigma_{X,Y} : X \otimes Y \simeq Y \otimes X$ is called a *braiding* on the monoidal category $(\mathcal{C}, \otimes)$ if it satisfies conditions (1) and (2). A *braided monoidal category* is a monoidal category equipped with a braiding. We can summarize our discussion as follows: if $\mathcal{C}$ is an ordinary category, then endowing $\mathcal{C}$ with the structure of a braided monoidal category is equivalent to endowing the nerve $\mathrm{N}(\mathcal{C})$ with the structure of an $\mathbb{E}_2$-monoidal ∞-category.

Remark 5.1.2.5. It follows from Example 5.1.2.4 that if $\mathcal{C}$ is a braided monoidal category containing a sequence of objects $X_1, \ldots, X_n$, then the tensor product $X_1 \otimes \cdots \otimes X_n$ is the fiber of a local system of objects of $\mathcal{C}$ over the space $\mathrm{Rect}(\square^2 \times \{1, \ldots, n\}, \square^2)$. In other words, the tensor product $X_1 \otimes \cdots \otimes X_n$ is endowed with an action of the fundamental group $\pi_1 \mathrm{Conf}(\{1, \ldots, n\}, \mathbf{R}^2)$ of configurations of n distinct points in the plane $\mathbf{R}^2$ (see Lemma 5.1.1.3). The group $\pi_1 \mathrm{Conf}(\{1, \ldots, n\}, \mathbf{R}^2)$ is the Artin pure braid group on n strands. The action of $\pi_1 \mathrm{Conf}(\{1, \ldots, n\}, \mathbf{R}^2)$ on $X_1 \otimes \cdots \otimes X_n$ can be constructed by purely combinatorial means, by matching the standard generators of the Artin braid group with the isomorphisms σ_{X_i, X_j}. However, Theorem 5.1.2.2 provides a more illuminating geometric explanation of this phenomenon.

The proof of Theorem 5.1.2.2 will occupy our attention for the remainder of this section. We will need a few auxiliary constructions.

Construction 5.1.2.6. Fix integers $k, k' \geq 0$. We define a topological category $\mathcal{W}$ as follows:

(a) The objects of $\mathcal{W}$ are finite sequences $(\langle m_1 \rangle, \ldots, \langle m_b \rangle)$ of objects of $\mathfrak{Fin}_*$.

(b) Given a pair of objects of $W = (\langle m_1 \rangle, \ldots, \langle m_b \rangle), W' = (\langle m'_1 \rangle, \ldots, \langle m'_{b'} \rangle)$ of $\mathcal{W}$, the mapping space $\mathrm{Map}_{\mathcal{W}}(W, W')$ is given by

$$\coprod_{\beta : \langle b \rangle \to \langle b' \rangle} \mathrm{Map}^{\beta}_{t\mathbb{E}_k^{\otimes}}(\langle b \rangle, \langle b' \rangle) \times \prod_{\beta(i) = j} \mathrm{Map}_{t\mathbb{E}_{k'}^{\otimes}}(\langle m_i \rangle, \langle m'_j \rangle).$$

Here the coproduct is taken over all maps of pointed finite sets $\beta : \langle b \rangle \to \langle b' \rangle$, and $\mathrm{Map}^{\beta}_{t\mathbb{E}_k^{\otimes}}(\langle b \rangle, \langle b' \rangle)$ denotes the inverse image of β under the map $\mathrm{Map}_{t\mathbb{E}_k^{\otimes}}(\langle b \rangle, \langle b' \rangle) \to \mathrm{Hom}_{\mathfrak{Fin}_*}(\langle b \rangle, \langle b' \rangle)$.

We observe that there is a canonical isomorphism of $\mathrm{N}(\mathcal{W})$ with the wreath product $\mathbb{E}_k^{\otimes} \wr \mathbb{E}_{k'}^{\otimes}$, defined in §2.4.4.

There are evident topological functors

$$^t\mathbb{E}_k^{\otimes} \times^t \mathbb{E}_k^{\otimes} \to \mathcal{W} \to^t \mathbb{E}_{k+k'}^{\otimes},$$

given on objects by the formulas

$$(\langle b \rangle, \langle m \rangle) \mapsto (\langle m \rangle, \langle m \rangle, \ldots, \langle m \rangle) \qquad (\langle m_1 \rangle, \ldots, \langle m_b \rangle) \mapsto \langle m_1 + \cdots + m_b \rangle,$$

respectively. Passing to the homotopy coherent nerve, we obtain functors

$$\mathbb{E}_k^{\otimes} \times \mathbb{E}_{k'}^{\otimes} \xrightarrow{\theta'} \mathbb{E}_k^{\otimes} \wr \mathbb{E}_{k'}^{\otimes} \simeq \mathrm{N}(\mathcal{W}) \xrightarrow{\theta} \mathbb{E}_{k+k'}^{\otimes}.$$

Here θ' is the functor described in Remark 2.4.4.2. We will reduce the proof of Theorem 5.1.2.2 to the following:

Proposition 5.1.2.7. *Let $k \geq 0$ be a nonnegative integer. Then the map $\theta : \mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes} \to \mathbb{E}_{k+1}^{\otimes}$ is a weak approximation to the ∞-operad $\mathbb{E}_{k+1}^{\otimes}$.*

Let us assume Proposition 5.1.2.7 for the moment.

Corollary 5.1.2.8. *Let $k \geq 0$ be a nonnegative integer. Then the map $\theta : \mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes} \to \mathbb{E}_{k+1}^{\otimes}$ described above induces a weak equivalence of ∞-preoperads $(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes}, M) \to \mathbb{E}_{k+1}^{\otimes, \natural}$. Here M denotes the collection of all inert morphisms in $\mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes}$ (see Construction 2.4.4.1).*

Proof. Let θ_0 denote the composition of θ with the forgetful functor $\mathbb{E}_{k+1}^{\otimes}$. We observe that for every object $X \in \mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes}$ and every inert morphism $\alpha : \theta_0(X) \to \langle n \rangle$ in $\mathrm{N}(\mathfrak{Fin}_*)$, there exists a θ_0-coCartesian morphism $\overline{\alpha} : X \to Y$ lying over α: if $X = (\langle m_1 \rangle, \ldots, \langle m_b \rangle)$ then we can take $\overline{\alpha}$ to be the map

$$(\langle m_1 \rangle, \ldots, \langle m_b \rangle) \to (\langle m'_1 \rangle, \ldots, \langle m'_b \rangle)$$

determined by a collection of inert morphisms $\beta_i : \langle m_i \rangle \to \langle m'_i \rangle$ with $\beta_i^{-1} \langle m'_i \rangle^{\circ} = \alpha^{-1} \langle n \rangle^{\circ} \cap \langle m_i \rangle^{\circ}$.

Note that the ∞-category $\mathcal{E} = (\mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes}) \times_{\mathrm{N}(\mathfrak{Fin}_*)} \{\langle 1 \rangle\}$ has a final object $(\langle 1 \rangle)$, so that θ induces a weak homotopy equivalence $\mathcal{E} \to \mathbb{E}_{k+1}$ (since $\mathbb{E}_{k+1}$ is a contractible Kan complex). Let θ_0 denote the composition of θ with the forgetful functor $\mathbb{E}_{k+1}^{\otimes}$. Let M_0 be the collection of all θ_0-coCartesian morphisms in $\mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes}$ which lie over an inert morphism $\langle m \rangle \to \langle 1 \rangle$ in $\mathrm{N}(\mathfrak{Fin}_*)$. Let $\mathcal{C}^{\otimes}$ be an arbitrary ∞-operad. Combining Proposition 5.1.2.7 with Theorem 2.3.3.23, we learn that composition with θ induces an equivalence from

$\mathrm{Alg}_{\mathbb{E}_{k+1}}(\mathcal{C})$ to the full subcategory $\mathfrak{X} \subseteq \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathbb{E}_1^\otimes \wr \mathbb{E}_k^\otimes, \mathcal{C}^\otimes)$ spanned by those functors which carry every morphism in M_0 to an inert morphism in $\mathcal{C}^\otimes$ and every morphism in $\mathcal{E}$ to an equivalence in $\mathcal{C}$. To complete the proof, it will suffice to show that a functor $F \in \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathbb{E}_1^\otimes \wr \mathbb{E}_k^\otimes, \mathcal{C}^\otimes)$ belongs to $\mathfrak{X}$ if and only if F carries every inert morphism of $\mathbb{E}_1^\otimes \wr \mathbb{E}_k^\otimes$ to an inert morphism of $\mathcal{C}^\otimes$.

Suppose first that F carries inert morphisms to inert morphisms. Since M_0 consists of inert morphisms in $\mathbb{E}_1^\otimes \wr \mathbb{E}_k^\otimes$, it will suffice to show that F carries every morphism in $E \to E'$ in $\mathcal{E}$ to an equivalence in $\mathcal{C}$. We have a commutative diagram

$$E \xrightarrow{\qquad\qquad\qquad} E'$$

with maps labeled ϕ and ϕ' to $(\langle 1 \rangle)$

in $\mathbb{E}_1^\otimes \wr \mathbb{E}_k^\otimes$. By the two-out-of-three property, it will suffice to show that $F(\phi)$ and $F(\phi')$ are equivalences in $\mathcal{C}$, which follows from our assumption since ϕ and ϕ' are inert.

Conversely, suppose that $F \in \mathfrak{X}$; we wish to show that for every inert morphism $\alpha : X \to X'$ in $\mathbb{E}_1^\otimes \wr \mathbb{E}_k^\otimes$, the image $F(\alpha)$ is an inert morphism in $\mathcal{C}^\otimes$. Arguing as in Remark 2.1.2.9, we can reduce to the case where $X' \in \mathcal{E}$. Then α can be written as a composition $\alpha' \circ \alpha''$, where $\alpha'' \in M_0$ and α' is a morphism in $\mathcal{E}$. Our assumption that $F \in \mathfrak{X}$ guarantees that $F(\alpha')$ is an equivalence and that $F(\alpha'')$ is inert, so that $F(\alpha)$ is inert as desired. $\square$

Proof of Theorem 5.1.2.2. We proceed by induction on k. If $k = 0$, the desired result follows from Proposition 2.3.1.4, since the ∞-operad $\mathbb{E}_k^\otimes$ is unital. If $k = 1$, we consider the factorization

$$\mathbb{E}_1^{\otimes,\natural} \odot \mathbb{E}_k'^{\otimes,\natural} \to (\mathbb{E}_1^\otimes \wr \mathbb{E}_{k'}^\otimes, M) \to \mathbb{E}_{1+k}'^{\otimes,\natural}$$

and apply Corollary 5.1.2.8 together with Theorem 2.4.4.3. If $k > 1$, we have a commutative diagram

$$
\begin{array}{ccc}
\mathbb{E}_1^{\otimes,\natural} \odot \mathbb{E}_{k-1}^{\otimes,\natural} \odot \mathbb{E}_k'^{\otimes,\natural} & \longrightarrow & \mathbb{E}_k^{\otimes,\natural} \odot \mathbb{E}_k'^{\otimes,\natural} \\
\downarrow & & \downarrow \\
\mathbb{E}_1^{\otimes,\natural} \odot \mathbb{E}_{k+k'-1}^{\otimes,\natural} & \longrightarrow & \mathbb{E}_{k+k'}'^{\otimes,\natural}.
\end{array}
$$

The inductive hypothesis guarantees that the horizontal maps and the left vertical map are weak equivalences of ∞-preoperads, so that the right vertical map is a weak equivalence as well. $\square$

To prove Proposition 5.1.2.7, we first observe that the forgetful functor $\theta_0 : \mathbb{E}_1^\otimes \wr \mathbb{E}_k^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ satisfies condition $(*)$ of Proposition 2.3.3.11: that is, inert morphisms in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ can be lifted to θ_0-coCartesian morphisms in $\mathbb{E}_1^\otimes \wr \mathbb{E}_k^\otimes$. For every integer $n \geq 0$, we let $\mathrm{Tup}_n \subseteq \mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle n \rangle}$ be the subcategory whose objects are active maps $\langle m \rangle \to \langle n \rangle$ and whose morphisms are equivalences. Fix an object $X \in \mathbb{E}_1^\otimes \wr \mathbb{E}_k^\otimes$, write $X = (\langle n_1 \rangle, \langle n_2 \rangle, \ldots, \langle n_b \rangle)$, and let $n = n_1 + \cdots + n_b$. According to Proposition 2.3.3.14, it will suffice to show that the map

$$(\mathbb{E}_1^\otimes \wr \mathbb{E}_k^\otimes)_{/X} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle n \rangle}} \mathrm{Tup}_n \to (\mathbb{E}_{k+1}^\otimes)_{/\langle n \rangle} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle n \rangle}} \mathrm{Tup}_n$$

is a weak homotopy equivalence. We define subcategories

$$(\mathbb{E}_1^\otimes \wr \mathbb{E}_k^\otimes)^1_{/X} \subseteq (\mathbb{E}_1^\otimes \wr \mathbb{E}_k^\otimes)^0_{/X} \subseteq (\mathbb{E}_1^\otimes \wr \mathbb{E}_k^\otimes)_{/X}$$

as follows:

- The objects of $(\mathbb{E}_1^\otimes \wr \mathbb{E}_k^\otimes)^0_{/X}$ are morphisms $\alpha : (\langle m_1 \rangle, \langle m_2 \rangle, \ldots, \langle m_a \rangle) \to (\langle n_1 \rangle, \ldots, \langle n_b \rangle)$ such that the underlying maps $\langle a \rangle \to \langle b \rangle$ and $\langle m_1 + \cdots + m_a \rangle \to \langle n_1 + \cdots + n_b \rangle$ are active morphisms in $\mathcal{F}\mathrm{in}_*$. Such an object belongs to $(\mathbb{E}_1^\otimes \wr \mathbb{E}_k^\otimes)^1_{/X}$ if and only if each of the integers m_i is positive.

- The morphisms in $(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes})^0_{/X}$ are given by commutative diagrams

$$
\begin{array}{ccc}
(\langle m_1\rangle, \ldots, \langle m_a\rangle) & \xrightarrow{\quad\beta\quad} & (\langle m_1'\rangle, \ldots, \langle m_{a'}'\rangle) \\
 & \searrow^{\alpha} \qquad \swarrow_{\alpha'} & \\
 & (\langle n_1\rangle, \ldots, \langle n_b\rangle) &
\end{array}
$$

where α and α' belong to $(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes})^0_{/X}$ and β induces an active map $\langle a\rangle \to \langle a'\rangle$ in $\mathcal{F}\mathrm{in}_*$ and a bijection $\langle m_1 + \cdots + m_a\rangle \to \langle m_1' + \cdots + m_{a'}'\rangle$. Such a morphism belongs to $(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes})^1_{/X}$ if and only if α and α' are objects of $(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes})^1_{/X}$.

It is not difficult to see that the inclusions

$$(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes})^1_{/X} \subseteq (\mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes})_{/X} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle n\rangle}} \mathrm{Tup}_n$$

$$(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes})^1_{/X} \subseteq (\mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes})^0_{/X}$$

both admit right adjoints, and are therefore weak homotopy equivalences. By the two-out-of-three property, we conclude that the inclusion

$$(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes})^0_{/X} \hookrightarrow (\mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes})_{/X} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle n\rangle}} \mathrm{Tup}_n$$

is a weak homotopy equivalence. It will therefore suffice to show that the map

$$\phi_X : (\mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes})^0_{/X} \to (\mathbb{E}_{k+1}^{\otimes})_{/X} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle n\rangle}} \mathrm{Tup}_n$$

is a weak homotopy equivalence. We note that ϕ_X is equivalent to a product of maps $\{\phi_{(\langle n_i\rangle)}\}_{1 \le i \le b}$, which allows us to reduce to the case $b = 1$. In other words, we can reformulate Proposition 5.1.2.7 as follows:

Proposition 5.1.2.9. *For each $n \ge 0$, the functor $\theta : \mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes} \to \mathbb{E}_{k+1}^{\otimes}$ induces a weak homotopy equivalence of simplicial sets*

$$\phi : (\mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes})^0_{/(\langle n\rangle)} \to (\mathbb{E}_{k+1}^{\otimes})_{/\langle n\rangle} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle n\rangle}} \mathrm{Tup}_n \, .$$

The remainder of this section is devoted to the proof of Proposition 5.1.2.9.

Notation 5.1.2.10. For every topological category $\mathcal{C}$, we let $\mathrm{Sing}(\mathcal{C})$ denote the underlying simplicial category: that is, $\mathrm{Sing}(\mathcal{C})$ is the simplicial category having the same objects as $\mathcal{C}$, with morphism spaces given by

$$\mathrm{Map}_{\mathrm{Sing}(\mathcal{C})}(X, Y) = \mathrm{Sing}\,\mathrm{Map}_{\mathcal{C}}(X, Y).$$

If $F : \mathcal{C} \to \mathrm{Top}$ is a topological functor, we let $\mathrm{Sing}(F) : \mathrm{Sing}(\mathcal{C}) \to \mathcal{S}\mathrm{et}_{\Delta}$ denote the associated simplicial functor, given by $\mathrm{Sing}(F)(C) = \mathrm{Sing}(F(C))$. For every integer p and every pair of objects $X, Y \in^t \mathbb{E}_p^{\otimes}$, we let $\mathrm{Map}^{\mathrm{act}}_{{}^t\mathbb{E}_p^{\otimes}}(X, Y)$ denote the summand of the topological space $\mathrm{Map}_{{}^t\mathbb{E}_p^{\otimes}}(X, Y)$ spanned by those morphisms whose image in $\mathcal{F}\mathrm{in}_*$ is active.

Let $\mathcal{W}$ be the topological category of Construction 5.1.2.6 (so that $\mathrm{N}(\mathcal{W}) \simeq \mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes}$). We let $\mathcal{W}_0$ denote the topological subcategory of $\mathcal{W}$ whose morphisms are maps $(\langle m_1\rangle, \ldots, \langle m_b\rangle) \to (\langle m_1'\rangle, \ldots, \langle m_{b'}'\rangle)$ which induce an active map $\langle b\rangle \to \langle b'\rangle$ in $\mathcal{F}\mathrm{in}_*$ and a bijection $\langle m_1 + \cdots + m_b\rangle \to \langle m_1' + \cdots + m_{b'}'\rangle$ in $\mathcal{F}\mathrm{in}_*$. We define a topological functor $T : \mathcal{W}_0^{op} \to \mathrm{Top}$ as follows: for every object $W = (\langle m_1\rangle, \ldots, \langle m_b\rangle) \in \mathcal{W}_0$, we let $T(W)$ be the summand of $\mathrm{Map}_{\mathcal{W}}(W, (\langle n\rangle))$ corresponding to those maps $(\langle m_1\rangle, \ldots, \langle m_b\rangle) \to (\langle n\rangle)$ which induce active morphisms $\langle b\rangle \to \langle 1\rangle$ and $\langle m_1 + \cdots + m_b\rangle \to \langle n\rangle$ in $\mathcal{F}\mathrm{in}_*$. Let $v : \mathfrak{C}[\mathrm{N}(\mathcal{W}_0)] \to \mathrm{Sing}(\mathcal{W}_0)$ be the counit map. Unwinding the definitions, we obtain a canonical isomorphism of simplicial sets $(\mathbb{E}_1^{\otimes} \wr \mathbb{E}_k^{\otimes})^0_{/(\langle n\rangle)} \simeq \mathrm{Un}_v \mathrm{Sing}(T)$, where Un_v denotes the unstraightening functor of §T.2.2.1.

Let $\mathcal{E}$ denote the topological subcategory of ${}^t\mathbb{E}_{k+1}^{\otimes}$ spanned by those morphisms which induce equivalences in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. We define a functor $U : \mathcal{E}^{op} \to \mathrm{Top}$ by the formula $U(E) = \mathrm{Map}^{\mathrm{act}}_{{}^t\mathbb{E}_{k+1}^{\otimes}}(X, \langle n\rangle)$. Let $v' : \mathfrak{C}[\mathrm{N}(\mathcal{E})] \to \mathrm{Sing}(\mathcal{E})$ be the counit map. Unwinding the definitions, we have a canonical isomorphism of simplicial sets $(\mathbb{E}_{k+1}^{\otimes})_{/\langle n\rangle} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle n\rangle}} \mathrm{Tup}_n \simeq \mathrm{Un}_{v'} \mathrm{Sing}(U)$.

The topological functor $\xi : \mathcal{W} \to^t \mathbb{E}_{k+1}^{\otimes}$ introduced above restricts to a topological functor $\xi_0 : \mathcal{W}_0 \to \mathcal{E}$. We note that the map ϕ appearing in Proposition 5.1.2.8 can be identified with the map $\mathrm{Un}_v \, \mathrm{Sing}(T) \to \mathrm{Un}_{v'} \, \mathrm{Sing}(U)$ induced by a natural transformation of topological functors $T \to U \circ \xi_0$ (which is also determined by ξ). We will prove that ϕ is a weak homotopy equivalence by assembling T and U from simpler constituents.

Construction 5.1.2.11. We define a partially ordered set P as follows:

(a) The objects of P are pairs $(I, \sim)$, where $I \subseteq (-1, 1)$ is a finite union of open intervals and $\sim$ is an equivalence relation on the set $\pi_0 I$. For every point $x \in I$ we let $[x]$ denote the corresponding connected component of I. We assume that $\sim$ is convex in the following sense: if $x < y < z$ in I and $[x] \sim [z]$, then $[x] \sim [y] \sim [z]$.

(b) We have $(I, \sim) \leq (I', \sim')$ if and only if $I \subseteq I'$ and, for every pair of elements $x, y \in I$ such that $[x] \sim' [y]$, we have $[x] \sim [y]$.

For every element $(I, \sim) \in P$, we define subfunctors

$$T_{I,\sim} \subseteq T \qquad U_{I,\sim} \subseteq U$$

as follows:

- Let $W = (\langle m_1 \rangle, \ldots, \langle m_b \rangle)$ be an object of $\mathcal{W}_0$, so that $T(W)$ is the topological space

$$\mathrm{Map}_{t\mathbb{E}_1^{\otimes}}^{\mathrm{act}}(\langle b \rangle, \langle 1 \rangle) \times \prod_{1 \leq a \leq b} \mathrm{Map}_{t\mathbb{E}_k^{\otimes}}^{\mathrm{act}}(\langle m_a \rangle, \langle n \rangle)).$$

  We can identify a point of $T(W)$ with a sequence $(f, g_1, \ldots, g_b)$, where $f : \langle b \rangle^{\circ} \times \square^1 \to \square^1$ and $g_a : \langle m_a \rangle^{\circ} \times \square^k \to \langle n \rangle^{\circ} \times \square^k$ are rectilinear embeddings. We let $T_{I,\sim}(W)$ be the subset of $T(W)$ consisting of those sequences which satisfy the following pair of conditions:

  - The image of f is contained in I. It follows that f induces a map $\lambda : \langle b \rangle^{\circ} \to \pi_0 I$.
  - If $a, a' \in \langle b \rangle^{\circ}$ satisfy $\lambda(a) \sim \lambda(a')$, then either $a = a'$ or the maps g_a and $g_{a'}$ have disjoint images.

- Let $E = \langle m \rangle$ be an object of $\mathcal{E}$, so that $U(E)$ is the topological space $\mathrm{Map}_{t\mathbb{E}_{k+1}^{\otimes}}^{\mathrm{act}}(\langle m \rangle, \langle n \rangle) = \mathrm{Rect}(\langle m \rangle^{\circ} \times \square^{k+1}, \langle n \rangle^{\circ} \times \square^{k+1})$. Let us identify $\langle n \rangle^{\circ} \times \square^{k+1}$ with the product $\langle n \rangle^{\circ} \times \square^k \times (-1, 1)$, so that we have projection maps

$$\langle n \rangle^{\circ} \times \square^k \xleftarrow{p_0} \langle n \rangle^{\circ} \times \square^{k+1} \xrightarrow{p_1} (-1, 1).$$

  We let $U_{I,\sim}(E)$ denote the subspace of $U(E)$ consisting of those rectilinear embeddings $f : \langle m \rangle^{\circ} \times \square^{k+1} \to \langle n \rangle^{\circ} \times \square^{k+1}$ satisfying the following conditions:

  - The image of $p_1 \circ f$ is contained in $I \subseteq (-1, 1)$. Consequently, f induces a map $\lambda : \langle n \rangle^{\circ} \to \pi_0 I$.
  - If $i, j \in \langle n \rangle^{\circ}$ satisfy $\lambda(i) \sim \lambda(j)$, then either $i = j$ or the sets $\{i\} \times \square^{k+1}$ and $\{j\} \times \square^{k+1}$ have disjoint images in $\langle m \rangle^{\circ} \times \square^k$ (under the map $p_0 \circ f$).

It is easy to see that the functors $T_{I,\sim} : \mathcal{W}_0^{op} \to \mathrm{Top}$ and $U_{I,\sim} : \mathcal{E}^{op} \to \mathrm{Top}$ depend functorially on the pair $(I, \sim) \in P$. Moreover, the natural transformation $T \to U \circ \xi_0$ determined by ξ restricts to natural transformations $T_{I,\sim} \to U_{I,\sim} \circ \xi_0$. According to Theorem T.2.2.1.2, the unstraightening functor Un_v is a right Quillen equivalence from $(\mathcal{S}\mathrm{et}_\Delta)^{\mathrm{Sing}(\mathcal{W}_0)^{op}}$ (endowed with the projective model structure) to $(\mathcal{S}\mathrm{et}_\Delta)_{/\mathrm{N}(\mathcal{W}_0)}$ (endowed with the contravariant model structure); it follows that Un_v carries homotopy colimit diagrams between fibrant objects of $(\mathcal{S}\mathrm{et}_\Delta)^{\mathrm{Sing}(\mathcal{W}_0)^{op}}$ to homotopy colimit diagrams in $\mathcal{S}\mathrm{et}_\Delta$ (with respect to the usual model structure). Similarly, $\mathrm{Un}_{v'}$ carries homotopy colimit diagrams between fibrant objects of $(\mathcal{S}\mathrm{et}_\Delta)^{\mathrm{Sing}(\mathcal{E})^{op}}$ to homotopy colimit diagrams in $\mathcal{S}\mathrm{et}_\Delta$. Consequently, to prove Proposition 5.1.2.9 it will suffice to show the following:

Proposition 5.1.2.12. *The functor* $\mathrm{Sing}(T)$ *is a homotopy colimit of the diagram of functors* $\{\mathrm{Sing}(T_{I,\sim}) : \mathrm{Sing}(\mathcal{W}_0)^{op} \to \mathcal{S}\mathrm{et}_\Delta\}_{(I,\sim) \in P}$.

Proposition 5.1.2.13. *The functor* $\mathrm{Sing}(U)$ *is a homotopy colimit of the diagram of functors* $\{\mathrm{Sing}(U_{I,\sim}) : \mathrm{Sing}(\mathcal{E})^{op} \to \mathcal{S}\mathrm{et}_\Delta\}_{(I,\sim) \in P}$.

Proposition 5.1.2.14. *For every element* $(I,\sim) \in P$, *the topological functor* ξ *induces a weak homotopy equivalence of simplicial sets* $\mathrm{Un}_v \mathrm{Sing}(T_{I,\sim}) \to \mathrm{Un}_{v'} \mathrm{Sing}(U_{I,\sim})$.

Proof of Proposition 5.1.2.12. Fix an object $W = (\langle m_1 \rangle, \dots, \langle m_b \rangle) \in \mathcal{W}_0$; we wish to show that $\mathrm{Sing}\, T(W)$ is a homotopy colimit of the diagram $\{\mathrm{Sing}\, T_{I,\sim}(W)\}_{(I,\sim) \in P}$. Let $T'(W)$ denote the configuration space

$$\mathrm{Conf}(\langle b \rangle^\circ, (-1,1)) \times \prod_{1 \le a \le b} \mathrm{Conf}(\langle m_a \rangle^\circ, \langle n \rangle^\circ \times \square^k).$$

We can identify $T'(W)$ with the set of tuples $(f, g_1, \dots, g_b)$, where $f : \langle b \rangle^\circ \to (-1,1)$ and and $g_a : \langle m_a \rangle^\circ \to \langle n \rangle^\circ \times \square^k$ are injective maps. Given $(I,\sim) \in P$, let $T'_{I,\sim}(W)$ be the open subset of $T'(W)$ consisting of those tuples $(f, g_1, \dots, g_b)$ satisfying the following conditions:

- The image of the map f is contained in $I \subseteq (-1,1)$, so that f induces a map $\lambda : \langle b \rangle^\circ \to \pi_0 I$.

- If $a, a' \in \langle b \rangle^\circ$ satisfy $\lambda(a) \sim \lambda(a')$, then either $a = a'$ or the maps g_a and $g_{a'}$ have disjoint images.

Evaluation at the origins of $\square^1$ and $\square^k$ determines a map $T(W) \to T'(W)$ which restricts to maps $T_{I,\sim}(W) \to T'_{I,\sim}(W)$ for each $(I,\sim) \in P$. It follows from Lemma 5.1.1.3 that each of these maps is a homotopy equivalence. It will therefore suffice to show that $\mathrm{Sing}\, T'(W)$ is a homotopy colimit of the diagram of simplicial sets $\{\mathrm{Sing}\, T'_{I,\sim}(W)\}_{(I,\sim) \in P}$. According to Theorem A.3.1, it will suffice to prove the following:

(∗) Let $x = (f, g_1, \dots, g_b)$ be a point of $T'(W)$, and let $P_x = \{(I,\sim) \in P : x \in T'_{I,\sim}(W)\}$. Then the simplicial set $\mathrm{N}(P_x)$ is weakly contractible.

Let x be as in (∗), and let P'_x denote the subset of P_x consisting of those triples where f induces a surjection $\lambda : \langle b \rangle^\circ \to \pi_0 I$. The inclusion $\mathrm{N}(P'_x) \hookrightarrow \mathrm{N}(P_x)$ admits a right adjoint and is therefore a weak homotopy equivalence. It will therefore suffice to show that $\mathrm{N}(P'_x)$ is weakly contractible.

Let Q be the collection of all equivalence relations $\sim$ on the set $\langle b \rangle^\circ$ which have the following properties:

(i) If $f(a) \le f(a') \le f(a'')$ and $a \sim a''$, then $a \sim a' \sim a''$.

(ii) If $a \sim a'$ and the image of g_a intersects the image of $g_{a'}$, then $a = a'$.

We regard Q as a partially ordered set with respect to refinement. Pullback of equivalence relations along λ determines a forgetful functor $\mu : \mathrm{N}(P'_x) \to \mathrm{N}(Q)^{op}$. It is easy to see that μ is a Cartesian fibration. The simplicial set $\mathrm{N}(Q)$ is weakly contractible, since Q has a smallest element (given by the equivalence relation where $a \sim a'$ if and only if $a = a'$). We will complete the proof of (∗) by showing that the fibers of μ are weakly contractible, so that μ is left cofinal (Lemma T.4.1.3.2) and therefore a weak homotopy equivalence.

Fix an equivalence relation $\sim \in Q$. Unwinding the definitions, we see that $\mu^{-1}\{\sim\}$ can be identified with the nerve of the partially ordered set R consisting of those subsets $I \subseteq (-1,1)$ satisfying the following conditions:

(a) The set I is a finite union of open intervals.

(b) The set I contains the image of f.

(c) If $f(a)$ and $f(a')$ belong to the same connected component of I, then $a \sim a'$.

To see that $\mathrm{N}(R)$ is contractible, it suffices to observe that the partially ordered set R^{op} is filtered: this follows from the fact that R is nonempty (it contains $\bigcup_{a \in \langle b \rangle^\circ} (f(a) - \epsilon, f(a) + \epsilon)$ for sufficiently small $\epsilon > 0$) and is closed under pairwise intersections. $\qquad\square$

Proof of Proposition 5.1.2.13. The proof is essentially the same as that of Proposition 5.1.2.12, with some minor modifications. Fix an object $E = \langle m \rangle \in \mathcal{E}$. We wish to show that $\operatorname{Sing} U(E)$ is a homotopy colimit of the diagram $\{\operatorname{Sing} U_{I,\sim}(E)\}_{(I,\sim)\in P}$. Write $\langle n \rangle^{\circ} \times \square^{k+1}$ as a product $\langle n \rangle^{\circ} \times \square^{k} \times (-1, 1)$, and consider the projection maps

$$\langle n \rangle^{\circ} \times \square^{k} \xleftarrow{p_0} \langle n \rangle^{\circ} \times \square^{k+1} \xrightarrow{p_1} (-1, 1).$$

Let $U'(E) = \operatorname{Conf}(\langle m \rangle^{\circ}, \langle n \rangle^{\circ} \times \square^{k+1})$ be the collection of injective maps $f = (f_0, f_1) : \langle m \rangle^{\circ} \to (\langle n \rangle^{\circ} \times \square^{k}) \times (-1, 1)$. For every pair $(I, \sim) \in P$, we let $U'_{I,\chi}(E)$ denote the open subset of $U'(E)$ consisting of those functions $f = (f_0, f_1)$ which satisfy the following conditions:

- The image of f_1 is contained in $I \subseteq (-1, 1)$, so that f_1 induces a map $\lambda : \langle m \rangle^{\circ} \to \pi_0 I$.

- If $i, j \in \langle m \rangle^{\circ}$ satisfy $\lambda(i) \sim \lambda(j)$, then either $i = j$ or $f_0(i) \neq f_0(j)$.

Evaluation at the origin of $\square^{k+1}$ determines a map $U(E) \to U'(E)$ which restricts to maps $U_{i,\sim}(E) \to U'_{I,\sim}(E)$ for each $E \in \mathcal{E}$. It follows from Lemma 5.1.1.3 that each of these maps is a homotopy equivalence. It will therefore suffice to show that $\operatorname{Sing} U'(E)$ is a homotopy colimit of the diagram of simplicial sets $\{\operatorname{Sing} U'_{I,\sim}(E)\}_{(I,\sim)\in P}$. According to Theorem A.3.1, it will suffice to prove the following:

(∗) Let $f = (f_0, f_1)$ be a point of $U'(E)$, and let $P_x = \{(I, \sim) \in P : x \in U'_{I,\sim}(E)\}$. Then the simplicial set $\operatorname{N}(P_x)$ is weakly contractible.

Let f be as in (∗), and let P'_x denote the subset of P_x consisting of those triples where f_1 induces a surjection $\lambda : \langle m \rangle^{\circ} \to \pi_0 I$. The inclusion $\operatorname{N}(P'_x) \hookrightarrow \operatorname{N}(P_x)$ admits a right adjoint and is therefore a weak homotopy equivalence. It will therefore suffice to show that $\operatorname{N}(P'_x)$ is weakly contractible.

Let Q be the collection of all equivalence relations $\sim$ on the set $\langle m \rangle^{\circ}$ which have the following properties:

(i) If $f_1(i) \leq f_1(i') \leq f_1(i'')$ and $i \sim i''$, then $i \sim i' \sim i''$.

(ii) If $i \sim i'$ and $f_0(i) = f_0(i')$, then $i = i'$.

We regard Q as a partially ordered set with respect to refinement. Pullback of equivalence relations along λ determines a Cartesian fibration functor $\mu : \operatorname{N}(P'_x) \to \operatorname{N}(Q)^{op}$. The simplicial set $\operatorname{N}(Q)$ is weakly contractible, since Q has a smallest element (given by the equivalence relation where $i \sim j$ if and only if $f_1(i) = f_1(j)$). We will complete the proof of (∗) by showing that the fibers of μ are weakly contractible, so that μ is left cofinal (Lemma T.4.1.3.2) and therefore a weak homotopy equivalence.

Fix an equivalence relation $\sim \in Q$. Unwinding the definitions, we see that $\mu^{-1}\{\sim\}$ can be identified with the nerve of the partially ordered set R consisting of those subsets $I \subseteq (-1, 1)$ satisfying the following conditions:

(a) The set I is a finite union of open intervals.

(b) The set I contains the image of f_1.

(c) If $f_0(i)$ and $f_1(j)$ belong to the same connected component of I, then $i \sim j$.

To see that $\operatorname{N}(R)$ is contractible, it suffices to observe that the partially ordered set R^{op} is filtered: this follows from the fact that R is nonempty (it contains $\bigcup_{i \in \langle n \rangle^{\circ}}(f_1(i) - \epsilon, f_1(i) + \epsilon)$ for sufficiently small $\epsilon > 0$) and closed under pairwise intersections. □

Proof of Proposition 5.1.2.14. We define a functor $T^0_{I,\sim} : \mathcal{W}_0^{op} \to \operatorname{Top}$ by the formula $T^0_{I,\sim}(\langle m_1 \rangle, \ldots, \langle m_b \rangle) = \operatorname{Rect}(\langle b \rangle^{\circ} \times (-1, 1), I)$. There is an evident natural transformation $T_{I,\sim} \to T^0_{I,\sim}$, which induces a (projective) fibration of simplicial functors $\operatorname{Sing}(T_{I,\sim}) \to \operatorname{Sing}(T^0_{I,\sim})$ and therefore a right fibration of simplicial sets $\chi : \operatorname{Un}_v \operatorname{Sing}(T_{I,\sim}) \to \operatorname{Un}_v \operatorname{Sing}(T^0_{I,\sim})$. Unwinding the definitions, we can identify an object of $\operatorname{Un}_v \operatorname{Sing}(T^0_{I,\sim})$ with an object $W = (\langle m_1 \rangle, \ldots, \langle m_b \rangle) \in \mathcal{W}_0$ and a rectilinear embedding $f : \langle b \rangle^{\circ} \times (-1, 1) \to I$. Let $\mathcal{X} \subseteq \operatorname{Un}_v \operatorname{Sing}(T^0_{I,\sim})$ be the full subcategory of $\operatorname{Un}_v \operatorname{Sing}(T^0_{I,\sim})$ spanned by those pairs (W, f) for which f

induces a bijection $\langle b \rangle^{\circ} \to \pi_0 I$. It is not difficult to see that the inclusion $\mathfrak{X} \subseteq \mathrm{Un}_v \mathrm{Sing}(T^0_{I,\sim})$ admits a left adjoint and is therefore left cofinal. Since χ is a right fibration, we deduce that the inclusion

$$\mathfrak{X} \times_{\mathrm{Un}_v \mathrm{Sing}(T^0_{I,\sim})} \mathrm{Un}_v \mathrm{Sing}(T_{I,\sim}) \to \mathrm{Un}_v \mathrm{Sing}(T_{I,\sim})$$

is also left cofinal (Lemma T.4.1.3.2), and therefore a weak homotopy equivalence.

Define a functor $U^0_{I,\sim} : \mathcal{E} \to \mathrm{Top}$ by the formula $U^0_{I,\sim}(\langle m \rangle) = \mathrm{Hom}_{\mathrm{Set}}(\langle m \rangle^{\circ}, \pi_0 I)$. We have a commutative diagram

$$
\begin{array}{ccc}
\mathfrak{X} \times_{\mathrm{Un}_v \mathrm{Sing}(T^0_{I,\sim})}(\mathrm{Un}_v \mathrm{Sing}(T^0_{I,\sim})) & \xrightarrow{\ F\ } & \mathrm{Un}_{v'} \mathrm{Sing}(U_{I,\sim}) \\
\downarrow & & \downarrow \\
\mathfrak{X} & \xrightarrow{\ F_0\ } & \mathrm{Un}_{v'} \mathrm{Sing}(U^0_{I,\sim}).
\end{array}
$$

We will complete the proof by showing that the upper horizontal map is an equivalence of ∞-categories. Unwinding the definitions, we deduce easily that F_0 is an equivalence of ∞-categories: both the domain and codomain of F_0 are equivalent to the Kan complex $\mathrm{Tup}_{\pi_0 I}$. The vertical maps are right fibrations; it will therefore suffice to show that F induces a homotopy equivalence between the fibers of the vertical maps. This amounts to the following assertion:

(∗) For every object $W = (\langle m_1 \rangle, \ldots, \langle m_b \rangle) \in \mathcal{W}_0$ having image $E = \langle m_1 + \cdots + m_b \rangle = \langle m \rangle$ in $\mathcal{E}$, if $\eta \in T^0_{I,\sim}(W)$ determines an isomorphism $\langle b \rangle^{\circ} \simeq \pi_0 I$ having image $\eta' \in U^0_{I,\sim}(E)$, then the induced map

$$\gamma : \mathrm{Sing}(T_{I,\sim}(W) \times_{T^0_{I,\sim}(W)} \{\eta\}) \to \mathrm{Sing}(U_{I,\sim}(E) \times_{U^0_{I,\sim}(E)} \{\eta'\})$$

is a homotopy equivalence.

In fact, γ is the product of an identity map with the inclusion of a point into the product

$$\prod_{1 \leq i \leq m} \mathrm{Rect}((-1,1), I_i),$$

where I_i denotes the connected component of I which is the image of $i \in \langle m \rangle^{\circ}$ under the map $\eta' : \langle m \rangle^{\circ} \to \pi_0 I$. The desired result now follows from the fact that each embedding space $\mathrm{Rect}((-1,1), I_i)$ is contractible (Lemma 5.1.1.3). $\qquad\square$

5.1.3   Iterated Loop Spaces

Let X be a topological space equipped with a base point $*$, let Top denote the category of topological spaces, and let $\mathcal{K}\mathrm{an} \subseteq \mathcal{S}\mathrm{et}_\Delta$ be the category of Kan complexes. For each $n \geq 0$, let $\theta_X(\langle n \rangle) \simeq (\Omega^k X)^n$ denote the collection of n-tuples of maps $f_1, \ldots, f_n : [-1,1]^k \to X$ such that each f_i carries the boundary of $[-1,1]^k$ to the base point $* \in X$. The construction $\langle n \rangle \to \mathrm{Sing}_{\bullet} \theta_X(\langle n \rangle)$ determines a simplicial functor $\theta_X : \mathrm{Sing}^t \mathbb{E}^{\otimes}_k \to \mathcal{K}\mathrm{an}$ (which encodes the idea that the iterated loop space $\Omega^k X$ is acted on by the little cubes operad $\mathcal{C}_k$). This construction depends functorially on X. Restricting our attention to the case where $X = |K|$, where K is a (pointed) Kan complex, we obtain a simplicial functor

$$\mathcal{K}\mathrm{an}_{*/} \times \mathrm{Sing}^t \mathbb{E}^{\otimes}_k \to \mathcal{K}\mathrm{an}\,.$$

Passing to nerves and using the evident equivalence $\mathrm{N}(\mathcal{K}\mathrm{an}_{*/}) \to \mathcal{S}_*$, we obtain a functor

$$\mathrm{N}(\theta) : \mathcal{S}_* \times \mathbb{E}^{\otimes}_k \to \mathcal{S}\,.$$

For every pointed space K, the resulting map $\mathbb{E}^{\otimes}_k \to \mathcal{S}$ is evidently an $\mathbb{E}_k$-monoid object of $\mathcal{S}$ (in the sense of Definition 2.4.2.1). Consequently, $\mathrm{N}(\theta)$ is adjoint to a functor $\beta : \mathcal{S}_* \to \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$. We will refer to $\mathbb{E}_k$-monoid

objects of $\mathcal{S}$ simply as $\mathbb{E}_k$-*spaces*, and $\mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$ as the ∞-*category of* $\mathbb{E}_k$-*spaces*. The functor β implements the observation that for every pointed space X, the k-fold loop space of X is an $\mathbb{E}_k$-space. This observation has a converse: the functor β is *almost* an equivalence of ∞-categories. However, it fails to be an equivalence for two reasons:

(a) If X is a pointed space, then the k-fold loop space $\Omega^k X$ contains no information about the homotopy groups $\pi_i X$ for $i < k$. More precisely, if $f : X \to Y$ is a map of pointed spaces which induces isomorphisms $\pi_i X \to \pi_i Y$ for $i \geq k > 0$, then the induced map $\Omega^k X \to \Omega^k Y$ is a weak homotopy equivalence of spaces (which underlies a weak homotopy equivalence of $\mathbb{E}_k$-monoids). Consequently, the functor $\beta : \mathcal{S}_* \to \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$ fails to be conservative. To correct this problem, we need to restrict our attention to k-*connective* spaces: that is, pointed spaces X such that $\pi_i X \simeq *$ for $i < k$; in this case, there is no information about low-dimensional homotopy groups to be lost.

(b) Suppose that $k > 0$ and let $Y \in \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$; we will abuse notation by identifying Y with the space $Y(\langle 1 \rangle)$. Then Y carries an action of the ∞-operad $\mathbb{E}_1$: in particular, there is a multiplication map $Y \times Y \to Y$ which is unital and associative up to homotopy. This multiplication endows the set of connected components $\pi_0 Y$ with the structure of a monoid (which is commutative if $k > 1$). If $Y \simeq \Omega^k X$ lies in the image of the functor β, then we have a canonical isomorphism $\pi_0 Y \simeq \pi_k X$ (compatible with the monoid structures on each side). In particular, we deduce that the monoid $\pi_0 Y$ is actually a group (that is, Y is *grouplike* in the sense of Definition 5.1.3.2 below).

Remark 5.1.3.1. In the case $k = 0$, issues (a) and (b) do not arise: in fact, we have canonical equivalences of ∞-categories

$$\mathcal{S}_* \simeq \mathrm{Alg}_{\mathbb{E}_0}(\mathcal{S}) \simeq \mathrm{Mon}_{\mathbb{E}_0}(\mathcal{S})$$

(here we regard $\mathcal{S}$ as endowed with the Cartesian monoidal structure). The first equivalence results from Proposition 2.1.3.9, and the second from Proposition 2.4.2.5; the composition of these equivalences agrees with the map β defined above. For this reason, we will confine our attention to the case $k > 0$ in what follows.

We now introduce some terminology to address objection (b).

Definition 5.1.3.2. Let $\mathcal{X}$ be an ∞-topos, and let $\mathrm{Cut} : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{A}\mathrm{ss}^{\otimes}$ be defined as in Construction 4.1.2.5. We will say that an $\mathcal{A}\mathrm{ss}$-monoid object $X : \mathcal{A}\mathrm{ss}^{\otimes} \to \mathcal{X}$ is *grouplike* if the composition

$$\mathrm{N}(\mathbf{\Delta})^{op} \xrightarrow{\mathrm{Cut}} \mathcal{A}\mathrm{ss}^{\otimes} \xrightarrow{X} \mathcal{X}$$

is a groupoid object of $\mathcal{X}$ (see §T.6.1.2). Let $\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}^{\mathrm{gp}}(\mathcal{X})$ be the full subcategory of $\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{X})$ spanned by the grouplike $\mathcal{A}\mathrm{ss}$-monoid objects of $\mathcal{X}$.

We will say that an $\mathbb{E}_1$-monoid object $X : \mathbb{E}_1^{\otimes} \to \mathcal{X}$ is *grouplike* if it belongs to the essential image of $\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}^{\mathrm{gp}}(\mathcal{X})$ under the equivalence of ∞-categories $\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{X}) \to \mathrm{Mon}_{\mathbb{E}_1}(\mathcal{X})$ induced by the equivalence $\mathbb{E}_1 \to \mathcal{A}\mathrm{ss}$. We let $\mathrm{Mon}_{\mathbb{E}_1}^{\mathrm{gp}}(\mathcal{X}) \subseteq \mathrm{Mon}_{\mathbb{E}_1}(\mathcal{X})$ denote the full subcategory spanned by the grouplike $\mathbb{E}_1$-monoid objects of $\mathcal{X}$.

If $k > 0$, then we will say that an $\mathbb{E}_k$-monoid object $X : \mathbb{E}_k^{\otimes} \to \mathcal{X}$ is *grouplike* if the composite map $\mathbb{E}_1 \hookrightarrow \mathbb{E}_k^{\otimes} \xrightarrow{X} \mathcal{X}$ is an grouplike $\mathbb{E}_1$-monoid object of $\mathcal{X}$. We let $\mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}(\mathcal{X}) \subseteq \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{X})$ denote the full subcategory spanned by the grouplike $\mathbb{E}_k$-monoid objects.

Remark 5.1.3.3. Let X be an $\mathbb{E}_k$-monoid object of an ∞-topos $\mathcal{X}$, for $k > 0$, and let us abuse notation by identifying X with the underlying object $X(\langle 1 \rangle) \in \mathcal{X}$. Then X is equipped with a multiplication map $X \times X \to X$, which is associative up to homotopy. Using Lemma T.6.5.1.2, we deduce that $\tau_{\leq 0} X$ is an associative monoid in the homotopy category of $\tau_{\leq 0} \mathcal{X}$. Unwinding the definitions, we see that X is grouplike if and only if $\tau_{\leq 0} X$ is a group object of the homotopy category of $\tau_{\leq 0} \mathcal{X}$. In particular, the condition that X is grouplike does not depend on which of the natural embeddings $\mathbb{E}_1^{\otimes} \hookrightarrow \mathbb{E}_k^{\otimes}$ is chosen.

Remark 5.1.3.4. Let X be an $\mathbb{E}_k$-monoid object of an ∞-topos $\mathfrak{X}$ for $k > 0$. If X is 1-connective, then $\tau_{\leq 0} X$ is a final object of $h\tau_{\leq 0}\, \mathfrak{X}$, so that X is grouplike.

Remark 5.1.3.5. An $\mathbb{E}_k$-monoid object X of the ∞-category $\mathcal{S}$ of spaces is *grouplike* if and only if the monoid $\pi_0 X$ is a group. Note that the truncation functor $X \to \pi_0 X$ preserves colimits. Since the category of (commutative) groups is stable under colimits in the larger category of (commutative) monoids, we deduce that $\mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}(\mathcal{S})$ is stable under small colimits in $\mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$.

We now prove an abstract version of our main result:

Theorem 5.1.3.6. *Let $k > 0$, let $\mathfrak{X}$ be an ∞-topos, and let $\mathfrak{X}_*^{\geq k}$ denote the full subcategory of $\mathfrak{X}_*$ spanned by those pointed spaces which are k-connective. Then there is a canonical equivalence of ∞-categories $\alpha :$ $\mathfrak{X}_*^{\geq k} \simeq \mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}(\mathfrak{X})$.*

Proof. We first observe that the forgetful functor $\mathrm{Mon}_{\mathbb{E}_k}(\mathfrak{X}_*) \to \mathrm{Mon}_{\mathbb{E}_k}(\mathfrak{X})$ is an equivalence of ∞-categories. This map fits into a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Alg}_{\mathbb{E}_k}(\mathfrak{X}_*) & \longrightarrow & \mathrm{Alg}_{\mathbb{E}_k}(\mathfrak{X}) \\
\downarrow & & \downarrow \\
\mathrm{Mon}_{\mathbb{E}_k}(\mathfrak{X}_*) & \longrightarrow & \mathrm{Mon}_{\mathbb{E}_k}(\mathfrak{X}).
\end{array}
$$

The vertical maps are categorical equivalences by Proposition 2.4.2.5, so we are reduced to proving that the upper horizontal map is a categorical equivalence. The ∞-category $\mathrm{Alg}_{\mathbb{E}_k}(\mathfrak{X}_*)$ is equivalent to $\mathrm{Alg}_{\mathbb{E}_k}(\mathfrak{X})_{\mathbf{1}/}$, where $\mathbf{1}$ is a trivial $\mathbb{E}_k$-algebra in $\mathfrak{X}$; since $\mathbb{E}_k^{\otimes}$ is unital, it follows from Proposition 3.2.1.8 that the forgetful functor $\mathrm{Alg}_{\mathbb{E}_k}(\mathfrak{X})_{\mathbf{1}/} \to \mathrm{Alg}_{\mathbb{E}_k}(\mathfrak{X})$ is an equivalence of ∞-categories. It will therefore suffice to construct an equivalence $\alpha' : \mathfrak{X}_*^{\geq k} \to \mathrm{Mon}_{\mathbb{E}_k}(\mathfrak{X})$. The construction proceeds by recursion on k. Suppose first that $k > 0$, so we can write $k = k_- + k_+$ where $0 < k_-, k_+ < k$. The inductive hypothesis guarantees the existence of equivalences $\alpha'_- : \mathfrak{X}_*^{\geq k_-} \to \mathrm{Mon}_{\mathbb{E}_{k_-}}^{\mathrm{gp}}(\mathfrak{X}_*)$ and $\alpha'_+ : \mathfrak{X}_*^{\geq k_+} \to \mathrm{Mon}_{\mathbb{E}_{k_+}}^{\mathrm{gp}}(\mathfrak{X}_*)$. Note that a pointed object X of $\mathfrak{X}$ is k-connective if and only if $\Omega^{k''} X$ is k'-connective; moreover, any k'-connective $\mathbb{E}_{k''}$-monoid object of $\mathfrak{X}_*$ is automatically grouplike by Remark 5.1.3.4. It follows that α'_+ and α'_- induce an equivalence

$$
\gamma : \mathfrak{X}_*^{\geq k} \simeq \mathrm{Mon}_{\mathbb{E}_{k_+}}(\mathrm{Mon}_{\mathbb{E}_{k_-}}(\mathfrak{X}_*)).
$$

Let $\delta : \mathrm{Mon}_{\mathbb{E}_k}(\mathfrak{X}_*) \to \mathrm{Mon}_{\mathbb{E}_{k_+}}(\mathrm{Mon}_{\mathbb{E}_{k_-}}(\mathfrak{X}_*))$ be the map induced by the ∞-operad bifunctor $\mathbb{E}_{k_-}^{\otimes} \times \mathbb{E}_{k_+}^{\otimes} \to \mathbb{E}_k^{\otimes}$. Then δ fits into a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Alg}_{\mathbb{E}_k}(\mathfrak{X}_*) & \xrightarrow{\ \delta'\ } & \mathrm{Alg}_{\mathbb{E}_{k_+}}(\mathrm{Alg}_{\mathbb{E}_{k_-}}(\mathfrak{X}_*)) \\
\downarrow & & \downarrow \\
\mathrm{Mon}_{\mathbb{E}_k}(\mathfrak{X}_*) & \xrightarrow{\ \delta\ } & \mathrm{Mon}_{\mathbb{E}_{k_+}}(\mathrm{Mon}_{\mathbb{E}_{k_-}}(\mathfrak{X}_*)).
\end{array}
$$

The vertical maps are categorical equivalences by Proposition 2.4.2.5, and the map δ' is a categorical equivalence by Theorem 5.1.2.2; it follows that δ is likewise a categorical equivalence. Let δ^{-1} be a homotopy inverse to δ. We now complete the proof by setting $\alpha' = \delta^{-1} \circ \gamma$.

It remains to treat the case where $k = 1$. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_+^{op}, \mathfrak{X})$ spanned by those augmented simplicial objects $X_\bullet$ satisfying the following conditions:

  (i) The underlying map $f : X_0 \to X_{-1}$ is an effective epimorphism in $\mathfrak{X}$.

  (ii) The augmented simplicial object $X_\bullet$ is a Čech nerve of f.

(iii) The object $X_0 \in \mathfrak{X}$ is final.

Using Proposition T.4.3.2.15, we deduce that the construction $X_\bullet \mapsto f$ determines a trivial Kan fibration from $\mathcal{C}$ to the full subcategory of $\operatorname{Fun}(\Delta^1, \mathfrak{X})$ spanned by those morphisms $f : X_0 \to X_{-1}$ where X_0 is a final object of $\mathfrak{X}$ and f is an effective epimorphism. Let $\phi_0 : \mathfrak{X}_*^{\geq 1} \to \mathcal{C}$ be a section of this trivial Kan fibration. Let $\operatorname{Mon}(\mathfrak{X}) \subseteq \operatorname{Fun}(\operatorname{N}(\boldsymbol{\Delta}^{op}), \mathfrak{X})$ be the ∞-category of monoid objects of $\mathfrak{X}$, and $\operatorname{Mon}^{\mathrm{gp}}(\mathfrak{X}) \subseteq \operatorname{Mon}(\mathfrak{X})$ the full subcategory spanned by the grouplike monoid objects. Since $\mathfrak{X}$ is an ∞-topos, the restriction map $\phi_1 : \mathcal{C} \to \operatorname{Mon}^{\mathrm{gp}}(\mathfrak{X})$ is an equivalence of ∞-categories. Using Propositions 4.1.2.6, we deduce that the restriction functor $\operatorname{Mon}_{\mathcal{A}ss}^{\mathrm{gp}}(\mathfrak{X}) \to \operatorname{Mon}^{\mathrm{gp}}(\mathfrak{X})$ is an equivalence of ∞-categories which admits a homotopy inverse ϕ_2. Let $\phi_3 : \operatorname{Mon}_{\mathcal{A}ss}^{\mathrm{gp}}(\mathfrak{X}) \to \operatorname{Mon}_{\mathbb{E}_1}^{\mathrm{gp}}(\mathfrak{X})$ be the equivalence of ∞-categories induced by the categorical equivalence $\mathbb{E}_1^{\otimes} \to \mathcal{A}ss^{\otimes}$ of Example 5.1.0.7. We now define α to be the composite equivalence

$$\mathfrak{X}_*^{\geq 1} \overset{\phi_0}{\to} \mathcal{C} \overset{\phi_1}{\to} \operatorname{Mon}^{\mathrm{gp}}(\mathfrak{X}) \overset{\phi_2}{\to} \operatorname{Mon}_{\mathcal{A}ss}^{\mathrm{gp}}(\mathfrak{X}) \overset{\phi_3}{\to} \operatorname{Mon}_{\mathbb{E}_1}^{\mathrm{gp}}(\mathfrak{X}).$$

$\square$

Corollary 5.1.3.7. *The loop functor* $\Omega : \mathcal{S}_*^{\geq 1} \to \mathcal{S}$ *is conservative and preserves sifted colimits.*

Proof. Using Theorem 5.1.3.6, we may reduce to the problem of showing that the forgetful functor $\theta : \operatorname{Mon}_{\mathbb{E}_1}^{\mathrm{gp}}(\mathcal{S}) \to \mathcal{S}$ is conservative and preserves sifted colimits. Since $\operatorname{Mon}_{\mathbb{E}_1}^{\mathrm{gp}}(\mathcal{S})$ is stable under colimits in $\operatorname{Mon}_{\mathbb{E}_1}(\mathcal{S})$, it suffices to show that the forgetful functor $\operatorname{Mon}_{\mathbb{E}_1}(\mathcal{S}) \to \mathcal{S}$ is conservative and preserves sifted colimits. This follows from Proposition 2.4.2.5, Proposition 3.2.3.1, and Lemma 3.2.2.6. $\square$

Corollary 5.1.3.8. *For every integer* $n \geq 0$*, the loop functor* $\Omega : \mathcal{S}_*^{\geq n+1} \to \mathcal{S}_*^{\geq n}$ *is conservative and preserves sifted colimits.*

Corollary 5.1.3.9. *Let* $\operatorname{Sp}_{\geq 0}$ *denote the* ∞*-category of connective spectra. Then the functor* $\Omega_*^{\infty} : \operatorname{Sp}_{\geq 0} \to \mathcal{S}_*$ *is conservative and preserves sifted colimits.*

Proof. Write $\operatorname{Sp}_{\geq 0}$ as the limit of the tower $\cdots \to \mathcal{S}_*^{\geq 1} \overset{\Omega}{\to} \mathcal{S}_*^{\geq 0}$ and apply Corollary 5.1.3.8. $\square$

Remark 5.1.3.10. Let $\mathfrak{X}$ be an ∞-topos, and regard $\mathfrak{X}$ as endowed with the Cartesian symmetric monoidal structure. Theorem 5.1.3.6 guarantees the existence of an equivalence $\theta : \mathfrak{X}_*^{\geq 1} \simeq \operatorname{Mon}^{\mathrm{gp}}(\mathfrak{X}) \simeq \operatorname{Alg}^{\mathrm{gp}}(\mathfrak{X})$, where $\operatorname{Alg}^{\mathrm{gp}}(\mathfrak{X})$ denotes the essential image of $\operatorname{Mon}^{\mathrm{gp}}(\mathfrak{X})$ under the equivalence of ∞-categories $\operatorname{Mon}(\mathfrak{X}) \simeq \operatorname{Alg}(\mathfrak{X})$ supplied by Propositions 4.1.2.6 and 2.4.2.5. This equivalence fits into a commutative diagram

$$
\begin{array}{ccc}
\operatorname{Fun}(\Delta^1, \mathfrak{X}) \times_{\operatorname{Fun}(\{1\}, \mathfrak{X})} \mathfrak{X}_*^{\geq 1} & \overset{\overline{\theta}}{\longrightarrow} & \operatorname{LMod}^{\mathrm{gp}}(\mathfrak{X}) \\
\downarrow & & \downarrow \\
\mathfrak{X}_*^{\geq 1} & \overset{\theta}{\longrightarrow} & \operatorname{Alg}^{\mathrm{gp}}(\mathfrak{X}),
\end{array}
$$

where $\operatorname{LMod}^{\mathrm{gp}}(\mathfrak{X})$ denotes the fiber product $\operatorname{LMod}(\mathfrak{X}) \times_{\operatorname{Alg}(\mathfrak{X})} \operatorname{Alg}^{\mathrm{gp}}(\mathfrak{X})$ and $\overline{\theta}$ is an equivalence of ∞-categories. In other words, if $X \in \mathfrak{X}$ is a pointed connected object, then there is a canonical equivalence between the ∞-topos $\mathfrak{X}_{/X}$ and the ∞-category $\operatorname{LMod}_{\theta(X)}(\mathfrak{X})$ of $\theta(X)$-module objects of $\mathfrak{X}$.

To prove this, we let $\mathcal{D}$ denote the full subcategory of $\operatorname{Fun}(\Delta^1 \times \operatorname{N}(\boldsymbol{\Delta}_+)^{op}, \mathfrak{X})$ spanned by those functors F with the following properties:

(i) The functor F is a right Kan extension of its restriction to the full subcategory $\mathcal{K} \subseteq \Delta^1 \times \operatorname{N}(\boldsymbol{\Delta}_+)^{op}$ spanned by the objects $(0, [-1])$, $(1, [-1])$, and $(1, [0])$.

(ii) The object $F(1, [0]) \in \mathfrak{X}$ is final.

(iii) The augmentation map $F(1, [0]) \to F(1, [-1])$ is an effective epimorphism (equivalently, the object $F(1, [-1]) \in \mathfrak{X}$ is 1-connective).

It follows from Proposition T.4.3.2.15 that the restriction map $F \mapsto F|\mathcal{K}$ determines a trivial Kan fibration $\mathcal{D} \to \mathrm{Fun}(\Delta^1, \mathcal{X}) \times_{\mathrm{Fun}(\{1\}, \mathcal{X})} \mathcal{X}_*^{\geq 1}$. Recall that we have a canonical equivalence $\mathrm{LMod}(\mathcal{X}) \simeq \mathrm{LMon}(\mathcal{X})$ (Propositions 4.2.2.9 and 2.4.2.5). To construct the functor $\bar{\theta}$, it will suffice to show that the restriction functor $F \mapsto F|(\Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op})$ is a trivial Kan fibration from $\mathcal{D}$ onto $\mathrm{LMon}(\mathcal{X}) \times_{\mathrm{Mon}(\mathcal{X})} \mathrm{Mon}^{\mathrm{gp}}(\mathcal{X})$, where $\mathrm{LMon}(\mathcal{X})$ is defined as in Definition 4.2.2.2. Using Proposition T.4.3.2.8, we see that (i) is equivalent to the following pair of assertions:

(i_0) The restriction $F|(\{1\} \times \mathrm{N}(\boldsymbol{\Delta}_+)^{op})$ is a right Kan extension of its restriction to $\{1\} \times \mathrm{N}(\boldsymbol{\Delta}_+^{\leq 0})^{op}$.

(i_1) The functor F determines a Cartesian natural transformation from $F_0 = F|(\{0\} \times \mathrm{N}(\boldsymbol{\Delta}_+)^{op})$ to $F_1 = F|(\{1\} \times \mathrm{N}(\boldsymbol{\Delta}_+)^{op})$.

Assertions (i_0), (ii), and (iii) are equivalent to requirement that the functor F_1 belongs to the full subcategory $\mathcal{C} \subseteq \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_+)^{op}, \mathcal{X})$ appearing in the proof of Theorem 5.1.3.6. In particular, these conditions guarantee that F_1 is a colimit diagram. Combining this observation with Theorem T.6.1.3.9 allows us to replace (i_1) by the following pair of conditions:

(i_1') The functor F_0 is a colimit diagram.

(i_1'') The restriction $F|(\Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op})$ is a Cartesian transformation from $F_0|\mathrm{N}(\boldsymbol{\Delta})^{op}$ to $F_1|\mathrm{N}(\boldsymbol{\Delta})^{op}$.

It follows that $\mathcal{Y}$ can be identified with the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{N}(\boldsymbol{\Delta}_+)^{op})$ spanned by those functors F such that $F' = F|(\Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op})$ belongs to $\mathrm{LMon}(\mathcal{X}) \times_{\mathrm{Mon}(\mathcal{X})} \mathrm{Mon}^{\mathrm{gp}}(\mathcal{X})$ and F is a left Kan extension of F'. The desired result now follows from Proposition T.4.3.2.15.

Remark 5.1.3.11. In the situation of Remark 5.1.3.10, let X be a pointed 1-connective object of the ∞-topos $\mathcal{X}$. Under the equivalence $\mathcal{X}_{/X} \simeq \mathrm{LMod}_{\theta(X)}(\mathcal{X})$, the forgetful functor $\mathrm{LMod}_{\theta X}(\mathcal{X}) \to \mathcal{X}$ corresponds to the functor $(Y \to X) \mapsto (Y \times_X \mathbf{1})$ given by passing to the fiber over the base point $\eta : \mathbf{1} \to X$ (here $\mathbf{1}$ denotes the final object of $\mathcal{X}$). It follows that the free module functor $\mathcal{X} \to \mathrm{LMod}_{\theta(X)}(\mathcal{X})$ corresponds to the functor $\mathcal{X} \simeq \mathcal{X}_{/\mathbf{1}} \to \mathcal{X}_{/X}$ given by composition with η.

We note that the loop functor $\Omega : \mathcal{S}_*^{\geq 1} \to \mathcal{S}$ is corepresentable by the 1-sphere $S^1 \in \mathcal{S}_*^{\geq 1}$. It follows from Corollary 5.1.3.7 that S^1 is a compact projective object of $\mathcal{S}_*^{\geq 1}$. Since the collection of compact projective objects of $\mathcal{S}_*^{\geq 1}$ is stable under finite coproducts, we deduce the following:

Corollary 5.1.3.12. *Let F be a finitely generated free group, and BF its classifying space. Then BF is a compact projective object of $\mathcal{S}_*^{\geq 1}$.*

For each $n \geq 0$, let $F(n)$ denote the free group on n generators, and $BF(n)$ a classifying space for $F(n)$. Let $\mathcal{F}$ denote the full subcategory of the category of groups spanned by the objects $\{F(n)\}_{n \geq 0}$. We observe that the construction $F(n) \mapsto BF(n)$ determines a fully faithful embedding $i : \mathrm{N}(\mathcal{F}) \to \mathcal{S}_*^{\geq 1}$. Let $\mathcal{P}_\Sigma(\mathrm{N}(\mathcal{F}))$ be defined as in §T.5.5.8 (that is, $\mathcal{P}_\Sigma(\mathrm{N}(\mathcal{F}))$ is the ∞-category freely generated by $\mathrm{N}(\mathcal{F})$ under sifted colimits).

Remark 5.1.3.13. According to Corollary T.5.5.9.3, the ∞-category $\mathcal{P}_\Sigma(\mathrm{N}(\mathcal{F}))$ is equivalent to the underlying ∞-category of the simplicial model category $\mathbf{A}$ of simplicial groups.

It follows from Proposition T.5.5.8.15 that the fully faithful embedding i is equivalent to a composition

$$\mathrm{N}(\mathcal{F}) \xrightarrow{j} \mathcal{P}_\Sigma(\mathrm{N}(\mathcal{F})) \xrightarrow{F} \mathcal{S}_*^{\geq 1},$$

where F is a functor which preserves sifted colimits (moreover, the functor F is essentially unique).

Corollary 5.1.3.14. *The functor $F : \mathcal{P}_\Sigma(\mathrm{N}(\mathcal{F})) \to \mathcal{S}_*^{\geq 1}$ is an equivalence of ∞-categories.*

Remark 5.1.3.15. Combining Corollary 5.1.3.14 and Remark 5.1.3.13, we recover the following classical fact: the homotopy theory of pointed connected spaces is equivalent to the homotopy theory of simplicial groups. See [57] for an explicit combinatorial version of this equivalence.

Proof of Corollary 5.1.3.14. Since $i : \mathrm{N}(\mathcal{F}) \to \mathcal{S}_*^{\geq 1}$ is fully faithful and its essential image consists of compact projective objects (Corollary 5.1.3.12), Proposition T.5.5.8.22 implies that F is fully faithful. We observe that the functor i preserves finite coproducts, so that F preserves small colimits by virtue of Proposition T.5.5.8.15. Using Corollary T.5.5.2.9, we deduce that F admits a right adjoint G. Since F is fully faithful, G is a colocalization functor; to complete the proof, it will suffice to show that G is conservative.

Let $f : X \to Y$ be a morphism in $\mathcal{S}_*^{\geq 1}$ such that $G(f)$ is an equivalence; we wish to prove that f is an equivalence. Let $\mathbf{Z}$ be the free group on one generator, and $j\mathbf{Z}$ its image in $\mathcal{P}_\Sigma(\mathrm{N}(\mathcal{F}))$. Then f induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{S}_*^{\geq 1}}(S^1, X) \simeq \mathrm{Map}_{\mathcal{P}_\Sigma(\mathrm{N}(\mathcal{F}))}(j\mathbf{Z}, GX) \to \mathrm{Map}_{\mathcal{P}_\Sigma(\mathrm{N}(\mathcal{F}))}(j\mathbf{Z}, GY) \simeq \mathrm{Map}_{\mathcal{S}_*^{\geq 1}}(S^1, Y).$$

It follows that $\Omega(f) : \Omega X \to \Omega Y$ is a homotopy equivalence, so that f is a homotopy equivalence by virtue of Corollary 5.1.3.7. $\qquad\square$

We are now ready to prove a more precise version of Theorem 5.1.3.6:

Theorem 5.1.3.16. *Let $k > 0$, and let $\beta : \mathcal{S}_* \to \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$ be the functor described at the beginning of this section. Then $\beta|\, \mathcal{S}_*^{\geq k}$ is equivalent to the functor α constructed in the proof of Theorem 5.1.3.6, so that β induces an equivalence of ∞-categories $\mathcal{S}_*^{\geq k} \to \mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}(\mathcal{S})$.*

Proof. As before, we work by induction on k. Suppose first that $k = 1$, and let α^{-1} be a homotopy inverse to α. We wish to show that the composition $\theta : \alpha^{-1} \circ \beta$ is equivalent to the identity functor from $\mathcal{S}_*^{\geq 1}$ to itself. Let $F : \mathcal{P}_\Sigma(\mathrm{N}(\mathcal{F})) \to \mathcal{S}_*^{\geq 1}$ be the equivalence of ∞-categories of Corollary 5.1.3.14; it will suffice to construct an equivalence $F \simeq \theta \circ F$.

Let $\Omega : \mathcal{S}_*^{\geq 1} \to \mathcal{S}$ denote the loop space functor. It is easy to see that there is an equivalence $\Omega \circ \theta \simeq \Omega$. Using Corollary 5.1.3.7, we deduce that θ commutes with sifted colimits. In view of Proposition T.5.5.8.15, it will suffice to show that $F \circ j$ is equivalent to $\theta \circ F \circ j$ in the ∞-category $\mathrm{Fun}(\mathrm{N}(\mathcal{F}), \mathcal{S}_*^{\geq 1})$; here $j : \mathrm{N}(\mathcal{F}) \to \mathcal{P}_\Sigma(\mathrm{N}(\mathcal{F}))$ denotes the Yoneda embedding so that $F \circ j$ is equivalent to the classifying space functor $i : \mathrm{N}(\mathcal{F}) \to \mathcal{S}_*^{\geq 1}$.

The functor $\pi_0 : \mathcal{S} \to \mathrm{N}(\mathrm{Set})$ induces a functor from $U : \mathrm{Mon}_{\mathbb{E}_1}^{\mathrm{gp}}(\mathcal{S})$ to the nerve of the category $\mathcal{G}$ of groups. This functor admits a right adjoint T, given by the fully faithful embedding

$$\mathrm{N}(\mathcal{G}) \simeq \mathrm{Mon}_{\mathbb{E}_1}^{\mathrm{gp}}(\mathrm{N}(\mathrm{Set})) \subseteq \mathrm{Mon}_{\mathbb{E}_1}^{\mathrm{gp}}(\mathcal{S}).$$

The composition $U \circ \beta$ can be identified with the functor which carries a pointed space X to its fundamental group $\pi_1 X$, while $\alpha^{-1} \circ T$ carries a group G to a classifying space $BG \in \mathcal{A} \subseteq \mathcal{S}_*^{\geq 1}$. Consequently, on $\mathcal{S}_*^{\geq 1}$, the composition $\alpha^{-1} \circ T \circ U \circ \beta$ agrees with the truncation functor $\tau_{\leq 1}$, so there is a natural transformation of functors $v : \mathrm{id}_{\mathcal{S}_*^{\geq 1}} \to \alpha^{-1} \circ T \circ U \circ \beta$. Since T and U are adjoint, we also have a unit transformation $u : \theta \to \alpha^{-1} \circ T \circ U \circ \beta$. The natural transformation v is an equivalence when restricted to 1-truncated spaces, and the natural transformation u is an equivalence when restricted to spaces X such that ΩX is discrete. In particular, u and v are both equivalences on the essential image of the fully faithful embedding $i : \mathrm{N}(\mathcal{F}) \to \mathcal{S}_*^{\geq 1}$. It follows that u and v determines an equivalence of functors

$$F \circ j \simeq i \simeq \alpha^{-1} \circ T \circ U \circ \beta \circ i \simeq \alpha^{-1} \circ \beta \circ i \simeq \theta \circ F \circ j.$$

This completes the proof in the case $k = 1$.

Suppose now that $k > 1$. We observe that the functor β factors as a composition

$$\mathcal{S}_*^{\geq k} \xrightarrow{\beta'} \mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}(\mathcal{S}_*) \xrightarrow{\beta''} \mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}(\mathcal{S}),$$

where the functor β'' is an equivalence of ∞-categories (as in the proof of Theorem 5.1.3.6). Consequently, it will suffice to show that β' is equivalent to the functor α' constructed in the proof of Proposition 5.1.3.6. Write $k = k_- + k_+$, where $0 < k_-, k_+ < k$. By the inductive hypothesis, we may assume that the functors

$$\beta'_- : \mathcal{S}_*^{\geq 1} \to \mathrm{Mon}_{\mathbb{E}_{k_-}}(\mathcal{S}_*) \qquad \beta'_+ : \mathcal{S}_*^{\geq 1} \to \mathrm{Mon}_{\mathbb{E}_{k_+}}(\mathcal{S}_*)$$

are equivalent to the functors α'_- and α'_+ constructed in the proof of Proposition 5.1.3.6. The equivalence of α' and β' follows from the homotopy commutativity of the diagram

$$
\begin{array}{ccc}
\mathcal{S}^{\geq k}_* & \xrightarrow{\ \beta'_+\ } & \mathrm{Mon}_{\mathbb{E}_{k_+}}(\mathcal{S}^{\geq k_-}_*) \\
\downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \beta'_-} \\
\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathcal{S}_*) & \xrightarrow{\ \delta\ } & \mathrm{Mon}_{\mathbb{E}_{k_+}}(\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_{k_-}}(\mathcal{S}_*)).
\end{array}
$$

$\square$

Remark 5.1.3.17. Taking the limit of Theorem 5.1.3.16 as $k \to \infty$, we can identify the ∞-category $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathcal{S}) \simeq \varprojlim \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathcal{S})$ with the homotopy limit of the tower of ∞-categories

$$
\cdots \to \mathcal{S}^{\geq 2}_* \xrightarrow{\Omega} \mathcal{S}^{\geq 1}_* \xrightarrow{\Omega} \mathcal{S}^{\geq 0}_*,
$$

which is the ∞-category $\mathrm{Sp}^{\mathrm{cn}}$ of connective spectra.

5.1.4 Tensor Products of $\mathbb{E}_k$-Modules

Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $A \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$ be an $\mathbb{E}_k$-algebra object of $\mathcal{C}$. Under some mild assumptions, the ∞-category $\mathrm{Mod}^{\mathbb{E}_k}_A(\mathcal{C})$ of has the structure of an $\mathbb{E}_k$-monoidal ∞-category (see Corollary 3.4.4.6). In particular, if $k \geq 1$ then we have a tensor product operation

$$
\otimes : \mathrm{Mod}^{\mathbb{E}_k}_A(\mathcal{C}) \times \mathrm{Mod}^{\mathbb{E}_k}_A(\mathcal{C}) \to \mathrm{Mod}^{\mathbb{E}_k}_A(\mathcal{C}).
$$

Our goal in this section is to show that this operation is, in some sense, independent of n. More precisely, we show that there is a monoidal forgetful functor $\mathrm{Mod}^{\mathbb{E}_k}_A(\mathcal{C}) \to \mathrm{Mod}^{\mathbb{E}_1}_{A'}(\mathcal{C})$ is monoidal, where A' denotes the image of A in $\mathrm{Alg}(\mathcal{C})$. In other words, there is a forgetful functor from $\mathrm{Mod}^{\mathbb{E}_k}_A(\mathcal{C})$ to the ∞-category $_A\mathrm{BMod}_A(\mathcal{C})$ of A-A-bimodule objects of $\mathcal{C}$, and that the tensor product $\otimes$ corresponds (under this forgetful functor) to the relative tensor product $\otimes_A$ of §4.3.5.

Construction 5.1.4.1. Let $f : \mathcal{O}'^\otimes \to \mathcal{O}^\otimes$ be a map of coherent ∞-operads. Then f induces a map $F : \mathcal{K}_{\mathcal{O}'} \to \mathcal{K}_{\mathcal{O}}$ (see Notation 3.3.2.1). Suppose we are given a fibration of ∞-operads $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$, and set $\mathcal{C}'^\otimes = \mathcal{C}^\otimes \times_{\mathcal{O}^\otimes} \mathcal{O}'^\otimes$. Composition with F determines a map $\mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^\otimes \times_{\mathcal{O}^\otimes} \mathcal{O}'^\otimes \to \mathrm{Mod}^{\mathcal{O}'}(\mathcal{C}')^\otimes$. This map fits into a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^\otimes \times_{\mathcal{O}^\otimes} \mathcal{O}'^\otimes & \longrightarrow & \mathrm{Mod}^{\mathcal{O}'}(\mathcal{C}')^\otimes \\
\downarrow & & \downarrow \\
\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) & \longrightarrow & \mathrm{Alg}_{/\mathcal{O}'}(\mathcal{C}').
\end{array}
$$

Consequently, for every algebra object $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ having image $A' \in \mathrm{Alg}_{/\mathcal{O}'}(\mathcal{C}')$, we obtain a map $\mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C})^\otimes \times_{\mathcal{O}^\otimes} \mathcal{O}'^\otimes \to \mathrm{Mod}^{\mathcal{O}'}_{A'}(\mathcal{C}')^\otimes$.

We can now state our main result:

Theorem 5.1.4.2. *Let $k \geq 1$ be an integer and let $q : \mathcal{C}^\otimes \to \mathbb{E}_k^\otimes$ be a coCartesian fibration of ∞-operads. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects. Let $A \in \mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})$. Then:*

(1) *The map $\mathrm{Mod}^{\mathbb{E}_k}_A(\mathcal{C})^\otimes \to \mathbb{E}_k^\otimes$ is a coCartesian fibration of ∞-operads.*

(2) *Let A' denote the image of A in $\mathrm{Alg}_{\mathbb{E}_1/\mathbb{E}_k}(\mathcal{C})$. Then the map*

$$F : \mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})^{\otimes} \times_{\mathbb{E}_k^{\otimes}} \mathbb{E}_1^{\otimes} \to \mathrm{Mod}_{A'}^{\mathbb{E}_1}(\mathcal{C})$$

of Construction 5.1.4.1 is a monoidal functor.

Remark 5.1.4.3. We can regard Theorem 4.4.2.1 as a limiting case of Theorem 5.1.4.2, where we take $k = \infty$.

The remainder of this section is devoted to the proof of Theorem 5.1.4.2. Like the proof of Theorem 5.1.2.2, it rests on a calculation of the homotopy type of certain configuration spaces.

Notation 5.1.4.4. Fix an integer $a \geq 0$. We let $\mathcal{J}_a$ denote the subcategory of $\mathcal{F}\mathrm{in}_{*\langle a \rangle/}$ whose objects are injective maps $\langle a \rangle \to \langle m \rangle$ and whose morphisms are commutative diagrams

$$
\begin{array}{ccc}
 & \langle a \rangle & \\
 \swarrow & & \searrow \\
\langle m \rangle & \xrightarrow{\quad \alpha \quad} & \langle m' \rangle
\end{array}
$$

where α is a bijection. We define a category $\mathcal{J}'_a$ as follows:

- The objects of $\mathcal{J}'_a$ are sequences $(\langle m_1 \rangle, \ldots, \langle m_b \rangle, \rho)$, where $(\langle m_1 \rangle, \ldots, \langle m_b \rangle)$ is an object of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)^{\amalg}$ and $\rho : \langle a \rangle^{\circ} \to \langle m_1 + \cdots + m_b \rangle^{\circ}$ is an injection which induces an injection $\langle a \rangle^{\circ} \to \langle b \rangle^{\circ}$.

- A morphism in $\mathcal{J}'_a$ from $(\langle m_1 \rangle, \ldots, \langle m_b \rangle, \rho)$ to $(\langle m'_1 \rangle, \ldots, \langle m'_{b'} \rangle, \rho')$ is a morphism

$$(\langle m_1 \rangle, \ldots, \langle m_b \rangle) \to (\langle m' \rangle_1, \ldots, \langle m'_{b'} \rangle)$$

  which induces an active map $\langle b \rangle \to \langle b' \rangle$ and a bijection $\alpha : \langle m_1 + \cdots + m_b \rangle \to \langle m'_1 + \cdots + m'_{b'} \rangle$ such that $\rho' = \alpha \circ \rho$.

Let $\mathcal{W}$ be the topological category of Construction 5.1.2.6 (so that $\mathrm{N}(\mathcal{W}) \simeq \mathbb{E}_1^{\otimes} \wr \mathbb{E}_{k-1}^{\otimes}$). We let $\mathcal{W}_a$ denote the fiber product $\mathcal{W} \times_{\mathrm{hN}(\mathcal{F}\mathrm{in}_*)^{\amalg}} \mathcal{J}'_a$. We define a topological functor $T^a : \mathcal{W}_a^{op} \to \mathrm{Top}$ as follows: for every object $(W, \rho) \in \mathcal{W}_a$, we let $T(W)$ be the summand of $\mathrm{Map}_{\mathcal{W}}(W, (\langle 1 \rangle))$ corresponding to those maps $W = (\langle m_1 \rangle, \ldots, \langle m_b \rangle) \to (\langle 1 \rangle)$ which induce active morphisms $\langle b \rangle \to \langle 1 \rangle$ and $\langle m_1 + \cdots + m_b \rangle \to \langle 1 \rangle$ in $\mathcal{F}\mathrm{in}_*$. Let T_0^a denote the space $\mathrm{Rect}(\langle a \rangle^{\circ} \times \square^1, \square^1) \times \mathrm{Rect}(\square^{k-1}, \square^{k-1})^a$. For each $(W, \rho) \in \mathcal{W}_n$, composition with ρ determines a continuous map $T^a(W, \rho) \to T_0^a$.

Let $\mathcal{E}_a$ denote the topological category ${}^t\mathbb{E}_k^{\otimes} \times_{\mathcal{F}\mathrm{in}_*} \mathcal{J}_a$. We will identify objects of $\mathcal{E}_a$ with pairs $(\langle m \rangle, \rho)$, where $\langle m \rangle \in {}^t\mathbb{E}_k^{\otimes}$ and $\rho : \langle a \rangle \to \langle m \rangle$ is an injective map of finite linearly ordered sets. We define a topological functor $U^a : \mathcal{E}_a^{op} \to \mathrm{Top}$ by the formula $U^a(\langle m \rangle, \rho) = \mathrm{Map}^{\mathrm{act}}_{{}^t\mathbb{E}_{k+1}^{\otimes}}(\langle m \rangle, \langle 1 \rangle)$. Let U_0^a be the topological space $\mathrm{Map}^{\mathrm{act}}_{{}^t\mathbb{E}_k^{\otimes}}(\langle a \rangle, \langle 1 \rangle) = \mathrm{Rect}(\langle a \rangle^{\circ} \times \square^k, \square^k)$, so that for every object $(\langle m \rangle, \rho) \in \mathcal{E}_a$ composition with ρ determines a map of topological spaces $U^a(\langle m \rangle, \rho) \to U_0^a$.

Remark 5.1.4.5. The functors T^a and U^a of Notation 5.1.4.4 are given by the composition

$$\mathcal{W}_a^{op} \to \mathcal{W}^{op} \xrightarrow{T} \mathrm{Top}$$

$$\mathcal{E}_a^{\mathrm{op}} \to \mathcal{E}^{op} \xrightarrow{U} \mathrm{Top},$$

where T and U are the functors defined in Notation 5.1.2.10 in the special case $n = 1$. If $a = 0$, then we can identify T^a with T and U^a with U, and the topological spaces T_0^a and U_0^a consist of a single point.

Let $v : \mathfrak{C}[N(\mathcal{W}_a)] \to \mathrm{Sing}(\mathcal{W}_a)$ and $v' : \mathfrak{C}[N(\mathcal{E}_a)] \to \mathrm{Sing}(\mathcal{E}_a)$ be the counit maps. We let $*$ denote the simplicial category consisting of a single object, and regard the simplicial sets $\mathrm{Sing}(U_0)$ and $\mathrm{Sing}(T_0)$ as simplicial functors $* \to \mathcal{S}$. Let $v'' : \mathfrak{C}[\Delta^0] \to *$ be the canonical equivalence. The topological functor $\xi : \mathcal{W} \to^t \mathbb{E}_{k+1}^{\otimes}$ restricts to a topological functor $\xi_0 : \mathcal{W}_a \to \mathcal{E}_a$, and we have a commutative diagram of simplicial sets

$$
\begin{array}{ccc}
\mathrm{Un}_v\,\mathrm{Sing}(T^a) & \longrightarrow & \mathrm{Un}_{v''}\,\mathrm{Sing}(T_0^a) \\
\downarrow & & \downarrow \\
\mathrm{Un}_{v'}\,\mathrm{Sing}(U^a) & \longrightarrow & \mathrm{Un}_{v''}\,\mathrm{Sing}(U_0^a),
\end{array}
$$

where Un denotes the unstraightening functor of §T.2.2.1.

We will need the following result:

Proposition 5.1.4.6. *For every integer $a \geq 0$, the diagram of simplicial sets σ :*

$$
\begin{array}{ccc}
\mathrm{Un}_v\,\mathrm{Sing}(T^a) & \longrightarrow & \mathrm{Un}_{v''}\,\mathrm{Sing}(T_0^a) \\
\downarrow & & \downarrow \\
\mathrm{Un}_{v'}\,\mathrm{Sing}(U^a) & \longrightarrow & \mathrm{Un}_{v''}\,\mathrm{Sing}(U_0^a)
\end{array}
$$

is a homotopy pullback square (with respect to the usual model structure on Set_Δ).

Remark 5.1.4.7. In the situation of Proposition 5.1.4.6, the simplical sets $\mathrm{Un}_{v'}\,\mathrm{Sing}(U^a)$, $\mathrm{Un}_{v''}\,\mathrm{Sing}(U_0^a)$, and $\mathrm{Un}_{v''}\,\mathrm{Sing}(T_0^a)$ are Kan complexes, but $\mathrm{Un}_v\,\mathrm{Sing}(T^a)$ is merely an ∞-category. Consequently, σ is not a homotopy pullback diagram with respect to the Joyal model structure. Let S denote the homotopy fiber product $(\mathrm{Un}_{v'}\,\mathrm{Sing}(U^a)) \times_{\mathrm{Un}_{v''}\,\mathrm{Sing}(U_0^a)} (\mathrm{Un}_{v''}\,\mathrm{Sing}(T_0^a))$, so that σ determines a functor $f : \mathrm{Un}_v\,\mathrm{Sing}(T^a) \to S$. Proposition 5.1.4.6 is equivalent to the assertion that the homotopy fibers of f (computed with respect to the Joyal model structure) are weakly contractible.

Remark 5.1.4.8. In the special case $a = 0$, Proposition 5.1.4.6 follows from Proposition 5.1.2.9.

We will prove Proposition 5.1.4.6 at the end of this section. Let us now see that it implies Theorem 5.1.4.2.

Proof of Theorem 5.1.4.2. Let $q' : \mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})^{\otimes} \to \mathbb{E}_k^{\otimes}$ and $q'' : \mathrm{Mod}_A^{\mathbb{E}_1}(\mathcal{C})^{\otimes} \to \mathbb{E}_1^{\otimes}$ be the projection maps. Fix an object $\vec{M} \in \mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})_{\langle n \rangle}^{\otimes}$ and a morphism $\gamma : \langle n \rangle \to \langle m \rangle$ in $\mathbb{E}_k^{\otimes}$. To prove (1), we must show that γ can be lifted to a q'-coCartesian morphism $\overline{\gamma} : \vec{M} \to N$ in $\mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})^{\otimes}$. We may assume without loss of generality that γ is the image of a morphism $\gamma_0 : \langle n \rangle \to \langle m \rangle$ in $\mathbb{E}_1^{\otimes}$, so that $\overline{\gamma}$ determines a map $T(\overline{\gamma})$ in $\mathrm{Mod}_{A'}^{\mathbb{E}_1}(\mathcal{C})$. To prove (2), we must show that $F(\overline{\gamma})$ is q''-coCartesian.

For $1 \leq i \leq m$, let n_i denote the cardinality of the inverse image of $\{i\}$ under the map of pointed finite sets $\langle n \rangle \to \langle m \rangle$. Factor the map γ_0 as a composition

$$
\langle n \rangle \xrightarrow{\gamma_0'} \langle n' \rangle \xrightarrow{\gamma_0''} \langle m \rangle,
$$

where γ_0' is inert and γ_0'' is active. Replacing γ_0 with γ_0'', we can reduce to the case where γ_0 is active. Then γ_0 is an amalgam of active maps $\langle n_i \rangle \to \langle 1 \rangle$; we may therefore assume that $m = 1$. If $n = 0$, the desired result follows from Example 3.4.4.7. If $n = 1$, the map γ_0 is an equivalence and there is nothing to prove. If $n > 2$, then the map γ_0 factors as a composition

$$
\langle n \rangle \to \langle n - 1 \rangle \to \langle 1 \rangle
$$

each of which has fibers of cardinality $< n$, and the desired result follows from the inductive hypothesis. We may therefore assume without loss of generality that $n = 2$.

Let $\overline{\mathcal{D}} = \Delta^1 \times_{\mathbb{E}_k^\otimes} \mathcal{K}_{\mathbb{E}_k}$ and $\overline{\mathcal{D}}' = \Delta^1 \times_{\mathbb{E}_1} \mathcal{K}_{\mathbb{E}_1}$ be as in the statement of Theorem 3.4.4.3. Let $D' \in \overline{\mathcal{D}}'$ correspond to the identity map $\langle 1 \rangle \to \langle 1 \rangle$ in $\mathbb{E}_1$, and let D be the image of D' in $\overline{\mathcal{D}}$. Set $\mathcal{D} = \overline{\mathcal{D}} \times_{\Delta^1} \{0\}$ and $\mathcal{D}' = \overline{\mathcal{D}}' \times_{\Delta^1} \{0\}$. Let $\mathcal{D}_{/D}^{\mathrm{act}}$ be the full subcategory of $\overline{\mathcal{D}}_{/D} \times_{\overline{\mathcal{D}}} \mathcal{D}$ spanned by those diagrams

$$\begin{array}{ccc} \langle 2 \rangle & \xrightarrow{\;\alpha\;} & X \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} \\ \langle 1 \rangle & \longrightarrow & \langle 1 \rangle \end{array}$$

where β is active (here α is semi-inert), and let $\mathcal{D}_{/D'}^{\prime\,\mathrm{act}}$ be defined similarly.

According to Theorem 3.4.4.3, assertion (1) will follow if we can solve the lifting problem depicted in the diagram

$$\begin{array}{ccc} \mathcal{D}_{/D}^{\mathrm{act}} & \longrightarrow & \mathcal{C}^\otimes \\ \downarrow & {}^{\displaystyle f}\nearrow & \downarrow{\scriptstyle q} \\ (\mathcal{D}_{/D}^{\mathrm{act}})^\triangleright & \longrightarrow & \mathbb{E}_n^\otimes \end{array}$$

so that f is an operadic q-colimit diagram. Moreover, assertion (2) will follow if we show that the composite map $(\mathcal{D}_{/D'}^{\prime\,\mathrm{act}})^\triangleright \to (\mathcal{D}_{/D}^{\mathrm{act}})^\triangleright \xrightarrow{f} \mathcal{C}^\otimes$ is also an operadic q-colimit diagram.

We will prove the following:

($*$) The map of ∞-operads $\mathbb{E}_1 \to \mathbb{E}_k$ induces a left cofinal functor $\mathcal{D}_{/D'}^{\prime\,\mathrm{act}} \to \mathcal{D}_{/D}^{\mathrm{act}}$.

Assuming ($*$), we are reduced to the problem of solving the lifting problem

$$\begin{array}{ccc} \mathcal{D}_{/D'}^{\prime\,\mathrm{act}} & \longrightarrow & \mathcal{C}^\otimes \\ \downarrow & {}^{\displaystyle f'}\nearrow & \downarrow{\scriptstyle q} \\ (\mathcal{D}_{/D'}^{\prime\,\mathrm{act}})^\triangleright & \longrightarrow & \mathbb{E}_n^\otimes \end{array}$$

so that f' is an operadic q-colimit diagram. The equivalence $\mathbb{E}_1^\otimes \to \mathcal{A}\mathrm{ss}^\otimes$ of Example 5.1.0.7 induces an equivalence of $\mathcal{D}_{/D'}^{\prime\,\mathrm{act}}$ with the full subcategory $\mathcal{J}$ of $(\mathcal{A}\mathrm{ss}^\otimes)_{\langle 2 \rangle / / \langle 1 \rangle}$ spanned by those factorizations

$$\begin{array}{ccc} & \langle a \rangle & \\ {}^{\displaystyle\alpha}\nearrow & & \searrow{\scriptstyle\beta} \\ \langle 2 \rangle & \longrightarrow & \langle 1 \rangle \end{array}$$

where α is injective and β is active. The map β determines a linear ordering of $\langle a \rangle^\circ$. Let $\mathcal{J}_0$ be the full subcategory of $\mathcal{J}$ spanned by those diagrams for which the image of α contains the smallest and largest elements of $\langle a \rangle^\circ$. The inclusion $\mathcal{J}_0 \hookrightarrow \mathcal{J}$ admits a left adjoint, and is therefore left cofinal. It will therefore suffice to show that every lifting problem of the form

$$\begin{array}{ccc} \mathcal{J}_0 & \longrightarrow & \mathcal{C}^\otimes \\ \downarrow & {}^{\displaystyle g}\nearrow & \downarrow{\scriptstyle q} \\ \mathcal{J}_0^\triangleright & \longrightarrow & \mathbb{E}_n^\otimes \end{array}$$

admits a solution, where g is an operadic q-colimit diagram. This follows from Proposition 3.1.1.20, since there is an equivalence of ∞-categories $\mathcal{J}_0 \simeq \mathrm{N}(\mathbf{\Delta})^{op}$.

It remains to prove $(*)$. Let $\mathcal{W}$ be as in Construction 5.1.2.6, so that the inclusion $i : \mathbb{E}_1^{\otimes} \hookrightarrow \mathbb{E}_k^{\otimes}$ factors as a composition

$$\mathbb{E}_1^{\otimes} \overset{i'}{\hookrightarrow} \mathrm{N}(\mathcal{W}) \to \mathbb{E}_k^{\otimes}$$

(see Construction 5.1.2.6). Let $\mathcal{D}''$ denote the subcategory of

$$\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{W}))_{/\,\mathrm{id}_{(\langle 1 \rangle)}} \times_{\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{W}))_{/(\langle 1 \rangle)}} \{(\langle 1 \rangle, \langle 1 \rangle)\}$$

whose objects are diagrams

$$
\begin{array}{ccc}
(\langle 1 \rangle, \langle 1 \rangle) & \overset{\alpha}{\longrightarrow} & (\langle m_1 \rangle, \ldots, \langle m_b \rangle) \\
\downarrow & & \downarrow \beta \\
(\langle 1 \rangle) & \longrightarrow & (\langle 1 \rangle)
\end{array}
$$

where the maps $\langle 2 \rangle \to \langle b \rangle$ and $\langle 2 \rangle \to \langle m_1 + \cdots + m_b \rangle$ determined by α are injective, and the maps $\langle b \rangle \to \langle 1 \rangle$ and $\langle m_i \rangle \to \langle 1 \rangle$ determined by β are active. Morphisms in $\mathcal{D}''$ are given by diagrams

$$
\begin{array}{ccc}
(\langle 1 \rangle, \langle 1 \rangle) & \longrightarrow & (\langle m_1' \rangle, \ldots, \langle m_{b'}' \rangle) \\
\downarrow & & \downarrow \beta' \\
(\langle 1 \rangle, \langle 1 \rangle) & \longrightarrow & (\langle m_1 \rangle, \ldots, \langle m_b \rangle) \\
\downarrow & & \downarrow \\
(\langle 1 \rangle) & \longrightarrow & (\langle 1 \rangle)
\end{array}
$$

where β' induces an active map $\langle b' \rangle \to \langle b \rangle$. The map i' induces a functor $\mathcal{D}_{/D}^{\mathrm{act}} \to \mathcal{D}''$. This functor admits a left adjoint and is therefore left cofinal. Consequently, to prove $(*)$ it will suffice to show that i'' induces a left cofinal functor $Q : \mathcal{D}'' \to \mathcal{D}_{/D}^{\mathrm{act}}$.

Fix an object $E \in \mathcal{D}_{/D}^{\mathrm{act}}$. According to Theorem T.4.1.3.1, it will suffice to show that ∞-category $\mathfrak{X} = (\mathcal{D}_{/D}^{\mathrm{act}}) \times_{\mathcal{D}_{/D}^{\mathrm{act}}} \mathcal{D}''$ is weakly contractible. We can identify objects of $\mathfrak{X}$ with pairs $(\overline{E}, u : E \to Q(\overline{E}))$, where $\overline{E}$ is an object of $\mathcal{D}''$ and u is a morphism in $\mathcal{D}_{/D}^{\mathrm{act}}$. Let $\mathfrak{X}^0$ be the full subcategory of $\mathfrak{X}$ spanned by those objects for which u is an equivalence. Using Proposition 5.1.2.7 and Theorem T.4.1.3.1, we deduce that the inclusion $\mathfrak{X}^0 \to \mathfrak{X}$ is right cofinal, and therefore a weak homotopy equivalence. It will therefore suffice to show that $\mathfrak{X}^0$ is weakly contractible. The ∞-category $\mathfrak{X}^0$ can be identified with a homotopy fiber of the map $\mathrm{Un}_v(\mathrm{Sing}(T^2)) \to S$ appearing in Remark 5.1.4.7, and is therefore weakly contractible by Proposition 5.1.4.6. $\square$

We now turn to the proof of Proposition 5.1.4.6. We begin by defining a functor $\widetilde{T}^a : \mathcal{W}_a^{op} \to \mathrm{Top}$, given on objects by the formula $\widetilde{T}^a(W, \rho) = T^a(W) \times (\Box^k)^a$. Given a morphism $\alpha : (W, \rho) \to (W', \rho')$ in $\mathcal{W}_a^{op}$, we let

$$\widetilde{T}^a(\alpha) : T^a(W) \times (\Box^k)^a \to T^a(W') \times (\Box^k)^a$$

be given by

$$(f, \{p_i \in \Box^k\}_{1 \le i \le a}) \mapsto (T^a(\alpha)(f), \{\phi_i(p_i) \in \Box^k\}_{1 \le i \le a},$$

where $\phi_i : \Box^k \to \Box^k$ denotes the rectilinear embedding determined by restricting α to the cube indexed by $\rho(i)$. Define $\widetilde{U}^a : \mathcal{E}_a^{op} \to \mathrm{Top}$ similarly, so that we have natural transformations of functors

$$\widetilde{T}^a \to T^a \qquad \widetilde{U}^a \to U$$

which induce a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Un}_v \, \mathrm{Sing}(\widetilde{T}^a) & \longrightarrow & \mathrm{Un}_v \, \mathrm{Sing}(T^a) & \longrightarrow & \mathrm{Un}_{v''} \, \mathrm{Sing}(T_0^a) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Un}_{v'} \, \mathrm{Sing}(\widetilde{U}^a) & \longrightarrow & \mathrm{Un}_{v'} \, \mathrm{Sing}(U^a) & \longrightarrow & \mathrm{Un}_{v''} \, \mathrm{Sing}(U_0^a).
\end{array}$$

Since $(\square^k)^a$ is contractible, the horizontal maps on the left of the diagram are categorical equivalences (and therefore weak homotopy equivalences). Consequently, to prove Proposition 5.1.4.6, it will suffice to show that the outer rectangle is a homotopy pullback square.

Let $U_0'^a$ denote the open subset of $\mathrm{Conf}(\langle a \rangle^\circ, \square^k) \times U_0^a = \mathrm{Conf}(\langle a \rangle^\circ, \square^k) \times \mathrm{Rect}(\langle a \rangle^\circ \times \square^k, \square^k)$ consisting of those pairs (p, f), where $p : \langle a \rangle^\circ \to \square^k$ is an injective map and $f : \langle a \rangle^\circ \times \square^k \to \square^k$ is a rectilinear embedding satisfying $p(i) \in f(\{i\} \times \square^k)$ for $1 \le i \le a$. Let $T_0'^a$ denote the fiber product $T_0^a \times_{U_0^a} U_0'^a$. We regard $\mathrm{Sing}(T_0'^a)$ and $\mathrm{Sing}(U_0'^a)$ as simplicial functors $\ast \to \mathrm{Set}_\Delta$, so that we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Un}_v \, \mathrm{Sing}(\widetilde{T}^a) & \longrightarrow & \mathrm{Un}_{v''} \, \mathrm{Sing}(T_0'^a) & \longrightarrow & \mathrm{Un}_{v''} \, \mathrm{Sing}(T_0^a) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Un}_{v'} \, \mathrm{Sing}(\widetilde{U}^a) & \longrightarrow & \mathrm{Un}_{v''} \, \mathrm{Sing}(U_0'^a) & \longrightarrow & \mathrm{Un}_{v''} \, \mathrm{Sing}(U_0^a).
\end{array}$$

Here the left horizontal maps are trivial Kan fibrations. We are therefore reduced to showing that the square on the right is a homotopy pullback.

Let $U_0''^a$ denote the configuration space $\mathrm{Conf}(\langle a \rangle^\circ, \square^k)$ and let $T_0''^a$ denote the open subset of $U_0''^a$ consisting of configurations such that the composite map $\langle a \rangle^\circ \to \square^1$ is injective. There are canonical projection maps

$$U_0'^a \to U_0''^a \qquad T_0'^a \to T_0''^a$$

and it follows easily from Lemma 5.1.1.3 that these maps are homotopy equivalences. We regard $\mathrm{Sing}(T_0''^a)$ and $\mathrm{Sing}(U_0''^a)$ as simplicial functors $\ast \to \mathrm{Set}_\Delta$, so that we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Un}_v \, \mathrm{Sing}(\widetilde{T}^a) & \longrightarrow & \mathrm{Un}_{v''} \, \mathrm{Sing}(T_0'^a) & \longrightarrow & \mathrm{Un}_{v''} \, \mathrm{Sing}(T_0''^a) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Un}_{v'} \, \mathrm{Sing}(\widetilde{U}^a) & \longrightarrow & \mathrm{Un}_{v''} \, \mathrm{Sing}(U_0'^a) & \longrightarrow & \mathrm{Un}_{v''} \, \mathrm{Sing}(U_0''^a).
\end{array}$$

where the left horizontal maps are categorical equivalences. It will therefore suffice to show that the outer rectangle is a homotopy pullback square.

Fix a point $p \in T_0''^a$, which we identify with an injective map $\langle a \rangle^\circ \to \square^k$. We will show that the homotopy fiber of the map $\phi : \mathrm{Un}_v \, \mathrm{Sing}(\widetilde{T}^a) \to \mathrm{Un}_{v''} \, \mathrm{Sing}(T_0''^a)$ over the point p is weakly homotopy equivalent to the homotopy fiber of the map $\phi' : \mathrm{Un}_{v'} \, \mathrm{Sing}(\widetilde{U}^a) \to \mathrm{Un}_{v''} \, \mathrm{Sing}(U_0''^a)$ over p. For every object $(W, \rho) \in \mathcal{W}_a^{op}$, the map $\widetilde{T}^a(W, \rho) \to T_0''^a$ is a Serre fibration; let us denote its fiber over p by $\widehat{T}(W, \rho)$. Similarly, for $(E, \rho) \in \mathcal{E}_a^{op}$, the map $\widetilde{U}^a(E, \rho) \to U_0''^a$ is a Serre fibration, and we will denote its fiber by $\widehat{U}(E, \rho)$. We regard $\widehat{T}$ as a topological functor $\mathcal{W}_a^{op} \to \mathrm{Top}$ and $\widehat{U}$ as a topological functor $\mathcal{E}_a^{op} \to \mathrm{Top}$. Note that the homotopy fibers of ϕ and ϕ' can be identified with $\mathrm{Un}_v \, \mathrm{Sing}(\widehat{T})$ and $\mathrm{Un}_{v'} \, \mathrm{Sing}(\widehat{U})$, respectively. Consequently, we can reformulate Proposition 5.1.4.6 as follows:

Proposition 5.1.4.9. *For every $a \ge 0$ and every point $p \in \mathrm{Conf}(\langle a \rangle^\circ, \square^1 \times \square^{k-1})$ which determines an injective map $\langle a \rangle^\circ \to \square^1$, the induced map $\mathrm{Un}_v \, \mathrm{Sing}(\widehat{T}) \to \mathrm{Un}_{v'} \, \mathrm{Sing}(\widehat{U})$ is a weak homotopy equivalence of simplicial sets.*

Remark 5.1.4.10. In what follows, we let

$$T : \mathcal{W}_0^{op} \to \mathrm{Top} \qquad U : \mathcal{E}^{op} \to \mathrm{Top}$$

be as in Notation 5.1.2.10 (where we take the parameter n appearing in Notation 5.1.2.10 to be equal to 1). Unwinding the definitions, we see that for $W = (\langle m_1 \rangle, \ldots, \langle m_b \rangle) \in \mathcal{W}$ and $(W, \rho) \in \mathcal{W}_a$, we can identify $\widehat{T}(W, \rho)$ with the open subset of $T(W)$ consisting of those embeddings $f : \langle m_1 + \cdots + m_b \rangle^\circ \times \square^k \to \square^k$ such that $p(i) \in f(\{\rho(i)\} \times \square^k)$ for $1 \le i \le a$. Similarly, if $(\langle m \rangle, \rho) \in \mathcal{E}_a$, then $\widehat{U}(\langle m \rangle, \rho)$ can be identified with the open subset of $U(\langle m \rangle) = \mathrm{Rect}(\langle m \rangle^\circ \times \square^k, \square^k)$ consisting of those embeddings $f : \langle m \rangle^\circ \times \square^k \to \square^k$ such that $p(i) \in f(\{\rho(i)\} \times \square^k)$ for $1 \le i \le a$.

We will prove Proposition 5.1.4.9 using the same strategy as our proof of Proposition 5.1.2.9: namely, we will assemble $\widehat{T}$ and $\widehat{U}$ by forming homotopy colimits of simpler functors, for which the desired conclusion is easier to verify.

Construction 5.1.4.11. Let P be the partially ordered set defined in Construction 5.1.2.11. For each object $(I, \sim) \in P$, we let $T_{I,\sim} : \mathcal{W}_0^{op} \to \mathrm{Top}$ and $U_{I,\sim} : \mathcal{E}^{op} \to \mathrm{Top}$ be defined as in Construction 5.1.2.11 (here we take the parameter n appearing in Construction 5.1.2.11 to be 1). For every element $(I, \sim) \in P$, we define subfunctors

$$\widehat{T}_{I,\sim} \subseteq \widehat{T} \qquad \widehat{U}_{I,\sim} \subseteq \widehat{U}$$

by the formulas

$$\widehat{T}_{I,\sim}(W, \rho) = \widehat{T}(W, \rho) \times_{T(W)} T_{I,\sim}(W) \qquad \widehat{U}_{I,\sim}(E, \rho) = \widehat{U}(E, \rho) \times_{U(E)} U_{I,\sim}(E).$$

As in the proof of Proposition 5.1.2.9, we are reduced to proving the following trio of assertions:

Proposition 5.1.4.12. *The functor* $\mathrm{Sing}(\widehat{T})$ *is a homotopy colimit of the diagram of functors* $\{\mathrm{Sing}(\widehat{T}_{I,\sim}) : \mathrm{Sing}(\mathcal{W}_a)^{op} \to \mathcal{S}et_\Delta\}_{(I,\sim) \in P}$.

Proposition 5.1.4.13. *The functor* $\mathrm{Sing}(\widehat{U})$ *is a homotopy colimit of the diagram of functors* $\{\mathrm{Sing}(\widehat{U}_{I,\sim}) : \mathrm{Sing}(\mathcal{E}_a)^{op} \to \mathcal{S}et_\Delta\}_{(I,\sim) \in P}$.

Proposition 5.1.4.14. *For every element* $(I, \sim) \in P$, *the canonical map* $\mathrm{Un}_v \widehat{T}_{I,\sim} \to \mathrm{Un}_{v'} \widehat{U}_{I,\sim}$ *is a weak homotopy equivalence.*

Proof of Proposition 5.1.4.12. Fix an object $W = (\langle m_1 \rangle, \ldots, \langle m_b \rangle) \in \mathcal{W}_0$ and an object $(W, \rho) \in \mathcal{W}_a$; we wish to show that $\mathrm{Sing}\,\widehat{T}(W, \rho)$ is a homotopy colimit of the diagram $\{\mathrm{Sing}\,\widehat{T}_{I,\sim}(W, \rho)\}_{(I,\sim) \in P}$. For $1 \le i \le a$, we let $\rho_-(i)$ denote the image of i in $\langle b \rangle^\circ$, and $\rho_+(i)$ the corresponding element of $\langle m_{\rho_-(i)} \rangle^\circ$.

Let X denote the subspace of

$$\mathrm{Conf}(\langle b \rangle^\circ, (-1, 1)) \times \prod_{1 \le a \le b} \mathrm{Conf}(\langle m_a \rangle^\circ, \square^k)$$

consisting of tuples $(f, g_1, \ldots, g_b)$ such that, for $1 \le i \le a$, we have $p(i) = (f(\rho_-(i)), g_{\rho_-(i)}(\rho_+(i)))$. For each $(I, \sim) \in P$, we let $X_{I,\sim}$ denote the open subset of X consisting of those tuples $(f, g_1, \ldots, g_b)$ satisfying the following conditions:

- The image of the map f is contained in $I \subseteq (-1, 1)$, so that f induces a map $\lambda : \langle b \rangle^\circ \to \pi_0 I$.

- If $a, a' \in \langle b \rangle^\circ$ satisfy $\lambda(a) \sim \lambda(a')$, then either $a = a'$ or the maps g_a and $g_{a'}$ have disjoint images.

Lemma 5.1.1.3 determines a homotopy equivalence $\widetilde{T}(W, \rho) \to X$ which restricts to a homotopy equivalence $\widehat{T}_{I,\sim}(W, \rho) \to X_{I,\sim}$ for each $(I, \sim) \in P$. It will therefore suffice to show that $\mathrm{Sing}(X)$ is the homotopy colimit of the diagram of $\{\mathrm{Sing}\,X_{I,\sim}\}_{(I,\sim) \in P}$. According to Theorem A.3.1, it will suffice to prove the following:

($*$) Let $x = (f, g_1, \ldots, g_b)$ be a point of X, and let $P_x = \{(I, \sim) \in P : x \in X_{I,\sim}\}$. Then the simplicial set $\mathrm{N}(P_x)$ is weakly contractible.

The proof now proceeds exactly as in the proof of Proposition 5.1.2.12. Let x be as in ($*$), and let P_x' denote the subset of P_x consisting of those triples where f induces a surjection $\lambda : \langle b \rangle^\circ \to \pi_0 I$. The inclusion $\mathrm{N}(P_x') \hookrightarrow \mathrm{N}(P_x)$ admits a right adjoint and is therefore a weak homotopy equivalence. It will therefore suffice to show that $\mathrm{N}(P_x')$ is weakly contractible.

Let Q be the collection of all equivalence relations $\sim$ on the set $\langle b \rangle^\circ$ which have the following properties:

(i) If $f(i) \leq f(i') \leq f(i'')$ and $i \sim i''$, then $i \sim i' \sim i''$.

(ii) If $i \sim i'$ and the image of g_i intersects the image of $g_{i'}$, then $i = i'$.

We regard Q as a partially ordered set with respect to refinement. Pullback of equivalence relations along λ determines a forgetful functor $\mu : \mathrm{N}(P_x') \to \mathrm{N}(Q)^{op}$. It is easy to see that μ is a Cartesian fibration. The simplicial set $\mathrm{N}(Q)$ is weakly contractible, since Q has a smallest element (given by the equivalence relation where $i \sim i'$ if and only if $i = i'$). We will complete the proof of ($*$) by showing that the fibers of μ are weakly contractible, so that μ is left cofinal (Lemma T.4.1.3.2) and therefore a weak homotopy equivalence.

Fix an equivalence relation $\sim \in Q$. Unwinding the definitions, we see that $\mu^{-1}\{\sim\}$ can be identified with the nerve of the partially ordered set R consisting of those subsets $I \subseteq (-1, 1)$ satisfying the following conditions:

(a) The set I is a finite union of open intervals.

(b) The set I contains the image of f.

(c) If $f(a)$ and $f(a')$ belong to the same connected component of I, then $a \sim a'$.

To see that $\mathrm{N}(R)$ is contractible, it suffices to observe that the partially ordered set R^{op} is filtered: this follows from the fact that R is nonempty (it contains $\bigcup_{a \in \langle b \rangle^\circ} (f(a) - \epsilon, f(a) + \epsilon)$ for sufficiently small $\epsilon > 0$) and is closed under pairwise intersections. $\square$

Proof of Proposition 5.1.4.13. The proof is essentially the same as that of Proposition 5.1.4.12, with some minor modifications. Fix an object $E = \langle m \rangle \in \mathcal{E}$ and an lifting $(E, \rho) \in \mathcal{E}_a$; we wish to show that $\mathrm{Sing}\, \widehat{U}(W, \rho)$ is a homotopy colimit of the diagram $\{\mathrm{Sing}\, \widehat{U}_{I,\sim}(E, \rho)\}_{(I,\sim) \in P}$. Let X denote the subspace of $\mathrm{Conf}(\langle m \rangle^\circ, \square^k)$ consisting of injective maps $f : \langle m \rangle^\circ \to \square^k$ satisfying $f(\rho(i)) = p(i)$ for $1 \leq i \leq a$. We identify such a map f with a pair of maps (f_0, f_1), where $f_0 : \langle m \rangle^\circ \to \square^{k-1}$ and $f_1 : \langle m \rangle^\circ \to \square^1$. For each $(I, \sim) \in P$, we let $X_{I,\sim}$ denote the open subset of X consisting of those maps $f = (f_0, f_1)$ satisfying the following condition:

- The image of the map f_1 is contained in I, so that f_1 induces a map $\lambda : \langle m \rangle^\circ \to \pi_0 I$.

- If $i, j \in \langle m \rangle^\circ$ satisfy $\lambda(i) \sim \lambda(j)$, then either $i = j$ or $f_0(i) \neq f_0(j)$.

Lemma 5.1.1.3 determines a homotopy equivalence $\widetilde{U}(E, \rho) \to X$ which restricts to a homotopy equivalence $\widehat{T}_{I,\sim}(W, \rho) \to X_{I,\sim}$ for each $(I, \sim) \in P$. It will therefore suffice to show that $\mathrm{Sing}(X)$ is the homotopy colimit of the diagram of $\{\mathrm{Sing}\, X_{I,\sim}\}_{(I,\sim) \in P}$. According to Theorem A.3.1, it will suffice to prove the following:

($*$) Let $f = (f_0, f_1)$ be a point of X, and let $P_x = \{(I, \sim) \in P : f \in X_{I,\sim}\}$. Then the simplicial set $\mathrm{N}(P_x)$ is weakly contractible.

Let f be as in ($*$), and let P_x' denote the subset of P_x consisting of those triples where f_1 induces a surjection $\lambda : \langle m \rangle^\circ \to \pi_0 I$. The inclusion $\mathrm{N}(P_x') \hookrightarrow \mathrm{N}(P_x)$ admits a right adjoint and is therefore a weak homotopy equivalence. It will therefore suffice to show that $\mathrm{N}(P_x')$ is weakly contractible.

Let Q be the collection of all equivalence relations $\sim$ on the set $\langle m \rangle^\circ$ which have the following properties:

(i) If $f_1(i) \leq f_1(i') \leq f_1(i'')$ and $i \sim i''$, then $i \sim i' \sim i''$.

(ii) If $i \sim i'$ and $f_0(i) = f_0(i')$, then $i = i'$.

We regard Q as a partially ordered set with respect to refinement. Pullback of equivalence relations along λ determines a Cartesian fibration functor $\mu : \mathrm{N}(P_x') \to \mathrm{N}(Q)^{op}$. The simplicial set $\mathrm{N}(Q)$ is weakly contractible, since Q has a smallest element (given by the equivalence relation where $i \sim j$ if and only if $f_1(i) = f_1(j)$). We will complete the proof of $(*)$ by showing that the fibers of μ are weakly contractible, so that μ is left cofinal (Lemma T.4.1.3.2) and therefore a weak homotopy equivalence.

Fix an equivalence relation $\sim \in Q$. Unwinding the definitions, we see that $\mu^{-1}\{\sim\}$ can be identified with the nerve of the partially ordered set R consisting of those subsets $I \subseteq (-1, 1)$ satisfying the following conditions:

(a) The set I is a finite union of open intervals.

(b) The set I contains the image of f_1.

(c) If $f_0(i)$ and $f_1(j)$ belong to the same connected component of I, then $i \sim j$.

To see that $\mathrm{N}(R)$ is contractible, it suffices to observe that the partially ordered set R^{op} is filtered: this follows from the fact that R is nonempty (it contains $\bigcup_{i \in \langle n \rangle^\circ} (f_1(i) - \epsilon, f_1(i) + \epsilon)$ for sufficiently small $\epsilon > 0$) and closed under pairwise intersections. $\square$

Proof of Proposition 5.1.4.14. Identify $\square^k$ with the product $\square^{k-1} \times \square^1$, and let

$$\pi_0 : \square^k \to \square^{k-1} \qquad \pi_1 : \square^k \to \square^1$$

be the projection maps. We define a functor $\widehat{T}_{I,\sim}^0 : \mathcal{W}_a^{op} \to \mathrm{Top}$ by the formula

$$\widehat{T}_{I,\sim}^0((\langle m_1 \rangle, \ldots, \langle m_b \rangle), \rho) = \{f \in \mathrm{Rect}(\langle b \rangle^\circ \times \square^1, I) : \pi_1(p(i)) \in f(\{\rho_-(i)\} \times \square^1)\},$$

where $\rho_- : \langle a \rangle^\circ \to \langle b \rangle^\circ$ is defined as in the proof of Proposition 5.1.4.12. There is an evident natural transformation $\widehat{T}_{I,\sim} \to \widehat{T}_{I,\sim}^0$, which induces a (projective) fibration of simplicial functors $\mathrm{Sing}(\widehat{T}_{I,\sim}) \to \mathrm{Sing}(\widehat{T}_{I,\sim}^0)$ and therefore a right fibration of simplicial sets $\chi : \mathrm{Un}_v \, \mathrm{Sing}(\widehat{T}_{I,\sim}) \to \mathrm{Un}_v \, \mathrm{Sing}(\widehat{T}_{I,\sim}^0)$. Unwinding the definitions, we can identify an object of $\mathrm{Un}_v \, \mathrm{Sing}(\widehat{T}_{I,\sim}^0)$ with an object $W = (\langle m_1 \rangle, \ldots, \langle m_b \rangle) \in \mathcal{W}_0$, an injective map of pointed finite sets $\rho : \langle a \rangle \to \langle m_1 + \cdots + m_b \rangle$, and a rectilinear embedding $f : \langle b \rangle^\circ \times \square^1 \to I$. Let $\mathcal{X} \subseteq \mathrm{Un}_v \, \mathrm{Sing}(\widehat{T}_{I,\sim}^0)$ be the full subcategory of $\mathrm{Un}_v \, \mathrm{Sing}(\widehat{T}_{I,\sim}^0)$ spanned by those pairs (W, ρ, f) for which f induces a bijection $\langle b \rangle^\circ \to \pi_0 I$. It is not difficult to see that the inclusion $\mathcal{X} \subseteq \mathrm{Un}_v \, \mathrm{Sing}(\widehat{T}_{I,\sim}^0)$ admits a left adjoint and is therefore left cofinal. Since χ is a right fibration, we deduce that the inclusion

$$\mathcal{X} \times_{\mathrm{Un}_v \, \mathrm{Sing}(\widehat{T}_{I,\sim}^0)} \mathrm{Un}_v \, \mathrm{Sing}(\widehat{T}_{I,\sim}) \to \mathrm{Un}_v \, \mathrm{Sing}(\widehat{T}_{I,\sim})$$

is also left cofinal (Lemma T.4.1.3.2), and therefore a weak homotopy equivalence.

Define a functor $\widehat{U}_{I,\sim}^0 : \mathcal{E}_a \to \mathrm{Top}$ by the formula

$$U_{I,\sim}^0(\langle m \rangle, \rho) = \{\lambda \in \mathrm{Hom}_{\mathrm{Set}}(\langle m \rangle^\circ, \pi_0 I) : (\forall i \in \langle a \rangle^\circ) \pi_1(p(i)) \in \lambda(\rho(i))\}.$$

We have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{X} \times_{\mathrm{Un}_v \, \mathrm{Sing}(\widehat{T}_{I,\sim}^0)} (\mathrm{Un}_v \, \mathrm{Sing}(\widehat{T}_{I,\sim}^0)) & \xrightarrow{\;F\;} & \mathrm{Un}_{v'} \, \mathrm{Sing}(\widehat{U}_{I,\sim}) \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{\;\;F_0\;\;} & \mathrm{Un}_{v'} \, \mathrm{Sing}(\widehat{U}_{I,\sim}^0).
\end{array}
$$

We will complete the proof by showing that the upper horizontal map is an equivalence of ∞-categories. Unwinding the definitions, we deduce easily that F_0 is a categorical equivalence. The vertical maps are right fibrations; it will therefore suffice to show that F induces a homotopy equivalence between the fibers of the vertical maps. This amounts to the following assertion:

$(*)$ For every object $(W, \rho) = (((\langle m_1 \rangle, \ldots, \langle m_b \rangle), \rho) \in \mathcal{W}_a$ having image $(E, \rho) = (\langle m_1 + \cdots + m_b \rangle, \rho) = (\langle m \rangle, \rho)$ in $\mathcal{E}_a$, if $\eta \in \widehat{T}^0_{I, \sim}(W, \rho)$ determines an isomorphism $\langle b \rangle^\circ \simeq \pi_0 I$ having image $\eta' \in \widehat{U}^0_{I, \sim}(E, \rho)$, then the induced map

$$\gamma : \mathrm{Sing}(\widehat{T}_{I, \sim}(W) \times_{\widehat{T}^0_{I, \sim}(W)} \{\eta\}) \to \mathrm{Sing}(\widehat{U}_{I, \sim}(E) \times_{\widehat{U}^0_{I, \sim}(E)} \{\eta'\})$$

is a homotopy equivalence.

Let X denote the product $\prod_{1 \le i \le m} \mathrm{Rect}(\square^1, I_i)$, where I_i denotes the connected component of I which is the image of $i \in \langle m \rangle^\circ$ under the map $\eta' : \langle m \rangle^\circ \to \pi_0 I$. Let U denote the open subset of X consisting of those sequences of embeddings $(f_i)_{1 \le i \le m}$ having the property that $\pi_1(p(j)) \in f_{\rho(j)}$ for $1 \le j \le a$. We note that γ is the product of an identity map with the inclusion of a point into $\mathrm{Sing}(U)$. We complete the proof by observing that U is contractible (Lemma 5.1.1.3). $\qquad\square$

5.1.5 Coproducts and Tensor Products

Let $\mathcal{O}^\otimes$ be any ∞-operad, and let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category. Applying Construction 3.2.4.1 to the evident bifunctor of ∞-operads $\mathcal{O}^\otimes \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, we deduce that the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ admits a symmetric monoidal structure (see Proposition 3.2.4.3), given by pointwise tensor product: for $A, B \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ and $X \in \mathcal{O}$, we have $(A \otimes B)(X) \simeq A(X) \otimes B(X)$.

In the special case where $\mathcal{O}^\otimes = \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is the commutative ∞-operad, the tensor product $A \otimes B$ can be identified with the coproduct of A and B in the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) = \mathrm{CAlg}(\mathcal{C})$ (Proposition 3.2.4.7). For other ∞-operads, this is generally not the case. Suppose, for example, that $\mathcal{O}^\otimes$ is the associative ∞-operad, and that $\mathcal{C}$ is the (nerve of the) ordinary category $\mathrm{Vect}_{\mathbf{C}}$ of vector spaces over the field $\mathbf{C}$ of complex numbers. Then $\mathrm{Alg}_{\mathcal{A}\mathrm{ss}}(\mathcal{C})$ is equivalent the nerve of the category of associative $\mathbf{C}$-algebras. Given a pair of associative $\mathbf{C}$-algebras A and B, there is a diagram of associative algebras

$$A \to A \otimes_{\mathbf{C}} B \leftarrow B,$$

but this diagram does not exhibit $A \otimes_{\mathbf{C}} B$ as a coproduct of A and B. Instead, it exhibits $A \otimes_{\mathbf{C}} B$ as the quotient of the coproduct $A \coprod B$ by the (two-sided) ideal generated by commutators $[a, b] = ab - ba$, where $a \in A$ and $b \in B$. In other words, $A \otimes_{\mathbf{C}} B$ is freely generated by A and B subject to the condition that A and B commute in $A \otimes_{\mathbf{C}} B$.

Our goal in this section is to obtain an ∞-categorical generalization of the above assertion. We will replace the ordinary category $\mathrm{Vect}_{\mathbf{C}}$ by an arbitrary symmetric monoidal ∞-category $\mathcal{C}$, and the associative ∞-operad $\mathcal{A}\mathrm{ss}$ by a little k-cubes operad $\mathbb{E}_k$, for any $k \ge 0$ (we can recover the case of associative algebras by taking $k = 1$, by virtue of Example 5.1.0.7). Assume that $\mathcal{C}$ admits small colimits, and that the tensor product on $\mathcal{C}$ preserves small colimits separately in each variable. Then the forgetful functor $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\mathrm{Free} : \mathcal{C} \to \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$ (Corollary 3.1.3.5). Given a pair of objects $C, D \in \mathcal{C}$, the tensor product $\mathrm{Free}(C) \otimes \mathrm{Free}(D)$ is generally not equivalent to the coproduct $\mathrm{Free}(C) \coprod \mathrm{Free}(D) \simeq \mathrm{Free}(C \coprod D)$. To measure the difference, we note that every binary operation $f \in \mathrm{Mul}_{\mathbb{E}_k}(\{\langle 1 \rangle, \langle 1 \rangle\}, \langle 1 \rangle)$ gives rise to a map

$$\phi_f : C \otimes D \to \mathrm{Free}(C \coprod D) \otimes \mathrm{Free}(C \coprod D) \xrightarrow{f} \mathrm{Free}(C \coprod D).$$

Note that the composite map $C \otimes D \to \mathrm{Free}(C \coprod D) \xrightarrow{\psi} \mathrm{Free}(C) \otimes \mathrm{Free}(D)$ does not depend on f. The space of choices for the binary operation f is homotopy equivalent to the configuration space of pairs of points in $\mathbf{R}^k$ (Lemma 5.1.1.3), which is in turn homotopy equivalent to a sphere S^{k-1}. Allowing f to vary, we obtain a map

$$\phi : (C \otimes D) \otimes S^{k-1} \to \mathrm{Free}(C \coprod D)$$

in $\mathcal{C}$, where we regard $\mathcal{C}$ as tensored over the ∞-category $\mathcal{S}$ of spaces as explained in §T.4.4.4. Equivalently, we can view ϕ as a map

$$\mathrm{Free}(C \otimes D \otimes S^{k-1}) \to \mathrm{Free}(C \coprod D),$$

which fits into a diagram

$$\begin{array}{ccc} \mathrm{Free}(C \otimes D \otimes S^{k-1}) & \longrightarrow & \mathrm{Free}(C \coprod D) \\ \downarrow & & \downarrow \\ \mathrm{Free}(C \otimes D) & \longrightarrow & \mathrm{Free}(C) \otimes \mathrm{Free}(D). \end{array}$$

The commutativity of this diagram encodes the fact that $\psi \circ \phi_f$ is independent of f; equivalently, it reflects the idea that C and D "commute" inside the tensor product $\mathrm{Free}(C) \otimes \mathrm{Free}(D)$. The main result of this section can be formulated as follows:

Theorem 5.1.5.1. *Let $k \geq 0$, let $\mathcal{C}$ be a symmetric monoidal ∞-category which admits countable colimits, and assume that the tensor product on $\mathcal{C}$ preserves countable colimits separately in each variable. Let* $\mathrm{Free} :$ $\mathcal{C} \to \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$ *be a left adjoint to the forgetful functor. Then, for every pair of objects $C, D \in \mathcal{C}$, the construction sketched above gives rise to a pushout diagram*

$$\begin{array}{ccc} \mathrm{Free}(C \otimes D \otimes S^{k-1}) & \longrightarrow & \mathrm{Free}(C) \coprod \mathrm{Free}(D) \\ \downarrow & & \downarrow \\ \mathrm{Free}(C \otimes D) & \longrightarrow & \mathrm{Free}(C) \otimes \mathrm{Free}(D) \end{array}$$

in $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$.

Example 5.1.5.2. Suppose that $k = 0$. In this case, we can identify the ∞-category $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$ with the ∞-category $\mathcal{C}_{\mathbf{1}/}$ (Proposition 2.1.3.9; here $\mathbf{1}$ denotes the unit object of $\mathcal{C}$, and the free algebra functor $\mathrm{Free} : \mathcal{C} \to \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$ is given by the formula $C \mapsto \mathbf{1} \coprod C$. In this case, Theorem 5.1.5.1 asserts that the diagram

$$\begin{array}{ccc} \mathbf{1} & \longrightarrow & \mathbf{1} \coprod C \coprod D \\ \downarrow & & \downarrow \\ \mathbf{1} \coprod (C \otimes D) & \longrightarrow & (\mathbf{1} \coprod C) \otimes (\mathbf{1} \coprod D) \end{array}$$

is a pushout square. This follows immediately from the calculation

$$(\mathbf{1} \coprod C) \otimes (\mathbf{1} \coprod D) \simeq \mathbf{1} \coprod C \coprod D \coprod (C \otimes D).$$

Example 5.1.5.3. In the case $k = 1$, we can replace the operad $\mathbb{E}_1^{\otimes}$ with the associative ∞-operad $\mathcal{A}ss^{\otimes}$ (Example 5.1.0.7). In this case, Theorem 5.1.5.1 is equivalent to the assertion that the diagram

$$\mathrm{Free}(C \otimes D) \underset{g}{\overset{f}{\rightrightarrows}} \mathrm{Free}(C) \coprod \mathrm{Free}(D) \longrightarrow \mathrm{Free}(C) \otimes \mathrm{Free}(D)$$

is a coequalizer, where f and g are induced by the maps $C \otimes D \to \mathrm{Free}(C) \coprod \mathrm{Free}(D)$ given by multiplication on $\mathrm{Free}(C) \coprod \mathrm{Free}(D)$ in the two possible orders.

Example 5.1.5.4. We can also take $k = \infty$ in Theorem 5.1.5.1. In this case, the sphere S^{k-1} is contractible, so the left vertical map $\mathrm{Free}(C \otimes D \otimes S^{k-1}) \to \mathrm{Free}(C \otimes D)$ is an equivalence. Consequently, Theorem 5.1.5.1 reduces to the assertion that the right vertical map $\mathrm{Free}(C) \coprod \mathrm{Free}(D) \to \mathrm{Free}(C) \otimes \mathrm{Free}(D)$ is an equivalence. This follows from Proposition 3.2.4.7, since the ∞-operad $\mathbb{E}_k$ is equivalent to the commutative ∞-operad $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ (Corollary 5.1.1.5).

Let us now outline our approach to the proof of Theorem 5.1.5.1. The rough idea is to construct a functor $\mathcal{C} \times \mathcal{C} \to \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$ whose values can be computed in two different ways: the first computation will show that

this functor is given by the formula $(C, D) \mapsto \mathrm{Free}(C) \otimes \mathrm{Free}(D)$, while the second computation will show that it is given by

$$(C, D) \mapsto \mathrm{Free}(C \otimes D) \coprod_{\mathrm{Free}(C \otimes D) \otimes S^{k-1}} (\mathrm{Free}(C) \coprod \mathrm{Free}(D)).$$

The construction will use the formalism of operadic left Kan extensions developed in §3.1.2, and the comparison between the two calculations rests on a transitivity result for operadic left Kan extensions which is proven in §3.1.4. Our first step is to describe the tensor product of $\mathcal{O}$-algebras as an operadic left Kan extension.

Notation 5.1.5.5. We let Sub' denote the categorical mapping cylinder of the forgetful functor $\pi : \mathrm{Sub} \to \mathcal{F}\mathrm{in}_*$ of Definition 2.2.3.2. More precisely, the category Sub can be described as follows:

(i) An object of Sub' is either an object of Sub or an object of $\mathcal{F}\mathrm{in}_*$.

(ii) Morphisms in Sub' are given by the formulas

$$\mathrm{Hom}_{\mathrm{Sub}'}(\langle m \rangle, (\langle n \rangle, S, T)) = \emptyset.$$

$$\mathrm{Hom}_{\mathrm{Sub}'}(\langle m \rangle, \langle n \rangle) = \mathrm{Hom}_{\mathcal{F}\mathrm{in}_*}(\langle m \rangle, \langle n \rangle) \qquad \mathrm{Hom}_{\mathrm{Sub}'}((\langle m \rangle, S, T), \langle n \rangle) = \mathrm{Hom}_{\mathcal{F}\mathrm{in}_*}(\langle m \rangle, \langle n \rangle)$$

$$\mathrm{Hom}_{\mathrm{Sub}'}((\langle m \rangle, S, T), (\langle n \rangle, S', T')) = \mathrm{Hom}_{\mathrm{Sub}}((\langle m \rangle, S, T)(\langle n \rangle, S', T')).$$

The functors π_-, π_+, and π of Definition 2.2.3.2 extend naturally to retractions of Sub' onto the full subcategory $\mathcal{F}\mathrm{in}_* \subseteq \mathrm{Sub}'$; we will denote these retractions by π'_-, π'_+, and π'.

Let $\mathcal{O}^\otimes$ be an ∞-operad. We define a simplicial set $\mathcal{T}_\mathcal{O}$ equipped with a map $\mathcal{T}_\mathcal{O} \to \mathrm{N}(\mathrm{Sub}')$ so that the following universal property is satisfied: for every simplicial set K equipped with a map $K \to \mathrm{N}(\mathrm{Sub}')$, the set $\mathrm{Hom}_{\mathrm{N}(\mathrm{Sub}')}(K, \mathcal{T}_\mathcal{O})$ can be identified with the set of pairs of maps $e_-, e_+ : K \to \mathcal{O}^\otimes$ satisfying the following conditions:

(1) The diagram

is commutative.

(2) The maps e_- and e_+ coincide on $K \times_{\mathrm{N}(\mathrm{Sub}')} \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

We regard $\mathcal{T}_\mathcal{O}$ as an object of $(\mathrm{Set}_\Delta)_{/\Delta^1 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)}$ via the map

$$\mathcal{T}_\mathcal{O} \to \mathrm{N}(\mathrm{Sub}') \overset{\psi \times \pi'}{\to} \Delta^1 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*),$$

where $\psi : \mathrm{N}(\mathrm{Sub}') \to \Delta^1$ is the map given by 0 on the subcategory $\mathrm{Sub} \subseteq \mathrm{Sub}'$ and 1 on $\mathcal{F}\mathrm{in}_* \subseteq \mathrm{Sub}'$.

We observe that the fiber product $\mathcal{T}_\mathcal{O} \times_{\Delta^1} \{0\}$ is isomorphic with $\mathcal{O}^\otimes \boxplus \mathcal{O}^\otimes$, while $\mathcal{T}_\mathcal{O} \times_{\Delta^1} \{1\}$ is isomorphic to $\mathcal{O}^\otimes$.

Lemma 5.1.5.6. *Let $\mathcal{O}^\otimes$ be an ∞-operad. Then the map $q : \mathcal{T}_\mathcal{O} \to \Delta^1 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ of Notation 5.1.5.5 exhibits $\mathcal{T}_\mathcal{O}$ as a Δ^1-family of ∞-operads.*

Proof. It is easy to see that $\mathcal{T}_\mathcal{O}$ is an ∞-category and that the fibers $\mathcal{T}_\mathcal{O} \times_{\Delta^1} \{i\}$ are ∞-operads. To complete the proof, we must show the following:

(a) Every inert morphism in $\mathcal{T}_\mathcal{O} \times_{\Delta^1} \{i\}$ is q-coCartesian.

(b) Given an object $X \in \mathcal{T}_{\mathcal{O}} \times_{\Delta^1} \{i\}$ lying over $\langle n \rangle \in \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, any collection of inert morphisms $X \to X_j$ covering $\rho^j : \langle n \rangle \to \langle 1 \rangle$ for $1 \le j \le n$ exhibits X as a q-product of the objects $\{X_j\}_{1 \le j \le n}$.

Assertion (a) is obvious when $i \le 1$, and assertion (b) is obvious when $i = 0$. Let us prove that (b) holds when $i = 1$. Fix an object $Y \in \mathcal{T}_{\mathcal{O}} \times_{\Delta^1} \{0\}$ lying over $\langle m \rangle \in \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, and for every map $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$ let $\mathrm{Map}^{\alpha}_{\mathcal{T}_{\mathcal{O}}}(Y, X)$ denote the summand of $\mathrm{Map}_{\mathcal{T}_{\mathcal{O}}}(Y, X)$ lying over α. We wish to prove that the map

$$\phi : \mathrm{Map}^{\alpha}_{\mathcal{T}_{\mathcal{O}}}(Y, X) \to \prod_{1 \le j \le n} \mathrm{Map}^{\rho^j \alpha}_{\mathcal{T}_{\mathcal{O}}}(Y, X_j)$$

is a homotopy equivalence. We can identify Y with a quintuple $(\langle m \rangle, S, T, Y_-, Y_+) \in \mathcal{O}^{\otimes} \boxplus \mathcal{O}^{\otimes}$. The map α determines a pair of maps $\alpha_- : [S] \to \langle n \rangle$ and $\alpha_+ : [T] \to \langle n \rangle$. It now suffices to observe that ϕ is equivalent to a product of the maps

$$\mathrm{Map}^{\alpha_-}_{\mathcal{O}^{\otimes}}(Y_-, q_-(X)) \to \prod_{1 \le j \le n} \mathrm{Map}^{\rho^j \alpha_-}_{\mathcal{O}^{\otimes}}(Y_-, X_j)$$

$$\mathrm{Map}^{\alpha_+}_{\mathcal{O}^{\otimes}}(Y_+, q_+(X)) \to \prod_{1 \le j \le n} \mathrm{Map}^{\rho^j \alpha_+}_{\mathcal{O}^{\otimes}}(Y_+, X_j),$$

which are homotopy equivalences by virtue of our assumption that $\mathcal{O}^{\otimes}$ is an ∞-operad. $\square$

Construction 5.1.5.7. Let $\mathcal{O}^{\otimes}$ be an ∞-operad. There is a canonical map $H : \mathcal{O}^{\otimes} \times \Delta^1 \to \mathcal{T}_{\mathcal{O}}$, where $H_1 = H | \mathcal{O}^{\otimes} \times \{1\}$ is the canonical isomorphism $\mathcal{O}^{\otimes} \simeq \mathcal{T}_{\mathcal{O}} \times_{\Delta^1} \{1\}$ and $H_0 = H | \mathcal{O}^{\otimes} \times \{0\}$ is given on objects by the formula $(X \in \mathcal{O}^{\otimes}_{\langle n \rangle}) \mapsto (\langle n \rangle, \langle n \rangle, \langle n \rangle, X, X)$.

Suppose we are given a map of generalized ∞-operads $\psi : \mathcal{T}_{\mathcal{O}} \to \mathcal{D}^{\otimes}$, where $\mathcal{D}^{\otimes}$ is an ∞-operad. Restricting ψ to $\mathcal{T}_{\mathcal{O}} \times_{\Delta^1} \{1\}$, we obtain an ∞-operad map $\psi_{\pm} : \mathcal{O}^{\otimes} \to \mathcal{D}^{\otimes}$. Similarly, the restriction of ψ to $\mathcal{O}^{\otimes} \boxplus \mathcal{O}^{\otimes}$ yields a pair of ∞-operad maps $\psi_-, \psi_+ : \mathcal{O}^{\otimes} \to \mathcal{D}'^{\otimes}$. Let $p : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a coCartesian fibration of ∞-operads. We let $\mathrm{Alg}^-_{\mathcal{O}/\mathcal{D}}(\mathcal{C})$, $\mathrm{Alg}^+_{\mathcal{O}/\mathcal{D}}(\mathcal{C})$, and $\mathrm{Alg}^{\pm}_{\mathcal{O}/\mathcal{D}}(\mathcal{C})$ be the ∞-categories of $\mathcal{O}$-algebras in $\mathcal{C}$, defined using the ∞-operad morphisms ψ_-, ψ_+, and $\psi_{\pm}$, respectively. For every algebra object $A \in \mathrm{Fun}_{\mathcal{D}^{\otimes}}(\mathcal{O}^{\otimes} \boxplus \mathcal{O}^{\otimes}, \mathcal{C})$, we can choose a p-coCartesian natural transformation $\alpha : A \circ H_0 \to A'$ in $\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$ lifting the natural transformation $\psi \circ H$. This construction determines a functor

$$\mathrm{Alg}_{(\mathcal{T}_{\mathcal{O}})_0/\mathcal{D}}(\mathcal{C}) \to \mathrm{Alg}^{\pm}_{\mathcal{O}/\mathcal{D}}(\mathcal{C}).$$

Theorem 2.2.3.6 allows us to identify the ∞-category on the left hand side with $\mathrm{Alg}^-_{\mathcal{O}/\mathcal{D}}(\mathcal{C}) \times \mathrm{Alg}^+_{\mathcal{O}/\mathcal{D}}(\mathcal{C})$. Under this identification, we obtain a functor

$$\mathrm{Alg}^-_{\mathcal{O}/\mathcal{D}}(\mathcal{C}) \times \mathrm{Alg}^+_{\mathcal{O}/\mathcal{D}}(\mathcal{C}) \to \mathrm{Alg}^{\pm}_{\mathcal{O}/\mathcal{D}}(\mathcal{C});$$

we will refer to this functor as the *tensor product* and denote it by $\otimes$.

In the special case where $\mathcal{D}^{\otimes} = \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, we recover the usual tensor product operation on algebra objects of a symmetric monoidal ∞-category (see Proposition 3.2.4.3).

Proposition 5.1.5.8. *Suppose we are given an ∞-operad $\mathcal{O}^{\otimes}$, a map of generalized ∞-operads $\psi : \mathcal{T}_{\mathcal{O}} \to \mathcal{D}^{\otimes}$, and a coCartesian fibration of ∞-operads $q : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$. Then the tensor product functor $\otimes : \mathrm{Alg}^-_{\mathcal{O}/\mathcal{D}}(\mathcal{C}) \times \mathrm{Alg}^+_{\mathcal{O}/\mathcal{D}}(\mathcal{C}) \simeq \mathrm{Alg}_{(\mathcal{T}_{\mathcal{O}})_0/\mathcal{D}}(\mathcal{C}) \to \mathrm{Alg}^{\pm}_{\mathcal{O}}(\mathcal{C})$ described in Construction 5.1.5.7 is induced by operadic q-left Kan extension along the correspondence of ∞-operads $\mathcal{T}_{\mathcal{O}} \to \Delta^1 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.*

Proof. Let $\mathcal{A}$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{D}^{\otimes}}(\mathcal{T}_{\mathcal{O}}, \mathcal{C}^{\otimes})$ spanned by those maps of generalized ∞-operads which are operadic q-left Kan extensions. The assertion of Proposition 5.1.5.8 can be stated more precisely as follows:

(a) Every algebra object $A_0 \in \mathrm{Alg}_{(\mathcal{T}_{\mathcal{O}})_0/\mathcal{D}}(\mathcal{C})$ admits an operadic q-left Kan extension $A \in \mathcal{A}$.

(b) The restriction map $\mathcal{A} \to \mathrm{Alg}_{(\mathcal{T}_{\mathcal{O}})_0/\mathcal{D}}(\mathcal{C})$ is a trivial Kan fibration, and therefore admits a section s.

(c) The composition

$$\mathrm{Alg}_{(\mathcal{T}_{\mathcal{O}})_0/\mathcal{D}}(\mathcal{C}) \overset{s}{\to} \mathcal{A} \to \mathrm{Alg}_{\mathcal{O}}^{\pm}(\mathcal{C})$$

is equivalent to the tensor product functor described in Construction 5.1.5.7.

To prove assertion (a), we must show that for every object $X \in \mathcal{O}$, the induced diagram

$$(\mathcal{T}_{\mathcal{O}}^{\mathrm{act}})_{/X} \times_{\mathcal{T}_{\mathcal{O}}} (\mathcal{O}_{-}^{\otimes} \boxplus \mathcal{O}_{+}^{\otimes}) \to \mathcal{C}^{\otimes}$$

can be extended to an operadic q-colimit diagram lying over the obvious map

$$(\mathcal{T}_{\mathcal{O}}^{\mathrm{act}})_{/X} \times_{\mathcal{T}} (\mathcal{O}^{\otimes} \boxplus \mathcal{O}^{\otimes}))^{\triangleright} \to \mathcal{D}^{\otimes}$$

(Theorem 3.1.2.3). We observe that the ∞-category $(\mathcal{T}_{\mathcal{O}}^{\mathrm{act}})_{/X} \times_{\mathcal{T}_{\mathcal{O}}} (\mathcal{O}_{-}^{\otimes} \boxplus \mathcal{O}_{+}^{\otimes})$ has a final object, given by the quintuple $X' : (\langle 2 \rangle, \{0, *\}, \{1, *\}, X, X)$. It therefore suffices to show that $A_0(X')$ can be extended to an operadic q-colimit diagram $\{X'\}^{\triangleright} \to \mathcal{C}^{\otimes}$ lying over the active map $\langle 2 \rangle \to \langle 1 \rangle$. The existence of this extension follows from our assumption that q is a coCartesian fibration of ∞-operads (Proposition 3.1.1.13). Assertion (b) follows immediately from (a), by virtue of Lemma 3.1.3.14.

To prove (c), consider the functor $H : \mathcal{O}^{\otimes} \times \Delta^1 \to \mathcal{T}_{\mathcal{O}}$ of Construction 5.1.5.7. Composition with H induces a map of simplicial sets $h : \mathcal{A} \times \Delta^1 \to \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$. The proof of (a) shows that h can be regarded as a coCartesian natural transformation from the composite functor

$$h_0 : \mathcal{A} \to \mathrm{Alg}_{(\mathcal{T}_{\mathcal{O}})_0/\mathcal{D}}(\mathcal{C}) \overset{\circ H_0}{\to} \mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{\pm}(\mathcal{C})$$

to a functor $h_1 : \mathcal{A} \to \mathrm{Alg}_{\mathcal{O}}^{\pm}(\mathcal{C})$, so that h_1 can be identified with the tensor product of the forgetful functors $\mathcal{A} \to \mathrm{Alg}_{\mathcal{O}}^{-}(\mathcal{C})$ and $\mathcal{A} \to \mathrm{Alg}_{\mathcal{O}}^{+}(\mathcal{C})$. Composing this identification with the section s, we obtain the desired result. $\square$

For the discussion which follows, we fix an ∞-operad $\mathcal{O}^{\otimes}$ and a coCartesian fibration of ∞-operads $q : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$. Let $\psi : \mathcal{T}_{\mathcal{O}} \to \mathcal{D}^{\otimes}$ be a map of ∞-operad families so that the ∞-categories $\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{-}(\mathcal{C})$, $\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{+}(\mathcal{C})$, and $\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{\pm}(\mathcal{C})$ are defined as in Construction 5.1.5.7. Similarly, we can define ∞-categories $\mathrm{Fun}_{\mathcal{D}}^{-}(\mathcal{O}, \mathcal{C})$, $\mathrm{Fun}_{\mathcal{D}}^{+}(\mathcal{O}, \mathcal{C})$, and $\mathrm{Fun}_{\mathcal{D}}^{\pm}(\mathcal{O}, \mathcal{C})$. There are evident forgetful functors

$$\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{-}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{D}}^{-}(\mathcal{O}, \mathcal{C})$$

$$\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{+}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{D}}^{+}(\mathcal{O}, \mathcal{C})$$

$$\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{\pm}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{D}}^{\pm}(\mathcal{O}, \mathcal{C}).$$

Under some mild hypotheses, these forgetful functors admit left adjoints, which we will denote by F^{-}, F^{+}, and $F^{\pm}$. The construction $(X, Y) \mapsto F^{-}(X) \otimes F^{+}(Y)$ determines a functor from $\mathrm{Fun}_{\mathcal{D}}^{-}(\mathcal{O}, \mathcal{C}) \times \mathrm{Fun}_{\mathcal{D}}^{+}(\mathcal{O}, \mathcal{C})$ to $\mathrm{Alg}_{\mathcal{O}}^{\pm}(\mathcal{C})$. We now give a convenient description of this functor, using the theory of operadic q-left Kan extensions.

Remark 5.1.5.9. In the special case where $\mathcal{D}^{\otimes} = \mathcal{F}\mathrm{in}_*$ (so that $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category), the ∞-categories $\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{-}(\mathcal{C})$, $\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{+}(\mathcal{C})$, and $\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{\pm}(\mathcal{C})$ coincide and the superscripts become superfluous.

Notation 5.1.5.10. We let $\mathcal{T}_{\mathcal{O}}^0$ denote the subcategory of $\mathcal{T}_{\mathcal{O}}$ spanned by all those morphisms $f : X \to Y$ satisfying the following condition:

(∗) If X and Y belong to $\mathcal{O}^{\otimes} \boxplus \mathcal{O}^{\otimes} \subseteq \mathcal{T}_{\mathcal{O}}$, then the image of f in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is inert.

Remark 5.1.5.11. It follows easily from Lemma 5.1.5.6 that $\mathcal{T}_{\mathcal{O}}^0 \to \Delta^1 \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a Δ^1-family of ∞-operads, which we view as a correspondence of ∞-operads from $\mathcal{O}'^{\otimes} \boxplus \mathcal{O}'^{\otimes}$ to $\mathcal{O}^{\otimes}$; here $\mathcal{O}'^{\otimes}$ denotes the fiber product $\mathcal{O}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathcal{T}\mathrm{riv}^{\otimes}$.

Proposition 5.1.5.12. *Let κ be an uncountable regular cardinal, let $\mathcal{O}^{\otimes}$ be an ∞-operad which is essentially κ-small, let $q : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a coCartesian fibration of ∞-operads which is compatible with κ-small colimits and $\psi : \mathcal{T}_{\mathcal{O}} \to \mathcal{D}^{\otimes}$ a map of generalized ∞-operads. Then:*

(1) *For each symbol $\sigma \in \{-, +, \pm\}$, the forgetful functor $\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{\sigma}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{D}}^{\sigma}(\mathcal{O}, \mathcal{C})$ admits a left adjoint F^{σ}. If we let $\mathcal{O}'^{\otimes}$ denote the fiber product $\mathcal{O}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathcal{T}\mathrm{riv}^{\otimes}$, then F^{σ} is given by composing a homotopy inverse to the trivial Kan fibration $\mathrm{Alg}_{\mathcal{O}'}^{\sigma}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{D}}^{\sigma}(\mathcal{O}', \mathcal{C}) \simeq \mathrm{Fun}_{\mathcal{D}}^{\sigma}(\mathcal{O}, \mathcal{C})$ with the functor of operadic q-left Kan extension along the inclusion $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$.*

(2) *Let $F^{(2)} : \mathrm{Fun}_{\mathcal{D}}^{-}(\mathcal{O}, \mathcal{C}) \times \mathrm{Fun}_{\mathcal{D}}^{+}(\mathcal{O}, \mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}^{\pm}(\mathcal{C})$ be the functor given by the formula $X, Y \mapsto F(X) \otimes F(Y)$. Then $F^{(2)}$ is equivalent to the composition*

$$\mathrm{Fun}_{\mathcal{D}}^{-}(\mathcal{O}, \mathcal{C}) \times \mathrm{Fun}_{\mathcal{D}}^{+}(\mathcal{O}, \mathcal{C}) \simeq \mathrm{Fun}_{\mathcal{D}^{\otimes}}^{\mathrm{lax}}(\mathcal{O}'^{\otimes} \boxplus \mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \xrightarrow{f_{02}} \mathrm{Alg}_{\mathcal{O}}^{\pm}(\mathcal{C}),$$

where f_{02} denotes the functor given by operadic left Kan extension along the correspondence of ∞-operads $\mathcal{T}_{\mathcal{O}}^0$ of Notation 5.1.5.10.

Proof. Assertion (1) is a special case of Corollary 3.1.3.5. To prove (2), let $p : \Delta^2 \to \Delta^1$ be the map which collapses the edge $\Delta^{\{0,1\}} \subseteq \Delta^2$, and let $\mathcal{M}^{\otimes}$ be the subcategory of $\mathcal{T}_{\mathcal{O}} \times_{\Delta^1} \Delta^2$ spanned by those morphisms $f : X \to Y$ satisfying the following condition: if both X and Y belong to $\mathcal{M}^{\otimes} \times_{\Delta^2} \{0\}$, then the image of f in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is inert. The canonical map $\mathcal{M}^{\otimes} \to \Delta^2 \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a Δ^2-family of ∞-operads. Let $f_{ij} : \mathrm{Alg}_{\mathcal{M}_i/\mathcal{D}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{M}_j/\mathcal{D}}(\mathcal{C})$ be the functor given by operadic q-left Kan extension along the correspondence $\mathcal{M}^{\otimes} \times_{\Delta^2} \Delta^{\{i,j\}}$ for $0 \le i < j \le 2$. Note that for every object X of $\mathcal{M}_0^{\otimes}$, there exists a q-coCartesian morphism $X \to Y$, where $Y \in \mathcal{M}_1^{\otimes}$; here q denotes the projection $\mathcal{M}^{\otimes} \to \Delta^2$. It follows from Example B.3.4 that q is a flat inner fibration, so we have an equivalence of functors $f_{02} \simeq f_{12} \circ f_{01}$ (Theorem 3.1.4.1).

To study the functor f_{01}, we note that $\mathcal{M} \otimes_{\Delta^2} \Delta^{\{0,1\}}$ is the correspondence associated to the inclusion of ∞-operads

$$\mathcal{O}'^{\otimes} \boxtimes \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes} \boxtimes \mathcal{O}^{\otimes}.$$

We have a homotopy commutative diagram of ∞-categories

$$
\begin{array}{ccccc}
\mathrm{Alg}_{\mathcal{M}_0/\mathcal{D}}(\mathcal{C}) & \xrightarrow{f_{01}} & \mathrm{Alg}_{\mathcal{M}_1/\mathcal{D}}(\mathcal{C}) & \xrightarrow{f_{12}} & \mathrm{Alg}_{\mathcal{M}_2/\mathcal{D}}(\mathcal{C}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Fun}_{\mathcal{D}}^{-}(\mathcal{O}, \mathcal{C}) \times \mathrm{Fun}_{\mathcal{D}}^{+}(\mathcal{O}, \mathcal{C}) & \xrightarrow{g_{01}} & \mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{-}(\mathcal{C}) \times \mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{+}(\mathcal{C}) & \xrightarrow{g_{12}} & \mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{\pm}(\mathcal{C})
\end{array}
$$

where the vertical maps are categorical equivalences (Theorem 2.2.3.6), the map g_{01} can be identified with $F^{-} \times F^{+}$ (by virtue of (1)), and the map g_{12} can be identified with the tensor product $\otimes$ on $\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{\pm}(\mathcal{C})$ (Proposition 5.1.5.8). Composing these identifications, we obtain the desired description of f_{02}. $\qquad\square$

To deduce a version of Theorem 5.1.5.1 from Proposition 5.1.5.12, we would like to obtain a *different* description of the functor f_{02} given by operadic left Kan extension along $\mathcal{T}_{\mathcal{O}}^0$. This description will also be obtained from Theorem 3.1.4.1, but using a more interesting factorization of the correspondence $\mathcal{T}_{\mathcal{O}}^0$.

Notation 5.1.5.13. We define categories $\mathcal{J}_0$, $\mathcal{J}_1$, and $\mathcal{J}_2$ as follows:

(1) The category $\mathcal{J}_1$ has as objects triples $(\langle n \rangle, S, T)$, where S and T are subsets of $\langle n \rangle$ which contain the base point such that $\langle n \rangle = S \cup T$. A morphism from $(\langle n \rangle, S, T)$ to $(\langle n' \rangle, S', T')$ in $\mathcal{J}_1$ consists of a map $\alpha : \langle n \rangle \to \langle n' \rangle$ in $\mathcal{F}\mathrm{in}_*$ which restricts to inert morphisms $[S] \to [S']$, $[T] \to [T']$.

(2) The category $\mathcal{J}_0$ is the full subcategory of $\mathcal{J}_1$ spanned by those objects $(\langle n \rangle, S, T)$ for which $S \cap T = \{*\}$

(3) The category $\mathcal{J}_2$ coincides with $\mathcal{F}in_*$.

Let $\phi_{01} : \mathcal{J}_0 \to \mathcal{J}_1$ be the inclusion, let $\phi_{12} : \mathcal{J}_1 \to \mathcal{J}_2$ be the forgetful functor, and let $\phi_{02} = \phi_{12} \circ \phi_{01}$. We can assemble the categories $\mathcal{J}_i$ into one large category $\mathcal{J}$ as follows:

(i) An object of $\mathcal{J}$ is a pair (J, i), where $0 \leq i \leq 2$ and X is an object of $\mathcal{J}_i$.

(ii) Given a pair of objects (I, i), $(J, j) \in \mathcal{J}$, we have

$$\mathrm{Hom}_{\mathcal{J}}((I, i), (J, j)) = \begin{cases} \mathrm{Hom}_{\mathcal{J}_j}(\phi_{ij}(I), J) & \text{if } i < j \\ \mathrm{Hom}_{\mathcal{J}_j}(I, J) & \text{if } i = j \\ \emptyset & \text{if } i > j. \end{cases}$$

Let Sub' be the category defined in Notation 5.1.5.5. The inclusion $\mathcal{J}_0 \subseteq \mathrm{Sub}$ and equivalence $\mathcal{J}_2 \simeq \mathcal{F}in_*$ extend to a functor $r : \mathcal{J} \to \mathrm{Sub}'$, which is given on $\mathcal{J}_1$ by the formula $r(\langle n \rangle, S, T) = \langle n \rangle$. For every ∞-operad $\mathcal{O}^{\otimes}$, we define $\mathcal{M}[\mathcal{O}]^{\otimes}$ to be the fiber product $\mathrm{N}(\mathcal{J}) \times_{\mathrm{N}(\mathrm{Sub}')} \mathcal{T}_{\mathcal{O}}$. Let Φ denote the composite map

$$\mathcal{M}[\mathcal{O}]^{\otimes} \to \mathrm{N}(\mathcal{J}) \xrightarrow{r'} \Delta^2 \times \mathrm{N}(\mathcal{F}in_*).$$

Repeating the proof of Lemma 5.1.5.6, we obtain the following:

Lemma 5.1.5.14. *Let $\mathcal{O}^{\otimes}$ be an ∞-operad. The map $\Phi : \mathcal{M}[\mathcal{O}]^{\otimes} \to \Delta^2 \times \mathrm{N}(\mathcal{F}in_*)$ of Notation 5.1.5.13 exhibits $\mathcal{M}[\mathcal{O}]^{\otimes}$ as a Δ^2-family of ∞-operads.*

For $i \in \{0, 1, 2\}$, we let $\mathcal{M}[\mathcal{O}]^{\otimes}_i$ denote the fiber $\mathcal{M}[\mathcal{O}]^{\otimes} \times_{\Delta^2} \{i\}$. Let $\mathcal{O}'^{\otimes}$ denote the fiber product $\mathcal{O}^{\otimes} \times_{\mathrm{N}(\mathcal{F}in_*)} \mathcal{T}riv$. The fiber $\mathcal{M}[\mathcal{O}]^{\otimes}_0$ is isomorphic to $\mathcal{O}'^{\otimes} \boxtimes \mathcal{O}'^{\otimes}$, while $\mathcal{M}[\mathcal{O}]^{\otimes}_2$ is isomorphic to $\mathcal{O}^{\otimes}$ itself. We will denote the inner fiber $\mathcal{M}[\mathcal{O}]^{\otimes}_1$ by $\mathcal{Q}^{\otimes}$. This ∞-operad is more exotic: it in some sense encodes the "quadradic part" of the ∞-operad $\mathcal{O}^{\otimes}$. Note that the fiber product $\mathcal{M}[\mathcal{O}]^{\otimes} \times_{\Delta^2} \Delta^{\{0,2\}}$ is isomorphic to the correspondence $\mathcal{T}_{\mathcal{O}}^0$ of Notation 5.1.5.10.

To proceed further with our analysis, we need the following technical result, whose proof we defer until the end of this section:

Proposition 5.1.5.15. *Let $q : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}in_*)$ be a coherent ∞-operad. Then the map $\mathcal{M}[\mathcal{O}]^{\otimes} \to \Delta^2$ is a flat inner fibration.*

Note that if $\psi : \mathcal{T}_{\mathcal{O}} \to \mathcal{D}^{\otimes}$ is a map of generalized ∞-operads, then composition with ψ induces another map of generalized ∞-operads $\mathcal{M}[\mathcal{O}]^{\otimes} \to \mathcal{D}^{\otimes}$.

Corollary 5.1.5.16. *Let κ be an uncountable regular cardinal, let $\mathcal{O}^{\otimes}$ be an essentially κ-small ∞-operad, let $q : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a coCartesian fibration of ∞-operads which is compatible with κ-small colimits, and let $\psi : \mathcal{T}_{\mathcal{O}} \to \mathcal{D}^{\otimes}$ be an map of generalized ∞-operads.*

For $\sigma \in \{-, +, \pm\}$, let $F^{\sigma} : \mathrm{Fun}_{\mathcal{D}}^{\sigma}(\mathcal{O}, \mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{\sigma}(\mathcal{C})$ be a left adjoint to the restriction functor. Let Let $f_{01} : \mathrm{Fun}_{\mathcal{D}}^{-}(\mathcal{O}, \mathcal{C}) \times \mathrm{Fun}_{\mathcal{D}}^{+}(\mathcal{O}, \mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{M}[\mathcal{O}]_0/\mathcal{D}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{Q}/\mathcal{D}}(\mathcal{C})$ be given by operadic left Kan extension along the correspondence $\mathcal{M}[\mathcal{O}]^{\otimes} \times_{\Delta^2} \Delta^{\{0,1\}}$, and let $f_{12} : \mathrm{Alg}_{\mathcal{Q}/\mathcal{D}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{\pm}(\mathcal{C})$ be the functor given by left Kan extension along $\mathcal{M}[\mathcal{O}]^{\otimes} \times_{\Delta^2} \Delta^{\{1,2\}}$. If $\mathcal{O}^{\otimes}$ is coherent and $\mathcal{O}$ is a Kan complex, then the composition $f_{12} \circ f_{01}$ can be identified with the composite functor

$$\mathrm{Fun}_{\mathcal{D}}^{-}(\mathcal{O}, \mathcal{C}) \times \mathrm{Fun}_{\mathcal{D}}^{+}(\mathcal{O}, \mathcal{C}) \xrightarrow{F^{-} \times F^{+}} \mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{-}(\mathcal{C}) \times \mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{+}(\mathcal{C}) \xrightarrow{\otimes} \mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{\pm}(\mathcal{C}).$$

Proof. Combine Proposition 5.1.5.12, Theorem 3.1.4.1, and Proposition 5.1.5.15. $\square$

To apply Corollary 5.1.5.16, we need to understand the functors f_{01} and f_{12} better. To this end, we need to introduce some additional notation. Fix a map of generalized ∞-operads $\psi : \mathcal{T}_{\mathcal{O}} \to \mathcal{D}^{\otimes}$, so that ψ induces ∞-operad maps $\psi_-, \psi_+, \psi_{\pm} : \mathcal{O}^{\otimes} \to \mathcal{D}^{\otimes}$. We note that ψ also determines natural transformations

$$\psi_- \to \psi_{\pm} \leftarrow \psi_+.$$

If $q : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a coCartesian fibration of ∞-operads, then coCartesian transport along these transformations determines a pair of functors

$$\operatorname{Fun}^-_{\mathcal{D}}(\mathcal{O}, \mathcal{C}) \xrightarrow{u_-} \operatorname{Fun}^{\pm}_{\mathcal{D}}(\mathcal{O}, \mathcal{C}) \xleftarrow{u_+} \operatorname{Fun}^+_{\mathcal{D}}(\mathcal{O}, \mathcal{C}).$$

Let $\mathcal{Q}^{\otimes}_0$ denote the full subcategory of $\mathcal{Q}^{\otimes}$ spanned by those objects whose image in $\operatorname{N}(\mathcal{J}_1)$ belongs to the full subcategory $\operatorname{N}(\mathcal{J}_0) \subseteq \operatorname{N}(\mathcal{J}_1)$. There is an evident forgetful functor $\mathcal{Q}^{\otimes}_0 \to \mathcal{M}[\mathcal{O}]^{\otimes}_0$. If $\mathcal{O}^{\otimes}$ is unital, then this forgetful functor is a trivial Kan fibration. Choosing a section, we obtain a map of ∞-operads $i : \mathcal{M}[\mathcal{O}]^{\otimes}_0 \to \mathcal{Q}^{\otimes}$. There is also a natural transformation $\operatorname{id} \to i$ of functors $\mathcal{M}[\mathcal{O}]^{\otimes}_0 \to \mathcal{M}[\mathcal{O}]^{\otimes}$.

Composition with i induces a forgetful functor

$$\theta : \operatorname{Alg}_{\mathcal{Q}/\mathcal{D}}(\mathcal{C}) \to \operatorname{Alg}_{\mathcal{M}[\mathcal{O}]_0/\mathcal{D}}(\mathcal{C}) \simeq \operatorname{Fun}^{\pm}_{\mathcal{D}}(\mathcal{O}, \mathcal{C}) \times \operatorname{Fun}^{\pm}_{\mathcal{D}}(\mathcal{O}, \mathcal{C}).$$

Under the hypotheses of Corollary 5.1.5.16, the functor θ has a left adjoint $\theta^L : \operatorname{Fun}^{\pm}_{\mathcal{D}}(\mathcal{O}, \mathcal{C}) \times \operatorname{Fun}^{\pm}_{\mathcal{D}}(\mathcal{O}, \mathcal{C}) \to \operatorname{Alg}_{\mathcal{Q}/\mathcal{D}}(\mathcal{C})$, given by operadic q-left Kan extension along i. Let $F_{(2)} : \operatorname{Fun}^-_{\mathcal{D}}(\mathcal{O}, \mathcal{C}) \times \operatorname{Fun}^+_{\mathcal{D}}(\mathcal{O}, \mathcal{C}) \to \operatorname{Alg}_{\mathcal{Q}/\mathcal{D}}(\mathcal{C})$ be the composition of θ^L with $u_- \times u_+$. Since the composition $\theta \circ f_{01}$ is equivalent to $u_- \times u_+$, we obtain a natural transformation of functors $\alpha : F_{(2)} \to f_{01}$.

We would like to measure the failure of α to be an equivalence. To this end, consider the fully faithful embedding $\mathcal{T}\mathrm{riv}^{\otimes} \to \operatorname{N}(\mathcal{J}_1)$ given by the formula $\langle n \rangle \mapsto (\langle n \rangle, \langle n \rangle, \langle n \rangle)$. This embedding determines a map of ∞-operads $j : \mathcal{O}'^{\otimes} \to \mathcal{Q}^{\otimes}$. Composition with j induces a forgetful functor $j^* : \operatorname{Alg}_{\mathcal{Q}/\mathcal{D}}(\mathcal{C}) \to \operatorname{Alg}^{\pm}_{\mathcal{O}'/\mathcal{D}}(\mathcal{C}) \simeq \operatorname{Fun}^{\pm}_{\mathcal{D}}(\mathcal{O}, \mathcal{C})$. The hypotheses of Corollary 5.1.5.16 guarantee that j^* admits a left adjoint $j_!$, given by operadic q-left Kan extension along j.

Lemma 5.1.5.17. *Let $\mathcal{O}^{\otimes}$, $q : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$, and $\psi : \mathcal{T}_{\mathcal{O}} \to \mathcal{D}^{\otimes}$ be as in Corollary 5.1.5.16, and assume that $\mathcal{O}^{\otimes}$ is unital. Then the diagram*

$$
\begin{array}{ccc}
j_! j^* F_{(2)} & \xrightarrow{\;j_! j^* \alpha\;} & j_! j^* f_{01} \\
\downarrow & & \downarrow \\
F_{(2)} & \xrightarrow{\quad \alpha \quad} & f_{01}
\end{array}
$$

is a pushout diagram of functors from $\operatorname{Fun}^-_{\mathcal{D}}(\mathcal{O}, \mathcal{C}) \times \operatorname{Fun}^+_{\mathcal{D}}(\mathcal{O}, \mathcal{C})$ to $\operatorname{Alg}_{\mathcal{Q}/\mathcal{D}}(\mathcal{C})$.

The proof of Lemma 5.1.5.17 will be given at the end of this section. Let us accept Lemma 5.1.5.17 for the moment, and see how it leads to a version of Theorem 5.1.5.1. Note that the correspondence of ∞-operads $\mathcal{M}[\mathcal{O}]^{\otimes} \times_{\Delta^2} \Delta^{\{1,2\}}$ is associated to the forgetful map of ∞-operads $k : \mathcal{Q}^{\otimes} \to \mathcal{O}^{\otimes}$. Let $k^* : \operatorname{Alg}^{\pm}_{\mathcal{O}/\mathcal{D}}(\mathcal{C}) \to \operatorname{Alg}_{\mathcal{Q}/\mathcal{D}}(\mathcal{C})$ denote the induced map, so that k^* is right adjoint to the operadic left Kan extension functor $f_{12} : \operatorname{Alg}_{\mathcal{Q}/\mathcal{D}}(\mathcal{C}) \to \operatorname{Alg}^{\pm}_{\mathcal{O}/\mathcal{D}}(\mathcal{C})$. Since f_{12} preserves pushouts, we deduce from Lemma 5.1.5.17 the existence of a pushout diagram

$$
\begin{array}{ccc}
f_{12} j_! j^* F_{(2)} & \longrightarrow & f_{12} j_! j^* f_{01} \\
\downarrow & & \downarrow \\
j_{12} F_{(2)} & \longrightarrow & f_{12} f_{01}
\end{array}
$$

of functors from $\operatorname{Fun}(\mathcal{O}, \mathcal{C}) \times \operatorname{Fun}(\mathcal{O}, \mathcal{C})$ to $\operatorname{Alg}^{\pm}_{\mathcal{O}/\mathcal{D}}(\mathcal{C})$. Let us identify the terms in this diagram. The functor $f_{12} j_!$ is left adjoint to the forgetful functor $j^* k^* : \operatorname{Alg}^{\pm}_{\mathcal{O}/\mathcal{D}}(\mathcal{C}) \to \operatorname{Fun}^{\pm}_{\mathcal{D}}(\mathcal{O}, \mathcal{C})$, and is therefore equivalent to

$F^{\pm}$. The composition $j^* f_{01}$ can be identified with the tensor product functor $\otimes : \mathrm{Fun}_{\mathcal{D}}^-(\mathcal{O}, \mathcal{C}) \times \mathrm{Fun}_{\mathcal{D}}^+(\mathcal{O}, \mathcal{C}) \to \mathrm{Fun}_{\mathcal{D}}^{\pm}(\mathcal{O}, \mathcal{C})$ determined by the composite map $\mathcal{T}_{\mathcal{O}'} \to \mathcal{T}_{\mathcal{O}} \overset{\psi}{\to} \mathcal{D}^{\otimes}$. In the case where $\mathcal{O}^{\otimes}$ is coherent and $\mathcal{O}$ is a Kan complex, Corollary 5.1.5.16 allows us to identify $f_{12} f_{01}$ with the composition

$$\mathrm{Fun}_{\mathcal{D}}^-(\mathcal{O}, \mathcal{C}) \times \mathrm{Fun}_{\mathcal{D}}^+(\mathcal{O}, \mathcal{C}) \overset{F^- \times F^+}{\to} \mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^-(\mathcal{C}) \times \mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^+(\mathcal{C}) \overset{\otimes}{\longrightarrow} \mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{\pm}(\mathcal{C}).$$

Finally, $j_{12} F_{(2)}$ is the composition of $u_- \times u_+$ with a left adjoint to θk^*, which coincides with the composition

$$\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{\pm}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{D}}^{\pm}(\mathcal{O}, \mathcal{C}) \to \mathrm{Fun}_{\mathcal{D}}^{\pm}(\mathcal{O}, \mathcal{C}) \times \mathrm{Fun}_{\mathcal{D}}^{\pm}(\mathcal{O}, \mathcal{C}).$$

It follows that $j_{12} F_{(2)}$ can be identified with the composition of $F^{\pm}$ with the coproduct of the functors u_- and u_+. Combining this identifications, we arrive at the following conclusion:

Theorem 5.1.5.18. *Let $\mathcal{O}^{\otimes}$ be a coherent ∞-operad which is κ-small for some regular cardinal κ, let $q : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a coCartesian fibration of ∞-operads which is compatible with κ-small colimits, and let $\psi : \mathcal{T}_{\mathcal{O}} \to \mathcal{D}^{\otimes}$ be a map of generalized ∞-operads. Then, for every pair of objects $V \in \mathrm{Fun}_{\mathcal{D}}^-(\mathcal{O}, \mathcal{C}), W \in \mathrm{Fun}_{\mathcal{D}}^+(\mathcal{O}, \mathcal{C})$, there is a canonical pushout diagram*

$$
\begin{array}{ccc}
F^{\pm}(j^* F_{(2)}(X, Y)) & \longrightarrow & F^{\pm}(X \otimes Y) \\
\downarrow & & \downarrow \\
F^{\pm}(e_-(X) \coprod e_+(Y)) & \longrightarrow & F^-(X) \otimes F^+(Y)
\end{array}
$$

in the ∞-category $\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{\pm}(\mathcal{C})$. This diagram depends functorially on X and Y (in other words, it is given by a pushout diagram of functors from $\mathrm{Fun}_{\mathcal{D}}^-(\mathcal{O}, \mathcal{C}) \times \mathrm{Fun}_{\mathcal{D}}^+(\mathcal{O}, \mathcal{C})$ to $\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{\pm}(\mathcal{C})$).

Remark 5.1.5.19. In the special case where $\mathcal{D}^{\otimes} = \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, the superscripts in Theorem 5.1.5.18 are superfluous and the functors e_- and e_+ are equivalent to the identity. In this case, we obtain a pushout diagram

$$
\begin{array}{ccc}
F(j^* F_{(2)}(X, Y)) & \longrightarrow & F(X \otimes Y) \\
\downarrow & & \downarrow \\
F(X \coprod Y) & \longrightarrow & F(X) \otimes F(Y).
\end{array}
$$

We can now proceed with the proof of our main result.

Proof of Theorem 5.1.5.1. Let $\mathcal{O}^{\otimes}$ be the ∞-operad $\mathbb{E}_k^{\otimes}$, and let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. We will assume that $\mathcal{C}$ admits countable colimits and that the tensor product on $\mathcal{C}$ preserves countable colimits separately in each variable. Since $\mathcal{O}$ is a contractible Kan complex, evaluation at $\langle 1 \rangle \in \mathcal{O}$ induces a trivial Kan fibration $e : \mathrm{Fun}(\mathcal{O}, \mathcal{C}) \to \mathcal{C}$. Let $\mathrm{Free} : \mathcal{C} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ be the functor obtained by composing the free functor $F : \mathrm{Fun}(\mathcal{O}, \mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ of §5.1.5 with a section s of e, and let $f_{(2)} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be the composition

$$\mathcal{C} \times \mathcal{C} \overset{s \times s}{\longrightarrow} \mathrm{Fun}(\mathcal{O}, \mathcal{C}) \times \mathrm{Fun}(\mathcal{O}, \mathcal{C}) \overset{j^* F_{(2)}}{\longrightarrow} \mathrm{Fun}(\mathcal{O}, \mathcal{C}) \overset{e}{\to} \mathcal{C}.$$

Unwinding the definitions, we see that $f_{(2)}$ is the colimit of the functors $p_! : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ indexed by the binary operations

$$p \in \mathrm{Mul}_{\mathcal{O}}(\{\langle 1 \rangle, \langle 1 \rangle\}, \langle 1 \rangle) \simeq \mathrm{Rect}(\langle 2 \rangle^{\circ} \times \square^k, \square^k) \simeq S^{k-1}.$$

Because $\mathcal{C}^{\otimes}$ is symmetric monoidal, this diagram of functors is constant, and we can identify $f_{(2)}$ with the functor $(C, D) \mapsto C \otimes D \otimes S^{k-1}$. Invoking Theorem 5.1.5.18 (note that $\mathcal{O}^{\otimes}$ is coherent by Theorem 5.1.1.1

and is essentially κ-small, where κ is the first uncountable cardinal), we obtain the desired pushout diagram

$$
\begin{array}{ccc}
\mathrm{Free}(C \otimes D \otimes S^{k-1}) & \longrightarrow & \mathrm{Free}(C \otimes D) \\
\downarrow & & \downarrow \\
\mathrm{Free}(C \coprod D) & \longrightarrow & \mathrm{Free}(C) \otimes \mathrm{Free}(D)
\end{array}
$$

in the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. $\hfill\square$

We conclude this section with the proofs of Lemma 5.1.5.17 and Proposition 5.1.5.15.

Proof of Lemma 5.1.5.17. Let $\overline{\mathcal{C}}^{\otimes}$ denote the fiber product $\mathcal{C}^{\otimes} \times_{\mathcal{D}^{\otimes}} \mathcal{Q}^{\otimes}$, and let $q' : \overline{\mathcal{C}}^{\otimes} \to \mathcal{Q}^{\otimes}$ denote the projection map. We observe that $\mathrm{Alg}_{\mathcal{Q}/\mathcal{D}}(\mathcal{C})$ can be identified with a full subcategory of $\mathrm{Fun}_{\mathcal{Q}^{\otimes}}(\mathcal{Q}^{\otimes}, \overline{\mathcal{C}}^{\otimes})$. We will prove that for every pair of objects $X \in \mathrm{Fun}_{\mathcal{D}}^{-}(\mathcal{O}, \mathcal{C})$ and $Y \in \mathrm{Fun}_{\mathcal{D}}^{+}(\mathcal{O}, \mathcal{C})$, the diagram

$$
\begin{array}{ccc}
j_! j^* F_{(2)}(X, Y) & \longrightarrow & j_! j^* f_{01}(X, Y) \\
\downarrow & & \downarrow \\
F_{(2)}(X, Y) & \longrightarrow & f_{01}(X, Y)
\end{array}
$$

is a pushout in $\mathrm{Fun}_{\mathcal{Q}^{\otimes}}(\mathcal{Q}^{\otimes}, \overline{\mathcal{C}}^{\otimes})$. In view of Lemma 3.2.2.9, it will suffice to show that for each object $Q \in \mathcal{Q}^{\otimes}$, the diagram σ_Q :

$$
\begin{array}{ccc}
(j_! j^* F_{(2)}(X, Y))(Q) & \longrightarrow & (j_! j^* f_{01}(X, Y))(Q) \\
\downarrow & & \downarrow \\
F_{(2)}(X, Y)(Q) & \longrightarrow & f_{01}(X, Y)(Q)
\end{array}
$$

is a q'-colimit diagram in $\overline{\mathcal{C}}^{\otimes}$.

Since q' is a pullback of $q : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$, it is a coCartesian fibration. For every morphism $\beta : Q \to Q'$ in $\mathcal{Q}^{\otimes}$, let $\beta_! : \overline{\mathcal{C}}_Q^{\otimes} \to \overline{\mathcal{C}}_{Q'}^{\otimes}$ be the induced map. To prove that σ_Q is a q'-colimit diagram, it will suffice to show that each of the diagrams $\beta_!(\sigma_Q)$ is a pushout diagram in $\overline{\mathcal{C}}_{Q'}^{\otimes}$ (Proposition T.4.3.1.10). Let $\langle n \rangle$ denote the image of Q' in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, and choose inert morphisms $\gamma(i) : Q' \to Q'_i$ lying over $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \leq i \leq n$. Since q' is a coCartesian fibration of ∞-operads, the functors $\gamma(i)_!$ induce an equivalence

$$
\overline{\mathcal{C}}_{Q'}^{\otimes} \to \prod_{1 \leq i \leq n} \overline{\mathcal{C}}_{Q'_i}^{\otimes}.
$$

It follows that $\beta_!(\sigma_Q)$ is a pushout diagram if and only if each $\gamma(i)_! \beta_!(\sigma_Q)$ is a pushout diagram in $\overline{\mathcal{C}}_{Q'_i}^{\otimes}$. We may therefore replace β by $\gamma(i) \circ \beta$ and thereby reduce to the case $Q' \in \mathcal{Q}$. The map β factors as a composition

$$
Q \xrightarrow{\beta'} Q'' \xrightarrow{\beta''} Q'
$$

where β' is inert and β'' is active. Note that $\beta'_!(\sigma_Q)$ is equivalent to $\sigma_{Q''}$; we may therefore replace Q by Q'' and thereby reduce to the case where β is active. Together these conditions imply that Q belongs to the image of either $j : \mathcal{O}'^{\otimes} \to \mathcal{Q}^{\otimes}$ or the essential image of $i : \mathcal{O}'^{\otimes} \boxplus \mathcal{O}'^{\otimes} \to \mathcal{Q}^{\otimes}$. We consider each case in turn.

Suppose that Q belongs to the image of j. We claim in this case that the vertical maps in the diagram σ_Q are equivalences. It follows that $\beta_!(\sigma_Q)$ has the same property, and is therefore automatically a pushout diagram. Our claim is a special case of the following more general assertion: let U be an arbitrary object of $\mathrm{Alg}_{\mathcal{Q}/\mathcal{D}}(\mathcal{C})$: then the counit map $j_! j^* U \to U$ induces an equivalence $(j_! j^* U)(Q) \to U(Q)$. The functor

$j_!$ is computed by the formation of operadic q-left Kan extension: consequently, $(j_! j^* U)(Q)$ is an operadic q-colimit of the diagram

$$(\mathcal{M}[\mathcal{O}]_0^{\otimes})_{/Q}^{\mathrm{act}} \to \mathcal{Q}^{\otimes} \xrightarrow{U} \mathcal{C}^{\otimes}.$$

The desired assertion now follows from the observation that Q belongs to the image of j, so that $(\mathcal{M}[\mathcal{O}]_0^{\otimes})_{/Q}^{\mathrm{act}}$ contains Q as a final object.

Suppose instead that Q belongs to the essential image of i. We claim in this case that the horizontal maps in the diagram σ_Q are equivalences. As before, it follows that $\beta_! \sigma_Q$ has the same property and is therefore automatically a pushout diagram. To prove the claim, we first show that for *any* map $U \to V$ in $\mathrm{Alg}_{\mathcal{M}[\mathcal{O}]_0/\mathcal{D}}(\mathcal{C}) \simeq \mathrm{Fun}_{\mathcal{D}}^{\pm}(\mathcal{O}, \mathcal{C}) \times \mathrm{Fun}_{\mathcal{D}}^{\pm}(\mathcal{O}, \mathcal{C})$, the induced map $\xi : (j_! U)(Q) \to (j_! V)(Q)$ is an equivalence in $\mathcal{C}^{\otimes}$. To see this, we observe that $(j_! U)(Q)$ and $(j_! V)(Q)$ are given by operadic q-colimits of diagrams

$$\mathcal{M}[\mathcal{O}]_0^{\otimes} \times_{\mathcal{Q}^{\otimes}} (\mathcal{Q}^{\otimes})_{/Q}^{\mathrm{act}} \to \mathcal{M}[\mathcal{O}]_0^{\otimes} \to \mathcal{C}^{\otimes}.$$

To prove that ξ is an equivalence, it suffices to show that these diagrams are equivalent. The assumption that Q belongs to the image of i guarantees that every object of $\mathcal{M}[\mathcal{O}]_0^{\otimes} \times_{\mathcal{Q}^{\otimes}} (\mathcal{Q}^{\otimes})_{/Q}^{\mathrm{act}}$ lies over $\langle 0 \rangle \in \mathrm{N}(\mathcal{F}\mathrm{in}_*)$: the desired result now follows from the observation that every morphism in $\mathcal{C}_{\langle 0 \rangle}^{\otimes}$ is an equivalence, since $\mathcal{C}_{\langle 0 \rangle}^{\otimes}$ is a (contractible) Kan complex.

To complete the proof, we must show that the map $\xi' : F_{(2)}(X, Y)(Q) \to f_{01}(X, Y)(Q)$ is an equivalence whenever Q belongs to the image of i. Let $U \in \mathrm{Alg}_{\mathcal{M}[\mathcal{O}]_0/\mathcal{D}}(\mathcal{C})$ be a preimage of (X, Y) under the equivalence $\mathrm{Alg}_{\mathcal{M}[\mathcal{O}]_0}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{D}}^{-}(\mathcal{O}, \mathcal{C}) \times \mathrm{Fun}_{\mathcal{D}}^{+}(\mathcal{O}, \mathcal{C})$. Then $F_{(2)}(X, Y)(Q)$ and $f_{01}(X, Y)(Q)$ can be identified with the operadic q-colimits of diagrams

$$\mathcal{M}[\mathcal{O}]_0^{\otimes} \times_{\mathcal{Q}^{\otimes}} (\mathcal{Q}^{\otimes})_{/Q}^{\mathrm{act}} \to \mathcal{M}[\mathcal{O}]_0^{\otimes} \xrightarrow{U} \mathcal{C}^{\otimes}$$

$$\mathcal{M}[\mathcal{O}]_0^{\otimes} \times_{\mathcal{M}[\mathcal{O}]_{/Q}^{\otimes}} (\mathcal{M}[\mathcal{O}]^{\otimes})_{/Q}^{\mathrm{act}} \to \mathcal{M}[\mathcal{O}]_0^{\otimes} \xrightarrow{U} \mathcal{C}^{\otimes}.$$

To prove that ξ' is an equivalence, it suffices to show that the functor

$$\epsilon : \mathcal{M}[\mathcal{O}]_0^{\otimes} \times_{\mathcal{Q}^{\otimes}} (\mathcal{Q}^{\otimes})_{/Q}^{\mathrm{act}} \to \mathcal{M}[\mathcal{O}]_0^{\otimes} \times_{\mathcal{M}[\mathcal{O}]_{/Q}^{\otimes}} (\mathcal{M}[\mathcal{O}]^{\otimes})_{/Q}^{\mathrm{act}} \to \mathcal{M}[\mathcal{O}]_0^{\otimes}$$

is a categorical equivalence. Both the domain and codomain of ϵ are right-fibered over $(\mathcal{M}[\mathcal{O}]_0^{\otimes})^{\mathrm{act}}$: it will therefore suffice to show that ϵ induces a homotopy equivalence after passing to the fiber over any object $P \in \mathcal{M}[\mathcal{O}]_0^{\otimes}$ (Corollary T.2.4.4.4). Unwinding the definitions, we must show that the canonical map

$$\mathrm{Map}_{\mathcal{Q}^{\otimes}}(i(P), Q) \to \mathrm{Map}_{\mathcal{M}[\mathcal{O}]^{\otimes}}(P, Q)$$

is a homotopy equivalence. This follows from a simple calculation, using our assumptions that $\mathcal{O}^{\otimes}$ is unital and that Q belongs to the essential image of i. $\qquad \square$

Proof of Proposition 5.1.5.15. Let $\mathcal{O}^{\otimes}$ be a coherent ∞-operad such that $\mathcal{O}$ is a Kan complex; we wish to prove that the inner fibration $\mathcal{M}[\mathcal{O}]^{\otimes} \to \Delta^2$ is flat. Fix an object $X \in \mathcal{M}[\mathcal{O}]_0^{\otimes}$, corresponding to a pair of objects $X_-, X_+ \in \mathcal{O}^{\otimes}$, and let $Z \in \mathcal{O}^{\otimes} \simeq \mathcal{M}[\mathcal{O}]_2^{\otimes}$. Suppose we are given a morphism $X \to Z$ in $\mathcal{M}[\mathcal{O}]^{\otimes}$. We wish to prove that the ∞-category $\mathcal{C} = \mathcal{M}[\mathcal{O}]_{X//Z}^{\otimes} \times_{\Delta^2} \{1\}$ is weakly contractible. Let $\emptyset \in \mathcal{O}_{\langle 0 \rangle}^{\otimes}$ be a final object of $\mathcal{O}^{\otimes}$. Since $\mathcal{O}^{\otimes}$ is unital, $\emptyset$ is also an initial object of $\mathcal{O}^{\otimes}$. We can therefore choose a diagram σ :

$$\begin{array}{ccc} \emptyset & \longrightarrow & X_- \\ \downarrow & & \\ X_+ & & \end{array}$$

in $\mathcal{O}_{/Z}^{\otimes}$. Let $\overline{\mathcal{C}}[\sigma]$ denote the full subcategory of $\mathcal{O}_{\sigma//Z}^{\otimes}$ spanned by those diagrams

$$\begin{array}{ccc} \emptyset & \longrightarrow & X_- \\ \downarrow & & \downarrow{\scriptstyle f} \\ X_+ & \xrightarrow{\ g\ } & Q \end{array}$$

where f and g are both semi-inert (by virtue of (2), this is equivalent to the requirement that $q(f)$ and $q(g)$ are semi-inert morphisms in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$) and the map $q(X_-)\coprod q(X_+) \to q(Q)$ is a surjection. Using the fact that $\emptyset$ is an initial object of $\mathcal{O}^{\otimes}$, we obtain a trivial Kan fibration $\overline{\mathcal{C}}[\sigma] \to \mathcal{C}$. It will therefore suffice to show that $\overline{\mathcal{C}}[\sigma]$ is weakly contractible.

Let $q(X_+) = \langle m \rangle$. The proof will proceed by induction on m. If $m = 0$, then $\overline{\mathcal{C}}[\sigma]$ has an initial object and there is nothing to prove. Otherwise, the map $\emptyset \to X_+$ factors as a composition

$$\emptyset \to X_+' \xrightarrow{\ \alpha\ } X_+,$$

where $q(\alpha)$ is an inclusion $\langle m-1 \rangle \hookrightarrow \langle m \rangle$. Let $\tau : \Delta^1 \coprod_{\{0\}} \Delta^2 \to \mathcal{O}_{/Z}^{\otimes}$ denote the diagram $X_- \leftarrow \emptyset \to X_+' \to X_+$, and let $\tau_0 = \tau|(\Delta^1 \coprod_{\{0\}} \Delta^1)$. Let $\mathcal{D}$ denote the full subcategory of the fiber product

$$\mathrm{Fun}(\Delta^1, (\mathcal{O}_{/Z}^{\otimes})_{\tau_0/}) \times_{\mathrm{Fun}(\{1\},(\mathcal{O}_{/Z}^{\otimes})_{\tau_0/})} (\mathcal{O}_{/Z}^{\otimes})_{\tau/}$$

spanned by those diagrams

$$\begin{array}{ccc} \emptyset & \longrightarrow & X_- \\ \downarrow & & \downarrow \\ X_+' & \longrightarrow & Q' \\ \downarrow & & \downarrow \\ X_+ & \longrightarrow & Q \end{array}$$

in $\mathcal{O}_{/Z}^{\otimes}$ where the maps $X_- \to Q'$, $X_+' \to Q'$, and $Q' \to Q$ are semi-inert and the map $q(X_-)\coprod_{q(\emptyset)} q(X_+') \to q(Q')$ is surjective. Let $\mathcal{D}_0 \subseteq \mathcal{D}$ be the full subcategory spanned by those diagrams for which the map $X_+ \to Q$ is semi-inert and the map $q(Q')\coprod_{q(X_+')} q(X_+) \to q(Q)$ is surjective. We have canonical maps

$$\overline{\mathcal{C}}[\sigma] \xleftarrow{\ \phi\ } \mathcal{D}_0 \subseteq \mathcal{D} \xrightarrow{\ \psi\ } \overline{\mathcal{C}}[\tau_0].$$

The map ϕ admits a right adjoint and is therefore a weak homotopy equivalence, and the simplicial set $\overline{\mathcal{C}}[\tau_0]$ is weakly contractible by the inductive hypothesis. The inclusion $\mathcal{D}_0 \subseteq \mathcal{D}$ admits a right adjoint, and is therefore a weak homotopy equivalence. To complete the proof, it will suffice to show that ψ is a weak homotopy equivalence. We have a homotopy pullback diagram

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & (\mathcal{K}_{\mathcal{O}})_{\alpha//\mathrm{id}_Z} \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi'} \\ \overline{\mathcal{C}}[\tau_0] & \longrightarrow & \mathcal{O}_{X_+'//Z}^{\otimes} \,. \end{array}$$

The coherence of $\mathcal{O}^{\otimes}$ guarantees that the map $\mathcal{K}_{\mathcal{O}} \to \mathcal{O}^{\otimes}$ is a flat inner fibration (Theorem 3.3.2.2), so that ψ' satisfies the hypotheses of Lemma 3.3.2.8 (see Example 3.3.2.9). It follows that ψ is a weak homotopy equivalence, as desired. $\square$

5.2 Little Cubes and Manifold Topology

Fix an integer $k \geq 0$. In Definition 5.1.0.2 we introduced the ∞-operad $\mathbb{E}_k^{\otimes}$ of little k-cubes. The underlying ∞-category $\mathbb{E}_k$ has a unique object, which we can think of as an abstract open cube $\square^k$ of dimension k. The morphisms in $\mathbb{E}_k^{\otimes}$ are described by rectilinear embeddings from $\square^k$ to itself. There are a number of variants on Definition 5.1.0.2, where the condition that an embedding $i : \square^k \hookrightarrow \square^k$ be rectilinear is replaced by the requirement that i preserve some other structure. We will describe a number of these variants in §5.2.2. For our purposes, the main case of interest is that in which we require all of our cubes to be equipped with an open embedding into a topological manifold M of dimension k. The collection of such cubes can be organized into an ∞-operad $\mathbb{E}_M^{\otimes}$ which we will study in §5.2.4. The study of this ∞-operad will require some results from point-set topology concerning open immersions between topological manifolds, which we will review in §5.2.1.

The ∞-operad $\mathbb{E}_m^{\otimes}$ will play a central role in our discussion of topological chiral homology in §5.3. In the latter context it is sometimes convenient to work with *nonunital* $\mathbb{E}_M^{\otimes}$-algebras: that is, algebras over the closely related ∞-operad $(\mathbb{E}_M^{\otimes})_{\mathrm{nu}} \subseteq \mathbb{E}_M^{\otimes}$ obtained by removing all 0-ary operations. We will discuss the general theory of nonunital algebras in §5.2.3.

Convention 5.2.0.1. Unless otherwise specified, the word *manifold* will refer to a paracompact Hausdorff topological manifold of some fixed dimension k.

5.2.1 Embeddings of Topological Manifolds

In this section, we will review some classical results in point-set topology concerning embeddings between topological manifolds of the same dimension. We begin by stating a parametrized version of Brouwer's invariance of domain theorem (a proof will be given at the end of this section).

Theorem 5.2.1.1 (Brouwer). *Let M and N be manifolds of dimension k, and let S be an arbitrary topological space. Suppose we are given a continuous map $f : M \times X \to N \times X$ satisfying the following pair of conditions:*

(*i*) *The diagram*

$$
\begin{array}{ccc}
M \times X & \xrightarrow{\ \ f\ \ } & N \times X \\
& \searrow \quad \swarrow & \\
& X &
\end{array}
$$

   is commutative.

(*ii*) *The map f is injective.*

Then f is an open map.

Remark 5.2.1.2. When X is a single point, Theorem 5.2.1.1 was proven by Brouwer in [26].

Corollary 5.2.1.3. *Let M and N be manifolds of the same dimension, and let $f : M \times X \to N \times X$ be a continuous bijection which commutes with the projection to X. Then f is a homeomorphism.*

Let M and N be topological manifolds of the same dimension. We let $\mathrm{Emb}(M, N)$ denote the set of all open embeddings $M \hookrightarrow N$. We will regard $\mathrm{Emb}(M, N)$ as a topological space: it is a subspace of the collection of all continuous maps from M to N, which we endow with the compact-open topology. We let $\mathrm{Homeo}(M, N)$ denote the set of all homeomorphisms of M with N, regarded as a subspace of $\mathrm{Emb}(M, N)$. For $k \geq 0$, we let $\mathrm{Top}(k)$ denote the topological group $\mathrm{Homeo}(\mathbf{R}^k, \mathbf{R}^k)$ of homeomorphisms from $\mathbf{R}^k$ to itself.

Remark 5.2.1.4. Let M and N be topological manifolds of the same dimension, and let $\mathrm{Map}(M, N)$ denote the set of all continuous maps from M to N, endowed with the compact-open topology. Since M is locally compact, $\mathrm{Map}(M, N)$ classifies maps of topological spaces from M to N: that is, for any topological space X, giving a continuous map $X \to \mathrm{Map}(M, N)$ is equivalent to giving a continuous map $M \times X \to N$, which is in turn equivalent to giving a commutative diagram

$$\begin{array}{ccc} M \times X & \xrightarrow{\quad f \quad} & N \times X \\ & \searrow \qquad \swarrow & \\ & X. & \end{array}$$

Under this equivalence, continuous maps from X to $\mathrm{Emb}(M, N)$ correspond to commutative diagrams as above where f is injective (hence an open embedding, by Theorem 5.2.1.1), and continuous maps from X to $\mathrm{Homeo}(M, N)$ correspond to commutative diagrams as above where f is bijective (and therefore a homeomorphism, by Theorem 5.2.1.1). It follows that the space of embeddings $\mathrm{Emb}(M, M)$ has the structure of a topological monoid, and that $\mathrm{Homeo}(M, M)$ has the structure of a topological group.

In §5.2.2, we will need the to know that the topological monoid $\mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k)$ is grouplike: that is, the set of path components $\pi_0 \mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k)$ forms a group under composition. This is an immediate consequence of the following version of the Kister-Mazur theorem, whose proof we defer until the end of this section.

Theorem 5.2.1.5 (Kister-Mazur). *For each $k \geq 0$, the inclusion $\mathrm{Top}(k) \hookrightarrow \mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k)$ is a homotopy equivalence.*

We now describe some variants on the embedding spaces $\mathrm{Emb}(M, N)$ and their homotopy types.

Definition 5.2.1.6. Let M be a topological manifold of dimension k, let S be a finite set, and for every positive real number t let $B(t) \subset \mathbf{R}^k$ be as in Lemma 5.2.1.7. We let $\mathrm{Germ}(S, M)$ denote the simplicial set $\varinjlim_n \mathrm{Sing}\,\mathrm{Emb}(B(\frac{1}{2^n}) \times S, M)$. We will refer to $\mathrm{Germ}(S, M)$ as the *simplicial set of S-germs in M*.

Lemma 5.2.1.7. *Let M be a topological manifold of dimension k and S a finite set. For every positive real number t, let $B(t) \subset \mathbf{R}^k$ denote the open ball of radius t. For every pair of positive real numbers $s < t$, the restriction map $r : \mathrm{Emb}(B(t) \times S, M) \to \mathrm{Emb}(B(s) \times S, M)$ is a homotopy equivalence.*

Proof. This follows from the observation that the embedding $B(s) \hookrightarrow B(t)$ is isotopic to a homeomorphism. $\square$

By repeated application of Lemma 5.2.1.7 we deduce the following:

Proposition 5.2.1.8. *Let M be a topological manifold of dimension k and let S be a finite set. Then the obvious restriction map $\mathrm{Sing}\,\mathrm{Emb}(\mathbf{R}^k \times S, M) \to \mathrm{Germ}(S, M)$ is a homotopy equivalence of Kan complexes.*

Notation 5.2.1.9. Let M be a topological manifold of dimension k. Evaluation at the origin $0 \in \mathbf{R}^k$ induces a map $\theta : \mathrm{Emb}(\mathbf{R}^k, M) \to M$. We will denote the fiber of this map over a point $x \in M$ by $\mathrm{Emb}_x(\mathbf{R}^k, M)$. The map θ is a Serre fibration, so we have a fiber sequence of topological spaces

$$\mathrm{Emb}_x(\mathbf{R}^k, M) \to \mathrm{Emb}(\mathbf{R}^k, M) \to M.$$

We let $\mathrm{Germ}(M)$ denote the simplicial set $\mathrm{Germ}(\{*\}, M)$. Evaluation at 0 induces a Kan fibration $\mathrm{Germ}(M) \to \mathrm{Sing}\,M$; we will denote the fiber of this map over a point $x \in M$ by $\mathrm{Germ}_x(M)$. We have a map of fiber sequences

$$\begin{array}{ccccc} \mathrm{Sing}\,\mathrm{Emb}_x(\mathbf{R}^k, M) & \longrightarrow & \mathrm{Sing}\,\mathrm{Emb}(\mathbf{R}^k, M) & \longrightarrow & \mathrm{Sing}\,M \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi'} & & \downarrow{\scriptstyle\psi''} \\ \mathrm{Germ}_x(M) & \longrightarrow & \mathrm{Germ}(M) & \longrightarrow & \mathrm{Sing}\,M. \end{array}$$

Since ψ' is a homotopy equivalence (Proposition 5.2.1.8) and ψ'' is an isomorphism, we conclude that ψ is a homotopy equivalence.

The simplicial set $\mathrm{Germ}_0(\mathbf{R}^k)$ forms a simplicial group with respect to the operation of composition of germs. Since $\mathbf{R}^k$ is contractible, we have homotopy equivalences of simplicial monoids

$$\mathrm{Germ}_0(\mathbf{R}^k) \leftarrow \mathrm{Sing}\,\mathrm{Emb}_x(\mathbf{R}^k, \mathbf{R}^k) \rightarrow \mathrm{Sing}\,\mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k) \leftarrow \mathrm{Sing}\,\mathrm{Top}(k)$$

(see Theorem 5.2.1.5): in other words, $\mathrm{Germ}_0(\mathbf{R}^k)$ can be regarded as a model for the homotopy type of the topological group $\mathrm{Top}(k)$.

Remark 5.2.1.10. For any topological k-manifold M, the group $\mathrm{Germ}_0(\mathbf{R}^k)$ acts on $\mathrm{Germ}(M)$ by composition. This action is free, and we have a canonical isomorphism of simplicial sets $\mathrm{Germ}(M)/\mathrm{Germ}_0(\mathbf{R}^k) \simeq \mathrm{Sing}\,M$.

Remark 5.2.1.11. Let $j : U \to M$ be an open embedding of topological k-manifolds and S a finite set. Then evaluation at 0 determines a diagram of simplicial sets

$$\begin{array}{ccc} \mathrm{Sing}\,\mathrm{Emb}(\mathbf{R}^k \times S, U) & \longrightarrow & \mathrm{Sing}\,\mathrm{Emb}(\mathbf{R}^k \times S, M) \\ \downarrow & & \downarrow \\ \mathrm{Conf}(S, U) & \longrightarrow & \mathrm{Conf}(S, M). \end{array}$$

We claim that this diagram is homotopy Cartesian. In view of Proposition 5.2.1.8, it suffices to show that the equivalent diagram

$$\begin{array}{ccc} \mathrm{Germ}(S, U) & \longrightarrow & \mathrm{Germ}(S, M) \\ \downarrow & & \downarrow \\ \mathrm{Conf}(S, U) & \longrightarrow & \mathrm{Conf}(S, M), \end{array}$$

is homotopy Cartesian. This diagram is a pullback square and the vertical maps are Kan fibrations: in fact, the vertical maps are principal fibrations with structure group $\mathrm{Germ}_0(\mathbf{R}^k)^S$.

Taking $U = \mathbf{R}^k$, and S to consist of a single point, we have a larger diagram

$$\begin{array}{ccccc} \mathrm{Sing}\,\mathrm{Emb}_0(\mathbf{R}^k, \mathbf{R}^k) & \longrightarrow & \mathrm{Sing}\,\mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k) & \longrightarrow & \mathrm{Sing}\,\mathrm{Emb}(\mathbf{R}^k, M) \\ \downarrow & & \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \mathrm{Sing}\,\mathbf{R}^k & \longrightarrow & \mathrm{Sing}\,M. \end{array}$$

Since the horizontal maps on the left are homotopy equivalences of Kan complexes, we obtain a homotopy fiber sequence of Kan complexes

$$\mathrm{Sing}\,\mathrm{Emb}_0(\mathbf{R}^k, \mathbf{R}^k) \to \mathrm{Sing}\,\mathrm{Emb}(\mathbf{R}^k, M) \to \mathrm{Sing}\,M.$$

We conclude this section with the proofs of Theorems 5.2.1.1 and 5.2.1.5.

Proof of Theorem 5.2.1.1. Fix a continuous map $f : M \times X \to N \times X$ and an open set $U \subseteq M \times X$; we wish to show that $f(U)$ is open in $N \times X$. In other words, we wish to show that for each $u = (m, x) \in U$, the set $f(U)$ contains a neighborhood of $f(u) = (n, x)$ in $N \times S$. Since N is a manifold, there exists an open neighborhood $V \subseteq N$ containing n which is homeomorphic to Euclidean space $\mathbf{R}^k$. Replacing N by V (and shrinking M and X as necessary), we may assume that $N \simeq \mathbf{R}^k$. Similarly, we can replace M and X by small neighborhoods of m and s to reduce to the case where $M \simeq \mathbf{R}^k$ and $U = M \times X$.

We first treat the case where X consists of a single point. Let $D \subseteq M$ be a closed neighborhood of m homeomorphic to a (closed) k-dimensional disk, and regard N as an open subset of the k-sphere S^k. We have a long exact sequence of compactly supported cohomology groups

$$0 \simeq \mathrm{H}_c^{k-1}(S^k; \mathbf{Z}) \to \mathrm{H}_c^{k-1}(f(\partial D); \mathbf{Z}) \to \mathrm{H}_c^k(S^k - f(\partial D); \mathbf{Z}) \to \mathrm{H}_c^k(S^k; \mathbf{Z}) \to \mathrm{H}_c^k(f(\partial D); \mathbf{Z}) \simeq 0.$$

Since f is injective, $f(\partial D)$ is homeomorphic to a $(k-1)$-sphere. It follows that $\mathrm{H}_c^k(S^k - f(\partial D); \mathbf{Z})$ is a free $\mathbf{Z}$-module of rank 2, so that (by Poincare duality) the ordinary cohomology $\mathrm{H}^0(S^k - f(\partial D); \mathbf{Z})$ is also free of rank 2: in other words, the open set $S^k - f(\partial D)$ has exactly two connected components. We have another long exact sequence

$$0 \simeq \mathrm{H}_c^{k-1}(f(D); \mathbf{Z}) \to \mathrm{H}_c^k(S^k - f(D); \mathbf{Z}) \to \mathrm{H}_c^k(S^k; \mathbf{Z}) \to \mathrm{H}_c^k(f(D); \mathbf{Z}) \simeq 0.$$

This proves that $\mathrm{H}_c^k(S^k - f(D); \mathbf{Z})$ is free of rank 1 so that (by Poincare duality) $S^k - f(D)$ is connected. The set $S^k - f(\partial D)$ can be written as a union of connected sets $f(D - \partial D)$ and $S^k - f(D)$, which must therefore be the connected components of $S^k - f(\partial D)$. It follows that $f(D - \partial D)$ is open S^k so that $f(M)$ contains a neighborhood of $f(m)$ as desired.

Let us now treat the general case. Without loss of generality, we may assume that $f(u) = (0, x)$, where $x \in X$ and 0 denotes the origin of $\mathbf{R}^k$. Let $f_x : M \to N$ be the restriction of f to $M \times \{x\}$. The above argument shows that f_x is an open map, so that $f_x(M)$ contains a closed ball $\overline{B(\epsilon)} \subseteq \mathbf{R}^k$ for some positive radius ϵ. Let $S \subseteq M - \{m\}$ be the inverse image of the boundary $\partial \overline{B(\epsilon)}$, so that S is homeomorphic to the $(k-1)$-sphere. In particular, S is compact. Let $\pi : M \times X \to \mathbf{R}^k$ denote the composition of f with the projection map $N \times X \to N \simeq \mathbf{R}^k$. Shrinking X if necessary, we may suppose that the distance $d(f(s,x), f(s,y)) < \frac{\epsilon}{2}$ for all $s \in S$ and all $y \in X$. We will complete the proof by showing that $B(\frac{\epsilon}{2}) \times X$ is contained in the image of f. Supposing otherwise; then there exists $v \in B(\frac{\epsilon}{2})$ and $y \in X$ such that $(v, y) \notin f(M \times X)$. Then f_y defines a map from M to $\mathbf{R}^k - \{v\}$, so the restriction $f_y|S$ is nullhomotopic when regarded as a map from S to $\mathbf{R}^k - \{v\}$. However, this map is homotopic (via a straight-line homotopy) to $f_x|S$, which carries S homeomorphically onto $\partial \overline{B(\epsilon)} \subseteq \mathbf{R}^k - \{v\}$. It follows that the inclusion $\partial \overline{B(\epsilon)} \subseteq \mathbf{R}^k - \{v\}$ is nullhomotopic, which is impossible. $\square$

We now turn to the proof of Theorem 5.2.1.5. The main step is the following technical result:

Lemma 5.2.1.12. *Let X be a paracompact topological space, and suppose that there exists a continuous map $f_0 : \mathbf{R}^k \times X \to \mathbf{R}^k$ such that, for each $x \in X$, the restriction $f_{0,x} = f_0|\mathbf{R}^k \times \{x\}$ is injective. Then there exists an isotopy $f : \mathbf{R}^k \times X \times [0,1] \to \mathbf{R}^k$ with the following properties:*

(i) *The restriction $f|\mathbf{R}^k \times X \times \{0\}$ coincides with f_0.*

(ii) *For every pair $(x,t) \in X \times [0,1]$, the restricted map $f_{t,x} = f|\mathbf{R}^k \times \{x\} \times \{t\}$ is injective.*

(iii) *For each $x \in X$, the map $f_{1,x}$ is bijective.*

(iv) *Suppose $x \in X$ has the property that $f_{0,x}$ is bijective. Then $f_{t,x}$ is bijective for all $t \in [0,1]$.*

Proof. Let $w : X \to \mathbf{R}^k$ be given by the formula $w(x) = f_0(0, x)$. Replacing f_0 by the map $(v, x) \mapsto f_0(v, x) - w(x)$, we can reduce to the case where $w = 0$: that is, each of the maps $f_{0,x}$ carries the origin of $\mathbf{R}^k$ to itself.

For every continuous positive real-valued function $\epsilon : X \to \mathbf{R}_{>0}$, we let $B(\epsilon)$ denote the open subset of $\mathbf{R}^k \times X$ consisting of those pairs (v, x) such that $|v| < \epsilon(x)$. If r is a real number, we let $B(r) = B(\epsilon)$, where $\epsilon : X \to \mathbf{R}_{>0}$ is the constant function taking the value r.

Let $g^1 : \mathbf{R}^k \times X \to \mathbf{R}^k \times X$ be given by the formula $g^1(v, x) = (f_1(v, x), x)$. The image $g^1(B(1))$ is an open subset of $\mathbf{R}^k \times X$ (Theorem 5.2.1.1) which contains the zero section $\{0\} \times X$; it follows that $g^1(B(1))$ contains $B(\epsilon)$ for some positive real-valued continuous function $\epsilon : X \to \mathbf{R}_{>0}$. Replacing f_0 by the funciton $(v, x) \mapsto \frac{f_0(v, x)}{\epsilon(x)}$, we can assume that $B(1) \subseteq g^1(B(1))$.

We now proceed by defining a sequence of open embeddings $\{g^i : \mathbf{R}^k \times X \to \mathbf{R}^k \times X\}_{i \geq 2}$ and isotopies $\{h^i_t\}_{0 \leq t \leq 1}$ from g^i to g^{i+1}, so that the following conditions are satisfied:

(a) Each of the maps g^i is compatible with the projection to X.

(b) Each isotopy $\{h^i_t\}_{0 \leq t \leq 1}$ consists of open embeddings $\mathbf{R}^k \times X \to \mathbf{R}^k \times X$ which are compatible with the projection to X. Moreover, this isotopy is constant on the open set $B(i) \subseteq \mathbf{R}^k \times X$.

(c) For $i \geq 1$, we have $B(i) \subseteq g^i(B(i))$.

(d) Let $x \in X$ be such that the map $g^i_x : \mathbf{R}^k \to \mathbf{R}^k$ is a homeomorphism. Then $h^i_{t,x} : \mathbf{R}^k \to \mathbf{R}^k$ is a homeomorphism for all $t \in [0, 1]$.

Assuming that these requirements are met, we can obtain the desired isotopy f_t by the formula

$$f_t(v, x) = \begin{cases} \pi g^i(v, x) & \text{if } (|v| < i) \wedge (t > 1 - \frac{1}{2^{i-1}}) \\ \pi h^i_s(v, x) & \text{if } t = 1 + \frac{s-2}{2^i}, \end{cases}$$

where π denotes the projection from $\mathbf{R}^k \times X$ onto $\mathbf{R}^k$. We now proceed by induction on i. Assume that g^i has already been constructed; we will construct an isotopy h^i from g^i to another open embedding g^{i+1} to satisfy the above conditions. First, we need to establish a bit of notation.

For every pair of real numbers $r < s$, let $\{H(r, s)_t : \mathbf{R}^k \to \mathbf{R}^k\}_{0 \leq t \leq 1}$ be a continuous family of homeomorphisms satisfying the following conditions:

(i) The isotopy $\{H(r, s)_t\}$ is constant on $\{v \in \mathbf{R}^k : |v| < \frac{r}{2}\}$ and $\{v \in \mathbf{R}^k : |v| > s + 1\}$.

(ii) The map $H(r, s)$ restricts to a homeomorphism of $B(r)$ with $B(s)$.

We will assume that the homeomorphisms $\{H(r, s)_t\}$ are chosen to depend continuously on r, s, and t. Consequently, if $\epsilon < \epsilon'$ are positive real-valued functions on X, we obtain an isotopy $\{H(\epsilon, \epsilon')_t : \mathbf{R}^k \times X \to \mathbf{R}^k \times X\}$ by the formula $H(\epsilon, \epsilon')_t(v, x) = (H(\epsilon(x), \epsilon'(x))_t(v), x)$.

Since g^i is continuous and $\{0\} \times X \subseteq (g^i)^{-1}B(\frac{1}{2})$, there exists a real-valued function $\delta : X \to (0, 1)$ such that $g^i(B(\delta)) \subseteq B(\frac{1}{2})$. We define a homeomorphism $c : \mathbf{R}^k \times X \to \mathbf{R}^k \times X$ as follows:

$$c(v, x) = \begin{cases} (v, x) & \text{if } (v, x) \notin g^i(\mathbf{R}^k \times X) \\ g^i(H(\delta(x), i)_1^{-1}(w), x) & \text{if } (v, x) = g^i(w, x). \end{cases}$$

Since g^i carries $B(\delta)$ into $B(\frac{1}{2})$, we deduce that $c(g^i(v, x)) \in B(\frac{1}{2})$ if $(v, x) \in B(i)$. Note that c is the identity outside of the image $g^i B(i + 1)$; we can therefore choose a positive real valued function $\epsilon : X \to (i + 1, \infty)$ such that c is the identity outside of $B(\epsilon)$.

We now define h^i_t by the formula $h^i_t = c^{-1} \circ H(1, \epsilon)_t \circ c \circ g^i$ (here we identify the real number $1 \in \mathbf{R}$ with the constant function $X \to \mathbf{R}$ taking the value 1). It is clear that h^i_t is an isotopy from $g^i = g^i_0$ to another map $g^{i+1} = g^i_1$, satisfying conditions (a) and (d) above. Since $H(1, \epsilon)_t$ is the identity on $B(\frac{1}{2})$ and $c \circ g^i$ carries $B(i)$ into $B(\frac{1}{2})$, we deduce that h^i_t is constant on $B(i)$ so that (b) is satisfied. It remains only to verify (c): we must show that $g^{i+1} B(i + 1)$ contains $B(i + 1)$. In fact, we claim that $g^{i+1} B(i + 1)$ contains $B(\epsilon)$. Since c is supported in $B(\epsilon)$, it suffices to show that $(cg^{i+1})B(i + 1) = (H(1, \epsilon)_1 \circ c \circ g^i)B(i + 1)$ contains $B(\epsilon)$. For this we need only show that $(c \circ g^i)B(i + 1)$ contains $B(1) \subseteq B(i) \subseteq g^i B(i) \subseteq g^i B(i + 1)$. This is clear, since $H(\delta(x), i)_1$ induces a homeomorphism of $B(i + 1)$ with itself. $\qquad \square$

Proof of Theorem 5.2.1.5. For every compact set $K \subseteq \mathbf{R}^k$, the compact open topology on the set of continuous maps $\mathrm{Map}(K, \mathbf{R}^k)$ agrees with the topology induced by the metric $d_K(f, g) = \sup\{|f(v) - g(v)|, v \in K\}$.

Consequently, the compact open topology on the entire mapping space $\mathrm{Map}(\mathbf{R}^k, \mathbf{R}^k)$ is defined by the countable sequence of metrics $\{d_{\overline{B(n)}}\}_{n \geq 0}$ (here $\overline{B(n)}$ denotes the closed ball of radius n), or equivalently by the single metric

$$d(f, g) = \sum_{n \geq 0} \frac{1}{2^n} \inf\{1, d_{\overline{B(n)}}(f, g)\}.$$

It follows that $\mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k) \subseteq \mathrm{Map}(\mathbf{R}^k, \mathbf{R}^k)$ is metrizable and therefore paracompact. Applying Lemma 5.2.1.12 to the canonical pairing

$$f_0 : \mathbf{R}^k \times \mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k) \hookrightarrow \mathbf{R}^k \times \mathrm{Map}(\mathbf{R}^k, \mathbf{R}^k) \to \mathbf{R}^k,$$

we deduce the existence of an map $f : \mathbf{R}\,k \times \mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k) \times [0, 1] \to \mathbf{R}^k$ which is classified by a homotopy $\chi : \mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k) \times [0, 1] \to \mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k)$ from $\mathrm{id}_{\mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k)}$ to some map $s : \mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k) \to \mathrm{Homeo}(\mathbf{R}^k, \mathbf{R}^k)$. We claim that s is a homotopy inverse to the inclusion $i : \mathrm{Homeo}(\mathbf{R}^k, \mathbf{R}^k) \to \mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k)$. The homotopy χ shows that $i \circ s$ is homotopy to the identity on $\mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k)$, and the restriction of χ to $\mathrm{Homeo}(\mathbf{R}^k, \mathbf{R}^k) \times [0, 1]$ shows that $s \circ i$ is homotopic to the identity on $\mathrm{Homeo}(\mathbf{R}^k, \mathbf{R}^k)$. $\qquad\square$

5.2.2 Variations on the Little Cubes Operads

Fix an integer $k \geq 0$. In §5, we introduced a topological operad ${}^t\mathbb{E}_k$ whose n-ary operations are given by *rectilinear* open embeddings from $\square^k \times \langle n \rangle^\circ$ into $\square^k$. Our goal in this section is to introduce some variations on this construction, where we drop the requirement that our embeddings be rectilinear (or replace rectilinearity by some other condition). The main observation is that the resulting ∞-operads are closely related to the ∞-operad $\mathbb{E}_k^\otimes$ studied in §5.1.5 (see Proposition 5.2.2.8 below).

Definition 5.2.2.1. Fix an integer $k \geq 0$. We define a topological category ${}^t\mathbb{E}_{\mathrm{BTop}(k)}^\otimes$ as follows:

(1) The objects of ${}^t\mathbb{E}_{\mathrm{BTop}(k)}^\otimes$ are the objects $\langle n \rangle \in \mathcal{F}\mathrm{in}_*$.

(2) Given a pair of objects $\langle m \rangle, \langle n \rangle \in^t \mathbb{E}_{\mathrm{BTop}(k)}^\otimes$, the mapping space $\mathrm{Map}_{{}^t\mathbb{E}_{\mathrm{BTop}(k)}^\otimes}(\langle m \rangle, \langle n \rangle)$ is given by the disjoint union

$$\coprod_\alpha \prod_{1 \leq i \leq n} \mathrm{Emb}(\mathbf{R}^k \times \alpha^{-1}\{i\}, \mathbf{R}^k)$$

taken over all morphisms $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$.

We let $\mathrm{BTop}(k)^\otimes$ denote the ∞-category given by the homotopy coherent nerve $\mathrm{N}({}^t\mathbb{E}_{\mathrm{BTop}(k)})^\otimes$.

Remark 5.2.2.2. It follows from Proposition 2.1.1.27 that $\mathrm{BTop}(k)^\otimes$ is an ∞-operad.

Remark 5.2.2.3. Definition 5.2.2.1 is a close relative of Definition 5.1.0.2. In fact, choosing a homeomorphism $\mathbf{R}^k \simeq \square^k$, we obtain an inclusion of ∞-operads $\mathbb{E}_k^\otimes \to \mathrm{BTop}(k)^\otimes$.

Remark 5.2.2.4. The object $\langle 0 \rangle$ is initial in $\mathrm{BTop}(k)^\otimes$. It follows that $\mathrm{BTop}(k)^\otimes$ is a unital ∞-operad.

Example 5.2.2.5. Suppose that $k = 1$. Every open embedding $j : \mathbf{R}^k \times S \hookrightarrow \mathbf{R}^k$ determines a pair $(<, \epsilon)$, where $<$ is an element of the set of linear orderings of S (given by $s < s'$ if $j(0, s) < j(0, s')$) and $\epsilon : S \to \{\pm 1\}$ is a function defined so that $\epsilon(s) = 1$ if $j|(\mathbf{R}^k \times \{s\})$ is orientation preserving, and $\epsilon(s) = -1$ otherwise. This construction determines a homotopy equivalence $\mathrm{Emb}(\mathbf{R}^k \times S, \mathbf{R}^k) \to L(S) \times \{\pm 1\}^S$, where $L(S)$ denotes the set of linear orderings of S. It follows that $\mathrm{BTop}(k)^\otimes$ is equivalent to the nerve of its homotopy category and therefore arises from an operad in the category of sets via Construction 2.1.1.7. In fact, this is the operad which controls associative algebras with involution, as described in §2.3.4.

Notation 5.2.2.6. For each integer $k \geq 0$, we let $\mathrm{BTop}(k)$ denote the ∞-category $\mathrm{BTop}(k)^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \{\langle 1 \rangle\}$ underlying the ∞-operad $\mathrm{BTop}(k)^{\otimes}$. Then $\mathrm{BTop}(k)$ can be identified with the nerve of the topological category having a single object whose endomorphism monoid is the space $\mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k)$ of open embeddings from $\mathbf{R}^k$ to itself. It follows from the Kister-Mazur theorem (Theorem 5.2.1.5) that $\mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k)$ is a grouplike topological monoid, so that $\mathrm{BTop}(k)$ is a Kan complex. In fact, Theorem 5.2.1.5 shows that $\mathrm{BTop}(k)$ can be identified with a classifying space for the topological group $\mathrm{Top}(k)$ of homeomorphisms from $\mathbf{R}^k$ to itself.

Remark 5.2.2.7. We can modify Definition 5.2.2.1 by replacing the embedding spaces $\mathrm{Emb}(\mathbf{R}^k \times S, \mathbf{R}^k)$ by the products $\prod_{s \in S} \mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k)$. This yields another ∞-operad, which is canonically isomorphic to $\mathrm{BTop}(k)^{\mathrm{II}}$. The evident inclusions $\mathrm{Emb}(\mathbf{R}^k \times S, \mathbf{R}^k) \hookrightarrow \prod_{s \in S} \mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k)$ induce an inclusion of ∞-operads $\mathrm{BTop}(k)^{\otimes} \hookrightarrow \mathrm{BTop}(k)^{\mathrm{II}}$.

If $k > 0$, then the Kan complex $\mathrm{BTop}(k)$ is not contractible (nor even simply-connected, since an orientation-reversing homeomorphisms from $\mathbf{R}^k$ to itself cannot be isotopic to the identity), so the ∞-operad $\mathrm{BTop}(k)^{\otimes}$ is not reduced. Consequently, we can apply Theorem 2.3.4.4 to decompose $\mathrm{BTop}(k)^{\otimes}$ as the assembly of a family of reduced ∞-operads. The key to understanding this decomposition is the following observation:

Proposition 5.2.2.8. *Let k be a nonnegative integer, and choose a homeomorphism $\mathbf{R}^k \simeq \square^k$. The induced inclusion $f : \mathbb{E}_k^{\otimes} \to \mathrm{BTop}(k)^{\otimes}$ is an approximation to $\mathrm{BTop}(k)^{\otimes}$ (see Definition 2.3.3.6).*

Proof. Using Corollaries 2.3.3.16 and 2.3.3.17, we are reduced to proving that for every finite set S, the diagram

$$\begin{array}{ccc}
\mathrm{Sing}(\mathrm{Rect}(\square^k \times S, \square^k)) & \longrightarrow & \mathrm{Sing}\,\mathrm{Rect}(\square^k, \square^k)^S \\
\downarrow & & \downarrow \\
\mathrm{Sing}(\mathrm{Emb}(\mathbf{R}^k \times S), \mathbf{R}^k) & \longrightarrow & \mathrm{Sing}(\mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k))^S
\end{array}$$

is a homotopy pullback square of Kan complexes. Consider the larger diagram

$$\begin{array}{ccc}
\mathrm{Sing}(\mathrm{Rect}(\square^k \times S, \square^k)) & \longrightarrow & \mathrm{Sing}\,\mathrm{Rect}(\square^k, \square^k)^S \\
\downarrow & & \downarrow \\
\mathrm{Sing}(\mathrm{Emb}(\mathbf{R}^k \times S, \mathbf{R}^k)) & \longrightarrow & \mathrm{Sing}(\mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k))^S \\
\downarrow & & \downarrow \\
\mathrm{Germ}(S, \mathbf{R}^k) & \longrightarrow & \prod_{s \in S} \mathrm{Germ}(\{s\}, \mathbf{R}^k) \\
\downarrow & & \downarrow \\
\mathrm{Conf}(S, \mathbf{R}^k) & \longrightarrow & \prod_{s \in S} \mathrm{Conf}(\{s\}, \mathbf{R}^k).
\end{array}$$

The lower square is a pullback diagram in which the vertical maps are Kan fibrations, and therefore a homotopy pullback diagram. The middle square is a homotopy pullback diagram because the middle vertical maps are homotopy equivalences (Proposition 5.2.1.8). The outer rectangle is a homotopy pullback diagram because the vertical compositions are homotopy equivalences (Lemma 5.1.1.3). The desired result now follows from a diagram chase. $\square$

Remark 5.2.2.9. Fix a nonnegative integer k. The ∞-operad $\mathrm{BTop}(k)^{\otimes}$ is unital and its underlying ∞-category $\mathrm{BTop}(k)$ is a Kan complex (Notation 5.2.2.6). According to Theorem 2.3.4.4, there exists a reduced generalized ∞-operad $\mathcal{O}^{\otimes}$ and an assembly map $\mathcal{O}^{\otimes} \to \mathrm{BTop}(k)^{\otimes}$. Then $\mathcal{O}_{\langle 0 \rangle}^{\otimes} \simeq \mathcal{O} \simeq \mathrm{BTop}(k)$;

we may therefore assume without loss of generality that $\mathcal{O}^{\otimes} \to \mathrm{BTop}(k) \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a $\mathrm{BTop}(k)$-family of ∞-operads. Since $\mathbb{E}_k^{\otimes}$ is reduced, Theorem 2.3.4.4 guarantees that the inclusion $\mathbb{E}_k^{\otimes} \to \mathrm{BTop}(k)^{\otimes}$ factors (up to homotopy) through $\mathcal{O}^{\otimes}$. Without loss of generality, this map factors through $\mathcal{O}_x^{\otimes}$, where x denotes the unique vertex of $\mathrm{BTop}(k)$. The resulting map $\mathbb{E}_k^{\otimes} \to \mathcal{O}_x^{\otimes}$ is a approximation to $\mathcal{O}_x^{\otimes}$ (Proposition 5.2.2.8). Since both $\mathbb{E}_k^{\otimes}$ and $\mathcal{O}_x^{\otimes}$ are reduced, it is an equivalence (Corollary 2.3.3.24). We can summarize the situation as follows: the ∞-operad $\mathrm{BTop}(k)^{\otimes}$ is obtained by assembling a reduced $\mathrm{BTop}(k)$-family of ∞-operads, each of which is equivalent to $\mathbb{E}_k^{\otimes}$. More informally, we can regard this $\mathrm{BTop}(k)$-family as encoding an action of the loop space $\Omega \, \mathrm{BTop}(k) \simeq \mathrm{Sing}(\mathrm{Top}(k))$ on the ∞-operad $\mathbb{E}_k^{\otimes}$, so that $\mathrm{BTop}(k)^{\otimes}$ can be regarded as a semidirect product of the ∞-operad $\mathbb{E}_k^{\otimes}$ with the topological group $\mathrm{Top}(k)$ of homeomorphisms of $\mathbf{R}^k$ with itself.

We can summarize Remark 5.2.2.9 informally as follows: if $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category, then the ∞-category $\mathrm{Alg}_{\mathrm{BTop}(k)}(\mathcal{C})$ can be identified with the ∞-category of $\mathbb{E}_k$-algebra objects of $\mathcal{C}$ which are equipped with a compatible action of the topological group $\mathrm{Top}(k)$. The requirement that $\mathrm{Top}(k)$ act on an $\mathbb{E}_k$-algebra is rather strong: in practice, we often encounter situations where an algebra $A \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$ is acted on not by the whole of $\mathrm{Top}(k)$, but by some smaller group. Our next definition gives a convenient formulation of this situation.

Definition 5.2.2.10. Let B be a Kan complex equipped with a Kan fibration $B \to \mathrm{BTop}(k)$. We let $\mathbb{E}_B^{\otimes}$ denote the fiber product

$$\mathrm{BTop}(k)^{\otimes} \times_{\mathrm{BTop}(k)^{\mathrm{II}}} B^{\mathrm{II}}.$$

Remark 5.2.2.11. It follows immediately from the definitions that $\mathbb{E}_B^{\otimes}$ is a unital ∞-operad and that the map $\mathbb{E}_B^{\otimes} \to \mathrm{BTop}\, k^{\otimes}$ is an approximation of ∞-operads.

Warning 5.2.2.12. Our notation is slightly abusive. The ∞-operad $\mathbb{E}_B^{\otimes}$ depends not only on the Kan complex B, but also the integer k and the map $\theta : B \to \mathrm{BTop}(k)$. We can think of θ as classifying a topological fiber bundle over the geometric realization $|B|$, whose fibers are homeomorphic to $\mathbf{R}^k$.

Remark 5.2.2.13. Let $\mathcal{O}^{\otimes} \to \mathrm{BTop}(k) \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be the ∞-operad family of Remark 5.2.2.9. If $\theta : B \to \mathrm{BTop}(k)$ is any map of Kan complexes, then the fiber product $\mathcal{O}^{\otimes} \times_{\mathrm{BTop}(k)} B$ is a B-family of reduced unital ∞-operads. When θ is a Kan fibration (which we may assume without loss of generality), then this B-family of ∞-operads assembles to the unital ∞-operad $\mathbb{E}_B^{\otimes}$ (see §2.3.4). We can informally describe the situation as follows: an $\mathbb{E}_B$-algebra object of a symmetric monoidal ∞-category $\mathcal{C}$ is a (twisted) family of $\mathbb{E}_k$-algebra objects of $\mathcal{C}$, parametrized by Kan complex B (the nature of the twisting is determined by the map θ).

Remark 5.2.2.14. Let k and k' be integers. The homeomorphism $\mathbf{R}^{k+k'} \simeq \mathbf{R}^k \times \mathbf{R}^{k'}$ determines a map of Kan complexes $\mathrm{BTop}(k) \times \mathrm{BTop}(k') \to \mathrm{BTop}(k+k')$. This map induces a bifunctor of ∞-operads $\mathrm{BTop}(k)^{\mathrm{II}} \times \mathrm{BTop}(k')^{\mathrm{II}} \to \mathrm{BTop}(k+k')^{\mathrm{II}}$ which restricts to a functor $\mathrm{BTop}(k)^{\otimes} \times \mathrm{BTop}(k')^{\otimes} \to \mathrm{BTop}(k+k')^{\otimes}$. More generally, if we are given maps of Kan complexes $B \to \mathrm{BTop}\, k$ and $B' \to \mathrm{BTop}\, k'$, there is an induced bifunctor of ∞-operads

$$\theta_{B,B'} : \mathbb{E}_B^{\otimes} \times \mathbb{E}_{B'}^{\otimes} \to \mathbb{E}_{B \times B'}^{\otimes}$$

where we regard $B \times B'$ as equipped with the composite map $B \times B' \to \mathrm{BTop}\, k \times \mathrm{BTop}\, k' \to \mathrm{BTop}\, k + k'$ (classifying the sum of the bundles pulled back from B and B', respectively). The functor $\theta_{B,B'}$ exhibits $\mathbb{E}_{B \times B'}^{\otimes}$ as a tensor product of the ∞-operads $\mathbb{E}_B^{\otimes}$ and $\mathbb{E}_{B'}^{\otimes}$. To prove this, we observe that Remark 5.2.2.13 implies that the constructions $B \mapsto \mathbb{E}_B^{\otimes}$ and $B \mapsto \mathbb{E}_{B \times B'}^{\otimes}$ carry homotopy colimits of Kan complexes (over $\mathrm{Top}\, k$) to homotopy colimits of ∞-operads. Consequently, we may assume without loss of generality that $B \simeq \Delta^0$. Similarly, we may assume that $B' \simeq \Delta^0$. In this case, the bifunctor $\theta_{B,B'}$ is equivalent to bifunctor $\mathbb{E}_k^{\otimes} \times \mathbb{E}_{k'}^{\otimes} \to \mathbb{E}_{k+k'}^{\otimes}$ appearing in the statement of Theorem 5.1.2.2.

We conclude this section by illustrating Definition 5.2.2.10 with some examples. Another general class of examples will be discussed in §5.2.4.

Example 5.2.2.15. Let B be a contractible Kan complex equipped with a Kan fibration $B \to \mathrm{BTop}(k)$. Then $\mathbb{E}_B^\otimes$ is equivalent to the ∞-operad $\mathbb{E}_k^\otimes$.

Example 5.2.2.16. Fix $k \geq 0$, and choose a homeomorphism of $\mathbf{R}^k$ with the unit ball $B(1) \subseteq \mathbf{R}^k$. We will say that a map $f : B(1) \to B(1)$ is a *projective isometry* if there exists an element γ in the orthogonal group $\mathrm{O}(k)$, a positive real number λ, and a vector $v_0 \in B(1)$ such that f is given by the formula $f(w) = v_0 + \lambda\gamma(w)$. For every finite set S, we let $\mathrm{Isom}^+(B(1) \times S, B(1))$ denote the (closed) subspace of $\mathrm{Emb}(B(1) \times S, B(1))$ consisting of those open embeddings whose restriction to each ball $B(1) \times \{s\}$ is an orientation-preserving projective isometry. Let ${}^t\mathbb{E}_{\mathrm{SO}(k)}$ be the subcategory of ${}^t\mathbb{E}_{\mathrm{BTop}(k)}^\otimes$ having the same objects, with morphism spaces given by

$$\mathrm{Map}_{{}^t\mathbb{E}_{\mathrm{SO}(k)}}(\langle m \rangle, \langle n \rangle) = \coprod_\alpha \prod_{1 \leq i \leq n} \mathrm{Isom}^+(B(1) \times \alpha^{-1}\{i\}, B(1)).$$

Then $\mathcal{O}^\otimes = \mathrm{N}({}^t\mathbb{E}_{\mathrm{SO}(k)})$ is a unital ∞-operad. The inclusion $\mathcal{O}^\otimes \hookrightarrow \mathrm{BTop}(k)^\otimes$ is an approximation of $\mathrm{BTop}(k)^\otimes$ which induces an equivalence of $\mathcal{O}^\otimes$ with the ∞-operad $\mathbb{E}_B$, where B is a Kan complex which plays the role of a classifying space $\mathrm{BSO}(k)$ for the special orthogonal group $\mathrm{SO}(k)$ (and we arrange that the inclusion of topological groups $\mathrm{SO}(k) \to \mathrm{Top}(k)$ induces a Kan fibration $\mathrm{BSO}(k) \to \mathrm{BTop}(k)$). This recovers the operad of *framed disks* described, for example, in [124].

Variant 5.2.2.17. In Example 5.2.2.16, there is no need to restrict our attention to orientation preserving maps. If we instead allow *all* projective isometries, then we get another ∞-operad $\mathcal{O}^\otimes \simeq \mathbb{E}_B^\otimes$, where B is a classifying space for the full orthogonal group $\mathrm{O}(k)$.

Example 5.2.2.18. In the definition of $\mathrm{BTop}(k)^\otimes$, we have allowed arbitrary open embeddings between Euclidean spaces $\mathbf{R}^k$. We could instead restrict our attention to spaces of *smooth* open embeddings (which we regard as equipped with the Whitney topology, where convergence is given by uniform convergence of all derivatives on compact sets) to obtain an ∞-operad $\mathbb{E}_{\mathrm{Sm}}$. This can be identified with the ∞-operad $\mathbb{E}_B$, where B is a classifying space for the monoid of smooth embeddings from the open ball $B(1)$ to itself. Since every projective isometry is smooth, there is an obvious map $\mathcal{O}^\otimes \to \mathbb{E}_{\mathrm{Sm}}$, where $\mathcal{O}^\otimes$ is defined as in Variant 5.2.2.17. In fact, this map is an equivalence of ∞-operads: this follows from the fact that the inclusion from the orthogonal group $\mathrm{O}(k)$ into the space $\mathrm{Emb}^{\mathrm{sm}}(B(1), B(1))$ of smooth embeddings of $B(1)$ to itself is a homotopy equivalence (it has a homotopy inverse given by the composition $\mathrm{Emb}^{\mathrm{sm}}(B(1), B(1)) \to \mathrm{GL}_k(\mathbf{R}) \to \mathrm{O}(k)$, where the first map is given by taking the derivative at the origin and the second is a homotopy inverse to the inclusion $\mathrm{O}(k) \hookrightarrow \mathrm{GL}_k(\mathbf{R})$).

Many other variants on Example 5.2.2.18 are possible. For example, we can replace smooth manifolds with piecewise linear manifolds. We can also consider smooth or piecewise linear manifolds equipped with additional structures, such as orientations. We leave the details to the reader.

5.2.3 Nonunital $\mathbb{E}_k$-Algebras

Let A be an abelian group equipped with a commutative and associative multiplication $m : A \otimes A \to A$. A *unit* for the multiplication m is an element $1 \in A$ such that $1a = a$ for each $a \in A$. If there exists a unit for A, then that unit is unique and A is a commutative ring (with unit). Our goal in this section is to prove an analogous result, where the category of abelian groups is replaced by an arbitrary symmetric monoidal ∞-category $\mathcal{C}$ (Corollary 5.2.3.12). We begin with a discussion of nonunital algebras in general.

Definition 5.2.3.1. Let Surj denote the subcategory of $\mathcal{F}\mathrm{in}_*$ containing all objects of $\mathcal{F}\mathrm{in}_*$, such that a morphism $\alpha : \langle m \rangle \to \langle n \rangle$ belongs to Surj if and only if it is surjective. If $\mathcal{O}^\otimes$ is an ∞-operad, we let $\mathcal{O}_{\mathrm{nu}}^\otimes$ denote the fiber product $\mathcal{O}^\otimes \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\mathrm{Surj})$. Given a fibration of ∞-operads $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ and a map of ∞-operads $\mathcal{O}^\otimes \to \mathcal{D}^\otimes$, we let $\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{\mathrm{nu}}(\mathcal{C})$ denote the ∞-category $\mathrm{Alg}_{\mathcal{O}_{\mathrm{nu}}/\mathcal{D}}(\mathcal{C})$. We will refer to $\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{\mathrm{nu}}(\mathcal{C})$ as the ∞-*category of nonunital $\mathcal{O}$-algebra objects of* $\mathcal{C}$. In the special case where $\mathcal{D}^\otimes = \mathcal{O}^\otimes$, we will denote $\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{\mathrm{nu}}(\mathcal{C})$ by $\mathrm{Alg}_{/\mathcal{O}}^{\mathrm{nu}}(\mathcal{C})$. In the special case where $\mathcal{D} = \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, we will denote $\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}^{\mathrm{nu}}(\mathcal{C})$ by $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{nu}}(\mathcal{C})$.

Our goal is to show that if $\mathcal{O}^{\otimes}$ is a little k-cubes operad $\mathbb{E}_k$ for some $k \geq 1$, then the ∞-category $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{nu}}(\mathcal{C})$ of nonunital $\mathcal{O}$-algebra objects of $\mathcal{C}$ is not very different from the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ of unital $\mathcal{O}$-algebras objects of $\mathcal{C}$. More precisely, we will show that the restriction functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}^{\mathrm{nu}}(\mathcal{C})$ induces an equivalence of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ onto a subcategory $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{qu}}(\mathcal{C}) \subseteq \mathrm{Alg}_{\mathcal{O}}^{\mathrm{nu}}(\mathcal{C})$ whose objects are required to admit units up to homotopy and whose morphisms are required to preserve those units (see Definition 5.2.3.7 below). We first discuss the case $k = 1$, deferring the proof until Chapter 6. We will then use Theorem 5.1.2.2 to generalize to the case of an arbitrary $k \geq 1$.

Definition 5.2.3.2. Let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category with unit object $\mathbf{1}$, and let $A \in \mathrm{Alg}_{/\mathcal{A}\mathrm{ss}}^{\mathrm{nu}}(\mathcal{C})$ be a nonunital associative algebra object of $\mathcal{C}$. We will say that a morphism $u : \mathbf{1} \to A$ in $\mathcal{C}$ is a *quasi-unit* if the composite maps

$$A \simeq \mathbf{1} \otimes A \xrightarrow{u \otimes \mathrm{id}} A \otimes A \xrightarrow{m} A$$
$$A \simeq A \otimes \mathbf{1} \xrightarrow{\mathrm{id} \otimes u} A \otimes A \xrightarrow{m} A$$

are homotopic to the identity; here m denotes the multiplication on $\mathbf{A}$.

Remark 5.2.3.3. Let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category with unit object $\mathbf{1}$, and let $A \in \mathrm{Alg}_{/\mathcal{A}\mathrm{ss}}^{\mathrm{nu}}(\mathcal{C})$ be nonunital associative algebra objects of $\mathcal{C}$. Suppose that $u, v : \mathbf{1} \to A$ are quasi-units for A. Then the composite map

$$\mathbf{1} \simeq \mathbf{1} \otimes \mathbf{1} \xrightarrow{u \otimes v} A \otimes A \to A$$

is homotopic to both u and v, so that u and v are homotopic. In other words, a quasi-unit for A is uniquely determined up to homotopy (if such a quasi-unit exists).

Definition 5.2.3.4. Let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category with unit object $\mathbf{1}$, and let $A \in \mathrm{Alg}_{/\mathcal{A}\mathrm{ss}}^{\mathrm{nu}}(\mathcal{C})$ be a nonunital associative algebra object of $\mathcal{C}$. We will say that A is *quasi-unital* if there exists a quasi-unit $u : \mathbf{1} \to A$.

Given a morphism of nonunital associative algebras $f : A \to B$, we will say that f is *quasi-unital* if there exists a quasi-unit $u : \mathbf{1} \to A$ such that $f \circ u$ is a quasi-unit for B. Using Remark 5.2.3.3, we see that the same conclusion must hold for *any* quasi-unit of A. It follows that the collection of quasi-unital morphisms in $\mathrm{Alg}_{/\mathcal{A}\mathrm{ss}}^{\mathrm{nu}}(\mathcal{C})$ is closed under composition. We let $\mathrm{Alg}_{/\mathcal{A}\mathrm{ss}}^{\mathrm{qu}}(\mathcal{C})$ denote the subcategory of $\mathrm{Alg}_{/\mathcal{A}\mathrm{ss}}^{\mathrm{nu}}(\mathcal{C})$ spanned by the quasi-unital associative algebra objects of $\mathcal{C}$ and quasi-unital morphisms between them. We will refer to $\mathrm{Alg}_{/\mathcal{A}\mathrm{ss}}^{\mathrm{qu}}(\mathcal{C})$ as the ∞-*category of quasi-unital associative algebra objects of* $\mathcal{C}$.

We will need the following result:

Theorem 5.2.3.5. *Let $\mathcal{C}$ be a monoidal ∞-category. Then the restriction map $\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}_{/\mathcal{A}\mathrm{ss}}^{\mathrm{qu}}(\mathcal{C})$ is a trivial Kan fibration.*

Since the forgetful functor $\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}_{/\mathcal{A}\mathrm{ss}}^{\mathrm{qu}}(\mathcal{C})$ is obviously a categorical fibration, it will suffice to show that it is a categorical equivalence. This is an immediate consequence of Theorem 6.1.3.10, which we will prove in §6.1.3.

Remark 5.2.3.6. More informally, Theorem 5.2.3.5 asserts that if A is a nonunital associative algebra object of a monoidal ∞-category $\mathcal{C}$ and there exists a map $u : \mathbf{1} \to A$ which is a unit for A up to homotopy, then A can be promoted in an essentially unique way to a unital associative algebra.

We now formulate a generalization of Theorem 5.2.3.5.

Definition 5.2.3.7. Let $q : \mathcal{C}^{\otimes} \to \mathbb{E}_k^{\otimes}$ be a coCartesian fibration of ∞-operads. If $k = 1$, then the composite map $\mathcal{C}^{\otimes} \to \mathbb{E}_k^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ exhibits $\mathcal{C}^{\otimes}$ as a monoidal ∞-category. We let let $\mathrm{Alg}_{/\mathbb{E}_k}^{\mathrm{qu}}(\mathcal{C})$ denote the subcategory of $\mathrm{Alg}_{/\mathbb{E}_k}^{\mathrm{nu}}(\mathcal{C})$ corresponding to $\mathrm{Alg}_{/\mathcal{A}\mathrm{ss}}^{\mathrm{qu}}(\mathcal{C}) \subseteq \mathrm{Alg}_{/\mathcal{A}\mathrm{ss}}^{\mathrm{nu}}(\mathcal{C})$ under the equivalence $\mathrm{Alg}_{/\mathcal{A}\mathrm{ss}}^{\mathrm{nu}}(\mathcal{C}) \simeq \mathrm{Alg}_{/\mathbb{E}_k}^{\mathrm{nu}}(\mathcal{C})$ coming from Example 5.1.0.7. More generally, for any $k \geq 1$ we let $\mathrm{Alg}_{/\mathbb{E}_k}^{\mathrm{qu}}(\mathcal{C})$ denote the fiber product $\mathrm{Alg}_{/\mathbb{E}_k}^{\mathrm{nu}}(\mathcal{C}) \times_{\mathrm{Alg}_{/\mathbb{E}_1}^{\mathrm{nu}}(\mathcal{C}')} \mathrm{Alg}_{/\mathbb{E}_1}^{\mathrm{qu}}(\mathcal{C}')$, where $\mathcal{C}'^{\otimes} = \mathbb{E}_1^{\otimes} \times_{\mathbb{E}_k^{\otimes}} \mathcal{C}^{\otimes}$. We will refer to $\mathrm{Alg}_{/\mathbb{E}_k}^{\mathrm{qu}}(\mathcal{C})$ as the ∞-*category of quasi-unital $\mathbb{E}_k$-algebra objects of* $\mathcal{C}$.

Remark 5.2.3.8. We can make Definition 5.2.3.7 more explicit as follows. If $A \in \mathrm{Alg}^{\mathrm{nu}}_{/\mathbb{E}_k}(\mathcal{C})$ and $\mathbf{1}$ is the unit object of $\mathcal{C}$, we will say that a map $u : \mathbf{1} \to A$ is a *quasi-unit* for A if its homotopy class is both a left and right unit with respect to the multiplication on A, as in Definition 5.2.3.2. Then A belongs to $\mathrm{Alg}^{\mathrm{qu}}_{/\mathbb{E}_k}(\mathcal{C})$ if and only if it admits a quasi-unit u; in this case, u is determined uniquely up to homotopy. A morphism $f : A \to B$ belongs to $\mathrm{Alg}^{\mathrm{nu}}_{/\mathbb{E}_k}(\mathcal{C})$ belongs to $\mathrm{Alg}^{\mathrm{qu}}_{/\mathbb{E}_k}(\mathcal{C})$ if and only if A admits a quasi-unit $u : \mathbf{1} \to A$ and the composite map $f \circ u$ is a quasi-unit for B.

Remark 5.2.3.9. In the situation of Remark 5.2.3.8, a map $e : \mathbf{1} \to A$ is a quasi-unit for A if and only if each of the composite maps

$$A \simeq \mathbf{1} \otimes A \xrightarrow{e \otimes \mathrm{id}} A \otimes A \xrightarrow{m} A \qquad A \simeq A \otimes \mathbf{1} \xrightarrow{\mathrm{id} \otimes e} A \otimes A \xrightarrow{m} A$$

is homotopic to the identity. If $k > 1$, then the multiplication on A and the tensor product on $\mathcal{C}$ are commutative up to homotopy, so these conditions are equivalent to one another.

Let $q : \mathcal{C}^{\otimes} \to \mathbb{E}_k^{\otimes}$ be a coCartesian fibration of ∞-operads, and let $\theta : \mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C}) \to \mathrm{Alg}^{\mathrm{nu}}_{/\mathbb{E}_k}(\mathcal{C})$ be the restriction functor. Then θ carries $\mathbb{E}_k$-algebra objects of $\mathcal{C}$ to quasi-unital objects of $\mathrm{Alg}^{\mathrm{nu}}_{/\mathbb{E}_k}(\mathcal{C})$, and morphisms of $\mathbb{E}_k$-algebras to quasi-unital morphisms in $\mathrm{Alg}^{\mathrm{nu}}_{/\mathbb{E}_k}(\mathcal{C})$. Consequently, θ can be viewed as a functor from $\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})$ to $\mathrm{Alg}^{\mathrm{qu}}_{/\mathbb{E}_k}(\mathcal{C})$. The main result of this section is the following generalization of Theorem 5.2.3.5:

Theorem 5.2.3.10. *Let $k \geq 1$ and let $q : \mathcal{C}^{\otimes} \to \mathbb{E}_k^{\otimes}$ be a coCartesian fibration of ∞-operads. Then the forgetful functor $\theta : \mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C}) \to \mathrm{Alg}^{\mathrm{qu}}_{/\mathbb{E}_k}(\mathcal{C})$ is an equivalence of ∞-categories.*

The proof of Theorem 5.2.3.10 is somewhat elaborate, and will be given at the end of this section.

Remark 5.2.3.11. In the situation of Theorem 5.2.3.10, we may assume without loss of generality that $\mathcal{C}^{\otimes}$ is small (filtering $\mathcal{C}^{\otimes}$ if necessary). Using Proposition 6.3.1.10, we deduce the existence of a presentable $\mathbb{E}_k$-monoidal ∞-category $\mathcal{D}^{\otimes} \to \mathbb{E}_k^{\otimes}$ and a fully faithful $\mathbb{E}_k$-monoidal functor $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$. We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C}) & \longrightarrow & \mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{D}) \\
\downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta'} \\
\mathrm{Alg}^{\mathrm{qu}}_{/\mathbb{E}_k}(\mathcal{C}) & \longrightarrow & \mathrm{Alg}^{\mathrm{qu}}_{/\mathbb{E}_k}(\mathcal{D})
\end{array}
$$

where the horizontal maps are fully faithful embeddings, whose essential images consist of those (unital or nonunital) $\mathbb{E}_k$-algebra objects of $\mathcal{D}$ whose underlying object belongs to the essential image of the embedding $\mathcal{C} \hookrightarrow \mathcal{D}$. To prove that θ is a categorical equivalence, it suffices to show that θ' is a categorical equivalence. In other words, it suffices to prove Theorem 5.2.3.10 in the special case where $\mathcal{C}^{\otimes}$ is a presentable $\mathbb{E}_k$-monoidal ∞-category.

We will use Theorem 5.2.3.10 to deduce an analogous assertion regarding commutative algebras. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. We let $\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C})$ denote the ∞-category $\mathrm{Alg}^{\mathrm{nu}}_{\mathrm{Comm}}(\mathcal{C})$ of nonunital commutative algebra objects of $\mathcal{C}$. Definition 5.2.3.7 has an evident analogue for nonunital commutative algebras and maps between them: we will say that a nonunital commutative algebra $A \in \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C})$ is *quasi-unital* if there exists a map $e : \mathbf{1} \to A$ in $\mathcal{C}$ such that the composition

$$A \simeq \mathbf{1} \otimes A \xrightarrow{e \otimes \mathrm{id}} A \otimes A \to A$$

is homotopic to the identity (in the ∞-category $\mathcal{C}$). In this case, e is uniquely determined up to homotopy and we say that e is a quasi-unit for A; a morphism $f : A \to B$ in $\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C})$ is *quasi-unital* if A admits a quasi-unit $e : \mathbf{1} \to A$ such that $f \circ e$ is a quasi-unit for B. The collection of quasi-unital commutative algebras and quasi-unital morphisms between them can be organized into a subcategory $\mathrm{CAlg}^{\mathrm{qu}}(\mathcal{C}) \subseteq \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C})$.

Corollary 5.2.3.12. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Then the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}^{\mathrm{qu}}(\mathcal{C})$ is an equivalence of ∞-categories.*

Proof. In view of Corollary 5.1.1.5, we have an equivalence of ∞-operads $\varinjlim \mathbb{E}_k^{\otimes} \to \mathrm{Comm}^{\otimes} = \mathbb{E}_{\infty}^{\otimes}$. It will therefore suffice to show that the forgetful functor $\mathrm{Alg}_{\mathbb{E}_{\infty}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_{\infty}}^{\mathrm{qu}}(\mathcal{C})$ is an equivalence of ∞-categories. This map is the homotopy inverse limit of a tower of forgetful functors $\theta_k : \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_k}^{\mathrm{qu}}(\mathcal{C})$, each of which is an equivalence of ∞-categories by Theorem 5.2.3.10. $\square$

As a first step toward understanding the forgetful functor $\theta : \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{/\,\mathcal{O}}^{\mathrm{nu}}(\mathcal{C})$, let us study the left adjoint to θ. In classical algebra, if A is a nonunital ring, then we can canonically enlarge A to a unital ring by considering the product $A \oplus \mathbf{Z}$ endowed with the multiplication $(a, m)(b, n) = (ab + mb + na, mn)$. Our next result shows that this construction works quite generally:

Proposition 5.2.3.13. *Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad, let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads which is compatible with finite coproducts, and let $\theta : \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{/\,\mathcal{O}}^{\mathrm{nu}}(\mathcal{C})$ be the forgetful functor. Then:*

   (1) *For every object $A \in \mathrm{Alg}_{/\,\mathcal{O}}^{\mathrm{nu}}(\mathcal{C})$, there exists another object $A^{+} \in \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C})$ and a map $A \to \theta(A^{+})$ which exhibits A^{+} as a free $\mathcal{O}$-algebra generated by A.*

   (2) *A morphism $f : A \to \theta(A^{+})$ in $\mathrm{Alg}_{/\,\mathcal{O}}^{\mathrm{nu}}(\mathcal{C})$ exhibits A^{+} as a free $\mathcal{O}$-algebra generated by A if and only if, for every object $X \in \mathcal{O}$, the map $f_X : A(X) \to A^{+}(X)$ and the unit map $\mathbf{1}_X \to A^{+}(X)$ exhibit $A^{+}(X)$ as a coproduct of $A(X)$ and the unit object $\mathbf{1}_X$ in the ∞-category $\mathcal{C}_X$.*

   (3) *The functor θ admits a left adjoint.*

Proof. For every object $X \in \mathcal{O}$, the ∞-category $\mathcal{D} = \mathcal{O}_{\mathrm{nu}}^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{O}^{\otimes})_{/X}^{\mathrm{act}}$ can be written as a disjoint union of $\mathcal{D}_0 = (\mathcal{O}_{\mathrm{nu}}^{\otimes})_{/X}^{\mathrm{act}}$ with the full subcategory $\mathcal{D}_1 \subseteq \mathcal{D}$ spanned by those morphisms $X' \to X$ in $\mathcal{O}^{\otimes}$ where $X' \in \mathcal{O}_{\langle 0 \rangle}^{\otimes}$. The ∞-category $\mathcal{D}_0$ contains id_X as a final object. Since $\mathcal{O}^{\otimes}$ is unital, the ∞-category $\mathcal{D}_1$ is a contractible Kan complex containing a vertex $v : X_0 \to X$. It follows that the inclusion $\{\mathrm{id}_X, v\}$ is left cofinal in $\mathcal{D}$. Assertions (1) and (2) now follow from Proposition 3.1.3.3 (together with Propositions 3.1.1.15 and 3.1.1.16). Assertion (3) follows from (1) (Corollary 3.1.3.4). $\square$

In the stable setting, there is a close relationship between nonunital algebras and augmented algebras. To be more precise, we need to introduce a bit of terminology.

Definition 5.2.3.14. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads, and assume that $\mathcal{O}^{\otimes}$ is unital. An augmented $\mathcal{O}$-algebra object of $\mathcal{C}$ is a morphism $f : A \to A_0$ in $\mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C})$, where A_0 is a trivial algebra. We let $\mathrm{Alg}_{/\,\mathcal{O}}^{\mathrm{aug}}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C}))$ spanned by the augmented $\mathcal{O}$-algebra objects of $\mathcal{C}$.*

The following result will not play a role in the proof of Theorem 5.2.3.10, but is of some independent interest:

Proposition 5.2.3.15. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads. Assume that $\mathcal{O}^{\otimes}$ is unital and that q exhibits $\mathcal{C}$ as a stable $\mathcal{O}$-monoidal ∞-category. Let $F : \mathrm{Alg}_{/\,\mathcal{O}}^{\mathrm{nu}}(\mathcal{C}) \to \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C})$ be a left adjoint to the forgetful functor $\theta : \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{/\,\mathcal{O}}^{\mathrm{nu}}(\mathcal{C})$. Let $0 \in \mathrm{Alg}_{/\,\mathcal{O}}^{\mathrm{nu}}(\mathcal{C})$ be a final object, so that $F(0) \in \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{C})$ is a trivial algebra (Proposition 5.2.3.13). Then F induces an equivalence of ∞-categories*

$$T : \mathrm{Alg}_{/\,\mathcal{O}}^{\mathrm{nu}}(\mathcal{C}) \simeq \mathrm{Alg}_{/\,\mathcal{O}}^{\mathrm{nu}}(\mathcal{C})^{/0} \to \mathrm{Alg}_{/\,\mathcal{O}}^{\mathrm{aug}}(\mathcal{C}).$$

Proof. Let $p : \mathcal{M} \to \Delta^1$ be a correspondence associated to the adjunction $\mathrm{Alg}^{\mathrm{nu}}_{/\mathcal{O}}(\mathcal{C}) \underset{\theta}{\overset{F}{\rightleftarrows}} \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$. Let $\mathcal{D}$ denote the full subcategory of $\mathrm{Fun}_{\Delta^1}(\Delta^1 \times \Delta^1, \mathcal{M})$ spanned by those diagrams σ

$$
\begin{array}{ccc}
A & \xrightarrow{\;f\;} & A^+ \\
\downarrow & & \downarrow{\scriptstyle g} \\
A_0 & \xrightarrow{\;f_0\;} & A_0^+
\end{array}
$$

where A_0 is a final object of $\mathrm{Alg}^{\mathrm{nu}}_{/\mathcal{O}}(\mathcal{C})$ and the maps f and f' are p-coCartesian; this (together with Proposition 5.2.3.13) guarantees that $A_0^+ \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ is a trivial algebra so that g can be regarded as an augmented $\mathcal{O}$-algebra object of $\mathcal{C}$. Using Proposition T.4.3.2.15, we deduce that the restriction functor $\sigma \mapsto A$ determines a trivial Kan fibration $\mathcal{D} \to \mathrm{Alg}^{\mathrm{nu}}_{/\mathcal{O}}(\mathcal{C})$. By definition, the functor T is obtained by composing a section of this trivial Kan fibration with the restriction map $\phi : \mathcal{D} \to \mathrm{Alg}^{\mathrm{aug}}_{/\mathcal{O}}(\mathcal{C})$ given by $\sigma \mapsto g$. To complete the proof, it will suffice to show that ϕ is a trivial Kan fibration.

Let K denote the full subcategory of $\Delta^1 \times \Delta^1$ obtained by removing the object $(0,0)$, and let $\mathcal{D}_0$ be the full subcategory of $\mathrm{Fun}_{\Delta^1}(K, \mathcal{M})$ spanned by those diagrams

$$
A^+ \xrightarrow{\;g\;} A_0^+ \xleftarrow{\;f_0\;} A_0
$$

where A_0 is a final object of $\mathrm{Alg}^{\mathrm{nu}}_{/\mathcal{O}}(\mathcal{C})$ and A_0^+ is a trivial $\mathcal{O}$-algebra object of $\mathcal{C}$; note that this last condition is equivalent to the requirement that f_0 be p-coCartesian. The functor ϕ factors as a composition

$$
\mathcal{D} \xrightarrow{\;\phi'\;} \mathcal{D}_0 \xrightarrow{\;\phi''\;} \mathrm{Alg}^{\mathrm{aug}}_{\mathcal{O}}(\mathcal{C}).
$$

We will prove that ϕ' and ϕ'' are trivial Kan fibrations.

Let $\mathcal{D}_1$ be the full subcategory of $\mathrm{Fun}_{\Delta^1}(\Delta^1, \mathcal{M})$ spanned by the p-coCartesian morphisms $f_0 : A_0 \to A_0^+$ where A_0 is a final object of $\mathrm{Alg}^{\mathrm{nu}}_{/\mathcal{O}}(\mathcal{C})$. It follows from Proposition T.4.3.2.15 that the restriction map $f_0 \mapsto A_0$ determines a trivial Kan fibration from $\mathcal{D}_1$ to the contractible Kan complex of final objects in $\mathrm{Alg}^{\mathrm{nu}}_{/\mathcal{O}}(\mathcal{C})$, so that $\mathcal{D}_1$ is contractible. The restriction map $f_0 \mapsto A_0^+$ is a categorical fibration $\overline{\phi}''$ from $\mathcal{D}_1$ onto the contractible Kan complex of initial objects of $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$. It follows that $\overline{\phi}''$ is a trivial Kan fibration. The map ϕ'' is a pullback of $\overline{\phi}''$, and therefore also a trivial Kan fibration.

We now complete the proof by showing that ϕ' is a trivial Kan fibration. In view of Proposition T.4.3.2.15, it will suffice to show that a diagram $\sigma \in \mathrm{Fun}_{\Delta^1}(\Delta^1 \times \Delta^1, \mathcal{M})$ belongs to $\mathcal{D}$ if and only if $\sigma_0 = \sigma|K$ belongs to $\mathcal{D}_0$ and σ is a p-right Kan extension of σ_0. Unwinding the definitions (and using Corollary 3.2.2.5), we are reduced to showing that if we are given a diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\;f\;} & A^+ \\
\downarrow & & \downarrow{\scriptstyle g} \\
A_0 & \xrightarrow{\;f_0\;} & A_0^+
\end{array}
$$

where A_0 is a final object of $\mathrm{Alg}^{\mathrm{nu}}_{/\mathcal{O}}(\mathcal{C})$ and A_0^+ is a trivial algebra, then f is p-coCartesian if and only if the induced diagram

$$
\begin{array}{ccc}
A(X) & \xrightarrow{\;f_X\;} & A^+(X) \\
\downarrow & & \downarrow \\
A_0(X) & \longrightarrow & A_0^+(X)
\end{array}
$$

is a pullback square in $\mathcal{C}_X$, for each $X \in \mathcal{O}$. Since $\mathcal{C}_X$ is a stable ∞-category, this is equivalent to the requirement that the induced map $\psi : \operatorname{cofib}(f_X) \to A_0^+(X)$ is an equivalence. The map ψ fits into a commutative diagram

$$
\begin{array}{ccc}
\mathbf{1}_X & \longrightarrow & \mathbf{1}_X \\
\downarrow & & \downarrow \\
A^+(X) \longrightarrow \operatorname{cofib}(f) & \longrightarrow & A_0^+(X)
\end{array}
$$

where the vertical maps are given by the units for the algebra objects A^+ and A_0^+. Since $A_0^+(X)$ is a trivial algebra, the unit map $\mathbf{1}_X \to A_0^+(X)$ is an equivalence. Consequently, it suffices to show that f is p-coCartesian if and only if each of the composite maps $\mathbf{1}_X \to A^+(X) \to \operatorname{cofib}(f)$ is an equivalence. We have a pushout diagrm

$$
\begin{array}{ccc}
\mathbf{1}_X \coprod A(X) & \longrightarrow & A^+(X) \\
\downarrow & & \downarrow \\
\mathbf{1}_X & \longrightarrow & \operatorname{cofib}(f).
\end{array}
$$

Since $\mathcal{C}_X$ is stable, the lower horizontal map is an equivalence if and only if the upper horizontal map is an equivalence. The desired result now follows immediately from the criterion described in Proposition 5.2.3.13. $\qquad\square$

Let us now return to the proof of Theorem 5.2.3.10. The case $k = 1$ follows immediately from Theorem 5.2.3.5 (by virtue of Example 5.1.0.7). The proof of Theorem 5.2.3.10 in general will proceed by induction on k. For the remainder of this section, we will fix an integer $k \geq 1$, and assume that Theorem 5.2.3.10 has been verified for the ∞-operad $\mathbb{E}_k^\otimes$. Our goal is to prove that Theorem 5.2.3.10 is valid also for $\mathbb{E}_{k+1}^\otimes$. Fix a coCartesian fibration of ∞-operads $q : \mathcal{C}^\otimes \to \mathbb{E}_{k+1}^\otimes$; we wish to show that the forgetful functor $\theta : \operatorname{Alg}_{/\mathbb{E}_{k+1}}(\mathcal{C}) \to \operatorname{Alg}_{/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C})$ is an equivalence of ∞-categories. In view of Remark 5.2.3.11, we can assume that $\mathcal{C}^\otimes$ is a presentable $\mathbb{E}_{k+1}$-monoidal ∞-category.

We begin by constructing a left homotopy inverse to θ. Consider the bifunctor of ∞-operads $\mathbb{E}_1^\otimes \times \mathbb{E}_k^\otimes \to \mathbb{E}_{k+1}^\otimes$ of §5.1.2. Using this bifunctor, we can define $\mathbb{E}_1$-monoidal ∞-categories $\operatorname{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})^\otimes$ and $\operatorname{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}^{\mathrm{nu}}(\mathcal{C})^\otimes$. Moreover, the collection of quasi-unital $\mathbb{E}_k$-algebras and quasi-unital morphisms between them are stable under tensor products, so we can also consider an $\mathbb{E}_1$-monoidal subcategory $\operatorname{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C})^\otimes \subseteq \operatorname{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}^{\mathrm{nu}}(\mathcal{C})^\otimes$. By the same reasoning, we have $\mathbb{E}_k$-monoidal ∞-categories $\operatorname{Alg}_{\mathbb{E}_1/\mathbb{E}_{k+1}}(\mathcal{C})^\otimes$, $\operatorname{Alg}_{\mathbb{E}_1/\mathbb{E}_{k+1}}^{\mathrm{nu}}(\mathcal{C})^\otimes$, and $\operatorname{Alg}_{\mathbb{E}_1/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C})^\otimes$.

There is an evident forgetful functor $\operatorname{Alg}_{/\mathbb{E}_{k+1}}^{\mathrm{nu}}(\mathcal{C}) \to \operatorname{Alg}_{/\mathbb{E}_1}^{\mathrm{nu}}(\operatorname{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}^{\mathrm{nu}}(\mathcal{C}))$, which obviously restricts to a functor $\psi_0 : \operatorname{Alg}_{/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C}) \to \operatorname{Alg}_{/\mathbb{E}_1}^{\mathrm{nu}}(\operatorname{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C}))$. Using the inductive hypothesis (and Corollary T.2.4.4.4), we deduce that the evident categorical fibration $\operatorname{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})^\otimes \to \operatorname{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C})^\otimes$ is a categorical equivalence and therefore a trivial Kan fibration. It follows that the induced map

$$\operatorname{Alg}_{\mathbb{E}_1}^{\mathrm{nu}}(\operatorname{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})) \to \operatorname{Alg}_{/\mathbb{E}_1}^{\mathrm{nu}}(\operatorname{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C}))$$

is a trivial Kan fibration, which admits a section ψ_1. Let ψ_2 be the evident equivalence

$$\operatorname{Alg}_{/\mathbb{E}_1}^{\mathrm{nu}}(\operatorname{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})) \simeq \operatorname{Alg}_{/\mathbb{E}_k}(\operatorname{Alg}_{\mathbb{E}_1/\mathbb{E}_{k+1}}^{\mathrm{nu}}(\mathcal{C})).$$

We observe that the composition $\psi_2 \circ \psi_1 \circ \psi_0$ carries $\operatorname{Alg}_{/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C})$ into the subcategory

$$\operatorname{Alg}_{/\mathbb{E}_k}(\operatorname{Alg}_{\mathbb{E}_1/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C})) \subseteq \operatorname{Alg}_{/\mathbb{E}_k}(\operatorname{Alg}_{\mathbb{E}_1/\mathbb{E}_{k+1}}^{\mathrm{nu}}(\mathcal{C})).$$

Using the inductive hypothesis and Corollary T.2.4.4.4 again, we deduce that the forgetful functor

$$\operatorname{Alg}_{/\mathbb{E}_k}(\operatorname{Alg}_{\mathbb{E}_1/\mathbb{E}_{k+1}}(\mathcal{C})) \to \operatorname{Alg}_{/\mathbb{E}_k}(\operatorname{Alg}_{\mathbb{E}_1/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C}))$$

is a trivial Kan fibration, which admits a section ψ_3. Finally, Theorem 5.1.2.2 implies that the functor $\mathrm{Alg}_{/\mathbb{E}_{k+1}}(\mathcal{C}) \to \mathrm{Alg}_{/\mathbb{E}_1}(\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C}))$ is an equivalence of ∞-categories which admits a homotopy inverse ψ_4. Let ψ denote the composition $\psi_4\psi_3\psi_2\psi_1\psi_0$. Then ψ is a functor from $\mathrm{Alg}_{/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C})$ to $\mathrm{Alg}_{/\mathbb{E}_{k+1}}(\mathcal{C})$. The composition $\psi \circ \theta$ becomes homotopic to the identity after composing with the functor $\mathrm{Alg}_{/\mathbb{E}_{k+1}}(\mathcal{C}) \simeq \mathrm{Alg}_{/\mathbb{E}_1}^{\mathrm{qu}}(\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C})) \subseteq \mathrm{Alg}_{/\mathbb{E}_1}^{\mathrm{nu}}(\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}^{\mathrm{nu}}(\mathcal{C}))$, and is therefore homotopic to the identity on $\mathrm{Alg}_{/\mathbb{E}_{k+1}}(\mathcal{C})$.

To complete the proof of Theorem 5.2.3.10, it will suffice to show that the composition $\theta \circ \psi$ is equivalent to the identity functor from $\mathrm{Alg}_{/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C})$ to itself. This is substantially more difficult, and the proof will require a brief digression. In what follows, we will assume that the reader is familiar with the theory of centralizers of maps of $\mathbb{E}_k$-algebras developed in §4.3.3.7 (see Definition 6.1.4.1).

Definition 5.2.3.16. Let $\mathcal{C}^{\otimes} \to \mathbb{E}_k^{\otimes}$ be a coCartesian fibration of ∞-operads, let A and B be $\mathbb{E}_k$-algebra objects of $\mathcal{C}$, and let $u : \mathbf{1} \to A$ be a morphism in $\mathcal{C}$. We let $\mathrm{Map}_{\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})}^{u}(A, B)$ be the summand of the mapping space $\mathrm{Map}_{\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})}(A, B)$ given by those maps $f : A \to B$ such that $f \circ u$ is an invertible element in the monoid $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(\mathbf{1}, B)$.

Let $f : A \to B$ be a morphism in $\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})$ and let $u : \mathbf{1} \to A$ be as above. We will say that f is a *u-equivalence* if, for every object $C \in \mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})$, composition with f induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})}^{fu}(B, C) \to \mathrm{Map}_{\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})}^{u}(A, C).$$

Remark 5.2.3.17. Let M be an associative monoid. If x and y are commuting elements of M, then the product $xy = yx$ is invertible if and only if both x and y are invertible. In the situation of Definition 5.2.3.16, this guarantees that if $u : \mathbf{1} \to A$ and $v : \mathbf{1} \to A$ are morphisms in $\mathcal{C}$ such that u and v commute in the monoid $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(\mathbf{1}, A)$ and w denotes the product map $\mathbf{1} \simeq \mathbf{1} \otimes \mathbf{1} \xrightarrow{u \otimes v} A \otimes A \to A$, then we have $\mathrm{Map}_{\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})}^{w}(A, B) = \mathrm{Map}_{\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})}^{u}(A, B) \cap \mathrm{Map}_{\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})}^{v}(A, B)$ (where the intersection is formed in the mapping space $\mathrm{Map}_{\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})}(A, B)$). It follows that if $f : A \to B$ is a u-equivalence or a v-equivalence, then it is also a w-equivalence.

Remark 5.2.3.18. Let $\mathcal{C}^{\otimes} \to \mathbb{E}_k^{\otimes}$ be a presentable $\mathbb{E}_k$-monoidal ∞-category, and let $e : \mathbf{1} \to A$ be the unit map for an $\mathbb{E}_k$-algebra object $A \in \mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})$. We will abuse notation by identifying A with the underlying nonunital $\mathbb{E}_k$-algebra object, and let A^{+} be the free $\mathbb{E}_k$-algebra generated by this nonunital $\mathbb{E}_k$-algebra (see Proposition 5.2.3.13). Let e^{+} denote the composite map $\mathbf{1} \xrightarrow{e} A \to A^{+}$. Then the counit map $v : A^{+} \to A$ is an e^{+}-equivalence. To see this, it suffices to show that for every object $B \in \mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})$, composition with v induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})}(A, B) = \mathrm{Map}_{\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})}^{e}(A, B) \to \mathrm{Map}_{\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})}^{e^{+}}(A^{+}, B).$$

Note that any nonunital algebra morphism $f : A \to B$ carries e to an idempotent element $[f \circ e]$ of the monoid $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(\mathbf{1}, B)$, so $f \circ e$ is a quasi-unit for B if and only if $[f \circ e]$ is invertible. Consequently, the homotopy equivalence $\mathrm{Map}_{\mathrm{Alg}_{/\mathbb{E}_k}}(A^{+}, B) \simeq \mathrm{Map}_{\mathrm{Alg}_{/\mathbb{E}_k}^{\mathrm{nu}}}(A, B)$ induces an identification $\mathrm{Map}_{\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})}^{e^{+}}(A^{+}, B) \simeq \mathrm{Map}_{\mathrm{Alg}_{/\mathbb{E}_k}^{\mathrm{qu}}(\mathcal{C})}(A, B)$. The desired result now follows from the inductive hypothesis.

Lemma 5.2.3.19. *Let $q : \mathcal{C}^{\otimes} \to \mathbb{E}_{k+1}^{\otimes}$ be a presentable $\mathbb{E}_{k+1}$-monoidal ∞-category, so that $\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})$ inherits the structure of an $\mathbb{E}_1$-monoidal ∞-category. Let $f : A \to A'$ be a morphism in $\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})$, and let $u : \mathbf{1} \to A$ be a morphism in $\mathcal{C}$ such that f is a u-equivalence. Let $B \in \mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})$ and $v : \mathbf{1} \to B$ be an arbitrary morphism in $\mathcal{C}$. Then:*

(1) *The induced map $f \otimes \mathrm{id}_B$ is a $u \otimes v : \mathbf{1} \to A \otimes B$ equivalence.*

(2) *The induced map $\mathrm{id}_B \otimes f$ is a $v \otimes u : \mathbf{1} \to B \otimes A$-equivalence.*

Proof. We will prove (1); the proof of (2) is similar. Let $e_A : \mathbf{1} \to A$ and $e_B : \mathbf{1} \to B$ denote the units of A and B, respectively. We note that $u \otimes v$ is homotopic to the product of maps $e_A \otimes v$ and $u \otimes e_B$ which commute in the monoid $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(\mathbf{1}, A \otimes B)$. By virtue of Remark 5.2.3.17, it will suffice to show that $f \otimes \mathrm{id}_B$ is a w-equivalence, where $w = u \otimes e_B$.

Let w' be the composition of w with $f \otimes \mathrm{id}_B$, and let $C \in \mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})$. We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Map}^{w'}_{\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})}(A' \otimes B, C) & \longrightarrow & \mathrm{Map}^{w}_{\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})}(A \otimes B, C) \\
& \searrow \qquad \swarrow & \\
& \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})}(B, C) &
\end{array}
$$

and we wish to show that the horizontal map is a homotopy equivalence. It will suffice to show that this map induces a homotopy equivalence after passing to the homotopy fibers over any map $g : B \to C$. This is equivalent to the requirement that f induces a homotopy equivalence $\mathrm{Map}^{fu}_{\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})}(A', \mathfrak{Z}_{\mathbb{E}_k}(g)) \to \mathrm{Map}^{u}_{\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})}(A, \mathfrak{Z}_{\mathbb{E}_k}(g))$, which follows from our assumption that f is a u-equivalence. $\qquad\square$

Lemma 5.2.3.20. *Let $\mathcal{C}^{\otimes} \to \mathbb{E}_k^{\otimes}$ be a presentable $\mathbb{E}_k$-monoidal ∞-category, let $A \in \mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})$, and let $u : \mathbf{1} \to A$ be a morphism in the underlying ∞-category $\mathcal{C}$. Then there exists a morphism $f : A \to A[u^{-1}]$ in $\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})$ with the following universal properties:*

(1) *The map f is a u-equivalence.*

(2) *The composite map fu is a unit in the monoid $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(\mathbf{1}, A[u^{-1}])$.*

Proof. Let $P : \mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C}) \to \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$ be the functor described in §5.1.3. The inclusion $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathcal{S}) \subseteq \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$ admits a right adjoint G which can be described informally as follows: G carries an $\mathbb{E}_k$-space X to the subspace $X^{\mathrm{gp}} \subseteq X$ given by the union of those connected components of X which are invertible in $\pi_0 X$. Let $J : \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S}) \to \mathcal{S}$ be the forgetful functor, and let $\chi : \mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C}) \to \mathcal{S}$ be the functor corepresented by A. We can identify u with a point in the space $JP(A)$, which determines natural transformation of functors $\chi \to JP$. Let χ denote the fiber product $\chi \times_{JP} JGP$ in the ∞-category $\mathrm{Fun}(\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C}), \mathcal{S})$. Since χ, J, G, and P are all accessible functors which preserve small limits, the functor χ' is accessible and preserves small limits, and is therefore corepresentable by an object $A[u^{-1}] \in \mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})$ (Proposition T.5.5.2.7). The evident map $\chi' \to \chi$ induces a map $f : A \to A[u^{-1}]$ which is easily seen to have the desired properties. $\qquad\square$

Remark 5.2.3.21. Let $\mathcal{C}^{\otimes} \to \mathbb{E}_k^{\otimes}$ be as in Lemma 5.2.3.20, let $f : A \to B$ be a morphism in $\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})$ and let $u : \mathbf{1} \to A$ be a morphism in $\mathcal{C}$. Then f is a u-equivalence if and only if it induces an equivalence $A[u^{-1}] \to B[(fu)^{-1}]$ in the ∞-category $\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})$.

Example 5.2.3.22. Let $A \in \mathrm{Alg}^{\mathrm{qu}}_{/\mathbb{E}_k}(\mathcal{C})$ be a nonunital algebra equipped with a quasi-unit $e_A : \mathbf{1} \to A$. Let $A^+ \in \mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})$ be an algebra equipped with a nonunital algebra map $\beta : A \to A^+$ which exhibits A^+ as the free $\mathbb{E}_k$-algebra generated by A. Then the composite map $\gamma_0 : A \to A^+ \to A^+[(\beta e_A)^{-1}]$ is quasi-unital, and therefore (by the inductive hypothesis) lifts to an $\mathbb{E}_k$-algebra map $\gamma : \overline{A} \to A^+[(\beta e_A)^{-1}]$. Using the inductive hypothesis again, we deduce that γ is an equivalence in $\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})$, so that γ_0 is an equivalence of nonunital algebras.

We now return to the proof of Theorem 5.2.3.10 for a presentable $\mathbb{E}_{k+1}$-monoidal ∞-category $\mathcal{C}^{\otimes} \to \mathbb{E}_{k+1}^{\otimes}$. We will assume that Theorem 5.2.3.10 holds for the ∞-operad $\mathbb{E}_k^{\otimes}$, so that the forgetful functor $\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C}) \to \mathrm{Alg}^{\mathrm{qu}}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})$ is an equivalence of ∞-categories. Consequently, all of the notions defined above for $\mathbb{E}_k$-algebras make sense also in the context of quasi-unital $\mathbb{E}_k$-algebras; we will make use of this observation implicitly in what follows.

Let $\mathcal{D}$ denote the fiber product

$$\operatorname{Fun}(\partial\,\Delta^1, \operatorname{Alg}^{\mathrm{qu}}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})) \times_{\operatorname{Fun}(\partial\,\Delta^1, \operatorname{Alg}^{\mathrm{nu}}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C}))} \operatorname{Fun}(\Delta^1, \operatorname{Alg}^{\mathrm{nu}}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C}))$$

whose objects are nonunital maps $f : A \to B$ between quasi-unital $\mathbb{E}_k$-algebra objects of $\mathcal{C}$, and whose morphisms are given by commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow{\ f'\ } & B' \end{array}$$

where the vertical maps are quasi-unital. Let $\mathcal{D}_0$ denote the full subcategory $\operatorname{Fun}(\Delta^1, \operatorname{Alg}^{\mathrm{qu}}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})) \subseteq \mathcal{D}$ spanned by the quasi-unital maps $f : A \to B$. The inclusion $\mathcal{D}_0 \hookrightarrow \mathcal{D}$ admits a left adjoint L, given informally by the formula $(f : A \to B) \mapsto (A \to B[(fe_A)^{-1}])$, where $e_A : \mathbf{1} \to A$ denotes the unit of A. Using Remark 5.2.3.21, we deduce the following:

Lemma 5.2.3.23. *If α is a morphism in $\mathcal{D}$ corresponding to a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ A' & \xrightarrow{\ \ \ } & B', \end{array}$$

then $L(\alpha)$ is an equivalence if and only if the following pair of conditions is satisfied:

(i) *The map g is an equivalence.*

(ii) *The map g' is an fe_A-equivalence, where $e_A : \mathbf{1} \to A$ denotes a quasi-unit for A.*

Note that the $\mathbb{E}_1$-monoidal structure on $\operatorname{Alg}^{\mathrm{nu}}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})$ induces an $\mathbb{E}_1$-monoidal structure on the ∞-category $\mathcal{D}$.

Lemma 5.2.3.24. *The localization functor $L : \mathcal{D} \to \mathcal{D}_0 \subseteq \mathcal{D}$ is compatible with the $\mathbb{E}_1$-monoidal structure on $\mathcal{D}$. In other words, if $\alpha : D \to D'$ is an L-equivalence in $\mathcal{D}$ and E is any object of $\mathcal{D}$, then the induced maps $D \otimes E \to D' \otimes E$ and $E \otimes D \to E \otimes D'$ are again L-equivalences.*

Proof. Combine Lemmas 5.2.3.23 and 5.2.3.19. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Combining Lemma 5.2.3.24 with Proposition 2.2.1.9, we deduce that L can be promoted to an $\mathbb{E}_1$-monoidal functor from $\mathcal{D}$ to $\mathcal{D}_0$; in particular, L induces a functor $L' : \operatorname{Alg}^{\mathrm{nu}}_{/\mathbb{E}_1}(\mathcal{D}) \to \operatorname{Alg}^{\mathrm{nu}}_{/\mathbb{E}_1}(\mathcal{D}_0)$ which is left adjoint to the inclusion and therefore comes equipped with a natural transformation $\alpha : \operatorname{id}_{\operatorname{Alg}^{\mathrm{nu}}_{/\mathbb{E}_1}(\mathcal{D})} \to L'$.

We are now ready to complete the proof of Theorem 5.2.3.10. Let $G : \operatorname{Alg}_{/\mathbb{E}_{k+1}}(\mathcal{C}) \to \operatorname{Alg}^{\mathrm{nu}}_{/\mathbb{E}_{k+1}}(\mathcal{C})$ denote the forgetful functor, let F be a left adjoint to G (Proposition 5.2.3.15), and let $\beta : \operatorname{id}_{\operatorname{Alg}^{\mathrm{nu}}_{/\mathbb{E}_{k+1}}(\mathcal{C})} \to G \circ F$ be a unit transformation. Let $j : \operatorname{Alg}^{\mathrm{qu}}_{/\mathbb{E}_{k+1}}(\mathcal{C}) \to \operatorname{Alg}^{\mathrm{nu}}_{/\mathbb{E}_{k+1}}(\mathcal{C})$ be the inclusion functor and let $\xi : \operatorname{Alg}^{\mathrm{nu}}_{/\mathbb{E}_{k+1}}(\mathcal{C}) \to \operatorname{Alg}^{\mathrm{nu}}_{/\mathbb{E}_1}(\operatorname{Alg}^{\mathrm{nu}}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C}))$ be the forgetful functor. If $A \in \operatorname{Alg}^{\mathrm{nu}}_{/\mathbb{E}_{k+1}}(\mathcal{C})$ is quasi-unital, then $GF(A)$ is likewise quasi-unital. Consequently, the construction $A \mapsto \xi(\beta_A)$ induces a functor $\epsilon : \operatorname{Alg}^{\mathrm{qu}}_{/\mathbb{E}_{k+1}}(\mathcal{C}) \to \operatorname{Alg}^{\mathrm{nu}}_{/\mathbb{E}_1}(\mathcal{D})$. Let L' and $\alpha : \operatorname{id} \to L'$ be defined as above. The induced natural transformation $\epsilon \to L'\epsilon$ can be regarded as a functor from $\operatorname{Alg}^{\mathrm{qu}}_{/\mathbb{E}_{k+1}}(\mathcal{C})$ to $\operatorname{Fun}(\Delta^1 \times \Delta^1, \operatorname{Alg}^{\mathrm{nu}}_{/\mathbb{E}_1}(\operatorname{Alg}^{\mathrm{nu}}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})))$. This functor can be described informally as follows: it carries a quasi-unital algebra A to the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \beta_A\ } & F(A) \\ \downarrow & & \downarrow \\ A & \xrightarrow{\ \ \ } & F(A)[(\beta_A e_A)^{-1}], \end{array}$$

where $e_A : \mathbf{1} \to A$ denotes the quasi-unit of A. It follows from Example 5.2.3.22 that the lower horizontal map is an equivalence. Consequently, the above functor can be regarded as a natural transformation from ξGFj to ξj in the ∞-category $\mathrm{Fun}(\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C}), \mathrm{Alg}_{/\mathbb{E}_1}^{\mathrm{nu}}(\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C})))$. Composing with $\psi_4 \circ \psi_3 \circ \psi_2 \circ \psi_1$, we obtain a natural transformation $\delta : \psi\theta F \to \psi$ of functors from $\mathrm{Alg}_{/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C})$ to $\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C})$. Since $\psi\theta$ is homotopic to the identity, we can view δ as a natural transformation from $F|\mathrm{Alg}_{/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C})$ to ψ. This transformation is adjoint to a map of functors $\mathrm{id}_{\mathrm{Alg}_{/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C})} \to \theta \circ \psi$. It is easy to see that this transformation is an equivalence (using the fact that the forgetful functor $\mathrm{Alg}_{/\mathbb{E}_{k+1}}^{\mathrm{qu}}(\mathcal{C}) \to \mathcal{C}$ is conservative, by Lemma 3.2.2.6), so that ψ is a right homotopy inverse to θ. This completes the proof of Theorem 5.2.3.10.

5.2.4 Little Cubes in a Manifold

Let M be a topological space equipped with an $\mathbf{R}^k$-bundle $\zeta \to M$. Assuming that M is sufficiently nice, we can choose a Kan complex B such that X is homotopy equivalent to the geometric realization $|B|$, and the bundle ζ is classified by a Kan fibration of simplicial sets $\theta : B \to \mathrm{BTop}(k)$. In this case, we can apply the construction of Definition 5.2.2.10 to obtain an ∞-operad $\mathbb{E}_B^{\otimes}$. In the special case where M is a topological manifold of dimension k and ζ is the tangent bundle of M, we will denote this ∞-operad by $\mathbb{E}_M^{\otimes}$ (see Definition 5.2.4.1 below for a precise definition). We can think of $\mathbb{E}_M^{\otimes}$ as a variation on the ∞-operad $\mathbb{E}_k^{\otimes}$ whose objects are cubes $\square^k$ equipped with an open embedding into M, and whose morphisms are required to be compatible with these open embeddings (up to specified isotopy). We will also consider a more rigid version of the ∞-operad $\mathbb{E}_M^{\otimes}$, where the morphisms are required to be *strictly* compatible with the embeddings into M (rather than merely up to isotopy); this ∞-operad will be denoted by $\mathrm{N}(\mathrm{Disk}(M))^{\otimes}$ (Definition 5.2.4.6). The main result of this section is Theorem 5.2.4.9, which asserts that theory of $\mathbb{E}_M$-algebras is closely related to the more rigid theory of $\mathrm{N}(\mathrm{Disk}(M))^{\otimes}$-algebras.

Our first step is to define the ∞-operad $\mathbb{E}_M^{\otimes}$ more precisely.

Definition 5.2.4.1. Let M be a topological manifold of dimension k. We define a topological category $\mathcal{C}_M$ having two objects, which we will denote by M and $\mathbf{R}^k$, with mapping spaces given by the formulas

$$\mathrm{Map}_{\mathcal{C}_M}(\mathbf{R}^k, \mathbf{R}^k) = \mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k) \qquad \mathrm{Map}_{\mathcal{C}_M}(\mathbf{R}^k, M) = \mathrm{Emb}(\mathbf{R}^k, M)$$

$$\mathrm{Map}_{\mathcal{C}_M}(M, \mathbf{R}^k) = \emptyset \qquad \mathrm{Map}_{\mathcal{C}_M}(M, M) = \{\mathrm{id}_M\}.$$

We identify the Kan complex $\mathrm{BTop}(k)$ with a full subcategory of the nerve $\mathrm{N}(\mathcal{C}_M)$. Let B_M denote the Kan complex $\mathrm{BTop}(k) \times_{\mathrm{N}(\mathcal{C}_M)} \mathrm{N}(\mathcal{C}_M)_{/M}$. We let $\mathbb{E}_M^{\otimes}$ denote the ∞-operad $\mathrm{BTop}(k)^{\otimes} \times_{\mathrm{BTop}(k)^{\amalg}} B_M^{\amalg}$ In other words, we let $\mathbb{E}_M^{\otimes}$ denote the ∞-operad $\mathbb{E}_{B_M}^{\otimes}$ of Definition 5.2.2.10.

Remark 5.2.4.2. Let M be a topological manifold of dimension k, and let B_M be defined as in Definition 5.2.4.1. Then $\mathbb{E}_M^{\otimes}$ can be obtained as the assembly of a B_M-family of ∞-operads, each of which is equivalent to $\mathbb{E}_k^{\otimes}$ (Remark 5.2.2.13). To justify our notation, we will show that the Kan complex B_M is canonically homotopy equivalent to the (singular complex of) M. More precisely, we will construct a canonical chain of homotopy equivalences

$$B_M \leftarrow B_M' \to B_M'' \leftarrow \mathrm{Sing}(M).$$

To this end, we define topological categories $\mathcal{C}_M'$ and $\mathcal{C}_M''$, each of which consists of a pair of objects $\{\mathbf{R}^k, M\}$ with morphism spaces given by the formulas

$$\mathrm{Map}_{\mathcal{C}_M'}(\mathbf{R}^k, \mathbf{R}^k) = \mathrm{Emb}_0(\mathbf{R}^k, \mathbf{R}^k) \qquad \mathrm{Map}_{\mathcal{C}_M'}(\mathbf{R}^k, M) = \mathrm{Emb}(\mathbf{R}^k, M)$$

$$\mathrm{Map}_{\mathcal{C}_M''}(\mathbf{R}^k, \mathbf{R}^k) = \{0\} \qquad \mathrm{Map}_{\mathcal{C}_M''}(\mathbf{R}^k, M) = M$$

$$\mathrm{Map}_{\mathcal{C}_M'}(M, \mathbf{R}^k) = \emptyset = \mathrm{Map}_{\mathcal{C}_M''}(M, \mathbf{R}^k) \qquad \mathrm{Map}_{\mathcal{C}_M'}(M, M) = \{\mathrm{id}_M\} = \mathrm{Map}_{\mathcal{C}_M''}(M, M).$$

Here we let $\mathrm{Emb}_0(\mathbf{R}^k, \mathbf{R}^k)$ denote the closed subset of $\mathrm{Emb}(\mathbf{R}^k, \mathbf{R}^k)$ spanned by those open embeddings $f : \mathbf{R}^k \to \mathbf{R}^k$ such that $f(0) = 0$.

Let $\text{BTop}'(k)$ denote the full subcategory of $N(\mathcal{C}'_M)$ spanned by the object $\mathbf{R}^k$, let B'_M denote the fiber product $\text{BTop}(k) \times_{N(\mathcal{C}'_M)} N(\mathcal{C}'_M)_{/M}$, and let B''_M denote the fiber product $\{\mathbf{R}^k\} \times_{N(\mathcal{C}''_M)} N(\mathcal{C}''_M)_{/M}$. We have maps of topological categories $\mathcal{C}_M \xleftarrow{\theta} \mathcal{C}'_M \xrightarrow{\theta''} \mathcal{C}''_M$. The map θ is a weak equivalence of topological categories, and so induces a homotopy equivalence $B'_M \to B_M$. We claim that the induced map $\psi : B'_M \to B''_M$ is also a homotopy equivalence. We can identify vertices of B'_M with open embeddings $\mathbf{R}^k \to M$ and vertices of B''_M with points of M; since M is a k-manifold, the map ψ is surjective on vertices. Fix a vertex $(j : \mathbf{R}^k \hookrightarrow M) \in B'_M$. We have a map of homotopy fiber sequences

$$
\begin{array}{ccccc}
\text{Map}_{N(\mathcal{C}'_M)}(\mathbf{R}^k, \mathbf{R}^k) & \longrightarrow & \text{Map}_{N(\mathcal{C}'_M)}(\mathbf{R}^k, M) & \xrightarrow{\phi} & B'_M \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \text{Map}_{N(\mathcal{C}''_M)}(\mathbf{R}^k, M) & \longrightarrow & B''_M.
\end{array}
$$

It follows from Remark 5.2.1.11 that the left square is a homotopy pullback. It follows that the map of path spaces $\text{Map}_{B'_M}(j, j') \to \text{Map}_{B''_M}(\psi(j), \psi(j'))$ is a homotopy equivalence for every j' lying in the essential image of ϕ. Since the space $\text{BTop}'(k)$ is connected, the map ϕ is essentially surjective, so that ψ is a homotopy equivalence as desired.

We note there is a canonical homotopy equivalence $\text{Sing}(M) \to B''_M$ (adjoint to the weak homotopy equivalence appearing in Proposition T.2.2.2.7). Consequently, we obtain a canonical isomorphism $B_M \simeq B'_M \simeq B''_M \simeq \text{Sing}(M)$ in the homotopy category $\mathcal{H}$. It follows that $\mathbb{E}^{\otimes}_M$ can be identified with the colimit of a diagram ∞-operads parametrized by M, each of which is equivalent to $\mathbb{E}^{\otimes}_k$. This family is generally not constant: instead, it is twisted by the principal $\text{Top}(k)$-bundle given by the tangent bundle of M. In other words, if $\mathcal{C}^{\otimes}$ is an ∞-operad, then we can think of an object of $\text{Alg}_{\mathbb{E}_M}(\mathcal{C})$ as a (twisted) family of $\mathbb{E}_k$-algebra objects of $\mathcal{C}^{\otimes}$, parametrized by the points of M.

Example 5.2.4.3. Let M be the Euclidean space $\mathbf{R}^k$. Then the Kan complex B_M is contractible, so that $\mathbb{E}^{\otimes}_M$ is equivalent to the littles cubes operad $\mathbb{E}^{\otimes}_k$ (see Example 5.2.2.15). Since the ∞-operad $\mathbb{E}^{\otimes}_M$ depends functorially on M, we obtain another description of the "action up to homotopy" of the homeomorphism group $\text{Top}(k)$ on $\mathbb{E}^{\otimes}_k$ (at least if we view $\text{Top}(k)$ as a discrete group).

Example 5.2.4.4. Let M be a k-manifold which is given as a disjoint union of open submanifolds $M', M'' \subseteq M$. Then there is a canonical isomorphism of ∞-operads $\mathbb{E}^{\otimes}_M \simeq \mathbb{E}^{\otimes}_{M'} \boxplus \mathbb{E}^{\otimes}_{M''}$. Using Theorem 2.2.3.6, we deduce that the canonical map $\text{Alg}_{\mathbb{E}_M}(\mathcal{C}) \to \text{Alg}_{\mathbb{E}_{M'}}(\mathcal{C}) \times \text{Alg}_{\mathbb{E}_{M''}}(\mathcal{C})$ is an equivalence, for any ∞-operad $\mathcal{C}^{\otimes}$.

Example 5.2.4.5. Let M be a k-manifold and let M' be a k'-manifold. There is an evident map of Kan complexes $\phi : B_M \times B_{M'} \to B_{M \times M'}$, which induces a bifunctor of ∞-operads $\theta : \mathbb{E}^{\otimes}_M \times \mathbb{E}^{\otimes}_{M'} \to \mathbb{E}^{\otimes}_{M \times M'}$. Remark 5.2.4.2 implies that ϕ is a homotopy equivalence, so that θ exhibits $\mathbb{E}^{\otimes}_{M \times M'}$ as a tensor product of the ∞-operads $\mathbb{E}^{\otimes}_M$ and $\mathbb{E}^{\otimes}_{M'}$ (see Remark 5.2.2.14).

We now introduce a more rigid variant of the ∞-operad $\mathbb{E}^{\otimes}_M$.

Definition 5.2.4.6. Let M be a topological manifold of dimension k. Let $\text{Disk}(M)$ denote the collection of all open subsets $U \subseteq M$ which are homeomorphic to Euclidean space $\mathbf{R}^k$. We regard $\text{Disk}(M)$ as a partially ordered set (with respect to inclusions of open sets), and let $N(\text{Disk}(M))$ denote its nerve. Let $N(\text{Disk}(M))^{\otimes}$ denote the subcategory subset of $N(\text{Disk}(M))^{\amalg}$ spanned by those morphisms $(U_1, \ldots, U_m) \to (V_1, \ldots, V_n)$ with the following property: for every pair of distinct integers $1 \le i, j \le m$ having the same image $k \in \langle n \rangle^{\circ}$, the open subsets $U_i, U_j \subseteq V_k$ are disjoint.

Remark 5.2.4.7. Let M be a manifold of dimension. Then $\mathrm{N}(\mathrm{Disk}(M))^{\otimes}$ is the ∞-operad associated to the ordinary colored operad $\mathcal{O}$ whose objects are elements of $\mathrm{Disk}(M)$, with morphisms given by

$$\mathrm{Mul}_{\mathcal{O}}(\{U_1, \ldots, U_n\}, V) = \begin{cases} * & \text{if } U_1 \cup \ldots \cup U_n \subseteq V \text{ and } U_i \cap U_j = \emptyset \text{ for } i \neq j \\ \emptyset & \text{otherwise.} \end{cases}$$

In particular, $\mathrm{N}(\mathrm{Disk}(M))^{\otimes}$ is an ∞-operad (see Example 2.1.1.21).

Remark 5.2.4.8. Let $\mathrm{Disk}(M)'$ denote the category whose objects are open embeddings $\mathbf{R}^k \hookrightarrow M$, and whose morphisms are commutative diagrams

$$\begin{array}{ccc} \mathbf{R}^k & \xrightarrow{\quad f \quad} & \mathbf{R}^k \\ & \searrow \qquad \swarrow & \\ & M & \end{array}$$

where f is an open embedding. Then the forgetful functor $(j : \mathbf{R}^k \hookrightarrow M) \mapsto j(\mathbf{R}^k)$ determines an equivalence of categories from $\mathrm{Disk}(M)'$ to $\mathrm{Disk}(M)$. If we regard $\mathrm{Disk}(M)$ as a colored operad via the construction of Remark 5.2.4.7, then $\mathrm{Disk}(M)'$ inherits the structure of a colored operad, to which we can associate an ∞-operad $\mathrm{N}(\mathrm{Disk}(M)')^{\otimes}$ equipped with an equivalence $\phi : \mathrm{N}(\mathrm{Disk}(M)')^{\otimes} \to \mathrm{N}(\mathrm{Disk}(M))^{\otimes}$. The forgetful functor $(j : \mathbf{R}^k \hookrightarrow M) \mapsto \mathbf{R}^k$ determines a map of colored operads from $\mathrm{Disk}(M)'$ to ${}^t\mathbb{E}^{\otimes}_{\mathrm{BTop}(k)}$. Passing to nerves, we obtain a map of ∞-operads $\mathrm{N}(\mathrm{Disk}(M)')^{\otimes} \to \mathrm{BTop}(k)^{\otimes}$, which naturally factors through the map $\mathbb{E}^{\otimes}_M \to \mathrm{BTop}(k)^{\otimes}$. Composing with a homotopy inverse to ϕ, we get a map of ∞-operads $\mathrm{N}(\mathrm{Disk}(M))^{\otimes} \to \mathbb{E}^{\otimes}_M$.

We can describe the situation roughly as follows: the objects of the ∞-operads $\mathrm{N}(\mathrm{Disk}(M))^{\otimes}$ and $\mathbb{E}^{\otimes}_M$ are the same: copies of Euclidean space $\mathbf{R}^k$ equipped with an embedding in M. However, the morphisms are slightly different: an n-ary operation in $\mathbb{E}^{\otimes}_M$ is a diagram of open embeddings

$$\begin{array}{ccc} \coprod_{1 \leq i \leq n} \mathbf{R}^k & \xrightarrow{\hspace{3cm}} & \mathbf{R}^k \\ & \searrow \qquad \swarrow & \\ & M & \end{array}$$

which commutes up to (specified) isotopy, while an n-ary operation in $\mathrm{N}(\mathrm{Disk}(M))^{\otimes}$ is given by a diagram as above which commutes on the nose.

The map of ∞-operads $\psi : \mathrm{N}(\mathrm{Disk}(M))^{\otimes} \to \mathbb{E}^{\otimes}_M$ appearing in Remark 5.2.4.8 is not an equivalence. For example, the underlying ∞-category of $\mathbb{E}^{\otimes}_M$ is the Kan complex $B_M \simeq \mathrm{Sing}(M)$, while the underlying ∞-category of $\mathrm{N}(\mathrm{Disk}(M))^{\otimes}$ is the nerve of the partially ordered set $\mathrm{Disk}(M)$, which is certainly not a Kan complex. However, this is essentially the only difference: the map ψ exhibits $\mathbb{E}^{\otimes}_M$ as the ∞-operad obtained from $\mathrm{N}(\mathrm{Disk}(M))^{\otimes}$ by inverting each of the morphisms in $\mathrm{Disk}(M)$. More precisely, we have the following result:

Theorem 5.2.4.9. *Let M be a manifold and let $\mathcal{C}^{\otimes}$ be an ∞-operad. Composition with the map*

$$\mathrm{N}(\mathrm{Disk}(M))^{\otimes} \to \mathbb{E}^{\otimes}_M$$

of Remark 5.2.4.8 induces a fully faithful embedding $\theta : \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{N}(\mathrm{Disk}(M))}(\mathcal{C})$. The essential image of θ is the full subcategory of $\mathrm{Alg}_{\mathrm{N}(\mathrm{Disk}(M))}(\mathcal{C})$ spanned by the locally constant $\mathrm{N}(\mathrm{Disk}(M))^{\otimes}$-algebra objects of $\mathcal{C}$ (see Definition 4.2.4.1).

Theorem 5.2.4.9 is an immediate consequence of Proposition 2.3.4.5, together with the following pair of lemmas:

Lemma 5.2.4.10. *Let M be a manifold of dimension k. Then the map $\mathrm{N}(\mathrm{Disk}(M))^{\otimes} \to \mathbb{E}_M^{\otimes}$ induces a weak homotopy equivalence $\psi : \mathrm{N}(\mathrm{Disk}(M)) \to B_M$.*

Lemma 5.2.4.11. *The map of ∞-operads $\mathrm{Disk}(M)^{\otimes} \to \mathbb{E}_M^{\otimes}$ is a weak approximation to $\mathbb{E}_M^{\otimes}$.*

Proof of Lemma 5.2.4.10. The construction $U \mapsto B_U$ determines a functor χ from the category $\mathrm{Disk}(M)$ to the category of simplicial sets. Let X denote the relative nerve $\mathrm{N}_\chi(\mathrm{Disk}(M))$ (see §T.3.2.5), so that we have a coCartesian fibration $\theta : X \to \mathrm{N}(\mathrm{Disk}(M))$ whose over an object $U \in \mathrm{Disk}(M)$ is the Kan complex B_U. Remark 5.2.4.2 implies that the fibers of θ are contractible, so that θ is a trivial Kan fibration. The projection map θ has a section s, which carries an object $U \in \mathrm{Disk}(M)$ to a chart $\mathbf{R}^k \simeq U$ in B_U. The map ψ is obtained by composing the section s with the evident map $\psi' : X \to B_M$. Consequently, it will suffice to show that the map ψ' is a weak homotopy equivalence. According to Proposition T.3.3.4.5, this is equivalent to the requirement that B_M be a colimit of the diagram $\{U \mapsto B_U\}_{U \in \mathrm{Disk}(M)}$ in the ∞-category of spaces $\mathcal{S}$. Using Remark 5.2.4.2 again, we may reduce to showing that $\mathrm{Sing}\, M$ is a colimit of the diagram $\{U \mapsto \mathrm{Sing}\, U\}_{U \in \mathrm{Disk}(M)}$. In view of Theorem A.3.1, we need only show that for every point $x \in M$, the partially ordered set $P : \{U \in \mathrm{Disk}(M) : x \in U\}$ is weakly contractible. In fact, P^{op} is filtered: for every finite collection of open disks $U_i \subseteq M$ containing x, the intersection $\bigcap_i U_i$ is an open neighborhood of x which contains a smaller open neighborhood $V \simeq \mathbf{R}^k$ of x (because M is a topological manifold). $\square$

Proof of Lemma 5.2.4.11. Since the map $\mathbb{E}_M^{\otimes} \to \mathrm{BTop}(k)^{\otimes}$ is an approximation (Remark 5.2.2.11), it will suffice to show that the composite map

$$\gamma : \mathrm{N}(\mathrm{Disk}(M))^{\otimes} \to \mathbb{E}_M^{\otimes} \to \mathrm{BTop}(k)^{\otimes}$$

is a weak approximation to $\mathrm{BTop}(k)^{\otimes}$. To this end, fix an object $U \in \mathrm{Disk}(M)$ and an integer $m \geq 0$; wish to prove that the map

$$\psi : \mathrm{N}(\mathrm{Disk}(M))_{/U}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle 1 \rangle}} \{\langle m \rangle\} \to \mathrm{BTop}(k)_{/U}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)_{/\langle 1 \rangle}} \{\langle m \rangle\}$$

is a weak homotopy equivalence (Corollary 2.3.3.16). We can identify the domain of ψ with the nerve $\mathrm{N}(A)$, where $A \subseteq \mathrm{Disk}(M)^m$ denotes the partially ordered set of sequences $(V_1, \ldots, V_m) \in \mathrm{Disk}(M)^m$ such that $\bigcup V_i \subseteq U$ and $V_i \cap V_j = \emptyset$ for $i \neq j$.

It will now suffice to show that ψ induces a homotopy equivalence after passing to the homotopy fiber over the unique vertex of the Kan complex $\mathrm{BTop}(k)^m$. Unwinding the definitions, we must show that the canonical map

$$\mathrm{hocolim}_{(V_1, \ldots, V_m) \in A} \prod_{1 \leq i \leq m} \mathrm{Sing}\,\mathrm{Emb}(\mathbf{R}^k, V_i) \to \mathrm{Sing}\,\mathrm{Emb}(\mathbf{R}^k \times \langle m \rangle^{\circ}, U)$$

is a weak homotopy equivalence. Using Proposition 5.2.1.8, we can reduce to showing instead that the map

$$\mathrm{hocolim}_{(V_1, \ldots, V_m) \in A} \prod_{1 \leq i \leq m} \mathrm{Germ}(V_i) \to \mathrm{Germ}(\langle m \rangle^{\circ}, U)$$

is a homotopy equivalence. Both sides are acted on freely by the simplicial group $\mathrm{Germ}_0(\mathbf{R}^k)$. Consequently, it will suffice to show that we obtain a weak homotopy equivalence of quotients

$$\mathrm{hocolim}_{(V_1, \ldots, V_m) \in A} \prod_{1 \leq i \leq m} \mathrm{Conf}(\{i\}, V_i) \to \mathrm{Conf}(\langle m \rangle^{\circ}, U).$$

In view of Theorem A.3.1, it will suffice to show that for every injective map $\phi : \langle m \rangle^{\circ} \to U$, the partially ordered set $A_\phi = \{(V_1, \ldots, V_m) \in A : \phi(i) \in V_i\}$ has weakly contractible nerve. This is clear, since A_ϕ^{op} is filtered (because each point $\phi(i)$ has arbitrarily small neighborhoods homeomorphic to Euclidean space $\mathbf{R}^k$). $\square$

We can summarize Theorem 5.2.4.9 informally as follows. To give an $\mathbb{E}_M$-algebra object A of a symmetric monoidal ∞-category $\mathcal{C}$, we need to specify the following data:

(i) For every open disk $U \subseteq M$, an object $A(U) \in \mathcal{C}$.

(ii) For every collection of disjoint open disks $V_1, \ldots, V_n$ contained in an open disk $U \subseteq M$, a map $A(V_1) \otimes \ldots \otimes A(V_n) \to A(U)$, which is an equivalence when $n = 1$.

In §5.3.1, we will explain how to describe this data in another way: namely, as a cosheaf on the Ran space of M (see Definition 5.3.1.1). However, in the setting of the Ran space, it is much more convenient to work with a *nonunital* version of the theory of $\mathbb{E}_M$-algebras. Consequently, we will spend the remainder of this section explaining how to adapt the above ideas to the nonunital case. We associate to every k-manifold M an ∞-operad $(\mathbb{E}_M^{\otimes})_{\mathrm{nu}}$ as in Definition 5.2.3.1. It follows from Remark 2.3.3.9 and Proposition 2.3.4.8 that $(\mathbb{E}_M^{\otimes})_{\mathrm{nu}}$ is the assembly of the B_M-family of ∞-operads $(B_M \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)) \times_{B_M^{\amalg}} (\mathbb{E}_M^{\otimes})_{\mathrm{nu}}$, each fiber of which is equivalent to the nonunital little cubes operad $(\mathbb{E}_k^{\otimes})_{\mathrm{nu}}$. If $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category, we let $\mathrm{Alg}_{\mathbb{E}_M}^{\mathrm{nu}}(\mathcal{C})$ denote the ∞-category $\mathrm{Alg}_{(\mathbb{E}_M)_{\mathrm{nu}}}(\mathcal{C})$ of nonunital $\mathbb{E}_M$-algebra objects of $\mathcal{C}$. Our next goal is to show that the results of §5.2.3 can be generalized to the present setting: that is, for any symmetric monoidal ∞-category $\mathcal{C}$, we can identify $\mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C})$ with a subcategory of $\mathrm{Alg}_{\mathbb{E}_M}^{\mathrm{nu}}(\mathcal{C})$ (Proposition 5.2.4.14). Our first step is to identify the relevant subcategory more precisely.

Definition 5.2.4.12. If $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category and M is a manifold of dimension $k > 0$, we will say that a nonunital $\mathbb{E}_M$-algebra object $A \in \mathrm{Alg}_{\mathbb{E}_M}^{\mathrm{nu}}(\mathcal{C})$ is *quasi-unital* if, for every point $U \in B_M$, the restriction of A to the fiber $(\{U\} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)) \times_{B_M^{\amalg}} (\mathbb{E}_M^{\otimes})_{\mathrm{nu}} \simeq (\mathbb{E}_k^{\otimes})_{\mathrm{nu}}$ determines a quasi-unital $\mathbb{E}_k$-algebra object of $\mathcal{C}$, in the sense of Definition 5.2.3.7. Similarly, we will say that a map $f : A \to B$ of quasi-unital $\mathbb{E}_M$-algebra objects of $\mathcal{C}$ is *quasi-unital* if its restriction to each fiber $(\{U\} \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)) \times_{B_M^{\amalg}} (\mathbb{E}_M^{\otimes})_{\mathrm{nu}}$ determines a quasi-unital map of nonunital $\mathbb{E}_k$-algebras. We let $\mathrm{Alg}_{\mathbb{E}_M}^{\mathrm{qu}}(\mathcal{C})$ denote the subcategory fo $\mathrm{Alg}_{\mathbb{E}_M}^{\mathrm{nu}}(\mathcal{C})$ spanned by the quasi-unital $\mathbb{E}_M$-algebra objects of $\mathcal{C}$ and quasi-unital morphisms between them.

Remark 5.2.4.13. Let M be a manifold of dimension $k > 0$ and let A be a nonunital $\mathbb{E}_M$-algebra object of a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$. Fix a point $U \in B_M$, corresponding to an open embedding $\psi : \mathbf{R}^k \hookrightarrow M$. We will say that a map $u : \mathbf{1} \to A(U)$ in $\mathcal{C}$ is a *quasi-unit* for A if, for every pair of objects $V, W \in B_M$ and every morphism $\phi : U \oplus V \to W$, the composite map

$$A(V) \simeq \mathbf{1} \otimes A(V) \xrightarrow{u} A(U) \otimes A(V) \to A(W)$$

is homotopic to the map induced by the composition $U \to U \oplus V \xrightarrow{\phi} W$ in $\mathbb{E}_M$. Note that it suffices to check this condition in the special case where $V = W = U$ and, if $k > 1$, where ϕ is a single map (arbitrarily chosen). Unwinding the definition, we see that A is quasi-unital if and only if there exists a quasi-unit $u : \mathbf{1} \to A(U)$ for each $U \in B_M$. Similarly, a map $A \to B$ between quasi-unital $\mathbb{E}_M$-algebra objects is quasi-unital if, for every quasi-unit $u : \mathbf{1} \to A(U)$, the composite map $\mathbf{1} \xrightarrow{u} A(U) \to B(U)$ is a quasi-unit for B. Moreover, if M is connected, then it suffices to check these conditions for a single $U \in B_M$.

Proposition 5.2.4.14. *Let M be a manifold of dimension $k > 0$ and let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Then the restriction functor $\mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_M}^{\mathrm{qu}}(\mathcal{C})$ is an equivalence of ∞-categories.*

Proof. For every map of simplicial sets $K \to B_M$, let $\mathcal{O}_K^{\otimes}$ denote the K-family of ∞-operads

$$(K \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)) \times_{B_M^{\amalg}} \mathbb{E}_M$$

and set $\mathcal{O}_K'^{\otimes} = (K \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)) \times_{B_M^{\amalg}} (\mathbb{E}_M^{\otimes})_{\mathrm{nu}}$. Note that the projection map $q : \mathcal{O}_K^{\otimes} \to K$ is a coCartesian fibration. Let $\mathrm{Alg}_{\mathcal{O}_K}'(\mathcal{C})$ denote the full subcategory of $\mathrm{Alg}_{\mathcal{O}_K}(\mathcal{C})$ spanned by those ∞-operad maps which carry q-coCartesian morphisms to equivalences in $\mathcal{C}$, let $\mathrm{Alg}_{\mathcal{O}_K'}'(\mathcal{C})$ be defined similarly, and let $\mathrm{Alg}_{\mathcal{O}_K}^{\mathrm{qu}}(\mathcal{C})$ denote the subcategory of $\mathrm{Alg}_{\mathcal{O}_K'}'(\mathcal{C})$ spanned by those objects which restrict to quasi-unital $\mathcal{O}_{\{v\}}'^{\otimes} \simeq (\mathbb{E}_k^{\otimes})_{\mathrm{nu}}$-algebra objects of $\mathcal{C}$ and those morphisms which restrict to quasi-unital $\mathcal{O}_{\{v\}}'^{\otimes} \simeq (\mathbb{E}_k^{\otimes})_{\mathrm{nu}}$-algebra maps for every

vertex $v \in K$. There is an evident restriction map $\theta_K : \text{Alg}'_{\mathcal{O}_K}(\mathcal{C}) \to \text{Alg}^{\text{qu}}_{\mathcal{O}'_K}(\mathcal{C})$ fitting into a commutative diagram

$$
\begin{array}{ccc}
\text{Alg}_{\mathbb{E}_M}(\mathcal{C}) & \longrightarrow & \text{Alg}^{\text{qu}}_{\mathbb{E}_M}(\mathcal{C}) \\
\downarrow & & \downarrow \\
\text{Alg}'_{\mathcal{O}_K} & \xrightarrow{\theta_K} & \text{Alg}^{\text{qu}}_{\mathcal{O}'_K}(\mathcal{C}).
\end{array}
$$

If $K = B_M$, then the vertical maps are categorical equivalences. Consequently, it will suffice to prove that θ_K is an equivalence for every map of simplicial sets $K \to B_M$. The collection of simplicial sets K which satisfy this condition is clearly stable under homotopy colimits; we can therefore reduce to the case where K is a simplex, in which case the desired result follows from Theorem 5.2.3.10. $\qquad\square$

It follows from Lemma 5.2.4.11 and Remark 2.3.3.9 that for every manifold M, the map $\text{Disk}(M)^{\otimes}_{\text{nu}} \to (\mathbb{E}^{\otimes}_M)_{\text{nu}}$ is a weak approximation to $(\mathbb{E}^{\otimes}_M)_{\text{nu}}$. Combining this with Lemma 5.2.4.10 and Theorem 2.3.3.23, we deduce the following nonunital variant of Theorem 5.2.4.9:

Proposition 5.2.4.15. *Let M be a manifold and let $\mathcal{C}^{\otimes}$ be an ∞-operad. Then composition with map* $\text{N}(\text{Disk}(M))^{\otimes} \to \mathbb{E}^{\otimes}_M$ *of Remark 5.2.4.8 induces a fully faithful embedding*

$$\theta : \text{Alg}^{\text{nu}}_{\mathbb{E}_M}(\mathcal{C}) \to \text{Alg}^{\text{nu}}_{\text{N}(\text{Disk}(M))}(\mathcal{C}).$$

The essential image of θ is the full subcategory of $\text{Alg}^{\text{nu}}_{\text{N}(\text{Disk}(M))}(\mathcal{C})$ spanned by the locally constant objects.

Definition 5.2.4.16. Let M be a manifold of dimension $k > 0$ and let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. We will say that a locally constant $\text{Disk}(M)^{\otimes}_{\text{nu}}$-algebra object of $\mathcal{C}$ is *quasi-unital* if it corresponds to a quasi-unital $(\mathbb{E}^{\otimes}_M)_{\text{nu}}$-algebra object of $\mathcal{C}$ under the equivalence of Proposition 5.2.4.15. Similarly, we will say that a map $f : A \to B$ between locally constant quasi-unital $\text{Disk}(M)^{\otimes}_{\text{nu}}$-algebra objects of $\mathcal{C}$ is *quasi-unital* if it corresponds to a quasi-unital morphism in $\text{Alg}^{\text{nu}}_{\mathbb{E}_M}(\mathcal{C})$ under the equivalence of Proposition 5.2.4.15. We let $\text{Alg}^{\text{qu,loc}}_{\text{Disk}(M)}(\mathcal{C})$ denote the subcategory of $\text{Alg}_{\text{Disk}(M)}(\mathcal{C})$ spanned by the quasi-unital, locally constant $\text{Disk}(M)^{\otimes}$-algebra objects of $\mathcal{C}$ and quasi-unital morphisms between them.

Remark 5.2.4.17. Let $A \in \text{Alg}^{\text{nu}}_{\text{Disk}(M)}(\mathcal{C})$, let $W \in \text{Disk}(M)$ be an open disk in M, and let $U \subseteq W$ be an open disk with compact closure in W. We say that a map $\mathbf{1} \to A(U)$ in $\mathcal{C}$ is a *quasi-unit* for A if, for every disk $V \in \text{Disk}(M)$ such that $V \subseteq W$ and $V \cap U = \emptyset$, the diagram

$$
\begin{array}{ccc}
\mathbf{1} \otimes A(V) & \xrightarrow{u \otimes \text{id}} & A(U) \otimes A(V) \\
\downarrow & & \downarrow \\
A(V) & \longrightarrow & A(W)
\end{array}
$$

commutes up to homotopy. Note that if M has dimension at least 2, it suffices to check this condition for a single open disk V. Unwinding the definition, we see that A is quasi-unital if and only if there exists a quasi-unit $u : \mathbf{1} \to A(U)$ for every pair $U \subseteq W$ as above, and a map $f : A \to B$ in $\text{Alg}^{\text{nu}}_{\text{Disk}(M)}(\mathcal{C})$ is quasi-unital if and only if composition with f carries every quasi-unit $\mathbf{1} \to A(U)$ to a quasi-unit $\mathbf{1} \to B(U)$ (see Remark 5.2.4.13). In fact, it suffices to check these conditions for a single pair $U \subseteq W$ in each connected component of M.

Combining Proposition 5.2.4.14, Theorem 5.2.4.9, and Proposition 5.2.4.15, we arrive at the following:

Proposition 5.2.4.18. *Let M be a manifold of dimension $k > 0$ and $\mathcal{C}^{\otimes}$ a symmetric monoidal ∞-category. Then the restriction functor $\text{Alg}^{\text{loc}}_{\text{Disk}(M)}(\mathcal{C}) \to \text{Alg}^{\text{qu,loc}}_{\text{Disk}(M)}(\mathcal{C})$ is an equivalence of ∞-categories.*

In other words, there is no essential loss of information in passing from unital $\text{Disk}(M)^{\otimes}$-algebras to nonunital $\text{Disk}(M)^{\otimes}$-algebras, at least in the locally constant case. For this reason, we will confine our attention to nonunital algebras in §5.3.

5.3   Topological Chiral Homology

Let M be a topological manifold, and let $\mathbb{E}_M^\otimes$ be the ∞-operad introduced in Definition 5.2.4.1. Roughly speaking, we can think of an $\mathbb{E}_M$-algebra A object of a symmetric monoidal ∞-category $\mathcal{C}^\otimes$ as a family of $\mathbb{E}_k$-algebras A_x parametrized by the points $x \in M$ (more accurately, one should think of this family as "twisted" by the tangent bundle of M: that is, for every point $x \in M$ we should think of A_x as an algebra over an ∞-operad whose objects are little disks in the tangent space $T_{M,x}$ to M at x).

There is a convenient geometric way to encode this information. We define the *Ran space* $\mathrm{Ran}(M)$ of M to be the collection of all nonempty finite subsets of M (for a more detailed discussion of $\mathrm{Ran}(M)$, including a description of the topology on $\mathrm{Ran}(M)$, we refer the reader to 5.3.1). To every point $S \in \mathrm{Ran}(M)$, the tensor product $A_S = \bigotimes_{s \in S} A_s$ is an object of $\mathcal{C}$. We will see that these objects are the stalks of a $\mathcal{C}$-valued cosheaf $\mathcal{F}$ on the Ran space. We can regard $\mathcal{F}$ as a constructible cosheaf which is obtained by gluing together locally constant cosheaves along the locally closet subsets $\mathrm{Ran}^n(M) = \{S \in \mathrm{Ran}(M) : |S| = n\} \subseteq \mathrm{Ran}(M)$ for $n \geq 1$; the "gluing" data for these restrictions reflects the multiplicative structure of the algebras $\{A_x\}_{x \in M}$. In §5.3.4, we will see that the construction $A \mapsto \mathcal{F}$ determines an equivalence of ∞-categories from the ∞-category of (nonunital) $\mathbb{E}_M$-algebras in $\mathcal{C}$ to a suitable ∞-category of *factorizable* $\mathcal{C}$-valued cosheaves on $\mathrm{Ran}(M)$, which are constructible with respect to the above stratification (Theorem 5.3.4.10).

The description of an $\mathbb{E}_M$-algebra object A of $\mathcal{C}$ as a factorizable $\mathcal{C}$-valued cosheaf $\mathcal{F}$ on $\mathrm{Ran}(M)$ suggests an interesting invariant of A: namely, the object $\mathcal{F}(\mathrm{Ran}(M)) \in \mathcal{C}$ given by global sections of $\mathcal{F}$. In the case where M is connected, we will refer to the global sections $\mathcal{F}(\mathrm{Ran}(M))$ as the *topological chiral homology* of M with coefficients in A, which we will denote by $\int_M A$. We will give an independent definition of $\int_M A$ (which does not require the assumption that M is connected) in §5.3.2, and verify that it is equivalent to $\mathcal{F}(\mathrm{Ran}(M))$ for connected M in §5.3.4 (Theorem 5.3.4.14). The construction $A \mapsto \int_M A$ can be regarded as a generalization of Hochschild homology (Theorem 5.3.3.11) and has a number of excellent formal properties, which we will discuss in §5.3.3.

In §5.3.6, will use the theory of topological chiral homology to formulate and prove a nonabelian version of the Poincare duality theorem (Theorem 5.3.6.6). The proof will rely on general version of Verdier duality (Theorem 5.3.5.4), which we prove in §5.3.5.

Remark 5.3.0.1. We will regard Convention 5.2.0.1 as in force throughout this section: the word *manifold* will always refer to a paracompact, Hausdorff, topological manifold of some fixed dimension k.

5.3.1   The Ran Space

Definition 5.3.1.1. Let M be a manifold. We let $\mathrm{Ran}(M)$ denote the collection of nonempty finite subsets $S \subseteq M$. We will refer to $\mathrm{Ran}(M)$ as the *Ran space* of M.

The Ran space $\mathrm{Ran}(M)$ admits a natural topology, which we will define in a moment. Our goal in this section is to study the basic properties of $\mathrm{Ran}(M)$ as a topological space. Our principal results are Theorem 5.3.1.6, which asserts that $\mathrm{Ran}(M)$ is weakly contractible (provided that M is connected), and Proposition 5.3.1.14, which characterizes sheaves on $\mathrm{Ran}(M)$ which are constructible with respect to the natural filtration of $\mathrm{Ran}(M)$ by cardinality of finite sets.

Our first step is to define the topology on $\mathrm{Ran}(M)$. First, we need to introduce a bit of notation. Suppose that $\{U_i\}_{1 \leq i \leq n}$ is a nonempty collection of pairwise disjoint subsets of M. We let $\mathrm{Ran}(\{U_i\}) \subseteq \mathrm{Ran}(M)$ denote the collection of finite sets $S \subseteq M$ such that $S \subseteq \bigcup U_i$ and $S \cap U_i$ is nonempty for $1 \leq i \leq n$.

Definition 5.3.1.2. Let M be a manifold. We will regard the Ran space $\mathrm{Ran}(M)$ as equipped with the coarsest topology for which the subsets $\mathrm{Ran}(\{U_i\}) \subseteq \mathrm{Ran}(M)$ are open, for every nonempty finite collection of pairwise disjoint open sets $\{U_i\}$ of M.

Remark 5.3.1.3. If $\{U_i\}$ is a nonempty finite collection of pairwise disjoint open subsets of a manifold M, then the open subset $\mathrm{Ran}(\{U_i\}) \subseteq \mathrm{Ran}(M)$ is homeomorphic to a product $\prod_i \mathrm{Ran}(U_i)$, via the map $\{S_i \subseteq U_i\}) \mapsto (\bigcup_i S_i \subseteq M)$.

Remark 5.3.1.4. Let M be a manifold, and let $S = \{x_1, \ldots, x_n\}$ be a point of $\mathrm{Ran}(M)$. Then S has a basis of open neighborhoods in $\mathrm{Ran}(M)$ of the form $\mathrm{Ran}(\{U_i\})$, where the U_i range over all collections of disjoint open neighborhoods of the points x_i in M. Since M is a manifold, we may further assume that that each U_i is homeomorphic to Euclidean space.

Remark 5.3.1.5. If we choose a metric d on on the manifold M, then the topology on $\mathrm{Ran}(M)$ is described by a metric D, where

$$D(S, T) = \sup_{s \in S} \inf_{t \in T} d(s, t) + \sup_{t \in T} \inf_{s \in S} d(s, t).$$

It follows that $\mathrm{Ran}(M)$ is paracompact.

Our first main object in this section is to prove the following result of Beilinson and Drinfeld:

Theorem 5.3.1.6 (Beilinson-Drinfeld). *Let M be a connected manifold. Then $\mathrm{Ran}(M)$ is weakly contractible.*

We first formulate a relative version of Theorem 5.3.1.6 which is slightly easier to prove.

Notation 5.3.1.7. Let M be a manifold and S a finite subset of M. We let $\mathrm{Ran}(M)_S$ denote the closed subset of $\mathrm{Ran}(M)$ consisting of those nonempty finite subsets $T \subseteq \mathrm{Ran}(M)$ such that $S \subseteq T$.

Lemma 5.3.1.8 (Beilinson-Drinfeld). *Let M be a connected manifold and let S be a nonempty finite subset of M. Then $\mathrm{Ran}_S(M)$ is weakly contractible.*

Proof. We first prove that $\mathrm{Ran}(M)_S$ is path connected. Let T be a subset of M containing S. For each $t \in T$, choose a path $p_t : [0, 1] \to M$ such that $p_t(0) = t$ and $p_t(1) \in S$ (this is possible since M is connected and S is nonempty). Then the map $r \mapsto S \cup \{p_t(r)\}_{t \in T}$ determines a continuous path in $\mathrm{Ran}(M)_S$ joining T with S. We will complete the proof by showing that for each $n > 0$, every element $\eta \in \pi_n \mathrm{Ran}(M)_S$ is trivial; here we compute the homotopy group π_n with respect to the base point given by $S \in \mathrm{Ran}(M)_S$.

The topological space $\mathrm{Ran}(M)_S$ admits a continuous product $U : \mathrm{Ran}(M)_S \times \mathrm{Ran}(M)_S \to \mathrm{Ran}(M)_S$, given by the formula $U(T, T') = T \cup T'$. This product induces a map of homotopy groups

$$\phi : \pi_n \mathrm{Ran}(M)_S \times \pi_n \mathrm{Ran}(M)_S \to \pi_n \mathrm{Ran}(M)_S.$$

Since S is a unit with respect to the multiplication on $\mathrm{Ran}(M)_S$, we conclude that $\phi(\eta, 1) = \eta = \phi(1, \eta)$ (where we let 1 denote the unit element of the homotopy group $\pi_n \mathrm{Ran}(M)_S$). Because the composition of the diagonal embedding $\mathrm{Ran}(M)_S \to \mathrm{Ran}(M)_S \times \mathrm{Ran}(M)_S$ with U is the identity from $\mathrm{Ran}(M)_S$ to itself, we have also $\phi(\eta, \eta) = \eta$. It follows that

$$\eta = \phi(\eta, \eta) = \phi(\eta, 1)\phi(1, \eta) = \eta^2$$

so that $\eta = 1$ as desired. $\qquad\square$

Proof of Theorem 5.3.1.6. For every point $x \in M$, choose an open embedding $j_x : \mathbf{R}^k \hookrightarrow M$ such that $j_x(0) = x$. Let $U_x = j_x(B(1))$ be the image under j_x of the unit ball in $\mathbf{R}^k$, and let V_x be the open subset of $\mathrm{Ran}(M)$ consisting of those nonempty finite subsets $S \subseteq M$ such that $S \cap U_x \neq \emptyset$. Let $\mathcal{J}$ be the partially ordered set of all nonempty finite subsets of M (that is, $\mathcal{J}$ is the Ran space $\mathrm{Ran}(M)$, but viewed as a partially ordered set). We define a functor from $\mathcal{J}^{op}$ to the category of open subsets of $\mathrm{Ran}(M)$ by the formula

$$T \mapsto V_T = \bigcap_{x \in T} V_x.$$

For each $S \in \mathrm{Ran}(M)$, the partially ordered set $\{T \in \mathcal{J} : S \in V_T\}$ is nonempty and stable under finite unions, and therefore has weakly contractible nerve. It follows that $\mathrm{Sing}\,\mathrm{Ran}(M)$ is equivalent to the homotopy colimit of the diagram $\{\mathrm{Sing}\,V_T\}_{T \in \mathcal{J}^{op}}$ (Theorem A.3.1). We will prove that each of the spaces V_T is weakly contractible, so that this homotopy colimit is weakly homotopy equivalent to $\mathrm{N}(\mathcal{J}^{op})$ and therefore weakly contractible.

Fix $T \in \mathcal{J}$, and choose a continuous family of maps $\{h_r : \mathbf{R}^k \to \mathbf{R}^k\}_{0 \le r \le 1}$ with the following properties:

(i) For $0 \leq r \leq 1$, the map h_r is the identity outside of a ball $B(2) \subseteq \mathbf{R}^k$ of radius 2.

(ii) The map h_0 is the identity.

(iii) The map h_1 carries $B(1) \subseteq \mathbf{R}^k$ to the origin.

We now define a homotopy $\phi_T : \text{Ran}(M) \times [0,1] \to \text{Ran}(M)$ by the formula

$$\phi_T(S,r) = S \cup \bigcup_{x \in T} j_x h_t j_x^{-1}(S).$$

The homotopy ϕ_T leaves V_T and $\text{Ran}(M)_T$ setwise fixed, and carries $V_T \times \{1\}$ into $\text{Ran}(M)_T$. It follows that the inclusion $\text{Ran}(M)_T \subseteq V_T$ is a homotopy equivalence, so that V_T is weakly contractible by Lemma 5.3.1.8. $\square$

We now discuss a natural stratification of the Ran space.

Definition 5.3.1.9. Let M be a manifold. We let $\text{Ran}^{\leq n}(M)$ denote the subspace of $\text{Ran}(M)$ consisting of those subsets $S \subseteq M$ having cardinality $\leq n$, and $\text{Ran}^n(M)$ the subspace of $\text{Ran}^{\leq n}(M)$ consisting of those subsets $S \subseteq M$ having cardinality exactly n.

Remark 5.3.1.10. The set $\text{Ran}^{\leq n}(M)$ is closed in $\text{Ran}(M)$, and $\text{Ran}^n(M)$ is open in $\text{Ran}^{\leq n}(M)$.

Definition 5.3.1.11. Let M be a manifold and let $\mathcal{F} \in \text{Shv}(\text{Ran}(M))$ be a sheaf on $\text{Ran}(M)$. For each $n \geq 0$, let $i(n) : \text{Ran}^{\leq n}(M) \to \text{Ran}(M)$ denote the inclusion map. We will say that $\mathcal{F}$ is *constructible* if the following conditions are satisfied:

(1) The canonical map $\mathcal{F} \to \varprojlim_n i(n)_* i(n)^* \mathcal{F}$ is an equivalence.

(2) For each n, the restriction of $i(n)^* \mathcal{F}$ to the open subset $\text{Ran}^n(M) \subseteq \text{Ran}^{\leq n}(M)$ is locally constant.

Remark 5.3.1.12. Condition (2) of Definition 5.3.1.11 is equivalent to the requirement that $\mathcal{F}$ be $\mathbf{Z}_{\geq 0}$-constructible, where we regard $\text{Ran}(M)$ as $\mathbf{Z}_{\geq 0}$-stratified via the map $\text{Ran}(M) \to \mathbf{Z}_{\geq 0}$ given by $S \mapsto |S|$. We refer the reader to §A.5 for a general review of the theory of constructible sheaves. Here we are required to impose condition (1) because the partially ordered set $\mathbf{Z}_{\geq 0}$ does not satisfy the ascending chain condition.

Remark 5.3.1.13. We can endow the topological space $\text{Ran}(M)$ with another topology, where a set $U \subseteq \text{Ran}(M)$ is open if and only if its intersection with each $\text{Ran}^{\leq n}(M)$ is open (with respect to the topology of Definition 5.3.1.1). If $\mathcal{F}$ is a sheaf on $\text{Ran}(M)$ with respect to this second topology, then condition (1) of Definition 5.3.1.11 is automatic: this follows from Proposition T.7.1.5.8.

The following result gives a convenient characterization of constructible sheaves on the Ran space:

Proposition 5.3.1.14. *Let M be a manifold and $\mathcal{F} \in \text{Shv}(\text{Ran}(M))$. Then $\mathcal{F}$ is constructible if and only if it is hypercomplete and satisfies the following additional condition:*

($*$) *For every nonempty finite collection of disjoint disks $U_1, \ldots, U_n \subseteq M$ containing open subdisks $V_1 \subseteq U_1, \ldots, V_n \subseteq U_n$, the restriction map $\mathcal{F}(\text{Ran}(\{U_i\})) \to \mathcal{F}(\text{Ran}(\{V_i\}))$ is a homotopy equivalence.*

Proof. We first prove the "only if" direction. Suppose that $\mathcal{F}$ is constructible. To show that $\mathcal{F}$ is hypercomplete, we write $\mathcal{F}$ as a limit $\varprojlim i(n)_* i(n)^* \mathcal{F}$ as in Definition 5.3.1.11. It therefore suffices to show that each $i(n)^* \mathcal{F}$ is hypercomplete. This follows from the observation that $\text{Ran}^{\leq n}(M)$ is a paracompact topological space of finite covering dimension (Corollary T.7.2.1.12).

We now prove every constructible sheaf $\mathcal{F} \in \text{Shv}(\text{Ran}(M))$ satisfies ($*$). For $1 \leq i \leq n$, we invoke Theorem 5.2.1.5 to choose an isotopy $\{h_i^t : V_i \to U_i\}_{t \in \mathbf{R}}$ such that h_i^0 is the inclusion of V_i into U_i and h_i^1 is a homeomorphism. These isotopies determine an open embedding

$$H : \text{Ran}(\{V_i\}) \times \mathbf{R} \to \text{Ran}(\{U_i\}) \times \mathbf{R}.$$

Let $\mathcal{F}' \in \mathcal{S}\mathrm{hv}(\mathrm{Ran}(\{U_i\}) \times \mathbf{R})$ be the pullback of $\mathcal{F}$, so that $\mathcal{F}'$ is hypercomplete (see Lemma A.2.6 and Example A.2.8). It follows that $H^* \mathcal{F}'$ is hypercomplete. Since $\mathcal{F}$ is constructible, we deduce that $\mathcal{F}'$ is foliated. For $t \in \mathbf{R}$, let $\mathcal{F}'_t$ denote the restriction of $\mathcal{F}$ to $\mathrm{Ran}(\{V_i\}) \times \{t\}$. We have a commutative diagram of spaces

$$\mathcal{F}(\mathrm{Ran}(\{U_i\})) \xrightarrow{\quad\theta\quad} \mathcal{F}'((\mathrm{Ran}(\{V_i\}) \times \mathbf{R})$$
$$\theta' \searrow \qquad \swarrow \theta''$$
$$\mathcal{F}'_1(\mathrm{Ran}(\{V_i\}))$$

Since each h_i^1 is a homeomorphism, we deduce that θ' is a homotopy equivalence. Proposition A.2.5 guarantees that θ'' is a homotopy equivalence, so that θ is a homotopy equivalence by the two-out-of-three property. Applying Proposition A.2.5 again, we deduce that the composite map $\mathcal{F}(\mathrm{Ran}(\{U_i\})) \to \mathcal{F}'_0(\mathrm{Ran}(\{V_i\})) \simeq \mathcal{F}(\mathrm{Ran}(\{V_i\}))$ is a homotopy equivalence as desired.

We now prove the "if" direction of the proposition. Assume that $\mathcal{F}$ is hypercomplete and that $\mathcal{F}$ satisfies $(*)$; we wish to prove that $\mathcal{F}$ is constructible. We first show that the restriction of $\mathcal{F}$ to each $\mathrm{Ran}^n(M)$ is locally constant. Choose a point $S \in \mathrm{Ran}^n(M)$; we will show that $\mathcal{F} \,|\, \mathrm{Ran}^n(M)$ is constant in a neighborhood of S. Let $S = \{x_1, \ldots, x_n\}$, and choose disjoint open disks $U_1, \ldots, U_n \subseteq X$ such that $x_i \in U_i$. Let $W \subseteq \mathrm{Ran}^n(M)$ denote the collection of all subsets $S \subseteq M$ which contain exactly one point from each U_i. We will prove that $\mathcal{F} \,|\, \mathrm{Ran}^n(M)$ is constant on W. Let $X = \mathcal{F}(\mathrm{Ran}(\{U_i\}))$. Since $W \subseteq \mathrm{Ran}(\{U_i\})$, there is a canonical map from the constant sheaf on W taking the value X to $\mathcal{F} \,|\, W$; we will show that this map is an equivalence. Since $W \simeq U_1 \times \ldots \times U_n$ is a manifold, it has finite covering dimension so that $\mathcal{S}\mathrm{hv}(W)$ is hypercomplete. Consequently, to show that a morphism in $\mathcal{S}\mathrm{hv}(W)$ is an equivalence, it suffices to check after passing to the stalk at each point $\{y_1, \ldots, y_n\} \in W$. This stalk is given by $\varinjlim_V \mathcal{F}(V)$, where the colimit is taken over all open subsets $V \subseteq \mathrm{Ran}(M)$ containing $\{y_1, \ldots, y_n\}$. It follows from Remark 5.3.1.4 that it suffices to take the colimit over those open sets V of the form $\mathrm{Ran}(\{V_i\})$, where each $V_i \subseteq U_i$ is an open neighborhood of y_i. Condition $(*)$ guarantees that each of the maps $X \to \mathcal{F}(V)$ is a homotopy equivalence, so after passing to the filtered colimit we obtain a homotopy equivalence $X \to \varinjlim_V \mathcal{F}(V)$ as desired.

Let $\mathcal{G} = \varprojlim_n i(n)_* i(n)^* \mathcal{F}$ (using the notation of Definition 5.3.1.11). To complete the proof, it will suffice to show that the canonical map $\alpha : \mathcal{F} \to \mathcal{G}$ is an equivalence. Since each $i(n)^* \mathcal{F}$ is automatically hypercomplete (because $\mathrm{Ran}^{\leq n}(M)$ is a paracompact space of finite covering dimension), we see that $\mathcal{G}$ is hypercomplete. Using the results of §T.6.5.3, we deduce that the collection of those open sets $U \subseteq \mathrm{Ran}(M)$ such that α induces a homotopy equivalence $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is stable under the formation of unions of hypercoverings. It therefore suffices to show that α_U is an homotopy equivalence for some collection of open sets U which forms a basis for the topology of $\mathrm{Ran}(M)$. By virtue of Remark 5.3.1.4, we may assume that $U = \mathrm{Ran}(\{U_i\})$ for some collection of disjoint open disks $U_1, \ldots, U_n$.

For each integer m, let $\mathcal{F}^{\leq m} = i(m)^* \mathcal{F}$. We wish to prove that the map $\mathcal{F}(U) \to \varprojlim_m \mathcal{F}^{\leq m}(U \cap \mathrm{Ran}^{\leq m}(M))$ is a homotopy equivalence. In fact, we will prove that the individual maps $\mathcal{F}(U) \to \mathcal{F}^{\leq m}(U \cap \mathrm{Ran}^{\leq m}(M))$ are homotopy equivalences for $m \geq n$. Choose a point x_i in each disk U_i, and let $S = \{x_1, \ldots, x_n\}$. Let $\mathcal{F}_S$ denote the stalk of $\mathcal{F}$ at the point S. We have a commutative diagram of restriction maps

$$\mathcal{F}(U) \xrightarrow{\qquad\qquad} \mathcal{F}^{\leq m}(U \cap \mathrm{Ran}^{\leq m}(M))$$
$$\phi \searrow \qquad \swarrow \phi'$$
$$\mathcal{F}_S$$

where ϕ is a homotopy equivalence by the argument given above. By the two-out-of-three property, we are reduced to proving that ϕ' is a homotopy equivalence.

The set $U \cap \mathrm{Ran}^{\leq m}(M)$ admits a stratification by the linearly ordered set $[m]$, which carries a point $T \in \mathrm{Ran}(M)$ to the cardinality of T. Let $\mathcal{C} = \mathrm{Sing}^{[m]}(U \cap \mathrm{Ran}^{\leq m}(M))$. Since $\mathcal{F}$ is constructible, the sheaf $\mathcal{F} \,|\, (U \cap \mathrm{Ran}^{\leq m}(M))$ corresponds to some left fibration $q : \widetilde{\mathcal{C}} \to \mathcal{C}$ under the equivalence of ∞-categories

provided by Theorem A.9.3. Under this equivalence, we can identify $\mathcal{F}^{\leq m}(U \cap \mathrm{Ran}^{\leq m}(M))$ with the ∞-category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}, \widetilde{\mathcal{C}})$ of sections of q, while $\mathcal{F}_S$ corresponds to the fiber of $\widetilde{\mathcal{C}}_S$ of q over the point $S \in \mathcal{C}$. To prove that θ' is an equivalence, it suffices to show that S is an initial object of $\mathcal{C}$. To this end, choose homeomorphisms $\psi_i : \mathbf{R}^k \to U_i$ for $1 \leq i \leq n$ such that $\psi_i(0) = x_i$. We then have a map

$$c : [0,1] \times (U \cap \mathrm{Ran}^{\leq m}(M)) \to (U \cap \mathrm{Ran}^{\leq m}(M))$$

given by the formula $c(t, T) = \{\psi_i(tv) : \psi_i(v) \in T\}$. The continuous map c induces a natural transformation from the inclusion $\{S\} \hookrightarrow \mathcal{C}$ to the identity functor from $\mathcal{C}$ to itself, thereby proving that $S \in \mathcal{C}$ is initial as desired. $\square$

To apply Proposition 5.3.1.14, it is convenient to have the following characterization of hypercompleteness:

Proposition 5.3.1.15. *Let X be a topological space, $\mathcal{U}(X)$ the collection of open subsets of X, and $\mathcal{F} : \mathrm{N}(\mathcal{U}(X)^{op}) \to \mathcal{S}$ a presheaf on X. The following conditions are equivalent:*

(1) *The presheaf $\mathcal{F}$ is a hypercomplete sheaf on X.*

(2) *Let U be an open subset of X, $\mathcal{C}$ be a category, and $f : \mathcal{C} \to \mathcal{U}(U)$ a functor. Suppose that, for every point $x \in U$, the full subcategory $\mathcal{C}_x = \{C \in \mathcal{C} : x \in f(C)\} \subseteq \mathcal{C}$ has weakly contractible nerve. Then $\mathcal{F}$ exhibits $\mathcal{F}(U)$ as a limit of the diagram $\mathrm{N}(\mathcal{C})^{op} \to \mathrm{N}(\mathcal{U}(X)^{op}) \xrightarrow{\mathcal{F}} \mathcal{S}$.*

Lemma 5.3.1.16. *Let X be a topological space, and let $\mathcal{F} \in \mathrm{Shv}(X)$ be an ∞-connective sheaf satisfying the following condition:*

($*$) *Let A be a partially ordered set and $f : A \to \mathcal{U}(X)^{op}$ an order-preserving map such that, for every point $x \in X$, the full subcategory $A_x = \{a \in A : x \in f(a)\} \subseteq A$ is filtered. Then $\mathcal{F}$ exhibits $\mathcal{F}(X)$ as a limit of the diagram $\mathrm{N}(\mathcal{A}) \to \mathrm{N}(\mathcal{U}(X)^{op}) \xrightarrow{\mathcal{F}} \mathcal{S}$.*

Then the space $\mathcal{F}(X)$ is nonempty.

Proof. The functor $\mathcal{F} : \mathrm{N}(\mathcal{U}(X)^{op}) \to \mathcal{S}$ classifies a left fibration $q : \mathcal{E} \to \mathrm{N}(\mathcal{U}(X)^{op})$. We will construct a partially ordered set A and a map $\psi : \mathrm{N}(A) \to \mathcal{E}$ such that the composite map $\mathrm{N}(A) \to \mathrm{N}(\mathcal{U}(X)^{op})$ and each subset A_x is filterd. According to Corollary T.3.3.3.3, we can identify the limit $\varprojlim_{a \in A} \mathcal{F}(f(a))$ with the Kan complex $\mathrm{Fun}_{\mathrm{N}(\mathcal{U}(X)^{op})}(\mathrm{N}(A), \mathcal{E})$, which is nonempty by construction.

We will construct a sequence of partially ordered sets

$$\emptyset = A(0) \subseteq A(1) \subseteq \ldots$$

and compatible maps $\psi(n) : \mathrm{N}(A(n)) \to \mathcal{E}$ with the following properties:

(i) For every element $a \in A(n)$, the set $\{b \in A(n) : b < a\}$ is a finite subset of $A(n-1)$.

(ii) For every point $x \in X$ and every finite subset $S \subseteq A(n-1)_x$, there exists an upper bound for S in $A(n)_x$.

Assuming that this can be done, we can complete the proof by taking $A = \bigcup_n A(n)$ and ψ be the amalgamation of the maps $\psi(n)$.

The construction now proceeds by induction on n. Assume that $n > 0$ and that the map $\psi(n-1) : \mathrm{N}(A(n-1)) \to \mathcal{E}$ has already been constructed. Let K be the set of pairs (x, S), where $x \in X$ and S is a finite subset of $A(n-1)_x$ which is closed-downwards (that is, $a \leq a'$ and $a' \in S$ implies $a \in S$). We define $A(n)$ to be the disjoint union $A(n-1) \coprod K$. We regard $A(n)$ as a partially ordered set, where $a < b$ in $A(n)$ if and only $a, b \in A(n-1)$ and $a < b$ in $A(n-1)$, or $a \in A(n-1)$, $b = (x, S) \in K$, and $a \in S$. It is clear that $A(n)$ satisfies condition (i). It remains only to construct a map $\psi(n) : \mathrm{N}(A(n)) \to \mathcal{E}$ which extends $\psi(n-1)$

and satisfies (ii). Unwinding the definitions, we must show that for every pair $(x, S) \in K$, the extension problem

$$
\begin{array}{ccc}
N(S) & \xrightarrow{\psi'} & \mathcal{E} \\
\downarrow & {}^{\phi} \nearrow & \\
N(S)^{\triangleright} & &
\end{array}
$$

admits a solution, where ψ' denotes the restriction $\psi(n-1)| N(S)$ and ϕ carries the cone point of $N(S)^{\triangleright}$ to an object $E \in \mathcal{E}$ such that $x \in q(E) \in \mathcal{U}(X)$.

Since S is finite, the subset $U = \bigcap_{s \in S} q(\psi'(s))$ is an open subset of X containing the point x. The map ψ' determines a diagram $\alpha : N(S) \to \mathcal{E} \times_{N(\mathcal{U}(X)^{op})} \{U\} \simeq \mathcal{F}(U)$. To prove the existence of ϕ, it suffices to show that there exists a smaller open subset $V \subseteq U$ containing x such that the composite map $N(S) \xrightarrow{\alpha} \mathcal{F}(U) \to \mathcal{F}(V)$ is nullhomotopic. Since $N(S)$ is finite, it suffices to show α induces a nullhomotopic map from $N(S)$ into the stalk $\mathcal{F}_x = \varinjlim_{x \in V} \mathcal{F}(V)$. We conclude by observing that $\mathcal{F}_x$ is contractible (since $\mathcal{F}$ is assumed to be ∞-connective). $\square$

Proof of Proposition 5.3.1.15. Suppose first that (1) is satisfied; we will verify (2). Let $\chi : \mathcal{U}(X) \to \mathrm{Shv}(X)$ be the functor which carries an open set U to the sheaf χ_U given by the formula

$$
\chi_U(V) = \begin{cases} \Delta^0 & \text{if } V \subseteq U \\ \emptyset & \text{otherwise.} \end{cases}
$$

Let $\mathcal{G} = \varinjlim_{C \in \mathcal{C}} \chi_{f(C)}$. For every point $x \in U$, the stalk $\mathcal{G}_x$ is weakly homotopy equivalent to the nerve of the category $\mathcal{C}_x$, and for $x \notin U$ the stalk $\mathcal{G}_x$ is empty. If each $\mathcal{C}_x$ has weakly contractible nerve, then we conclude that the canonical map $\mathcal{G} \to \chi_U$ is ∞-connective, so that

$$
\mathcal{F}(U) \simeq \mathrm{Map}_{\mathrm{Shv}(X)}(\chi_U, \mathcal{F}) \simeq \mathrm{Map}_{\mathrm{Shv}(X)}(\mathcal{G}, \mathcal{F}) \simeq \varprojlim_{C \in \mathcal{C}} \mathrm{Map}_{\mathrm{Shv}(X)}(\chi_{f(C)}, \mathcal{F}) = \varprojlim_{C \in \mathcal{C}} \mathcal{F}(f(C)).
$$

Now suppose that (2) is satisfied. Let $S \subseteq \mathcal{U}(X)$ be a covering sieve on an open set $U \subseteq X$. Then for each $x \in U$, the partially ordered set $S_x = \{V \in S : x \in V\}$ is nonempty and stable under finite intersections, so that $N(S_x)^{op}$ is filtered and therefore weakly contractible. It follows from (2) that the map $\mathcal{F}(U) \to \varprojlim_{V \in S} \mathcal{F}(V)$ is a homotopy equivalence, so that $\mathcal{F}$ is a sheaf. It remains to show that $\mathcal{F}$ is hypercomplete. Choose an ∞-connective morphism $\alpha : \mathcal{F} \to \mathcal{F}'$, where $\mathcal{F}'$ is hypercomplete; we wish to show that α is an equivalence. The first part of the proof shows that $\mathcal{F}'$ also satisfies the condition stated in (2). Consequently, it will suffice to prove the following:

$(*)$ Let $\alpha : \mathcal{F} \to \mathcal{G}$ be an ∞-connective morphism in $\mathrm{Shv}(X)$, where $\mathcal{F}$ and $\mathcal{G}$ both satisfy (2). Then α is an equivalence.

To prove $(*)$, it suffices to show that for each open set $U \subseteq X$, α induces a homotopy equivalence $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$. We will show that α_U is n-connective for each $n \geq 0$, using induction on n. If $n > 0$, then we can conclude by applying the inductive hypothesis to the diagonal map $\beta : \mathcal{F} \to \mathcal{F} \times_{\mathcal{G}} \mathcal{F}$. It remains to consider the case $n = 0$: that is, to show that the map α_U is surjective on connected components. In other words, we must show that every map $\chi_U \to \mathcal{G}$ factors through α. This follows by applying Lemma 5.3.1.16 to the fiber product $\chi_U \times_{\mathcal{G}} \mathcal{F}$ (and restricting to the open set U). $\square$

5.3.2 Topological Chiral Homology

Let M be a k-manifold and $\mathcal{C}^{\otimes}$ a symmetric monoidal ∞-category. We can think of an $\mathbb{E}_M$-algebra $A \in \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C})$ as a family of $\mathbb{E}_k$-algebras $A_x \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$, parametrized by the points $x \in M$. In this section, we will explain how to extract form A a global invariant $\int_M A$, which we call the *topological chiral homology*

of M (with coefficients in A). Our construction is a homotopy-theoretic analogue of the Beilinson-Drinfeld theory of chiral homology for the chiral algebas of [14]. It is closely related to the notion of *blob homology* studied by Morrison and Walker ([112]).

The basic idea of the construction is simple. According to Theorem 5.2.4.9, we can think of an $\mathbb{E}_M$-algebra object A of a symmetric monoidal ∞-category $\mathcal{C}$ as a functor which assigns to every disjoint union of open disks $U \subseteq M$ an object $A(U) \in \mathcal{C}$, which carries disjoint unions to tensor products. Our goal is to formally extend the definition of A to *all* open subsets of M. Before we can give the definition, we need to establish some terminology.

Definition 5.3.2.1. Let M be a manifold and $\mathcal{U}(M)$ the partially ordered set of all open subsets of M. We can identify objects of the ∞-category $\mathrm{N}(\mathcal{U}(M))^{\amalg}$ with finite sequences $(U_1, \ldots, U_n)$ of open subsets of M. We let $\mathrm{N}(\mathcal{U}(M))^{\otimes}$ denote the subcategory of $\mathrm{N}(\mathcal{U}(M))^{\amalg}$ spanned by those morphisms $(U_1, \ldots, U_n) \to (V_1, \ldots, V_m)$ which cover a map $\alpha : \langle n \rangle \to \langle m \rangle$ in $\mathcal{F}\mathrm{in}_*$ and possess the following property: for $1 \le j \le m$, the sets $\{U_i\}_{\alpha(i)=j}$ are disjoint open subsets of V_j.

For every manifold M, the nerve $\mathrm{N}(\mathcal{U}(M)^{\otimes})$ is an ∞-operad which contains $\mathrm{N}(\mathrm{Disk}(M))^{\otimes}$ as a full subcategory.

Definition 5.3.2.2. We will say that a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is *sifted-complete* if the underlying ∞-category $\mathcal{C}$ admits small sifted colimits and the tensor product functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small sifted colimits.

Remark 5.3.2.3. If a simplicial set K is sifted, then the requirement that the tensor product $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserve sifted colimits is equivalent to the requirement that it preserve sifted colimits separately in each variable.

Example 5.3.2.4. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Assume that the underlying ∞-category $\mathcal{C}$ admits small colimits, and that the tensor product on $\mathcal{C}$ preserves small colimits separately in each variable. Let $\mathcal{O}^{\otimes}$ be an arbitrary small ∞-operad, so that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ inherits a symmetric monoidal structure (given by pointwise tensor product). The ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ itself admits small colimits (Corollary 3.2.3.3), but the tensor product on $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ generally does not preserve colimits in each variable. However, it does preserve *sifted* colimits separately in each variable: this follows from Proposition 3.2.3.1. Consequently, $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is a sifted-complete symmetric monoidal ∞-category.

The main existence result we will need is the following:

Theorem 5.3.2.5. *Let M be a manifold and let $q : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a sifted-complete symmetric monoidal ∞-category. For every algebra object $A \in \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C})$, the restriction $A|\,\mathrm{N}(\mathrm{Disk}(M))^{\otimes}$ admits an operadic left Kan extension to $\mathrm{N}(\mathcal{U}(M)^{\otimes})$.*

Assuming Theorem 5.3.2.5 for the moment, we can give the definition of topological chircal homology.

Definition 5.3.2.6. Let M be a manifold and let $\mathcal{C}^{\otimes}$ be a sifted-complete symmetric monoidal ∞-category. We let $\int : \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{N}(\mathcal{U}(M))}(\mathcal{C})$ be the functor given by restriction to $\mathrm{N}(\mathrm{Disk}(M))^{\otimes}$ followed by operadic left Kan extension along the inclusion $\mathrm{N}(\mathrm{Disk}(M))^{\otimes} \to \mathrm{N}(\mathcal{U}(M)^{\otimes})$. If $A \in \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C})$ and U is an open subset of M, we will denote the value of $\int(A)$ on the open set $U \subseteq M$ by $\int_U A \in \mathcal{C}$. We will refer to $\int_U A$ as the *topological chiral homology of U with coefficients in A*.

Remark 5.3.2.7. To describe the content of Definition 5.3.2.6 more concretely, it is useful to introduce a bit of notation. If M is a manifold, we let $\mathrm{Disj}(M)$ denote the partially ordered subset of $\mathcal{U}(M)$ spanned by those open subsets $U \subseteq M$ which are homeomorphic to $S \times \mathbf{R}^k$ for some finite set S. In the situation of Definition 5.3.2.6, the algebra object A determines a functor $\theta : \mathrm{N}(\mathrm{Disj}(M)) \to \mathcal{C}$, given informally by the formula

$$V_1 \cup \cdots \cup V_n \mapsto A(V_1) \otimes \cdots \otimes A(V_n)$$

(here the V_i denote pairwise disjoint open disks in M). The topological chiral homology $\int_M A \in \mathcal{C}$ is then given by the colimit of the diagram θ.

Example 5.3.2.8. Let $U \subseteq M$ be an open subset homeomorphic to Euclidean space. Then there is a canonical equivalence $A(U) \simeq \int_U A$.

Remark 5.3.2.9. Suppose that we have a map of ∞-operads $\psi : \mathbb{E}_M^{\otimes} \to \mathcal{O}^{\otimes}$, where $\mathcal{O}^{\otimes}$ is some other ∞-operad. Let $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. Then we will abuse notation by denoting the topological chiral homology $\int_M(\psi \circ A)$ simply by $\int_M A$. This abuse is consistent with the notation of Definition 6.3.3.5 in the following sense: if $A \in \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C})$, then the topological chiral homology $\int_U A$ of U with coefficients in A is equivalent to the topological chiral homology $\int_U(A|\mathbb{E}_U^{\otimes})$ of U with coefficients in the induced $\mathbb{E}_U$-algebra.

Example 5.3.2.10. Let $A \in \mathrm{Alg}_{\mathrm{BTop}(k)}(\mathcal{C})$. Then Remark 5.3.2.9 allows us to define the topological chiral homology $\int_M A$ of *any* k-manifold with coefficients in A. Similarly, if $A \in \mathrm{Alg}_{\mathbb{E}_{\mathrm{Sm}}}(\mathcal{C})$ (see Example 5.2.2.18), then $\int_M A$ is defined for any smooth k-manifold M. Many other variations on this theme are possible: roughly speaking, if A is an $\mathbb{E}_k$-algebra object of $\mathcal{C}$ equipped with a compatible action of some group G mapping to $\mathrm{Top}(k)$, then $\int_M A$ is well-defined if we are provided with a reduction of the structure group of M to G.

In order to prove Theorem 5.3.2.5 (and to establish the basic formal properties of topological chiral homology), we need to have good control over colimits indexed by partially ordered sets of the form $\mathrm{Disj}(M)$, where M is a manifold (see Remark 5.3.2.7). We will obtain this control by introducing a less rigid version of the ∞-category $\mathrm{N}(\mathrm{Disj}(M))$, where we allow open disks in M to "move".

Definition 5.3.2.11. Fix an integer $k \geq 0$. We let $\mathbf{Man}(k)$ denote the topological category whose objects are k-manifolds, with morphism spaces given by $\mathrm{Map}_{\mathbf{Man}(k)}(N, M) = \mathrm{Emb}(N, M)$. If M is a k-manifold, we let $\mathbf{D}(M)$ denote the full subcategory of the ∞-category $\mathrm{N}(\mathbf{Man}(k))_{/M}$ spanned by those objects of the form $j : N \to M$, where N is homeomorphic to $S \times \mathbf{R}^k$ for some finite set S.

Remark 5.3.2.12. An object of the ∞-category $\mathbf{D}(M)$ can be identified with a finite collection of open embeddings $\{\psi_i : \mathbf{R}^k \hookrightarrow M\}_{1 \leq i \leq n}$ having disjoint images. Up to equivalence, this object depends only on the sequence of images $(\psi_1(\mathbf{R}^k), \ldots, \psi_n(\mathbf{R}^k))$, which we can identify with an object of the category $\mathrm{Disj}(M)$. However, the morphisms in these two categories are somewhat different: a morphism in $\mathbf{D}(M)$ is given by a diagram

$$\coprod_{1 \leq i \leq m} \mathbf{R}^k \xrightarrow{\hspace{3cm}} \coprod_{1 \leq j \leq n} \mathbf{R}^k$$
$$\{\phi_i\} \searrow \qquad \swarrow \{\psi_j\}$$
$$M$$

which commutes up to (specified) isotopy, which does not guarantee an inclusion of images $\bigcup \phi_i(\mathbf{R}^k) \subseteq \bigcup \psi_j(\mathbf{R}^k)$. Nevertheless, there is an evident functor $\gamma : \mathrm{N}(\mathrm{Disj}(M)) \to \mathbf{D}(M)$, defined by choosing a parametrization of each open disk in M (up to equivalence, the functor γ is independent of these choices).

The fundamental result we will need is the following:

Proposition 5.3.2.13. *Let M be a k-manifold. Then:*

(1) *The functor $\gamma : \mathrm{N}(\mathrm{Disj}(M)) \to \mathbf{D}(M)$, described in Remark 5.3.2.12, is left cofinal.*

(2) *Let $\mathrm{Disj}(M)_{\mathrm{nu}}$ denote the subcategory of $\mathrm{Disj}(M)$ whose objects are nonempty open sets $U \in \mathrm{Disj}(M)$ and whose morphisms are inclusions $U \hookrightarrow V$ such that the induced map $\pi_0 U \to \pi_0 V$ is surjective. If M is connected, then the induced functor $\mathrm{N}(\mathrm{Disj}(M)^{\mathrm{nu}}) \to \mathbf{D}(M)$ is left cofinal.*

The second assertion of Proposition 5.3.2.13 will require the following technical result:

Lemma 5.3.2.14. *Let M be a connected manifold, let S be a finite subset of M, and let $\mathrm{Disj}(M)_S^{\mathrm{nu}}$ denote the full subcategory of $\mathrm{Disj}(M)^{\mathrm{nu}}$ spanned by those objects $V \in \mathrm{Disj}(M)^{\mathrm{nu}}$ such that $S \subseteq V$. Then the simplicial set $\mathrm{N}(\mathrm{Disj}(M)^{\mathrm{nu}})$ is weakly contractible.*

Proof. For every object $V \in \mathrm{Disj}(M)^{\mathrm{nu}}_S$, let $\psi(V)$ denote the subset of $\mathrm{Ran}(M)$ consisting of those subsets T with the following properties:

(i) We have inclusions $S \subseteq T \subseteq V$.

(ii) The map $T \to \pi_0 V$ is surjective.

For every point $T \in \mathrm{Ran}(M)_S$, let $\mathcal{C}_T$ denote the full subcategory of $\mathrm{Disj}(M)^{\mathrm{nu}}_S$ spanned by those objects V such that $T \in \psi(V)$. Each of the category $\mathcal{C}_T^{op}$ is filtered (for every finite collection $V_1, \ldots, V_n \in \mathcal{C}_T$, we can choose $V \in \mathcal{C}_T$ such that $V \subseteq \bigcap V_i$ and each of the maps $\pi_0 V \to \pi_0 V_i$ is surjective: namely, take V to be a union of sufficiently small open disks containing the points of T). It follows from Theorem A.3.1 that the Kan complex $\mathrm{Sing}\,\mathrm{Ran}(M)_S$ is equivalent to the homotopy colimit of the diagram $\{\psi(V)\}_{V \in \mathrm{Disk}(M)^{\mathrm{nu}}_S}$. For each $V \in \mathrm{Disj}(M)^{\mathrm{nu}}_S$, write V as a disjoint union of open disks $U_1 \cup \ldots \cup U_m$. Then $\psi(V)$ is homeomorphic to a product $\prod_{1 \leq i \leq m} \mathrm{Ran}(U_m)_{S \cap U_m}$, and is therefore weakly contractible by Lemmas 5.3.1.8 and 5.3.1.6. It follows that the Kan complex $\mathrm{Sing}(\mathrm{Ran}(M)_S)$ is weakly homotopy equivalent to the nerve of the category $\mathrm{Disj}(M)^{\mathrm{nu}}$. The desired result now follows from the weak contractiblity of $\mathrm{Sing}\,\mathrm{Ran}(M)_S$ (Lemmas 5.3.1.8 and 5.3.1.6). $\qquad\square$

Proof of Proposition 5.3.2.13. We first give the proof of (1). Let $S = \{1, \ldots, n\}$, let $U = S \times \mathbf{R}^k$, and let $\psi : U \to M$ be an open embedding corresponding to an object of $\mathbf{D}(M)$. According to Theorem T.4.1.3.1, it will suffice to show that the ∞-category $\mathcal{C} = \mathrm{N}(\mathrm{Disj}(M)) \times_{\mathbf{D}(M)} \mathbf{D}(M)_{\psi/}$ is weakly contractible. We observe that the projection map $\mathcal{C} \to \mathrm{N}(\mathrm{Disj}(M))$ is a left fibration, associated to a functor $\chi : \mathrm{N}(\mathrm{Disj}(M)) \to \mathcal{S}$ which carries each object $V \in \mathrm{Disj}(M)$ to the homotopy fiber of the map of Kan complexes $\mathrm{Sing}\,\mathrm{Emb}(U, V) \to \mathrm{Sing}\,\mathrm{Emb}(U, M)$. According to Proposition T.3.3.4.5, it will suffice to show that the colimit $\varinjlim(\chi)$ is contractible. Since colimits in $\mathcal{S}$ are universal, it will suffice to show that $\mathrm{Sing}\,\mathrm{Emb}(U, M)$ is a colimit of the diagram $\{\mathrm{Sing}\,\mathrm{Emb}(U, V)\}_{V \in \mathrm{Disj}(M)}$. Using Theorem T.6.1.3.9 and Remark 5.2.1.11, we are reduced to showing that $\mathrm{Sing}\,\mathrm{Conf}(S, M)$ is a colimit of the diagram $\{\mathrm{Sing}\,\mathrm{Conf}(S, V)\}_{V \in \mathrm{Disj}(M)}$. According to Theorem A.3.1, it will suffice to show that for every injective map $j : S \hookrightarrow M$, the partially ordered set $\mathrm{Disj}(M)_S = \{V \in \mathrm{Disj}(M) : j(S) \subseteq V\}$ has weakly contractible nerve. This is clear, since $\mathrm{Disj}(M)^{op}_S$ is filtered: every open neighborhood of $j(S)$ contains a union of sufficiently small open disks around the points $\{j(s)\}_{s \in S}$.

The proof of (2) is identical except for the last step: we must instead show that that for every injective map $j : S \hookrightarrow M$, the category $\mathrm{Disj}(M)^{\mathrm{nu}}_S = \{V \in \mathrm{Disj}(M)^{\mathrm{nu}} : j(S) \subseteq V\}$ has weakly contractible nerve, which follows from Lemma 5.3.2.14. $\qquad\square$

The advantage of the ∞-category $\mathbf{D}(M)$ over the more rigid ∞-category $\mathrm{N}(\mathrm{Disj}(M))$ is summarized in the following result:

Proposition 5.3.2.15. *For every manifold M, the ∞-category $\mathbf{D}(M)$ is sifted.*

For later use, it will be convenient to prove a slightly more general form of Proposition 5.3.2.15. Let $\pi : \widetilde{M} \to M$ be a covering map between manifolds whose fibers are finite. Since any finite covering of a disk is homeomorphic to a disjoint union of disks, the construction $U \mapsto \pi^{-1}U$ determines a functor $\mathbf{D}(M) \to \mathbf{D}(\widetilde{M})$. When $\widetilde{M} = M \coprod M$, this can be identified with the diagonal map $\mathbf{D}(M) \to \mathbf{D}(M) \times \mathbf{D}(M)$. Proposition 5.3.2.15 is therefore an immediate consequence of the following:

Proposition 5.3.2.16. *Let $\pi : \widetilde{M} \to M$ be a covering map between manifolds which has finite fibers. Then the induced map $\pi^{-1} : \mathbf{D}(M) \to \mathbf{D}(\widetilde{M})$ is left cofinal.*

Proof. We have a commutative diagram

$$
\begin{array}{ccc}
 & \mathrm{N}(\mathrm{Disj}(M)) & \\
 {}^{\gamma}\swarrow & & \searrow^{\theta} \\
\mathbf{D}(M) \xrightarrow{\quad\quad \pi^{-1} \quad\quad} & & \mathbf{D}(\widetilde{M}),
\end{array}
$$

where γ is left cofinal by virtue of Proposition 5.3.2.13. It will therefore suffice to show that θ is left cofinal (Proposition T.4.1.1.3). Fix an object $\phi : U \to \widetilde{M}$ of $\mathbf{D}(\widetilde{M})$. According to Theorem T.4.1.3.1, it will suffice to show that the ∞-category $\mathcal{C} = \mathbf{D}(\widetilde{M})_{\phi/} \times_{\mathbf{D}(\widetilde{M})} \mathrm{N}(\mathrm{Disj}(M))$ is weakly contractible. There is an evident left fibration $\mathcal{C} \to \mathrm{N}(\mathrm{Disj}(M))$, classified by a functor $\chi : \mathrm{N}(\mathrm{Disj}(M)) \to \mathcal{S}$ which carries an object $V \in \mathrm{Disj}(M)$ to the homotopy fiber of the map

$$\mathrm{Sing}(\mathrm{Emb}(U, \pi^{-1}V)) \to \mathrm{Sing}(\mathrm{Emb}(U, \widetilde{M}))$$

over the vertex given by (ϕ, ψ). Using Proposition T.3.3.4.5, we can identify the weak homotopy type of $\mathcal{C}$ with the colimit $\varinjlim(\chi) \in \mathcal{S}$. Consequently, it will suffice to show that $\varinjlim(\chi)$ is contractible. Since colimits in $\mathcal{S}$ are universal, it will suffice to show that $\mathrm{Sing}(\mathrm{Emb}(U, \widetilde{M}))$ is a colimit of the diagram $\chi' : \mathrm{N}(\mathrm{Disj}(M)) \to \mathcal{S}$ given by the formula $\chi'(V) = \mathrm{Sing}(\mathrm{Emb}(U, \pi^{-1}V))$.

Let $S \subseteq U$ be a set which contains on point from each connected component of U and let $\chi'' : \mathrm{N}(\mathrm{Disj}(M)) \to \mathcal{S}$ be the functor given by the formula $V \mapsto \mathrm{Sing}(\mathrm{Conf}(S, \pi^{-1}V))$. There is an evident restriction functor of diagrams $\chi' \to \chi''$. Using Remark 5.2.1.11 and Theorem T.6.1.3.9, we are reduced to proving that the canonical map $\varinjlim(\chi'') \to \mathrm{Sing}(\mathrm{Conf}(S, \widetilde{M}))$ is a homotopy equivalence. In view of Theorem A.3.1, it will suffice to show that for every point $j \in \mathrm{Conf}(S, \widetilde{M})$ the full subcategory $\mathrm{Disj}(M)_j$ of $\mathrm{Disj}(M)$ spanned by those objects $V \in \mathrm{Disj}(M)$ such that $j(S) \subseteq \pi^{-1}V$ is weakly contractible. This is clear, since $\mathrm{Disj}(M)_j^{op}$ is filtered. $\qquad\square$

Armed with Proposition 5.3.2.15, we are ready to prove that topological chiral homology is well-defined.

Proof of Theorem 5.3.2.5. According to Theorem 3.1.2.3, it will suffice to show that for each open set $U \subseteq M$ the induced diagram

$$\mathrm{N}(\mathrm{Disj}(U)) \xrightarrow{\theta} \mathbf{D}(U) \xrightarrow{\beta} \mathbb{E}_M \xrightarrow{A} \mathcal{C}^{\otimes}$$

can be extended to an operadic colimit diagram in $\mathcal{C}^{\otimes}$. Since θ is left cofinal (Proposition 5.3.2.13), it suffices to show that $A \circ \beta$ can be extended to an operadic colimit diagram in $\mathcal{C}^{\otimes}$. Choose a q-coCartesian natural transformation from $A \circ \beta$ to a functor $\chi : \mathbf{D}(()U) \to \mathcal{C}$, given informally by the formula $\chi(\{\psi_i : V_i \hookrightarrow U\}_{1 \leq i \leq n}) = A(\psi_1) \otimes \cdots \otimes A(\psi_n)$. In view of Proposition 3.1.1.15, it will suffice to show that χ can be extended to an operadic colimit diagram in $\mathcal{C}$. Since $\mathbf{D}(U)$ is sifted (Proposition 5.3.2.15) and the tensor product on $\mathcal{C}$ preserves sifted colimits separately in each variable, it suffices to show that χ can be extended to a colimit diagram in $\mathcal{C}$ (Proposition 3.1.1.16). This colimit exists because $\mathcal{C}$ admits sifted colimits and $\mathbf{D}(U)$ is sifted. $\qquad\square$

We close this section with the following result concerning the functorial behavior of topological chiral homology:

Proposition 5.3.2.17. *Let M be a manifold, and let $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a symmetric monoidal functor. Assume $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are sifted-complete and that the underlying functor $F : \mathcal{C} \to \mathcal{D}$ preserves sifted colimits. Then:*

(1) *If $A \in \mathrm{Alg}_{\mathrm{N}(\mathcal{U}(M))}(\mathcal{C})$ has the property that $A_0 = A| \mathrm{N}(\mathrm{Disk}(M))^{\otimes}$ is locally constant and A is an operadic left Kan extension of A_0, then FA is an operadic left Kan extension of FA_0.*

(2) *For any locally constant algebra $A \in \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C})$, the canonical map $\int_M FA \to F(\int_M A)$ is an equivalence in $\mathcal{C}$.*

Proof. We first prove (1). Since A_0 is locally constant, we can assume that A_0 factors as a composition $\mathrm{N}(\mathrm{Disk}(M))^{\otimes} \to \mathbb{E}_M^{\otimes} \xrightarrow{A_0'} \mathcal{C}^{\otimes}$ (Theorem 5.2.4.9). We wish to prove that for every object $U \in \mathcal{U}(M)$, the diagram FA exhibits $FA(U) \in \mathcal{D}$ as an operadic colimit of the composite diagram

$$\mathrm{Disj}(M) \xrightarrow{\alpha} \mathbf{D}(M) \xrightarrow{\beta} \mathbb{E}_M^{\otimes} \xrightarrow{A_0'} \mathcal{C}^{\otimes} \xrightarrow{F} \mathcal{D}^{\otimes} .$$

Since α is left cofinal (Proposition 5.3.2.13), it will suffice to show that FA exhibits $FA(U)$ as an operadic colimit of $F \circ A'_0 \circ \beta$.

Let $p : \mathcal{C}^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ exhibit $\mathcal{C}^\otimes$ as a symmetric monoidal ∞-category, and let $q : \mathcal{D}^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ exhibit $\mathcal{D}^\otimes$ as a symmetric monoidal ∞-category. Choose a p-coCartesian natural transformation α from $A'_0 \circ \beta$ to a map $\phi : \mathbf{D}(M) \to \mathcal{C}$. Since F is a symmetric monoidal functor, $F(\alpha)$ is a q-coCartesian natural transformation from $F \circ A'_0 \circ \beta$ to $F \circ \phi$. It will therefore suffice to show that FA exhibits $FA(U)$ as a colimit of the diagram $F \circ \phi$ in the ∞-category $\mathcal{D}$ (Propositions 3.1.1.15 and 3.1.1.16). Since $F|\mathcal{C}$ preserves sifted colimits and the ∞-category $\mathbf{D}(M)$ is sifted (Proposition 5.3.2.15), it suffices to show that $A(U)$ is a colimit of the diagram ϕ. Using Propositions 3.1.1.15 and 3.1.1.16 again, we are reduced to proving that $A(U)$ is an operadic colimit of the diagram $A'_0 \circ \beta$, which (since α is left cofinal) follows from our assumption that A is an operadic left Kan extension of A_0. This completes the proof of (1). Assertion (2) is an immediate consequence. $\square$

5.3.3   Properties of Topological Chiral Homology

Our goal in this section is to establish four basic facts about the theory of topological chiral homology. In what follows, we will assume that $\mathcal{C}^\otimes$ is a sifted-complete symmetric monoidal ∞-category and M a topological manifold of dimension k.

(1) For a fixed algebra $A \in \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C})$, the construction $U \mapsto \int_U A$ carries disjoint unions of open subsets of M to tensor products in the ∞-category $\mathcal{C}$ (Theorem 5.3.3.1).

(2) For a fixed open set $U \subseteq M$, the construction $A \mapsto \int_U A$ carries tensor products of $\mathbb{E}_M$-algebra objects of $\mathcal{C}$ to tensor products in $\mathcal{C}$ (Theorem 5.3.3.2).

(3) If $A \in \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C})$ arises from a family $\{A_x\}_{x \in M}$ of *commutative* algebra objects of $\mathcal{C}$, then $\int_U A$ can be identified with image in $\mathcal{C}$ of the colimit $\varinjlim_{x \in U}(A_x) \in \mathrm{CAlg}(\mathcal{C})$ (Theorem 5.3.3.8).

(4) If $k = 1$ and M is the circle S^1, then we can view an algebra object $A \in \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C})$ as an associative algebra object of $\mathcal{C}$ (equipped with an automorphism θ given by monodromy around the circle). In this case, the topological chiral homology $\int_M A$ can be identified with the (θ-twisted) Hochschild homology of A, which is computed by an analogue of the usual cyclic bar complex (Theorem 5.3.3.11).

We begin with assertion (1). The functor $\int$ of Definition 5.3.2.6 carries $\mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C})$ into $\mathrm{Alg}_{\mathrm{N}(\mathcal{U}(M))}(\mathcal{C})$. Consequently, whenever $U_1, \ldots, U_m$ are disjoint open subsets of $U \subseteq M$, we have a multiplication map

$$\int_{U_1} A \otimes \cdots \otimes \int_{U_m} A \to \int_U A.$$

Theorem 5.3.3.1. *Let M be a manifold and $\mathcal{C}^\otimes$ a sifted-complete symmetric monoidal ∞-category. Then for every object $A \in \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C})$ and every collection of pairwise disjoint open subsets $U_1, \ldots, U_m \subseteq M$, the map*

$$\int_{U_1} A \otimes \cdots \otimes \int_{U_m} A \to \int_{\bigcup U_i} A$$

is an equivalence in $\mathcal{C}$.

Proof. It follows from Proposition 5.3.2.13 that for each open set $U \subseteq M$, the topological chiral homology $\int_U A$ is the colimit of a diagram $\psi_U : \mathbf{D}(()U) \to \mathcal{C}$ given informally by the formula $\psi_U(V_1 \cup \ldots \cup V_n) = A(V_1) \otimes \cdots \otimes A(V_n)$. Since each $\mathbf{D}(U_i)$ is sifted (Proposition 5.3.2.15) and the tensor product on $\mathcal{C}$ preserves sifted colimits separately in each variable, we can identify the tensor product $\int_{U_1} A \otimes \cdots \otimes \int_{U_m} A$ with the colimit $\varinjlim_{\mathbf{D}(()U_1) \times \ldots \times \mathbf{D}(()U_n)}(\psi_{U_1} \otimes \cdots \otimes \psi_{U_m})$. Let $W = \bigcup U_i$. The tensor product functor $\psi_{U_1} \otimes \cdots \otimes \psi_{U_m}$ can be identified with the pullback of ψ_W along the evident map

$$\alpha : \mathbf{D}(U_1) \times \cdots \times \mathbf{D}(U_m) \to \mathbf{D}(W)$$

$$(V_1 \subseteq U_1, \ldots, V_m \subseteq U_m) \mapsto V_1 \cup \ldots \cup V_m.$$

Consequently, we are reduced to proving that the α induces an equivalence

$$\varinjlim(\alpha \circ \psi_W) \to \varinjlim \psi_W.$$

It will suffice to show that α is left cofinal. This follows by applying Proposition T.4.1.1.3 to the commutative diagram

$$
\begin{array}{ccc}
\mathrm{N}(\mathrm{Disj}(U_1) \times \ldots \times \mathrm{Disj}(U_n)) & \longrightarrow & \mathbf{D}(U_1) \times \ldots \times \mathbf{D}(U_m) \\
\downarrow & & \downarrow{\scriptstyle \alpha} \\
\mathrm{N}(\mathrm{Disj}(W)) & \longrightarrow & \mathbf{D}(W);
\end{array}
$$

note that the horizontal maps are left cofinal by Proposition 5.3.2.13, and the map β is an isomorphism of simplicial sets. $\qquad\square$

To formulate assertion (2) more precisely, suppose we are given a pair of algebras $A, B \in \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C})$. Let $\int(A), \int(B) \in \mathrm{Alg}_{\mathrm{N}(\mathcal{U}(M))}(\mathcal{C})$ be given by operadic left Kan extension. Then $(\int(A) \otimes \int(B)) | \mathrm{N}(\mathrm{Disk}(M))^{\otimes}$ is an extension of $(A \otimes B) | \mathrm{N}(\mathrm{Disk}(M))^{\otimes}$, so we have a canonical map $\int(A \otimes B) \to \int(A) \otimes \int(B)$. We then have the following:

Theorem 5.3.3.2. *Let M be a manifold and $\mathcal{C}^{\otimes}$ a sifted-complete symmetric monoidal ∞-category. Then for every pair of locally constant algebras $A, B \in \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C})$, the canonical map $\theta : \int_M (A \otimes B) \to \int_M A \otimes \int_M B$ is an equivalence in $\mathcal{C}$.*

We will deduce Theorem 5.3.3.2 from a more general result for covering spaces.

Construction 5.3.3.3. Let M be a k-manifold and let $\pi : \widetilde{M} \to M$ be a covering map with finite fibers, so that we have ∞-operads $p : \mathbb{E}_M^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and $\widetilde{p} : \mathbb{E}_{\widetilde{M}}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. For every finite set S equipped with an embedding $j : \mathbf{R}^k \times S \hookrightarrow M$, the inverse image $(\mathbf{R}^k \times S) \times_M \widetilde{M}$ has the form $\mathbf{R}^k \times \widetilde{S}$, for some finite covering $\widetilde{S}$ of S (since the space $\mathbf{R}^k$ is simply connected). Moreover, there is an evident map $\widetilde{j} : \mathbf{R}^k \times \widetilde{S} \to \widetilde{M}$. The construction $(S_*, j) \mapsto (\widetilde{S}_*, \widetilde{j})$ determines a functor $U : \mathbb{E}_M^{\otimes} \to \mathbb{E}_{\widetilde{M}}^{\otimes}$. The evident projections $\widetilde{S} \to S$ determine a natural transformation $\alpha : \widetilde{p} \circ U \to p$, which we can view as a map $\mathbb{E}_M^{\otimes} \times \Delta^1 \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

Let $q : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a symmetric monoidal ∞-category. Composition with U determines a functor.

$$\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathbb{E}_{\widetilde{M}}^{\otimes}, \mathcal{C}^{\otimes}) \overset{\circ U}{\to} \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathbb{E}_M^{\otimes} \times \{0\}, \mathcal{C}^{\otimes})$$

Since q is a coCartesian transformation, the natural transformation α it determines a functor

$$\alpha_! : \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathbb{E}_M^{\otimes} \times \{0\}, \mathcal{C}^{\otimes}) \to \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathbb{E}_M^{\otimes} \times \{1\}, \mathcal{C}^{\otimes}).$$

Composing these functors and restricting to $\mathrm{Alg}_{\mathbb{E}_{\widetilde{M}}}(\mathcal{C}) \subseteq \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathbb{E}_{\widetilde{M}}^{\otimes}, \mathcal{C}^{\otimes})$, we obtain a functor

$$\pi_* : \mathrm{Alg}_{\mathbb{E}_{\widetilde{M}}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C}).$$

Remark 5.3.3.4. Let $\pi : \widetilde{M} \to M$ and $\mathcal{C}^{\otimes}$ be as in Construction 5.3.3.3. We can informally think of an object $A \in \mathrm{Alg}_{\mathbb{E}_{\widetilde{M}}}(\mathcal{C})$ as a family of $\mathbb{E}_k$-algebras $A_y \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$, indexed by the points of $\widetilde{M}$. In terms of this description, we can identify $\pi_* A \in \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C})$ with the family given by

$$(\pi_* A)_x = \bigotimes_{\pi(y)=x} A_y.$$

Example 5.3.3.5. Let M be a manifold, let S be a finite set, and let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Let $\pi : M \times S \to M$ denote the projection map, and let $A \in \mathrm{Alg}_{\mathbb{E}_{M \times S}}(\mathcal{C})$. Then $\pi_* A \simeq \bigotimes_{s \in S} A_s$, where $\{A_s\}_{s \in S}$ denotes the image of A under the equivalence

$$\mathrm{Alg}_{\mathbb{E}_{M \times S}}(\mathcal{C}) \simeq \prod_{s \in S} \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C})$$

of Example 5.2.4.4.

Let $\pi : \widetilde{M} \to M$ be as in Construction 5.3.3.3. The construction $U \mapsto \pi^{-1}U$ determines a functors $\mathbf{D}(M) \to \mathbf{D}(\widetilde{M})$ and $\mathrm{Disj}(M)) \to \mathrm{Disj}(\widetilde{M})$. For any algebra object $A \in \mathrm{Alg}_{\mathbb{E}_{\widetilde{M}}}(\mathcal{C})$, the composite functors

$$\mathrm{Disj}(M) \to (\mathbb{E}_M^{\otimes})^{\mathrm{act}} \overset{\pi_* A}{\to} (\mathcal{C}^{\otimes})^{\mathrm{act}} \to \mathcal{C}$$

$$\mathrm{Disj}(M) \overset{\pi^{-1}}{\to} \mathrm{Disj}(\widetilde{M}) \to (\mathbb{E}_{\widetilde{M}}^{\otimes})^{\mathrm{act}} \overset{A}{\to} (\mathcal{C}^{\otimes})^{\mathrm{act}} \to \mathcal{C}$$

are homotopic to one another. We therefore obtain a canonical map $\int_M (\pi_* A) \to \int_{\widetilde{M}} A$ (provided that both sides are defined).

Theorem 5.3.3.6. *Let $\pi : \widetilde{M} \to M$ be a covering map between manifolds which has finite fibers, and let $\mathcal{C}^{\otimes}$ be a sifted-complete symmetric monoidal ∞-category. For any algebra object $A \in \mathrm{Alg}_{\mathbb{E}_{\widetilde{M}}}(\mathcal{C})$, the canonical map*

$$\int_M (\pi_* A) \to \int_{\widetilde{M}} A$$

is an equivalence in $\mathcal{C}$.

Proof. Proposition 5.3.2.13 allows us to identify $\int_M (\pi_* A)$ with the colimit of a diagram $\phi : \mathbf{D}(M) \to (\mathbb{E}_M^{\otimes})^{\mathrm{act}} \overset{\pi_* A}{\to} (\mathcal{C}^{\otimes})^{\mathrm{act}} \to \mathcal{C}$ and $\int_{\widetilde{M}} A$ with the colimit of a diagram $\psi : \mathbf{D}(\widetilde{M}) \to (\mathbb{E}_{\widetilde{M}}^{\otimes})^{\mathrm{act}} \overset{A}{\to} (\mathcal{C}^{\otimes})^{\mathrm{act}} \to \mathcal{C}$. The desired result now follows from the observation that $\pi^{-1} : \mathbf{D}(M) \to \mathbf{D}(\widetilde{M})$ is left cofinal (Proposition 5.3.2.16). $\square$

Proof of Theorem 5.3.3.2. Let $A, B \in \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C})$. We may assume without loss of generality that A and B are given by the restriction of an algebra $C \in \mathrm{Alg}_{\mathbb{E}_{M \coprod M}}(\mathcal{C})$ (Example 5.2.4.4). We have a commutative diagram

$$\int_M (A \otimes B) \overset{\theta}{\longrightarrow} (\int_M A) \otimes (\int_M B)$$

with θ'' from $\int_M(A \otimes B)$ and θ' to $(\int_M A)\otimes(\int_M B)$, meeting at $\int_{M \coprod M} C$.

The map θ' is an equivalence by Theorem 5.3.3.1, and the map θ'' is an equivalence by Theorem 5.3.3.6 (see Example 5.3.3.5). It follows that θ is an equivalence, as desired. $\square$

The proof of assertion (3) is based on the following simple observation:

Lemma 5.3.3.7. *Let M be a manifold and $\mathcal{C}$ an ∞-category which admits small colimits. Regard $\mathcal{C}$ as endowed with the coCartesian symmetric monoidal structure (see §2.4.3). Then, for every object $A \in \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C})$, the functor $\int A$ exhibits the topological chiral homology $\int_M A$ as the colimit of the diagram $A | \mathrm{N}(\mathrm{Disk}(M)) : \mathrm{N}(\mathrm{Disk}(M)) \to \mathcal{C}$.*

Proof. Let $\chi : \mathrm{N}(\mathrm{Disj}(M)) \to \mathcal{C}$ be the functor given informally by the formula $\chi(U_1 \cup \ldots \cup U_n) = A(U_1) \coprod \cdots \coprod A(U_n)$, where the U_i are disjoint open disks in M. We observe that χ is a left Kan extension of $\chi | \mathrm{N}(\mathrm{Disk}(M))$, so that $\int_M A \simeq \mathrm{colim} \chi \simeq \mathrm{colim}(\chi | \mathrm{N}(\mathrm{Disk}(M)))$ (see Lemma T.4.3.2.7). $\square$

Theorem 5.3.3.8. *Let M be a manifold and $\mathcal{C}^{\otimes}$ a sifted-complete symmetric monoidal ∞-category. Regard the Kan complex B_M as the underlying ∞-category of the ∞-operad B_M^{II}, and let $A \in \mathrm{Alg}_{B_M}(\mathcal{C})$ so that $\int_M A$ is well-defined (see Remark 5.3.2.9). Composing A with the diagonal map $B_M \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to B_M^{\mathrm{II}}$, we obtain a functor $\psi : B_M \to \mathrm{CAlg}(\mathcal{C})$. Let $A' = \mathrm{colim}(\psi) \in \mathrm{CAlg}(\mathcal{C})$. Then there is a canonical equivalence $\int_M A \simeq A'(\langle 1 \rangle)$ in the ∞-category $\mathcal{C}$.*

Remark 5.3.3.9. Let A be as in the statement of Theorem 5.3.3.8. It follows from Theorem 2.4.3.18 that A is determined by the functor ψ, up to canonical equivalence. In other words, we may identify $A \in \mathrm{Alg}_{B_M}(\mathcal{C})$ with a family of commutative algebra objects of $\mathcal{C}$ parametrized by the Kan complex B_M (which is homotopy equivalent to $\mathrm{Sing}(M)$, by virtue of Remark 5.2.4.2). Theorem 5.3.3.8 asserts that in this case, the colimit of this family of commutative algebras is computed by the formalism of topological chiral homology.

Proof of Theorem 5.3.3.8. Let $\phi : \mathrm{Disk}(M)^{\otimes} \times \mathcal{F}\mathrm{in}_* \to \mathrm{Disk}(M)^{\otimes}$ be the functor given by the construction

$$((U_1, \ldots, U_m), \langle n \rangle) \mapsto (U'_1, \ldots, U'_{mn}),$$

where $U'_{mi+j} = U_j$. Composing ϕ with the map $\mathrm{N}(\mathrm{Disk}(M))^{\otimes} \to B_M^{\mathrm{II}} \xrightarrow{A} \mathcal{C}^{\otimes}$, we obtain a locally constant algebra object $\overline{A} \in \mathrm{Alg}_{\mathrm{N}(\mathrm{Disk}(M))}(\mathrm{CAlg}(\mathcal{C}))$, where $\mathrm{CAlg}(\mathcal{C})$ is endowed with the symmetric monoidal structure given by pointwise tensor product (see Example 3.2.4.4). Since the symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{C})$ is coCartesian (Proposition 3.2.4.7), the colimit $\varinjlim(\psi)$ can be identified with the topological chiral homology $\int_M \overline{A} \in \mathrm{CAlg}(\mathcal{C})$. Let $\theta : \mathrm{CAlg}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$ denote the forgetful functor. We wish to prove the existence of a canonical equivalence $\theta(\int_M \overline{A}) \simeq \int_M \theta(\overline{A})$. In view of Proposition 5.3.2.17, it suffices to observe that θ is a symmetric monoidal functor and that the underlying functor $\mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ preserves sifted colimits (Proposition 3.2.3.1). $\square$

If M is an arbitrary k-manifold, we can view an $\mathbb{E}_M$-algebra object of a symmetric monoidal ∞-category $\mathcal{C}$ as a family of $\mathbb{E}_k$-algebras $\{A_x\}_{x \in M}$ parametrized by the points of M. In general, this family is "twisted" by the tangent bundle of M. In the special case where $M = S^1$, the tangent bundle T_M is trivial, so we can think of an $\mathbb{E}_M$-algebra as a family of associative algebras parametrized by the circle: that is, as an associative algebra A equipped with an automorphism σ (given by monodromy around the circle). Our final goal in this section is to show that in this case, the topological chiral homology $\int_{S^1} A$ coincides with the Hochschild homology of the A-bimodule corresponding to σ.

Fix an object of $\mathbf{D}(S^1)$ corresponding to a single disk $\psi : \mathbf{R} \hookrightarrow S^1$. An object of $\mathbf{D}(S^1)_{\psi/}$ is given by a diagram

which commutes up to isotopy, where U is a finite union of disks. The set of components $\pi_0(S^1 - \psi'(U))$ is finite (equal to the number of components of U). Fix an orientation of the circle. We define a linear ordering $\leq$ on $\pi_0(S^1 - \psi'(U))$ as follows: if $x, y \in S^1$ belong to different components of $S^1 - \psi'(U)$, then we write $x < y$ if the three points $(x, y, \psi'(j(0)))$ are arranged in a clockwise order around the circle, and $y < x$ otherwise. This construction determines a functor from $\mathbf{D}(S^1)_{\psi/}$ to (the nerve of) the category of nonempty finite linearly ordered sets, which is equivalent to $\mathbf{\Delta}^{op}$. A simple calculation yields the following:

Lemma 5.3.3.10. *Let $M = S^1$, and let $\psi : \mathbf{R} \hookrightarrow S^1$ be any open embedding. Then the above construction determines an equivalence of ∞-categories $\theta : \mathbf{D}(M)_{\psi/} \to \mathrm{N}(\mathbf{\Delta}^{op})$.*

We can now formulate the relationship between Hochschild homology and topological chiral homology precisely as follows:

Theorem 5.3.3.11. *Let $q : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a sifted-complete symmetric monoidal category. Let $A \in \mathrm{Alg}_{\mathbb{E}_{S^1}}(\mathcal{C})$ be an algebra determining a diagram $\chi : \mathbf{D}(S^1) \to \mathcal{C}$ whose colimit is $\int_{S^1} A$. Choose an open embedding $\psi : \mathbf{R} \hookrightarrow S^1$. Then the restriction $\chi|\mathbf{D}(S^1)_{\psi/}$ is equivalent to a composition*

$$\mathbf{D}(S^1)_{\psi/} \xrightarrow{\theta} \mathrm{N}(\boldsymbol{\Delta}^{op}) \xrightarrow{B_\bullet} \mathcal{C},$$

where θ is the equivalence of Lemma 5.3.3.10 and $B_\bullet$ is a simplicial object of $\mathcal{C}$. Moreover, there is a canonical equivalence $\int_{S^1} A \simeq |B_\bullet|$.

Lemma 5.3.3.12. *Let $\mathcal{C}$ be a nonempty ∞-category. Then $\mathcal{C}$ is sifted if and only if, for each object $C \in \mathcal{C}$, the projection map $\theta_C : \mathcal{C}_{C/} \to \mathcal{C}$ is left cofinal.*

Proof. According to Theorem T.4.1.3.1, the projection map θ_C is left cofinal if and only if, for every object $D \in \mathcal{C}$, the ∞-category $\mathcal{C}_{C/} \times_{\mathcal{C}} \mathcal{C}_{D/}$ is weakly contractible. Using the evident isomorphism $\mathcal{C}_{C/} \times_{\mathcal{C}} \mathcal{C}_{D/} \simeq \mathcal{C} \times_{(\mathcal{C} \times \mathcal{C})} (\mathcal{C} \times \mathcal{C})_{(C,D)/}$, we see that this is equivalent to the left cofinality of the diagonal map $\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ (Theorem T.4.1.3.1). $\qquad\square$

Proof of Theorem 5.3.3.11. The first assertion follows from Lemma 5.3.3.10. The second follows from the observation that $\mathbf{D}(S^1)_{\psi/} \to \mathbf{D}(S^1)$ is a left cofinal map, by virtue of Lemma 5.3.3.12 and Proposition 5.3.2.15. $\qquad\square$

Remark 5.3.3.13. In the situation of Theorem 5.3.3.11, let us view A as an associative algebra object of $\mathcal{C}$ equipped with an automorphism σ. We can describe the simplicial object $B_\bullet$ informally as follows. For each $n \geq 0$, the object $B_n \in \mathcal{C}$ can be identified with the tensor power $A^{\otimes(n+1)}$. For $0 \leq i < n$, the ith face map from B_n to B_{n-1} is given by the composition

$$B_n \simeq A^{\otimes i} \otimes (A \otimes A) \otimes A^{\otimes(n-1-i)} \to A^{\otimes i} \otimes A \otimes A^{\otimes(n-1-i)} \simeq B_{n-1},$$

where the middle map involves the multiplication on A. The nth face map is given instead by the composition

$$B_n \simeq (A \otimes A^{\otimes(n-1)}) \otimes A \simeq A \otimes (A \otimes A^{\otimes n-1}) \simeq (A \otimes A) \otimes A^{\otimes n-1} \to A \otimes A^{\otimes^{n-1}} \simeq B_{n-1}.$$

Example 5.3.3.14. Let $\mathcal{E}$ denote the homotopy category of the ∞-operad $\mathrm{BTop}(1)^{\otimes}$, so that $\mathrm{N}(\mathcal{E})$ is the ∞-operad describing associative algebras with involution (see Example 5.2.2.5). Then $\mathrm{N}(\mathcal{E})$ contains a subcategory equivalent to the associative ∞-operad $\mathcal{A}\mathrm{ss}^{\otimes}$. Since the circle S^1 is orientable, the canonical map $\mathbb{E}_{S^1}^{\otimes} \to \mathrm{BTop}(1)^{\otimes} \to \mathrm{N}(\mathcal{E})$ factors through this subcategory. We obtain by composition a functor $\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_{S^1}}(\mathcal{C})$ for any symmetric monoidal ∞-category $\mathcal{C}$. If $\mathcal{C}$ admits sifted colimits and the tensor product on $\mathcal{C}$ preserves sifted colimits, we can then define the topological chiral homology $\int_{S^1} A$. It follows from Theorem 5.3.3.11 that this topological chiral homology can be computed in very simple terms: namely, it is given by the geometric realization of a simplicial object $B_\bullet$ of $\mathcal{C}$ consisting of iterated tensor powers of the algebra A. In fact, in this case, we can say more: the simplicial object $B_\bullet$ can be canonically promoted to a *cyclic object* of $\mathcal{C}$. The geometric realization of this cyclic object provides the usual bar resolution for computing the Hochschild homology of A.

5.3.4   Factorizable Cosheaves and Ran Integration

Let M be a manifold and let A be an $\mathbb{E}_M$-algebra object of a sifted-complete symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$. We refer to the object $\int_U A \in \mathcal{C}$ introduced in Definition 5.3.2.6 as the topological chiral homology of U with coefficients in A, which is intended to suggest that (like ordinary homology) it enjoys some form of codescent with respect to open coverings in M. However, the situation is more subtle: the functor $U \mapsto \int_U A$ is not generally a cosheaf on the manifold M itself (except in the situation described in Lemma 5.3.3.7). However, it can be used to construct a cosheaf on the Ran space $\mathrm{Ran}(M)$ introduced in §5.3.1. In other words, we can view topological chiral homology as given by the procedure of integration over the Ran space (Theorem 5.3.4.14).

We begin with a review of the theory of cosheaves.

Definition 5.3.4.1. Let $\mathcal{C}$ be an ∞-category, X a topological space, and $\mathcal{U}(X)$ the partially ordered set of open subsets of X. We will say that a functor $\mathcal{F} : \mathrm{N}(\mathcal{U}(X)) \to \mathcal{C}$ is a *cosheaf* on X if, for every object $C \in \mathcal{C}$, the induced map

$$\mathcal{F}_C : \mathrm{N}(\mathcal{U}(X))^{op} \stackrel{\mathcal{F}}{\to} \mathcal{C}^{op} \stackrel{e_C}{\to} \mathcal{S}$$

is a sheaf on X, where $e_C : \mathcal{C}^{op} \to \mathcal{S}$ denotes the functor represented by $\mathcal{C}$. We will say that a cosheaf $\mathcal{F} : \mathrm{N}(\mathcal{U}(X)) \to \mathcal{C}$ is *hypercomplete* if each of the sheaves $\mathcal{F}_C \in \mathrm{Shv}(X)$ is hypercomplete. If X is the Ran space of a manifold M, we will say that $\mathcal{F}$ is *constructible* if each of the sheaves $\mathcal{F}_C$ is constructible in the sense of Definition 5.3.1.11.

Remark 5.3.4.2. Let X be a topological space. It follows from Proposition 5.3.1.15 that a functor $\mathcal{F} : \mathrm{N}(\mathcal{U}(X)) \to \mathcal{C}$ is a hypercomplete cosheaf on X if and only if, for every open set $U \subseteq X$ and every functor $f : \mathcal{J} \to \mathcal{U}(U)$ with the property that $\mathcal{J}_x = \{J \in \mathcal{J} : x \in f(J)\}$ has weakly contractible nerve for each $x \in U$, the functor $\mathcal{F}$ exhibits $\mathcal{F}(U)$ as a colimit of the diagram $\{\mathcal{F}(f(J))\}_{J \in \mathcal{J}}$.

In particular, if $g : \mathcal{C} \to \mathcal{D}$ is a functor which preserves small colimits, then composition with g carries hypercomplete cosheaves to hypercomplete cosheaves. Similarly, if $\mathcal{C} = \mathcal{P}(\mathcal{E})$ for some small ∞-category $\mathcal{E}$, a functor $\mathcal{F} : \mathrm{N}(\mathcal{U}(X)) \to \mathcal{C}$ is a hypercomplete cosheaf if and only if, for every $E \in \mathcal{E}$, the functor $U \mapsto \mathcal{F}(U)(E)$ determines a cosheaf of spaces $\mathrm{N}(\mathcal{U}(X)) \to \mathcal{S}$.

Our first goal in this section is to show that, if M is a manifold, then we can identify $\mathbb{E}_M$-algebras with a suitable class of cosheaves on the Ran space $\mathrm{Ran}(M)$. To describe this class more precisely, we need to introduce a bit of terminology.

Definition 5.3.4.3. Let M be a manifold, and let U be a subset of $\mathrm{Ran}(M)$. The *support* $\mathrm{Supp}\, U$ of U is the union $\bigcup_{S \in U} S \subseteq M$. We will say that a pair of subsets $U, V \subseteq \mathrm{Ran}(M)$ are *independent* if $\mathrm{Supp}\, U \cap \mathrm{Supp}\, V = \emptyset$.

Definition 5.3.4.4. If U and V are subsets in $\mathrm{Ran}(M)$, we let $U \star V$ denote the set $\{S \cup T : S \in U, T \in V\} \subseteq \mathrm{Ran}(M)$.

Remark 5.3.4.5. If U is an open subset of $\mathrm{Ran}(X)$, then $\mathrm{Supp}\, U$ is an open subset of X.

Example 5.3.4.6. If $\{U_i\}_{1 \le i \le n}$ is a nonempty finite collection of disjoint open subsets of a manifold M, then the open set $\mathrm{Ran}(\{U_i\}) \subseteq \mathrm{Ran}(M)$ defined in §5.3.1 can be identified with $\mathrm{Ran}(U_1) \star \mathrm{Ran}(U_2) \star \cdots \star \mathrm{Ran}(U_n)$.

Remark 5.3.4.7. If U and V are open in $\mathrm{Ran}(M)$, then $U \star V$ is also open in $\mathrm{Ran}(M)$.

Remark 5.3.4.8. We will generally consider the set $U \star V$ only in the case where U and V are independent subsets of $\mathrm{Ran}(M)$. In this case, the canonical map $U \times V \to U \star V$ given by the formula $(S, T) \mapsto S \cup T$ is a homeomorphism.

Definition 5.3.4.9. Let M be a manifold. We define a category $\mathrm{Fact}(M)^{\otimes}$ as follows:

(1) The objects of $\mathrm{Fact}(M)^{\otimes}$ are finite sequences $(U_1, \ldots, U_n)$ of open subsets $U_i \subseteq \mathrm{Ran}(M)$.

(2) A morphism from $(U_1, \ldots, U_m)$ to $(V_1, \ldots, V_n)$ in $\mathrm{Fact}\, M$ is a surjective map $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$ with the following property: for $1 \le i \le n$, the sets $\{U_j\}_{\alpha(j)=i}$ are pairwise independent and $\star_{\alpha(j)=i} U_j \subseteq V_i$.

We let $\mathrm{Fact}(M) \subseteq \mathrm{Fact}(M)^{\otimes}$ denote the fiber product $\mathrm{Fact}(M)^{\otimes} \times_{\mathcal{F}\mathrm{in}_*} \{\langle 1 \rangle\}$, so that $\mathrm{Fact}(M)$ is the category whose objects are open subsets of $\mathrm{Ran}(M)$ and whose morphisms are inclusions of open sets.

The ∞-category $\mathrm{N}(\mathrm{Fact}(M)^{\otimes})$ is an ∞-operad. Moreover, there is a canonical map of ∞-operads $\Psi : \mathrm{N}(\mathrm{Disk}(M))_{\mathrm{nu}}^{\otimes} \to \mathrm{N}(\mathrm{Fact}(M)^{\otimes})$, given on objects by the formula $(U_1, \ldots, U_n) \mapsto (\mathrm{Ran}(U_1), \ldots, \mathrm{Ran}(U_n))$.

We can now state our main result:

Theorem 5.3.4.10. *Let M be a manifold and let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Assume that $\mathcal{C}$ admits small colimits and that the tensor product on $\mathcal{C}$ preserves small colimits separately in each variable. Then the operation of operadic left Kan extension along the inclusion $\Psi : \mathrm{N}(\mathrm{Disk}(M))_{\mathrm{nu}}^{\otimes} \to \mathrm{N}(\mathrm{Fact}(M)^{\otimes})$ determines a fully faithful embedding $F : \mathrm{Alg}_{\mathrm{N}(\mathrm{Disk}(M))}^{\mathrm{nu}}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{N}(\mathrm{Fact}(M))}(\mathcal{C})$. Moreover, the essential image of the full subcategory $\mathrm{Alg}_{\mathrm{Disk}(M)}^{\mathrm{nu,loc}}(\mathcal{C})$ spanned by the locally constant objects of $\mathrm{Alg}_{\mathrm{N}(\mathrm{Disk}(M))}^{\mathrm{u}}(\mathcal{C})$ is the full subcategory of $\mathrm{Alg}_{\mathrm{Fact}(M)}(\mathcal{C})$ spanned by those objects A satisfying the following conditions:*

(1) *The restriction of A to $\mathrm{N}(\mathrm{Fact}(M))$ is a constructible cosheaf on $\mathrm{Ran}(M)$, in the sense of Definition 5.3.4.1.*

(2) *Let $U, V \subseteq \mathrm{Ran}(M)$ be independent open sets. Then the induced map $A(U) \otimes A(V) \to A(U \star V)$ is an equivalence in $\mathcal{C}$.*

Remark 5.3.4.11. In view of Proposition 5.2.4.15, we can formulate Theorem 5.3.4.10 more informally as follows: giving a nonunital $\mathbb{E}_M$-algebra object of the ∞-category $\mathcal{C}$ is equivalent to giving a constructible $\mathcal{C}$-valued cosheaf $\mathcal{F}$ on the Ran space $\mathrm{Ran}(M)$, with the additional feature that $\mathcal{F}(U \star V) \simeq \mathcal{F}(U) \otimes \mathcal{F}(V)$ when U and V are independent subsets of $\mathrm{Ran}(M)$. Following Beilinson and Drinfeld, we will refer a cosheaf with this property as a *factorizable* cosheaf on $\mathrm{Ran}(M)$.

Remark 5.3.4.12. If M is a manifold, let $\mathrm{Ran}^{+}(M)$ denote the collection of *all* finite subsets of M, so that $\mathrm{Ran}^{+}(M) = \mathrm{Ran}(M) \cup \{\emptyset\}$. We regard $\mathrm{Ran}^{+}(M)$ as a topological space, taking as a basis those subsets of the form $\mathrm{Ran}^{+}(U)$ where U is an open subset of M. Note that this topology is usually not Hausdorff: for example, $\emptyset \in \mathrm{Ran}^{+}(M)$ belongs to every nonempty open subset of M. It is possible to obtain a unital analogue of the easy part of Theorem 5.3.4.10: pperadic left Kan extensions embeds the ∞-category $\mathrm{Alg}_{\mathrm{N}(\mathrm{Disk}(M))}(\mathcal{C})$ as a full subcategory of an ∞-category of factorizable cosheaves on $\mathrm{Ran}^{+}(M)$. However, we do not know if there if there is a simple description of the essential image of the locally constant $\mathrm{N}(\mathrm{Disk}(M))$-algebras.

The proof of Theorem 5.3.4.10 rests on the following basic calculation:

Lemma 5.3.4.13. *Let M be a k-manifold, let $D \in (\mathbb{E}_M^{\otimes})_{\mathrm{nu}}$ be an object (corresponding to a nonempty finite collection of open embeddings $\{\psi_i : \mathbf{R}^k \to M\}_{1 \leq i \leq m}$), let $\chi : \mathrm{N}(\mathrm{Disj}(M)_{\mathrm{nu}}) \to \mathcal{S}$ be a functor classified by the left fibration $\mathrm{N}(\mathrm{Disj}(M)_{\mathrm{nu}}) \times_{\mathbb{E}_M^{\otimes}} ((\mathbb{E}_M^{\otimes})_{\mathrm{nu}})_{D/}^{\mathrm{act}}$ (here $\mathrm{Disj}(M)_{\mathrm{nu}}$ is defined as in Proposition 5.3.2.13), and let $\overline{\chi} : \mathrm{N}(\mathrm{Fact}(M)) \to \mathcal{S}$ be a left Kan extension of χ. Then $\overline{\chi}$ is a hypercomplete $\mathcal{S}$-valued cosheaf on $\mathrm{Ran}(M)$.*

Proof. Recall that a natural transformation of functors $\alpha : F \to G$ from an ∞-category $\mathcal{C}$ to $\mathcal{S}$ is said to be *Cartesian* if, for every morphism $C \to D$ in $\mathcal{C}$, the induced diagram

$$
\begin{array}{ccc}
F(C) & \longrightarrow & F(D) \\
\downarrow & & \downarrow \\
G(C) & \longrightarrow & G(D)
\end{array}
$$

is a pullback square in $\mathcal{S}$. Let D' be the image of D in $\mathrm{BTop}(k)^{\otimes}$, and let $\chi' : \mathrm{N}(\mathrm{Disj}(M)_{\mathrm{nu}}) \to \mathcal{S}$ be a functor classified by the left fibration $\mathrm{N}(\mathrm{Disj}(M)_{\mathrm{nu}}) \times_{\mathrm{BTop}(k)^{\otimes}} (\mathrm{BTop}(k)^{\otimes})_{/D'}^{\mathrm{act}}$. There is an evident natural transformation of functors $\beta : \chi \to \chi'$, which induces a natural transformation $\overline{\beta} : \overline{\chi} \to \overline{\chi}'$. It is easy to see that β is a Cartesian natural transformation. Let $S = \{1, \ldots, m\}$, so that we can identify χ' with the functor which assigns to $V \in \mathrm{Disj}(M)_{\mathrm{nu}}$ the summand $\mathrm{Sing}\,\mathrm{Emb}'(S \times \mathbf{R}^k, V) \subseteq \mathrm{Sing}\,\mathrm{Emb}(S \times \mathbf{R}^k, V)$ consisting of those open embeddings $j : S \times \mathbf{R}^k \to V$ which are surjective on connected components. Let $\chi'' : \mathrm{N}(\mathrm{Disj}(M)_{\mathrm{nu}}) \to \mathcal{S}$ be the functor given by the formula $V \mapsto \mathrm{Sing}\,\mathrm{Conf}'(S, V)$, where $\mathrm{Sing}\,\mathrm{Conf}'(S, V) \subseteq \mathrm{Sing}\,\mathrm{Conf}(S, V)$ is the summand consisting of injective maps $i : S \to V$ which are surjective on connected components. We have an evident natural transformation of functors $\gamma : \chi' \to \chi''$. Using Remark 5.2.1.10, we deduce that γ is Cartesian, so that $\alpha = \gamma \circ \beta$ is a Cartesian natural transformation from χ to χ''.

Let $\phi : \mathrm{Conf}(S, M) \to \mathrm{Ran}(M)$ be the continuous map which assigns to each configuration $i : S \to M$ its image $i(S) \subseteq M$ (so that ϕ exhibits $\mathrm{Conf}(S, M)$ as a finite covering space of $\mathrm{Ran}^m(M) \subseteq \mathrm{Ran}(M)$). Let $\overline{\chi}'' : \mathrm{N}(\mathrm{Fact}(M)) \to \mathcal{S}$ be the functor given by the formula $U \mapsto \mathrm{Sing}(\phi^{-1}U)$. We observe that $\overline{\chi}''$ is canonically equivalent to χ''. We claim that $\overline{\chi}''$ is a left Kan extension of χ''. To prove this, it suffices to show that for every open subset $U \subseteq \mathrm{Ran}(M)$, the map $\overline{\chi}''$ exhibits $\mathrm{Sing}(\phi^{-1}U)$ as a colimit of the diagram $\{\chi''(V)\}_{V \in \mathcal{J}}$, where $\mathcal{J} \subseteq \mathrm{Disj}(M)_{\mathrm{nu}}$ is the full subcategory spanned by those unions of disks $V = U_1 \cup \ldots \cup U_n$ such that $\mathrm{Ran}(\{U_i\}) \subseteq U$. For each $x \in \phi^{-1}(U)$, let $\mathcal{J}_x$ denote the full subcategory of $\mathcal{J}$ spanned by those open sets V such that the map $x : S \to M$ factors through a map $S \to V$ which is surjective on connected components. In view of Theorem A.3.1, it will suffice to show that $\mathcal{J}_x$ has weakly contractible nerve. In fact, we claim that $\mathcal{J}_x^{op}$ is filtered: this follows from the observation that every open neighborhood of $x(S)$ contains an open set of the form $U_1 \cup \ldots \cup U_m$, where the U_i are a collection of small disjoint disks containing the elements of $x(S)$.

The map α induces a natural transformation $\overline{\alpha} : \overline{\chi} \to \overline{\chi}''$. Using Theorem T.6.1.3.9, we deduce that $\overline{\alpha}$ is also a Cartesian natural transformation. We wish to show that $\overline{\chi}$ satisfies the criterion of Remark 5.3.4.2. In other words, we wish to show that if $U \subseteq \mathrm{Ran}(M)$ is an open subset and $f : \mathcal{I} \to \mathrm{Fact}(M)$ is a diagram such that each $f(I) \subseteq U$ and the full subcategory $\mathcal{I}_x = \{I \in \mathcal{I} : x \in f(I)\}$ has weakly contractible nerve for each $x \in U$, then $\overline{\chi}$ exhibits $\overline{\chi}(U)$ as a colimit of the diagram $\{\overline{\chi}(f(I))\}_{I \in \mathcal{I}}$. By virtue of Theorem T.6.1.3.9, it will suffice to show that $\overline{\chi}''$ exhibits $\overline{\chi}''(U)$ as a colimit of the diagram $\{\overline{\chi}''(f(I))\}_{I \in \mathcal{I}}$. This is an immediate consequence of Theorem A.3.1. □

Proof of Theorem 5.3.4.10. The existence of the functor F follows from Corollary 3.1.3.5. Let A_0 be a nonunital $\mathrm{N}(\mathrm{Disk}(M))^\otimes$-algebra object of $\mathcal{C}$. Using Corollary 3.1.3.5, Proposition 3.1.1.15, and Proposition 3.1.1.16, we see that $A = F(A_0)$ can be described as an algebra which assigns to each $U \subseteq \mathrm{Ran}(M)$ a colimit of the diagram

$$\chi_U : \mathrm{N}(\mathrm{Disk}(M))^\otimes \times_{\mathrm{N}(\mathrm{Fact}(M)^\otimes)} \mathrm{N}(\mathrm{Fact}(M)^\otimes)^{\mathrm{act}}_{/U} \to \mathcal{C}\,.$$

The domain of this functor can be identified with the nerve of the category $\mathcal{C}_U$ whose objects are finite collections of disjoint disks $V_1, \ldots, V_n \subseteq M$ such that $\mathrm{Ran}(\{V_i\}) \subseteq U$. In particular, if $U = \mathrm{Ran}(U')$ for some open disk $U' \subseteq M$, then the one-element sequence (U') is a final object of $\mathcal{C}_U$. It follows that the canonical map $A_0 \to A \,|\, \mathrm{N}(\mathrm{Disk}(M))^\otimes$ is an equivalence, so that the functor F is fully faithful.

We next show that if $A = F(A_0)$ for some $A_0 \in \mathrm{Alg}^{\mathrm{loc,nu}}_{\mathrm{N}(\mathrm{Disk}(M))}(\mathcal{C})$, then A satisfies conditions (1) and (2). To prove that A satisfies (2), we observe that if $U, V \subseteq \mathrm{Ran}(M)$ are independent then we have a canonical equivalence $\mathcal{C}_{U \star V} \simeq \mathcal{C}_U \star \mathcal{C}_V$. Under this equivalence, the functor $\chi_{U \star V}$ is given by the tensor product of the functors χ_U and χ_V. The map $A(U) \otimes A(V) \to A(U \star V)$ is a homotopy inverse to the equivalence

$$\varinjlim_{\mathrm{N}(\mathcal{C}_{U \star V})} \chi_{U \star V} \simeq \varinjlim_{\mathrm{N}(\mathcal{C}_U) \times \mathrm{N}(\mathcal{C}_V)} \chi_U \otimes \chi_V \to (\varinjlim_{\mathrm{N}(\mathcal{C}_U)} \chi_U) \otimes (\varinjlim_{\mathrm{N}(\mathcal{C}_V)} \chi_V)$$

provided by our assumption that the tensor product on $\mathcal{C}$ preserves small colimits separately in each variable.

We next show that $A \,|\, \mathrm{N}(\mathrm{Fact}(M))$ is a hypercomplete cosheaf on $\mathrm{Ran}(M)$. By virtue of Proposition 5.2.4.15, we can assume that A_0 factors as a composition

$$\mathrm{Disk}(M)^\otimes_{\mathrm{nu}} \to (\mathbb{E}^\otimes_M)_{\mathrm{nu}} \xrightarrow{A'_0} \mathcal{C}^\otimes\,.$$

Let $\mathcal{D}$ be the subcategory of $(\mathbb{E}^\otimes_M)_{\mathrm{nu}}$ spanned by the active morphisms. As explained in §2.2.4, the ∞-category $\mathcal{D}$ admits a symmetric monoidal structure and we may assume that A'_0 factors as a composition

$$(\mathbb{E}^\otimes_M)_{\mathrm{nu}} \to \mathcal{D}^\otimes \xrightarrow{A''_0} \mathcal{C}^\otimes,$$

where A''_0 is a symmetric monoidal functor. Corollary 6.3.1.12 implies that the $\mathcal{P}(\mathcal{D})$ inherits a symmetric monoidal structure, and that A''_0 factors (up to homotopy) as a composition

$$\mathcal{D}^\otimes \to \mathcal{P}(\mathcal{D})^\otimes \xrightarrow{T} \mathcal{C}^\otimes$$

where T is a symmetric monoidal functor such that the underlying functor $T_{\langle 1 \rangle} : \mathcal{P}(\mathcal{D}) \to \mathcal{C}$ preserves small colimits. Let B_0 denote the composite map

$$\mathrm{Disk}(M)^{\otimes}_{\mathrm{nu}} \to (\mathbb{E}^{\otimes}_M)_{\mathrm{nu}} \to \mathcal{D}^{\otimes} \to \mathcal{P}(\mathcal{D})^{\otimes},$$

and let $B \in \mathrm{Alg}_{\mathrm{Fact}(M)}(\mathcal{P}(\mathcal{D}))$ be an operadic left Kan extension of B_0, so that $A_0 \simeq T \circ B_0$ and $A \simeq T \circ B$. Since $T_{\langle 1 \rangle}$ preserves small colimits, it will suffice to show that $B \,|\, \mathrm{N}(\mathrm{Fact}(M))$ is a hypercomplete $\mathcal{P}(\mathcal{D})$-valued cosheaf on $\mathrm{Ran}(M)$ (Remark 5.3.4.2). Fix an object $D \in \mathcal{D}$, and let $e_D : \mathcal{P}(\mathcal{D}) \to \mathcal{S}$ be the functor given by evaluation on D. In view of Remark 5.3.4.2, it will suffice to show that $e_D \circ (B \,|\, \mathrm{N}(\mathrm{Fact}(M)))$ is a hypercomplete $\mathcal{S}$-valued cosheaf on $\mathrm{Ran}(M)$. The desired result is now a translation of Lemma 5.3.4.13.

To complete the proof that A satisfies (1), it suffices to show that for each $C \in \mathcal{C}$, the functor $U \mapsto \mathrm{Map}_{\mathcal{C}}(A(U), C)$ satisfies condition $(*)$ of Proposition 5.3.1.14. Let $U_1, \ldots, U_n \subseteq M$ be disjoint disks containing smaller disks $V_1, \ldots, V_n \subseteq M$; it will suffice to show that the corestriction map $A(\mathrm{Ran}(\{V_i\})) \to A(\mathrm{Ran}(\{U_i\}))$ is an equivalence in $\mathcal{C}$. Since A satisfies (2), we can reduce to the case where $n = 1$. In this case, we have a commutative diagram

$$
\begin{array}{ccc}
A_0(V_1) & \xrightarrow{\ \beta\ } & A_0(U_1) \\
\downarrow & & \downarrow \\
A(\mathrm{Ran}(V_1)) & \xrightarrow{\ \beta'\ } & A(\mathrm{Ran}(U_1)).
\end{array}
$$

The vertical maps are equivalences (since F is fully faithful), and the map β is an equivalence because A_0 is locally constant.

Now suppose that $A \in \mathrm{Alg}_{\mathrm{N}(\mathrm{Fact}(M))}(\mathcal{C})$ satisfies conditions (1) and (2); we wish to prove that A lies in the essential image of $F \,|\, \mathrm{Alg}^{\mathrm{loc},\mathrm{nu}}_{\mathrm{N}(\mathrm{Disk}(M))}(\mathcal{C})$. Let $A_0 = A \,|\, \mathrm{N}(\mathrm{Disk}(M))^{\otimes}$. Since A satisfies (1), Proposition 5.3.1.14 guarantees that A_0 is locally constant; it will therefore suffice to show that the canonical map $F(A_0) \to A$ is an equivalence in the ∞-category $\mathrm{Alg}_{\mathrm{N}(\mathrm{Fact}(M))}(\mathcal{C})$. It will suffice to show that for every open set $U \subseteq \mathrm{Ran}(M)$ and every object $C \in \mathcal{C}$, the induced map $\alpha_U : \mathrm{Map}_{\mathcal{C}}(A(U), C) \to \mathrm{Map}_{\mathcal{C}}(F(A_0)(U), C)$ is a homotopy equivalence of spaces. Since A and $F(A_0)$ both satisfy condition (1), the collection of open sets U such that α_U is a homotopy equivalence is stable under unions of hypercovers. Consequently, Remark 5.3.1.4 allows us to assume that $U = \mathrm{Ran}(V_1) \star \cdots \star \mathrm{Ran}(V_n)$ for some collection of disjoint open disks $V_1, \ldots, V_n \subseteq M$. We claim that $\beta : F(A_0)(U) \to A(U)$ is an equivalence. Since A and $F(A_0)$ both satisfy (2), it suffices to prove this result after replacing U by $\mathrm{Ran}(V_i)$ for $1 \leq i \leq n$. We may therefore assume that $U = \mathrm{Ran}(V)$ for some open disk $V \subseteq M$. In this case, we have a commutative diagram

$$
\begin{array}{ccc}
 & A_0(V) & \\
{}^{\beta'}\nearrow & & \searrow^{\beta''} \\
F(A_0)(U) & \xrightarrow[\quad\beta\quad]{} & A(U).
\end{array}
$$

The map β' is an equivalence by the first part of the proof, and β'' is an equivalence by construction. The two-out-of-three property shows that β is also an equivalence, as desired. $\square$

The construction of topological chiral homology is quite closely related to the left Kan extension functor F studied in Theorem 5.3.4.10. Let M be a manifold, let $A \in \mathrm{Alg}_{\mathrm{N}(\mathrm{Disk}(M))}(\mathcal{C})$, and let $A_0 = A \,|\, \mathrm{N}(\mathrm{Disk}(M))^{\otimes}_{\mathrm{nu}}$. Evaluating $\Psi(A_0)$ on the Ran space $\mathrm{Ran}(M)$, we obtain an object of $\mathcal{C}$ which we will denote by $\int^{\mathrm{nu}}_M A$. Unwinding the definition, we see that $\int^{\mathrm{nu}}_M A$ can be identified with the colimit $\varinjlim_{V \in \mathrm{Disj}(M)_{\mathrm{nu}}} \chi(V)$, where $\chi : \mathrm{N}(\mathrm{Disj}(M)) \to \mathcal{C}$ is the functor given informally by the formula $\chi(U_1 \cup \ldots \cup U_n) = A(U_1) \otimes \cdots \otimes A(U_n)$. The topological chiral homology $\int_M A$ is given by the colimit $\varinjlim_{V \in \mathrm{Disj}(M)} \chi(V)$. The inclusion of $\mathrm{Disj}(M)_{\mathrm{nu}}$ into $\mathrm{Disj}(M)$ induces a map $\int^{\mathrm{nu}}_M A \to \int_M A$. We now have the following result:

Theorem 5.3.4.14. *Let M be a manifold and $\mathcal{C}^{\otimes}$ a symmetric monoidal ∞-category. Assume that $\mathcal{C}$ admits small colimits and that the tensor product on $\mathcal{C}$ preserves colimits separately in each variable, and let $A \in \mathrm{Alg}_{\mathrm{N(Disk}(M))}(\mathcal{C})$. Suppose that M is connected and that A is locally constant. Then the canonical map $\int_M^{\mathrm{nu}} A \to \int_M A$ is an equivalence in $\mathcal{C}$.*

Proof. The map A determines a diagram $\psi : \mathrm{N(Disj}(M)) \to \mathcal{C}$, given informally by the formula $\psi(U_1 \cup \ldots \cup U_n) = A(U_1) \otimes \cdots \otimes A(U_n)$. We wish to prove that the canonical map $\theta : \varinjlim(\psi \,|\, \mathrm{N(Disj}(M)^{\mathrm{nu}})) \to \varinjlim(\psi)$ is an equivalence. Since A is locally constant, we can use Theorem 5.2.4.9 to reduce to the case where A factors as a composition $\mathrm{Disk}(M)^{\otimes} \to \mathbb{E}_M \xrightarrow{A'} \mathcal{C}^{\otimes}$. In this case, ψ factors as a composition $\mathrm{N(Disj}(M)) \to \mathbf{D}(M) \xrightarrow{\psi'} \mathcal{C}$, so we have a commutative diagram

$$
\begin{array}{ccc}
\varinjlim(\psi \,|\, \mathrm{N(Disj}(M)^{\mathrm{nu}})) & \xrightarrow{\quad\theta\quad} & \varinjlim(\psi) \\
& \searrow{\theta'} \quad \swarrow{\theta''} & \\
& \varinjlim(\psi'). &
\end{array}
$$

Proposition 5.3.2.13 guarantees that θ' and θ'' are equivalences in $\mathcal{C}$, so that θ is an equivalence by the two-out-of-three property. $\square$

Theorem 5.3.4.14 can be regarded as making the functor Ψ of Theorem 5.3.4.10 more explicit: if A_0 is a locally constant quasi-unital $\mathrm{N(Disk}(M))^{\otimes}$-algebra and M is connected, then the global sections of the associated factorizable cosheaf can be computed by the topological chiral homology construction of Definition 5.3.2.6. We can also read this theorem in the other direction. If A is a locally constant $\mathrm{N(Disk}(M))^{\otimes}$-algebra, the the functor $U \mapsto \int_U A$ does not determine a cosheaf $\mathrm{N}(\mathcal{U}(M)) \to \mathcal{C}$ in the sense of Definition 5.3.4.1. However, when U is connected, the topological chiral homology $\int_U A$ *can* be computed as the global sections of a sheaf on the Ran space $\mathrm{Ran}(U)$. This is a reflection of a more subtle sense in which the construction $U \mapsto \int_U A$ behaves "locally in U." We close this section with a brief informal discussion.

Let M be a manifold of dimension k, and let $N \subseteq M$ be a submanifold of dimension $k - d$ which has a trivial neighborhood of the form $N \times \mathbf{R}^d$. Let $A \in \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C})$ and let $\int(A)$ denote the associated $\mathrm{N}(\mathcal{U}(M))^{\otimes}$-algebra object of $\mathcal{C}$. Restricting $\int(A)$ to open subsets of M of the form $N \times V$, where V is a union of finitely many open disks in $\mathbf{R}^d$, we obtain another algebra $A_N \in \mathrm{Alg}_{\mathrm{N(Disk}(\mathbf{R}^d))}(\mathcal{C})$. This algebra is locally constant, and can therefore be identified with an $\mathbb{E}_d$-algebra object of $\mathcal{C}$ (Theorem 5.2.4.9). We will denote this algebra by $\int_N A$.

Warning 5.3.4.15. This notation is slightly abusive: the $\mathbb{E}_d$-algebra $\int_N A$ depends not only on the closed submanifold $N \subseteq M$ but also on a trivialization of a neighborhood of N.

Suppose now that $d = 1$, and that $N \subseteq M$ is a hypersurface which separates the connected manifold M into two components. Let M_+ denote the union of one of these components with the neighborhood $N \times \mathbf{R}$ of N, and M_- the union of the other component with $N \times \mathbf{R}$ of N. After choosing appropriate conventions regarding the orientation of $\mathbf{R}$, we can endow the topological chiral homology $\int_{M_+} A$ with the structure of a right module over $\int_N A$ (which we will identify with an associative algebra object of $\mathcal{C}$), and $\int_{M_-} A$ with the structure of a left module over $\int_N A$. There is a canonical map

$$
\left(\int_{M_+} A \right) \otimes_{\int_N A} \left(\int_{M_-} A \right) \to \int_M A,
$$

which can be shown to be an equivalence. In other words, we can recover the topological chiral homology $\int_M A$ of the entire manifold M if we understand the topological chiral homologies of M_+ and M_-, together with their interface along the hypersurface N.

Using more elaborate versions of this analysis, one can compute $\int_M A$ using any sufficiently nice decomposition of M into manifolds with corners (for example, from a triangulation of M). This can be made precise using the formalism of extended topological quantum field theories (see [98] for a sketch).

Example 5.3.4.16. Let $M = \mathbf{R}^k$, so that the ∞-operad $\mathbb{E}_M^{\otimes}$ is equivalent to $\mathbb{E}_k^{\otimes}$. Let $N = S^{k-1}$ denote the unit sphere in $\mathbf{R}^k$. We choose a trivialization of the normal bundle to N in M, which assigns to each point $x \in S^{k-1} \subseteq \mathbf{R}^k$ the "inward pointing" normal vector given by $-x$ itself. According to the above discussion, we can associate to any algebra object $A \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$ an $\mathbb{E}_1$-algebra object of $\mathcal{C}$, which we will denote by $B = \int_{S^{k-1}} A$. Using Example 5.1.0.7, we can identify B with an associative algebra object of $\mathcal{C}$. One can show that this associative algebra has the following property: there is an equivalence of ∞-categories $\theta : \mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C}) \simeq \mathrm{LMod}_B(\mathcal{C})$ which fits into a commutative diagram of ∞-categories

$$
\begin{array}{ccc}
\mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C}) & \xrightarrow{\ \theta\ } & \mathrm{LMod}_B(\mathcal{C}) \\
& \searrow \quad \swarrow & \\
& \mathcal{C} &
\end{array}
$$

which are right-tensored over $\mathcal{C}$ (in view of Theorem 6.3.5.5, the existence of such a diagram characterizes the object $B \in \mathrm{Alg}(\mathcal{C})$ up to canonical equivalence). Under the equivalence θ, the left B-module B corresponds to the object $F(\mathbf{1}) \in \mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})$ appearing in the statement of Theorem 8.3.5.1.

5.3.5   Verdier Duality

Our goal in this section is to prove the following result:

Theorem 5.3.5.1 (Verdier Duality). *Let $\mathcal{C}$ be a stable ∞-category which admits small limits and colimits, and let X be a locally compact Hausdorff space. There is a canonical equivalence of ∞-categories*

$$
\mathbb{D} : \mathrm{Shv}(X; \mathcal{C})^{op} \simeq \mathrm{Shv}(X; \mathcal{C}^{op}).
$$

Remark 5.3.5.2. Let k be a field and let $\mathbf{A}$ denote the category of chain complexes of k-vector spaces. Then $\mathbf{A}$ has the structure of a simplicial category; we let $\mathcal{C} = \mathrm{N}(\mathbf{A})$ denote the nerve of $\mathbf{A}$ (that is, the derived ∞-category of the abelian category of k-vector spaces; see Definition 1.3.2.7). Vector space duality induces a simplicial functor $\mathbf{A}^{op} \to \mathbf{A}$, which in turn gives rise to a functor $\mathcal{C}^{op} \to \mathcal{C}$. This functor preserves limits, and therefore induces a functor $\mathrm{Shv}(X; \mathcal{C}^{op}) \to \mathrm{Shv}(X; \mathcal{C})$ for any locally compact Hausdorff space X. Composing this map with the equivalence $\mathbb{D}$ of Theorem 5.3.5.1, we obtain a functor $\mathbb{D}' : \mathrm{Shv}(X; \mathcal{C})^{op} \to \mathrm{Shv}(X, \mathcal{C})$: that is, a contravariant functor from $\mathrm{Shv}(X; \mathcal{C})$ to itself.

It is the functor $\mathbb{D}'$ which is usually referred to as Verdier duality. Note that $\mathbb{D}'$ is not an equivalence of ∞-categories: it is obtained by composing the equivalence $\mathbb{D}$ with vector space duality, which fails to be an equivalence unless suitable finiteness restrictions are imposed. We refer the reader to [155] for further discussion.

The first step in the proof of Theorem 5.3.5.1 is to choose a convenient model for the ∞-category $\mathrm{Shv}(X; \mathcal{C})$ of $\mathcal{C}$-valued sheaves on X. Let $\mathcal{K}(X)$ denote the collection of all compact subsets of X, regarded as a partially ordered set with respect to inclusion. Recall (Definition T.7.3.4.1) that a $\mathcal{K}$-*sheaf* on X (with values in an ∞-category $\mathcal{C}$) is a functor $\mathcal{F} : \mathrm{N}(\mathcal{K}(X))^{op} \to \mathcal{C}$ with the following properties:

(i) The object $\mathcal{F}(\emptyset) \in \mathcal{C}$ is final.

(ii) For every pair of compact sets $K, K' \subseteq X$, the diagram

$$
\begin{array}{ccc}
\mathcal{F}(K \cup K') & \longrightarrow & \mathcal{F}(K) \\
\downarrow & & \downarrow \\
\mathcal{F}(K') & \longrightarrow & \mathcal{F}(K \cap K')
\end{array}
$$

is a pullback square in $\mathcal{C}$.

(iii) For every compact set $K \subseteq X$, the canonical map $\varinjlim_{K'} \mathcal{F}(K') \to \mathcal{F}(K)$ is an equivalence, where K' ranges over all compact subsets of X which contain a neighborhood of K.

We let $\mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathcal{K}(X)^{op}, \mathcal{C})$ spanned by the $\mathcal{K}$-sheaves. We now have the following:

Lemma 5.3.5.3. *Let X be a locally compact Hausdorff space and $\mathcal{C}$ a stable ∞-category which admits small limits and colimits. Then there is a canonical equivalence of ∞-categories $\mathrm{Shv}(X; \mathcal{C}) \simeq \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C})$.*

Proof. Since $\mathcal{C}$ is stable, filtered colimits in $\mathcal{C}$ are left exact. The desired result is now a consequence of Theorem T.7.3.4.9 (note that Theorem T.7.3.4.9 is stated under the hypothesis that $\mathcal{C}$ is presentable, but this hypothesis is used only to guarantee the existence of small limits and colimits in $\mathcal{C}$). $\square$

Using Lemma 5.3.5.3, we can reformulate Theorem 5.3.5.1 as follows:

Theorem 5.3.5.4. *Let X be a locally compact Hausdorff space and let $\mathcal{C}$ be a stable ∞-category which admits small limits and colimits. Then there is a canonical equivalence of ∞-categories*

$$\mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C})^{op} \simeq \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op}).$$

We will prove Theorem 5.3.5.4 by introducing an ∞-category which is equivalent to both $\mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C})^{op}$ and $\mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op})$.

Notation 5.3.5.5. Fix a locally compact Hausdorff space X. We define a partially ordered set M as follows:

(1) The objects of M are pairs (i, S) where $0 \leq i \leq 2$ and S is a subset of X such that S is compact if $i = 0$ and $X - S$ is compact if $i = 2$.

(2) We have $(i, S) \leq (j, T)$ if either $i \leq j$ and $S \subseteq T$, or $i = 0$ and $j = 2$.

Remark 5.3.5.6. The projection $(i, S) \mapsto i$ determines a map of partially ordered sets $\phi : M \to [2]$. For $0 \leq i \leq 2$, we let M_i denote the fiber $\phi^{-1}\{i\}$. We have canonical isomorphisms $M_0 \simeq \mathcal{K}(X)$ and $M_2 \simeq \mathcal{K}(X)^{op}$, while M_1 can be identified with the partially ordered set of *all* subsets of X.

The proof of Theorem 5.3.5.4 rests on the following:

Proposition 5.3.5.7. *Let X be a locally compact Hausdorff space, $\mathcal{C}$ a stable ∞-category which admits small limits and colimits, and let M be the partially ordered set of Notation 5.3.5.5. Let $F : \mathrm{N}(M) \to \mathcal{C}$ be a functor. The following conditions are equivalent:*

(1) *The restriction $(F | \mathrm{N}(M_0))^{op}$ determines a $\mathcal{K}$-sheaf $\mathrm{N}(\mathcal{K}(X))^{op} \to \mathcal{C}^{op}$, the restriction $F | \mathrm{N}(M_1)$ is zero, and F is a left Kan extension of the restriction $F | \mathrm{N}(M_0 \cup M_1)$.*

(2) *The restriction $F | \mathrm{N}(M_2)$ determines a $\mathcal{K}$-sheaf $\mathrm{N}(\mathcal{K}(X))^{op} \to \mathcal{C}$, the restriction $F | \mathrm{N}(M_1)$ is zero, and F is a right Kan extension of $F | \mathrm{N}(M_1 \cup M_2)$.*

Assuming Proposition 5.3.5.7 for the moment, we can give the proof of Theorem 5.3.5.4.

Proof of Theorem 5.3.5.4. Let $\mathcal{E}(\mathcal{C})$ be the full subcategory of $\mathrm{Fun}(\mathrm{N}(M), \mathcal{C})$ spanned by those functors which satisfy the equivalent conditions of Proposition 5.3.5.7. The inclusions $M_0 \hookrightarrow M \hookleftarrow M_2$ determine restriction functors

$$\mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op}) \xleftarrow{\theta} \mathcal{E}(\mathcal{C})^{op} \xrightarrow{\theta'} \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C})^{op}.$$

Note that a functor $F \in \mathrm{Fun}(\mathrm{N}(M), \mathcal{C})$ belongs to $\mathcal{E}(\mathcal{C})$ if and only if $F | \mathrm{N}(M_0)$ belongs to $\mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op})$, $F | \mathrm{N}(M_0 \cup M_1)$ is a right Kan extension of $F | \mathrm{N}(M_0)$, and F is a left Kan extension of $F | \mathrm{N}(M_0 \cup M_1)$. Applying Proposition T.4.3.2.15, we deduce that θ is a trivial Kan fibration. The same argument shows that θ' is a trivial Kan fibration, so that θ and θ' determine an equivalence $\mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op}) \simeq \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C})^{op}$. $\square$

Remark 5.3.5.8. The construction $(i, S) \mapsto (2-i, X-S)$ determines an order-reversing bijection from the partially ordered set M to itself. Composition with this involution induces an isomorphism $\mathcal{E}(\mathcal{C})^{op} \simeq \mathcal{E}(\mathcal{C})^{op}$, which interchanges the restriction functors θ and θ' appearing in the proof of Theorem 5.3.5.4. It follows that the equivalence of Theorem 5.3.5.4 is symmetric in $\mathcal{C}$ and $\mathcal{C}^{op}$ (up to coherent homotopy).

We will give the proof of Proposition 5.3.5.7 at the end of this section. For the moment, we will concentrate on the problem of making the equivalence of Theorem 5.3.5.1 more explicit.

Definition 5.3.5.9. Let X be a locally compact Hausdorff space and let $\mathcal{C}$ be a pointed ∞-category which admits small limits and colimits. Let $\mathcal{F}$ be a $\mathcal{C}$-valued sheaf on X. For every compact set $K \subseteq X$, we let $\Gamma_K(X; \mathcal{C})$ denote the fiber product $\mathcal{F}(X) \times_{\mathcal{F}(X-K)} 0$, where 0 denotes a zero object of $\mathcal{C}$. For every open set $U \subseteq X$, we let $\Gamma_c(U; \mathcal{F})$ denote the filtered colimit $\varinjlim_{K \subseteq U} \Gamma_K(M; \mathcal{F})$, where K ranges over all compact subsets of U. The construction $U \mapsto \Gamma_c(U; \mathcal{F})$ determines a functor $\mathrm{N}(\mathcal{U}(X)) \to \mathcal{C}$, which we will denote by $\Gamma_c(\bullet; \mathcal{F})$.

Proposition 5.3.5.10. *In the situation of Definition 5.3.5.9, suppose that the ∞-category $\mathcal{C}$ is stable. Then the equivalence $\mathbb{D}$ of Theorem 5.3.5.1 is given by the formula $\mathbb{D}(\mathcal{F})(U) = \Gamma_c(U; \mathcal{F})$.*

Remark 5.3.5.11. Proposition 5.3.5.10 is an abstract formulation of the following more classical fact: conjugation by Verdier duality exchanges cohomology with compactly supported cohomology.

Proof. It follows from the proof of Theorem T.7.3.4.9 that the equivalence

$$\theta : \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op})^{op} \simeq \mathrm{Shv}(X; \mathcal{C}^{op})^{op}$$

of Lemma 5.3.5.3 is given by the formula $\theta(\mathcal{G})(U) = \varinjlim_{K \subseteq U} \mathcal{G}(K)$. Consequently, it will suffice to show that the composition of the equivalence $\psi : \mathrm{Shv}(X; \mathcal{C}) \to \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C})$ of Lemma 5.3.5.3 with the equivalence $\psi' : \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C}) \to \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op})^{op}$ is given by the formula $(\psi' \circ \psi)(\mathcal{F})(K) = \Gamma_K(X; \mathcal{F})$. To prove this, we need to introduce a bit of notation.

Let M' denote the partially ordered set of pairs (i, S), where $0 \leq i \leq 2$ and S is a subset of X such that S is compact if $i = 0$ and $X - S$ is either open or compact if $i = 2$; we let $(i, S) \leq (j, T)$ if $i \leq j$ and $S \subseteq T$ or if $i = 0$ and $j = 2$. We will regard the set M of Notation 5.3.5.5 as a partially ordered subset of M'. For $0 \leq i \leq 2$, let M'_i denote the subset $\{(j, S) \in M' : j = i\} \subseteq M'$. Let $\mathcal{D}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}(M'), \mathcal{C})$ spanned by those functors F which satisfy the following conditions:

(i) The restriction $F | \mathrm{N}(M_2)$ is a $\mathcal{K}$-sheaf on X.

(ii) The restriction $F | \mathrm{N}(M'_2)$ is a right Kan extension of $F | \mathrm{N}(M_2)$.

(iii) The restriction $F | \mathrm{N}(M'_1)$ is zero.

(iv) The restriction $F | \mathrm{N}(M')$ is a right Kan extension of $F | \mathrm{N}(M'_1 \cup M'_2)$.

Note that condition (ii) is equivalent to the requirement that $F | \mathrm{N}(M'_1 \cup M'_2)$ is a right Kan extension of $F | \mathrm{N}(M_1 \cup M_2)$. It follows from Proposition T.4.3.2.8 that condition (iv) is equivalent to the requirement that $F | \mathrm{N}(M)$ is a right Kan extension of $F | \mathrm{N}(M_1 \cup M_2)$. Consequently, the inclusion $M \hookrightarrow M'$ induces a restriction functor $\mathcal{D} \to \mathcal{E}$, where $\mathcal{E} \subseteq \mathrm{Fun}(\mathrm{N}(M), \mathcal{C})$ is defined as in the proof of Theorem 5.3.5.4. Using Theorem T.7.3.4.9 and Proposition T.4.3.2.15, we deduce that the restriction functor $\mathcal{D} \to \mathrm{Fun}(\mathrm{N}(\mathcal{U}(X))^{op}, \mathcal{C})$ is a trivial Kan fibration onto the full subcategory $\theta : \mathrm{Shv}(X; \mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{N}(\mathcal{U}(X))^{op}, \mathcal{C})$; moreover, the composition $\psi' \circ \psi$ is given by composing a homotopy inverse to θ with the restriction functor $\mathcal{D} \to \mathrm{Fun}(\mathrm{N}(M_0), \mathcal{C}) \simeq \mathrm{Fun}(\mathrm{N}(\mathcal{K}(X))^{op}, \mathcal{C}^{op})^{op}$.

We define a map of simplicial sets $\phi : \mathrm{N}(M_0) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{N}(M'))$ so that ϕ carries an object $(0, K) \in M_0$ to the diagram

$$
\begin{array}{ccc}
(0, K) & \longrightarrow & (1, K) \\
\downarrow & & \downarrow \\
(2, \emptyset) & \longrightarrow & (2, K).
\end{array}
$$

It follows from Theorem T.4.1.3.1 that for each $(0, K) \in M_0$, the image $\phi(0, K)$ can be regarded as a left cofinal map $\Lambda_2^2 \to \mathrm{N}(M')_{(0,K)/} \times_{\mathrm{N}(M')} \mathrm{N}(M_1' \cup M_2')$. Consequently, if $F \in \mathcal{D}$ then condition (iv) is equivalent to the requirement that the composition of F with each $\phi(0, K)$ yields a pullback diagram

$$
\begin{array}{ccc}
F(0, K) & \longrightarrow & F(1, K) \\
\downarrow & & \downarrow \\
F(2, \emptyset) & \longrightarrow & F(2, K)
\end{array}
$$

in the ∞-category $\mathcal{C}$. Since $F(1, K)$ is a zero object of $\mathcal{C}$ (condition (iii)), we can identify $F(0, K)$ with the fiber of the map $F(2, \emptyset) \to F(2, K)$. Taking F to be a preimage of $\mathcal{F} \in \mathrm{Shv}(X; \mathcal{C})$ under the functor θ, we obtain the desired equivalence

$$
(\psi' \circ \psi)(\mathcal{F})(K) \simeq \mathrm{fib}(\mathcal{F}(X) \to \mathcal{F}(X - K)) = \Gamma_K(X; \mathcal{F}).
$$

$\square$

Corollary 5.3.5.12. *Let X be a locally compact Hausdorff space, let $\mathcal{C}$ be a stable ∞-category which admits small limits and colimits, and let $\mathcal{F} \in \mathrm{Shv}(X; \mathcal{C})$ be a $\mathcal{C}$-valued sheaf on X. Then the functor $\Gamma_c(\bullet; \mathcal{F})$ is a $\mathcal{C}$-valued cosheaf on X.*

We will need the following consequence of Corollary 5.3.5.12 in the next section.

Corollary 5.3.5.13. *Let M be a manifold and let $\mathcal{F} \in \mathrm{Shv}(M; \mathrm{Sp})$ be a Sp-valued sheaf on M. Then:*

(1) *The functor $\mathcal{F}$ exhibits $\Gamma_c(M; \mathcal{F})$ as a colimit of the diagram $\{\Gamma_c(U; \mathcal{F})\}_{U \in \mathrm{Disk}(M)}$.*

(2) *The functor $\mathcal{F}$ exhibits $\Gamma_c(M; \mathcal{F})$ as a colimit of the diagram $\{\Gamma_c(U; \mathcal{F})\}_{U \in \mathrm{Disj}(M)}$.*

Proof. We will give the proof of (1); the proof of (2) is similar. According to Corollary 5.3.5.12, the functor $U \mapsto \Gamma_c(U; \mathcal{F})$ is a cosheaf of spectra on M. Since every open subset of M is a paracompact topological space of finite covering dimension, the ∞-topos $\mathrm{Shv}(M)$ is hypercomplete so that $\mathcal{F}$ is automatically hypercomplete. According to Remark 5.3.4.2, it will suffice to show that for every point $x \in M$, the category $\mathrm{Disk}(M)_x = \{U \in \mathrm{Disk}(M) : x \in U\}$ has weakly contractible nerve. This follows from the observation that $\mathrm{Disk}(M)_x^{op}$ is filtered (since every open neighborhood of M contains an open set $U \in \mathrm{Disk}(M)_x$). $\square$

We conclude this section by giving the proof of Proposition 5.3.5.7.

Proof of Proposition 5.3.5.7. We will prove that condition (2) implies (1); the converse follows by symmetry, in view of Remark 5.3.5.8. Let $F : \mathrm{N}(M) \to \mathcal{C}$ be a functor satisfying condition (2), and let M' and $\mathcal{D} \subseteq \mathrm{Fun}(\mathrm{N}(M'), \mathcal{C})$ be defined as in the proof of Proposition 5.3.5.10. Using Proposition T.4.3.2.15, we deduce that F can be extended to a functor $F' : \mathrm{N}(M') \to \mathcal{C}$ belonging to $\mathcal{D}$. It follows from Theorem T.7.3.4.9 that the inclusion $\mathcal{U}(X)^{op} \subseteq M_2'$ determines a restriction functor $\mathcal{D} \to \mathrm{Shv}(X; \mathcal{C})$; let $\mathcal{F} \in \mathrm{Shv}(X; \mathcal{C})$ be the image of F' under this restriction functor. The proof of Proposition 5.3.5.10 shows that $\mathcal{G} = F | \mathrm{N}(M_0)$ is given informally by the formula $\mathcal{G}(K) = \Gamma_K(X; \mathcal{F})$.

We first show that $\mathcal{G}^{op}$ is a $\mathcal{C}^{op}$-valued K-sheaf on X. For this, we must verify the following:

(i) The object $\mathcal{G}(\emptyset) \simeq \Gamma_\emptyset(X; \mathcal{F})$ is zero. This is clear, since the restriction map $\mathcal{F}(X) \to \mathcal{F}(X - \emptyset)$ is an equivalence.

(ii) Let K and K' be compact subsets of X. Then the diagram σ:

$$
\begin{array}{ccc}
\mathcal{G}(K \cap K') & \longrightarrow & \mathcal{G}(K) \\
\downarrow & & \downarrow \\
\mathcal{G}(K') & \longrightarrow & \mathcal{G}(K \cup K')
\end{array}
$$

is a pushout square in $\mathcal{C}$. Since $\mathcal{C}$ is stable, this is equivalent to the requirement that σ is a pullback square. This follows from the observation that σ is the fiber of a map between the squares

$$
\begin{array}{ccc}
\mathcal{F}(X) \longrightarrow \mathcal{F}(X) \\
\downarrow \qquad\qquad \downarrow \\
\mathcal{F}(X) \longrightarrow \mathcal{F}(X)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{F}(X - (K \cap K')) \longrightarrow \mathcal{F}(X - K) \\
\downarrow \qquad\qquad\qquad \downarrow \\
\mathcal{F}(X - K') \longrightarrow \mathcal{F}(X - (K \cup K')).
\end{array}
$$

The left square is obviously a pullback, and the right is a pullback since $\mathcal{F}$ is a sheaf.

(iii) For every compact subset $K \subseteq X$, the canonical map $\theta : \mathcal{G}(K) \to \varprojlim_{K'} \mathcal{G}(K')$ is an equivalence in $\mathcal{C}$, where K' ranges over the partially ordered set A of all compact subsets of X which contain a neighborhood of K. We have a map of fiber sequences

$$
\begin{array}{ccc}
\mathcal{G}(K) & \xrightarrow{\ \theta\ } & \varprojlim_{K' \in A} \mathcal{G}(K') \\
\downarrow & & \downarrow \\
\mathcal{F}(X) & \xrightarrow{\ \theta'\ } & \varprojlim_{K' \in A} \mathcal{F}(X) \\
\downarrow & & \downarrow \\
\mathcal{F}(X - K) & \xrightarrow{\ \theta''\ } & \varprojlim_{K' \in A} \mathcal{F}(X - K').
\end{array}
$$

It therefore suffices to show that θ' and θ'' are equivalences. The map θ' is an equivalence because the partially ordered set A has weakly contractible nerve (in fact, both A and A^{op} are filtered). The map θ'' is an equivalence because $\mathcal{F}$ is a sheaf and the collection $\{X - K'\}_{K' \in A}$ is a covering sieve on $X - K$.

To complete the proof, we will show that F is a left Kan extension of $F|\,\mathrm{N}(M_0 \cup M_1)$. Let $M'' \subseteq M_0 \cup M_1$ be the subset consisting of objects of the form (i, S), where $0 \le i \le 1$ and $S \subseteq X$ is compact. We note that $F|\,\mathrm{N}(M_0 \cup M_1)$ is a left Kan extension of $F|\,\mathrm{N}(M'')$. In view of Proposition T.4.3.2.8, it will suffice to show that F is a left Kan extension of $F|\,\mathrm{N}(M'')$ at every element $(2, S) \in M_2$. We will prove the stronger assertion that $F'|\,\mathrm{N}(M'' \cup M_2')$ is a left Kan extension of $F|\,\mathrm{N}(M'')$. To prove this, we let B denote the subset of M_2' consisting of pairs $(2, X - U)$ where $U \subseteq X$ is an open set with compact closure. In view of Proposition T.4.3.2.8, it suffices to prove the following:

(a) The functor $F'|\,\mathrm{N}(M'' \cup M_2')$ is a left Kan extension of $F'|\,\mathrm{N}(M'' \cup B)$.

(b) The functor $F'|\,\mathrm{N}(M'' \cup B)$ is a left Kan extension of $F|\,\mathrm{N}(M'')$.

To prove (a), we note that Theorem T.7.3.4.9 guarantees that $F'|\,\mathrm{N}(M_2')$ is a left Kan extension of $F'|\,\mathrm{N}(M''')$ (note that, if K is a compact subset of X, then the collection of open neighborhoods of U of K with compact closure is cofinal in the collection of all open neighborhoods of K in X). To complete the proof, it suffices to observe that for every object $(2, X - K) \in M_2' - B$, the inclusion $\mathrm{N}(M''')_{/(2, X - K)} \subseteq \mathrm{N}(M'' \cup M''')_{/(2, X - K)}$ is left cofinal. In view of Theorem T.4.1.3.1, this is equivalent to the requirement that for every object $(i, S) \in M''$, the partially ordered set $P = \{(2, X - U) \in B : (i, S) \le (2, X - U) \le (2, X - K)\}$ has weakly contractible nerve. This is clear, since P is nonempty and stable under finite unions (and therefore filtered). This completes the proof of (a).

To prove (b), fix an open subset $U \subseteq X$ with compact closure; we wish to prove that $F'(2, X - U)$ is a colimit of the diagram $F|\,\mathrm{N}(M'')_{/(2, X - U)}$. For every compact set $K \subseteq X$, let M_K'' denote the subset of M'' consisting of those pairs (i, S) with $(0, K) \le (i, S) \le (2, X - U)$. Then $\mathrm{N}(M'')_{/(2, X - U)}$ is a filtered colimit of the simplicial sets $\mathrm{N}(M_K'')$, where K ranges over the collection of compact subsets of X which contain

U. It follows that $\operatorname{colim}(F'|\operatorname{N}(M'')_{/(2,X-U)})$ can be identified with the filtered colimit of the diagram $\{\operatorname{colim}(F'|\operatorname{N}(M_K''))\}_K$ (see §T.4.2.3). Consequently, it will suffice to prove that for every compact set K containing U, the diagram F' exhibits $F'(2, X-U)$ as a colimit of $F'|\operatorname{N}(M_K'')$. Theorem T.4.1.3.1 guarantees that the diagram $(K, 0) \leftarrow (K-U, 0) \rightarrow (K-U, 1)$ is left cofinal in $\operatorname{N}(M_K'')$. Consequently, we are reduced to proving that the diagram

$$
\begin{array}{ccc}
F'(0, K-U) & \longrightarrow & F'(1, K-U) \\
\downarrow & & \downarrow \\
F'(0, K) & \longrightarrow & F'(2, X-U)
\end{array}
$$

is a pushout square in $\mathcal{C}$. Form a larger commutative diagram

$$
\begin{array}{ccccc}
F'(0, K-U) & \longrightarrow & F(1, K-U) & & \\
\downarrow & & \downarrow & & \\
F'(0, K) & \longrightarrow & Z & \longrightarrow & F(1, K) \\
\downarrow & & \downarrow & & \downarrow \\
F(2, \emptyset) & \longrightarrow & F(2, K-U) & \longrightarrow & F(2, K) \\
& & \downarrow & & \downarrow \\
& & F(2, X-U) & \longrightarrow & F(2, X),
\end{array}
$$

where the middle right square is a pullback. Since F' is a right Kan extension of $F'|\operatorname{N}(M_1 \cup M_2')$, the proof of Proposition 5.3.5.10 shows that the middle horizontal rectangle is also a pullback square. It follows that the lower middle square is a pullback. Since the left vertical rectangle is a pullback diagram (Proposition 5.3.5.10 again), we deduce that the upper left square is a pullback. Since $\mathcal{C}$ is stable, we deduce that the upper left square is a pushout diagram. To complete the proof of (b), it suffices to show that the composite map $Z \rightarrow F(2, K-U) \rightarrow F(2, X-U)$ is an equivalence. We note that $F(1, K-U)$ and $F(2, X) \simeq \mathcal{F}(\emptyset)$ are zero objects of $\mathcal{C}$, so the composite map $F(1, K-U) \rightarrow F(2, K) \rightarrow F(X)$ is an equivalence. It will therefore suffice to show that the right vertical rectangle is a pullback square. Since the middle right square is a pullback by construction, we are reduced to proving that the lower right square is a pullback. This is the diagram

$$
\begin{array}{ccc}
\mathcal{F}((X-K) \cup U) & \longrightarrow & \mathcal{F}(X-K) \\
\downarrow & & \downarrow \\
\mathcal{F}(U) & \longrightarrow & \mathcal{F}(\emptyset),
\end{array}
$$

which is a pullback square because $\mathcal{F}$ is a sheaf and the open sets $U, X - K \subseteq X$ are disjoint. $\qquad\square$

5.3.6   Nonabelian Poincare Duality

Let M be an oriented k-manifold. Poincare duality provides a canonical isomorphism

$$
\operatorname{H}_c^m(M; A) \simeq \operatorname{H}_{k-m}(M; A)
$$

for any abelian group A (or, more generally, for any local system of abelian groups on M). Our goal in this section is to establish an analogue of this statement for *nonabelian* cohomology: that is, cohomology with coefficients in a local system of *spaces* on M. To formulate this analogue, we will need to replace the right hand side by the topological chiral homology $\int_M A$ of M with coefficients in an appropriate $\mathbb{E}_M$-algebra.

Remark 5.3.6.1. The ideas described in this section are closely related to results of Segal, McDuff, and Salvatore on configuration spaces (see [130], [110], and [123]). In particular, a special case of our main result (Theorem 5.3.6.6) can be found in [123].

Definition 5.3.6.2. Let M be a manifold, and let $p : E \to M$ be a Serre fibration equipped with a distinguished section $s : M \to E$. Given a commutative diagram

$$
\begin{array}{ccc}
|\Delta^n| \times M & \xrightarrow{\;\;f\;\;} & E \\
 & \searrow \quad \swarrow {\scriptstyle p} & \\
 & M, &
\end{array}
$$

we will say that f is *trivial* on an open set $U \subseteq M$ if the restriction $f|(|\Delta^n| \times U)$ is given by the composition

$$|\Delta^n| \times U \to U \subseteq M \xrightarrow{s} E.$$

We define the *support* of f to be the smallest closed set K such that f is trivial on $M - K$. Given an open set $U \subseteq M$, we let $\Gamma(U; E)$ denote the simplicial set whose n-simplices are maps f as above, and $\Gamma_c(U; E)$ the simplicial subset spanned by those simplices such that the support of f is a compact subset of U (in this case, f is determined by its restriction $f|(|\Delta^n| \times U)$)

The construction $(U_1, \ldots, U_n) \mapsto \Gamma_c(U_1; E) \times \ldots \times \Gamma_c(U_n; E)$ determines a functor from $\mathcal{U}(M)^\otimes$ to the simplicial category of Kan complexes. Passing to nerves, we obtain a functor $\mathrm{N}(\mathcal{U}\,M^\otimes) \to \mathcal{S}$, which we view as a $\mathrm{N}(\mathcal{U}(M)^\otimes)$-monoid object of $\mathcal{S}$. Let us regard the ∞-category $\mathcal{S}$ as endowed with the Cartesian monoidal structure, so that this monoid object lifts in an essentially unique way to a $\mathrm{N}(\mathcal{U}(M)^\otimes)$-algebra object of $\mathcal{S}$ (Proposition 2.4.2.5). We will denote this algebra by $E_!$.

Remark 5.3.6.3. Let $p : E \to M$ be as in Definition 5.3.6.2. Every inclusion of open disks $U \subseteq V$ in M is isotopic to a homeomorphism (Theorem 5.2.1.5), so the inclusion $\Gamma_c(U; E) \to \Gamma_c(V; E)$ is a homotopy equivalence. It follows that the restriction $E_!| \mathrm{N}(\mathrm{Disk}(M))^\otimes$ is a locally constant object of $\mathrm{Alg}_{\mathrm{N}(\mathrm{Disk}(M))}(\mathcal{S})$, and is therefore equivalent to the restriction $E^!| \mathrm{N}(\mathrm{Disk}(M))^\otimes$ for some essentially unique $\mathbb{E}_M$-algebra $E^! \in \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{S})$ (Theorem 5.2.4.9).

Remark 5.3.6.4. Let M be a manifold and let $p : E \to M$ be a Serre fibration equipped with a section s. Then the functor $U \mapsto \Gamma(U; E)$ determines a sheaf $\mathcal{F}$ on M with values in the ∞-category $\mathcal{S}_*$ of pointed spaces (Proposition T.7.1.3.14). Using Remark 5.3.6.14 and Lemma 5.3.6.15, we can identify the functor $U \mapsto \Gamma_c(U; \mathcal{F})$ of Definition 5.3.5.9 with the functor $U \mapsto \Gamma_c(U; E)$.

Remark 5.3.6.5. Let $p : E \to M$ be as in Definition 5.3.6.2. Since p is a Serre fibration, the inverse image $U \times_M E$ is weakly homotopy equivalent to a product $U \times K$ for every open disk $\mathbf{R}^k \simeq U \subseteq M$, for some pointed topological space K. For every positive real number r, let X_r denote the simplicial subset of $\Gamma_c(U; E)$ whose n-simplices correspond to maps which are supported in the closed ball $\overline{B(r)} \subseteq \mathbf{R}^k \simeq U$. Then each X_r is homotopy equivalent to the iterated loop space $\mathrm{Sing}(\Omega^k K)$. Since there exist compactly supported isotopies of $\mathbf{R}^k$ carrying $B(r)$ to $B(s)$ for $0 < r < s$, we deduce that the inclusion $X_r \subseteq X_s$ is a homotopy equivalence for each $r < s$. It follows that $\Gamma_c(U; E) = \varinjlim_r X_r$ is weakly homotopy equivalent to X_r for every real number r.

In other words, we can think of $E^! : \mathbb{E}_M^\otimes \to \mathcal{S}^\times$ as an algebra which assigns to each open disk $j : U \hookrightarrow M$ the k-fold loop space of F, where F is the fiber of the Serre fibration $p : E \to M$ over any point in the image of j.

We can now state our main result as follows:

Theorem 5.3.6.6 (Nonabelian Poincare Duality). *Let M be a k-manifold, and let $p : E \to M$ be a Serre fibration whose fibers are k-connective, which is equipped with a section $s : M \to E$. Then $E_!$ exhibits $\Gamma_c(M; E)$ as the colimit of the diagram $E_!| \mathrm{N}(\mathrm{Disj}(M))$. In other words, $\Gamma_c(M; E)$ is the topological chiral homology $\int_M E^!$, where $E^! \in \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{S})$ is the algebra described in Remark 5.3.6.3.*

Remark 5.3.6.7. The assumption that $p : E \to M$ have k-connective fibers is essential. For example, suppose that $E = M \coprod M$ and that the section $s : M \to E$ is given by the inclusion of the second factor. If M is compact, then the inclusion of the second factor determines a vertex $\eta \in \Gamma_c(M; E)$. The support of η is the whole of the manifold M: in particular, η does not lie in the essential image of any of the extension maps $i : \Gamma_c(U; E) \to \Gamma_c(M; E)$ where U is a proper open subset of M. In particular, if U is a disjoint union of open disks, then η cannot lie in the essential image of i unless $k = 0$ or M is empty.

Remark 5.3.6.8. Theorem 5.3.6.6 implies in particular that any compactly supported section s' of $p : E \to M$ is homotopic to a section whose support is contained in the union of disjoint disks in M. It is easy to see this directly, at least when M admits a triangulation. Indeed, let $M_0 \subseteq M$ be the $(k-1)$-skeleton of this triangulation, so that the open set $M - M_0$ consists of the interiors of the k-simplices of the triangulation and is thus a union of disjoint open disks in M. Since the fibers of p are k-connective, the space of sections of p over the $(k-1)$-dimensional space M_0 is connected. Consequently, we can adjust s' by a homotopy so that it agrees with s on a small neighborhood of M_0 in M, and is therefore supported in $M - M_0$.

Remark 5.3.6.9. Theorem 5.3.6.6 can be rephrased in terms of the *embedding calculus* developed by Weiss (see [160]). Let $p : E \to M$ be a Serre fibration, and regard the functor $U \mapsto \Gamma_c(U; E)$ as a precosheaf $\mathcal{F}$ on M with values in $\mathcal{S}et_\Delta$. Applying the formalism of the embedding calculus, we obtain a sequence of polynomial approximations

$$\mathcal{F}^{\leq 0} \to \mathcal{F}^{\leq 1} \to \mathcal{F}^{\leq 2} \to \cdots,$$

where each $\mathcal{F}^{\leq n}$ is a left Kan extension of the restriction of $\mathcal{F}$ to those open subsets of M which are homeomorphic to a union of at most n disks. Theorem 5.3.6.6 asserts that, when the fibers of p are sufficiently connected, the canonical map

$$\text{hocolim}_n \, \mathcal{F}^{\leq n} \to \mathcal{F}$$

is a weak equivalence. In other words, the functor $U \mapsto \Gamma_c(U; E)$ can be recovered as the limit of its polynomial approximations.

Example 5.3.6.10. Let M be the circle S^1, let X be a connected pointed space, and let $E = X \times S^1$, equipped with the projection map $p : E \to M$. Then $E^! \in \text{Alg}_{\mathbb{E}_{S^1}}(\mathcal{S})$ is the $\mathbb{E}_{S^1}$-algebra determined by the associative algebra object $\text{Sing}(\Omega X) \in \text{Alg}_{\mathcal{A}ss}(\mathcal{S})$. Since M is compact, we can identify $\Gamma_c(S^1; E)$ with the singular complex of the space $LX = \text{Map}(S^1, X)$ of *all* sections of p. In view of Example 5.3.3.14, Theorem 5.3.6.6 recovers the following classical observation: the free loop space LX is equivalent to the Hochschild homology of the based loop space ΩX (regarded as an associative algebra with respect to composition of loops).

Remark 5.3.6.11. Let M be a k-manifold. We will say than an algebra $A \in \text{Alg}_{\mathbb{E}_M}(\mathcal{S})$ is *grouplike* if, for every open disk $U \subseteq M$, the restriction $A|\mathbb{E}_U \in \text{Alg}_{\mathbb{E}_U}(\mathcal{S}) \simeq \text{Alg}_{\mathbb{E}_k}(\mathcal{S})$ is grouplike in the sense of Definition 5.1.3.2 (by convention, this condition is vacuous if $k = 0$). For every fibration $E \to M$, the associated algebra $E^! \in \text{Alg}_{\mathbb{E}_M}(\mathcal{S})$ is grouplike. In fact, the converse holds as well: every grouplike object of $\text{Alg}_{\mathbb{E}_M}(\mathcal{S})$ has the form $E^!$, for an essentially unique Serre fibration $E \to M$ with k-connective fibers.

To prove this, we need to introduce a bit of notation. For each open set $U \subseteq M$, let $\mathcal{A}_U$ denote the simplicial category whose objects are Serre fibrations $p : E \to U$ equipped with a section s, where the pair (U, E) is a relative CW complex and the fibers of p are k-connective; an n-simplex of $\text{Map}_{\mathcal{C}_U}(E, E')$ is a commutative diagram

$$
\begin{array}{ccc}
E \times \Delta^n & \xrightarrow{\ \ f\ \ } & E' \\
& \searrow{\scriptstyle p} \quad \swarrow{\scriptstyle p'} & \\
& U, &
\end{array}
$$

such that f respects the preferred sections of p and p'. Let $\mathcal{B}_U$ denote the full subcategory of $\text{Alg}_{\mathbb{E}_U}(\mathcal{S})$ spanned by the grouplike objects. The construction $E \mapsto E^!$ determines a functor $\theta_U : \text{N}(\mathcal{A}_U) \to \mathcal{B}_U$, which

we claim is an equivalence of ∞-categories. If $U \simeq \mathbf{R}^k$ is an open disk in M, then this assertion follows from Theorem 5.1.3.6 (at least if $k > 0$; the case $k = 0$ is trivial). Let $\mathcal{J}$ denote the collection of all open subsets $U \subseteq M$ which are homeomorphic to $\mathbf{R}^k$, partially ordered by inclusion. This collection of open sets satisfies the following condition:

($*$) For every point $x \in M$, the subset $\mathcal{J}_x = \{U \in \mathcal{J} : x \in U\}$ has weakly contractible nerve (in fact, $\mathcal{J}_x^{op}$ is filtered, since every open subset of M containing x contains an open disk around x).

We have a commutative diagram of ∞-categories

$$
\begin{array}{ccc}
\mathrm{N}(\mathcal{A}_M) & \xrightarrow{\ \theta_M\ } & \mathcal{B}_M \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\
\varprojlim_{U \in \mathcal{J}^{op}} \mathrm{N}(\mathcal{A}_U) & \longrightarrow & \varprojlim_{U \in \mathcal{J}^{op}} \mathcal{B}_U
\end{array}
$$

(here the limits are taken in the ∞-category $\mathcal{C}\mathrm{at}_\infty$). Here the lower horizontal map is an equivalence of ∞-categories. Consequently, to prove that θ_M is an equivalence of ∞ categories, it suffices to show that the vertical maps are equivalences of ∞-categories. We consider each in turn.

For each $U \subseteq M$, let $\mathcal{C}_U$ denote the simplicial category whose objects are Kan fibrations $p : X \to \mathrm{Sing}(U)$. The functor $E \mapsto \mathrm{Sing}(E)$ determines an equivalence of ∞-categories $\mathrm{N}(\mathcal{A}_U) \to \mathrm{N}(\mathcal{C}_U)_*$. Consequently, to show that ϕ is a categorical equivalence, it will suffice to show that the associated map $\mathrm{N}(\mathcal{C}_M) \to \varprojlim_{U \in \mathcal{J}^{op}} \mathrm{N}(\mathcal{C}_U)$ is a categorical equivalence. This is equivalent to the requirement that $\mathrm{Sing}(M)$ is a colimit of the diagram $\{\mathrm{Sing}(U)\}_{U \in \mathcal{J}}$ in the ∞-category $\mathcal{S}$, which follows from ($*$) and Theorem A.3.1.

To prove that ψ is a categorical equivalence, it suffices to show that $\mathrm{Alg}_{\mathbb{E}_M}(\mathcal{S})$ is a limit of the diagram $\{\mathrm{Alg}_{\mathbb{E}_U}(\mathcal{S})\}_{U \in \mathcal{J}^{op}}$. For each $U \subseteq M$, let $\mathcal{D}_U$ denote the ∞-category $\mathrm{Alg}_{\mathcal{O}_U}(\mathcal{S})$, where $\mathcal{O}_U^\otimes$ denotes the generalized ∞-operad $\mathrm{BTop}(k)^\otimes \times_{\mathrm{BTop}(k)^{\amalg}} (B_U \times \mathrm{N}(\mathcal{F}\mathrm{in}_*))$. It follows that the restriction functor $A \mapsto A | \mathcal{O}^\otimes$ determines an equivalence of ∞-categories $\mathrm{Alg}_{\mathbb{E}_U}(\mathcal{S}) \to \mathcal{D}_U$. It will therefore suffices to show that $\mathcal{D}_M$ is a limit of the diagram of ∞-categories $\{\mathcal{D}_U\}_{U \in \mathcal{J}^{op}}$. To prove this, we show that the functor $U \mapsto \mathcal{O}_U^\otimes$ exhibits the generalized ∞-operad $\mathcal{O}_M^\otimes$ as a homotopy colimit of the generalized ∞-operads $\{\mathcal{O}_U^\otimes\}_{U \in \mathcal{J}}$. For this, it is sufficient to show that the Kan complex B_M is a homotopy colimit of the diagram $\{B_U\}_{U \in \mathcal{J}}$, which follows from Remark 5.2.4.2, ($*$), and Theorem A.3.1.

Remark 5.3.6.12. In proving Theorem 5.3.6.6, it is sufficient to treat the case where the manifold M is connected. To see this, we note that for every open set $U \subseteq M$, we have a map $\theta_U : \int_U E^! \to \Gamma_c(U; E)$. Assume that θ_U is a homotopy equivalence whenever U is connected. We will prove that θ_U is a homotopy equivalence whenever the set of connected components $\pi_0(U)$ is finite. It will then follow that θ_U is an equivalence for every open set $U \subseteq M$, since the construction $U \mapsto \theta_U$ commutes with filtered colimits; in particular, it will follow that θ_M is a homotopy equivalence.

To carry out the argument, let $U \subseteq M$ be an open set with finitely many connected components $U_1, \ldots, U_n$, so that we have a commutative diagram

$$
\begin{array}{ccc}
\prod_{1 \le i \le n} \int_{U_i} E^! & \xrightarrow{\ \theta_{U_1} \times \cdots \times \theta_{U_n}\ } & \prod_{1 \le i \le n} \Gamma_c(U_i, E) \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\
\int_U E^! & \xrightarrow{\ \ \ \theta_U\ \ \ } & \Gamma_c(U, E).
\end{array}
$$

The map $\theta_{U_1} \times \cdots \times \theta_{U_n}$ is a homotopy equivalence since each U_i is connected, the map ϕ is a homotopy equivalence by Theorem 5.3.3.1, and the map ψ is an isomorphism of Kan complexes; it follows that θ_U is a homotopy equivalence as desired.

Notation 5.3.6.13. Let $p : E \to M$ be as in Definition 5.3.6.2. Given a compact set $K \subseteq M$, we let $\Gamma_K(M; E)$ denote the simplicial set whose n-simplices are commutative diagrams

$$(|\Delta^n| \times M) \coprod_{|\Delta^n| \times (M-K) \times \{0\}} (|\Delta^n| \times (M-K) \times [0,1])$$

$$E \xleftarrow{\quad f \quad} \xrightarrow{\qquad\qquad p \qquad\qquad} M$$

such that $f|(|\Delta^n| \times (M - K) \times \{1\})$ is given by the composition

$$|\Delta^n| \times V \times \{1\} \to (M - K) \subseteq M \xrightarrow{s} E.$$

In other words, an n-simplex of $\Gamma_K(M; E)$ is an n-parameter family of sections of E, together with a nullhomotopy of this family of sections on the open set $M - K$.

Note that any n-simplex of $\Gamma_c(M; E)$ which is trivial on $M - K$ extends canonically to an n-simplex of $\Gamma_K(M; E)$, by choosing the nullhomotopy to be constant. In particular, if $U \subseteq M$ is any open set, then we obtain a canonical map

$$\Gamma_c(U; E) \to \varinjlim_{K \subseteq U} \Gamma_K(M; E),$$

where the colimit is taken over the (filtered) collection of all compact subsets of U.

Remark 5.3.6.14. The simplicial set $\Gamma_K(M; E)$ can be identified with the homotopy fiber of the restriction map $\mathcal{F}(M) \to \mathcal{F}(M - K)$, where $\mathcal{F} \in \mathcal{S}hv(M)$ is the sheaf associated to the fibration $p : E \to M$.

Lemma 5.3.6.15. *Let $p : E \to M$ be a Serre fibration equipped with a section s (as in Definition 5.3.6.2), let $U \subseteq M$ be an open set. Then the canonical map*

$$\Gamma_c(U; E) \to \varinjlim_{K \subseteq U} \Gamma_K(M; E)$$

is a homotopy equivalence.

Proof. It will suffice to show that if $A \subseteq B$ is an inclusion of finite simplicial sets and we are given a commutative diagram

$$
\begin{array}{ccc}
A & \longrightarrow & \Gamma_c(U; E) \\
\downarrow & \nearrow^{f'} & \downarrow \\
B & \xrightarrow{f} & \varinjlim_{K \subseteq U} \Gamma_K(M; E),
\end{array}
$$

then, after modifying f by a homotopy that is constant on A, there exists a dotted arrow f' as indicated in the diagram (automatically unique, since the right vertical map is a monomorphism). Since B is finite, we may assume that f factors through $\Gamma_K(M; E)$ for some compact subset $K \subseteq U$. Such a factorization determines a pair (F, h), where $F : |B| \times M \to E$ is a map of spaces over M and $h : |B| \times (M-K) \times [0,1] \to E$ is a fiberwise homotopy of $F|(|B| \times (M - K))$ to the composite map $|B| \times (M - K) \to M \xrightarrow{s} E$. Choose a continuous map $\lambda : M \to [0,1]$ which is supported in a compact subset K' of U and takes the value 1 in a neighborhood of K. Let $F' : |B| \times M \to E$ be the map defined by the formula

$$F'(b, x) = \begin{cases} F(b, x) & \text{if } x \in K \\ h(b, x, 1 - \lambda(x)) & \text{if } x \notin K. \end{cases}$$

Then F' determines a map $B \to \Gamma_c(U; E)$ such that the composite map $B \to \Gamma_c(U; E) \to \varinjlim_{K \subseteq U} \Gamma_K(M; E)$ is homotopic to f relative to A, as desired. $\qquad\square$

We now proceed with the proof of Theorem 5.3.6.6. If M is homeomorphic to Euclidean space $\mathbf{R}^k$, then $\mathrm{Disj}(M)$ contains M as a final object and Theorem 5.3.6.6 is obvious. Combining this observation with Remark 5.3.6.12, we obtain an immediate proof in the case $k = 0$. If $k = 1$, then we may assume (by virtue of Remark 5.3.6.12) that M is homeomorphic to either an open interval (in which case there is nothing to prove) or to the circle S^1. The latter case requires some argument:

Proof of Theorem 5.3.6.6 for $M = S^1$. Choose a small open disk $U \subseteq S^1$ and a parametrization $\psi : \mathbf{R} \simeq U$, and let $\chi : \mathbf{D}(S^1)_{/\psi} \to \mathcal{S}$ be the diagram determined by $E^!$. According to Theorem 5.3.3.11, the functor χ is equivalent to a composition $\mathbf{D}(S^1)_{/\psi} \to \mathrm{N}(\Delta^{op}) \xrightarrow{B_\bullet} \mathcal{S}$ for some simplicial object $B_\bullet$ of $\mathcal{S}$, and the topological chiral homology $\int_{S^1} E^!$ can be identified with the geometric realization $|B_\bullet|$. We wish to show that the canonical map $\theta : |B_\bullet| \to \Gamma_c(S^1; E)$ is an equivalence in $\mathcal{S}$. Since $\mathcal{S}$ is an ∞-topos, it will suffice to verify the following pair of assertions:

(a) The map $\theta_0 : B_0 \to \Gamma_c(S^1; E)$ is an effective epimorphism. In other words, θ_0 induces a surjection $\pi_0 B_0 = \pi_0 \Gamma_c(U; E) \to \pi_0 \Gamma_c(S^1; E)$.

(b) The map θ exhibits $B_\bullet$ as a Čech nerve of θ_0. That is, for each $n \geq 0$, the canonical map

$$B_n \to B_0 \times_{\Gamma_c(S^1;E)} \cdots \times_{\Gamma_c(S^1;E)} B_0$$

is a homotopy equivalence (here the fiber products are taken in the ∞-category $\mathcal{S}$).

To prove (a), let $s : S^1 \to E$ denote our given section of the Serre fibration $p : E \to S^1$, and let $f : S^1 \to E$ denote any other section of p. Choose a point $x \in U$. Since $S^1 - \{x\}$ is contractible and the fibers of p are connected, there exists a (fiberwise) homotopy $h : (S^1 - \{x\}) \times [0,1] \to E$ from $f|(S^1 - \{x\})$ to $s|(S^1 - \{x\})$. Let $\lambda : S^1 \to [0,1]$ be a continuous function which vanishes in a neighborhood of x, and takes the value 1 outside a compact subset of U. Let $h' : S^1 \times [0,1] \to E$ be the map defined by

$$h'(y,t) = \begin{cases} f(x) & \text{if } y = x \\ h(y, t\lambda(y)) & \text{if } y \neq x. \end{cases}$$

Then h' determines a homotopy from f to another section $f' = h'|(S^1 \times \{1\})$, whose support is a compact subset of U.

We now prove (b). Choose a collection of open disks $U_1, \ldots, U_n \subseteq S^1$ which are disjoint from one another and from U. Then the closed set $S^1 - (U \cup U_1 \cup \ldots \cup U_n)$ is a disjoint union of connected components $A_0, \ldots, A_n$. Unwinding the definitions, we are required to show that the simplicial set $\Gamma_c(U \cup U_1 \cup \ldots \cup U_n; E)$ is a homotopy product of the simplicial sets $\Gamma_c(S^1 - A_i; E)$ in the model category $(\mathcal{S}et_\Delta)_{/\Gamma_c(S^1;E)}$. For each index i, let $\mathcal{U}_i$ denote the collection of all open subsets of S^1 that contain A_i, and let $\mathcal{U} = \bigcap \mathcal{U}_i$. It follows from Lemma 5.3.6.15 that we have canonical homotopy equivalences

$$\Gamma_c(S^1 - A_i; E) \to \varinjlim_{V \in \mathcal{U}_i} \Gamma_{S^1 - V}(S^1; E)$$

$$\Gamma_c(U \cup U_1 \cup \ldots \cup U_n; E) \to \varinjlim_{V \in \mathcal{U}} \Gamma_{S^1 - V}(S^1; E).$$

Note that for each $V \in \mathcal{U}_i$, the forgetful map $\Gamma_{S^1 - V}(S^1; E) \to \Gamma_c(S^1; E)$ is a Kan fibration. It follows that each $\varinjlim_{V \in \mathcal{U}_i} \Gamma_{S^1 - V}(S^1; E)$ is a fibrant object of $(\mathcal{S}et_\Delta)_{/\Gamma_c(S^1;E)}$, so the relevant homotopy product coincides with the actual product $\prod_{0 \leq i \leq n} \varinjlim_{V_i \in \mathcal{U}_i} \Gamma_{S^1 - V_i}(S^1; E)$ (formed in the category $(\mathcal{S}et_\Delta)_{/\Gamma_c(S^1;E)}$. Let $\mathcal{V}$ denote the partially ordered set of sequences $(V_0, \ldots, V_n) \in \mathcal{U}_0 \times \cdots \times \mathcal{U}_n$ such that $V_i \cap V_j = \emptyset$ for $i \neq j$. We observe that the inclusion $\mathcal{V} \subseteq (\mathcal{U}_0 \times \cdots \times \mathcal{U}_n)$ is right cofinal, and the construction $(V_0, \ldots, V_n) \mapsto \bigcup V_i$ is a right cofinal map from $\mathcal{V}$ to $\mathcal{U}$. Consequently, we obtain isomorphisms

$$\varinjlim_{V \in \mathcal{U}} \Gamma_{S^1 - V}(S^1; E) \simeq \varinjlim_{(V_0, \ldots, V_n)} \Gamma_{S^1 - \bigcup V_i}(S^1; E)$$

$$\prod_{0 \leq i \leq n} \varinjlim_{V_i \in \mathcal{U}_i} \Gamma_{S^1 - V_i}(S^1; E) \simeq \varinjlim_{(V_0, \ldots, V_n) \in \mathcal{V}} \prod_{0 \leq i \leq n} \Gamma_{S^1 - V_i}(S^1; E);$$

here the product is taken in the category $(\mathcal{S}et_\Delta)_{/\Gamma_c(S^1; E)}$. To complete the proof, it suffices to show that for each $(V_0, V_1, \ldots, V_n) \in \mathcal{V}$, the canonical map

$$\theta : \Gamma_{S^1 - \bigcup V_i}(S^1; E) \to \prod_{0 \leq i \leq n} \Gamma_{S^1 - V_i}(S^1; E)$$

is a homotopy equivalence.

We now complete the proof by observing that θ is an isomorphism (since the open sets V_i are assumed to be pairwise disjoint). $\square$

Our proof of Theorem 5.3.6.6 in higher dimensions will use a rather different method. We first consider the following *linear* version of Theorem 5.3.6.6, which is an easy consequence of the version of Verdier duality presented in §5.3.5.

Proposition 5.3.6.16. *Let M be a k-manifold, let $\mathcal{F} \in \mathrm{Shv}(M; \mathrm{Sp})$ be a locally constant Sp-valued sheaf on M, and let $\mathcal{F}' \in \mathrm{Shv}(M; \mathcal{S}_*)$ be the sheaf of pointed spaces given by the formula $\mathcal{F}'(U) = \Omega_*^\infty \mathcal{F}(U)$. Assume that for every open disk $U \subseteq M$, the spectrum $\mathcal{F}(U)$ is k-connective. Then $\mathcal{F}'$ exhibits $\Gamma_c(M; \mathcal{F}')$ as a colimit of the diagram $\{\Gamma_c(U; \mathcal{F}')\}_{U \in \mathrm{Disj}(M)}$ in the ∞-category $\mathcal{S}_*$.*

Proof. It follows from Corollary 5.3.5.13 that $\mathcal{F}$ exhibits $\Gamma_c(M; \mathcal{F})$ as a colimit of the diagram

$$\{\Gamma_c(U; \mathcal{F})\}_{U \in \mathrm{Disj}(M)}$$

in the ∞-category Sp of spectra. It will therefore suffice to show that the functor Ω_*^∞ preserves the colimit of the diagram $\{\Gamma_c(U; \mathcal{F})\}_{U \in \mathrm{Disj}(M)}$.

Let us regard the ∞-category Sp as endowed with its Cartesian symmetric monoidal structure, which (by virtue of Proposition 2.4.3.19) is also the coCartesian symmetric monoidal structure. The functor $U \mapsto \Gamma_c(U; \mathcal{F})$ determines a functor $\mathrm{N}(\mathrm{Disk}(M)) \to \mathrm{Sp}$, which extends to a map of ∞-operads $\mathrm{N}(\mathrm{Disk}(M))^{\amalg} \to \mathrm{Sp}^{\amalg}$ and therefore determines an algebra $A \in \mathrm{Alg}_{\mathrm{N}(\mathrm{Disk}(M))}(\mathrm{Sp})$. Since $\mathcal{F}$ is locally constant, the algebra A is locally constant and is therefore equivalent to a composition

$$\mathrm{N}(\mathrm{Disk}(M))^{\otimes} \to \mathbb{E}_M^{\otimes} \overset{B}{\to} \mathrm{Sp}^{\times}.$$

Let $A' : \mathrm{N}(\mathrm{Disk}(M))^{\otimes} \to \mathrm{Sp}$ and $B' : \mathbb{E}_M \to \mathrm{Sp}$ be the associated monoid objects of Sp (see Proposition 2.4.2.5). We wish to show that Ω_*^∞ preserves the colimit of the diagram $\{A'(U)\}_{U \in \mathrm{Disj}(M)}$. In view of Proposition 5.3.2.13, it will suffice to prove that Ω_*^∞ preserves the colimit of the diagram $B'|\mathbf{D}(M)$. For every open set $U = U_1 \cup \ldots \cup U_n$ of $\mathbf{D}(M)$, the spectrum $B'(U) \simeq \prod_{1 \leq i \leq n} B'(U_i) \simeq \prod_{1 \leq i \leq n} \Omega^k \mathcal{F}(U_i)$ is connective. Since the ∞-category $\mathbf{D}(M)$ is sifted (Proposition 5.3.2.15), the desired result follows from Corollary 5.1.3.9. $\square$

Recall that if $\mathcal{X}$ is an ∞-topos, then colimits in $\mathcal{X}$ are universal: that is, for every morphism $f : X \to Y$ in $\mathcal{X}$, the fiber product construction $Z \mapsto X \times_Y Z$ determines a colimit-preserving functor from $\mathcal{X}_{/Y}$ to $\mathcal{X}_{/X}$. In other words, the fiber product $X \times_Y Z$ is a colimit-preserving functor of Z. The same argument shows that $X \times_Y Z$ is a colimit-preserving functor of X. However, the dependence of the fiber product $X \times_Y Z$ on Y is more subtle. The following result, which asserts that the construction $Y \mapsto X \times_Y Z$ commutes with colimits in many situations.

Lemma 5.3.6.17. *Let $\mathcal{X}$ be an ∞-topos, and let $\mathcal{X}_*^{\geq 1}$ denote the full subcategory of $\mathcal{X}_*$ spanned by the pointed connected objects. Let $\mathcal{C}$ denote the ∞-category $\mathrm{Fun}(\Lambda_2^2, \mathcal{X}) \times_{\mathrm{Fun}(\{2\}, \mathcal{X})} \mathcal{X}_*^{\geq 1}$ whose objects are diagrams $X \to Z \leftarrow Y$ in $\mathcal{X}$, where Z is a pointed connected object of $\mathcal{X}$. Let $F : \mathcal{C} \to \mathcal{X}$ be the functor*

$$\mathcal{C} \longrightarrow \mathrm{Fun}(\Lambda_2^2, \mathcal{X}) \overset{\varprojlim}{\longrightarrow} \mathcal{X}$$

given informally by the formula $(X \to Z \leftarrow Y) \mapsto X \times_Z Y$. The F preserves sifted colimits.

Proof. Let $\mathcal{C}'$ denote the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta}_+^{op}), \mathfrak{X})$ spanned by those functors G which corresponding to diagrams of augmented simplicial objects

$$
\begin{array}{ccc}
W_\bullet & \longrightarrow & X_\bullet \\
\downarrow & & \downarrow \\
Y_\bullet & \longrightarrow & Z_\bullet
\end{array}
$$

which satisfy the following conditions:

(i) The object Z_0 is final.

(ii) The augmentation map $Z_0 \to Z_{-1}$ is an effective epimorphism (equivalently, Z_{-1} is a connected object of $\mathfrak{X}$).

(iii) Let K denote the full subcategory of $\Delta^1 \times \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta}_+^{op})$ spanned by the objects $(1, 0, [-1])$, $(0, 1, [-1])$, $(1, 1, [-1])$, and $(1, 1, [0])$. Then G is a right Kan extension of $G|K$. In particular, the diagram

$$
\begin{array}{ccc}
W_{-1} & \longrightarrow & X_{-1} \\
\downarrow & & \downarrow \\
Y_{-1} & \longrightarrow & Z_{-1}
\end{array}
$$

is a pullback square.

It follows from Proposition T.4.3.2.15 that the restriction map $G \mapsto G|K$ induces a trivial Kan fibration $q : \mathcal{C}' \to \mathcal{C}$. Note that the functor F is given by composing a section of q with the evaluation functor $G \mapsto G(0, 0, [-1])$. To prove that F commutes with sifted colimits, it will suffice to show that $\mathcal{C}'$ is stable under sifted colimits in $\mathrm{Fun}(\Delta^1 \times \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta}_+^{op}), \mathfrak{X})$.

Let $\mathcal{D}$ be the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta}^{op}), \mathfrak{X})$ spanned by those diagrams of simplicial objects

$$
\begin{array}{ccc}
W_\bullet & \longrightarrow & X_\bullet \\
\downarrow & & \downarrow \\
Y_\bullet & \longrightarrow & Z_\bullet
\end{array}
$$

satisfying the following conditions:

(i') The simplicial object $Z_\bullet$ is a group object of $\mathfrak{X}$ (that is, $Z_\bullet$ is a groupoid object of $\mathfrak{X}$ and Z_0 is final in $\mathfrak{X}$; equivalently, for each $n \geq 0$ the natural map $Z_n \to Z_1^n$ is an equivalence).

(ii') For each integer n and each inclusion $[0] \hookrightarrow [n]$, the induced maps

$$X_n \to X_0 \times Z_n \quad Y_n \to Y_0 \times Z_n \quad W_n \to X_0 \times Y_0 \times Z_n$$

are equivalences.

Since the product functor $\mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ commutes with sifted colimits (Proposition T.5.5.8.6), we deduce that $\mathcal{D}$ is stable under sifted colimits in $\mathrm{Fun}(\Delta^1 \times \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta}^{op}), \mathfrak{X})$. Let $\mathcal{D}' \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta}_+^{op}), \mathfrak{X})$ be the full subcategory spanned by those functors G such that G is a left Kan extension of $G_0 = G|(\Delta^1 \times \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta}_+^{op}))$ and $G_0 \in \mathcal{D}$. Then $\mathcal{D}'$ is stable under sifted colimits in $\mathrm{Fun}(\Delta^1 \times \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta}_+^{op}), \mathfrak{X})$. We will complete the proof by showing that $\mathcal{D}' = \mathcal{C}'$.

Suppose first that $G \in \mathcal{C}'$, corresponding to a commutative diagram of augmented simplicial objects

$$
\begin{array}{ccc}
W_\bullet & \longrightarrow & X_\bullet \\
\downarrow & & \downarrow \\
Y_\bullet & \longrightarrow & Z_\bullet.
\end{array}
$$

Condition (iii) guarantees that $Z_\bullet$ is a Čech nerve of the augmentation map $Z_0 \to Z_{-1}$. Since this augmentation map is an effective epimorphism (by virtue of (ii)), we deduce that the augmented simplicial object $Z_\bullet$ is a colimit diagram. Condition (iii) guarantees that the natural maps $X_n \to Z_n \times_{Z_{-1}} X_{-1}$ is are equivalences. Since colimits in $\mathcal{X}$ are universal, we deduce that $X_\bullet$ is also a colimit diagram. The same argument shows that $Y_\bullet$ and $W_\bullet$ are colimit diagrams, so that G is a left Kan extension of $G_0 = G|(\Delta^1 \times \Delta^1 \times N(\Delta^{op}))$. To complete the proof that $G \in \mathcal{D}'$, it suffices to show that G_0 satisfies conditions (i') and (ii'). Condition (i') follows easily from (i) and (iii), and condition (ii') follows from (iii).

Conversely, suppose that $G \in \mathcal{D}'$; we wish to show that G satisfies conditions (i), (ii), and (iii). Condition (i) follows immediately from (i'), and condition (ii) from the fact that $Z_\bullet$ is a colimit diagram. It remains to prove (iii). Let K' denote the full subcategory of $\Delta^1 \times \Delta^1 \times N(\Delta^{op}_+)$ spanned by the objects $(0, 1, [-1])$, $(1, 0, [-1])$, and $\{(1, 1, [n])\}_{n \geq -1}$. Since $\mathcal{X}$ is an ∞-topos and $Z_\bullet$ is the colimit of a groupoid object of $\mathcal{X}$, it is a Čech nerve of the augmentation map $Z_0 \to Z_{-1}$. This immediately implies that $G|K'$ is a right Kan extension of $G|K$. To complete the proof, it will suffice to show that G is a right Kan extension of $G|K'$ (Proposition T.4.3.2.8).

We first claim that G is a right Kan extension of $G|K'$ at $(0, 1, [n])$ for each $n \geq 0$. Equivalently, we claim that each of the maps

$$
\begin{array}{ccc}
X_n & \longrightarrow & X_{-1} \\
\downarrow & & \downarrow \\
Z_n & \longrightarrow & Z_{-1}
\end{array}
$$

is a pullback diagram. Since $X_\bullet$ and $Z_\bullet$ are both colimit diagrams, it will suffice to show that the map $X_\bullet \to Z_\bullet$ is a Cartesian transformation of simplicial objects (Theorem T.6.1.3.9): in other words, it will suffice to show that for every morphism $[m] \to [n]$ in Δ, the analogous diagram

$$
\begin{array}{ccc}
X_n & \longrightarrow & X_m \\
\downarrow & & \downarrow \\
Z_n & \longrightarrow & Z_m
\end{array}
$$

is a pullback square. Choosing a map $[0] \hookrightarrow [m]$, we obtain a larger diagram

$$
\begin{array}{ccccc}
X_n & \longrightarrow & X_m & \longrightarrow & X_0 \\
\downarrow & & \downarrow & & \downarrow \\
Z_n & \longrightarrow & Z_m & \longrightarrow & Z_0.
\end{array}
$$

Since Z_0 is a final object of $\mathcal{X}$, condition (ii') implies that the right square and the outer rectangle are pullback diagrams, so that the left square is a pullback diagram as well. A similar argument shows that $Y_\bullet \to Z_\bullet$ and $W_\bullet \to Z_\bullet$ are Cartesian transformations, so that G is a right Kan extension of $G|K'$ at $(1, 0, [n])$ and $(0, 0, [n])$ for each $n \geq 0$.

To complete the proof, we must show that G is a right Kan extension of $G|K'$ at $(0, 0, [-1])$: in other

words, that the diagram σ :

$$
\begin{array}{ccc}
W_{-1} & \longrightarrow & X_{-1} \\
\downarrow & & \downarrow \\
Y_{-1} & \longrightarrow & Z_{-1}
\end{array}
$$

is a pullback square. Since the map $\epsilon : Z_0 \to Z_{-1}$ is an effective epimorphism, it suffices to show that the diagram σ becomes a pullback square after base change along ϵ. In other words, we need only show that the diagram

$$
\begin{array}{ccc}
W_0 & \longrightarrow & X_0 \\
\downarrow & & \downarrow \\
Y_0 & \longrightarrow & Z_0
\end{array}
$$

is a pullback square, which follows immediately from (ii'). $\hfill\square$

Proof of Theorem 5.3.6.6 for $k \geq 2$. Replacing E by $|\operatorname{Sing}(E)|$, we can assume without loss of generality that E is the geometric realization of a simplicial set X equipped with a Kan fibration $X \to \operatorname{Sing}(M)$. We wish to prove that the canonical map $\int_M E^! \to \Gamma_c(M; E)$ is a homotopy equivalence. For this, it suffices to show that $\tau_{\leq m}(\int_M E^!) \to \tau_{\leq m}\Gamma_c(M; E)$ is a homotopy equivalence for every integer $m \geq 0$. Since the truncation functor $\tau_{\leq m} : \mathcal{S} \to \tau_{\leq m}\mathcal{S}$ preserves small colimits and finite products, Proposition 5.3.2.17 allows us to identify the left hand side with the topological chiral homology $\int_M(\tau_{\leq m}E^!)$ in the ∞-category $\tau_{\leq m}\mathcal{S}$.

Regard X as an object of the ∞-topos $\mathcal{X} = \mathcal{S}_{/\operatorname{Sing}(M)}$, let X' be an $(m+k)$-truncation of X, and let $E' = |X'|$. The map $X \to X'$ induces a map $E^! \to E'^!$ which is an equivalence on m-truncations, and therefore induces an equivalence $\tau_{\leq m}(\int_M E^!) \to \tau_{\leq m}(\int_M E'^!)$. This equivalence fits into a commutative diagram

$$
\begin{array}{ccc}
\tau_{\leq m} \int_M E^! & \xrightarrow{\ \alpha\ } & \tau_{\leq m}\Gamma_c(M; E) \\
\downarrow & & \downarrow{\scriptstyle \beta} \\
\tau_{\leq m} \int_M E'^! & \xrightarrow{\ \alpha'\ } & \tau_{\leq m}\Gamma_c(M; E'),
\end{array}
$$

where β is also an equivalence (since M has dimension k). Consequently, to prove that α is an equivalence, it suffices to prove that α' is an equivalence. We may therefore replace X by X' and thereby reduce to the case where X is an n-truncated object of $\mathcal{X}$ for some $n \gg 0$.

The proof now proceeds by induction on n. If $n < k$, then X is both k-connective and $(k-1)$-truncated, and is therefore equivalent to the final object of $\mathcal{X}$. In this case, both $\int_M E^!$ and $\Gamma_c(M; E)$ are contractible and there is nothing to prove. Assume therefore that $n \geq k \geq 2$. Let $A = \pi_n X$, regarded as an object of the topos of discrete objects $\operatorname{Disc}\mathcal{X}_{/X}$. Since X is a 2-connective object of $\mathcal{X}$, this topos is equivalent to the topos of discrete objects $\operatorname{Disc}\mathcal{X}$ of local systems of sets on the manifold M. We will abuse notation by identifying A with its image under this equivalence; let $K(A, n+1)$ denote the associated Eilenberg-MacLane objects of $\mathcal{X}$. Let $Y = \tau_{\leq n-1}X$, so that X is an n-gerbe over Y banded by A and therefore fits into a pullback square

$$
\begin{array}{ccc}
X & \longrightarrow & \mathbf{1} \\
\downarrow & & \downarrow \\
Y & \longrightarrow & K(A, n+1)
\end{array}
$$

Let $E_0 = |Y|$ and $E_1 = |K(A, n+1)|$, so that we have a fiber sequence $E \to E_0 \to E_1$ of Serre fibrations

over M. We then have a commutative diagram

$$
\begin{array}{ccc}
\int_M E^! & \xrightarrow{\ \alpha\ } & \int_M E_0^! & \longrightarrow & \int_M E_1^! \\
\downarrow & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} \\
\Gamma_c(M;E) & \longrightarrow & \Gamma_c(M;E_0) & \longrightarrow & \Gamma_c(M;E_1)
\end{array}
$$

where α_0 is a homotopy equivalence by the inductive hypothesis, and α_1 is a homotopy equivalence by Proposition 5.3.6.16. Consequently, to prove that α is a homotopy equivalence, it suffices to prove that the upper line is a fiber sequence. The algebras $E^!$, $E_0^!$, and $E_1^!$ determine functors $\chi, \chi_0, \chi_1 : \mathbf{D}(M) \to \mathcal{S}_*$, which fit into a pullback square

$$
\begin{array}{ccc}
\chi & \longrightarrow & * \\
\downarrow & & \downarrow \\
\chi_0 & \longrightarrow & \chi_1.
\end{array}
$$

To complete the proof, it suffices to show that the induced square of colimits

$$
\begin{array}{ccc}
\varinjlim(\chi) & \longrightarrow & * \\
\downarrow & & \downarrow \\
\varinjlim(\chi_0) & \longrightarrow & \varinjlim(\chi_1)
\end{array}
$$

is again a pullback diagram. Since $n \geq k$, the object $K(A, n+1)$ is $(k+1)$-connective, so that χ_1 takes values in connected spaces. The desired result now follows from Lemma 5.3.6.17, since $\mathbf{D}(M)$ is sifted (Proposition 5.3.2.15). $\qquad\square$

Chapter 6

Algebraic Structures on ∞-Categories

Let $\mathcal{C}$ be a monoidal ∞-category and let $M \in \mathcal{C}$. Given an associative algebra object $A \in \mathrm{Alg}(\mathcal{C})$, we can define an *action* of A on M to be an object $\overline{M} \in \mathrm{LMod}_A(\mathcal{C})$ lying over the object $M \in \mathcal{C}$: in other words, a specification of a left A-module structure on M. In §6.1, we will see that there is often an associative algebra object $\mathrm{End}(M) \in \mathrm{Alg}(\mathcal{C})$ which is *universal* among associative algebra objects of $\mathcal{C}$ with an action on M. We will refer to $\mathrm{End}(M)$ as an *endomorphism algebra* of M.

In §6.2, we specialize to the case where $\mathcal{C}$ is the ∞-category Cat_∞, so that M can be identified with a small ∞-category $\mathcal{M}$. In this case, the endomorphism algebra $\mathrm{End}(M)$ can be identified with the ∞-category $\mathrm{Fun}(\mathcal{M}, \mathcal{M})$ of functors from $\mathcal{M}$ to itself. The ∞-category $\mathcal{M}$ is naturally left-tensored over $\mathrm{Fun}(\mathcal{M}, \mathcal{M})$. Consequently, for every algebra object $T \in \mathrm{Alg}(\mathrm{Fun}(\mathcal{M}, \mathcal{M}))$, there is an associated ∞-category $\mathrm{Mod}_T(\mathcal{M})$ of left T-modules, equipped with a forgetful functor $\mathrm{Mod}_T(\mathcal{M}) \to \mathcal{M}$. In general, we say that an ∞-category $\mathcal{N}$ equipped with a forgetful functor $G : \mathcal{N} \to \mathcal{M}$ is *monadic over* $\mathcal{M}$ if there exists an algebra object $T \in \mathrm{Alg}(\mathrm{Fun}(\mathcal{M}, \mathcal{M}))$ such that $\mathcal{N}$ is equivalent to $\mathrm{Mod}_T(\mathcal{M})$ (via an equivalence which is compatible with the forgetful functor to $\mathcal{M}$). Our main result is an ∞-categorical generalization of the Barr-Beck theorem, which gives necessary and sufficient conditions for $\mathcal{N}$ to be monadic over $\mathcal{M}$ (Theorem 6.2.2.5).

Let us now return to the case of a general monoidal ∞-category $\mathcal{C}$. We can regard the construction $(M \in \mathcal{M}) \to (\mathrm{End}(M) \in \mathrm{Alg}(\mathcal{C}))$ as functorially assigning to every ∞-category $\mathcal{M}$ left-tensored over $\mathcal{C}$ and every object $M \in \mathcal{M}$ an algebra object of $\mathcal{C}$. In §6.3, we will consider the left adjoint to this construction, which associates to an algebra $A \in \mathrm{Alg}(\mathcal{C})$ the ∞-category $\mathrm{RMod}_A(\mathcal{C})$ (which is equipped with a distinguished object, given by the right action of the associative algebra A on itself). Under some mild hypotheses, one can show that this functor determines a fully faithful embedding from $\mathrm{Alg}(\mathcal{C})$ into an ∞-category $\mathrm{Cat}_\infty^{\mathrm{Mod}}{}_{\mathfrak{M}/}$ whose objects are ∞-categories left-tensored over $\mathcal{C}$, equipped with a distinguished object (Theorem 6.3.5.5). In other words, there is a very close connection between algebra objects of $\mathcal{C}$ and their ∞-categories of modules.

6.1   Endomorphism Objects

Let M be an abelian group, and let $\mathrm{End}(M)$ denote the set of group homomorphisms from M to itself. Then $\mathrm{End}(M)$ is endowed with the structure of an associative ring, where addition is given pointwise and multiplication is given by composition of homomorphisms. Moreover, we can characterize $\mathrm{End}(M)$ by the following universal property: for any associative ring R, giving a ring homomorphism $\phi : R \to \mathrm{End}(M)$ is equivalent to endowing M with the structure of an R-module.

In this section, we will generalize the above ideas by replacing the ordinary category of abelian groups (with its tensor product) by an arbitrary monoidal ∞-category $\mathcal{C}$. If $M \in \mathcal{C}$ is an object, then an *endomorphism object* for M is an object $E \in \mathcal{C}$ equipped with a map $a : E \otimes M \to M$ which enjoys the following universal property: for every other object $C \in \mathcal{C}$, composition with a induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(C, E) \to \mathrm{Map}_{\mathcal{C}}(C \otimes M, M)$. We will show that if an endomorphism object E exists, then E

has the structure of an associative algebra object of $\mathcal{C}$. Moreover, it is universal among associative algebras which act on M in the following sense: for every associative algebra object $A \in \mathrm{Alg}(\mathcal{C})$, there is a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Alg}(\mathcal{C})}(A, E) \simeq \{A\} \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{LMod}(\mathcal{C}) \times_{\mathcal{C}} \{M\}.$$

We will prove these assertions in §6.1.2 by identifying E with the final object of a monoidal ∞-category $\mathcal{C}[M]$ determined by $\mathcal{C}$ and M. To describe the relevant constructions, it will be convenient to view a monoidal ∞-category not as a fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ but instead in terms of the associated fibration $\mathcal{C}^{\circledast} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$ (see Notation 4.2.2.16). We will explain this shift in perspective in §6.1.1, and show that it entails no essential loss of information.

In the second half of this section, we will describe some applications of the theory of endomorphism algebras. We begin in §6.1.3 by defining the ∞-category $\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$ of nonunital associative algebra objects of $\mathcal{C}$: roughly speaking, these are given by objects $A \in \mathcal{C}$ equipped with a multiplication $A \otimes A \to A$ which is associative up to coherent homotopy, but does not necessarily admit a unit. Our main result (proven in §6.1.3) is Theorem 6.1.3.10: roughly speaking, it asserts that if $A \in \mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$ is a nonunital associative algebra which admits a unit $\mathbf{1} \to A$ up to homotopy, then A can be promoted to a unital associative algebra object of $\mathcal{C}$ in an essentially unique way. The rough idea of the proof is to realize A as the endomorphism algebra of itself as a (nonunital) right A-module. In §6.1.4 we consider a rather different application: we use the theory of endomorphism objects to prove (a version of) Deligne's conjecture regarding the action of the little 2-cubes operad on the Hochschild cochains of a differential graded algebra.

6.1.1   Simplicial Models for Planar ∞-Operads

Suppose we are given a coCartesian fibration of ∞-operads $p : \mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$. According to Example 2.4.2.4, p is classified by an $\mathcal{A}\mathrm{ss}$-monoid object $\chi : \mathcal{A}\mathrm{ss}^{\otimes} \to \mathcal{C}\mathrm{at}_{\infty}$. Proposition 4.1.2.6 implies that χ is determined (up to canonical equivalence) by the composite map

$$\mathrm{N}(\boldsymbol{\Delta})^{op} \xrightarrow{\mathrm{Cut}} \mathcal{A}\mathrm{ss}^{\otimes} \xrightarrow{\chi} \mathcal{C}\mathrm{at}_{\infty} .$$

This composite map classifies the coCartesian fibration $\mathcal{C}^{\circledast} \simeq \mathcal{C}^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$. This suggests an alternative to Definition 4.1.1.10: we could instead define a monoidal ∞-category to be a coCartesian fibration $\mathcal{C}^{\circledast} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$ which is classified by a monoid object $\mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathcal{C}\mathrm{at}_{\infty}$. Our goal in this section is to prove a generalization of this statement, which applies to an arbitrary fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$.

We begin by recalling Notation 4.2.2.16: given a planar ∞-operad $\mathcal{O}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$, we let $\mathcal{O}^{\circledast}$ denote the fiber product $\mathrm{N}(\boldsymbol{\Delta})^{op} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathcal{O}^{\otimes}$. We say that a morphism in $\mathcal{O}^{\circledast}$ is *inert* if its image in $\mathcal{O}^{\otimes}$ is inert. We let $\mathcal{O}^{\circledast,\natural}$ denote the marked simplicial set $(\mathcal{O}^{\circledast}, M)$, where M is the collection of inert morphisms in $\mathcal{O}^{\circledast}$. In particular, we let $\mathrm{N}(\boldsymbol{\Delta})^{op,\natural}$ denote the marked simplicial set $(\mathrm{N}(\boldsymbol{\Delta})^{op}, M)$, where M is the collection of all morphisms $[m] \to [n]$ in $\boldsymbol{\Delta}$ which factor as a composition $[m] \simeq \{i, i+1, \ldots, i+m\} \hookrightarrow [n]$.

The main result of this section can be stated as follows:

Proposition 6.1.1.1. *The construction $\overline{X} \mapsto \overline{X} \times_{\mathcal{A}\mathrm{ss}^{\otimes,\natural}} \mathrm{N}(\boldsymbol{\Delta})^{op,\natural}$ determines a right Quillen equivalence $(\mathcal{P}\mathrm{Op}_{\infty})_{/\mathcal{A}\mathrm{ss}^{\otimes,\natural}} \to (\mathcal{P}\mathrm{Op}_{\infty})_{/\mathrm{N}(\boldsymbol{\Delta})^{op,\natural}}$. Here $\mathcal{P}\mathrm{Op}_{\infty}$ denotes the category of ∞-preoperads (see §2.1.4), endowed with the ∞-operadic model structure.*

Remark 6.1.1.2. Proposition 4.1.2.6 implies that the map $\mathrm{N}(\boldsymbol{\Delta})^{op,\natural} \to \mathcal{A}\mathrm{ss}^{\otimes,\natural}$ is a weak equivalence of ∞-preoperads. However, Proposition 6.1.1.1 does not follow formally from this, because $\mathrm{N}(\boldsymbol{\Delta})^{op,\natural}$ is not a fibrant object of $\mathcal{P}\mathrm{Op}_{\infty}$ and the ∞-operadic model structure on $\mathcal{P}\mathrm{Op}_{\infty}$ is not right proper.

Definition 6.1.1.3. Let $p : \mathcal{O}^{\circledast} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$ be a categorical fibration. We will say that a morphism α in $\mathcal{O}^{\circledast}$ is *inert* if it is p-coCartesian and $p(\alpha)$ is an inert morphism in $\mathrm{N}(\boldsymbol{\Delta})^{op}$. Let M be the collection of inert morphisms in $\mathcal{O}^{\circledast}$. We will say that $\mathcal{O}^{\circledast} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$ is a $\boldsymbol{\Delta}$-*planar ∞-operad* if $(\mathcal{O}^{\circledast}, M)$ is a fibrant object of $(\mathcal{P}\mathrm{Op}_{\infty})_{/\mathrm{N}(\boldsymbol{\Delta})^{op,\natural}}$. In this case, we will refer to a morphism in $\mathcal{O}^{\circledast}$ as *inert* if it belongs to M. We will say that $p : \mathcal{O}^{\circledast} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$ is a $\boldsymbol{\Delta}$-*monoidal ∞-category* if it is a $\boldsymbol{\Delta}$-planar ∞-operad and the map p is a coCartesian fibration.

Remark 6.1.1.4. If $(\mathcal{O}^{\circledast}, M)$ is any fibrant object of $(\mathcal{P}\mathrm{Op}_{\infty})_{/\,\mathrm{N}(\boldsymbol{\Delta})^{op,\natural}}$, then the underlying map $p : \mathcal{O}^{\circledast} \to \mathrm{N}(\mathcal{F}\mathrm{in}_{*})$ is a categorical fibration and M is the collection of morphisms α of $\mathcal{O}^{\circledast}$ such that $p(\alpha)$ is inert and α is p-coCartesian (Remark 6.1.1.19). In other words, we can identify Δ-planar ∞-operads with fibrant objects of $(\mathcal{P}\mathrm{Op}_{\infty})_{/\,\mathrm{N}(\boldsymbol{\Delta})^{op,\natural}}$.

Remark 6.1.1.5. We will obtain a more explicit description of the class of Δ-planar ∞-operads below (see Definition 6.1.1.11 and Proposition 6.1.1.13).

Remark 6.1.1.6. The collection of Δ-planar ∞-operads (planar ∞-operads) admits the structure of a simplicial category, where we define $\mathrm{Map}(\mathcal{C}^{\circledast}, \mathcal{D}^{\circledast})$ ($\mathrm{Map}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$) to be the subcategory of $\mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathcal{C}^{\circledast}, \mathcal{D}^{\circledast})$ ($\mathrm{Fun}_{\mathcal{A}\mathrm{ss}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$) spanned by those functors which preserve inert morphisms and those morphisms which are equivalences. It follows from Proposition 6.1.1.1 that the pullback functor

$$\mathcal{O}^{\otimes} \mapsto \mathrm{N}(\boldsymbol{\Delta})^{op} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_{*})} \mathcal{O}^{\otimes}$$

determines a weak equivalence from the simplicial category of planar ∞-operads to the simplicial category of Δ-planar ∞-operads.

Definition 6.1.1.7. Let $\mathcal{C}^{\circledast} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$ be a Δ-planar ∞-operad. We let $^{\Delta}\mathrm{Alg}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{C}^{\circledast})$ spanned by those functors which preserve inert morphisms.

Remark 6.1.1.8. The notation of Definition 6.1.1.7 is compatible with that of Definition 4.2.2.10 if the Δ-planar ∞-operad $\mathcal{O}^{\circledast}$ is given by the fiber product $\mathcal{O}^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathrm{N}(\boldsymbol{\Delta})^{op}$, for some planar ∞-operad $\mathcal{O}^{\otimes}$ (this can always be arranged after replacing $\mathcal{O}^{\circledast}$ by an equivalent Δ-planar ∞-operad, by virtue of Proposition 6.1.1.1).

We will deduce Proposition 6.1.1.1 from a much more general assertion (Theorem 6.1.1.10). To formulate it, we need a bit of terminology.

Definition 6.1.1.9. Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_{*})$ be an ∞-operad and let $f : \mathcal{E} \to \mathcal{O}^{\otimes}$ be an approximation to $\mathcal{O}^{\otimes}$. We will say that a morphism α in $\mathcal{E}$ is *inert* if $f(\alpha)$ is an inert morphism in $\mathcal{O}^{\otimes}$. We let $\mathcal{E}^{\natural}$ denote the marked simplicial set $(\mathcal{E}, M)$, where M is the collection of inert morphisms in $\mathcal{E}$. We will regard $(\mathcal{E}, M)$ as an ∞-preoperad via the composite map $p \circ f$.

We can now formulate the main result of this section (which immediately implies Proposition 6.1.1.1, by virtue of Proposition 4.1.2.10):

Theorem 6.1.1.10. *Let $\mathcal{O}^{\otimes}$ be an ∞-operad and let $f : \mathcal{E} \to \mathcal{O}^{\otimes}$ be an approximation to $\mathcal{O}^{\otimes}$. Assume that f induces an equivalence of ∞-categories $\mathcal{E} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_{*})} \{\langle 1 \rangle\} \to \mathcal{O}$. Then composition with f induces a left Quillen equivalence*

$$(\mathcal{P}\mathrm{Op}_{\infty})_{/\,\mathcal{E}^{\natural}} \to (\mathcal{P}\mathrm{Op}_{\infty})_{/\,\mathcal{O}^{\otimes,\natural}}.$$

Here $\mathcal{P}\mathrm{Op}_{\infty}$ denotes the category of ∞-preoperads, endowed with the ∞-operadic model structure of Proposition 2.1.4.6.

The proof of Theorem 6.1.1.10 will hinge on having a good characterization of the fibrant objects of $\mathcal{P}\mathrm{Op}_{\infty/\,\mathcal{E}^{\natural}}$.

Definition 6.1.1.11. Let $f : \mathcal{E} \to \mathcal{O}^{\otimes}$ be an approximation to an ∞-operad $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_{*})$, and let $q : \overline{\mathcal{E}} \to \mathcal{E}$ be a categorical fibration of simplicial sets. We will say that q is *fibrous* if the following conditions are satisfied:

(1) The map q is an inner fibration.

(2) For every object $\overline{E} \in \overline{\mathcal{E}}$ and every inert morphism $\alpha : q(\overline{E}) \to E'$, there exists a q-coCartesian morphism $\overline{\alpha} : \overline{E} \to \overline{E}'$ lifting α.

(3) Let $\overline{E} \in \overline{\mathcal{E}}$, let $E = q(\overline{E})$, and let $\langle n \rangle = (p \circ f)(E)$. For $1 \leq i \leq n$, choose an inert morphism $\alpha_i : E \to E_i$ in $\mathcal{E}$ covering $\rho^i : \langle n \rangle \to \langle 1 \rangle$, and a q-coCartesian morphism $\overline{\alpha}_i : \overline{E} \to \overline{E}_i$ in $\overline{\mathcal{E}}$ covering α_i. Then the morphisms $\overline{\alpha}_i$ exhibit $\overline{E}$ as a q-product of the objects $\{\overline{E}_i\}_{1 \leq i \leq n}$.

(4) Let $E \in \mathcal{E}$, let $\langle n \rangle = (p \circ f)(E)$, and choose inert morphisms $\alpha_i : E \to E_i$ in $\mathcal{E}$ covering the maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$. Then the morphisms α_i induce an equivalence of ∞-categories $\overline{\mathcal{E}}_E \to \prod_{1 \leq i \leq n} \overline{\mathcal{E}}_{E_i}$.

Remark 6.1.1.12. In the situation of Definition 6.1.1.11, conditions (1) and (2) imply that q is a categorical fibration (Corollary T.2.4.6.5).

The key ingredient in the proof of Theorem 6.1.1.10 is the following result:

Proposition 6.1.1.13. *Let $f : \mathcal{E} \to \mathcal{O}^{\otimes}$ be an approximation to an ∞-operad $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, and let $q : \overline{\mathcal{E}} \to \mathcal{E}$ be a map of simplicial sets. The following conditions are equivalent:*

(a) *There exists a fibration of ∞-operads $\overline{\mathcal{O}}^{\otimes} \to \mathcal{O}^{\otimes}$ and a categorical equivalence $\overline{\mathcal{E}} \to \mathcal{E} \times_{\mathcal{O}^{\otimes}} \overline{\mathcal{O}}^{\otimes}$ (compatible with q).*

(b) *There exists a collection of edges M of $\overline{\mathcal{E}}$ such that q induces a fibration of ∞-preoperads $(\overline{\mathcal{E}}, M) \to \mathcal{E}^{\natural}$.*

(c) *The map q is fibrous.*

We will prove Proposition 6.1.1.13 at the end of this section.

Proof of Theorem 6.1.1.10. Let $f : \mathcal{E} \to \mathcal{O}^{\otimes}$ be an approximation to an ∞-operad $\mathcal{O}^{\otimes}$ which induces an equivalence $\mathcal{E} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \{\langle 1 \rangle\} \to \mathcal{O}$. Let $F : (\mathcal{P}\mathrm{Op}_{\infty})_{/\mathcal{E}^{\natural}} \to (\mathcal{P}\mathrm{Op}_{\infty})_{/\mathcal{O}^{\otimes,\natural}}$ be the functor given by composition with f, and let G be its right adjoint (given by $\overline{X} \mapsto \mathcal{E}^{\natural} \times_{\mathcal{O}^{\otimes,\natural}} \overline{X}$). We wish to show that the Quillen adjunction (F, G) is a Quillen equivalence. The implication $(b) \Rightarrow (a)$ of Proposition 6.1.1.13 shows that the right derived functor RG is essentially surjective. It will therefore suffice to show that RG is right homotopy inverse to the left derived functor LF. Unwinding the definitions, we must show that if $\overline{X}$ is a fibrant object of $(\mathcal{P}\mathrm{Op}_{\infty})_{/\mathcal{O}^{\otimes,\natural}}$, then the induced map $\mathcal{E}^{\natural} \times_{\mathcal{O}^{\otimes,\natural}} \overline{X} \to \overline{X}$ is a weak equivalence of ∞-preoperads. Since $\overline{X}$ is fibrant, it has the form $\overline{\mathcal{O}}^{\otimes,\natural}$, where $\overline{\mathcal{O}}^{\otimes} \to \mathcal{O}^{\otimes}$ is a fibration of ∞-operads. Using Theorem 2.3.3.23, we are reduced to proving that the map $\mathcal{E} \times_{\mathcal{O}^{\otimes}} \overline{\mathcal{O}}^{\otimes} \to \overline{\mathcal{O}}^{\otimes}$ is an approximation to $\overline{\mathcal{O}}^{\otimes}$, which follows from Remark 2.3.3.9. $\qquad\square$

Variant 6.1.1.14. Let $\gamma : \mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1 \to \mathcal{L}\mathcal{M}^{\otimes}$ be the approximation to the ∞-operad $\mathcal{L}\mathcal{M}^{\otimes}$ described in Remark 4.2.2.8. The map γ satisfies the hypotheses of Theorem 6.1.1.10, so that composition with γ induces a left Quillen equivalence

$$(\mathcal{P}\mathrm{Op}_{\infty})_{/(\mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1)^{\natural}} \to (\mathcal{P}\mathrm{Op}_{\infty})_{/\mathcal{L}\mathcal{M}^{\otimes,\natural}}.$$

Given a fibrous map $p : \mathcal{M}^{\circledast} \to \mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1$, then $\mathcal{O}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$ is a Δ-planar ∞-operad. We will denote the fiber $\mathcal{M}^{\circledast} \times_{\mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1} \{([0], 0)\}$ by $\mathcal{M}$, and say that p *exhibits the ∞-category $\mathcal{M}$ as weakly enriched over the Δ-planar ∞-operad $\mathcal{O}^{\circledast}$.* If p is a coCartesian fibration, then $\mathcal{O}^{\circledast}$ is a Δ-monoidal ∞-category; in this case, we will say that p *exhibits $\mathcal{M}$ as left-tensored over the Δ-monoidal ∞-category $\mathcal{O}^{\circledast}$.* Note that this terminology is consistent with that of Definitions 4.2.1.12 and 4.2.1.19.

Suppose that $\mathcal{M}^{\circledast} \to \mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1$ exhibits $\mathcal{M} = \mathcal{M}^{\circledast}_{[0],0}$ as weakly enriched over the Δ-planar ∞-operad $\mathcal{C}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$. We let $^{\Delta}\mathrm{LMod}(\mathcal{M})$ denote the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1}(\mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1, \mathcal{M}^{\circledast})$ spanned by those functors which preserve inert morphisms. There is an evident forgetful functor $^{\Delta}\mathrm{LMod}(\mathcal{M}) \to {}^{\Delta}\mathrm{Alg}(\mathcal{C})$. For each object $A \in {}^{\Delta}\mathrm{Alg}(\mathcal{C})$, we will denote the fiber product $^{\Delta}\mathrm{LMod}(\mathcal{M}) \times_{^{\Delta}\mathrm{Alg}(\mathcal{C})} \{A\}$ by $^{\Delta}\mathrm{LMod}_A(\mathcal{M})$. We note that these notations are consistent with those of Definition 4.2.2.10, in the case where $\mathcal{O}^{\circledast} = \mathcal{O}^{\otimes} \times_{\mathcal{L}\mathcal{M}^{\otimes}} (\mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1)$ for some fibration of ∞-operads $\mathcal{O}^{\otimes} \to \mathcal{L}\mathcal{M}^{\otimes}$.

We now turn to the proof of Proposition 6.1.1.13. We will need a mild generalization of the construction of monoidal envelopes described in §2.2.4.

Construction 6.1.1.15. Let $f : \mathcal{E} \to \mathcal{O}^{\otimes}$ be an approximation to an ∞-operad $\mathcal{O}^{\otimes}$ and suppose we are given a fibrous map of simplicial sets $q : \overline{\mathcal{E}} \to \mathcal{E}$. We let $\mathrm{Env}_{\mathcal{E}}(\overline{\mathcal{E}})$ denote the full subcategory of $\overline{\mathcal{E}} \times_{\mathrm{Fun}(\{0\},\mathcal{E})} \mathrm{Fun}(\Delta^1,\mathcal{E})$ spanned by those pairs $(\overline{E}, \alpha : q(\overline{E}) \to E')$ such that $f(\alpha)$ is an active morphism of $\mathcal{O}^{\otimes}$. Evaluation at $\{1\} \subseteq \Delta^1$ determines a forgetful functor $\mathrm{Env}_{\mathcal{E}}(\overline{\mathcal{E}}) \to \mathcal{E}$.

In the special case where $\mathcal{E}$ is itself an ∞-operad, Construction 6.1.1.15 reduces to Construction 2.2.4.1. We now generalize a few of the properties established in §2.2.4 to the present context.

Lemma 6.1.1.16. *Let $f : \mathcal{E} \to \mathcal{O}^{\otimes}$ be an approximation to an ∞-operad $\mathcal{O}^{\otimes}$ and let $q : \overline{\mathcal{E}} \to \mathcal{E}$ be fibrous. Then the forgetful functor $q' : \mathrm{Env}_{\mathcal{E}}(\overline{\mathcal{E}}) \to \mathcal{E}$ is a coCartesian fibration. Moreover, a morphism α in $\mathrm{Env}_{\mathcal{E}}(\overline{\mathcal{E}})$ is q'-coCartesian if and only if its image α_0 in $\overline{\mathcal{E}}$ is q-coCartesian and $q(\alpha_0)$ is f-inert.*

Proof. Note that evaluation at $\{1\} \subseteq \Delta^1$ induces a coCartesian fibration $\overline{\mathcal{E}} \times_{\mathrm{Fun}(\{0\},\mathcal{E})} \mathrm{Fun}(\Delta^1,\mathcal{E}) \to \mathcal{E}$ (Corollary T.2.4.7.12). In view of Lemma 2.2.4.11 and Remark 2.2.4.12, it will suffice to show the following:

(∗) The inclusion $\mathrm{Env}_{\mathcal{E}}(\overline{\mathcal{E}}) \hookrightarrow \overline{\mathcal{E}} \times_{\mathrm{Fun}(\{0\},\mathcal{E})} \mathrm{Fun}(\Delta^1,\mathcal{E})$ admits a left adjoint. Moreover, a morphism $\alpha : X \to Y$ in $\overline{\mathcal{E}} \times_{\mathrm{Fun}(\{0\},\mathcal{E})} \mathrm{Fun}(\Delta^1,\mathcal{E})$ exhibits Y as an $\mathrm{Env}_{\mathcal{E}}(\overline{\mathcal{E}})$-localization of X if and only if $Y \in \mathrm{Env}_{\mathcal{E}}(\overline{\mathcal{E}})$, the image of α in $\overline{\mathcal{E}}$ is f-inert, and the image of α in $\mathcal{E}$ is an equivalence.

According to Proposition 2.1.2.4, the f-active and f-inert morphisms determine a factorization system on $\mathcal{E}$. Let $\mathcal{A}$ be the full subcategory of $\mathrm{Fun}(\Delta^1,\mathcal{E})$ spanned by the f-active morphisms. Lemma T.5.2.8.19 that the inclusion $\mathcal{A} \subseteq \mathrm{Fun}(\Delta^1,\mathcal{E})$ admits a left adjoint, and that a morphism $\alpha : g \to g'$ in $\mathrm{Fun}(\Delta^1,\mathcal{E})$ corresponding to a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{u} & X' \\
\downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u'} \\
Y & \xrightarrow{v'} & Y'
\end{array}
$$

in $\mathcal{O}^{\otimes}$ exhibits g' as an $\mathcal{A}$-localization of v if and only if v' is f-active, u is f-inert, and u' is an equivalence. The desired result now follows from Lemma 2.2.4.13. $\qquad\square$

Lemma 6.1.1.17. *Let $f : \mathcal{E} \to \mathcal{O}^{\otimes}$ be an approximation to an ∞-operad $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and let $q : \overline{\mathcal{E}} \to \mathcal{E}$ be fibrous. Suppose we are given an object $E \in \mathcal{E}$ and a finite collection of f-inert morphisms $\{\alpha_a : E \to E_a\}_{a \in A}$ with the following property: the maps α_a lie over inert morphisms $\langle n \rangle \to \langle n_a \rangle$ in $\mathcal{F}\mathrm{in}_*$ which determine a bijection $\coprod_a \langle n_a \rangle^{\circ} \to \langle n \rangle^{\circ}$. Then:*

(a) *For any collection of objects $\overline{E}_a \in \overline{\mathcal{E}}$ lying over E_a, there exists an object $\overline{E} \in \overline{\mathcal{E}}$ and a collection of morphisms $\overline{\alpha}_a : \overline{E} \to \overline{E}_a$ which exhibit $\overline{E}$ as a q-product of the objects $\overline{E}_a$.*

(b) *For any object $\overline{E} \in \overline{\mathcal{E}}$ lifting E and any collection of morphisms $\overline{\alpha}_a : \overline{E} \to \overline{E}_a$ lifting α_a, the morphisms $\overline{\alpha}_a$ exhibit $\overline{E}$ as a q-product of the objects $\overline{E}_a$ if and only if each $\overline{\alpha}_a$ is q-coCartesian.*

Proof. We first prove (a). For $a \in A$ and $1 \leq i \leq n_a$, choose a locally $(p \circ f)$-coCartesian morphism $\beta_{a,i} : E_a \to E_{a,i}$ in $\mathcal{E}$ covering the map $\rho^i : \langle n_a \rangle \to \langle 1 \rangle$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. These maps determine a map $u_a : K_a^{\triangleleft} \to \mathcal{E}$, where K_a denotes the discrete simplicial set $\{1, \ldots, n_a\}$. Each $\beta_{a,i}$ can be lifted to a q-coCartesian morphism $\overline{E}_a \to \overline{E}_{a,i}$ in $\overline{\mathcal{E}}$, and these liftings determine a diagram $\overline{u}_a : K_a^{\triangleleft} \to \mathcal{E}$. Since q is fibrous, each of the diagrams $\overline{u}_a$ is a q-limit diagram.

Let $L = \coprod_{a \in A} K_a^{\triangleleft}$, let L_0 be the full subcategory of L spanned by the cone points of each $K_a^{\triangleleft}$, and let L_1 be the full subcategory of L given by $\coprod_{a \in A} K_a$. Let $u : L \to \overline{\mathcal{E}}$ be the coproduct of the maps u_a and let $\overline{u} : L \to \overline{\mathcal{E}}$ be defined similarly; we note that $\overline{u}$ is a q-right Kan extension of $\overline{u}|L_1$. The maps α_a determine a diagram $v : L_0^{\triangleleft} \to \mathcal{E}$. Since L_0 is left cofinal in L, we can amalgamate u and v to obtain a map $w : L^{\triangleleft} \to \mathcal{E}$. Using the assumption that q is fibrous, we can lift w to a diagram $\overline{w} : L^{\triangleleft} \to \overline{\mathcal{E}}$ which is a q-right Kan extension of $\overline{u}|L_1$. Since $\overline{u}$ is already a q-right Kan extension of $\overline{u}|L_1$, we may assume without loss of

generality that $\overline{w}|L = \overline{u}$. Then $\overline{w}$ is a q-limit diagram. Since L_0 is right cofinal in L, it follows that $\overline{w}|L_0^\triangleleft$ is a q-limit diagram, which proves (a).

The above argument shows that an arbitrary diagram $\overline{v} : L_0^\triangleleft \to \mathcal{E}$ lifting v (corresponding to a collection of morphisms $\overline{\alpha}_a : \overline{E} \to \overline{E}_a$) is a q-limit diagram if and only if, for every index $a \in A$ and $1 \leq i \leq n_a$, the composite map $\overline{\beta}_{a,i} \circ \overline{\alpha}_a : \overline{E} \to \overline{E}_{a,i}$ is q-coCartesian, where $\overline{\beta}_{a,i}$ is a q-coCartesian morphism $\overline{E}_a \to \overline{E}_{a,i}$ lifting $\beta_{a,i}$. To prove (b), we wish to show that each of the maps $\overline{\alpha}_a$ is q-coCartesian if and only if each of the compositions $\overline{\beta}_{a,i} \circ \overline{\alpha}_a$ is q-coCartesian. The "only if" direction is obvious. To prove the converse, let us suppose that each $\overline{\beta}_{a,i} \circ \overline{\alpha}_a$ is q-coCartesian. For each $a \in A$, we can factor $\overline{\alpha}_a$ as a composition

$$\overline{E} \overset{\overline{\alpha}'_a}{\to} \overline{E}'_a \overset{\overline{\alpha}''_a}{\to} \overline{E}_a$$

where $\overline{\alpha}'_a$ is a q-coCartesian morphism lifting α_a and $\overline{\alpha}''_a$ is a morphism lifting the identity id_{E_a}. Since $\overline{\alpha}'_a$ and $\overline{\beta}_{a,i} \circ \overline{\alpha}_a$ are q-coCartesian, we conclude that each composition $\overline{\beta}_{a,i} \circ \overline{\alpha}''_a$ is q-coCartesian. In other words, the functor $\overline{\mathcal{E}}_{E_a} \to \overline{\mathcal{E}}_{E_{a,i}}$ determined by $\beta_{a,i}$ carries $\overline{\alpha}''_a$ to an equivalence, for each $1 \leq i \leq n_a$. Since q is fibrous, we have $\overline{\mathcal{E}}_{E_a} \simeq \prod_{1 \leq i \leq n_a} \overline{\mathcal{E}}_{E_{a,i}}$; it follows that $\overline{\alpha}''_a$ is an equivalence, so that $\overline{\alpha}_a$ is q-coCartesian as desired. $\qquad\square$

Lemma 6.1.1.18. *Let $f : \mathcal{E} \to \mathcal{O}^\otimes$ be an approximation to an ∞-operad $p : \mathcal{O}^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and let $q : \overline{\mathcal{E}} \to \mathcal{E}$ be fibrous. Then the coCartesian fibration $q' : \mathrm{Env}_\mathcal{E}(\overline{\mathcal{E}}) \to \mathcal{E}$ of Lemma 6.1.1.16 is classified by a functor $\chi : \mathcal{E} \to \mathcal{C}\mathrm{at}_\infty$ which belongs to $\mathrm{Mon}_\mathcal{E}(\mathcal{C}\mathrm{at}_\infty)$.*

Proof. Fix an object $E \in \mathcal{E}$ lying over $\langle n \rangle \in \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, and choose locally $(p \circ f)$-coCartesian morphisms $\alpha_i : E \to E_i$ covering the maps $\rho^i : \langle n \rangle \to \langle i \rangle$ for $1 \leq i \leq n$. Let $K = \{1, \ldots, n\}$, regarded as a discrete simplicial set, so that the maps α_i determine a diagram $u : K^\triangleleft \to \mathcal{E}$ carrying the cone point to the object E. Let $\mathfrak{X}(\overline{\mathcal{E}})$ be the full subcategory of $\mathrm{Fun}(K^\triangleleft, \mathrm{Env}_\mathcal{E}(\overline{\mathcal{E}})) \times_{\mathrm{Fun}(K^\triangleleft, \mathcal{E})} \{u\}$ spanned by those functors $K^\triangleleft \to \mathrm{Env}_\mathcal{E}(\overline{\mathcal{E}})$ lifting u which carry each edge $\{i\}^\triangleleft$ of $K^\triangleleft$ to a q'-coCartesian morphism in $\mathrm{Env}_\mathcal{E}(\overline{\mathcal{E}})$. Proposition T.4.3.2.15 implies that the restriction map $\theta : \mathfrak{X}(\overline{\mathcal{E}}) \to \mathrm{Env}_\mathcal{E}(\overline{\mathcal{E}})_E$ is a trivial Kan fibration.

We wish to show that the coCartesian fibration q' induces an equivalence of ∞-categories $\mathrm{Env}_\mathcal{E}(\overline{\mathcal{E}})_E \to \prod_{1 \leq i \leq n} \mathrm{Env}_\mathcal{E}(\overline{\mathcal{E}})_{E_i}$. Unwinding the definitions, this map is given by composing a section of θ with the restriction map $\phi : \mathfrak{X}(\overline{\mathcal{E}}) \to \mathfrak{X}_0(\overline{\mathcal{E}})$, where

$$\mathfrak{X}_0(\overline{\mathcal{E}}) = \mathrm{Fun}(K, \mathrm{Env}_\mathcal{E}(\overline{\mathcal{E}})) \times_{\mathrm{Fun}(K, \mathcal{E})} \{(E_1, \ldots, E_n)\} \simeq \mathrm{Env}_\mathcal{E}(\overline{\mathcal{E}})_{E_1} \times \cdots \mathrm{Env}_\mathcal{E}(\overline{\mathcal{E}})_{E_n}.$$

It will therefore suffice to show that ϕ is a trivial Kan fibration.

We begin by treating the case where $\overline{\mathcal{E}} = \mathcal{E}$. We may assume without loss of generality that $f : \mathcal{E} \to \mathcal{O}^\otimes$ is a categorical fibration. In this case, Proposition T.4.3.2.15 implies that composition with f induces a trivial Kan fibration $\mathrm{Env}_\mathcal{E}(\overline{\mathcal{E}}) \to \mathcal{E} \times_{\mathcal{O}^\otimes} \mathrm{Env}_\mathcal{O}(\mathcal{O})^\otimes$, and the desired result follows immediately from Proposition 2.2.4.4.

We now treat the general case. Note that ϕ factors as a composition

$$\mathfrak{X}(\overline{\mathcal{E}}) \overset{\phi'}{\to} \mathfrak{X}(\mathcal{E}) \times_{\mathfrak{X}_0(\mathcal{E})} \mathfrak{X}_0(\overline{\mathcal{E}}) \overset{\phi''}{\to} \mathfrak{X}_0(\overline{\mathcal{E}}).$$

The special case treated above shows that ϕ'' is a trivial Kan fibration. It will therefore suffice to show that ϕ' is a trivial Kan fibration. This follows from Lemma 6.1.1.17 and Proposition T.4.3.2.15. $\qquad\square$

Proof of Proposition 6.1.1.13. We first show that $(a) \Rightarrow (b)$. Suppose there exists a fibration of ∞-operads $\phi : \overline{\mathcal{O}}^\otimes \to \mathcal{O}^\otimes$ and a categorical equivalence $u : \overline{\mathcal{E}} \to \mathcal{E} \times_{\mathcal{O}^\otimes} \overline{\mathcal{O}}^\otimes$. We wish to show that the map $q : (\overline{\mathcal{E}}, M) \to \mathcal{E}^\natural$ is a fibration of ∞-preoperads. Choose a trivial cofibration of ∞-preoperads $i : (X, M_X) \to (Y, M_Y)$. We wish to show that every lifting problem

$$\begin{array}{ccc} (X, M_X) & \overset{v}{\longrightarrow} & (\overline{\mathcal{E}}, M) \\ \downarrow & \nearrow & \downarrow q \\ (Y, M_Y) & \longrightarrow & \mathcal{E}^\natural \end{array}$$

admits a solution. Since $\overline{\mathcal{O}}^{\otimes,\natural} \to \mathcal{O}^{\otimes,\natural}$ is a fibration of ∞-preoperads, the composite map $u \circ v$ can be extended to a map $\phi : Y \to \mathcal{E} \times_{\mathcal{O}^\otimes} \overline{\mathcal{O}}^\otimes$ which carries every morphism in M_Y to an inert morphism in $\overline{\mathcal{O}}^\otimes$. Applying Proposition T.A.2.3.1 (in the category $(\mathcal{S}et_\Delta)_{/\mathcal{E}}$, equipped with the Joyal model structure) we can extend v to a map $\overline{v} : Y \to \overline{\mathcal{E}}$ such that $u \circ \overline{v}$ is homotopic to ϕ. It follows that $\overline{v}$ carries each morphism in M_Y to a q-coCartesian morphism in $\overline{\mathcal{E}}$ and therefore provides the desired solution.

We now prove that $(b) \Rightarrow (c)$. Let $\mathfrak{P}_\mathcal{E}$ denote the categorical pattern on $\mathcal{E}$ given by $(M, T, \{p_\alpha : \langle n \rangle^{\circ \triangleleft} \to \mathcal{E}\}_{\alpha \in A})$ where M is the collection of all f-inert morphisms in $\mathcal{E}$, T is the collection of all 2-simplices in $\mathcal{E}$, and A is the collection of all diagrams $p_\alpha : \langle n \rangle^{\circ \triangleleft} \to \mathcal{E}$ which carry the cone point of $\langle n \rangle^{\circ \triangleleft}$ to an object $E \in \mathcal{E}$ lying over $\langle n \rangle \in \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and carry each edge $\{i\}^\triangleleft$ to an f-inert morphism $E \to E_i$ in $\mathcal{E}$ lying over the map $\rho^i : \langle n \rangle \to \langle 1 \rangle$. It now suffices to observe that q is fibrous if and only if it is $\mathfrak{P}_\mathcal{E}$-fibered, and that every $\mathfrak{P}$-anodyne map (see Definition B.1.1) determines a weak equivalence of ∞-preoperads.

We complete the proof by showing that $(c) \Rightarrow (a)$. Assume that $q : \overline{\mathcal{E}} \to \mathcal{E}$ is fibrous. Let $q' : \mathrm{Env}_\mathcal{E}(\overline{\mathcal{E}}) \to \mathcal{E}$ be the map given by Construction 6.1.1.15. Then q' is a coCartesian fibration (Lemma 6.1.1.16) and is classified by a map $\chi : \mathcal{E} \to \mathcal{C}\mathrm{at}_\infty$ which belongs to $\mathrm{Mon}_\mathcal{E}(\mathcal{C}\mathrm{at}_\infty)$ (Lemma 6.1.1.18). Using Proposition 4.1.2.11, we may assume that χ factors as a composition

$$\mathcal{E} \xrightarrow{f} \mathcal{O}^\otimes \xrightarrow{\chi'} \mathcal{C}\mathrm{at}_\infty,$$

where $\chi' \in \mathrm{Mon}_\mathcal{O}(\mathcal{C}\mathrm{at}_\infty)$. The map χ' classifies a coCartesian fibration of ∞-operads $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ and there is a categorical equivalence $\theta : \mathrm{Env}_\mathcal{E}(\overline{\mathcal{E}}) \to \mathcal{E} \times_{\mathcal{O}^\otimes} \mathcal{C}^\otimes$. Let $\mathrm{Env}_\mathcal{E}^0(\overline{\mathcal{E}})$ denote the full subcategory of $\mathcal{U}_\mathcal{E}(\overline{\mathcal{E}})$ spanned by those pairs $(\overline{E}, \alpha : q(\overline{E}) \to E')$ where α is an equivalence, and let $\mathcal{C}_0 \subseteq \mathcal{C}$ denote the essential image of $\mathrm{Env}_\mathcal{E}^0(\overline{\mathcal{E}}) \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \{\langle 1 \rangle\}$ under the equivalence $\mathrm{Env}_\mathcal{E}(\overline{\mathcal{E}}) \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \{\langle 1 \rangle\} \simeq \mathcal{C}$. Let $\mathcal{C}_0^\otimes$ denote the full subcategory of $\mathcal{C}^\otimes$ determined by $\mathcal{C}_0 \subseteq \mathcal{C}$ (see the introduction to §2.2.1), so that we have a fibration of ∞-operads $\mathcal{C}_0^\otimes \to \mathcal{O}^\otimes$. The functor θ restricts to an an equivalence of ∞-categories $\mathrm{Env}_\mathcal{E}^0(\overline{\mathcal{E}}) \to \mathcal{E} \times_{\mathcal{O}^\otimes} \mathcal{C}_0^\otimes$. Composing with a section of the trivial Kan fibration $\mathrm{Env}_\mathcal{E}^0(\overline{\mathcal{E}}) \to \overline{\mathcal{E}}$, we obtain the desired categorical equivalence $\overline{\mathcal{E}} \to \mathcal{E} \times_{\mathcal{O}^\otimes} \mathcal{C}_0^\otimes$. $\qquad\square$

Remark 6.1.1.19. Suppose that $q : \overline{\mathcal{E}} \to \mathcal{E}$ satisfies the equivalent conditions of Proposition 6.1.1.13. It follows from the proof that there is a *unique* collection morphisms M for which the map $(\overline{\mathcal{E}}, M) \to \mathcal{E}^\natural$ is a fibration: namely, the collection of morphisms α such that α is q-coCartesian and $q(\alpha)$ is f-inert.

6.1.2 Endomorphism ∞-Categories

Let $\mathcal{C}$ be a monoidal category containing an object M, and suppose that there exists an *endomorphism object* $\mathrm{End}(M)$: that is, an object of $\mathcal{C}$ equipped with evaluation map $e : \mathrm{End}(M) \otimes M \to M$ which induces a bijection $\mathrm{Hom}_\mathcal{C}(Y, \mathrm{End}(M)) \to \mathrm{Hom}_\mathcal{C}(Y \otimes M, M)$ for every $Y \in \mathcal{C}$ (such an object always exists if the monoidal category $\mathcal{C}$ is *closed*). Then $\mathrm{End}(M)$ is automatically equipped with the structure of an algebra object of $\mathcal{C}$: the unit map $1_\mathcal{C} \to \mathrm{End}(M)$ classifies the identity map $\mathrm{id}_M : M \to M$, while the multiplication $\mathrm{End}(M) \otimes \mathrm{End}(M) \to \mathrm{End}(M)$ classifies the composition

$$\mathrm{End}(M) \otimes (\mathrm{End}(M) \otimes M) \to \mathrm{End}(M) \otimes M \to M.$$

Our goal in this section is to obtain an ∞-categorical generalization of the above construction. With an eye toward later applications, we will treat a slightly more general problem. Suppose that $\mathcal{C}$ is a monoidal ∞-category, $\mathcal{M}$ an ∞-category which is left-tensored over $\mathcal{C}$, and M is an object of $\mathcal{M}$. Let us say that an object $C \in \mathcal{C}$ *acts on* M if we are given a map $C \otimes M \to M$. We would like to extract an object $\mathrm{End}(M) \in \mathcal{C}$ which is *universal* among objects which act on M, and show that $\mathrm{End}(M)$ is an algebra object of $\mathcal{C}$ (and further, that M can be promoted to an object of $\mathrm{LMod}_{\mathrm{End}(M)}(\mathcal{M})$).

Our first goal is to give a careful formulation of the universal property desired of $\mathrm{End}(M)$. We would like to have, for every algebra object $A \in \mathrm{Alg}(\mathcal{C})$, a homotopy equivalence of $\mathrm{Map}_{\mathrm{Alg}(\mathcal{C})}(A, \mathrm{End}(M))$ with a suitable classifying space for actions of A on M. The natural candidate for this latter space is the fiber product

$\{A\} \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{Mod}(\mathcal{M}) \times_{\mathcal{M}} \{M\}$, which can be viewed as a fiber of the projection map $\theta : \mathrm{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\} \to \mathrm{Alg}(\mathcal{C})$. We will see below that θ is a right fibration (Corollary 6.1.2.42). Our problem is therefore to find an object $\mathrm{End}(M) \in \mathrm{Alg}(\mathcal{C})$ which *represents* the right fibration θ, in the sense that we have a an equivalence $\mathrm{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\} \simeq \mathrm{Alg}(\mathcal{C})_{/\mathrm{End}(M)}$ of right fibrations over $\mathrm{Alg}(\mathcal{C})$ (see §T.4.4.4).

The next step is to realize the ∞-category $\mathrm{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\}$ as (equivalent to) the ∞-category of *algebra objects* in a suitable monoidal ∞-category $\mathcal{C}[M]$. Roughly speaking, we will think of objects of $\mathcal{C}[M]$ as pairs (C, η), where $C \in \mathcal{C}$ and $\eta : C \otimes M \to M$ is a morphism in $\mathcal{M}$. The monoidal structure on $\mathcal{C}[M]$ may be described informally by the formula $(C, \eta) \otimes (C', \eta') = (C \otimes C', \eta'')$, where η'' denotes the composition

$$C \otimes C' \otimes M \xrightarrow{\mathrm{id} \otimes \eta'} C \otimes M \xrightarrow{\eta} M.$$

The desired object $\mathrm{End}(M)$ can be viewed as a *final* object of $\mathcal{C}[M]$. Provided that this final object exists, it automatically has the structure of an algebra object of $\mathcal{C}[M]$ (Corollary 3.2.2.4). The image of $\mathrm{End}(M)$ under the (monoidal) forgetful functor $\mathcal{C}[M] \to \mathcal{C}$ will therefore inherit the structure of an algebra object of $\mathcal{C}$.

We are now ready to begin with a detailed definition of the monoidal ∞-category $\mathcal{C}[M]$. For technical reasons, it will be convenient to work in the setting of Δ-monoidal ∞-categories (see §6.1.1).

Definition 6.1.2.1. Let $p : \mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$ be a map which exhibits $\mathcal{M} = \mathcal{M}^{\circledast}_{0,[0]}$ as weakly enriched over the Δ-planar ∞-operad $\mathcal{C}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$. An *enriched morphism* of $\mathcal{M}$ is a diagram

$$M \xleftarrow{\alpha} X \xrightarrow{\beta} N$$

in $\mathcal{M}^{\circledast}$ satisfying the following conditions:

- The image $p(\alpha)$ is the morphism $(0, [1]) \to (0, [0])$ in $\Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$ determined by the embedding $[0] \simeq \{0\} \hookrightarrow [1]$ in $\boldsymbol{\Delta}$.

- The map β is inert, and $p(\beta)$ is the morphism $(0, [1]) \to (0, [0])$ in $\Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$ determined by the embedding $[0] \simeq \{1\} \hookrightarrow [1]$ in $\boldsymbol{\Delta}$.

We let $\mathrm{Str}\, \mathcal{M}^{\mathrm{en}}_{[1]}$ denote the full subcategory of $\mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}}(\Lambda^2_0, \mathcal{M}^{\circledast})$ spanned by the enriched morphisms of $\mathcal{M}$.

There is an evident pair of evaluation functors $\mathrm{Str}\, \mathcal{M}^{\mathrm{en}}_{[1]} \to \mathcal{M}$. Given an object $M \in \mathcal{M}$, we let $\mathcal{C}[M]$ denote the fiber product $\{M\} \times_{\mathcal{M}} \mathrm{Str}\, \mathcal{M}^{\mathrm{en}}_{[1]} \times_{\mathcal{M}} \{M\}$. We will refer to $\mathcal{C}[M]$ as the *endomorphism ∞-category of M*.

Remark 6.1.2.2. The notation of Definition 1.3.2.11 is abusive: the ∞-categories $\mathrm{Str}\, \mathcal{M}^{\mathrm{en}}_{[1]}$ and $\mathcal{C}[M]$ depend not only on $\mathcal{M}$ and M, but also on the weak enrichment $p : \mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$.

Remark 6.1.2.3. Let $p : \mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$ be a map which exhibits $\mathcal{M} = \mathcal{M}^{\circledast}_{0,[0]}$ as weakly enriched over the Δ-planar ∞-operad $\mathcal{C}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$, and let $M \in \mathcal{M}$ be an object. By definition, a object of $\mathcal{C}[M]$ consists of the following data:

- An object $X \in \mathcal{M}^{\circledast}_{0,[1]}$, which we can think of as consisting of a pair of objects $C \in \mathcal{C}$, $M' \in \mathcal{M}$.

- An inert morphism $\beta : X \to M$ covering the map $(0, [1]) \to (0, [0])$ given by $[0] \simeq \{1\} \hookrightarrow [1]$ in $\boldsymbol{\Delta}$. We can think of β as giving an equivalence $M' \simeq M$.

- A morphism $\alpha : X \to M$ covering the map $(0, [1]) \to (0, [0])$ given by $[0] \simeq \{0\} \hookrightarrow [1]$ in $\boldsymbol{\Delta}$. If p is a coCartesian fibration (so that $\mathcal{M}$ is left-tensored over the Δ-monoidal ∞-category $\mathcal{C}^{\circledast}$), then we can think of α as given by a map $C \otimes M' \to M$.

It follows that if p is a coCartesian fibration, then we can think of objects of $\mathcal{C}[M]$ as pairs (C, η), where $C \in \mathcal{C}$ and $\eta : C \otimes M \to M$ is a morphism in $\mathcal{M}$.

Our next goal is to show that in the situation of Definition 6.1.2.1, the ∞-category $\mathcal{C}[M]$ is the underlying ∞-category of Δ-planar ∞-operad $\mathcal{C}[M]^{\circledast}$. We can think of the fiber $\mathcal{C}[M]^{\circledast}_{[n]}$ as a sequence of object $C_1, C_2, \ldots, C_n \in \mathcal{C}$ equipped with maps $\eta_i : C_i \otimes M \to M$ (at least in the case where p is a coCartesian fibration). We begin by describing this ∞-category in a way which is more evidently functorial in the linearly ordered set $[n]$. Before proceeding, we need a simple observation:

Remark 6.1.2.4. Let $p : \mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$ be a map which exhibits $\mathcal{M} = \mathcal{M}^{\circledast}_{0,[0]}$ as weakly enriched over the Δ-planar ∞-operad $\mathcal{C}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$. Then the inclusion $\mathcal{C}^{\circledast} \hookrightarrow \mathcal{M}^{\circledast}$ admits a left adjoint L. For every object $M \in \mathcal{C}^{\circledast}_{0,[n]}$, the localization map $M \to LM$ is an inert morphism in $\mathcal{M}^{\circledast}$ covering the map $(0, [n]) \to (1, [n])$ in $\Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$.

Notation 6.1.2.5. Let $n \geq 0$ be an integer. We let $\mathrm{Po}_{[n]}$ denote the set $\{(i,j) \in [n] \times [n] : i \leq j\}$. We will regard $\mathrm{Po}_{[n]}$ as partially ordered, where $(i,j) \leq (i',j')$ if $i' \leq i \leq j \leq j'$. In other words, we have $(i,j) \leq (i',j')$ if $\{i, i+1, \ldots, j\} \subseteq \{i', i'+1, \ldots, j'\}$.
  We define a functor $\Phi_{[n]} : \mathrm{Po}_{[n]} \to \boldsymbol{\Delta}$ by the formula $\Phi_{[n]}(i,j) = [j-i] \simeq \{i, i+1, \ldots, j\}$.

Definition 6.1.2.6. Let $p : \mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$ be a map which exhibits $\mathcal{M} = \mathcal{M}^{\circledast}_{0,[0]}$ as weakly enriched over the Δ-planar ∞-operad $\mathcal{C}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$. An *enriched n-string in* $\mathcal{M}$ is a map $\sigma : \mathrm{N}(\mathrm{Po}_{[n]})^{op} \to \mathcal{M}^{\circledast}$ with the following properties:

(1) The composition $p \circ \sigma$ is given by the composition

$$\mathrm{N}(\mathrm{Po}_{[n]})^{op} \overset{\Phi_{[n]}}{\to} \mathrm{N}(\boldsymbol{\Delta})^{op} \simeq \{0\} \times \mathrm{N}(\boldsymbol{\Delta})^{op} \hookrightarrow \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$$

  where $\Phi_{[n]}$ is defined as in Notation 6.1.2.5 and LCut as in Construction 4.2.2.6.

(2) For $i \leq i' \leq j$, the map $\sigma(i',j) \to \sigma(i,j)$ is an inert morphism in $\mathcal{M}^{\circledast}$.

(3) Let $L : \mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast}$ be a left adjoint to the inclusion (see Remark 6.1.2.4). For $i \leq i' \leq j' \leq j$, the map $L\sigma(i',j') \to L\sigma(i,j)$ is an inert morphism in $\mathcal{C}^{\circledast}$.

We let $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[n]}$ denote the full subcategory of $\mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathrm{N}(\mathrm{Po}_{[n]})^{op}, \mathcal{M}^{\circledast})$ spanned by the enriched n-strings in $\mathcal{M}$. We will refer to $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[n]}$ as the ∞-*category of enriched n-strings in* $\mathcal{M}$.

Warning 6.1.2.7. As with Definition 1.3.2.11, our notation is abusive: the ∞-category $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[n]}$ depends not only on the ∞-category $\mathcal{M}$, but also on the weak enrichment $\mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$.

Example 6.1.2.8. When $n = 1$, condition (3) of Definition 6.1.2.6 is automatic. Consequently, an enriched n-string in $\mathcal{M}$ is just an enriched morphism in $\mathcal{M}$, in the sense of Definition 1.3.2.11. In particular, the notations of Definitions 1.3.2.11 and 6.1.2.6 are compatible with one another.

Remark 6.1.2.9. In the situation of Definition 6.1.2.6, let us say that an *n-string* in $\mathcal{M}$ is a sequence of objects $M_0, M_1, \ldots, M_n \in \mathcal{M}$. Every enriched n-string $\sigma \in \mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[n]}$ determined an n-string $\{M_i\}_{0 \leq i \leq n}$ by the formula $M_i = \sigma(i,i)$. This construction determines a forgetful functor $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[n]} \to \mathcal{M}^{[n]}$.

Example 6.1.2.10. If $n = 0$ in Definition 6.1.2.6, then $\mathrm{Po}_{[n]} = \{(0,0)\}$ and the forgetful functor $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[n]} \to \mathcal{M}^{[0]} \simeq \mathcal{M}$ is an isomorphism. In other words, an enriched 0-string in $\mathcal{M}$ is just an object of $\mathcal{M}$.

Remark 6.1.2.11. Let $f : [m] \to [n]$ be a morphism in $\mathbf{\Delta}$. Then f induces a map of partially ordered sets $\mathrm{Po}_f : \mathrm{Po}_{[m]} \to \mathrm{Po}_{[n]}$, given by $(i,j) \mapsto (f(i), f(j))$. Suppose further that f is an inert morphism in $\mathbf{\Delta}$: that is, f induces a bijection from $[m]$ to $\{i, i+1, \ldots, j\} \subseteq [n]$ for some $0 \le i \le j \le n$. Then the diagram

$$
\begin{array}{ccc}
\mathrm{Po}_{[m]} & \xrightarrow{\ \mathrm{Po}_f\ } & \mathrm{Po}_{[n]} \\
& \searrow{\scriptstyle \Phi_{[m]}} \quad \swarrow & \\
& \mathbf{\Delta} &
\end{array}
$$

is commutative. It follows that for any $p : \mathcal{M}^\circledast \to \Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}$ as in Definition 6.1.2.6, composition with Po_f induces a forgetful functor $\operatorname{Str}\mathcal{M}^{\mathrm{en}}_{[n]} \to \operatorname{Str}\mathcal{M}^{\mathrm{en}}[m]$.

Notation 6.1.2.12. Let $p : \mathcal{M}^\circledast \to \Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}$ be as in Definition 6.1.2.6, and let S be a convex subset of $[n]$ for some $n \ge 0$. Then S is the image of a unique injective map $f : [m] \to [n]$ in $\mathbf{\Delta}$. We let $\operatorname{Str}\mathcal{M}^{\mathrm{en}}_S$ denote the ∞-category $\operatorname{Str}\mathcal{M}^{\mathrm{en}}_{[m]}$, so that Remark 6.1.2.11 produces a forgetful functor $\operatorname{Str}\mathcal{M}^{\mathrm{en}}_{[n]} \to \operatorname{Str}\mathcal{M}^{\mathrm{en}}_S$.

The following result expresses the idea that an enriched n-string can be obtained by "composing" a chain of enriched morphisms (of length n):

Proposition 6.1.2.13 (Segal Condition). *Let $p : \mathcal{M}^\circledast \to \Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}$ be a map which exhibits $\mathcal{M} = \mathcal{M}^\circledast_{0,[0]}$ as weakly enriched over the $\mathbf{\Delta}$-planar ∞-operad $\mathcal{C}^\circledast = \mathcal{M}^\circledast \times_{\Delta^1} \{1\}$. For each $n \ge 0$, the ∞-category $\operatorname{Str}\mathcal{M}^{\mathrm{en}}_{[n]}$ is a homotopy limit of the diagram of ∞-categories*

$$
\begin{array}{ccccccc}
\operatorname{Str}\mathcal{M}^{\mathrm{en}}_{\{0,1\}} & & \cdots & & \operatorname{Str}\mathcal{M}^{\mathrm{en}}_{\{m-1,m\}} & \\
\swarrow \qquad \searrow & & \swarrow \qquad \searrow & & \swarrow \qquad \searrow & \\
\operatorname{Str}\mathcal{M}^{\mathrm{en}}_{\{0\}} \qquad\qquad \operatorname{Str}\mathcal{M}^{\mathrm{en}}_{\{1\}} & & \operatorname{Str}\mathcal{M}^{\mathrm{en}}_{\{n-1\}} & & & \operatorname{Str}\mathcal{M}^{\mathrm{en}}_{\{n\}}.
\end{array}
$$

Proof. Let $\mathrm{Po}^0_{[n]}$ denote the subset of $\mathrm{Po}_{[n]}$ consisting of those pairs (i,j) where $i \le j \le i+1$, and let $\mathcal{X}$ be the full subcategory of $\operatorname{Fun}_{\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}}(\mathrm{N}(\mathrm{Po}^0_{[n]})^{op}, \mathcal{M}^\circledast)$ spanned by those functors whose restriction to $\mathrm{N}(\mathrm{Po}_{[1]})^{op}$ belongs to $\operatorname{Str}\mathcal{M}^{\mathrm{en}}_{[1]}$, for every map $[1] \simeq \{i, i+1\} \hookrightarrow [n]$. Then $\mathcal{X}$ is a model for the homotopy limit in question; it will therefore suffice to show that the forgetful functor $\theta : \operatorname{Str}\mathcal{M}^{\mathrm{en}}_{[n]} \to \mathcal{X}$ is an equivalence of ∞-categories.

Let $\operatorname{Str}\mathcal{M}^{\mathrm{en},+}_{[n]}$ be the full subcategory of $\operatorname{Fun}_{\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}}(\Delta^1 \times \mathrm{N}(\mathrm{Po}_{[n]})^{op}, \mathcal{M}^\circledast)$ spanned by those functors F^+ satisfying the following pair of conditions:

(i) The restriction $F = F^+|(\{0\} \times \mathrm{N}(\mathrm{Po}_{[n]})^{op})$ belongs to $\operatorname{Str}\mathcal{M}^{\mathrm{en}}_{[n]}$.

(ii) For every object $(i,j) \in \mathrm{Po}_{[n]}$, the map $F^+(0, i, j) \to F^+(1, i, j)$ is p-coCartesian. In other words, F^+ is a p-left Kan extension of F.

We define the full subcategory $\mathcal{X}^+ \subseteq \operatorname{Fun}_{\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}}(\Delta^1 \times \mathrm{N}(\mathrm{Po}^0_{[n]})^{op}, \mathcal{M}^\circledast)$ similarly. We have a commutative diagram of restriction maps

$$
\begin{array}{ccc}
\operatorname{Str}\mathcal{M}^{\mathrm{en},+}_{[n]} & \xrightarrow{\ \theta^+\ } & \mathcal{X}^+ \\
\downarrow & & \downarrow \\
\operatorname{Str}\mathcal{M}^{\mathrm{en}}_{[n]} & \longrightarrow & \mathcal{X}.
\end{array}
$$

Proposition T.4.3.2.15 implies that the vertical maps in this diagram are trivial Kan fibrations. It will therefore suffice to show that θ^+ is a trivial Kan fibration. According to Proposition T.4.3.2.15, this is a consequence of the following pair of assertions:

(a) For every functor $F_0^+ \in \mathfrak{X}^+$, there exists a functor $F^+ \in \mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}}(\Delta^1 \times \mathrm{N}(\mathrm{Po}_{[n]})^{op}, \mathcal{M}^{\circledast})$ such that F^+ is a p-right Kan extension of F_0^+.

(b) An arbitrary functor $F^+ \in \mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}}(\Delta^1 \times \mathrm{N}(\mathrm{Po}_{[n]})^{op}, \mathcal{M}^{\circledast})$ belongs to $\mathrm{Str}\,\mathcal{M}_{[n]}^{\mathrm{en},+}$ if and only if $F_0^+ = F^+|(\Delta^1 \times \mathrm{N}(\mathrm{Po}_{[n]}^0)^{op})$ belongs to $\mathfrak{X}^+$, and F^+ is a p-right Kan extension of F_0^+.

We begin by proving (a). Fix an object $(a, i, j) \in \Delta^1 \times \mathrm{N}(\mathrm{Po}_{[n]})^{op}$, and let

$$\mathcal{J} = (\Delta^1 \times \mathrm{N}(\mathrm{Po}_{[n]}^0)^{op}) \times_{\Delta^1 \times \mathrm{N}(\mathrm{Po}_{[n]})^{op}} (\Delta^1 \times \mathrm{N}(\mathrm{Po}_{[n]})^{op})_{(a,i,j)/}.$$

According to Lemma T.4.3.2.13, it will suffice to show that $f = F_0^+|\mathcal{J}$ can be extended to a p-limit diagram $\overline{f} : \mathcal{J}^{\triangleleft} \to \mathcal{M}^{\circledast}$ lifting the evident map $\mathcal{J}^{\triangleleft} \to \Delta^1 \times \mathrm{N}(\mathrm{Po}_{[n]})^{op} \to \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$.

We first treat the case where $a = 1$. Let S be the full subcategory of $\mathcal{J}$ spanned by those triples of the form $(1, i', i'+1)$ where $i \leq i' < j$. The inclusion $S \to \mathcal{J}$ is right cofinal. It therefore suffices to show that the collection of objects $\{f(1, i', i'+1)\}_{i \leq i' < j}$ admit a p-product in $\mathcal{M}^{\circledast}$, which follows from our assumption that $\mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$ is a Δ-planar ∞-operad.

Assume now that $a = 0$. For $i \leq k \leq j$, we let $\mathcal{J}(k)$ denote the full subcategory of $\mathcal{J}$ spanned by those triples (b, i', j') where such that either $b = 1$ or $k \leq i' \leq j' \leq j$. We will prove that $f|\mathcal{J}(k)$ can be extended to a p-limit diagram $\overline{f}_k : \mathcal{J}(k)^{\triangleleft} \to \mathcal{M}^{\circledast}$ (compatible with the map $\mathcal{J}^{\triangleleft} \to \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$) using descending induction on k. When $k = j$, let S denote the discrete simplicial set with vertices $\{(0, j, j)\} \cup \{(1, i', i'+1)\}_{i \leq i' < j}$. We note that the inclusion $S \to \mathcal{J}$ is right cofinal; it will therefore suffice to show that there exists a p-product for the collection of objects $\{f(0, j, j)\} \cup \{f(1, i', i'+1)\}_{i \leq i' < j}$, which follows from the assumption that the map $\mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$ is fibrous (see Proposition 6.1.1.13). Let us therefore assume that $k < j$ and that the extension $\overline{f}_{k+1}$ has been constructed. Let $\mathcal{J}(k)'$ denote the full subcategory of $\mathcal{J}(k)$ obtained by removing the object $(0, k, k)$. Since $F \in \mathfrak{X}^+$, the functor $f|\mathcal{J}(k)'$ is a p-right Kan extension of $f|\mathcal{J}(k+1)$. It follows from Lemma T.4.3.2.7 that $\overline{f}_{k+1}$ extends (in an essentially unique fashion) to a p-limit diagram $\overline{f}_k' : \mathcal{J}(k)'^{op} \to \mathcal{M}^{\circledast}$. The inclusion $\mathcal{J}(k)' \hookrightarrow \mathcal{J}(k)$ is right cofinal, so that we can further extend $\overline{f}_k'$ (again in an essentially unique way) to a p-limit diagram $\overline{f}_k : \mathcal{J}^{\triangleleft} \to \mathcal{M}^{\circledast}$. This completes the proof of (a). Moreover, the proof yields the following version of (b):

(b') Let $F^+ \in \mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}}(\Delta^1 \times \mathrm{N}(\mathrm{Po}_{[n]})^{op}, \mathcal{M}^{\circledast})$ be such that $F_0^+ = F^+|(\Delta^1 \times \mathrm{N}(\mathrm{Po}_{[n]}^0)^{op})$ belongs to $\mathfrak{X}^+$. Then F^+ is a p-right Kan extension of F_0^+ if and only if, for each $0 \leq i \leq j \leq n$, the maps

$$F^+(0, i, j) \to F^+(0, j, j) \qquad F(0, i, j) \to F(1, i', i'+1) \qquad F(1, i, j) \to F(1, i', i'+1)$$

are inert, where $i \leq i' < j$.

We now prove (b). Assume first that $F^+ \in \mathrm{Str}\,\mathcal{M}_{[n]}^{\mathrm{en},+}$. It is clear that $F_0^+ = F^+|(\Delta^1 \times \mathrm{N}(\mathrm{Po}_{[n]}^0)^{op})$ belongs to $\mathfrak{X}^+$; we wish to prove that F^+ is a p-right Kan extension of F_0^+. We will show that F^+ satisfies the criteria of (b'). Fix $0 \leq i \leq j \leq n$. Since $F = F^+|(\{0\} \times \mathrm{N}(\mathrm{Po}_{[n]})^{op})$ belongs to $\mathrm{Str}\,\mathcal{M}_{[n]}^{\mathrm{en}}$, the map $F^+(0, i, j) \to F^+(0, j, j)$ is inert. For $i \leq i' < j$, consider the composition

$$F^+(0, i, j) \xrightarrow{\alpha} F^+(1, i, j) \xrightarrow{\beta} F^+(1, i', i'+1).$$

We wish to show that β and $\beta \circ \alpha$ are inert. Since α is inert by (ii), it will suffice to show that β is inert. Let $L : \mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast}$ be a left adjoint to the inclusion. Condition (ii) implies that $\beta \simeq L(\beta')$, where β' is the map $F^+(0, i, j) \to F^+(0, i', i'+1)$. The condition that $F \in \mathrm{Str}\,\mathcal{M}_{[n]}^{\mathrm{en}}$ guarantees that $L(\beta')$ is inert as desired.

We now prove the converse direction of (b). Let $F^+ \in \mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}}(\Delta^1 \times \mathrm{N}(\mathrm{Po}_{[n]})^{op}, \mathcal{M}^{\circledast})$. Assume that $F_0^+ = F^+|(\Delta^1 \times \mathrm{N}(\mathrm{Po}_{[n]}^0)^{op})$ belongs to $\mathfrak{X}^+$ and that F^+ is a p-right Kan extension of F_0^+; we wish to prove that F^+ satisfies conditions (i) and (ii). We begin by verifying (ii). Choose $0 \leq i \leq j \leq n$; we wish to show that the map $\alpha : F^+(0, i, j) \to F^+(1, i, j)$ is inert. Condition (b) implies that for $i \leq k < j$, the

map $\beta_j : F^+(1, i, j) \to F^+(1, k, k+1)$ is inert; it will therefore suffice to show that each composition $\beta_j \circ \alpha$ is inert, which follows from (b').

We now complete the proof by showing that $F = F^+|(\{0\} \times \mathrm{N}(\mathrm{Po}_{[n]})^{op})$ belongs to $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[n]}$. Fix $0 \leq i \leq i' \leq j$; we claim that the map $\alpha : F^+(0, i, j) \to F^+(0, i', j)$ is an inert morphism in $\mathcal{M}^{\circledast}$. Condition (b') guarantees that we have inert maps $\beta : F^+(0, i', j) \to F^+(0, j, j)$ and $\gamma_k : F^+(0, i', j) \to F^+(0, k, k+1)$ for $i' \leq k < j$. It will therefore suffice to show that the compositions $\gamma_k \circ \alpha$ and $\beta \circ \alpha$ are inert, which also follows from (b'). To complete the proof that F is an enriched n-string, it suffices to show that LF carries each morphism in $\mathrm{Po}_{[n]}$ to an inert morphism in $\mathcal{C}^{\circledast}$. Fix $0 \leq i \leq i' \leq j' \leq j \leq n$; we wish to show that the map $\delta : LF(i, j) \to LF(i', j')$ is inert. Using (ii), we can identify δ with the map $F^+(1, i, j) \to F^+(1, i', j')$. Condition (b') implies that the maps $F^+(1, i', j') \to F^+(1, k, k+1)$ are inert for $i' \leq k < j'$. It will therefore suffice to show that each of the composite maps $F^+(1, i, j) \to F^+(1, k, k+1)$ is inert, which also follows from (b'). $\square$

We now study the relationship between the ∞-categories $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[n]}$ as $[n]$ varies. As we will see, they can be identified with the fibers of a certain categorical fibration $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}} \to \mathrm{N}(\mathbf{\Delta})^{op}$.

Notation 6.1.2.14. We define a category Po as follows:

- The objects of Po are triples $([n], i, j)$ where $[n] \in \mathbf{\Delta}$ and $0 \leq i \leq j \leq n$.

- Given a pair of objects $([n], i, j), ([n'], i', j') \in \mathrm{Po}$, a morphism from $([n], i, j)$ to $([n'], i', j')$ is a map of linearly ordered sets $\alpha : [n] \to [n']$ such that $i' \leq \alpha(i) \leq \alpha(j) \leq j'$ (equivalently, $\alpha\{i, i+1, \ldots, j\} \subseteq \{i', i'+1, \ldots, j'\}$).

- Composition of morphisms in Po is defined in the obvious way.

We let Po' denote the full subcategory of Po spanned by those objects of the form $([n], i, i)$.

Remark 6.1.2.15. There is an evident forgetful functor $\theta : \mathrm{Po} \to \mathbf{\Delta}$. The fiber of θ over an object $[n]$ is the category associated to the partially ordered set $\mathrm{Po}_{[n]}$ of Notation 6.1.2.5.

The functor θ exhibits Po as cofibered in categories of $\mathbf{\Delta}$: in other words, the induced map $\mathrm{N}(\mathrm{Po}) \to \mathrm{N}(\mathbf{\Delta})$ is a coCartesian fibration of simplicial sets. A morphism $\alpha : ([n], i, j) \to ([n'], i', j')$ is θ-coCartesian if and only if $i' = \alpha(i)$ and $j' = \alpha(j)$.

Remark 6.1.2.16. The functors $\Phi_{[n]} : \mathrm{Po}_{[n]} \to \mathbf{\Delta}$ introduced in Notation 6.1.2.5 are given by the restriction of a single functor $\Phi : \mathrm{Po} \to \mathbf{\Delta}$, which is given on objects by the formula $\Phi([n], i, j) = [j-i] \simeq \{i, i+1, \ldots, j\}$.

Construction 6.1.2.17. We regard the ∞-category $\mathrm{N}(\mathrm{Po})^{op}$ as equipped with forgetful functors $\mathrm{N}(\mathrm{Po})^{op} \to \mathrm{N}(\mathbf{\Delta})^{op}$ (given by the forgetful functor $([n], i, j) \mapsto [n]$) and $\mathrm{N}(\mathrm{Po})^{op} \to \Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}$, given by the composition

$$\mathrm{N}(\mathrm{Po})^{op} \xrightarrow{\Phi} \mathrm{N}(\mathbf{\Delta})^{op} \simeq \{0\} \times \mathrm{N}(\mathbf{\Delta})^{op} \hookrightarrow \Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}.$$

Let $p : \mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}$ be a map which exhibits $\mathcal{M} = \mathcal{M}^{\circledast}_{0, [0]}$ as weakly enriched over the Δ-planar ∞-operad $\mathcal{C}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$. We define maps of simplicial sets $\overline{\mathrm{Str}\,\mathcal{M}}^{\mathrm{en}} \to \mathrm{Str}\,\mathcal{M} \to \mathrm{N}(\mathbf{\Delta})^{op}$ so that the following universal properties are satisfied: for every map of simplicial sets $K \to \mathrm{N}(\mathbf{\Delta})^{op}$, there are canonical bijections

$$\mathrm{Fun}_{\mathrm{N}(\mathbf{\Delta})^{op}}(K, \overline{\mathrm{Str}\,\mathcal{M}}^{\mathrm{en}}) \simeq \mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}}(K \times_{\mathrm{N}(\mathbf{\Delta})^{op}} \mathrm{N}(\mathrm{Po})^{op}, \mathcal{M}^{\circledast})$$

$$\mathrm{Fun}_{\mathrm{N}(\mathbf{\Delta})^{op}}(K, \mathrm{Str}\,\mathcal{M}) \simeq \mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}}(K \times_{\mathrm{N}(\mathbf{\Delta})^{op}} \mathrm{N}(\mathrm{Po}')^{op}, \mathcal{M}^{\circledast}).$$

Unwinding the definitions, we see that the fiber of $\overline{\mathrm{Str}\,\mathcal{M}}^{\mathrm{en}}$ over an object $[n] \in \mathbf{\Delta}^{op}$ can be identified with $\mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}}(\mathrm{N}(\mathrm{Po}_{[n]})^{op}, \mathcal{M}^{\circledast})$. We let $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}$ denote the full simplicial subset of $\overline{\mathrm{Str}\,\mathcal{M}}^{\mathrm{en}}$ spanned by those vertices which are enriched n-strings in $\mathcal{M}$ (see Definition 6.1.2.6).

Remark 6.1.2.18. Let $p : \mathcal{M}^{\circledast} \to \Delta^1 \times N(\mathbf{\Delta})^{op}$ be as in Construction 6.1.2.17. For each $n \geq 0$, there are canonical isomorphisms

$$\operatorname{Str} \mathcal{M}^{\mathrm{en}} \times_{N(\mathbf{\Delta})^{op}} \{[n]\} \simeq \operatorname{Str} \mathcal{M}^{\mathrm{en}}_{[n]} \qquad \operatorname{Str} \mathcal{M} \times_{N(\mathbf{\Delta})^{op}} \{[n]\} \simeq \mathcal{M}^{[n]} .$$

Remark 6.1.2.19. The ∞-category $\operatorname{Str} \mathcal{M}$ depends only on the ∞-category $\mathcal{M}$, and not on the weak enrichment $\mathcal{M}^{\circledast} \to \Delta^1 \times N(\mathbf{\Delta})^{op}$.

Since the map $N(\mathrm{Po}')^{op} \to N(\mathbf{\Delta})^{op}$ is a right fibration of simplicial sets, Corollary T.3.2.2.12 immediately implies the following:

Proposition 6.1.2.20. *Let $\mathcal{M}$ be an ∞-category. Then the map $q_0 : \operatorname{Str} \mathcal{M} \to N(\mathbf{\Delta})^{op}$ is a coCartesian fibration. Moreover, a morphism α in $\operatorname{Str} \mathcal{M}$ is q_0-coCartesian if and only if, for every morphism β in $N(\mathrm{Po}^0)^{op}$ lying over $q_0(\alpha)$, the image of β (under α) is an equivalence in $\mathcal{M}$.*

Remark 6.1.2.21. In the situation of Proposition 6.1.2.20, every map $f : [n] \to [m]$ of linearly ordered sets determines a functor $\operatorname{Str} \mathcal{M}_{[m]} \to \operatorname{Str} \mathcal{M}_{[n]}$, which can be identified with the functor $\mathcal{M}^{[m]} \to \mathcal{M}^{[n]}$ given by composition with f.

The analogous result for ∞-categories of enriched strings is a bit more difficult to state.

Lemma 6.1.2.22. *Let $p : \mathcal{M}^{\circledast} \to \Delta^1 \times N(\mathbf{\Delta})^{op}$ be a map which exhibits $\mathcal{M} = \mathcal{M}^{\circledast}_{0,[0]}$ as weakly enriched over the $\mathbf{\Delta}$-planar ∞-operad $\mathcal{C}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$. Then:*

(1) *The projection map $\overline{q} : \overline{\operatorname{Str} \mathcal{M}}^{\mathrm{en}} \to N(\mathbf{\Delta})^{op}$ is a categorical fibration.*

(2) *For every object $X \in \overline{\operatorname{Str} \mathcal{M}}^{\mathrm{en}}$ and every inert morphism $\alpha : \overline{q}(X) \to [n]$ in $N(\mathbf{\Delta})^{op}$, there exists a $\overline{q}$-coCartesian morphism $\overline{\alpha} : X \to Y$ in $\overline{\operatorname{Str} \mathcal{M}}^{\mathrm{en}}$ with $\overline{q}(\overline{\alpha}) = \alpha$.*

(3) *Let $\overline{\alpha} : X \to Y$ be a morphism in $\overline{\operatorname{Str} \mathcal{M}}^{\mathrm{en}}$ such that $\alpha = \overline{q}(\overline{\alpha})$ is an inert morphism $[m] \to [n]$ in $N(\mathbf{\Delta})^{op}$. Then $\overline{\alpha}$ is $\overline{q}$-coCartesian if and only if, for every pair of integers $0 \leq i \leq j \leq n$, the induced map $X(\alpha(i), \alpha(j)) \to Y(i, j)$ is an equivalence in $\mathcal{M}^{\circledast}$.*

(4) *Let $\overline{\alpha} : X \to Y$ satisfy the condition of (3). If $X \in \operatorname{Str} \mathcal{M}^{\mathrm{en}}$, then $Y \in \operatorname{Str} \mathcal{M}^{\mathrm{en}}$.*

(5) *Assume that p is a coCartesian fibration. Then the map $\overline{q}$ is a coCartesian fibration. Moreover, a morphism $\overline{\alpha} : X \to Y$ in $N(\mathbf{\Delta})^{op}$ is $\overline{q}$-coCartesian if and only if, for every pair of integers $0 \leq i \leq j \leq n$, the induced map $X(\alpha(i), \alpha(j)) \to Y(i, j)$ is an inert morphism in $\mathcal{M}^{\circledast}$, where $\alpha = \overline{q}(\overline{\alpha})$.*

(6) *Assume that p is a coCartesian fibration, and let $\overline{\alpha} : X \to Y$ be a morphism in $\overline{\operatorname{Str} \mathcal{M}}^{\mathrm{en}}$ satisfying the condition of (5). If $X \in \operatorname{Str} \mathcal{M}^{\mathrm{en}}$, then $Y \in \operatorname{Str} \mathcal{M}^{\mathrm{en}}$.*

Proof. The map $N(\mathrm{Po})^{op} \to N(\mathbf{\Delta})^{op}$ is a Cartesian fibration and therefore a flat categorical fibration (Example B.3.11). Assertion (1) now follows from Proposition B.4.5. Suppose that we are given an object $X \in \overline{\operatorname{Str} \mathcal{M}}^{\mathrm{en}}_{[m]}$ and an inert morphism $\alpha : [m] \to [n]$ in $N(\mathbf{\Delta})^{op}$. Let $Y_0 \in \overline{\operatorname{Str} \mathcal{M}}^{\mathrm{en}}_{[n]}$ be the composition of X with the map $N(\mathrm{Po}_{[n]})^{op} \to N(\mathrm{Po}_{[m]})^{op}$ determined by α. There is an evident map $\overline{\alpha}_0 : X \to Y_0$ covering α, which is q-coCartesian by Lemma B.4.8. This proves (2). Note that the map $\overline{\alpha}_0$ satisfies the criterion of (3). If $\overline{\alpha} : X \to Y'$ is any morphism covering α, then the induced map $Y_0 \to Y$ is an equivalence if and only if $X(\alpha(i), \alpha(j)) \to Y(i, j)$ is an equivalence for all $0 \leq i \leq j \leq n$, which proves (3). For assertion (4), we may assume without loss of generality that $Y = Y_0$ in which case the result is obvious. Assertion (5) is a special case of Corollary T.3.2.2.12.

It remains to prove (6). Assume that p is a coCartesian fibration and let $\overline{\alpha} : X \to Y$ be a $\overline{q}$-coCartesian morphism which covers a map $\alpha : [m] \to [n]$ in $N(\mathbf{\Delta})^{op}$. Assume further that X is an enriched m-string; we

wish to show that Y is an enriched n-string. Let $0 \le i \le i' \le j \le n$; we wish to show that $u : Y(i,j) \to Y(i',j)$ is an inert morphism in $\mathcal{M}^{\otimes}$. We have a commutative diagram

$$
\begin{array}{ccc}
X(\alpha(i),\alpha(j)) & \xrightarrow{\ v\ } & Y(i,j) \\
\downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u} \\
X(\alpha(i'),\alpha(j)) & \xrightarrow{\ v'\ } & Y(i',j).
\end{array}
$$

Since v is $\overline{q}$-coCartesian (by (5)), it will suffice to show that $u \circ v \simeq v' \circ u'$ is $\overline{q}$-coCartesian. This follows from the fact that u' and v' are $\overline{q}$-coCartesian (for v' this follows from (5), for u' it follows from our assumption that X is an enriched m-string).

Let $L : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes}$ be a left adjoint to the inclusion. To complete the proof, we must show that for $0 \le i \le i' \le j' \le j \le n$, the map $f : LY(i,j) \to LY(i',j')$ is an inert morphism in $\mathcal{C}^{\otimes}$. The argument proceeds as above. Consider the diagram

$$
\begin{array}{ccc}
LX(\alpha(i),\alpha(j)) & \xrightarrow{\ g\ } & LY(i,j) \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
LX(\alpha(i'),\alpha(j')) & \xrightarrow{\ g'\ } & LY(i',j').
\end{array}
$$

Since L carries $\overline{q}$-coCartesian morphisms in $\mathcal{M}^{\otimes}$ to $\overline{q}$-coCartesian morphisms in $\mathcal{C}^{\otimes}$, the morphisms g and g' are $\overline{q}$-coCartesian. Consequently, to prove that f is inert, it will suffice to show that $f \circ g \simeq g' \circ f'$ is $\overline{q}$-coCartesian. We are therefore reduced to showing that f' is $\overline{q}$-coCartesian, which follows from our assumption that X is an enriched m-string. $\qquad \square$

From Lemma 6.1.2.22 we immediately deduce the following:

Proposition 6.1.2.23. *Let $p : \mathcal{M}^{\otimes} \to \Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}$ be a map which exhibits $\mathcal{M} = \mathcal{M}^{\otimes}_{0,[0]}$ as weakly enriched over the Δ-planar ∞-operad $\mathcal{C}^{\otimes} = \mathcal{M}^{\otimes} \times_{\Delta^1} \{1\}$. Then:*

(1) *The projection map $q : \operatorname{Str}\mathcal{M}^{\mathrm{en}} \to \mathrm{N}(\mathbf{\Delta})^{op}$ is a categorical fibration.*

(2) *For every object $X \in \operatorname{Str}\mathcal{M}^{\mathrm{en}}$ and every inert morphism $\alpha : \overline{q}(X) \to [n]$ in $\mathrm{N}(\mathbf{\Delta})^{op}$, there exists a q-coCartesian morphism $\overline{\alpha} : X \to Y$ in $\overline{\operatorname{Str}\mathcal{M}}^{\mathrm{en}}$ with $q(\overline{\alpha}) = \alpha$.*

(3) *Let $\overline{\alpha} : X \to Y$ be a morphism in $\operatorname{Str}\mathcal{M}^{\mathrm{en}}$ such that $\alpha = q(\overline{\alpha})$ is an inert morphism $[m] \to [n]$ in $\mathrm{N}(\mathbf{\Delta})^{op}$. Then $\overline{\alpha}$ is q-coCartesian if and only if, for every pair of integers $0 \le i \le j \le n$, the induced map $X(\alpha(i),\alpha(j)) \to Y(i,j)$ is an equivalence in $\mathcal{M}^{\otimes}$.*

(4) *Assume that p is a coCartesian fibration. Then the map q is a coCartesian fibration. Moreover, a morphism $\overline{\alpha} : X \to Y$ in $\mathrm{N}(\mathbf{\Delta})^{op}$ is q-coCartesian if and only if, for every pair of integers $0 \le i \le j \le n$, the induced map $X(\alpha(i),\alpha(j)) \to Y(i,j)$ is an inert morphism in $\mathcal{M}^{\otimes}$, where $\alpha = q(\overline{\alpha})$.*

Lemma 6.1.2.24. *Let $p : \mathcal{M}^{\otimes} \to \Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}$ be a map which exhibits $\mathcal{M} = \mathcal{M}^{\otimes}_{0,[0]}$ as weakly enriched over the Δ-planar ∞-operad $\mathcal{C}^{\otimes} = \mathcal{M}^{\otimes} \times_{\Delta^1} \{1\}$, and let $q : \operatorname{Str}\mathcal{M}^{\mathrm{en}} \to \mathrm{N}(\mathbf{\Delta})^{op}$ be the induced map. Let $n \ge 0$, let $\operatorname{Po}^0_{[n]} \subseteq \operatorname{Po}_{[n]}$ be defined as in the proof of Proposition 6.1.2.13, and let $\operatorname{Po}^1_{[n]} = \operatorname{Po}^0_{[n]} \cup \{(0,n)\} \subseteq \operatorname{Po}_{[n]}$. Suppose we are given a diagram*

$$
\begin{array}{ccc}
\mathrm{N}(\operatorname{Po}^1_{[n]})^{op} & \xrightarrow{\ X\ } & \operatorname{Str}\mathcal{M}^{\mathrm{en}} \\
\downarrow & & \downarrow{\scriptstyle q} \\
\mathrm{N}(\operatorname{Po}_{[n]})^{op} & \longrightarrow & \mathrm{N}(\mathbf{\Delta})^{op}
\end{array}
$$

where X carries each morphism in $\operatorname{Po}^1_{[n]}$ to a q-coCartesian morphism in $\operatorname{Str}\mathcal{M}$. Then X is a q-limit diagram.

Proof. Let $\mathcal{J}$ denote the fiber product $\mathrm{N}(\mathrm{Po})^{op} \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} \mathrm{N}(\mathrm{Po}^1_{[n]})^{op}$, so that the objects of $\mathcal{J}$ can be identified with sequences $0 \leq i \leq i' \leq j' \leq j \leq n$ such that $(i,j) \in \mathrm{Po}^1_{[n]}$, and let $\mathcal{J}_0$ be the full subcategory spanned by those objects where $(i,j) \in \mathrm{Po}^0_{[n]}$. The diagram X determines a functor $F : \mathcal{J} \to \mathcal{M}^{\circledast}$. According to Proposition B.4.9, it will suffice to show that F is a p-right Kan extension of $F_0 = F| \mathcal{J}_0$. Fix an object $\sigma \in \mathcal{J}$ which does not belong to $\mathcal{J}_0$, necessarily of the form $0 \leq i \leq i' \leq j' \leq j \leq n$, where $i = 0$ and $j = n$; we wish to prove that F is a p-right Kan extension of F_0 at σ. Replacing $[n]$ by the interval $\{i', i'+1, \ldots, j'\}$, we can reduce to the case where $i' = 0$ and $j' = n$. Then σ is an initial object of $\mathcal{J}$, and we wish to prove that F exhibits $F(\sigma)$ as a p-limit of the diagram F_0. Let $\mathcal{J}_1$ be the full subcategory of $\mathcal{J}_0$ spanned by those sequences $0 \leq i \leq i' \leq j' \leq j \leq 0$ where $i = i'$ and $j = j'$. The inclusion $\mathcal{J}_1 \subseteq \mathcal{J}_0$ admits a right adjoint, so that $\mathcal{J}_1 \to \mathcal{J}_0$ is right cofinal. It will therefore suffice to show that F exhibits $F(\sigma)$ as a p-limit of the diagram $F_1 = F| \mathcal{J}_1$.

Let $L : \mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast}$ be a left adjoint to the inclusion and consider the natural transformation $F \to LF$, which we identify with a functor $F^+ : \Delta^1 \times \mathcal{J} \to \mathcal{M}^{\circledast}$. It now suffices to show that F^+ exhibits $F^+(0, \sigma)$ as a p-limit of the diagram $F^+ | \Delta^1 \times \mathcal{J}_1$. This follows from assertion (b) in the proof of Proposition 6.1.2.13. $\square$

Lemma 6.1.2.25. *Let $\mathcal{M}$ be an ∞-category and let $q_0 : \mathrm{Str}\, \mathcal{M} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$ be the induced map. Let $n \geq 0$, $\mathrm{Po}^0_{[n]}$, and $\mathrm{Po}^1_{[n]}$ be as in the statement of Lemma 6.1.2.20, and suppose we are given a diagram*

$$
\begin{array}{ccc}
\mathrm{N}(\mathrm{Po}^1_{[n]})^{op} & \xrightarrow{\ X\ } & \mathrm{Str}\, \mathcal{M} \\
\downarrow & & \downarrow {\scriptstyle q_0} \\
\mathrm{N}(\mathrm{Po}_{[n]})^{op} & \longrightarrow & \mathrm{N}(\boldsymbol{\Delta})^{op}
\end{array}
$$

where X carries each morphism in $\mathrm{Po}^1_{[n]}$ to a q_0-coCartesian morphism in $\mathrm{Str}\, \mathcal{M}$. Then X is a q_0-limit diagram.

Proof. Let $\mathcal{J}$ denote the fiber product $\mathrm{N}(\mathrm{Po}')^{op} \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} \mathrm{N}(\mathrm{Po}^1_{[n]})^{op}$, so that the objects of $\mathcal{J}$ can be identified with sequences $0 \leq i \leq k \leq j \leq n$ such that $(i,j) \in \mathrm{Po}^1_{[n]}$, and let $\mathcal{J}_0$ be the full subcategory spanned by those objects where $(i,j) \in \mathrm{Po}^0_{[n]}$. The diagram X determines a functor $F : \mathcal{J} \to \mathcal{M}$. According to Proposition B.4.9, it will suffice to show that F is a right Kan extension of $F_0 = F|\mathcal{J}_0$. Fix an object $\sigma \in \mathcal{J}$ given by $0 \leq i \leq k \leq j \leq n$ and let $\mathcal{J}' = \mathcal{J}_0 \times_{\mathcal{J}} \mathcal{J}_{\sigma/}$. We wish to show that F exhibits $F(\sigma)$ as a limit of the diagram $F|\mathcal{J}'$. Since F carries every morphism in $\mathcal{J}$ to an equivalence in $\mathcal{M}$, it will suffice to show that $\mathcal{J}'$ is weakly contractible. This is clear, since $\mathcal{J}'$ has a final object. $\square$

Remark 6.1.2.26. In the situation of Lemma 6.1.2.25, we will say that a morphism α in $\mathrm{Str}\, \mathcal{M}$ is *inert* if $q_0(\alpha)$ is inert and α is q_0-Cartesian.

Lemma 6.1.2.27. *Let $p : \mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$ be a map which exhibits $\mathcal{M} = \mathcal{M}^{\circledast}_{0,[0]}$ as weakly enriched over the $\boldsymbol{\Delta}$-planar ∞-operad $\mathcal{C}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$. Consider the diagram*

$$
\begin{array}{ccc}
\mathrm{Str}\, \mathcal{M}^{\mathrm{en}} & \xrightarrow{\ f\ } & \mathrm{Str}\, \mathcal{M} \\
 & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle q_0} & \\
 & \mathrm{N}(\boldsymbol{\Delta})^{op}. &
\end{array}
$$

Then:

(1) *The map $f : \mathrm{Str}\, \mathcal{M}^{\mathrm{en}} \to \mathrm{Str}\, \mathcal{M}$ is a categorical fibration.*

(2) *Let $X \in \mathrm{Str}\, \mathcal{M}^{\mathrm{en}}$ and suppose we are given an inert morphism $\alpha : f(X) \to Y$ in $\mathrm{Str}\, \mathcal{M}$. Then there exists an f-coCartesian morphism $\overline{\alpha} : X \to \overline{Y}$ lifting α.*

(3) *Let $\alpha : f(X) \to Y$ be as in (2), and let $\alpha_0 = q_0(\alpha) : [m] \to [n]$ be the induced morphism in $\mathrm{N}(\boldsymbol{\Delta})^{op}$. An arbitrary morphism $\overline{\alpha} : X \to \overline{Y}$ lifting α is f-coCartesian if and only if $\overline{\alpha}$ induces an equivalence $X(\alpha_0(i), \alpha_0(j)) \to \overline{Y}(i, j)$ for $0 \le i \le j \le n$.*

(4) *Assume that p is a coCartesian fibration, let $X \in \mathrm{Str}\,\mathcal{M}^{\mathrm{en}}$, and suppose we are given a q_0-coCartesian morphism $\alpha : f(X) \to Y$ in $\mathrm{Str}\,\mathcal{M}$. Then there exists an f-coCartesian morphism $\overline{\alpha} : X \to \overline{Y}$ lifting α.*

(5) *Let $\alpha : f(X) \to Y$ be as in (4), and let $\alpha_0 = q_0(\alpha) : [m] \to [n]$ be the induced morphism in $\mathrm{N}(\boldsymbol{\Delta})^{op}$. An arbitrary morphism $\overline{\alpha} : X \to \overline{Y}$ lifting α is f-coCartesian if and only if $\overline{\alpha}$ induces a p-coCartesian map $X(\alpha_0(i), \alpha_0(j)) \to \overline{Y}(i, j)$ for $0 \le i \le j \le n$.*

(6) *Let $n \ge 0$, $\mathrm{Po}^0_{[n]}$, and $\mathrm{Po}^1_{[n]}$ be as in the statement of Lemma 6.1.2.20, and suppose we are given a diagram*

$$
\begin{array}{ccc}
\mathrm{N}(\mathrm{Po}^1_{[n]})^{op} & \xrightarrow{\quad X \quad} & \mathrm{Str}\,\mathcal{M}^{\mathrm{en}} \\
\downarrow & & \downarrow{\scriptstyle q} \\
\mathrm{N}(\mathrm{Po}_{[n]})^{op} & \longrightarrow & \mathrm{N}(\boldsymbol{\Delta})^{op}.
\end{array}
$$

If X carries every morphism in $\mathrm{N}(\mathrm{Po}^1_{[n]})^{op}$ to a q-coCartesian morphism in $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}$, then X is an f-limit diagram.

Proof. To prove (1), it suffices to show that the map $\overline{f} : \overline{\mathrm{Str}\,\mathcal{M}}^{\mathrm{en}} \to \mathrm{Str}\,\mathcal{M}$ is a categorical fibration: that is, that $\overline{f}$ has the right lifting property with respect to every cofibration of simplicial sets $i : A \to B$ which is also a categorical equivalence. This is equivalent to the requirement that, for every map $B \to \mathrm{N}(\boldsymbol{\Delta})^{op}$, the morphism p has the right lifting property with respect to the induced map

$$
j : (A \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} \mathrm{N}(\mathrm{Po})^{op}) \coprod_{A \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} \mathrm{N}(\mathrm{Po}')^{op}} (B \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} \mathrm{N}(\mathrm{Po}')^{op}) \to B \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} \mathrm{N}(\mathrm{Po})^{op}.
$$

Since p is a categorical fibration and j is a cofibration, it suffices to show that j is a categorical equivalence. Because the Joyal model structure is left proper, we are reduced to showing that the inclusions

$$
A \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} \mathrm{N}(\mathrm{Po})^{op} \to B \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} \mathrm{N}(\mathrm{Po})^{op}
$$

$$
A \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} \mathrm{N}(\mathrm{Po}')^{op} \to B \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} \mathrm{N}(\mathrm{Po}')^{op}
$$

are categorical equivalences. This follows from Proposition T.3.3.1.3, since the forgetful functors

$$
\mathrm{N}(\mathrm{Po})^{op} \to \mathrm{N}(\boldsymbol{\Delta})^{op} \qquad \mathrm{N}(\mathrm{Po}')^{op} \to \mathrm{N}(\boldsymbol{\Delta})^{op}
$$

are Cartesian fibrations.

Assertions (2), (3), (4), and (5) follow by combining Propositions 6.1.2.20, 6.1.2.23, and PT.2.4.1.3. Similarly, assertion (6) follows from Lemmas 6.1.2.24 and 6.1.2.20 together with Proposition T.2.4.1.3. $\square$

Definition 6.1.2.28. Let $p : \mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$ be a map which exhibits $\mathcal{M} = \mathcal{M}^{\circledast}_{0,[0]}$ as weakly enriched over the Δ-planar ∞-operad $\mathcal{C}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$. For every object $M \in \mathcal{M}$, we let $\mathcal{C}[M]^{\circledast}$ denote the fiber product $\mathrm{N}(\boldsymbol{\Delta})^{op} \times_{\mathrm{Str}\,\mathcal{M}} \mathrm{Str}\,\mathcal{M}^{\mathrm{en}}$, where the map $\mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathrm{Str}\,\mathcal{M}$ corresponds to the constant functor $\mathrm{N}(\mathrm{Po}')^{op} \to \mathcal{M}$ taking the value $M \in \mathcal{M}$.

Remark 6.1.2.29. In the situation of Definition 6.1.2.28, consider the projection map $\theta : \mathcal{C}[M]^{\circledast} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$. The inverse image $\theta^{-1}\{[1]\}$ is isomorphic to the ∞-category $\mathcal{C}[M]$ of Definition 6.1.2.1.

Proposition 6.1.2.30. *Let $p : \mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}$ be a map which exhibits $\mathcal{M} = \mathcal{M}^{\circledast}_{0,[0]}$ as weakly enriched over the $\mathbf{\Delta}$-planar ∞-operad $\mathcal{C}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$. For every object $M \in \mathcal{M}$, the projection map $\theta : \mathcal{C}[M]^{\circledast} \to \mathrm{N}(\mathbf{\Delta})^{op}$ exhibits $\mathcal{C}[M]^{\circledast}$ as a $\mathbf{\Delta}$-planar ∞-operad. If p is a coCartesian fibration, then θ exhibits $\mathcal{C}[M]^{\circledast}$ as a $\mathbf{\Delta}$-monoidal ∞-category.*

Proof. We verify that the functor θ satisfies the requirements of Definition 6.1.1.11. Conditions (1) and (2) follow immediately from Lemma 6.1.2.27 (which also implies that θ is a coCartesian fibration if p is a coCartesian fibration), and condition (4) follows from Proposition 6.1.2.13. To verify condition (3), consider an object $X \in \mathcal{C}[M]^{\circledast}_{[n]}$ and a collection of θ-coCartesian morphisms $\alpha_i : X \to X_i$ covering the maps $[n] \to [1]$ in $\mathrm{N}(\mathbf{\Delta})^{op}$ given by the inclusions $[1] \simeq \{i-1, i\} \hookrightarrow [n]$ in $\mathbf{\Delta}$ for $1 \leq i \leq n$. We wish to prove that the maps α_i exhibit X as a θ-product of the objects X_i. Assume for notational simplicity that $n \neq 1$ (otherwise the result is obvious). Let $\mathrm{Po}^1_{[n]}$ be as in the statement of Lemma 6.1.2.20 and let $\mathrm{Po}^2_{[n]} \subseteq \mathrm{Po}^1_{[n]}$ be the subset consisting of pairs (i,j) where either $(i,j) = (0,n)$ or $j = i+1$. The morphisms $\{\alpha_i\}_{1 \leq i \leq n}$ determine a diagram $X_0 : \mathrm{N}(\mathrm{Po}^2_{[n]})^{op} \to \mathrm{Str}\, \mathcal{M}^{\mathrm{en}}$ which fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{N}(\mathrm{Po}^2_{[n]})^{op} & \xrightarrow{\;X_0\;} & \mathrm{Str}\,\mathcal{M}^{\mathrm{en}} \\ \downarrow & \nearrow \; X \; & \downarrow f \\ \mathrm{N}(\mathrm{Po}^1_{[n]})^{op} & \longrightarrow & \mathrm{Str}\,\mathcal{M}. \end{array}$$

We wish to prove that X_0 determines a θ-limit diagram in $\mathcal{C}[M]^{\circledast}$. Using Lemma T.4.3.2.13, we can choose an f-left Kan extension X of X_0 as indicated in the diagram. Lemma 6.1.2.27 implies that X is an f-limit diagram, so that X determines a θ-limit diagram X' in $\mathcal{C}[M]^{\circledast}$. To complete the proof, it will suffice to show that $X' | \mathrm{N}(\mathrm{Po}^1_{[n]})^{op}$ is a θ-limit diagram. In view of Lemma T.4.3.2.7, it will suffice to show that X' is a θ-right Kan extension of $X' | \mathrm{N}(\mathrm{Po}^2_{[n]})^{op}$. Unwinding the definitions, it will suffice to show that every object $Y \in \mathcal{C}[M]^{\circledast}_{[0]}$ is θ-final. To this end, choose any object $Z \in \mathcal{C}[M]^{\circledast}_{[n]}$ and any morphism $\alpha : [n] \to [0]$ in $\mathrm{N}(\mathbf{\Delta})^{op}$; we wish to show that mapping space

$$\mathrm{Map}^{\alpha}_{\mathcal{C}[M]^{\circledast}}(Z, *) = \mathrm{Map}_{\mathcal{C}[M]^{\circledast}}(Z, Y) \times_{\mathrm{Hom}_{\mathbf{\Delta}}([0],[n])} \{\alpha\}$$

is weakly contractible. Since α is inert, we can choose a θ-coCartesian morphism $Z \to Z_0$ in $\mathcal{C}[M]^{\circledast}$ lying over α. It will therefore suffice to show that the mapping space $\mathrm{Map}_{\mathcal{C}[M]^{\circledast}_{[0]}}(Z_0, Y)$ is contractible. This is clear, since $\mathcal{C}[M]^{\circledast}_{[0]}$ is isomorphic to Δ^0. $\qquad\square$

Remark 6.1.2.31. In the situation of Definition 6.1.2.1, the ∞-category $\mathcal{C}[M] \times_{\mathcal{C}} \{\mathbf{1}_{\mathcal{C}}\}$ can be identified with the Kan complex $\mathcal{M}^{/M} \times_{\mathcal{M}} \{M\} \simeq \mathrm{Map}_{\mathcal{M}}(M, M)$. The monoidal structure on $\mathcal{C}[M]$ induces a coherently associative multiplication on $\mathrm{Map}_{\mathcal{M}}(M, M)$, which simply encodes the composition in the ∞-category $\mathcal{M}$. In particular, if $\widetilde{1}_{\mathcal{C}}$ is an object of $\mathcal{C}[M] \times_{\mathcal{C}} \{\mathbf{1}_{\mathcal{C}}\}$ which classifies an *equivalence* from M to itself, then $\widetilde{1}_{\mathcal{C}}$ is an *invertible object* of $\mathcal{C}[M]$ (see Remark 4.1.1.19).

In what follows, it will be convenient to work with a slight variant of the ∞-category $\mathcal{C}[M]^{\circledast}$.

Definition 6.1.2.32. Let $p : \mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}$ be a map which exhibits $\mathcal{M} = \mathcal{M}^{\circledast}_{0,[0]}$ as weakly enriched over the $\mathbf{\Delta}$-planar ∞-operad $\mathcal{C}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$. We define maps of simplicial sets $\overline{\mathrm{Str}\,\mathcal{M}}^{\mathrm{en},+} \to \mathrm{N}(\mathbf{\Delta})^{op}$ so that the following universal property is satisfied: for every map of simplicial sets $K \to \mathrm{N}(\mathbf{\Delta})^{op}$, there are canonical bijections

$$\mathrm{Fun}_{\mathrm{N}(\mathbf{\Delta})^{op}}(K, \overline{\mathrm{Str}\,\mathcal{M}}^{\mathrm{en},+}) \simeq \mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}}(\Delta^1 \times K \times_{\mathrm{N}(\mathbf{\Delta})^{op}} \mathrm{N}(\mathrm{Po})^{op}, \mathcal{M}^{\circledast})$$

Unwinding the definitions, we see that the fiber of $\overline{\mathrm{Str}\,\mathcal{M}}^{\mathrm{en},+}$ over an object $[n] \in \mathbf{\Delta}^{op}$ can be identified with $\mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}}(\Delta^1 \times \mathrm{N}(\mathrm{Po}_{[n]})^{op}, \mathcal{M}^{\circledast})$. We let $\mathrm{Str}\,\mathcal{M}^{\mathrm{en},+}$ denote the full simplicial subset of $\overline{\mathrm{Str}\,\mathcal{M}}^{\mathrm{en},+}$ spanned

by those vertices which correspond to functors $F : \Delta^1 \times \mathrm{N}(\mathrm{Po}_{[n]})^{op} \to \mathcal{M}^{\circledast}$ such that $F_0 = F|(\{0\} \times \mathrm{N}(\mathrm{Po}_{[n]})^{op}$ is an enriched n-string and F is a p-left Kan extension of F_0. The inclusion $\{0\} \times \mathrm{N}(\mathrm{Po})^{op} \hookrightarrow \Delta^1 \times \mathrm{N}(\mathrm{Po})^{op}$ induces a trivial Kan fibration $u : \mathrm{Str}\,\mathcal{M}^{\mathrm{en},+} \to \mathrm{Str}\,\mathcal{M}$. If $M \in \mathcal{M}$ is an object, we let $\mathcal{C}^+[M]^{\circledast}$ denote the fiber product $\mathcal{C}[M]^{\circledast} \times_{\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}} \mathrm{Str}\,\mathcal{M}^{\mathrm{en},+}$, so that u induces a trivial Kan fibration $\mathcal{C}^+[M]^{\circledast} \to \mathcal{C}[M]^{\circledast}$.

Remark 6.1.2.33. Consider the functor $i : \mathrm{N}(\mathbf{\Delta})^{op} \to \Delta^1 \times \mathrm{N}(\mathrm{Po})^{op}$, given by $[n] \mapsto (1, ([n], 0, n))$. In the situation of Definition 6.1.2.32, let q^+ denote the projection map $\mathcal{C}^+[M]^{\circledast} \to \mathrm{N}(\mathbf{\Delta})^{op}$. It follows from Proposition 6.1.2.30 that q^+ exhibits $\mathcal{C}^+[M]^{\circledast}$ as a Δ-planar ∞-operad. Composition with i induces a categorical fibration $\theta : \mathcal{C}^+[M]^{\circledast} \to \mathcal{C}^{\circledast}$, and Lemma 6.1.2.27 implies that θ is a map of Δ-planar ∞-operads (that is, θ carries inert morphisms in $\mathcal{C}^+[M]^{\circledast}$ to inert morhisms in $\mathcal{C}^{\circledast}$). If $p : \mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}$ is a coCartesian fibration, then q^+ is a coCartesian fibration and θ carries q^+-coCartesian morphisms to p-coCartesian morphisms (that is, θ is a map of Δ-monoidal ∞-categories).

We now come to the main result of this section, which describes the ∞-category of algebra objects of $\mathcal{C}[M]^{\circledast}$.

Theorem 6.1.2.34. *Let $p : \mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}$ be a map which exhibits $\mathcal{M} = \mathcal{M}^{\circledast}_{0,[0]}$ as weakly enriched over the Δ-planar ∞-operad $\mathcal{C}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$. Let $s : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathrm{N}(\mathrm{Po})^{op}$ be the functor given on objects by $[n] \mapsto ([n], 0, n)$. Then, for each object $M \in \mathcal{M}$, composition with s induces a categorical equivalence*

$$\theta : {}^{\Delta}\mathrm{Alg}(\mathcal{C}^+[M]) \to {}^{\Delta}\mathrm{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\}.$$

Remark 6.1.2.35. In the situation of Theorem 6.1.2.34, we obtain an equivalence of ∞-categories

$$^{\Delta}\mathrm{Alg}(\mathcal{C}[M]) \to {}^{\Delta}\mathrm{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\}$$

(well-defined up to a contractible space of choices) by composing a section of the trivial Kan fibration $^{\Delta}\mathrm{Alg}(\mathcal{C}^+[M]) \to {}^{\Delta}\mathrm{Alg}(\mathcal{C}[M])$ with the categorical equivalence

$$\theta : {}^{\Delta}\mathrm{Alg}(\mathcal{C}^+[M]) \to {}^{\Delta}\mathrm{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\}.$$

Proof. Let $\mathcal{X} \subseteq \mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}}(\Delta^1 \times \mathrm{N}(\mathrm{Po})^{op}, \mathcal{M}^{\circledast})$ be the full subcategory spanned by those functors F satisfying the following conditions:

(a) If $0 \le i \le i' \le j \le n$, then the map $F(0, [n], i, j) \to F(0, [n], i', j)$ is inert.

(b) If $0 \le i \le i' \le j' \le n$, then the map $F(1, [n], i, j) \to F(1, [n], i', j')$ is inert.

(c) For $0 \le i \le j \le n$, the map $F(0, [n], i, j) \to F(1, [n], i, j)$ is inert.

(d) If $\alpha : [m] \to [n]$ is an inert morphism in $\mathbf{\Delta}$, then $F(0, [n], \alpha(i), \alpha(j)) \to F(0, [m], i, j)$ is an equivalence in $\mathcal{M}^{\circledast}$.

Note that conditions (a) and (d) imply the following:

(a') For every inert morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$ satisfying $\alpha(m) = n$, the induced map $F(0, [n], 0, n) \to F(0, [m], 0, m)$ is inert.

Similarly, (b) and (d) imply:

(b') For every inert morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$, the induced map $F(1, [n], 0, n) \to F(1, [m], 0, m)$ is inert.

Finally, (c) and (d) imply:

(d') If $\alpha : [m] \to [n]$ is an inert morphism in $\mathbf{\Delta}$, then $F(1, [n], \alpha(i), \alpha(j)) \to F(1, [m], i, j)$ is an equivalence in $\mathcal{M}^{\circledast}$.

It follows that composition with s induces a forgetful functor $\theta' : \mathfrak{X} \to {}^{\Delta}\mathrm{LMod}(\mathfrak{M})$. Let $\mathfrak{X}_0 \subseteq \mathrm{Fun}(\mathrm{N}(\mathrm{Po}')^{op}, \mathfrak{M})$ be the full subcategory spanned by those functors which carry each morphism in $\mathrm{N}(\mathrm{Po}')^{op}$ to an equivalence in $\mathfrak{M}$. Let $\overline{M} \in \mathfrak{X}_0$ be the constant functor taking the value $M \in \mathfrak{M}$. Unwinding the definitions, we have a canonical isomorphism ${}^{\Delta}\mathrm{Alg}(\mathfrak{C}^+[M]) \simeq \mathfrak{X} \times_{\mathfrak{X}_0} \overline{M}$. In other words, we can identify θ' with the map between vertical fibers determined by the diagram

$$
\begin{array}{ccc}
\mathfrak{X} & \xrightarrow{\phi} & {}^{\Delta}\mathrm{LMod}(\mathfrak{M}) \\
\downarrow & & \downarrow \\
\mathfrak{X}_0 & \xrightarrow{\phi_0} & \mathfrak{M}\,.
\end{array}
$$

It will therefore suffice to show that the functors ϕ and ϕ_0 are categorical equivalences. The map ϕ_0 is an equivalence since $\mathrm{N}(\mathrm{Po}')^{op}$ is weakly contractible (it has a final object, given by $([0], 0, 0)$). It will therefore suffice to show that ϕ is a categorical equivalence. Let $\mathfrak{X}'$ be the full subcategory of $\mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}}(\{0\} \times \mathrm{N}(\mathrm{Po})^{op}, \mathfrak{M}^{\otimes})$ spanned by those functors F which satisfy (a'), (b'), (d), (d'), and the following weaker version of (c):

(c') For $n \geq 0$, the map $F(0, [n], 0, n) \to F(1, [n], 0, n)$ is inert.

A functor $F \in \mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}}(\{0\} \times \mathrm{N}(\mathrm{Po})^{op}, \mathfrak{M}^{\otimes})$ belongs to $\mathfrak{X}'$ if and only if $F_0 = F|(\Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op})$ determines an object of ${}^{\Delta}\mathrm{LMod}(\mathfrak{M})$ and F is a p-right Kan extension of F_0. Then $\mathfrak{X}'$ is a full subcategory of $\mathfrak{X}$ and Proposition T.4.3.2.15 guarantees that the restriction map $\phi | \mathfrak{X}'$ is a trivial Kan fibration. We will complete the proof by showing that $\mathfrak{X} = \mathfrak{X}'$. In other words, we will show that a functor $F \in \mathfrak{X}'$ also satisfies conditions (a), (b), and (c). To prove (a), consider $0 \leq i \leq i' \leq j \leq n$. Condition (d) guarantees that the induced map $F(0, [n], i, j) \to F(0, [n], i', j)$ is equivalent to the map $F([j-i], 0, j-i) \to F([j-i'], 0, j-i')$ and is therefore inert by (a'). To prove (b), we assume that $0 \leq i \leq i' \leq j' \leq j \leq n$ and note that (d') implies that $F(0, [n], i, j) \to F(0, [n], i', j')$ is equivalent to the map $LF([j-i], 0, j-i) \to LF([j'-i'], 0, j'-i')$ which is inert by (b'). It remains to verify (c). Fix $0 \leq i \leq j \leq n$; we wish to show that the map $u : F(0, [n], i, j) \to F(1, [n], i, j)$ is inert. Using (b), we are reduced to proving that the composite map $F(0, [n], i, j) \to F(1, [n], i, j) \simeq F(1, [j-i], 0, j-i)$ is inert. This map factors as a composition

$$
F(0, [n], i, j) \xrightarrow{u'} F(0, [j-i], 0, j-i) \xrightarrow{u''} F(1, [j-i], 0, j-i)
$$

where u' is an equivalence by (a) and u'' is inert by virtue of (c'). $\qquad\square$

Our next goal is to analyze the relationship between the Δ-planar ∞-operads $\mathfrak{C}[M]^{\otimes}$ and $\mathfrak{C}^{\otimes}$. In the situation of Definition 6.1.2.32, we let $\mathfrak{C}^+[M]$ denote the fiber product $\mathfrak{C}^+[M]^{\otimes} \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} \{[1]\}$. Our starting point is the following observation:

Lemma 6.1.2.36. *Let $p : \mathfrak{M}^{\otimes} \to \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$ be a map which exhibits $\mathfrak{M} = \mathfrak{M}^{\otimes}_{0,[0]}$ as weakly enriched over the Δ-planar ∞-operad $\mathfrak{C}^{\otimes} = \mathfrak{M}^{\otimes} \times_{\Delta^1} \{1\}$. For every object $M \in \mathfrak{M}$, the categorical fibration $\theta : \mathfrak{C}^+[M]^{\otimes} \to \mathfrak{C}^{\otimes}$ of Remark 6.1.2.33 induces a right fibration of ∞-categories $\theta_{[1]} : \mathfrak{C}^+[M] \to \mathfrak{C}$.*

Proof. Let $\mathcal{J}$ be the full subcategory of the product $\Delta^1 \times \mathrm{N}(\mathrm{Po}_{[1]})^{op}$ obtained by removing the vertex $(1, [1], 0, 0)$ and let $\mathcal{J}_0 \subset \mathcal{J}$ be the full subcategory obtained by removing the vertex $(0, [1], 0, 0)$. Let $\mathfrak{X}_0$ be the full subcategory of $\mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathcal{J}_0, \mathfrak{M}^{\otimes}) \times_{\mathfrak{M}} \{M\}$ spanned by the p-limit diagrams and let $\mathfrak{X} = \mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathcal{J}, \mathfrak{M}^{\otimes}) \times_{\mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathcal{J}_0, \mathfrak{M}^{\otimes})} \mathfrak{X}_0$. The map $\theta_{[1]}$ factors as a composition of restriction functors

$$
\mathfrak{C}^+[M] \xrightarrow{\phi} \mathfrak{X} \times_{\mathfrak{M}} \{M\} \xrightarrow{\theta'} \mathfrak{X}_0 \xrightarrow{\psi} \mathfrak{C}\,.
$$

where ϕ and ψ are trivial Kan fibrations by Proposition T.4.3.2.15. It will therefore suffice to show that θ' is a right fibration. But θ' is a pullback of the right fibration $(\mathfrak{M}^{\otimes})^{/M} \to \mathfrak{M}^{\otimes}$. $\qquad\square$

Let $\mathcal{M}^{\circledast}$ and M be as above. For each $n \geq 0$ we can identify the map $\theta_{[n]} : \mathcal{C}^{+}[M]^{\circledast}_{[n]} \to \mathcal{C}^{\circledast}_{[n]}$ with the nth power of the right fibration $\theta_{[1]}$ of Lemma 6.1.2.36. The forgetful functor $\mathcal{C}^{+}[M]^{\circledast} \to \mathcal{C}^{\circledast}$ is not a right fibration, but we have the following closely related result:

Lemma 6.1.2.37. *Let $p : \mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$ be a map which exhibits $\mathcal{M} = \mathcal{M}^{\circledast}_{0,[0]}$ as weakly enriched over the $\boldsymbol{\Delta}$-planar ∞-operad $\mathcal{C}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$. Let $M \in \mathcal{M}$ be an object and let $\theta : \mathcal{C}^{+}[M]^{\circledast} \to \mathcal{C}^{\circledast}$ be as in Remark 6.1.2.33. For each $n \geq 0$, every morphism f in $\mathcal{C}^{+}[M]^{\circledast}_{[m]}$ is θ-Cartesian.*

Proof. We begin with an auxiliary construction. Define a map of simplicial sets $\overline{\mathcal{C}}^{\circledast} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$ so that the following universal property is satisfied: for every map of simplicial sets $K \to \mathrm{N}(\boldsymbol{\Delta})^{op}$, there is a canonical bijection
$$\mathrm{Hom}_{(\mathrm{Set}_{\Delta})/\mathrm{N}(\boldsymbol{\Delta})^{op}}(K, \overline{\mathcal{C}}^{\circledast}) \to \mathrm{Hom}_{(\mathrm{Set}_{\Delta})/\mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathrm{N}(\mathrm{Po})^{op} \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} K, \mathcal{C}^{\circledast}).$$

For each $n \geq 0$, the fiber $\overline{\mathcal{C}}^{\circledast}_{[n]}$ can be identified with $\mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathrm{N}(\mathrm{Po}_{[n]})^{op}, \mathcal{C}^{\circledast})$; we let $\widetilde{\mathcal{C}}^{\circledast}$ denote the full subcategory of $\overline{\mathcal{C}}^{\circledast}$ spanned by those objects which correspond to functors $F : \mathrm{N}(\mathrm{Po}_{[n]})^{op} \to \mathcal{C}^{\circledast}$ which are p-left Kan extensions of $F|\{([n], 0, n)\}$. Composition with the functor $s : \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathrm{N}(\mathrm{Po})^{op}$ given by $[n] \mapsto ([n], 0, n)$ induces a restriction functor $r : \overline{\mathcal{C}}^{\circledast} \to \mathcal{C}^{\circledast}$.

We claim that r induces a trivial Kan fibration $\widetilde{\mathcal{C}}^{\circledast} \to \mathcal{C}^{\circledast}$. To prove this, we associate to every map of simplicial sets $K \to \mathrm{N}(\boldsymbol{\Delta})^{op}$ the fiber product $\mathrm{N}(\mathrm{Po})^{op}_K = K \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} \mathrm{N}(\mathrm{Po})^{op}$. Let M_K be the collection of all morphisms in $\mathrm{N}(\mathrm{Po})^{op}_K$ whose image in K is degenerate. Then s determines a map of marked simplicial sets $u_K : K^{\flat} \to (\mathrm{N}(\mathrm{Po})^{op}_K, M_K)$. When K is a simplex, Proposition T.4.3.2.15 implies that u_K is a weak equivalence of marked simplicial sets. It follows by standard devissage argument that u_K is a weak equivalence of marked simplicial sets for every K; in particular, every map $\Delta^n \to \mathrm{N}(\boldsymbol{\Delta})^{op}$ induces a trivial cofibration
$$\Delta^{n,\flat} \coprod_{(\partial \Delta^n)^{\flat}} (\mathrm{N}(\mathrm{Po})^{op}_{\partial \Delta^n}, M_{\partial \Delta^n}) \hookrightarrow (\mathrm{N}(\mathrm{Po})^{op}_{\partial \Delta^n}, M_{\partial \Delta^n})$$
from which it follows immediately that the map $\widetilde{\mathcal{C}}^{\circledast} \to \mathcal{C}^{\circledast}$ has the right lifting property with respect to $\partial \Delta^n \hookrightarrow \Delta^n$.

The map θ is a pullback of the restriction functor $\overline{\theta} : \mathrm{Str}\,\mathcal{M}^{\mathrm{en},+} \to \mathrm{Str}\,\mathcal{M} \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} \mathcal{C}^{\circledast}$. It will therefore suffice to show that f is $\overline{\theta}$-Cartesian. Let $\phi : \mathrm{Str}\,\mathcal{M} \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} \mathcal{C}^{\circledast} \to \mathcal{C}^{\circledast}$, so that ϕ is a pullback of the map $\phi_0 : \mathrm{Str}\,\mathcal{M} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$. The image of f in $\mathrm{Str}\,\mathcal{M}$ is an equivalence and therefore ϕ_0-Cartesian. It follows that $\overline{\theta}(f)$ is ϕ-Cartesian. Consequently, to prove that f is $\overline{\theta}$-Cartesian, it will suffice to show that f is $(\phi \circ \overline{\theta})$-Cartesian (Proposition T.2.4.1.3). The map $\phi \circ \overline{\theta}$ factors as a composition
$$\mathrm{Str}\,\mathcal{M}^{\mathrm{en},+} \xrightarrow{\psi} \widetilde{\mathcal{C}}^{\circledast} \to \mathcal{C}^{\circledast},$$
where the second map is a trivial Kan fibration. It will therefore suffice to show that f is ψ-Cartesian.

Let $f : X \to Y$ be a morphism in $\mathrm{Str}\,\mathcal{M}^{\mathrm{en},+}_{[n]}$. Set $\mathcal{J} = \Delta^1 \times \mathrm{N}(\mathrm{Po}_{[n]})^{op}$ and let $\mathcal{J}_0 = \{1\} \times \mathrm{N}(\mathrm{Po}_{[n]})^{op} \subseteq \mathcal{J}$. We can identify X and Y with functors $\mathcal{J} \to \mathcal{M}^{\circledast}$ and f with a functor $F : \Delta^1 \times \mathcal{J} \to \mathcal{M}^{\circledast}$. According to Proposition B.4.9, it will suffice to show that F is a p-right Kan extension of $F_0 = F|\mathcal{I}$, where $\mathcal{I} = (\Delta^1 \times \mathcal{J}_0) \coprod_{\{1\} \times \mathcal{J}_0} (\{1\} \times \mathcal{J})$. Fix an object of E of $\Delta^1 \times \mathcal{J}$ not belonging to $\mathcal{I}$ having image $([n], i, j)$ in $\boldsymbol{\Delta}$; we will prove that F is a p-right Kan extension of F_0 at E. That is, we show that F exhibits $F(E)$ as a p-limit of the diagram $F_0|(\mathcal{I}_{X/})$. Unwinding the definitions, we must show that the diagram

$$
\begin{array}{ccc}
X(0, [n], i, j) & \longrightarrow & Y(0, [n], i, j) \\
\downarrow & & \downarrow \\
X(1, [n], i, j) & \longrightarrow & Y(1, [n], i, j)
\end{array}
$$

is a p-limit square. Since $Y \in \operatorname{Str} \mathcal{M}^{\mathrm{en},+}_{[n]}$, it exhibits $Y(0, [n], i, j)$ as a p-product of $Y(1, [n], i, j)$ with $Y(0, [n], j, j)$. It will therefore suffice to show that F exhibits $X(0, [n], i, j)$ as a p-product of $X(1, [n], i, j)$ with $Y(0, [n], j, j) \simeq M \simeq X(0, [n], j, j)$, which follows from our assumption that $X \in \operatorname{Str} \mathcal{M}^{\mathrm{en},+}_{[n]}$. $\qquad\square$

Remark 6.1.2.38. In the situation of Lemma 6.1.2.36, the functor $\theta_{[1]} : \mathcal{C}^+[M] \to \mathcal{C}$ is conservative. It follows that for each $n \geq 0$, the functor $\theta_{[n]} : \mathcal{C}^+[M]^{\circledast}_{[n]} \to \mathcal{C}^{\circledast}_{[n]}$ is conservative, so that θ is itself conservative. In particular, we note that a morphism α in $\mathcal{C}^+[M]^{\circledast}$ is inert if and only if $\theta(\alpha)$ is an inert morphism in $\mathcal{C}^{\circledast}$.

Proposition 6.1.2.39. *Let $p : \mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$ be a map which exhibits $\mathcal{M} = \mathcal{M}^{\circledast}_{0,[0]}$ as weakly enriched over the $\boldsymbol{\Delta}$-planar ∞-operad $\mathcal{C}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$. For each object $M \in \mathcal{M}$, the forgetful functor $\theta : \mathcal{C}^+[M]^{\circledast} \to \mathcal{C}^{\circledast}$ induces a right fibration $^{\Delta}\mathrm{Alg}(\mathcal{C}^+[M]) \to {}^{\Delta}\mathrm{Alg}(\mathcal{C})$.*

Proof. Combine Remark 6.1.2.38 with Lemmas 6.1.2.36 and 6.1.2.37. $\qquad\square$

Corollary 6.1.2.40. *Let $p : \mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$ be a coCartesian fibration which exhibits $\mathcal{M} = \mathcal{M}^{\circledast}_{0,[0]}$ as left-tensored over the $\boldsymbol{\Delta}$-monoidal ∞-category $\mathcal{C}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$. Assume that the ∞-category $\mathcal{C}[M]$ has a final object A. Then:*

(1) *The object A can be promoted to an object of $^{\Delta}\mathrm{Alg}(\mathcal{C}^+[M])$ in an essentially unique way. We will abuse notation by denoting this object also by A.*

Let $\theta : {}^{\Delta}\mathrm{Alg}(\mathcal{C}^+[M]) \to {}^{\Delta}\mathrm{Alg}(\mathcal{C})$ be the forgetful functor. Then:

(2) *There is a canonical equivalence of ∞-categories $^{\Delta}\mathrm{Alg}(\mathcal{C})_{/\theta(A)} \simeq {}^{\Delta}\mathrm{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\}$.*

Proof. Proposition 6.1.2.30 implies that the forgetful functor $\mathcal{C}[M]^{\circledast} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$ exhibits $\mathcal{C}[M]^{\circledast}$ as a $\boldsymbol{\Delta}$-monoidal ∞-category, so that $\mathcal{C}[M]^{\circledast} \simeq \mathcal{O}^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} \mathrm{N}(\boldsymbol{\Delta})^{op}$ for some coCartesian fibration of ∞-operads $\mathcal{O}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$. Combining Corollary 3.2.2.5 with Proposition 4.1.2.15, we conclude that A can be promoted to an object of $^{\Delta}\mathrm{Alg}(\mathcal{C}[M])$, which is automatically a final object of $^{\Delta}\mathrm{Alg}(\mathcal{C}[M])$ (and therefore unique up to equivalence). Since the map $^{\Delta}\mathrm{Alg}(\mathcal{C}^+[M]) \to {}^{\Delta}\mathrm{Alg}(\mathcal{C}[M])$ is a trivial Kan fibration; we may lift A to a final object of $^{\Delta}\mathrm{Alg}(\mathcal{C}^+[M])$, which we will also denote by A. This proves (1). We have a diagram of maps

$$^{\Delta}\mathrm{Alg}(\mathcal{C})_{/\theta(A)} \leftarrow {}^{\Delta}\mathrm{Alg}(\mathcal{C}^+[M])_{/A} \to {}^{\Delta}\mathrm{Alg}(\mathcal{C}^+[M]) \to {}^{\Delta}\mathrm{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\}$$

which are equivalences of ∞-categories by virtue of Proposition 6.1.2.39 and Theorem 6.1.2.34. $\qquad\square$

Corollary 6.1.2.41. *Let $p : \mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta})^{op}$ be a coCartesian fibration which exhibits $\mathcal{M} = \mathcal{M}^{\circledast}_{0,[0]}$ as left-tensored over the $\boldsymbol{\Delta}$-monoidal ∞-category $\mathcal{C}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$. Let $\overline{M} \in {}^{\Delta}\mathrm{LMod}(\mathcal{M})$ be a left module object having images $M \in \mathcal{M}$ and $A \in {}^{\Delta}\mathrm{Alg}(\mathcal{C})$. Suppose that the multiplication map $A \otimes M \to M$ exhibits A as a classifying object for endomorphisms of M. Then, for every algebra object $B \in {}^{\Delta}\mathrm{Alg}(\mathcal{C})$, we have a canonical isomorphism $\mathrm{Map}_{\Delta\mathrm{Alg}(\mathcal{C})}(B, A) \simeq {}^{\Delta}\mathrm{LMod}_B(\mathcal{M}) \times_{\mathcal{M}} \{M\}$ in the homotopy category $\mathcal{H}$ of spaces.*

Corollary 6.1.2.42. *Let $\mathcal{C}$ be a $\boldsymbol{\Delta}$-monoidal ∞-category, $\mathcal{M}$ an ∞-category which is left-tensored over $\mathcal{C}$, and $M \in \mathcal{M}$ an object. Then the forgetful functor*

$$^{\Delta}\mathrm{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\} \to {}^{\Delta}\mathrm{Alg}(\mathcal{C})$$

is a right fibration.

Proof. Combine Proposition 6.1.2.39 with Theorem 6.1.2.34. $\qquad\square$

6.1.3   Nonunital Associative Algebras and their Modules

Recall that a *nonunital ring* is an abelian group $(A, +)$ equipped with a bilinear and associative multiplication $m : A \times A \to A$. Every associative ring determines a nonunital ring, simply by forgetting the multiplicative identity element. On the other hand, if A is an associative ring, then the ring structure on A is uniquely determined by underlying nonunital ring of A. In other words, if A is a nonunital ring which admits a multiplicative identity 1, then 1 is uniquely determined. The proof is simple: if 1 and $1'$ are both identities for the multiplication on A, then $1 = 11' = 1'$. Our goal in this section is to prove an ∞-categorical version of this result. More precisely, we will show that if A is a nonunital algebra object of a monoidal ∞-category $\mathcal{C}$ which admits a (two-sided) unit up to homotopy, then A can be extended to an algebra object of $\mathcal{C}$ in an essentially unique way (Theorem 6.1.3.10). In ordinary category theory, this is a tautology. However, in the ∞-categorical setting the result is not as obvious, since the unit of an algebra object A of $\mathcal{C}$ is required to satisfy a hierarchy of coherence conditions with respect to the multiplication on A.

We begin by recalling some basic definitions. Let $\mathrm{Comm}_{\mathrm{nu}}^{\otimes}$ denote the nonunital commutative ∞-operad: that is, the subcategory of $N(\mathcal{F}\mathrm{in}_*)$ whose morphisms are given by *surjective* maps $\langle m \rangle \to \langle n \rangle$ of pointed finite sets (see Definition 5.2.3.1). We let $\mathcal{A}\mathrm{ss}_{\mathrm{nu}}^{\otimes}$ denote the fiber product $\mathrm{Comm}_{\mathrm{nu}}^{\otimes} \times_{\mathrm{Comm}^{\otimes}} \mathcal{A}\mathrm{ss}^{\otimes}$; we will refer to $\mathcal{A}\mathrm{ss}_{\mathrm{nu}}^{\otimes}$ as the *nonunital associative ∞-operad*. Given a planar ∞-operad $\mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$, we let $\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$ denote the ∞-category $\mathrm{Alg}_{\mathcal{A}\mathrm{ss}_{\mathrm{nu}} / \mathcal{A}\mathrm{ss}}(\mathcal{C})$; we will refer to $\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$ as the *∞-category of nonunital associative algebra objects of* $\mathcal{C}$. For many purposes, it is convenient to have a slightly different model for the theory of nonunital associative algebras.

Definition 6.1.3.1. We will say that a morphism $[m] \to [n]$ in $\mathbf{\Delta}_s$ is *inert* if it determines an inert morphism in the ∞-operad $\mathcal{A}\mathrm{ss}_{\mathrm{nu}}^{\otimes}$: that is, if it induces a bijection $[m] \simeq \{i, i+1, \ldots, i+m\} \subseteq [n]$. If $\mathcal{C}^{\circledast} \to N(\mathbf{\Delta})^{op}$ is a $\mathbf{\Delta}$-planar ∞-operad, we let $^{\mathbf{\Delta}}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}_{N(\mathbf{\Delta})^{op}}(N(\mathbf{\Delta}_s)^{op}, \mathcal{C}^{\circledast})$ spanned by those functors which carry inert morphisms in $N(\mathbf{\Delta}_s)^{op}$ to inert morphisms in $\mathcal{C}^{\circledast}$.

Proposition 6.1.3.2. *Let $\mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ be a planar ∞-operad and let $\mathcal{C}^{\circledast} = \mathcal{C}^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} N(\mathbf{\Delta})^{op}$ be the associated $\mathbf{\Delta}$-planar ∞-operad. Then composition with the map $N(\mathbf{\Delta}_s)^{op} \to \mathcal{A}\mathrm{ss}_{\mathrm{nu}}^{\otimes}$ of Proposition 6.1.3.3 induces an equivalence of ∞-categories $\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}) \to {}^{\mathbf{\Delta}}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$.*

Proof. Combine Proposition 6.1.3.3 with Theorem 2.3.3.23. $\square$

Proposition 6.1.3.3. *Let $\mathbf{\Delta}_s$ denote the subcategory of $\mathbf{\Delta}$ whose morphisms are given by injective maps $[m] \to [n]$ of linearly ordered finite sets. The functor $\mathrm{Cut} : N(\mathbf{\Delta})^{op} \to \mathcal{A}\mathrm{ss}^{\otimes}$ of Construction 4.1.2.5 determines an approximation $N(\mathbf{\Delta}_s)^{op} \to \mathcal{A}\mathrm{ss}_{\mathrm{nu}}^{\otimes}$ to the ∞-operad $\mathcal{A}\mathrm{ss}_{\mathrm{nu}}^{\otimes}$.*

Proof. Combine Proposition 4.1.2.10 with Remark 2.3.3.9. $\square$

Remark 6.1.3.4. The definition of a nonunital associative algebra makes sense in the more general setting of *nonunital* planar ∞-operads. We will have no need for this additional generality.

Definition 6.1.3.5. Let $\mathcal{C}^{\circledast} \to N(\mathbf{\Delta})^{op}$ be a $\mathbf{\Delta}$-monoidal ∞-category, and let $A \in {}^{\mathbf{\Delta}}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$ be a nonunital algebra object of $\mathcal{C}$. Let $\mathbf{1}$ denote the unit object of $\mathcal{C}$. A map $u : \mathbf{1} \to A$ is a *right unit* if the composition

$$A \simeq A \otimes \mathbf{1} \overset{u}{\to} A \otimes A \to A$$

is homotopic to the identity in $\mathcal{C}$. Similarly, we will say that u is a *left unit* if the composition

$$A \simeq \mathbf{1} \otimes A \overset{u}{\to} A \otimes A \to A$$

is homotopic to the identity in $\mathcal{C}$. We will say that u is a *quasi-unit* if it is both a left unit and a right unit. We will say that A is *quasi-unital* if there exists a quasi-unit $u : \mathbf{1} \to A$.

Remark 6.1.3.6. Let $\mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ be a monoidal ∞-category, and let $A \in \mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$. Then A is quasi-unital in the sense of Definition 5.2.3.2 if and only if the image of A in $^{\mathbf{\Delta}}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$ is quasi-unital in the sense of Definition 6.1.3.5.

Remark 6.1.3.7. Let A be as in Definition 6.1.3.5, and suppose that A admits a left unit $u : \mathbf{1} \to A$ and a right unit $v : \mathbf{1} \to A$. Then the composite map

$$\mathbf{1} \simeq \mathbf{1} \simeq \mathbf{1} \overset{u \otimes v}{\to} A \otimes A \to A$$

is homotopic to both u and v, so that u and v are homotopic to each other. It follows that A is quasi-unital if and only if it admits both a left and a right unit; in this case, the quasi-unit of A is determined uniquely up to homotopy.

Definition 6.1.3.8. Let $\mathcal{C}^{\otimes} \to \mathrm{N}(\mathbf{\Delta})^{op}$ be a Δ-monoidal ∞-category, and let $A \in {}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$ be a nonunital algebra object of $\mathcal{C}$, and let $u : \mathbf{1} \to A$ be a quasi-unit of A. We will say that a morphism $f : A \to B$ in ${}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$ is *quasi-unital* if $f \circ u$ is a quasi-unit for B (in particular, this implies that B is quasi-unital). We let ${}^{\Delta}\mathrm{Alg}^{\mathrm{qu}}(\mathcal{C})$ denote the subcategory of ${}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$ spanned by the quasi-unital objects of ${}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$ and quasi-unital morphisms between them. We will refer to ${}^{\Delta}\mathrm{Alg}^{\mathrm{qu}}(\mathcal{C})$ as the ∞-*category of quasi-unital algebra morphisms in* $\mathcal{C}$.

Remark 6.1.3.9. Let $\mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ be a monoidal ∞-category. Then the equivalence of ∞-categories $\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}) \to {}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$ restricts to an equivalence of ∞-categories $\mathrm{Alg}^{\mathrm{qu}}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C}) \to {}^{\Delta}\mathrm{Alg}^{\mathrm{qu}}(\mathcal{C})$. Here $\mathrm{Alg}^{\mathrm{qu}}_{/\mathcal{A}\mathrm{ss}}(\mathcal{C})$ is the ∞-category introduced in Definition 5.2.3.4.

In view of Remark 6.1.3.9, Theorem 5.2.3.5 (which was asserted without proof in §5.2.3) can be reformulated as follows:

Theorem 6.1.3.10. *Let* $\mathcal{C}^{\otimes} \to \mathrm{N}(\mathbf{\Delta})^{op}$ *be a* Δ-*monoidal* ∞-*category. Then the restriction functor*

$$^{\Delta}\mathrm{Alg}(\mathcal{C}) \to {}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$$

induces a trivial Kan fibration ${}^{\Delta}\mathrm{Alg}(\mathcal{C}) \to {}^{\Delta}\mathrm{Alg}^{\mathrm{qu}}(\mathcal{C})$.

We will prove Theorem 6.1.3.10 at the end of this section. The basic idea is as follows: if A is a quasi-unital associative algebra, then A can be identified with the algebra of (left) A-module endomorphisms of itself. To make this idea precise, we will need a good theory of nonunital modules over nonunital algebras.

Recall that if A is a nonunital ring, then a *nonunital left A-module* is an abelian group M equipped with a bilinear multiplication map $A \times M \to M$ which satisfies the associativity formula $a(bm) = (ab)m$. Note that if A admits a unit, then this condition does not imply that M is an A-module, because it does not imply that the unit element of A acts by the identity on M. For example, there is a trivial nonunital A-module structure on any abelian group M, given by the zero map $A \times M \overset{0}{\to} M$.

We now adapt the theory of nonunital left modules to the ∞-categorical context. Let $\mathcal{LM}^{\otimes}_{\mathrm{nu}}$ denote the ∞-operad $\mathcal{LM}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{Comm}^{\otimes}_{\mathrm{nu}}$. Given a fibration of ∞-operads $\mathcal{O}^{\otimes} \to \mathcal{LM}^{\otimes}$, corresponding to an ∞-category $\mathcal{M} = \mathcal{O}_{\mathfrak{m}}$ weakly enriched over a planar ∞-operad $\mathcal{C}^{\otimes} = \mathcal{O}^{\otimes}_{\mathfrak{a}}$, we let $\mathrm{LMod}^{\mathrm{nu}}(\mathcal{M})$ denote the the ∞-category $\mathrm{Alg}_{\mathcal{LM}_{\mathrm{nu}} / \mathcal{LM}}(\mathcal{O})$. We will refer to $\mathrm{LMod}^{\mathrm{nu}}(\mathcal{M})$ as the ∞-*category of nonunital left module objects of* $\mathcal{M}$. We have the following analogue of Proposition 6.1.3.3, which follows immediately from Remarks 4.2.2.8 and 2.3.3.9:

Proposition 6.1.3.11. *The functor* $\gamma : \Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{LM}^{\otimes}$ *of Remark 4.2.2.8 determines an approximation* $\Delta^1 \times \mathrm{N}(\mathbf{\Delta}_s)^{op} \to \mathcal{LM}^{\otimes}_{\mathrm{nu}}$ *to the* ∞-*operad* $\mathcal{LM}^{\otimes}_{\mathrm{nu}}$.

Definition 6.1.3.12. We will say that a morphism in $\Delta^1 \times \mathrm{N}(\mathbf{\Delta}_s)^{op}$ is inert if its image in $\mathcal{LM}^{\otimes}_{\mathrm{nu}}$ is inert. If $\mathcal{M}^{\otimes} \to \Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}$ is a map which exhibits $\mathcal{M} = \mathcal{M}^{\otimes}_{(0,[0])}$ as weakly enriched over the Δ-planar ∞-operad $\mathcal{M}^{\otimes} \times_{\Delta^1} \{1\}$, then we let ${}^{\Delta}\mathrm{LMod}^{\mathrm{nu}}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}}(\Delta^1 \times \mathrm{N}(\mathbf{\Delta}_s)^{op}, \mathcal{M}^{\otimes})$ spanned by those functors which carry inert morphisms in $\Delta^1 \, \mathrm{N}(\mathbf{\Delta}_s)^{op}$ to inert morphisms in $\mathcal{C}^{\otimes}$.

Combining Proposition 6.1.3.11 with Theorem 2.3.3.23, we obtain:

Proposition 6.1.3.13. *Let $\mathcal{O}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a fibration of ∞-operads which exhibits $\mathcal{M} = \mathcal{O}_{\mathfrak{m}}$ as weakly enriched over the Δ-planar ∞-operad $\mathcal{C}^{\otimes} = \mathcal{O}_{\mathfrak{a}}^{\otimes}$, and let $\mathcal{M}^{\circledast} = \mathcal{O}^{\otimes} \times_{\mathcal{LM}^{\otimes}} (\Delta^1 \times \mathrm{N}(\Delta)^{op})$. Then composition with the map $\Delta^1 \times \mathrm{N}(\Delta_s)^{op} \to \mathcal{LM}_{\mathrm{nu}}^{\otimes}$ induces an equivalence of ∞-categories $\mathrm{LMod}^{\mathrm{nu}}(\mathcal{M}) \to {}^{\Delta}\mathrm{LMod}^{\mathrm{nu}}(\mathcal{M})$.*

Remark 6.1.3.14. In the situation of Definition 6.1.3.12, there are evident forgetful functors

$$\mathcal{M} \leftarrow {}^{\Delta}\mathrm{LMod}^{\mathrm{nu}}(\mathcal{M}) \to {}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}).$$

We will generally abuse notation by identifying an object of ${}^{\Delta}\mathrm{LMod}^{\mathrm{nu}}(\mathcal{M})$ with its image in $\mathcal{M}$. If A is a nonunital algebra object of $\mathcal{C}$, we let ${}^{\Delta}\mathrm{LMod}_A^{\mathrm{nu}}(\mathcal{C})$ denote the fiber product ${}^{\Delta}\mathrm{LMod}^{\mathrm{nu}}(\mathcal{M}) \times_{{}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})} \{A\}$. If $A \in {}^{\Delta}\mathrm{Alg}(\mathcal{C})$, we will generally abuse notation by writing ${}^{\Delta}\mathrm{LMod}_A^{\mathrm{nu}}(\mathcal{C})$ for ${}^{\Delta}\mathrm{LMod}_{\theta(A)}^{\mathrm{nu}}(\mathcal{C})$, where $\theta(A)$ denotes the image of A under the forgetful functor $\theta : {}^{\Delta}\mathrm{Alg}(\mathcal{C}) \to {}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$.

Definition 6.1.3.15. Let $\mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\Delta)^{op}$ be a coCartesian fibration which exhibits $\mathcal{M} = \mathcal{M}_{(0,[0])}^{\circledast}$ as left-tensored over the Δ-monoidal ∞-category $\mathcal{C}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$, and let $M \in {}^{\Delta}\mathrm{LMod}^{\mathrm{nu}}(\mathcal{M})$ be a nonunital module. We will say that M is *quasi-unital* if the following conditions are satisfied:

(1) The image of M in ${}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$ is a quasi-unital algebra object $A \in {}^{\Delta}\mathrm{Alg}^{\mathrm{qu}}(\mathcal{C})$.

(2) If $u : \mathbf{1} \to A$ is a quasi-unit for A, then the composite map

$$\psi : M \simeq \mathbf{1} \otimes M \to A \otimes M \to M$$

is homotopic to the identity (as a morphism in the ∞-category $\mathcal{M}$).

Remark 6.1.3.16. In view of Remark 6.1.3.7, the condition of Definition 6.1.3.15 does not depend on the choice of a quasi-unit $u : 1_{\mathcal{C}} \to A$.

Remark 6.1.3.17. In the situation of Definition 6.1.3.15, the condition that ψ be homotopic to the identity is equivalent to the (apparently weaker) condition that ψ be an equivalence. For suppose that ψ is an equivalence. Since the composition

$$\mathbf{1} \simeq \mathbf{1} \otimes \mathbf{1} \xrightarrow{u \otimes u} A \otimes A \longrightarrow A$$

is homotopic to u, we conclude that ψ^2 is homotopic to ψ (that is, ψ^2 and ψ belong to the same connected component of $\mathrm{Map}_{\mathcal{M}}(M, M)$). If ψ is invertible in the homotopy category $\mathrm{h}\mathcal{M}$, this forces ψ to be homotopic to the identity.

We have the following counterpart of Theorem 6.1.3.10:

Proposition 6.1.3.18. *Let $p : \mathcal{M}^{\circledast} \to \Delta^1 \times \mathrm{N}(\Delta)^{op}$ be a coCartesian fibration which exhibits $\mathcal{M} = \mathcal{M}_{(0,[0])}^{\circledast}$ as left-tensored over the Δ-monoidal ∞-category $\mathcal{C}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$. Fix an object $A \in {}^{\Delta}\mathrm{Alg}(\mathcal{C})$. Then the canonical map*

$$\theta : {}^{\Delta}\mathrm{LMod}_A(\mathcal{M}) \to {}^{\Delta}\mathrm{LMod}_A^{\mathrm{qu}}(\mathcal{M})$$

is a trivial Kan fibration.

Proof. It is clear that θ is a categorical fibration. It will therefore suffice to show that θ is a categorical equivalence. We may assume without loss of generality that $\mathcal{M}^{\circledast} = (\Delta^1 \times \mathrm{N}(\Delta)^{op}) \times_{\mathcal{LM}^{\otimes}} \mathcal{M}^{\otimes}$ for some coCartesian fibration of ∞-operads $\mathcal{M}^{\otimes} \to \mathcal{LM}^{\otimes}$. Let $\mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast}$ be defined as in Notation 4.2.2.16, and let $\mathcal{N} = \mathrm{N}(\Delta)^{op} \times_{\mathcal{C}^{\circledast}} \mathcal{M}^{\circledast}$ where $\mathrm{N}(\Delta)^{op}$ maps to $\mathcal{C}^{\circledast}$ via the algebra object A. Let $q : \mathcal{N} \to \mathrm{N}(\Delta)^{op}$ be the canonical map, so that q is a locally coCartesian fibration (Lemma 4.2.2.19).

We define a subcategory $\mathcal{I} \subseteq [1] \times \Delta^{op}$ as follows:

• Every object of $[1] \times \Delta^{op}$ belongs to $\mathcal{I}$.

• A morphism $\alpha : (i, [m]) \to (j, [n])$ in $[1] \times \Delta^{op}$ belongs to $\mathcal{I}$ if and only if either $i = 0$ or the map $[n] \to [m]$ is injective.

For $i = 0, 1$, we let $\mathcal{I}_i$ denote the full subcategory of $\mathcal{I}$ spanned by the objects $\{(i, [n])\}_{n \geq 0}$, so that $\mathcal{I}_0 \simeq \boldsymbol{\Delta}^{op}$ and $\mathcal{I}_1 \simeq \boldsymbol{\Delta}_s^{op}$. There is an evident forgetful functor $\mathcal{I} \to \boldsymbol{\Delta}^{op}$. We observe that $^\triangle \mathrm{LMod}_A(\mathcal{M})$ and $^\triangle \mathrm{LMod}_A^{\mathrm{nu}}(\mathcal{M})$ can be identified with full subcategories of $\mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathrm{N}(\mathcal{I}_0), \mathcal{N})$ and $\mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathrm{N}(\mathcal{I}_0\, 1), \mathcal{N})$, respectively.

Let $\mathfrak{X}$ denote the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathrm{N}(\mathcal{I}), \mathcal{N})$ spanned by those functors F with the following properties:

(i) The restriction $F_0 = F | \mathrm{N}(\mathcal{I}_0)$ belongs to $^\triangle \mathrm{LMod}_A(\mathcal{M})$.

(ii) The functor F is a q-left Kan extension of F_-.

Note that conditions (i) and (ii) immediately imply:

(iii) The restriction $F_1 = F | \mathrm{N}(\mathcal{I}_1)$ belongs to $^\triangle \mathrm{LMod}_A^{\mathrm{qu}}(\mathcal{M})$.

Conversely, conditions (ii) and (iii) imply (i) (since every inert morphism in $\mathrm{N}(\boldsymbol{\Delta})^{op}$ belongs to $\mathrm{N}(\boldsymbol{\Delta}_s)^{op}$).

The map θ factors as a composition

$$^\triangle \mathrm{LMod}_A(\mathcal{M}) \xrightarrow{\theta'} \mathfrak{X} \xrightarrow{\theta''} {}^\triangle \mathrm{LMod}_A^{\mathrm{qu}}(\mathcal{M}).$$

Proposition T.4.3.2.15 implies that θ' is the section of a trivial Kan fibration $\mathfrak{X} \to {}^\triangle \mathrm{LMod}_A(\mathcal{M})$, and therefore a categorical equivalence. We will complete the proof by showing that θ'' is a trivial Kan fibration. According to Proposition T.4.3.2.15, it will suffice to prove:

(a) For every $F_1 \in {}^\triangle \mathrm{LMod}_A^{\mathrm{qu}}(\mathcal{M}) \subseteq \mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathrm{N}(\mathcal{I}_1), \mathcal{N})$, there exists a functor $F \in \mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathrm{N}(\mathcal{I}), \mathcal{N})$ which is a p-right Kan extension of F_1.

(b) If $F \in \mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathrm{N}(\mathcal{I}), \mathcal{N})$ is a functor such that $F_1 = F | \mathrm{N}(\mathcal{I}_1)$ belongs to $^\triangle \mathrm{LMod}_A^{\mathrm{qu}}(\mathcal{M})$, then $F \in \mathfrak{X}$ if and only if F is a p-right Kan extension of F^0.

We begin by proving (a). Fix a functor $F_1 \in {}^\triangle \mathrm{LMod}_A^{\mathrm{qu}}(\mathcal{M}) \subseteq \mathrm{Fun}_{\mathrm{N}(\boldsymbol{\Delta})^{op}}(\mathrm{N}(\mathcal{I}_1), \mathcal{N})$ and an object $E = (0, [n]) \in \mathcal{I}_0$. Let $\mathcal{J} = \mathcal{I}_1 \times_{\mathcal{I}} \mathcal{I}_{E/}$. According to Lemma T.4.3.2.13, it will suffice to show that $f = F_1 | \mathrm{N}(\mathcal{J})$ can be extended to a q-limit diagram $\mathrm{N}(\mathcal{J})^{\triangleleft} \to \mathcal{N}$ (compatible with the evident map $\mathrm{N}(\mathcal{J})^{\triangleleft} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$).

Let $\mathcal{J}_0$ denote the full subcategory of $\mathcal{J}$ spanned by those maps $(0, [n]) \to (1, [m])$ for which the image of the underlying map $[m] \to [n]$ contains n. We claim that the inclusion $\mathrm{N}(\mathcal{J}_0) \subseteq \mathrm{N}(\mathcal{J})$ is right cofinal. In view of Theorem T.4.1.3.1, it will suffice to show that, for every morphism $E \to E' = (1, [m])$ in $\mathcal{I}$, the category $\mathcal{Z} = \mathcal{J}_0 \times_{\mathcal{J}_{E/}} \mathcal{J}_{E//E'}$ has a weakly contractible nerve. Let $\gamma : [m] \to [n]$ be the underlying map of linearly ordered sets. If $\gamma(m) = n$ then $\mathcal{Z}$ has a final object and there is nothing to prove. Assume therefore that $\gamma(m) < n$. Unwinding the definitions, we can identify $\mathcal{Z}$ with a product of categories $\{\mathcal{E}_i^{op}\}_{0 \leq i \leq n}$, where

$$\mathcal{E}_i \simeq \begin{cases} (\boldsymbol{\Delta}_{s,+})_{\gamma^{-1}(i)/} & \text{if } i < n \\ \boldsymbol{\Delta}_s & \text{if } i = n. \end{cases}$$

The categories $\mathcal{E}_i$ have initial objects for $i < n$, and $\mathcal{E}_n$ has weakly contractible nerve (because the inclusion $\mathrm{N}(\boldsymbol{\Delta}_s) \subseteq \mathrm{N}(\boldsymbol{\Delta})$ is right cofinal (Lemma T.5.5.8.4), right cofinal maps are weak homotopy equivalences (Proposition T.4.1.1.3), and $\mathrm{N}(\boldsymbol{\Delta})$ is weakly contractible (Lemma T.5.5.8.4 and Proposition T.5.5.8.7)). It follows that $\mathrm{N}(\mathcal{Z}) \simeq \prod_{0 \leq i \leq n} \mathrm{N}(\mathcal{E}_i)^{op}$ is likewise weakly contractible. We are therefore reduced to proving:

(a') There exists a q-limit diagram $\overline{g} : \mathrm{N}(\mathcal{J}_0)^{\triangleleft} \to \mathcal{N}$ rendering the following diagram commutative:

$$
\begin{array}{ccc}
\mathrm{N}(\mathcal{J}_0) & \xrightarrow{\ g\ } & \mathcal{N} \\
\big\downarrow & \overline{g}\ \nearrow & \big\downarrow p \\
\mathrm{N}(\mathcal{J}_0)^{\triangleleft} & \longrightarrow & \mathrm{N}(\boldsymbol{\Delta})^{op},
\end{array}
$$

where g is given by the restriction of F_1.

We now observe that, for every morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$ for which $\alpha(m) = n$ classifying a map $\Delta^1 \to N(\mathbf{\Delta})^{op}$, the pullback $\mathcal{N} \times_{N(\mathbf{\Delta})^{op}} \Delta^1$ is equivalent to a product $\mathcal{M} \times \Delta^1$. It follows that for every object $N \in \mathcal{N}_{[m]}$, there exists a locally q-Cartesian morphism $\overline{\alpha} : N' \to N$ in $\mathcal{N}$ covering α. Remark 4.2.2.22 implies that $\overline{\alpha}$ is q-Cartesian.

Let $h_1 : N(\mathcal{J}_0)^{\triangleleft} \to N(\mathbf{\Delta})^{op}$ denote the composition $N(\mathcal{J}_0)^{\triangleleft} \to N(\mathcal{J}) \to N(\mathbf{\Delta})^{op}$, so that we have a natural transformation $h : \Delta^1 \times N(\mathcal{J}_0)^{\triangleleft} \to N(\mathbf{\Delta})^{op}$ from $h_0 = h|\{0\} \times N(\mathcal{J}_0)^{\triangleleft}$ to h_1, where h_0 is the constant functor taking the value $[n]$. For each object $x \in N(\mathcal{J}_0)^{\triangleleft}$, the restriction of h to $\Delta^1 \times \{x\}$ classifies a morphism $\alpha : [m] \to [n]$ satisfying $\alpha(m) = n$. It follows that we can lift $h|(\Delta^1 \times N(\mathcal{J}_0))$ to a p-Cartesian transformation $\widetilde{h} : g' \to g$. Using Proposition T.4.3.1.9, we are reduced to proving that the diagram $g' : N(\mathcal{J}_0) \to \mathcal{N}_{[n]}$ can be extended to a q-limit diagram in $\mathcal{N}_{[n]}$. In what follows, we will abuse notations by identifying $\mathcal{N}_{[n]}$ with $\mathcal{M}$, so that g' can be regarded as a diagram in $\mathcal{M}$. Let $M = f_1([0]) \in \mathcal{M}$. Unwinding the definitions, we see that the values assumed by g' can be identified with M, and the morphisms between these values are given by iterated multiplication by the unit $1_{\mathcal{C}} \to A$. Since M is assumed to be quasi-unital, it follows that g' carries every morphism in $\mathcal{J}_0$ to an equivalence in $\mathcal{N}_{[n]}$. The simplicial set $N(\mathcal{J}_0)$ is weakly contractible, since it is isomorphic to the product $(\prod_{0 \leq i < n} N(\mathbf{\Delta}_{s,+})^{op}) \times N(\mathbf{\Delta}_s)^{op}$. Applying Corollary T.4.4.4.10, we deduce that g' admits a colimit $\overline{g}' : N(\mathcal{J}_0)^{\triangleright} \to \mathcal{N}_{[n]}$, and that $\overline{g}'$ carries $\{v\}^{\triangleright}$ to an equivalence in $\mathcal{N}_{[n]}$. Since the ∞-category $\mathcal{M}_{[n]}^{\circledast}$ is equivalent to the product $\mathcal{C}_{[n]}^{\circledast} \times \mathcal{M}$, we conclude that $\overline{g}'$ is a $q'_{[n]}$-limit diagram, where $q'_{[n]} : \mathcal{M}_{[n]}^{\circledast} \to \mathcal{C}_{[n]}^{\circledast}$ denotes the projection. Applying Corollary T.4.3.1.15 to the diagram

$$\mathcal{M}^{\circledast} \xrightarrow{\;\;\;q'\;\;\;} \mathcal{C}^{\circledast}$$
$$N(\mathbf{\Delta})^{op},$$

we conclude that $\overline{g}'$ is a q'-limit diagram. Since q is a pullback of q', we deduce that $\overline{g}'$ is a q-limit diagram, as desired. This completes the proof of (a). Moreover, the proof shows that an arbitrary extension F of F_1 is a q-right Kan extension at $(0, [n])$ if and only if, for some $\gamma : [m] \to [n]$ in $\mathbf{\Delta}$ with $\gamma(m) = n$, the map $F(0, [n]) \to F(1, [m])$ is locally q-Cartesian; moreover, this condition is independent of the choice of γ. In particular, we can take $\gamma = \mathrm{id}_{[n]}$ to conclude that F is a q-right Kan extension of F_1 if and only if F satisfies condition (ii), which proves (b). $\qquad\square$

We now return to Theorem 6.1.3.10. Before giving the proof, let us sketch the main idea. Suppose that A is a nonunital ring, and we wish to promote A to an associative ring. Let $M = A$, regarded as a (nonunital) right module over itself. Left multiplication induces a homomorphism of nonunital algebras $\phi : A \to \mathrm{Hom}_A(M, M)$. If A admits a left unit 1, then A is freely generated by 1 as a right A-module, so that evaluation at 1 induces an isomorphism $\mathrm{Hom}_A(M, M) \simeq M$. Under this isomorphism, ϕ corresponds to the map $a \mapsto a1$. If the element $1 \in A$ is also a right unit, then ϕ is an isomorphism. On the other hand, $\mathrm{End}_A(M, M)$ is manifestly an associative ring. To translate this sketch into the setting of higher category theory, we will need the following lemma, which will be proven at the end of this section:

Lemma 6.1.3.19. *Let $\mathcal{C}$ be a $\mathbf{\Delta}$-monoidal ∞-category, and let $A \in {}^{\mathbf{\Delta}}\mathrm{Alg}^{\mathrm{qu}}(\mathcal{C})$. There exists an ∞-category $\mathcal{M}$ which is left-tensored over $\mathcal{C}$ and an object $M \in {}^{\mathbf{\Delta}}\mathrm{LMod}_A^{\mathrm{qu}}(\mathcal{M})$ for which the underlying map $A \otimes M \to M$ exhibits A as a classifying object for morphisms from M to M.*

We will also need some nonunital analogues of the results of §6.1.2.

Lemma 6.1.3.20. *Let $p : \mathcal{M}^{\circledast} \to \Delta^1 \times N(\mathbf{\Delta})^{op}$ be a map which exhibits $\mathcal{M} = \mathcal{M}_{0,[0]}^{\circledast}$ as weakly enriched over the $\mathbf{\Delta}$-planar ∞-operad $\mathcal{C}^{\circledast} = \mathcal{M}^{\circledast} \times_{\Delta^1} \{1\}$. Let $s : N(\mathbf{\Delta})^{op} \to N(\mathrm{Po})^{op}$ be the functor given on objects by $[n] \mapsto ([n], 0, n)$. Then, for each object $M \in \mathcal{M}$, composition with s induces a categorical equivalence*

$$\theta : {}^{\mathbf{\Delta}}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}^+[M]) \to {}^{\mathbf{\Delta}}\mathrm{LMod}^{\mathrm{nu}}(\mathcal{M}) \times_{\mathcal{M}} \{M\}.$$

Proof. The proof is essentially identical to that of Theorem 6.1.2.34. Let $\mathrm{Po}_s = \mathrm{Po} \times_{\mathbf{\Delta}} \mathbf{\Delta}_s$, and let $\mathfrak{X} \subseteq$ $\mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}}(\Delta^1 \times \mathrm{N}(\mathrm{Po}_s)^{op}, \mathcal{M}^{\circledast})$ be the full subcategory spanned by those functors F satisfying the following conditions:

(a) If $0 \leq i \leq i' \leq j \leq n$, then the map $F(0, [n], i, j) \to F(0, [n], i', j)$ is inert.

(b) If $0 \leq i \leq i' \leq j' \leq n$, then the map $F(1, [n], i, j) \to F(1, [n], i', j')$ is inert.

(c) For $0 \leq i \leq j \leq n$, the map $F(0, [n], i, j) \to F(1, [n], i, j)$ is inert.

(d) If $\alpha : [m] \to [n]$ is an inert morphism in $\mathbf{\Delta}$, then $F(0, [n], \alpha(i), \alpha(j)) \to F(0, [m], i, j)$ is an equivalence in $\mathcal{M}^{\circledast}$.

Note that conditions (a) and (d) imply the following:

(a') For every inert morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}_s$ satisfying $\alpha(m) = n$, the induced map $F(0, [n], 0, n) \to F(0, [m], 0, m)$ is inert.

Similarly, (b) and (d) imply:

(b') For every inert morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}_s$, the induced map $F(1, [n], 0, n) \to F(1, [m], 0, m)$ is inert.

Finally, (c) and (d) imply:

(d') If $\alpha : [m] \to [n]$ is an inert morphism in $\mathbf{\Delta}_s$, then $F(1, [n], \alpha(i), \alpha(j)) \to F(1, [m], i, j)$ is an equivalence in $\mathcal{M}^{\circledast}$.

It follows that composition with s induces a forgetful functor $\theta' : \mathfrak{X} \to {}^{\Delta}\mathrm{LMod}^{\mathrm{nu}}(\mathcal{M})$. Let $\mathrm{Po}'_s = \mathrm{Po}' \cap \mathrm{Po}_s$, and let $\mathfrak{X}_0 \subseteq \mathrm{Fun}(\mathrm{N}(\mathrm{Po}'_s)^{op}, \mathcal{M})$ be the full subcategory spanned by those functors which carry each morphism in $\mathrm{N}(\mathrm{Po}'_s)^{op}$ to an equivalence in $\mathcal{M}$. Let $\overline{M} \in \mathfrak{X}_0$ be the constant functor taking the value $M \in \mathcal{M}$. Unwinding the definitions, we have a canonical isomorphism ${}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}^+[M]) \simeq \mathfrak{X} \times_{\mathfrak{X}_0} \overline{M}$. In other words, we can identify θ' with the map between vertical fibers determined by the diagram

$$
\begin{array}{ccc}
\mathfrak{X} & \xrightarrow{\phi} & {}^{\Delta}\mathrm{LMod}^{\mathrm{nu}}(\mathcal{M}) \\
\downarrow & & \downarrow \\
\mathfrak{X}_0 & \xrightarrow{\phi_0} & \mathcal{M} \, .
\end{array}
$$

It will therefore suffice to show that the functors ϕ and ϕ_0 are categorical equivalences. The map ϕ_0 is an equivalence since $\mathrm{N}(\mathrm{Po}'_s)^{op}$ is weakly contractible (it has a final object, given by $([0], 0, 0)$). It will therefore suffice to show that ϕ is a categorical equivalence. Let $\mathfrak{X}'$ be the full subcategory of $\mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}}(\{0\} \times \mathrm{N}(\mathrm{Po}_s)^{op}, \mathcal{M}^{\circledast})$ spanned by those functors F which satisfy (a'), (b'), (d), (d'), and the following weaker version of (c):

(c') For $n \geq 0$, the map $F(0, [n], 0, n) \to F(1, [n], 0, n)$ is inert.

A functor $F \in \mathrm{Fun}_{\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}}(\{0\} \times \mathrm{N}(\mathrm{Po}_s)^{op}, \mathcal{M}^{\circledast})$ belongs to $\mathfrak{X}'$ if and only if $F_0 = F|(\Delta^1 \times \mathrm{N}(\mathbf{\Delta}_s)^{op})$ determines an object of ${}^{\Delta}\mathrm{LMod}^{\mathrm{nu}}(\mathcal{M})$ and F is a p-right Kan extension of F_0. Then $\mathfrak{X}'$ is a full subcategory of $\mathfrak{X}$ and Proposition T.4.3.2.15 guarantees that the restriction map $\phi| \mathfrak{X}'$ is a trivial Kan fibration. We will complete the proof by showing that $\mathfrak{X} = \mathfrak{X}'$. In other words, we will show that a functor $F \in \mathfrak{X}'$ also satisfies conditions (a), (b), and (c). To prove (a), consider $0 \leq i \leq i' \leq j \leq n$. Condition (d) guarantees that the induced map $F(0, [n], i, j) \to F(0, [n], i', j)$ is equivalent to the map $F([j-i], 0, j-i) \to F([j-i'], 0, j-i')$ and is therefore inert by (a'). To prove (b), we assume that $0 \leq i \leq i' \leq j' \leq j \leq n$ and note that (d') implies that $F(0, [n], i, j) \to F(0, [n], i', j')$ is equivalent to the map $LF([j-i], 0, j-i) \to LF([j'-i'], 0, j'-i')$

which is inert by (b'). It remains to verify (c). Fix $0 \leq i \leq j \leq n$; we wish to show that the map $u : F(0, [n], i, j) \to F(1, [n], i, j)$ is inert. Using (b), we are reduced to proving that the composite map $F(0, [n], i, j) \to F(1, [n], i, j) \simeq F(1, [j-i], 0, j-i)$ is inert. This map factors as a composition

$$F(0, [n], i, j) \xrightarrow{u'} F(0, [j-i], 0, j-i) \xrightarrow{u''} F(1, [j-i], 0, j-i)$$

where u' is an equivalence by (a) and u'' is inert by virtue of (c'). $\square$

Lemma 6.1.3.21. *Let $\mathcal{M}$ be an ∞-category weakly enriched over a Δ-planar ∞-operad $\mathcal{C}^{\circledast}$. For each object $M \in \mathcal{M}$, the forgetful functor $\theta : \mathcal{C}^{+}[M]^{\circledast} \to \mathcal{C}^{\circledast}$ induces a right fibration ${}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}^{+}[M]) \to {}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$.*

Proof. Combine Lemmas 6.1.2.36 and 6.1.2.37. $\square$

Lemma 6.1.3.22. *Let $\mathcal{C}$ be a Δ-monoidal ∞-category, $\mathcal{M}$ an ∞-category which is left-tensored over $\mathcal{C}$, and $M \in \mathcal{M}$ an object. Then the forgetful functor*

$$ {}^{\Delta}\mathrm{LMod}^{\mathrm{nu}}(\mathcal{M}) \times_{\mathcal{M}} \{M\} \to {}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}) $$

is a right fibration.

Proof. Combine Lemma 6.1.3.21 and 6.1.3.20. $\square$

Lemma 6.1.3.23. *Let $\mathcal{M}$ be an ∞-category left-tensored over a Δ-monoidal ∞-category $\mathcal{C}^{\circledast}$. Assume that the ∞-category $\mathcal{C}[M]$ has a final object A. Then:*

(1) *The object A can be promoted to an object of ${}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}^{+}[M])$ in an essentially unique way. We will abuse notation by denoting this object also by A.*

Let $\theta : {}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}^{+}[M]) \to {}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$ be the forgetful functor. Then:

(2) *There is a canonical equivalence of ∞-categories ${}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})_{/\theta(A)} \simeq {}^{\Delta}\mathrm{LMod}^{\mathrm{nu}}(\mathcal{M}) \times_{\mathcal{M}} \{M\}$.*

Proof. Proposition 6.1.2.30 implies that the forgetful functor $\mathcal{C}[M]^{\circledast} \to \mathrm{N}(\mathbf{\Delta})^{op}$ exhibits $\mathcal{C}[M]^{\circledast}$ as a Δ-monoidal ∞-category, so that $\mathcal{C}[M]^{\circledast} \simeq \mathcal{O}^{\otimes} \times_{\mathcal{A}ss^{\otimes}} \mathrm{N}(\mathbf{\Delta})^{op}$ for some coCartesian fibration of ∞-operads $\mathcal{O}^{\otimes} \to \mathcal{A}ss^{\otimes}$. Combining Corollary 3.2.2.5 with Proposition 6.1.3.2, we deduce that A can be lifted to an object of ${}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}[M])$, and that any such lifting is a final object of ${}^{\Delta}\mathrm{Alg}(\mathcal{C}[M])$ (and therefore uniquely determines up to equivalence. Since the map ${}^{\Delta}\mathrm{Alg}(\mathcal{C}^{+}[M]) \to {}^{\Delta}\mathrm{Alg}(\mathcal{C}[M])$ is a trivial Kan fibration; we may lift A to a final object of ${}^{\Delta}\mathrm{Alg}(\mathcal{C}^{+}[M])$, which we will also denote by A. This proves (1). We have a diagram of maps

$$ {}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})_{/\theta(A)} \leftarrow {}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}^{+}[M])_{/A} \to {}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}^{+}[M]) \to {}^{\Delta}\mathrm{LMod}^{\mathrm{nu}}(\mathcal{M}) \times_{\mathcal{M}} \{M\} $$

which are equivalences of ∞-categories by virtue of Lemmas 6.1.3.21 and 6.1.3.20. $\square$

Corollary 6.1.3.24. *Let $\mathcal{M}$ be an ∞-category left-tensored over a Δ-monoidal ∞-category $\mathcal{C}^{\circledast}$. Let $\overline{M} \in {}^{\Delta}\mathrm{LMod}^{\mathrm{nu}}(\mathcal{M})$ be a nonunital left module object having images $M \in \mathcal{M}$ and $A \in {}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$. Suppose that the multiplication map $A \otimes M \to M$ exhibits A as a classifying object for endomorphisms of M. Then, for every algebra object $B \in {}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$, we have a canonical isomorphism $\mathrm{Map}_{{}^{\Delta}\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})}(B, A) \simeq {}^{\Delta}\mathrm{LMod}_{B}^{\mathrm{nu}}(\mathcal{M}) \times_{\mathcal{M}} \{M\}$ in the homotopy category $\mathcal{H}$ of spaces.*

Proof of Theorem 6.1.3.10. The restriction functor $\theta : {}^{\Delta}\mathrm{Alg}(\mathcal{C}) \to {}^{\Delta}\mathrm{Alg}^{\mathrm{qu}}(\mathcal{C})$ is evidently a categorical fibration. It will therefore suffice to show that θ is a categorical equivalence. We first show that θ is essentially surjective. Let A_0 be a quasi-unital algebra object of $\mathcal{C}$. According to Lemma 6.1.3.19, we can find an ∞-category $\mathcal{M}$ which is left-tensored over $\mathcal{C}$ and a quasi-unital module $\overline{M}_0 \in {}^{\Delta}\mathrm{LMod}_{A_0}^{\mathrm{qu}}(\mathcal{M}) \subseteq {}^{\Delta}\mathrm{LMod}^{\mathrm{nu}}(\mathcal{M})$

which exhibits A_0 as a classifying object for morphisms from M to M; here we let M denote the image of $\overline{M}_0$ under the forgetful functor $^\Delta\mathrm{LMod}^{\mathrm{qu}}(\mathcal{M}) \to \mathcal{M}$. We have a commutative diagram

$$\begin{array}{ccc}
^\Delta\mathrm{LMod}(\mathcal{M}) \times_\mathcal{M} \{M\} & \longrightarrow & ^\Delta\mathrm{Alg}(\mathcal{C}^+[M]) & \longrightarrow & ^\Delta\mathrm{Alg}(\mathcal{C}) \\
\downarrow & & \downarrow{\scriptstyle\theta'} & & \downarrow{\scriptstyle\theta} \\
^\Delta\mathrm{LMod}^{\mathrm{nu}}(\mathcal{M}) \times_\mathcal{M} \{M\} & \longrightarrow & ^\Delta\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}^+[M]) & \longrightarrow & ^\Delta\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}).
\end{array}$$

Let $\overline{A}_0$ be the image of $\overline{M}_0$ in $^\Delta\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}^+[M])$. To prove that A_0 belongs to the essential image of θ, it will suffice to show that $\overline{A}_0$ belongs to the essential image of θ'.

Let $C \in \mathcal{C}^+[M]$ denote the image of $\overline{A}_0$. By assumption, C is a final object of $\mathcal{C}^+[M]$. Using Corollary 3.2.2.5 and Proposition 6.1.3.2, we deduce that $\overline{A}_0$ is a final object of $\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}^+[M])$. On the other hand, Corollary 3.2.2.5 and Proposition 4.1.2.15 imply that $\mathrm{Alg}(\mathcal{C}^+[M])$ has a final object $\overline{A}$. Using Corollary 3.2.2.5 and Proposition 6.1.3.2 again, we deduce that $\theta'(\overline{A})$ is a final object of $^\Delta\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}^+[M])$ and therefore equivalent to $\overline{A}_0$, which proves that $\overline{A}_0$ belongs to the essential image of θ'.

We now prove that θ is fully faithful. Fix objects $A, B \in {}^\Delta\mathrm{Alg}(\mathcal{C})$, and set $A_0 = \theta(A) \in {}^\Delta\mathrm{Alg}^{\mathrm{qu}}(\mathcal{C}), B_0 = \theta(B) \in {}^\Delta\mathrm{Alg}^{\mathrm{qu}}(\mathcal{C})$. We wish to prove that the map $\mathrm{Map}_{^\Delta\mathrm{Alg}(\mathcal{C})}(B, A) \to \mathrm{Map}_{^\Delta\mathrm{Alg}^{\mathrm{qu}}(\mathcal{C})}(B_0, A_0)$ is a homotopy equivalence. Use Lemma 6.1.3.19 to choose $\overline{M}_0 \in {}^\Delta\mathrm{LMod}^{\mathrm{qu}}_{A_0}(\mathcal{M})$ as above. Consider the diagram

$$\begin{array}{ccccc}
^\Delta\mathrm{LMod}_A(\mathcal{M}) & \longleftarrow & ^\Delta\mathrm{LMod}_A(\mathcal{M}) \times_\mathcal{M} \{M\} & \longrightarrow & \{A\} \times_{^\Delta\mathrm{Alg}(\mathcal{C})} {}^\Delta\mathrm{Alg}(\mathcal{C}^+[M]) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\psi} \\
^\Delta\mathrm{LMod}^{\mathrm{qu}}_{A_0}(\mathcal{M}) & \longleftarrow & ^\Delta\mathrm{LMod}^{\mathrm{qu}}_{A_0}(\mathcal{M}) \times_\mathcal{M} \{M\} & \longrightarrow & \{A_0\} \times_{^\Delta\mathrm{Alg}^{\mathrm{qu}}(\mathcal{C})} \mathrm{Alg}^{\mathrm{qu}}(\mathcal{C}^+[M]).
\end{array}$$

The left square is a pullback, and the left vertical map is a trivial Kan fibration (Proposition 6.1.3.18). The horizontal maps on the right are both categorical equivalences (Theorem 6.1.2.34 and Lemma 6.1.3.20). Using the two-out-of-three property, we deduce that ψ is a categorical equivalence. Since ψ is also a categorical fibration, it is a trivial Kan fibration; we may therefore choose $\overline{M} \in {}^\Delta\mathrm{LMod}_A(\mathcal{M})$ lifting $\overline{M}_0$.

According to Corollary 6.1.2.41 and Lemma 6.1.3.24, we have canonical homotopy equivalences

$$\phi : \mathrm{Map}_{^\Delta\mathrm{Alg}(\mathcal{C})}(B, A) \simeq {}^\Delta\mathrm{LMod}_B(\mathcal{M}) \times_\mathcal{M} \{M\}$$

$$\phi^{\mathrm{nu}} : \mathrm{Map}_{^\Delta\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})}(B_0, A_0) \simeq {}^\Delta\mathrm{LMod}^{\mathrm{nu}}_{B_0}(\mathcal{M}) \times_\mathcal{M} \{M\}.$$

Let $f : B_0 \to A_0$ be a map of nonunital algebras, and let $\overline{N}_0$ be the corresponding object of $^\Delta\mathrm{LMod}^{\mathrm{nu}}_{B_0}(\mathcal{M}) \times_\mathcal{M} \{M\}$. Let $u : \mathbf{1} \to B$ be the unit map for the algebra B. Then f is quasi-unital if and only if the composition $u : \mathbf{1} \to B \overset{f}{\to} A$ is homotopic to the unit of A; here we abuse notation by identifying A and B with their images in $\mathcal{C}$. Since A is a classifying object for morphisms from M to itself, we can identify u with the induced map $u' : \mathbf{1} \otimes M \to M$, so that f is quasi-unital if and only if u' is homotopic to the identity, which is equivalent to the condition that $\overline{N}_0$ be quasi-unital. It follows that ϕ^{nu} restricts to a homotopy equivalence

$$\phi^{\mathrm{qu}} : \mathrm{Map}_{^\Delta\mathrm{Alg}^{\mathrm{qu}}(\mathcal{C})}(B_0, A_0) \simeq {}^\Delta\mathrm{LMod}^{\mathrm{qu}}_{B_0}(\mathcal{M}) \times_\mathcal{M} \{M\}.$$

We wish to prove that θ induces a homotopy equivalence $\mathrm{Map}_{^\Delta\mathrm{Alg}(\mathcal{C})}(B, A) \to \mathrm{Map}_{^\Delta\mathrm{Alg}^{\mathrm{qu}}(\mathcal{C})}(B_0, A_0)$. In view of the above identifications, it will suffice to show that the restriction map

$$g : {}^\Delta\mathrm{LMod}_B(\mathcal{M}) \times_\mathcal{M} \{M\} \to {}^\Delta\mathrm{LMod}^{\mathrm{qu}}_{B_0}(\mathcal{M}) \times_\mathcal{M} \{M\}$$

is a homotopy equivalence. Proposition 6.1.3.18 implies that g is a trivial Kan fibration. $\qquad\square$

We now turn to the proof of Lemma 6.1.3.19. The idea is to define $\mathcal{M}$ to be the ∞-category of nonunital *right* A-modules in $\mathcal{C}$, and $M \in \mathcal{M}$ to be A itself, regarded as a right A-module.

Proof of Lemma 6.1.3.19. We may assume without loss of generality that $\mathcal{C}^{\circledast} = \mathcal{C}^{\otimes} \times_{\mathcal{A}ss^{\otimes}} N(\boldsymbol{\Delta})^{op}$ for some monoidal ∞-category $\mathcal{C}^{\otimes}$, and that A is the image of an object of $\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$ (which we will also denote by A). Let us regard $\mathcal{C}$ as bitensored over itself. Using a variant on the constructions described in §4.3.2, we can view the ∞-category $\mathcal{M} = \mathrm{RMod}_A^{\mathrm{nu}}(\mathcal{C})$ of nonunital right A-modules as an ∞-category which is left-tensored over $\mathcal{C}^{\otimes}$, and therefore left-tensored over the Δ-monoidal ∞-category $\mathcal{C}^{\circledast}$. Let $M \in \mathcal{M}$ be the object corresponding to A (regarded as a nonunital right module over itself in the obvious way), so that M inherits an evident (nonunital) left action of A. We claim that the action map $\theta : A \otimes M \to M$ exhibits A as a morphism object $\mathrm{Mor}_{\mathcal{M}}(M, M)$. To prove this, it suffices to show that for every object $C \in \mathcal{C}$, composition with θ induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(C, A) \to \mathrm{Map}_{\mathcal{M}}(C \otimes M, M)$.

We must show that for every Kan complex K, θ induces a bijection

$$[K, \mathrm{Map}_{\mathcal{C}}(C, A)] \to [K, \mathrm{Map}_{\mathcal{M}}(C \otimes M, M)];$$

here $[K, X]$ denotes the set of maps from K to X in the homotopy category $\mathcal{H}$ of spaces. Replacing $\mathcal{C}$ by $\mathrm{Fun}(K, \mathcal{C})$ and A by the nonunital algebra $A' \in \mathrm{Alg}^{\mathrm{nu}}(\mathrm{Fun}(K, \mathcal{C})) \simeq \mathrm{Fun}(K, \mathrm{Alg}^{\mathrm{nu}}(\mathcal{C}))$ corresponding to the constant map $K \to \{A\} \subseteq \mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$, we can reduce to the case where $K = \Delta^0$. In other words, it will suffice to show composition with θ induces a bijection of sets $q : \pi_0 \mathrm{Map}_{\mathcal{C}}(C, A) \to \pi_0 \mathrm{Map}_{\mathcal{M}}(C \otimes M, M)$.

Our next step is to construct an inverse to q. Let $u : \mathbf{1} \to A$ be a quasi-unit. Let $\phi : C \otimes M \to M$ be an arbitrary morphism in $\mathcal{M}$. Then ϕ determines a map $C \otimes A \to A$ in $\mathcal{C}$. Let $q'(\phi)$ denote the composition

$$C \simeq C \otimes \mathbf{1} \xrightarrow{u} C \otimes A \to A.$$

We may view q' as a map of sets from $\pi_0 \mathrm{Map}_{\mathcal{M}}(C \otimes M, M)$ to $\pi_0 \mathrm{Map}_{\mathcal{C}}(C, A)$. The composition $q' \circ q : \pi_0 \mathrm{Map}_{\mathcal{C}}(C, A) \to \pi_0 \mathrm{Map}_{\mathcal{C}}(C, A)$ is induced by the map $r_u : A \to A$ given by right multiplication by u. Since u is a *right* unit of A, we deduce that $q' \circ q$ is the identity. In particular, q is injective.

To complete the proof, it will suffice to show that q is surjective. According to Theorem 6.1.2.34, if C is an object of $\mathcal{C}$, then giving a map $\phi : C \otimes M \to M$ in $\mathcal{M}$ is equivalent to lifting C to an object $\widetilde{C} \in \mathcal{C}^+[M]$. In particular, the left action of A on M gives rise to a canonical element $\widetilde{A} \in \mathcal{C}^+[M]$. Then ϕ belongs to the image of q if and only if there exists a map $\widetilde{C} \to \widetilde{A}$ in $\mathcal{C}^+[M]$. Consequently, the surjectivity of q is equivalent to the following assertion:

($*$) For every object $\widetilde{C} \in \mathcal{C}^+[M]$, there exists a morphism $\widetilde{C} \to \widetilde{A}$ in $\mathcal{C}[M]$.

Proposition 6.1.2.39 asserts that the forgetful functor $\mathcal{C}^+[M] \to \mathcal{C}$ is a right fibration. Consequently, the quasi-unit $u : \mathbf{1} \to A$ can be lifted to a map $\widetilde{u} : E \to \widetilde{A}$ in $\mathcal{C}^+[M]$. Since u is a *left* unit of A, the object E classifies a map $v : \mathbf{1} \otimes M \to M$ which is an equivalence in $\mathcal{C}$, so that v is an equivalence in $\mathcal{M}$. It follows from Remark 6.1.2.31 that E is an invertible object of $\mathcal{C}^+[M]$ (see Remark 4.1.1.19), so that the functor $\widetilde{C} \mapsto \widetilde{C} \otimes E$ is an equivalence from $\mathcal{C}^+[M]$ to itself. Consequently, condition ($*$) is equivalent to:

($*'$) For every object $\widetilde{C} \in \mathcal{C}^+[M]$, there exists a morphism $\widetilde{C} \otimes E \to \widetilde{A}$ in $\mathcal{C}^+[M]$.

In view of the existence of $\widetilde{u} : E \to \widetilde{A}$, it will suffice to prove the following slightly stronger assertion:

($*''$) For every object $\widetilde{C} \in \mathcal{C}^+[M]$, there exists a morphism $\widetilde{C} \otimes \widetilde{A} \to \widetilde{A}$ in $\mathcal{C}^+[M]$.

Applying Theorem 6.1.2.34 again, we see that ($*''$) is equivalent to the following assertion: for every map $\phi : C \otimes M \to M$, there exists a commutative diagram

$$\begin{array}{ccc} C \otimes A \otimes M & \longrightarrow & C \otimes M \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ A \otimes M & \longrightarrow & M; \end{array}$$

in $\mathcal{M}$, where the horizontal arrows are given by the canonical action of A on M. This is a straightforward consequence of the constructions of $\mathcal{M}$ and M. $\qquad\square$

6.1.4 Deligne's Conjecture

Let A be an associative algebra over a field k with multiplication m. Then the cyclic bar complex

$$\ldots \longrightarrow A \otimes_k A \otimes_k A \overset{m \otimes \mathrm{id} - \mathrm{id} \otimes m}{\longrightarrow} A \otimes_k A$$

provides a resolution of A by free $(A \otimes_k A^{op})$-modules; we will denote this resolution by $P_\bullet$. The cochain complex $\mathrm{HC}^*(A) = \mathrm{Hom}_{A \otimes_k A^{op}}(P_\bullet, A)$ is called the *Hochschild cochain complex* of the algebra A. The cohomologies of the Hochschild cochain complex (which are the Ext-groups $\mathrm{Ext}^i_{A \otimes_k A^{op}}(A, A)$) are called the *Hochschild cohomology* groups of the algebra A. A famous conjecture of Deligne asserts that the Hochschild cochain complex $\mathrm{HC}^*(A)$ carries an action of the little 2-disks operad: in other words, that we can regard $\mathrm{HC}^*(A)$ as an $\mathbb{E}_2$-algebra object in the ∞-category of chain complexes over k. This conjecture has subsequently been proven by many authors in many different ways (see, for example, [109], [89], and [142]). In this section we will outline a proof of Deligne's conjecture, using the formalism developed in this book. Our basic strategy can be outlined as follows:

(1) Let $\mathcal{C}^\otimes$ be an arbitrary presentable symmetric monoidal ∞-category, and let $f : A \to B$ be a morphism of $\mathbb{E}_k$-algebra objects of $\mathcal{C}$. We will prove that there exists another $\mathbb{E}_k$-algebra $\mathfrak{Z}_{\mathbb{E}_k}(f)$ of $\mathcal{C}$, which is universal with respect to the existence of a commutative diagram

$$
\begin{array}{ccc}
 & \mathfrak{Z}_{\mathbb{E}_k}(f) \otimes A & \\
 \overset{u}{\nearrow} & & \searrow \\
A & \xrightarrow{\hspace{2cm} f \hspace{2cm}} & B
\end{array}
$$

in the ∞-category $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$, where u is induced by the unit map $\mathbf{1} \to \mathfrak{Z}_{\mathbb{E}_k}(f)$ (Theorem 6.1.4.12).

(2) In the case where $A = B$ and f is the identity map, we will denote $\mathfrak{Z}_{\mathbb{E}_k}(f)$ by $\mathfrak{Z}_{\mathbb{E}_k}(A)$. We will see that $\mathfrak{Z}_{\mathbb{E}_k}(A)$ has the structure of an $\mathbb{E}_{k+1}$-algebra object of $\mathcal{C}$. (More generally, the centralizer construction is functorial in the sense that there are canonical maps of $\mathbb{E}_k$-algebras $\mathfrak{Z}_{\mathbb{E}_k}(f) \otimes \mathfrak{Z}_{\mathbb{E}_k}(g) \to \mathfrak{Z}_{\mathbb{E}_k}(f \circ g)$; in the special case $f = g = \mathrm{id}_A$ this gives rise to an associative algebra structure on the $\mathbb{E}_k$-algebra $\mathfrak{Z}_{\mathbb{E}_k}(A)$, which promotes $\mathfrak{Z}_{\mathbb{E}_k}(A)$ to an $\mathbb{E}_{k+1}$-algebra by Theorem 5.1.2.2.)

(3) We will show that the image of $\mathfrak{Z}_{\mathbb{E}_k}(f)$ in $\mathcal{C}$ can be identified with a classifying object for morphisms from A to B in the ∞-category $\mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})$ (Theorem 6.1.4.27). In the special case where $k = 1$ and $f = \mathrm{id}_A$, we can identify this with a classifying object for endomorphisms of A as an A-bimodule (see Theorem 4.3.4.28), thereby recovering the usual definition of Hochschild cohomology.

We begin with a very general discussion of centralizers.

Definition 6.1.4.1. Let $q : \mathcal{C}^\otimes \to \mathcal{LM}$ be a coCartesian fibration of ∞-operads, so that q exhibits $\mathcal{M} = \mathcal{C}_\mathfrak{m}$ as left-tensored over the monoidal ∞-category $\mathcal{C}_\mathfrak{a}^\otimes$. Let $\mathbf{1}$ denote the unit object of $\mathcal{C}_\mathfrak{a}$, and suppose we are given a morphism $f : \mathbf{1} \otimes M \to N$ in $\mathcal{M}$. A *centralizer of f* is final object of the ∞-category

$$(\mathcal{C}_\mathfrak{a})_{\mathbf{1}/} \times_{\mathcal{M}_{\mathbf{1} \otimes M/}} \mathcal{M}_{\mathbf{1} \otimes M//N} \, .$$

We will denote such an object, if it exists, by $\mathfrak{Z}(f)$. We will refer to $\mathfrak{Z}(f)$ as the *centralizer* of the morphism f.

Remark 6.1.4.2. We will generally abuse notation by identifying $\mathfrak{Z}(f)$ with its image in the ∞-category $\mathcal{C}$. By construction, this object is equipped with a map $\mathfrak{Z}(f) \otimes M \to N$ which fits into a commutative diagram

$$
\begin{array}{ccc}
 & \mathfrak{Z}(f) \otimes M & \\
 \nearrow & & \searrow \\
\mathbf{1} \otimes M & \xrightarrow{\hspace{1.5cm} f \hspace{1.5cm}} & N.
\end{array}
$$

Remark 6.1.4.3. In the situation of Definition 6.1.4.1, we can lift M to an object $\overline{M} \in \mathrm{LMod}_A(\mathcal{M})$, where A is a trivial algebra object of $\mathcal{C}_{\mathfrak{a}}^{\otimes}$. Then we can identify centralizers for a morphism $f : M \to N$ with morphism objects $\mathrm{Mor}_{\mathcal{M}_{M/}}(M, N)$ computed in the $\mathcal{LM}$-monoidal ∞-category $\mathcal{C}_{\overline{M}_{\mathcal{LM}}/}^{\otimes}$ (see Definition 4.2.1.28).

Definition 6.1.4.4. Let $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a coCartesian fibration of ∞-operads, and let $M \in \mathcal{M} = \mathcal{C}_{\mathfrak{m}}$ be an object. A *center* of M is a final object of the fiber product $\mathrm{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\}$. If such an object exists, we will denote it by $\mathfrak{Z}(M)$.

Remark 6.1.4.5. In the situation of Definition 6.1.4.4, we will often abuse notation by identifying the center $\mathfrak{Z}(M)$ with its image in the ∞-category $\mathrm{Alg}(\mathcal{C})$ of associative algebra objects of the monoidal ∞-category $\mathcal{C}_{\mathfrak{a}}$.

Our first goal is to show that, as our notation suggests, the theory of centers is closely related to the theory of centralizers. Namely, we have the following:

Proposition 6.1.4.6. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a coCartesian fibration of ∞-operads, and let $M \in \mathcal{M} = \mathcal{C}_{\mathfrak{m}}$. Suppose that there exists a centralizer of the canonical equivalence $e : \mathbf{1} \otimes M \to M$. Then there exists a center of M. Moreover, a lifting $\overline{M} \in \mathrm{LMod}_A(\mathcal{M})$ of M exhibits A as a center of M if and only if the diagram*

$$
\begin{array}{ccc}
 & A \otimes M & \\
 & & \\
\mathbf{1} \otimes M & \xrightarrow{\quad e \quad} & M
\end{array}
$$

exhibits A as a centralizer of e.

The proof will require a few preliminaries.

Lemma 6.1.4.7. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a fibration of ∞-operads which exhibits the ∞-category $\mathcal{M} = \mathcal{C}_{\mathfrak{m}}$ as weakly enriched over the planar ∞-operad $\mathcal{C}_{\mathfrak{a}}^{\otimes}$. Assume that $\mathcal{C}^{\otimes}$ admits a $\mathfrak{a}$-unit object (Definition 3.2.1.1), and let $\theta : \mathrm{LMod}(\mathcal{M}) \to \mathcal{M}$ be the forgetful functor.*

(1) *The functor θ admits a left adjoint L.*

(2) *Let $\mathrm{LMod}'(\mathcal{M}) \subseteq \mathrm{LMod}(\mathcal{M})$ be the essential image of L. Then θ induces a trivial Kan fibration $\mathrm{LMod}'(\mathcal{M}) \to \mathcal{M}$. In particular, L is fully faithful.*

(3) *An object $(A, M) \in \mathrm{LMod}(\mathcal{M})$ belongs to $\mathrm{LMod}'(\mathcal{M})$ if and only if A is a trivial algebra object of $\mathcal{C}_{\mathfrak{a}}$ (see §refunitr).*

Proof. Let us identify the ∞-operad Triv with the full subcategory of $\mathcal{LM}$ spanned by those objects having the form $(\langle n \rangle, \langle n \rangle^{\circ})$. The functor θ factors as a composition

$$\mathrm{LMod}(\mathcal{M}) \xrightarrow{\theta'} \mathrm{Alg}_{\mathrm{Triv}\,/\,\mathcal{LM}}(\mathcal{C}) \xrightarrow{\theta''} \mathcal{M},$$

where θ'' is a trivial Kan fibration (Example 2.1.3.5). To prove (1), it will suffice to show that θ' admits a left adjoint. We claim that this left adjoint exists, and is given by operadic q-left Kan extension along the inclusion $\mathrm{Triv}^{\otimes} \subset \mathcal{LM}^{\otimes}$. According to Proposition 3.1.3.3, it suffices to verify that for each $\overline{M} \in \mathrm{Alg}_{\mathrm{Triv}\,/\,\mathcal{LM}}(\mathcal{C})$ and every object of the form $X \in \mathcal{LM}$, the map $\mathrm{Triv}^{\otimes} \times_{\mathcal{LM}^{\otimes}} (\mathcal{LM}^{\otimes})_{/X}^{\mathrm{act}} \to \mathcal{C}^{\otimes}$ can be extended to an operadic q-colimit diagram (lying over the natural map $(\mathrm{Triv}^{\otimes} \times_{\mathcal{LM}^{\otimes}} (\mathcal{LM}^{\otimes})^{\mathrm{act}})_{/X}^{\triangleright} \to \mathcal{LM}^{\otimes})$. If $X = \mathfrak{m}$, then $X \in \mathrm{Triv}^{\otimes}$ and the result is obvious. If $X = \mathfrak{a}$, then the desired result follows from our assumption that $\mathcal{C}^{\otimes}$ admits an $\mathfrak{a}$-unit. This proves (1). Moreover, we see that if $\overline{A} = (A, M) \in \mathrm{LMod}(\mathcal{M})$, then a map $f : \overline{M} \to \overline{A} | \mathrm{Triv}^{\otimes}$ exhibits $\overline{A}$ as a free $\mathcal{LM}$-algebra generated by $\overline{M}$ if and only $\overline{A} | \mathcal{A}\mathrm{ss}^{\otimes}$ is a trivial algebra and f is an equivalence. It follows that the unit map $\mathrm{id} \to \theta \circ L$ is an equivalence, so that L is a fully faithful embedding whose essential image $\mathrm{LMod}'(\mathcal{M})$ is as described in assertion (3). To complete the proof, we observe that $\theta | \mathrm{LMod}'(\mathcal{M})$ is an equivalence of ∞-categories and also a categorical fibration, and therefore a trivial Kan fibration. $\qquad\square$

Lemma 6.1.4.8. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a coCartesian fibration of ∞-operads which exhibits the ∞-category $\mathcal{M} = \mathcal{C}_{\mathfrak{m}}$ as left-tensored over the monoidal ∞-category $\mathcal{C}_{\mathfrak{a}}^{\otimes}$. Let $A \in \mathrm{Alg}(\mathcal{C}_{\mathfrak{a}})$ be a trivial algebra object, and let $\overline{M} \in \mathrm{LMod}_A(\mathcal{M})$. Let $\mathcal{C}'^{\otimes} = \mathcal{C}_{\overline{M}_{\mathcal{LM}}/}^{\otimes}$ be defined as in Notation 2.2.2.3, let $\mathcal{M}' = \mathcal{C}'_{\mathfrak{m}} \simeq \mathcal{M}_{M/}$, and let $M' = \mathrm{id}_M \in \mathcal{M}'$. Then the forgetful functor*

$$\theta : \mathrm{LMod}(\mathcal{M}') \times_{\mathcal{M}'} \{M'\} \to \mathrm{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\}$$

is a trivial Kan fibration. In particular, M has a center in $\mathcal{C}^{\otimes}$ if and only if M' has a center in $\mathcal{C}'^{\otimes}$.

Proof. Note that $\overline{M} : \mathcal{LM}^{\otimes} \to \mathcal{C}^{\otimes}$ is a coCartesian section of q, so that $\mathcal{C}'^{\otimes} \to \mathcal{LM}^{\otimes}$ is a coCartesian fibration of ∞-operads (Theorem 2.2.2.4). Since θ is evidently a categorical fibration, it will suffice to show that θ is a trivial Kan fibration. To this end, we let $\mathcal{A}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{LMod}(\mathcal{M}))$ spanned by those morphisms $(N, B) \to (N', B')$ which exhibit (N, B) as an $\mathrm{LMod}'(\mathcal{M})$-colocalization of (N', B'), where $\mathrm{LMod}'(\mathcal{M})$ is the full subcategory of $\mathrm{LMod}(\mathcal{M})$ described in Lemma 6.1.4.7 (in other words, a morphism $(N, B) \to (N', B')$ belongs to $\mathcal{A}$ if and only if B is a trivial algebra and the map $N \to N'$ is an equivalence in $\mathcal{M}$). Evaluation at $\{1\} \subseteq \Delta^1$ and $\mathfrak{m} \in \mathcal{LM}$ induces a functor $e : \mathcal{A} \to \mathcal{M}$. The map θ factors as a composition

$$\mathrm{LMod}(\mathcal{M}') \times_{\mathcal{M}'} \{M'\} \xrightarrow{\theta'} \mathcal{A} \times_{\mathcal{M}} \{M\} \xrightarrow{\theta''} \mathrm{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\},$$

where θ'' is a pullback of the trivial Kan fibration $\mathcal{A} \to \mathrm{LMod}(\mathcal{M})$ given by evaluation at $\{1\}$. We conclude by observing that θ' is also an equivalence of ∞-categories. $\square$

Lemma 6.1.4.9. *Let $\mathcal{M}$ be an ∞-category left-tensored over a monoidal ∞-category $\mathcal{C}^{\otimes}$, let $M \in \mathcal{M}$ be an object, and suppose that there exists a morphism object $\mathrm{Mor}_{\mathcal{M}}(M, M)$. Then there exists a center $\mathfrak{Z}(M)$. Moreover, an object $\overline{M} \in \mathrm{LMod}_A(\mathcal{M})$ lifting M exhibits A as a center of M if and only if the canonical map $A \otimes M \to M$ exhibits A as a morphism object $\mathrm{Mor}_{\mathcal{M}}(M, M)$.*

Proof. Combine Corollary 6.1.2.42, Corollary 3.2.2.5, and Proposition 4.2.2.11. $\square$

Proof of Proposition 6.1.4.6. Combine Lemma 6.1.4.8, Lemma 6.1.4.9, and Remark 6.1.4.3. $\square$

We are primarily interested in studying centralizers in the setting of $\mathcal{O}^{\otimes}$-algebra objects of a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$. To emphasize the role of $\mathcal{O}^{\otimes}$, it is convenient to introduce a special notation for this situation:

Definition 6.1.4.10. Let $\mathcal{O}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be ∞-operads, and let $p : \mathcal{O}^{\otimes} \times \mathcal{LM}^{\otimes} \to \mathcal{D}^{\otimes}$ be a bifunctor of ∞-operads. Suppose that $q : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a coCartesian fibration of ∞-operads. Then we have an induced coCartesian fibration of ∞-operads $q' : \mathrm{Alg}_{\mathcal{O}/\mathcal{D}}(\mathcal{C})^{\otimes} \to \mathcal{LM}^{\otimes}$. If $f : A \to B$ is a morphism in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{\mathfrak{m}}$, then we let $\mathfrak{Z}_{\mathcal{O}}(f)$ denote the centralizer of f (as a morphism in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes}$), provided that this centralizer exists. If $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{\mathfrak{m}}$, we let $\mathfrak{Z}_{\mathcal{O}}(A)$ denote the center of A, provided that such a center exists.

Remark 6.1.4.11. The primary case of interest to us is that in which $\mathcal{D}^{\otimes} = \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, so that $\mathcal{C}^{\otimes}$ can be regarded as a symmetric monoidal ∞-category and the map $p : \mathcal{O}^{\otimes} \times \mathcal{LM}^{\otimes} \to \mathcal{D}^{\otimes}$ is uniquely determined. In this case, we will denote $\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}(\mathcal{C})_{\mathfrak{m}} \simeq \mathrm{Alg}_{\mathcal{O}/\mathcal{D}}(\mathcal{C})_{\mathfrak{a}}$ simply by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. If $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$, we can identify the center $\mathfrak{Z}_{\mathcal{O}}(A)$ (if it exists) with an associative algebra object of the symmetric monoidal ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. If $\mathcal{O}^{\otimes}$ is a little cubes operad, then Theorem 5.1.2.2 and Example 5.1.0.7 provide equivalences of ∞-categories

$$\mathrm{Alg}_{\mathbb{E}_{k+1}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})) \leftarrow \mathrm{Alg}(\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})),$$

so we can identify $\mathfrak{Z}_{\mathbb{E}_k}(A)$ with an $\mathbb{E}_{k+1}$-algebra object of $\mathcal{C}$.

In the situation of Definition 6.1.4.10, it is generally not possible to prove the existence of centralizers by direct application of Lemma 6.1.4.9: the tensor product of $\mathcal{O}$-algebra objects usually does not commute with colimits in either variable, so there generally does not exist a morphism object $\mathrm{Mor}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{\mathfrak{m}}}(A, B)$ for a pair of algebras $A, B \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{\mathfrak{m}}$. Nevertheless, if $\mathcal{O}$ is coherent, then we will show that the centralizer $\mathfrak{Z}_{\mathcal{O}}(f)$ of a morphism $f : A \to B$ exists under very general conditions:

Theorem 6.1.4.12. *Let $\mathcal{O}^{\otimes}$ be a coherent ∞-operad, let $p : \mathcal{O}^{\otimes} \times \mathcal{LM}^{\otimes} \to \mathcal{D}^{\otimes}$ be a bifunctor of ∞-operads, and let $q : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ exhibit $\mathcal{C}^{\otimes}$ as a presentable $\mathcal{D}$-monoidal ∞-category. Then, for every morphism $f : A \to B$ in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{\mathfrak{m}}$, there exists a centralizer $\mathfrak{Z}_{\mathcal{O}}(f)$.*

Corollary 6.1.4.13. *Let $k \geq 0$, and let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Assume that $\mathcal{C}$ is presentable and that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. Then:*

(1) *For every morphism $f : A \to B$ in $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$, there exists a centralizer $\mathfrak{Z}_{\mathbb{E}_k}(f) \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$.*

(2) *For every object $A \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$, there exists a center*

$$\mathfrak{Z}_{\mathbb{E}_k}(A) \in \mathrm{Alg}(\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})) \simeq \mathrm{Alg}_{\mathbb{E}_{k+1}}(\mathcal{C}).$$

Proof. Combine Theorems 6.1.4.12 and 5.1.1.1. $\square$

Example 6.1.4.14 (Koszul Duality)**.** Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Assume that $\mathcal{C}$ is presentable and that the tensor product on $\mathcal{C}$ preserves small colimits separately in each variable. An *augmented $\mathbb{E}_k$-algebra* in $\mathcal{C}$ is a map of $\mathbb{E}_k$-algebras $\epsilon : A \to \mathbf{1}$, where $\mathbf{1}$ is a trivial $\mathbb{E}_k$-algebra object of $\mathcal{C}$. It follows from Theorem 6.1.4.27 that ϵ admits a centralizer $\mathfrak{Z}(\epsilon)$. We will refer to this centralizer as the *Koszul dual* of ϵ, and denote it by $A^{\vee}$. By definition, $A^{\vee}$ is universal among $\mathbb{E}_k$-algebras such that $A \otimes A^{\vee}$ is equipped with an augmentation $\epsilon' : A \otimes A^{\vee} \to \mathbf{1}$ compatible with the augmentation ϵ on A. In this case, the composite map

$$\epsilon^{\vee} : A^{\vee} \simeq \mathbf{1} \otimes A^{\vee} \to A \otimes A^{\vee} \xrightarrow{\epsilon'} \mathbf{1}$$

is an augmentation on $A^{\vee}$, so we can again regard $A^{\vee}$ as an augmented $\mathbb{E}_k$-algebra object of $\mathcal{C}$. In many cases, the relationship between A and $A^{\vee}$ is symmetric: that is, ϵ' also exhibits A as a centralizer of $\epsilon^{\vee}$. We will discuss this construction in more detail elsewhere.

Example 6.1.4.15 (The Drinfeld Center)**.** Let $\mathcal{C}$ be an $\mathbb{E}_k$-monoidal ∞-category. Using Example 2.4.2.4 and Proposition 2.4.2.5, we can view $\mathcal{C}$ as an $\mathbb{E}_k$-algebra object of the ∞-category $\mathcal{Cat}_{\infty}$ (which we regard as endowed with the Cartesian symmetric monoidal structure). Corollary 6.1.4.13 guarantees the existence of a center $\mathfrak{Z}_{\mathbb{E}_k}(\mathcal{C})$, which we can view as an $\mathbb{E}_{k+1}$-monoidal ∞-category. In the special case where $k = 1$ and $\mathcal{C}$ is (the nerve of) an ordinary monoidal category, the center $\mathfrak{Z}_{\mathbb{E}_1}(\mathcal{C})$ is also equivalent to the nerve of an ordinary category $\mathcal{Z}$. Example 5.1.2.4 guarantees that $\mathcal{Z}$ admits the structure of a braided monoidal category. This braided monoidal category $\mathcal{Z}$ is called the *Drinfeld center* of the monoidal category underlying $\mathcal{C}$ (see, for example, [81]). Consequently, we can view the construction $\mathcal{C} \mapsto \mathfrak{Z}_{\mathbb{E}_k}(\mathcal{C})$ as a higher-categorical generalization of the theory of the Drinfeld center.

Our goal for the remainder of this section is to provide a proof of Theorem 6.1.4.12. The idea is to change ∞-categories to maneuver into a situation where Lemma 6.1.4.9 can be applied. To carry out this strategy, we will need to introduce a bit of notation.

Definition 6.1.4.16. Let $\mathcal{O}^{\otimes}$ be an ∞-operad and S an ∞-category. A *coCartesian S-family of $\mathcal{O}$-operads* is a map $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \times S$ with the following properties:

(i) The map q is a categorical fibration.

(ii) The underlying map $\mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{Fin}_*) \times S$ exhibits $\mathcal{C}^{\otimes}$ as an S-family of ∞-operads, in the sense of Definition 2.3.2.10.

(iii) For every object $C \in \mathcal{C}^{\otimes}$ with $q(C) = (X, s) \in \mathcal{O}^{\otimes} \times S$ and every morphism $f : s \to s'$ in S, there exists a q-coCartesian morphism $C \to C'$ in $\mathcal{C}^{\otimes}$ lifting the morphism (id_X, f).

Remark 6.1.4.17. Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \times S$ be a coCartesian S-family of $\mathcal{O}$-operads. Condition (iii) of Definition 6.1.4.16 guarantees that the underlying map $\mathcal{C}^{\otimes} \to S$ is a coCartesian fibration, classified by some map $\chi : S \to \mathcal{C}\mathrm{at}_{\infty}$. The map q itself determines a natural transformation from χ to the constant functor χ_0 taking the value $\mathcal{O}^{\otimes}$, so that χ determines a functor $\overline{\chi} : S \to \mathcal{C}\mathrm{at}_{\infty}^{/\mathcal{O}^{\otimes}}$. This construction determines a bijective correspondence between equivalences classes of S-families of $\mathcal{O}$-operads and equivalence classes of functors from S to the ∞-category $(\mathrm{Op}_{\infty})^{/\mathcal{O}^{\otimes}}$, where Op_{∞} is the ∞-category of ∞-operads (see Definition 2.1.4.1).

Definition 6.1.4.18. Let $\mathcal{O}^{\otimes}$ be an ∞-operad, S an ∞-category, and $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \times S$ be a coCartesian S-family of $\mathcal{O}$-operads. We define a simplicial set $\mathrm{Alg}_{/\mathcal{O}}^{S}(\mathcal{C})$ equipped with a map $\mathrm{Alg}_{/\mathcal{O}}^{S}(\mathcal{C}) \to S$ so that the following universal property is satisfied: for every map of simplicial sets $T \to S$, there is a canonical bijection of $\mathrm{Fun}_S(T, \mathrm{Alg}_{/\mathcal{O}}^{S}(\mathcal{C}))$ with the subset of $\mathrm{Fun}_{\mathcal{O}^{\otimes} \times S}(\mathcal{O}^{\otimes} \times T, \mathcal{C})$ spanned by those maps with the property that for each vertex $t \in T$, the induced map $\mathcal{O}^{\otimes} \to \mathcal{C}_t^{\otimes}$ belongs to $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}_t)$.

Remark 6.1.4.19. If $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \times S$ is as in Definition 6.1.4.18, then the induced map $q' : \mathrm{Alg}_{/\mathcal{O}}^{S}(\mathcal{C}) \to S$ is a coCartesian fibration. We will refer to a section $A : S \to \mathrm{Alg}_{/\mathcal{O}}^{S}(\mathcal{C})$ of q' as an S-*family of $\mathcal{O}$-algebra objects of* $\mathcal{C}$. We will say that an S-family of $\mathcal{O}$-algebra objects of $\mathcal{C}$ is *coCartesian* if A carries each morphism in S to a q'-coCartesian morphism in $\mathrm{Alg}_{/\mathcal{O}}^{S}(\mathcal{C})$.

Definition 6.1.4.20. Let $\mathcal{O}^{\otimes}$ be a coherent ∞-operad, let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \times S$ be a coCartesian S-family of $\mathcal{O}$-operads, and let $\mathcal{C}_0^{\otimes}$ denote the product $\mathcal{O}^{\otimes} \times S$. If A is an S-family of algebra objects of $\mathcal{C}$, we let $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$ denote the fiber product

$$\overline{\mathcal{M}\mathrm{od}}^{\mathcal{O}}(\mathcal{C})^{\otimes} \times_{\mathrm{^pAlg}_{\mathcal{O}}(\mathcal{C})} S,$$

where $\overline{\mathcal{M}\mathrm{od}}^{\mathcal{O}}(\mathcal{C})$ and $\mathrm{^pAlg}_{\mathcal{O}}(\mathcal{C})$ are defined as in §3.3.3 and the map $S \to \mathrm{^pAlg}_{\mathcal{O}}(\mathcal{C})$ is determined by A. We let $\mathrm{Mod}_A^{\mathcal{O},S}(\mathcal{C})$ denote the fiber product $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}) \times_{\mathrm{Mod}_{qA}^{\mathcal{O}}(\mathcal{C}_0)} \mathcal{C}_0$.

Remark 6.1.4.21. Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \times S$ be as in Definition 6.1.4.20 and A is an S-family of ∞-operads. Then the ∞-category $\mathrm{Mod}_A^{\mathcal{O},S}(\mathcal{C})^{\otimes}$ is equipped with an evident forgetful functor $\mathrm{Mod}_A^{\mathcal{O},S}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes} \times S$. For every object $s \in S$, the fiber $\mathrm{Mod}_A^{\mathcal{O},S}(\mathcal{C})_s^{\otimes} = \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes} \times_S \{s\}$ is canonically isomorphic to the ∞-operad $\mathrm{Mod}_{A_s}^{\mathcal{O}}(\mathcal{C}_s)^{\otimes}$ defined in §3.3.3.

We will need the following technical result, whose proof will be given at the end of this section.

Proposition 6.1.4.22. *Let* $\mathcal{O}^{\otimes}$ *be a coherent* ∞-*operad,* $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \times S$ *a coCartesian* S-*family of* $\mathcal{O}$-*operads, and* $A \in \mathrm{Fun}_S(S, \mathrm{Alg}_{/\mathcal{O}}^{S}(\mathcal{C}))$ *a coCartesian* S-*family of* $\mathcal{O}$-*algebras. Then:*

(1) *The forgetful functor* $q' : \mathrm{Mod}_A^{\mathcal{O},S}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes} \times S$ *is a coCartesian* S-*family of* ∞-*operads.*

(2) *Let* $\overline{f}$ *be a morphism in* $\mathrm{Mod}_A^{\mathcal{O},S}(\mathcal{C})^{\otimes}$ *whose image in* $\mathcal{O}^{\otimes}$ *is degenerate. Then* $\overline{f}$ *is* q'-*coCartesian if and only if its image in* $\mathcal{C}^{\otimes}$ *is* q-*coCartesian.*

Remark 6.1.4.23. In the situation of Proposition 6.1.4.22, suppose that $\mathcal{O}^{\otimes}$ is the 0-cubes operad $\mathbb{E}_0$. Let $\mathcal{C}$ denote the fiber product $\mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{O}$. Then the forgetful functor $\theta : \mathrm{Mod}_A^{\mathcal{O},S}(\mathcal{C}) \to \mathcal{C}$ is a trivial Kan fibration. To prove this, it suffices to show that θ is a categorical equivalence (since it is evidently a categorical fibration). According to Corollary T.2.4.4.4, it suffices to show that θ induces a categorical equivalence after passing to the fiber over each vertex of S, which follows from Proposition 3.3.3.19.

Suppose now that $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \times S$ is a coCartesian S-family of $\mathcal{O}$-operads and that A is a coCartesian S-family of $\mathcal{O}$-algebra objects of $\mathcal{C}$. Then A determines an S-family of $\mathcal{O}$-algebra objects of $\mathrm{Mod}_A^{\mathcal{O},S}(\mathcal{C})$, which we will denote also by A. Note that, for each $s \in S$, $A_s \in \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Mod}_{A_s}^{\mathcal{O}}(\mathcal{C}_s))$ is a trivial algebra and

therefore initial in $\mathrm{Alg}_{/\,\mathbb{O}}(\mathrm{Mod}_{A_s}^{\mathbb{O}}(\mathcal{C}_s))$ (Proposition 3.2.1.8). Let $\mathrm{Alg}_{/\,\mathbb{O}}^{S}(\mathcal{C})^{A_S/}$ be defined as in §T.4.2.2 and let $\mathrm{Alg}_{/\,\mathbb{O}}^{S}(\mathrm{Mod}_A^{\mathbb{O},S}(\mathcal{C}))^{A_S/}$ be defined similarly. We have a commutative diagram

$$\mathrm{Alg}_{/\,\mathbb{O}}^{S}(\mathrm{Mod}_A^{\mathbb{O},S}(\mathcal{C}))^{A_S/} \xrightarrow{\ \theta\ } \mathrm{Alg}_{/\,\mathbb{O}}^{S}(\mathcal{C})^{A_S/}$$
$$S.$$

The vertical maps are coCartesian fibrations and θ preserves coCartesian morphisms. Using Corollary 3.4.1.7, we deduce that θ induces a categorical equivalence after passing to the fiber over each object of S. Applying Corollary T.2.4.4.4, we deduce the following:

Proposition 6.1.4.24. *Let $\mathbb{O}^{\otimes}$ be a coherent ∞-operad, $q : \mathcal{C}^{\otimes} \to \mathbb{O}^{\otimes} \times S$ a coCartesian S-family of $\mathbb{O}$-operads, and A a coCartesian S-family of $\mathbb{O}$-algebra objects of $\mathcal{C}$. Then the forgetful functor*

$$\theta : \mathrm{Alg}_{\mathbb{O}}^{S}(\mathrm{Mod}_A^{\mathbb{O},S}(\mathcal{C}))^{A_S/} \to \mathrm{Alg}_{/\,\mathbb{O}}^{S}(\mathcal{C})^{A_S/}$$

is an equivalence of ∞-categories.

Proposition 6.1.4.24 provides a mechanism for reducing questions about centralizers of arbitrary algebra morphisms $f : A \to B$ to the special case where A is a trivial algebra.

Remark 6.1.4.25. Let $\mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a coCartesian fibration of ∞-operads which exhibits an ∞-category $\mathcal{M} = \mathcal{C}_{\mathfrak{m}}$ as left-tensored over the monoidal ∞-category $\mathcal{C}_{\mathfrak{a}}$. Let $\mathbf{1}$ denote the unit object of $\mathcal{C}_{\mathfrak{a}}$, let $f : M_0 \to M$ be a morphism in $\mathcal{M}$, and consider the fiber product $\mathcal{X} = \mathcal{C}_{\mathfrak{a}} \times_{\mathcal{M}} \mathcal{M}_{/M}$, where the map $\mathcal{C}_{\mathfrak{a}} \to \mathcal{M}$ is given by tensor product with M_0. We will identify the tensor product $\mathbf{1} \otimes M_0$ with M_0, so that the pair $(\mathbf{1}, f : M_0 \to M)$ can be identified with an object $X \in \mathcal{X}$. The undercategory $\mathcal{X}_{X/}$ can be identified with the fiber product $(\mathcal{C}_{\mathfrak{a}})_{\mathbf{1}/} \times_{\mathcal{M}_{M_0/}} \mathcal{M}_{M_0//M}$. Assume that the ∞-category $\mathcal{X}$ has a final object (in other words, there exists a morphism object $\mathrm{Mor}_{\mathcal{M}}(M_0, M)$ in $\mathcal{C}_{\mathfrak{a}}$). Using Proposition T.1.2.13.8, we deduce that the forgetful functor $\mathcal{X}_{X/} \to \mathcal{X}$ induces an equivalence between the full subcategories spanned by the final objects of $\mathcal{X}_{X/}$ and $\mathcal{X}$. In other words:

(i) A map $\epsilon : \mathbf{1} \to Z$ in $\mathcal{C}_{\mathfrak{a}}$ together with a commutative diagram

$$\begin{array}{ccc} & Z \otimes M_0 & \\ {\scriptstyle \epsilon \otimes \mathrm{id}_{M_0}}\nearrow & & \searrow {\scriptstyle g} \\ M_0 & \xrightarrow{\ f\ } & M \end{array}$$

in $\mathcal{M}$ is a centralizer of f if and only if the underlying morphism g exhibits Z as a morphism object $\mathrm{Mor}_{\mathcal{M}}(M_0, M)$.

(ii) For any object $Z \in \mathcal{A}_{\mathfrak{a}}$ and any morphism $Z \otimes M_0 \to M$ which exhibits Z as a morphism object $\mathrm{Mor}_{\mathcal{M}}(M_0, M)$, there exists a map $\mathbf{1} \to Z$ and a commutative diagram

$$\begin{array}{ccc} & Z \otimes M_0 & \\ {\scriptstyle \epsilon \otimes \mathrm{id}_{M_0}}\nearrow & & \searrow {\scriptstyle g} \\ M_0 & \xrightarrow{\ f\ } & M \end{array}$$

satisfying the conditions of (i).

Proposition 6.1.4.26. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \times \mathcal{LM}^{\otimes}$ be a coCartesian $\mathcal{LM}^{\otimes}$-family of $\mathcal{O}^{\otimes}$-operads, and assume that the induced map $\mathrm{Alg}^{\mathcal{LM}^{\otimes}}_{/\,\mathcal{O}}(\mathcal{C}) \to \mathcal{LM}^{\otimes}$ is a coCartesian fibration of ∞-operads (this is automatic if, for example, the map $\mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ is a coCartesian fibration of ∞-operads). For every object $X \in \mathcal{O}^{\otimes}$, we let $\mathcal{C}_{X,\mathfrak{a}}$ denote the fiber of q over the vertex $(X, \mathfrak{a})$, and define $\mathcal{C}_{X,\mathfrak{m}}$ similarly. Let $f : A_0 \to A$ be a morphism in $\mathrm{Alg}^{\mathcal{LM}^{\otimes}}_{/\,\mathcal{O}}(\mathcal{C})_{\mathfrak{m}}$. Assume that:*

(i) *The ∞-operad $\mathcal{O}^{\otimes}$ is unital.*

(ii) *The algebra object A_0 is trivial (see §3.2.1).*

(iii) *For every object $X \in \mathcal{O}^{\otimes}$, there exists a morphism object $\mathrm{Mor}_{\mathcal{C}_{X,\mathfrak{m}}}(A_0(X), A(X)) \in \mathcal{C}_{X,\mathfrak{a}}$.*

Then:

(1) *There exists a centralizer $\mathfrak{Z}_f (\in) \mathrm{Alg}^{\mathcal{LM}^{\otimes}}_{\mathcal{O}}(\mathcal{C})_{\mathfrak{a}}$.*

(2) *Let $Z \in \mathrm{Alg}^{\mathcal{LM}^{\otimes}}_{/\,\mathcal{O}}(\mathcal{C})_{\mathfrak{a}}$ be an algebra object. Then a commutative diagram*

exhibits Z as a centralizer of f if and only if, for every object $X \in \mathcal{O}$, the induced map $g_X : \mathfrak{Z}(X) \otimes A_0(X) \to A(X)$ exhibits $\mathfrak{Z}(X)$ as a morphism object $\mathrm{Mor}_{\mathcal{C}_{X,\mathfrak{m}}}(A_0(X), A(X))$.

Proof. Let $\mathbf{1} \in \mathrm{Alg}^{\mathcal{LM}^{\otimes}}_{/\,\mathcal{O}}(\mathcal{C})_{\mathfrak{a}}$ be a trivial algebra; we will abuse notation by identifying the tensor product $\mathbf{1} \otimes A_0$ with A_0. To prove (1), we must show that the ∞-category

$$\mathcal{A} = (\mathrm{Alg}^{\mathcal{LM}^{\otimes}}_{/\,\mathcal{O}}(\mathcal{C})_{\mathfrak{a}})_{\mathbf{1}/} \times_{(\mathrm{Alg}^{\mathcal{LM}^{\otimes}}_{/\,\mathcal{O}}(\mathcal{C})_{\mathfrak{m}})_{A_0/}} (\mathrm{Alg}^{\mathcal{LM}^{\otimes}}_{/\,\mathcal{O}}(\mathcal{C})_{\mathfrak{m}})_{A_0//A}$$

has a final object. Let $\mathcal{C}^{\otimes}_{\mathfrak{a}}$ denote the fiber product $\mathcal{C}^{\otimes} \times_{\mathcal{LM}^{\otimes}} \{\mathfrak{a}\}$, define $\mathcal{C}^{\otimes}_{\mathfrak{m}}$ similarly, and set

$$\mathcal{E}^{\otimes} = (\mathcal{C}^{\otimes}_{\mathfrak{a}})_{\mathbf{1}_{\mathcal{O}}/} \times_{(\mathcal{C}^{\otimes}_{\mathfrak{m}})_{A_{0\,\mathcal{O}}/}} (\mathcal{C}^{\otimes}_{\mathfrak{m}})_{A_{0\,\mathcal{O}}//A_{\mathcal{O}}}$$

(see §2.2.2 for an explanation of this notation). Using Theorem 2.2.2.4 (and assumption (ii)), we deduce that the evident forgetful functor $\mathcal{E}^{\otimes} \to \mathcal{O}^{\otimes}$ is a coCartesian fibration of ∞-operads; moreover, we have a canonical isomorphism $\mathcal{A} \simeq \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{E})$. For each object $X \in \mathcal{O}^{\otimes}$, the ∞-category $\mathcal{E}_X = \mathcal{E}^{\otimes} \times_{\mathcal{O}^{\otimes}} \{X\}$ is equivalent to the fiber product

$$(\mathcal{C}_{X,\mathfrak{a}})_{\mathbf{1}(X)/} \times_{(\mathcal{C}_{X,\mathfrak{m}})_{A_0(X)/}} (\mathcal{C}_{X,\mathfrak{m}})_{A_0(X)//A(X)},$$

which has a final object by virtue of assumption (iii) and Remark 6.1.4.25. It follows that $\mathcal{A}$ has a final object; moreover, an object $A \in \mathcal{A} \simeq \mathrm{Alg}_{/\,\mathcal{O}}(\mathcal{E})$ is final if and only if each $A(X)$ is a final object of $\mathcal{E}_X$. This proves (1), and reduces assertion (2) to the contents of Remark 6.1.4.25. $\square$

We now apply Proposition 6.1.4.26 to the study of centralizers in general. Fix a coherent ∞-operad $\mathcal{O}^{\otimes}$, a bifunctor of ∞-operads $\mathcal{O}^{\otimes} \times \mathcal{LM}^{\otimes} \to \mathcal{D}^{\otimes}$, and a coCartesian fibration of ∞-operads $q : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$. Let $A \in \mathrm{Alg}_{\mathcal{O}/\mathcal{D}}(\mathcal{C})_{\mathfrak{m}}$, and let $\overline{A} \in \mathrm{LMod}(\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}(\mathcal{C}))$ be such that $\overline{A}_{\mathfrak{m}} = A$ and $\overline{A}_{\mathfrak{a}}$ is a trivial algebra. We can regard $\overline{A}$ as a coCartesian $\mathcal{LM}^{\otimes}$-family of $\mathcal{O}$-algebra objects of $\mathcal{C}^{\otimes} \times_{\mathcal{D}^{\otimes}} (\mathcal{O}^{\otimes} \times \mathcal{LM}^{\otimes})$. Let $\overline{\mathcal{C}}^{\otimes} = \mathrm{Mod}^{\mathcal{O},\mathcal{LM}^{\otimes}}_{\overline{A}}(\mathcal{C})$ be the coCartesian S-family of $\mathcal{O}$-operads given by Proposition 6.1.4.22. Since $\overline{A}_{\mathfrak{a}}$ is trivial, the forgetful functor $\overline{\mathcal{C}}^{\otimes}_{\mathfrak{a}} \to \mathcal{C}^{\otimes}_{\mathfrak{a}} = \mathcal{C}^{\otimes} \times_{\mathcal{D}^{\otimes}} (\mathcal{O}^{\otimes} \times \{\mathfrak{a}\})$ is an equivalence of $\mathcal{O}$-operads, and induces

an equivalence of ∞-categories $\mathrm{Alg}_{/\,\mathcal{O}}(\overline{\mathcal{C}})_{\mathfrak{a}} \to \mathrm{Alg}_{\mathcal{O}\,/\,\mathcal{D}}(\mathcal{C})_{\mathfrak{a}}$. It follows from Proposition 6.1.4.24 that every morphism $f : A \to B$ in $\mathrm{Alg}_{\mathcal{O}\,/\,\mathcal{D}}(\mathcal{C})_{\mathfrak{m}}$ is equivalent to $\theta(f')$, where $f' : A \to B'$ is a morphism in $\mathrm{Alg}_{/\,\mathcal{O}}(\overline{\mathcal{C}})_{\mathfrak{m}}$; here we abuse notation by identifying A with the associated trivial $\mathcal{O}$-algebra object of $\overline{\mathcal{C}}_{\mathfrak{m}}^{\otimes}$. It follows from Proposition 6.1.4.24 that the forgetful functor θ induces an identification between centralizers of f in $\mathrm{Alg}_{\mathcal{O}\,/\,\mathcal{D}}(\mathcal{C})_{\mathfrak{a}}$ and centralizers of f' in $\mathrm{Alg}_{/\,\mathcal{O}}(\overline{\mathcal{C}})_{\mathfrak{a}}$. Combining this observation with Proposition 6.1.4.26, we obtain the following result:

Theorem 6.1.4.27. *Let $\mathcal{O}^{\otimes}$ be a coherent ∞-operad, $\mathcal{O}^{\otimes} \times \mathcal{LM}^{\otimes} \to \mathcal{D}^{\otimes}$ a bifunctor of ∞-operads, and $q : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ a coCartesian fibration of ∞-operads. Let $f : A \to B$ be a morphism in $\mathrm{Alg}_{\mathcal{O}\,/\,\mathcal{D}}(\mathcal{C})_{\mathfrak{m}}$ and let $\overline{\mathcal{C}}^{\otimes}$ and $f' : A \to B'$ be defined as above. Assume that:*

$(*)$ *For every object $X \in \mathcal{O}$, there exists a morphism object $\mathrm{Mor}_{\overline{\mathcal{C}}_{X,\mathfrak{m}}}(A(X), B'(X)) \in \overline{\mathcal{C}}_{X,\mathfrak{a}}$.*

Then:

(1) *There exists a centralizer $\mathfrak{Z}(f) \in \mathrm{Alg}_{\mathcal{O}\,/\,\mathcal{D}}(\mathcal{C})_{\mathfrak{a}}$.*

(2) *Let Z be an arbitrary object of $\mathrm{Alg}_{\mathcal{O}\,/\,\mathcal{D}}(\mathcal{C})_{\mathfrak{a}}$, and let σ :*

$$
\begin{array}{ccc}
 & Z \otimes A & \\
 \nearrow & & \searrow{\scriptstyle g} \\
 A & \xrightarrow{\quad f \quad} & B
\end{array}
$$
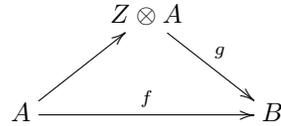
be a commutative diagram in $\mathrm{Alg}_{\mathcal{O}\,/\,\mathcal{D}}(\mathcal{C})_{\mathfrak{m}}$. Let Z' be a preimage of Z in $\mathrm{Alg}_{\mathcal{O}}(\overline{\mathcal{C}})_{\mathfrak{a}}$, so that σ lifts (up to homotopy) to a commutative diagram
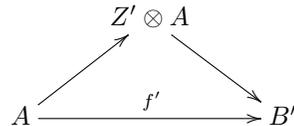
$$
\begin{array}{ccc}
 & Z' \otimes A & \\
 \nearrow & & \searrow \\
 A & \xrightarrow{\quad f' \quad} & B'
\end{array}
$$

in $\mathrm{Alg}_{/\,\mathcal{O}}(\overline{\mathcal{C}})_{\mathfrak{m}}$. Then σ exhibits Z as a centralizer of f if and only if, for every object $X \in \mathcal{O}$, the induced map $Z'(X) \otimes A(X) \to B'(X)$ exhibits $Z'(X)$ as a morphism object $\mathrm{Mor}_{\overline{\mathcal{C}}_{X,\mathfrak{m}}}(A(X), B'(X)) \in \overline{\mathcal{C}}_{X,\mathfrak{a}}$.

Corollary 6.1.4.28. *In the situation of Theorem 6.1.4.27, suppose that $\mathcal{O}^{\otimes}$ is the 0-cubes ∞-operad $\mathbb{E}_0$. Then we can identify centralizers of a morphism $f : A \to B$ in $\mathrm{Alg}_{\mathcal{O}\,/\,\mathcal{D}}(\mathcal{C})_{\mathfrak{m}}$ with morphism objects $\mathrm{Mor}_{\mathcal{C}_{\mathfrak{m}}}(A, B)$ in $\mathcal{C}_{\mathfrak{a}}$.*

Proof. Combine Theorem 6.1.4.27 with Remark 6.1.4.23. $\square$

Remark 6.1.4.29. More informally, we can state Theorem 6.1.4.27 as follows: the centralizer of a morphism $f : A \to B$ can be identified with the classifying object for A-module maps from A to B. In particular, the center $\mathfrak{Z}(A)$ can be identified with the endomorphism algebra of A, regarded as a module over itself.

We now return to the proof of our main result.

Proof of Theorem 6.1.4.12. Combine Theorem 6.1.4.27, Proposition 4.2.1.33, and Theorem 3.4.4.2. $\square$

We conclude this section with the proof of Proposition 6.1.4.22. First, we need a lemma.

Lemma 6.1.4.30. *Let $n \geq 2$, and let $\mathcal{C} \to \Delta^n$ be an inner fibration of ∞-categories. Let $q : \mathcal{D} \to \mathcal{E}$ be another inner fibration of ∞-categories. Every lifting problem of the form*

$$
\begin{array}{ccc}
\Lambda_0^n \times_{\Delta^n} \mathcal{C} & \xrightarrow{g} & \mathcal{D} \\
\downarrow & \nearrow & \downarrow q \\
\mathcal{C} & \longrightarrow & \mathcal{E}
\end{array}
$$

admits a solution, provided that $g|\Delta^{\{0,1\}} \times_{\Delta^n} \mathcal{C}$ is a q-left Kan extension of $g|\{0\} \times_{\Delta^n} \mathcal{C}$.

Proof. We first define a map $r : \Delta^n \times \Delta^1 \to \Delta^n$, which is given on vertices by the formula

$$
r(i,j) = \begin{cases} 0 & \text{if } (i,j) = (1,0) \\ i & \text{otherwise,} \end{cases}
$$

and let $j : \Delta^n \to \Delta^n \times \Delta^1$ be the map (id, j_0), where j_0 carries the first two vertices of Δ^n to $\{0\} \subseteq \Delta^1$ and the remaining vertices to $\{1\} \subseteq \Delta^1$.

Let $K = (\Lambda_0^n \times \Delta^1) \coprod_{\Lambda_0^n \times \{0\}} (\Delta^n \times \{0\})$, let $\mathcal{C}' = (\Delta^n \times \Delta^1) \times_{\Delta^n} \mathcal{C}$, and let $\mathcal{C}_0' = K \times_{\Delta^n} \mathcal{C}$. We will show that there exists a solution to the lifting problem

$$
\begin{array}{ccc}
\mathcal{C}_0' & \longrightarrow & \mathcal{D} \\
g' \downarrow & \nearrow & \downarrow q \\
\mathcal{C}' & \longrightarrow & \mathcal{E} .
\end{array}
$$

Composing this solution with the map $\mathcal{C} \to \mathcal{C}'$ induced by j, we will obtain the desired result.

For every simplicial subset $L \subseteq \Delta^n$, let $\mathcal{C}_L'$ denote the fiber product

$$
((L \times \Delta^1) \coprod_{L \times \{0\}} (\Delta^n \times \{0\})) \times_{\Delta^n} \mathcal{C},
$$

and let X_L denote the full subcategory of $\mathrm{Fun}_{\mathcal{E}}(\mathcal{C}_L', \mathcal{D}) \times_{\mathrm{Fun}_{\mathcal{E}}(\mathcal{C}_\emptyset', \mathcal{D})} \{g'|\mathcal{C}_\emptyset'\}$ spanned by those functors F with the following property: for each vertex $v \in L$, the restriction of F to $(\{v\} \times \Delta^1) \times_{\Delta^n} \mathcal{C}$ is a q-left Kan extension of $F|(\{v\} \times \{0\}) \times_{\Delta^n} \mathcal{C}$.

To complete the proof, it will suffice to show that the restriction map $X_{\Delta^n} \to X_{\Lambda_0^n}$ is surjective on vertices. We will prove the following stronger assertion:

(∗) For every inclusion $L' \subseteq L$ of simplicial subsets of Δ^n, the restriction map $\theta_{L',L} : X_L \to X_{L'}$ is a trivial Kan fibration.

The proof proceeds by induction on the number of nondegenerate simplices of L. If $L' = L$, then $\theta_{L',L}$ is an isomorphism and there is nothing to prove. Otherwise, choose a nondegenerate simplex σ of L which does not belong to L', and let L_0 be the simplicial subset of L obtained by removing σ. The inductive hypothesis guarantees that the map θ_{L',L_0} is a trivial Kan fibration. Consequently, to show that $\theta_{L',L}$ is a trivial Kan fibration, it will suffice to show that $\theta_{L_0,L}$ is a trivial Kan fibration. Note that $\theta_{L_0,L}$ is a pullback of the map $\theta_{\partial \sigma, \sigma}$: we may therefore assume without loss of generality that $L = \sigma$ is a simplex of Δ^n.

Since the map $\theta_{L',L}$ is evidently a categorical fibration, it will suffice to show that each $\theta_{L',L}$ is a categorical equivalence. We may assume by the inductive hypothesis that $\theta_{\emptyset,L'}$ is a categorical equivalence. By a two-out-of-three argument, we may reduce to the problem of showing that $\theta_{\emptyset,L'} \circ \theta_{L',L} = \theta_{\emptyset,L}$ is a categorical equivalence. In other words, we may assume that L' is empty. We are now reduced to the problem of showing that the map $X_\sigma \to X_\emptyset$ is a trivial Kan fibration, which follows from Proposition T.4.3.2.15. $\square$

Proof of Proposition 6.1.4.22. It follows from Remark 3.3.3.15 that q' is a categorical fibration and the induced map $\mathrm{Mod}_A^{\mathcal{O},S}(\mathcal{C})^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times S$ exhibits $\mathrm{Mod}_A^{\mathcal{O},S}(\mathcal{C})^{\otimes}$ as an S-family of ∞-operads (note that the projection $\mathrm{Mod}_A^{\mathcal{O},S}(\mathcal{C})^{\otimes} \to \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ is an equivalence of ∞-categories). To complete the proof of (1), it will suffice to show that q' satisfies condition (iii) of Definition 6.1.4.16. That is, we must show that if M is an object of $\mathrm{Mod}_A^{\mathcal{O},S}(\mathcal{C})^{\otimes}$ having image (X, s) in $\mathcal{O}^{\otimes} \times S$ and $f : s \to s'$ is a morphism in S, then (id_X, f) can be lifted to a q'-coCartesian morphism $M \to M'$ in $\mathrm{Mod}_A^{\mathcal{O},S}(\mathcal{C})^{\otimes}$. Let $\mathcal{A}$ be the full subcategory of $(\mathcal{O}^{\otimes})^{X/}$ spanned by the semi-inert morphisms $X \to Y$ in $\mathcal{O}^{\otimes}$, and let $\mathcal{A}_0$ be the full subcategory of $\mathcal{A}$ spanned by the null morphisms. The object $M \in \mathrm{Mod}_A^{\mathcal{O},S}(\mathcal{C})^{\otimes}$ determines a functor $F : \mathcal{A} \to \mathcal{C}^{\otimes}$. Let F_0 denote the composite map $\mathcal{A} \xrightarrow{F} \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$. Since q exhibit $\mathcal{C}^{\otimes}$ as a coCartesian S-family of ∞-operads, there exists a q-coCartesian natural transformation $H : \mathcal{A} \times \Delta^1 \to \mathcal{C}^{\otimes}$ from F to another map F', such that $q \circ H$ is the product map $\mathcal{A} \times \Delta^1 \xrightarrow{F_0 \times f} \mathcal{O}^{\otimes} \times S$. Let $H' : \mathcal{A}_0 \times \Delta^1 \to \mathcal{C}^{\otimes}$ be the composition $\mathcal{A}_0 \times \Delta^1 \to \mathcal{O}^{\otimes} \times S \xrightarrow{A} \mathcal{C}^{\otimes}$. Since A is a coCartesian S-family of $\mathcal{O}$-algebras, the functors $H|\mathcal{A}_0 \times \Delta^1$ and H' are equivalent; we may therefore modify H by a homotopy (fixed on $\mathcal{A} \times \{0\}$) and thereby assume that $H|\mathcal{A}_0 \times \Delta^1 = H'$, so that H determines a morphism $\alpha : M \to M'$ in $\mathrm{Mod}_A^{\mathcal{O},S}(\mathcal{C})^{\otimes}$ lying over (id_X, f). To complete the proof of (1), it will suffice to show that α is q'-coCartesian.

Let $\mathcal{C}_0^{\otimes} = \mathcal{O}^{\otimes} \times S$. We have a commutative diagram of ∞-categories

$$
\begin{array}{ccc}
\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes} & \xrightarrow{\;r\;} & \overline{\mathcal{M}\mathrm{od}}^{\mathcal{O}}(\mathcal{C})^{\otimes} \times_{\overline{\mathcal{M}\mathrm{od}}^{\mathcal{O}}(\mathcal{C}_0)^{\otimes}} \mathcal{C}_0^{\otimes} \\
\downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle p'} \qquad \searrow{\scriptstyle p} \\
\mathcal{O}^{\otimes} \times S & \longrightarrow & \mathcal{O}^{\otimes} \times {}^{\mathrm{P}}\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \times_{{}^{\mathrm{P}}\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}_0)} \mathcal{C}_0^{\otimes} \xrightarrow{\;p''\;} \mathcal{C}_0^{\otimes}
\end{array}
$$

Since the upper square is a pullback diagram, it will suffice to show that $r(\alpha)$ is p'-coCartesian. In view of Proposition T.2.4.1.3, we are reduced to showing that $r(\alpha)$ is p-coCartesian and that $(p' \circ r)(\alpha)$ is p''-coCartesian.

To prove that $r(\alpha)$ is p-coCartesian, we must show that every lifting problem of the form

$$
\begin{array}{ccc}
\Lambda_0^n & \xrightarrow{\;g\;} & \overline{\mathcal{M}\mathrm{od}}^{\mathcal{O}}(\mathcal{C})^{\otimes} \times_{\overline{\mathcal{M}\mathrm{od}}^{\mathcal{O}}(\mathcal{C})^{\otimes}} \mathcal{C}_0 \\
\downarrow & \nearrow & \downarrow \\
\Delta^n & \longrightarrow & \mathcal{C}_0^{\otimes}
\end{array}
$$

admits a solution, provided that $n \geq 2$ and that g carries the initial edge of Λ_0^n to the morphism determined by H. Unwinding the definitions, this amounts to solving a lifting problem of the form

$$
\begin{array}{ccc}
\Lambda_0^n \times_{\mathcal{O}^{\otimes}} \mathcal{K}_{\mathcal{O}} & \xrightarrow{\;G\;} & \mathcal{C}^{\otimes} \\
\downarrow & \nearrow & \downarrow \\
\Delta^n \times_{\mathcal{O}^{\otimes}} \mathcal{K}_{\mathcal{O}} & \longrightarrow & \mathcal{O}^{\otimes} \times S.
\end{array}
$$

The existence of a solution to this lifting problem is guaranteed by Lemma 6.1.4.30. The assertion that $(p' \circ r)(\alpha)$ is p''-coCartesian can be proven in the same way. This completes the proof of (1).

Let $\overline{f} : M \to M''$ be as in (2), let f be the image of $\overline{f}$ in S, and let $\widetilde{f} : M \to M'$ be the q'-coCartesian map constructed above. We have a commutative diagram

$$
\begin{array}{ccc}
 & M' & \\
\widetilde{f} \nearrow & & \searrow g \\
M & \xrightarrow[\overline{f}]{} & M''.
\end{array}
$$

Let $\theta : \mathrm{Mod}_{\widetilde{A}}^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$ be the forgetful functor. By construction, $\theta(\overline{f})$ is a q-coCartesian morphism in $\mathcal{C}^{\otimes}$, so that $\theta(\widetilde{f})$ is q-coCartesian if and only if $\theta(g)$ is an equivalence. We note that $\overline{f}$ is q'-coCartesian if and only if the map g is an equivalence. The "only if" direction of (2) is now obvious, and the converse follows from Remark 6.1.4.21 together with Corollary 3.4.3.4. $\qquad\square$

6.2 Monads and the Barr-Beck Theorem

Suppose given a pair of adjoint functors $\mathcal{C}\mathrel{\substack{\xrightarrow{F}\\ \xleftarrow[G]{}}}\mathcal{D}$ between ordinary categories. Then:

(A) The composition $T = G \circ F$ has the structure of a *monad* on $\mathcal{C}$; that is, an algebra object of the category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ of endofunctors of $\mathcal{C}$. Here the unit map $\mathrm{id}_{\mathcal{C}} \to T$ given by the unit of the adjunction between F and G, and the product is given by the composition

$$T \circ T = G \circ (F \circ G) \circ F \to G \circ \mathrm{id}_{\mathcal{D}} \circ F = T$$

where the second map is given by a compatible counit v for the adjunction between F and G.

(B) For every object $D \in \mathcal{D}$, the object $G(D)$ has the structure of a module over the monad T, given by the map $TG(D) = ((G \circ F) \circ G)(D) = (G \circ (F \circ G))(D) \xrightarrow{v} G(D)$. This construction determines a functor θ from $\mathcal{D}$ to the category of T-modules in $\mathcal{C}$.

(C) In many cases, the functor θ is an equivalence of categories. The *Barr-Beck theorem* provides necessary and sufficient conditions on the functor G to guarantee that this is the case. We refer the reader to [99] for a detailed statement (or to Theorem 6.2.2.5 for our ∞-categorical version, which subsumes the classical statement).

Our goal in this section is to obtain ∞-categorical generalizations of assertions (A) through (C). Our first step is to define the notion of a *monad* on an ∞-category $\mathcal{M}$. We begin by defining a monoidal structure on the ∞-category $\mathrm{End}(\mathcal{M}) = \mathrm{Fun}(\mathcal{M}, \mathcal{M})$.

Notation 6.2.0.1. Let $\mathcal{M}$ be a simplicial set, and let **LM** be the colored operad introduced in §4.2.1.1. Composition of functors endows $\mathrm{Fun}(\mathcal{M}, \mathcal{M})$ with the structure of a simplicial monoid, and $\mathcal{M}$ can be regarded as a left module over $\mathrm{Fun}(\mathcal{M}, \mathcal{M})$. Let **LM** be the colored operad of Definition 4.2.1.1. The action of $\mathrm{Fun}(\mathcal{M}, \mathcal{M})$ on $\mathcal{M}$ determines a **LM**-algebra object E in the ordinary category Set_{Δ}, given by $E(\mathfrak{m}) = \mathcal{M}$ and $E(\mathfrak{a}) = \mathrm{Fun}(\mathcal{M}, \mathcal{M})$. We will identify E with a functor $\mathbf{LM}^{\otimes} \to \mathrm{Set}_{\Delta}$. We let $\overline{\mathrm{End}}(\mathcal{M})^{\otimes}$ denote the relative nerve $\mathrm{N}_E(\mathbf{LM}^{\otimes})$, so that there is an evident forgetful map $\overline{\mathrm{End}}(\mathcal{M})^{\otimes} \to \mathcal{LM}^{\otimes}$. We let $\mathrm{End}(\mathcal{M})^{\otimes}$ denote the fiber product $\mathcal{Ass}^{\otimes} \times_{\mathcal{LM}^{\otimes}} \overline{\mathrm{End}}(\mathcal{M})^{\otimes}$.

Proposition 6.2.0.2. *Let $\mathcal{M}$ be an ∞-category. Then:*

(1) *The map $p : \mathrm{End}(\mathcal{M})^{\otimes} \to \mathcal{Ass}^{\otimes}$ is a coCartesian fibration of ∞-operads, which exhibits $\mathrm{End}(\mathcal{M}) = \mathrm{End}(\mathcal{M})_{\langle 1 \rangle}^{\otimes} \simeq \mathrm{Fun}(\mathcal{M}, \mathcal{M})$ as a monoidal ∞-category.*

(2) *The map $q : \overline{\mathrm{End}}(\mathcal{M})^{\otimes} \to \mathcal{LM}^{\otimes}$ is a coCartesian fibration of ∞-operads, which exhibits $\mathcal{M} \simeq \overline{\mathrm{End}}(\mathcal{M})_{\mathfrak{m}}$ as an ∞-category left-tensored over $\mathrm{End}(\mathcal{M})^{\otimes}$.*

Proof. Proposition T.3.2.5.21 implies that p and q are coCartesian fibrations. The desired result now follows from the observation that for each object $(\langle n \rangle, S)$ in $\mathcal{LM}^{\otimes}$, the fiber $q^{-1}\{(\langle n \rangle, S)\}$ is isomorphic to the product $(\prod_{i \in S} \mathcal{M}) \times (\prod_{i \in \langle n \rangle - S} \mathrm{Fun}(\mathcal{M}, \mathcal{M}))$, and that these identifications are compatible with the functors associated inert morphisms in $\mathcal{LM}^{\otimes}$. $\qquad\square$

We will refer to the monoidal structure on $\mathrm{Fun}(\mathcal{M}, \mathcal{M})$ supplied by Proposition 6.2.0.2 as the *composition monoidal structure*.

Definition 6.2.0.3. Let $\mathcal{M}$ be an ∞-category. A *monad* on $\mathcal{M}$ is an algebra object of $\mathrm{Fun}(\mathcal{M}, \mathcal{M})$ (with respect to the composition monoidal structure). If T is a monad on $\mathcal{M}$, we let $\mathrm{LMod}_T(\mathcal{M})$ denote the associated ∞-category of (left) T-modules in $\mathcal{M}$.

Remark 6.2.0.4. More informally, a monad on an ∞-category $\mathcal{M}$ consists of an endofunctor $T : \mathcal{M} \to \mathcal{M}$ equipped with maps $1 \to T$ and $T \circ T \to T$ which satisfy the usual unit and associativity conditions, up to coherent homotopy. A *T-module* is then an object $M \in \mathcal{M}$ equipped with a structure map $T(M) \to M$ which is compatible with the algebra structure on T, again up to coherent homotopy.

Remark 6.2.0.5. Let $\mathcal{M}$ be an ∞-category, which we regard as an object of Cat_∞, and regard Cat_∞ as endowed with the Cartesian monoidal structure. Giving a monoidal ∞-category $\mathcal{C}$ over which $\mathcal{M}$ is left-tensored is equivalent to extending $\mathcal{M}$ to a $\mathcal{LM}$-monoid objects of Cat_∞. In view of Theorem 6.1.2.34, this is equivalent to producing an algebra object of the monoidal ∞-category $\mathcal{C}^+[\mathcal{M}]$. In particular, $\overline{\mathrm{End}}(\mathcal{M})^\otimes$ determines an algebra object of $A \in \mathrm{Alg}(\mathcal{C}^+[\mathcal{M}])$.

For every ∞-category $\mathcal{D}$, the action of $\mathrm{End}(\mathcal{M})$ on $\mathcal{M}$ determines a homotopy equivalence (even an iso-morphism of simplicial sets) $\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{D}, \mathrm{End}(\mathcal{M})) \to \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{D} \times \mathcal{M}, \mathcal{M})$. Combining Proposition 6.1.2.39 with Corollary 3.2.2.4, we deduce that A is a final object of $\mathrm{Alg}(\mathcal{C}^+[\mathcal{M}])$. In other words, $\mathrm{End}^\otimes(\mathcal{M})$ is *universal* among monoidal ∞-categories which act on $\mathcal{M}$.

We can now state a preliminary version of the ∞-categorical Barr-Beck theorem.

Theorem 6.2.0.6. *Suppose we are given a pair of adjoint functors*

$$\mathcal{C} \xrightarrow[\;\;G\;\;]{\;\;F\;\;} \mathcal{D}$$

between ∞-categories. Assume further that G is conservative, the ∞-category $\mathcal{D}$ admits geometric realizations of simplicial objects, and G preserves geometric realizations of simplicial objects. Then there exists a monad T on $\mathcal{C}$ and an equivalence $G' : \mathcal{D} \simeq \mathrm{LMod}_T(\mathcal{C})$ such that G is homotopic to the composition of G' with the forgetful functor $\mathrm{LMod}_T(\mathcal{C}) \to \mathcal{C}$.

Remark 6.2.0.7. In the situation of Theorem 6.2.0.6, Proposition 4.2.4.2 implies that the image of T in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ can be identified with the composition $G \circ F$, and is therefore uniquely determined by G. In fact, we will see in §6.2.2 that T is uniquely determined as an object of $\mathrm{Alg}(\mathrm{Fun}(\mathcal{M}, \mathcal{M}))$.

We will prove Theorem 6.2.0.6 in §6.2.2. In fact, we will prove a slightly stronger result. It is not necessary to assume that the functor G preserves geometric realizations of *all* simplicial objects in $\mathcal{C}$. It is sufficient (and also necessary) that G preserves geometric realizations for the special class of *G-split* simplicial objects, which we study in §6.2.1.

The Barr-Beck theorem is an extremely useful result in higher category theory. For example, one can often show that a functor $U : \mathcal{C} \to \mathcal{C}'$ is an equivalence by exhibiting a forgetful functor $G : \mathcal{C}' \to \mathcal{D}$ and showing that G and $G \circ U$ exhibit both $\mathcal{C}$ and $\mathcal{C}'$ as representations of the same monad on $\mathcal{D}$ (see Corollary 6.2.2.14 for a formalization of this strategy, which was already used to prove rectification theorems for commutative and associative algebras in Chapter 4). In §6.2.4 we will describe another typical application, to the higher-categorical theory of descent. This discussion will require some general observations concerning the functoriality of the formation of adjoint functors, which is the topic of §6.2.3.

6.2.1   Split Simplicial Objects

Suppose we are given a chain complex of abelian groups

$$\cdots \to A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0.$$

An *augmentation* on A_* is a map of abelian groups $d_0 : A_0 \to A_{-1}$ such that $d_0 \circ d_1 = 0$. We say that A_* is a *split exact resolution of* A_{-1} if there exists a contracting chain homotopy for the extended chain complex

$$\cdots \to A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} A_{-1}.$$

That is, A_* is a split exact resolution of A_{-1} if there exist maps $\{h_n : A_{n-1} \to A_n\}_{n \geq 0}$ such that $d_0 h_0 = \mathrm{id}_{A_{-1}}$ and $d_{n+1} h_{n+1} + h_n d_n = \mathrm{id}_{A_n}$ for $n \geq 0$. In this case, if we let $Z_n \subseteq A_n$ be the kernel of the differential d_n, then we have canonical isomorphisms $A_n \simeq Z_n \oplus Z_{n-1}$ for each $n \geq 0$, where the inclusion $Z_{n-1} \hookrightarrow A_n$ is given by $h_n|Z_{n-1}$. This implies that the chain complex A_* is exact. In fact, a much stronger assertion is true: for any additive functor F, the induced chain complex

$$\cdots \to F(A_2) \to F(A_1) \to F(A_0) \to F(A_{-1})$$

is exact, even if F is not an exact functor.

In the situation above, we can use the Dold-Kan correspondence (Theorem 1.2.3.7) to identify A_* with the normalized chain complex of a simplicial abelian group $C_\bullet$. It is possible to formulate the condition that A_* is a split exact resolution of an abelian group A_{-1} in terms of the simplicial abelian group $C_\bullet$. Moreover, this formulation makes sense not only for simplicial objects of the category of abelian groups, but for simplicial objects of an arbitrary ∞-category.

Notation 6.2.1.1. The category $\boldsymbol{\Delta}_{-\infty}$ is defined as follows: the objects of $\boldsymbol{\Delta}_{-\infty}$ are integers $n \geq -1$, and $\mathrm{Hom}_{\boldsymbol{\Delta}_{-\infty}}([m], [n])$ is the set of nondecreasing maps maps $\alpha : [m] \cup \{-\infty\} \to [n] \cup \{-\infty\}$ such that $\alpha(-\infty) = -\infty$ (which we regard as a least element of both $[m] \cup \{-\infty\}$ and $[n] \cup \{-\infty\}$). We have inclusions of subcategories $\boldsymbol{\Delta} \subseteq \boldsymbol{\Delta}_+ \subseteq \boldsymbol{\Delta}_{-\infty}$, where the latter identifies $\boldsymbol{\Delta}_+$ with the subcategory of $\boldsymbol{\Delta}_{-\infty}$ having the same objects, and a map $\alpha : [m] \cup \{-\infty\} \to [n] \cup \{-\infty\}$ belongs to $\boldsymbol{\Delta}_+$ if and only if $\alpha^{-1}(-\infty) = \{-\infty\}$.

Definition 6.2.1.2. Let $\mathcal{C}$ be an ∞-category. We will say that an augmented simplicial object $U : \mathrm{N}(\boldsymbol{\Delta}_+)^{op} \to \mathcal{C}$ is *split* if U extends to a functor $\mathrm{N}(\boldsymbol{\Delta}_{-\infty})^{op} \to \mathcal{C}$. We will say that a simplicial object $U : \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathcal{C}$ is *split* if it extends to a split augmented simplicial object. Given a functor $G : \mathcal{D} \to \mathcal{C}$, we will say that an (augmented) simplicial object U of $\mathcal{D}$ is *G-split* if $G \circ U$ is split, when regarded as an (augmented) simplicial object of $\mathcal{C}$.

Remark 6.2.1.3. According to Lemma T.6.1.3.16, every split augmented simplicial object is a colimit diagram. (In [97], we used a slightly different notation, employing a category $\boldsymbol{\Delta}_\infty$ in place of the category $\boldsymbol{\Delta}_{-\infty}$ defined above. However, these two categories are equivalent to one another, via the functor which reverses order.)

Remark 6.2.1.4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories, and let V be a split simplicial object of $\mathcal{C}$. Then $F \circ V$ is a split simplicial object of $\mathcal{D}$. It follows from Lemma T.6.1.3.16 implies that V admits a colimit in $\mathcal{C}$, and that F preserves that colimit.

Example 6.2.1.5. We define a functor $\phi : \mathrm{N}(\boldsymbol{\Delta}_{-\infty})^{op} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$ by the formula $\phi(\{-\infty\} \cup [n]) \simeq \{-\infty\} \star [n] \star [0] \simeq [n+1]$, and let $\Phi : \mathrm{N}(\boldsymbol{\Delta}_{-\infty})^{op} \to \mathcal{LM}^{\otimes}$ be the composition of ϕ with the functor $\mathrm{LCut} : \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathcal{LM}^{\otimes}$ of Construction 4.2.2.6. Note that $\Phi(\{-\infty\}) = \mathfrak{m}$. Since $\{-\infty\}$ is an initial object of $\boldsymbol{\Delta}_{-\infty}$, there is a canonical natural transformation $\alpha : \Phi \to \Phi_0$, where $\Phi_0 : \mathrm{N}(\boldsymbol{\Delta}_{-\infty})^{op} \to \mathcal{LM}^{\otimes}$ is the constant functor taking the value $\mathfrak{m}$.

Let $q : \mathcal{M}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a coCartesian fibration of ∞-operads, corresponding to an ∞-category $\mathcal{M}$ left-tensored over a monoidal ∞-category $\mathcal{C}^{\otimes} = \mathcal{M}_{\mathfrak{a}}^{\otimes}$. Let $X : \mathcal{LM}^{\otimes} \to \mathcal{M}^{\otimes}$ be an object of $\mathrm{LMod}(\mathcal{M})$. We can identify X with a pair (A, M), where A is an algebra object of $\mathcal{C}$ and $M \in \mathcal{M}$ has the structure of a left module over A. The composition $X \circ \Phi$ is a functor $\mathrm{N}(\boldsymbol{\Delta}_{-\infty})^{op} \to \mathcal{M}^{\otimes}$. Since q is a coCartesian fibration, we can lift α to a q-coCartesian natural transformation $\overline{\alpha} : (X \circ \Phi) \to X'$ for some functor $X' : \mathrm{N}(\boldsymbol{\Delta}_{-\infty})^{op} \to \mathcal{M}$. Unwinding the definitions, we see that $X'| \mathrm{N}(\boldsymbol{\Delta})^{op}$ is the bar construction $\mathrm{Bar}_A(A, M)_\bullet$ of Construction 4.3.5.7. It follows that $\mathrm{Bar}_A(A, M)_\bullet$ is a split simplicial object of $\mathcal{M}$. Moreover, Remark 6.2.1.3 implies that $X'| \mathrm{N}(\boldsymbol{\Delta}_+)^{op}$ is a colimit diagram, which produces another proof that $|\mathrm{Bar}_A(A, M)_\bullet| \simeq X'(\{-\infty\}) = M$.

The construction $[n] \mapsto [0] \star [n] \simeq [n+1]$ determines a functor ρ from $\boldsymbol{\Delta}_+$ to itself. There is an evident natural transformation $\alpha : \mathrm{id}_{\boldsymbol{\Delta}_+} \to \rho$. Let j denote the inclusion $\boldsymbol{\Delta}_+ \hookrightarrow \boldsymbol{\Delta}_{-\infty}$. There is another natural transformation of functors $\beta : j \circ \rho \to j$, which carries an object $[n] \in \boldsymbol{\Delta}_+$ to the map $f : [n+1] \cup \{-\infty\} \to [n] \cup \{-\infty\}$ such that $f(0) = \infty$ and $f(k) = k-1$ for $0 < k \le n+1$. The composition of $j(\alpha)$ and β is the identity natural transformation from the functor j to itself. We can view $j(\alpha)$ and β as determining a map of simplicial sets $\mathrm{N}(\boldsymbol{\Delta}_+) \times \Delta^2 \to \mathrm{N}(\boldsymbol{\Delta}_\infty)$.

Proposition 6.2.1.6. *The diagram*

$$
\begin{array}{ccc}
\mathrm{N}(\boldsymbol{\Delta}_+) \times \Lambda_0^2 & \longrightarrow & \mathrm{N}(\boldsymbol{\Delta}_+) \\
\downarrow & & \downarrow \\
\mathrm{N}(\boldsymbol{\Delta}_+) \times \Delta^2 & \longrightarrow & \mathrm{N}(\boldsymbol{\Delta}_{-\infty})
\end{array}
$$

is a homotopy pushout square of simplicial sets (with respect to the Joyal model structure).

Proof. Let $f : [m] \cup \{-\infty\} \to [n] \cup \{-\infty\}$ be a morphism in $\boldsymbol{\Delta}_{-\infty}$. We define the *complexity* of $c(f)$ to be the cardinality of the set $\{i \in [m] : f(i) = \infty\}$. If σ is an k-simplex of $\mathrm{N}(\boldsymbol{\Delta}_{-\infty})$ corresponding to a sequence of maps

$$
[n_0] \cup \{-\infty\} \xrightarrow{f_1} \cdots \xrightarrow{f_k} [n_k] \cup \{-\infty\},
$$

then we define the complexity $c(\sigma)$ to be the sum $c(f_1) + \cdots + c(f_k)$. For every nonnegative integer m, let $X(m)$ denote the simplicial subset of $\mathrm{N}(\boldsymbol{\Delta}_{-\infty})$ spanned by those simplices σ such that $c(\sigma) \le m$. We note that $X(0)$ can be identified with $\mathrm{N}(\boldsymbol{\Delta}_+)$, and that $\mathrm{N}(\boldsymbol{\Delta}_{-\infty}) = \bigcup_m X(m)$.

Proposition 6.2.1.6 follows immediately from the following pair of assertions:

(a) The diagram

$$
\begin{array}{ccc}
\mathrm{N}(\boldsymbol{\Delta}_+) \times \Lambda_0^2 & \longrightarrow & \mathrm{N}(\boldsymbol{\Delta}_+) \\
\downarrow & & \downarrow \\
\mathrm{N}(\boldsymbol{\Delta}_+) \times \Delta^2 & \longrightarrow & X(1)
\end{array}
$$

is a homotopy pushout square of simplicial sets (with respect to the Joyal model structure).

(b) For $n \ge 2$, the inclusion $X(n-1) \hookrightarrow X(n)$ is a categorical equivalence.

The proofs of both (a) and (b) will require some auxiliary constructions. Let $F_1, \ldots, F_k$ be a sequence of functors from $\boldsymbol{\Delta}_+$ to itself (in practice, each of these functors will be either ρ or the identity). We define a category $\mathcal{C}[F_1, \ldots, F_k]$ as follows:

(i) An object of $\mathcal{C}[F_1, \ldots, F_k]$ is a pair $([n], i)$, where $[n] \in \boldsymbol{\Delta}_+$ and $0 \le i \le k$.

(ii) If $i \le j$, a morphism from $([m], i) \to ([n], j)$ in $\mathcal{C}[F_1, \ldots, F_k]$ is a map $[m] \to F_{i+1} \cdots F_j[n]$ in $\boldsymbol{\Delta}_+$. If $i > j$, there are no morphisms from $([m], i)$ to $([n], j)$ in $\mathcal{C}[F_1, \ldots, F_k]$.

We now prove (a). The natural transformation $\beta : j\rho \to j$ determines a map $\mathrm{N}(\mathcal{C}[\rho]) \to \mathrm{N}(\boldsymbol{\Delta}_\infty)$, which factors through the simplicial subset $X(1)$. This map fits into a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{N}(\boldsymbol{\Delta}_+) \times \Lambda_0^2 & \longrightarrow & \mathrm{N}(\mathcal{C}[\mathrm{id}]) & \longrightarrow & \mathrm{N}(\boldsymbol{\Delta}_+) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{N}(\boldsymbol{\Delta}_+) \times \Delta^2 & \longrightarrow & \mathrm{N}(\mathcal{C}[\rho]) & \longrightarrow & X(1)
\end{array}
$$

where the right hand square is a pushout square and therefore (since the vertical maps are all cofibrations) a homotopy pushout square. It will therefore suffice to show that the left square is a homotopy pushout diagram. This square fits into a larger diagram

$$
\begin{array}{ccc}
N(\mathbf{\Delta}_+) \times \Delta^{\{0,1\}} & \xrightarrow{\ \theta\ } & N(\mathbf{\Delta}_+) \\
\downarrow & & \downarrow \\
N(\mathbf{\Delta}_+) \times \Lambda_0^2 & \longrightarrow & N(\mathcal{C}[\mathrm{id}]) \\
\downarrow & & \downarrow \\
N(\mathbf{\Delta}_+) \times \Delta^2 & \longrightarrow & N(\mathcal{C}[\rho])
\end{array}
$$

where the upper square is a pushout diagram and therefore (since the vertical maps are again cofibrations) a homotopy pushout square; here the map θ classifies the natural transformation $\alpha : \mathrm{id} \to \rho$. It therefore suffices to show that the outer square is a homotopy pushout. This square fits into a larger diagram

$$
\begin{array}{ccccc}
N(\mathbf{\Delta}_+) \times \{1\} & \longrightarrow & N(\mathbf{\Delta}_+) \times \Delta^{\{0,1\}} & \xrightarrow{\ \theta\ } & N(\mathbf{\Delta}_+) \\
\downarrow & & \downarrow & & \downarrow \\
N(\mathbf{\Delta}_+) \times \Delta^{\{1,2\}} & \longrightarrow & N(\mathbf{\Delta}_+) \times \Delta^2 & \longrightarrow & N(\mathcal{C}[\rho]).
\end{array}
$$

The left square is obviously a homotopy pushout diagram, and the outer rectangle is a homotopy pushout diagram by Proposition T.3.2.2.7. It follows that the right square is a homotopy pushout diagram, as desired.

We now prove (*b*). Fix $n \geq 2$. For every subset $S \subseteq \{1, \ldots, n\}$, let $\mathcal{C}_S = \mathcal{C}[F_1, \ldots, F_n]$ where

$$
F_i = \begin{cases} \rho & \text{if } i \in S \\ \mathrm{id}_{\mathbf{\Delta}_+} & \text{if } i \notin S. \end{cases}
$$

The natural transformation $\alpha : \mathrm{id}_{\mathbf{\Delta}_+} \to \rho$ determines inclusion functors $\mathcal{C}_S \to \mathcal{C}_{S'}$ for $S \subseteq S'$. Let $Y = N(\mathcal{C}_{\{1,\ldots,n\}})$ and let $Y_S = N(\mathcal{C}_S)$ for $S \subseteq \{1, \ldots, n\}$, so that we can identify each Y_S with a simplicial subset of Y. Let P be the collection of all subsets $S \subseteq \{1, \ldots, n\}$ and $P_0 \subset P$ the collection of all *proper* subsets $S \subset \{1, \ldots, n\}$. Let $Y' = \bigcup_{S \in P_0} Y_S \subseteq Y$. The diagram $S \mapsto Y_S$ is projectively cofibrant, so we can identify Y' with a homotopy colimit of the diagram $\{Y_S\}_{S \in P_0}$. There is an evident pushout diagram of simplicial sets

$$
\begin{array}{ccc}
Y' & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X(n-1) & \longrightarrow & X(n).
\end{array}
$$

Consequently, to prove that the inclusion $X(n-1) \hookrightarrow X(n)$ is a categorical equivalence, it will suffice to show that the inclusion $Y' \hookrightarrow Y$ is a categorical equivalence. In other words, we are reduced to proving that the diagram $\{Y_S\}_{S \in P}$ is a homotopy colimit (with respect to the Joyal model structure).

There is an obvious forgetful functor $Y \to \Delta^n$. For every simplicial subset $K \subseteq \Delta^n$, let $Z_K : P \to \mathcal{S}et_\Delta$ be the functor given by the formula $Z_K(S) = Y_S \times_{\Delta^n} K$. We wish to prove that Z_{Δ^n} is a homotopy colimit diagram. Let $K_0 = \Delta^{\{0,1\}} \coprod_{\{1\}} \Delta^{\{1,2\}} \coprod_{\{2\}} \cdots \coprod_{\{n-1\}} \Delta^{\{n-1,n\}}$, so that the inclusion $K_0 \hookrightarrow \Delta^n$ is a categorical equivalence. Since each of the projection maps $Y_S \to \Delta^n$ is a Cartesian fibration, we deduce that the inclusion $Z_{K_0} \to Z_{\Delta^n}$ is a weak equivalence of diagrams (Proposition T.3.3.1.3). It will therefore suffice to show that Z_{K_0} is a homotopy colimit diagram. Since the collection of homotopy colimit diagrams in $\mathcal{S}et_\Delta$ is stable under the formation of homotopy pushout squares, we can reduce further to proving that Z_K is a homotopy colimit diagram when $K = \{i\} \subseteq \Delta^n$ for $0 \leq i \leq n$ or $K = \Delta^{\{i-1,i\}} \subseteq \Delta^n$ for $0 < i \leq n$.

In the first case, this is obvious: the diagram $Z_{\{i\}}$ is isomorphic to the constant diagram taking the value $\mathrm{N}(\boldsymbol{\Delta}_+)$, which is a homotopy colimit diagram because the simplicial set $\mathrm{N}(P_0)$ is weakly contractible (since P_0 has a least element $\emptyset \subset \{1, \ldots, n\}$). To handle the second, let us suppose that $K = \Delta^{\{i-1,i\}}$, so that Z_K is given by the formula

$$Z_K(S) = \begin{cases} \mathrm{N}(\mathcal{C}[\rho]) & \text{if } i \in S \\ \mathrm{N}(\mathcal{C}[\mathrm{id}]) & \text{otherwise.} \end{cases}$$

Let P_1 be the subset of P_0 consisting of the subsets $\emptyset, \{i\} \subset \{1, \ldots, n\}$. We observe that $Z_K|P_0$ is a homotopy left Kan extension of $Z_K|P_1$. Consequently, to prove that Z_K is a homotopy colimit diagram, it suffices to show that Z_K exhibits $Z_K(\{1, \ldots, n\})$ as a homotopy colimit of the restriction $Z_K|P_1$. Since P_1 has a final object $\{i\}$, this follows from the observation that the map $Z_K(\{i\}) \to Z_K(\{1, \ldots, n\})$ is a categorical equivalence (in fact, an isomorphism) of simplicial sets. $\square$

For any ∞-category $\mathcal{C}$, composition with the functor $\rho : \boldsymbol{\Delta}_+ \to \boldsymbol{\Delta}_+$ determines a functor from the ∞-category of augmented simplicial objects of $\mathcal{C}$ to itself. We will denote this functor by T. The natural transformation $\mathrm{id} \to \rho$ determines a natural map $TX_\bullet \to X_\bullet$, for every augmented simplicial object $X_\bullet$ of $\mathcal{C}$. Proposition 6.2.1.6 immediately implies the following:

Corollary 6.2.1.7. *Let $\mathcal{C}$ be an ∞-category, and let $X_\bullet$ be an augmented simplicial object of $\mathcal{C}$. The following conditions are equivalent:*

(1) *The augmented simplicial object $X_\bullet$ is split.*

(2) *The canonical map $TX_\bullet \to X_\bullet$ admits a right homotopy inverse.*

Remark 6.2.1.8. More precisely, we see that if $X_\bullet$ is an augmented simplicial object of an ∞-category $\mathcal{C}$, then extending $X_\bullet$ to a functor $\mathrm{N}(\boldsymbol{\Delta}_{-\infty})^{op} \to \mathcal{C}$ is equivalent to giving a natural transformation $X_\bullet \to TX_\bullet$ together with a homotopy of the composite map

$$X_\bullet \to TX_\bullet \to X_\bullet$$

with the identity on $X_\bullet$.

Example 6.2.1.9. Let $\mathcal{C}$ be an abelian category. Then Dold-Kan correspondence establishes an equivalence of categories between the category $\mathrm{Fun}(\boldsymbol{\Delta}^{op}, \mathcal{C})$ of simplicial objects of $\mathcal{C}$ and the category $\mathrm{Ch}_{\geq 0}(\mathcal{C})$ of nonnegatively graded chain complexes in $\mathcal{C}$ (see, for example, [159]). Let $X_\bullet$ be a simplicial object of $\mathcal{C}$, and let

$$\cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$$

be the corresponding (normalized) chain complex. Then extending $X_\bullet$ to an augmented simplicial object $\overline{X}_\bullet : \boldsymbol{\Delta}_+^{op} \to \mathcal{C}$ is equivalent to providing an augmentation on the chain complex C_*: that is, providing a map $d_0 : C_0 \to C_{-1}$ such that $d_0 \circ d_1 = 0$. In this case, the augmented simplicial object $T\overline{X}_\bullet$ corresponds to the cone

$$\cdots \to C_3 \oplus C_2 \to C_2 \oplus C_1 \to C_1 \oplus C_0 \to C_0.$$

Giving a section of the canonical map $T\overline{X}_\bullet \to \overline{X}_\bullet$ is equivalent to giving a collection of maps $h_n : C_{n-1} \to C_n$ satisfying $d_0 h_0 = \mathrm{id}_{C_{-1}}$ and $d_{n+1} h_{n+1} + h_n d_n = \mathrm{id}_{C_n}$ for $n \geq 0$. In other words, extending $X_\bullet$ to a split simplicial object of $\mathcal{C}$ is equivalent to giving a contracting homotopy which exhibits C_* as a split exact resolution of some object $C_{-1} \in \mathcal{C}$.

For later use, we record some other consequences of Proposition 6.2.1.6.

Corollary 6.2.1.10. *Let $\mathcal{C}$ be an ∞-category, and let $\mathfrak{X}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_+^{op}), \mathcal{C})$ spanned by the split augmented simplicial objects of $\mathcal{C}$ (that is, $\mathfrak{X}$ is the essential image of the restriction functor $\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_{-\infty}^{op}), \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_+^{op}), \mathcal{C}))$. Then $\mathfrak{X}$ is stable under retracts in $\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_+^{op}), \mathcal{C})$.*

Proof. Suppose we have a commutative diagram

of augmented simplicial objects of $\mathcal{C}$, where $X_\bullet$ is split. We wish to prove that $Y_\bullet$ is split. According to Corollary 6.2.1.7, there exists a map of augmented simplicial objects $\gamma : X_\bullet \to TX_\bullet$ such that the composition $X_\bullet \xrightarrow{\gamma} TX_\bullet \to X_\bullet$ is homotopic to the identity. Let γ' denote the composition

$$Y_\bullet \xrightarrow{\alpha} X_\bullet \xrightarrow{\gamma} TX_\bullet \xrightarrow{T\beta} TY_\bullet.$$

A simple diagram chase shows that the composition $Y_\bullet \xrightarrow{\gamma'} TY_\bullet \to Y_\bullet$ is homotopic to the identity, so that $Y_\bullet$ splits as required. $\square$

Remark 6.2.1.11. The functor ρ carries $\boldsymbol{\Delta}_+$ into the subcategory $\boldsymbol{\Delta} \subseteq \boldsymbol{\Delta}_+$. Consequently, the translation functor T factors through a functor $\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}^{op}), \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}^{op}_+), \mathcal{C})$. We will abuse notation and denote this functor also by T.

Corollary 6.2.1.12. *Let $\mathcal{C}$ be an ∞-category, and let $X_\bullet$ be an augmented simplicial object of $\mathcal{C}$. The following conditions are equivalent:*

(1) *The augmented simplicial object $X_\bullet$ is split.*

(2) *There exists a simplicial object $U_\bullet$ of $\mathcal{C}$ such that $X_\bullet$ is a retract of $TU_\bullet$ (in the ∞-category of augmented simplicial objects of $\mathcal{C}$).*

Proof. The implication $(1) \Rightarrow (2)$ follows from Corollary 6.2.1.7. We will prove that $(2) \Rightarrow (1)$. Using Corollary 6.2.1.10, we may assume that $X_\bullet = TU_\bullet$. The desired result in this case follows from the observation that the shift functor $\rho : \boldsymbol{\Delta}_+ \to \boldsymbol{\Delta}$ extends naturally to a functor $\boldsymbol{\Delta}_{-\infty} \to \boldsymbol{\Delta}$, determined by the natural identification of linearly ordered sets $[n+1] = [0] \star [n] \simeq [n] \cup \{-\infty\}$. $\square$

6.2.2 The Barr-Beck Theorem

Let $G : \mathcal{D} \to \mathcal{C}$ be a functor between ∞-categories which admits a left adjoint F. Our goal in this section is to show that the composition $G \circ F \in \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ can be promoted to a monad $T \in \mathrm{Alg}(\mathrm{End}(\mathcal{C}))$ on $\mathcal{C}$ (Definition 6.2.0.3), that G factors through a functor $G' : \mathcal{D} \to \mathrm{LMod}_T(\mathcal{C})$ of (left) T-module objects of $\mathcal{C}$, and to give necessary and sufficient conditions for the functor G' to be an equivalence of ∞-categories (Theorem 6.2.2.5).

We begin by noting that for any pair of ∞-categories $\mathcal{C}$ and $\mathcal{D}$, the ∞-category $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ carries a left action of the simplicial monoid $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$. We regard the ∞-category as left-tensored over the monoidal ∞-category $\mathrm{End}(\mathcal{C})^{\otimes}$. The essential observation is the following:

Lemma 6.2.2.1. *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor between ∞-categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $v : F \circ G \to \mathrm{id}_{\mathcal{D}}$ be the counit of an adjunction between F and G. Then v induces a map*

$$(G \circ F) \circ G = G \circ (F \circ G) \xrightarrow{v} G \circ \mathrm{id}_{\mathcal{D}} = G$$

which exhibits $G \circ F \in \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ as a classifying object for endomorphisms of $G \in \mathrm{Fun}(\mathcal{D}, \mathcal{C})$.

Proof. We wish to show that for every functor $U : \mathcal{C} \to \mathcal{C}$, composition with v induces a homotopy equivalence $\alpha : \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}, \mathcal{C})}(U, G \circ F) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{D}, \mathcal{C})}(U \circ G, G)$. Let $u : \mathrm{id}_{\mathcal{C}} \to G \circ F$ be the unit for an adjunction

which is compatible with v. Then u determines a map $\beta : \mathrm{Map}_{\mathrm{Fun}(\mathcal{D},\mathcal{C})}(U \circ G, G) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{C})}(U, G \circ F)$ given by the composition

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{D},\mathcal{C})}(U \circ G, G) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{C})}(U \circ G \circ F, G \circ F) \overset{\circ u}{\to} \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{C})}(U, G \circ F).$$

From the compatibility between u and v, it follows readily that β is a homotopy inverse to α. $\square$

Definition 6.2.2.2. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor between ∞-categories. A *endomorphism monad* consists of a monad $T \in \mathrm{Alg}(\mathrm{End}(\mathcal{C}))$ together with a left T-module $\overline{G} \in \mathrm{LMod}_T(\mathrm{Fun}(\mathcal{D}, \mathcal{C}))$ whose image in $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ coincides with G, such that the composition map $T \circ G \to G$ exhibits T as a classifying object for endomorphisms of G.

In the situation of Definition 6.2.2.2, we will say that $\overline{G}$ *exhibits T as an endomorphism monad of G*. Combining Lemma 6.2.2.1, Theorem 6.1.2.34, and Corollary 3.2.2.5, we obtain the following result:

Proposition 6.2.2.3. *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor between ∞-categories. Assume that there exists a functor $F : \mathcal{C} \to \mathcal{D}$ and a natural transformation $u : \mathrm{id}_{\mathcal{C}} \to G \circ F$ which is the unit for an adjunction between G and F. Then:*

(1) *There exists an endomorphism monad for G.*

(2) *Let T be a monad on $\mathcal{C}$ and let $\overline{G} \in \mathrm{LMod}_T(\mathrm{Fun}(\mathcal{D}, \mathcal{C}))$ be a lifting of G. Then $\overline{G}$ exhibits T as an endomorphism monad of G if and only if the composite map*

$$T \overset{\mathrm{id}_T \times u}{\to} T \circ G \circ F \overset{a \times \mathrm{id}_F}{\to} G \circ F$$

is an equivalence in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$, where $a : T \circ G \to G$ denotes the action of T on G.

Suppose that $G : \mathcal{D} \to \mathcal{C}$ is a functor between ∞-categories which is a left module over a monad $T \in \mathrm{Alg}(\mathrm{End}(\mathcal{C}))$. The left action of T on G determines a functor $G' : \mathcal{D} \to \mathrm{LMod}_T(\mathcal{C})$ such that G is obtained by composing G' with the forgetful functor $\mathrm{LMod}_T(\mathcal{C}) \to \mathcal{C}$.

Definition 6.2.2.4. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor between ∞-categories. Assume that G has a left adjoint F, so that G admits an endomorphism monad T (Proposition 6.2.2.3). We will say that $\mathcal{D}$ is *monadic over* $\mathcal{C}$ if the induced functor $G' : \mathcal{D} \to \mathrm{LMod}_T(\mathcal{C})$ is an equivalence of ∞-categories.

If $G : \mathcal{D} \to \mathcal{C}$ exhibits the ∞-category $\mathcal{D}$ as monadic over the ∞-category $\mathcal{C}$, then we can think of G as a forgetful functor: an object $D \in \mathcal{D}$ can be identified with its image $G(D) \in \mathcal{C}$, together with the data of an action of the endomorphism monad T of the functor G.

We are now ready to state the main result of this section (which can be regarded as a more precise version of Theorem 6.2.0.6):

Theorem 6.2.2.5 (∞-Categorical Barr-Beck Theorem)**.** *et $G : \mathcal{D} \to \mathcal{C}$ be a functor between ∞-categories which admits a left adjoint $F : \mathcal{C} \to \mathcal{D}$. The following are equivalent:*

(a) *The functor G exhibits $\mathcal{D}$ as monadic over $\mathcal{C}$.*

(b) *There exists a monoidal ∞-category $\mathcal{E}^{\otimes}$, a left action of $\mathcal{E}^{\otimes}$ on $\mathcal{C}$, an algebra object $A \in \mathrm{Alg}(\mathcal{E})$ and an equivalence $G' : \mathcal{D} \simeq \mathrm{LMod}_A(\mathcal{C})$ such that G is equivalent to the composition of G' with the forgetful functor $\mathrm{LMod}_A(\mathcal{C}) \to \mathcal{C}$.*

(c) *The functor G satisfies the following conditions:*

    (1) *The functor $G : \mathcal{D} \to \mathcal{C}$ is* conservative*; that is, a morphism $f : D \to D'$ in $\mathcal{D}$ is an equivalence if and only if $G(f)$ is an equivalence in $\mathcal{C}$.*

    (2) *Let V be a simplicial object of $\mathcal{D}$ which is G-split. Then V admits a colimit in $\mathcal{D}$, and that colimit is preserved by G.*

Remark 6.2.2.6. For a proof of Theorem 6.2.2.5 in the setting of classical category theory, we refer the reader to [9].

Remark 6.2.2.7. Hypotheses (1) and (2) of Theorem 6.2.2.5 can be rephrased as the following single condition:

($*$) Let $V : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{D}$ be a simplicial object of $\mathcal{D}$ which is G-split. Then V has a colimit in $\mathcal{D}$. Moreover, an arbitrary extension $\overline{V} : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{D}$ is a colimit diagram if and only if $G \circ \overline{V}$ is a colimit diagram.

It is clear that (1) and (2) imply ($*$), and that ($*$) implies (2). To prove that ($*$) implies (1), let us consider an arbitrary morphism $f : D' \to D$ in $\mathcal{D}$. The map f determines an augmented simplicial object $\overline{V}$ of $\mathcal{D}$, with

$$\overline{V}([n]) = \begin{cases} D' & \text{if } n \geq 0 \\ D & \text{if } n = -1. \end{cases}$$

The underlying simplicial object $V = \overline{V} | \mathrm{N}(\mathbf{\Delta})^{op}$ is constant, and therefore G-split. Since $\mathrm{N}(\mathbf{\Delta})$ is contractible, $\overline{V}$ is a colimit diagram if and only if f is an equivalence, and $G \circ \overline{V}$ is a colimit diagram if and only if $G(f)$ is an equivalence. If ($*$) is satisfied, then these conditions are equivalent, so that G is conservative as desired.

Remark 6.2.2.8. Let $\mathcal{C}$ be an ∞-category, and regard $\mathcal{C}$ as left-tensored over the monoidal ∞-category $\mathrm{End}(\mathcal{C})$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{LMod}(\mathcal{C}) & \xrightarrow{\;\;\theta\;\;} & \mathcal{C} \times \mathrm{Alg}(\mathrm{End}(\mathcal{C})) \\ & {}_{p}\searrow \qquad \swarrow_{p'} & \\ & \mathrm{Alg}(\mathrm{End}(\mathcal{C})), & \end{array}$$

where θ is determined by the p and the forgetful functor $\mathrm{LMod}(\mathcal{C}) \to \mathcal{C}$. The functor p' is obviously a Cartesian fibration, and Corollary 4.2.3.2 implies that p is a Cartesian fibration as well. Proposition 4.2.3.1 implies that θ carries p-Cartesian edges to p'-Cartesian edges. Consequently, θ classifies a natural transformation of functors $F \to F'$, where $F : \mathrm{Alg}(\mathrm{End}(\mathcal{C}))^{op} \to \mathcal{C}\mathrm{at}_\infty$ is the functor classified by p (so that $F(A) \simeq \mathrm{LMod}_T(\mathcal{C})$ for every monad $T \in \mathrm{Alg}(\mathrm{End}(\mathcal{C}))$) and F' is the constant functor taking the value $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$. We may identify this transformation with a functor $\alpha : \mathrm{Alg}(\mathrm{End}(\mathcal{C}))^{op} \to \mathcal{C}\mathrm{at}_\infty^{/\mathcal{C}}$. We can interpret Theorem 6.2.2.5 as describing the essential image of the functor α: namely, a functor $G : \mathcal{D} \to \mathcal{C}$ belongs to the essential image of α if and only if G admits a left adjoint and satisfies conditions (1) and (2) of Theorem 6.2.2.5. With more effort, one can show that the functor α is fully faithful. In other words, we may identify monads on $\mathcal{C}$ with (certain) ∞-categories lying over $\mathcal{C}$.

Example 6.2.2.9. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor between ∞-categories which exhibits $\mathcal{D}$ as monadic over $\mathcal{C}$. Assume that $\mathcal{D}$ and $\mathcal{C}$ admit finite limits. The functor G is left exact, and therefore induces a functor $g : \mathrm{Sp}(\mathcal{D}) \to \mathrm{Sp}(\mathcal{C})$. Suppose that g admits a left adjoint. Then g exhibits $\mathrm{Sp}(\mathcal{D})$ as monadic over $\mathrm{Sp}(\mathcal{C})$. To prove this, we show that g satisfies the hypotheses of Theorem 6.2.2.5:

(1) The functor g is conservative.

(2) If $U_\bullet$ is a simplicial object of $\mathrm{Sp}(\mathcal{D})$ which is g-split, then $U_\bullet$ admits a colimit in $\mathrm{Sp}(\mathcal{D})$, and that colimit is preserved by g.

Assertion (1) follows immediately from our assumption that G is conservative. We now prove (2). Let $U_\bullet$ be a g-split simplicial object of $\mathrm{Sp}(\mathcal{D})$. For each $K \in \mathcal{S}_*^{\mathrm{fin}}$, $U_\bullet(K)$ is a G-split simplicial object of $\mathcal{D}$. It follows from Theorem 6.2.2.5 that $U_\bullet(K)$ admits a colimit $V(K) \in \mathcal{D}$, which is preserved by the functor G. The construction $K \mapsto V(K)$ determines a functor $V : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{D}$, where $G \circ V \simeq |g(U_\bullet)|$ is reduced and excisive. Since G is left exact and conservative, we deduce that V is left exact and excisive. It follows that $V \in \mathrm{Sp}(\mathcal{D})$. Since V is a colimit of the diagram $U_\bullet$ in $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{D})$, it is a colimit of the diagram $U_\bullet$ in $\mathrm{Sp}(\mathcal{D})$. By construction, this colimit is preserved by the functor g.

We now turn to the proof of Theorem 6.2.2.5. The implication $(a) \Rightarrow (b)$ is obvious (take $\mathcal{E} = \mathrm{End}(\mathcal{C})$ and A to be an endomorphism monad of G). The implication $(b) \Rightarrow (c)$ follows immediately from Corollary 4.2.3.2 together with the following:

Lemma 6.2.2.10. *Let $\mathcal{C}$ be a monoidal ∞-category, $\mathcal{M}$ an ∞-category which is left-tensored over $\mathcal{C}$, A an algebra object, and $\theta : \mathrm{LMod}_A(\mathcal{M}) \to \mathcal{M}$ the forgetful functor. Then:*

(1) *Every θ-split simplicial object of $\mathrm{LMod}_A(\mathcal{M})$ admits a colimit in $\mathcal{M}$.*

(2) *The functor θ preserves colimits of θ-split simplicial objects.*

Proof. Let $\mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes}$ be defined as in Notation 4.2.2.16. In view of Proposition 4.2.2.11, it will suffice to prove the analogues of (1) and (2) for the forgetful functor $^{\Delta}\mathrm{LMod}_A(\mathcal{M}) \to \mathcal{M}$. Note that a simplicial object of $^{\Delta}\mathrm{LMod}_A(\mathcal{M})$ can be viewed as a *bisimplicial object* of $\mathcal{M}^{\otimes}$. In order to avoid confusion, we let K denote the simplicial set $\mathrm{N}(\mathbf{\Delta})^{op}$ when we wish to emphasize the role of $\mathrm{N}(\mathbf{\Delta})^{op}$ as indexing simplicial objects.

Form a pullback diagram

$$
\begin{array}{ccc}
\mathcal{N} & \longrightarrow & \mathcal{M}^{\otimes} \\
\downarrow{\scriptstyle p} & & \downarrow \\
\mathrm{N}(\mathbf{\Delta})^{op} & \xrightarrow{A \circ \mathrm{Cut}} & \mathcal{C}^{\otimes},
\end{array}
$$

so that p is a locally coCartesian fibration (Lemma 4.2.2.19). Let $u : K \to {}^{\Delta}\mathrm{LMod}_A(\mathcal{M})$ be a θ'-split simplicial object, corresponding to a map $V : K \times \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{N}$. We observe that every fiber of p is equivalent to $\mathcal{M}$, and each of the induced maps $V_{[n]} : K \times \{[n]\} \to \mathcal{N}_{[n]}$ can be identified with the composition of V with the forgetful functor θ'. It follows that each of the simplicial objects $V_{[n]}$ is split. Using Lemma T.6.1.3.16, we deduce that each $V_{[n]}$ admits a colimit $\overline{V}_{[n]} : K^{\triangleright} \to \mathcal{N}_{[n]}$, and that these colimits are preserved by each of the associated functors $\mathcal{N}_{[n]} \to \mathcal{N}_{[m]}$ (Remark 6.2.1.4). Applying Proposition T.4.3.1.10, we conclude that each $\overline{V}_{[n]}$ is a p-colimit diagram. We now invoke Lemma 3.2.2.9 to deduce the existence of a map $\overline{V} : K^{\triangleright} \times \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{N}$ which induces a p-colimit diagram on each fiber $K^{\triangleright} \times \{[n]\}$. Moreover, Lemma 3.2.2.9 implies that $\overline{V}$ defines a colimit diagram $\overline{v} : K^{\triangleright} \to \mathrm{Map}_{\mathrm{N}(\mathbf{\Delta})^{op}}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{N})$. Since $v = \overline{v}|K$ factors through the full subcategory $^{\Delta}\mathrm{LMod}_A(\mathcal{M}) \subseteq \mathrm{Map}_{\mathrm{N}(\mathbf{\Delta})^{op}}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{N})$, it is easy to see that $\overline{v}$ factors through $^{\Delta}\mathrm{LMod}_A(\mathcal{M})$. It follows that $\overline{v}$ is a colimit of v in $^{\Delta}\mathrm{LMod}_A(\mathcal{M})$. This proves (1), and (2) follows from the observation that $\theta' \circ \overline{v} = \overline{V}|K^{\triangleright} \times \{[0]\}$ is a colimit diagram in $\mathcal{M}$ by construction. $\qquad\square$

The difficult part of Theorem 6.2.2.5 is the proof that (c) implies (a). Assume that $G : \mathcal{D} \to \mathcal{C}$ is a functor which admits a left adjoint and therefore an endomorphism monad T, and let $G' : \mathcal{D} \to \mathrm{LMod}_T(\mathcal{C})$ be the induced functor. Note that G' is an equivalence of ∞-categories if and only if it is conservative and admits a fully faithful left adjoint. The first condition is immediate if G is conservative (since G is the composition of G' with the forgetful functor $\mathrm{LMod}_T(\mathcal{C}) \to \mathcal{C}$). Consequently, to complete the proof of Theorem 6.2.2.5 it will suffice to verify the following:

Lemma 6.2.2.11. *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor between ∞-categories which admits a left adjoint F and let T be an endomorphism monad for G. Assume that:*

$(\ast)$ *Every G-split simplicial object of $\mathcal{D}$ admits a colimit in $\mathcal{D}$, and this colimit is preserved by G.*

Then the canonical map $G' : \mathcal{D} \to \mathrm{LMod}_T(\mathcal{C})$ admits a fully faithful left adjoint.

Proof. Let $\mathfrak{X} \subseteq \mathrm{LMod}_T(\mathcal{C})$ be the full subcategory spanned by those left T-modules $M \in \mathrm{LMod}_T(\mathcal{C})$ such that the functor

$$
D \mapsto \mathrm{Map}_{\mathrm{LMod}_T(\mathcal{C})}(M, G'(D)) \in \mathcal{S}
$$

is corepresentable by an object $F'(M) \in \mathcal{D}$. In this case, $F'(M)$ is well-defined up to canonical equivalence and the construction $M \mapsto F'(M)$ determines a functor $F' : \mathfrak{X} \to \mathcal{D}$, which we can regard as a partially-defined left adjoint to G'. Let $G'' : \mathrm{LMod}_T(\mathcal{C}) \to \mathcal{C}$ be the forgetful functor, and let $F'' : \mathcal{C} \to \mathrm{LMod}_T(\mathcal{C})$ be a

left adjoint to G'', given informally by $C \mapsto T(C)$. Let $\mathfrak{X}_0$ denote the full subcategory of $\mathfrak{X}$ spanned by those objects M such that the unit map $u_M : G''(M) \to (G''G'F')(M) \simeq GF'(M)$ is an equivalence in $\mathcal{C}$. For every object $C \in \mathcal{C}$, the object $F''(C)$ belongs to $\mathfrak{X}$ and we have a canonical equivalence $F(C) \simeq F'(F''(C))$. Moreover, $u_{F''(C)}$ can be identified with the canonical map

$$T(C) \simeq G''F''(C) \to G''G'F'F''(C) \simeq GF(C)$$

which is an equivalence by virtue of our assumption that T is an endomorphism monad for G. It follows that $\mathfrak{X}_0$ contains the essential image of F''. Let M be an arbitrary object of $\mathrm{LMod}_T(\mathcal{C})$, and let $M_\bullet$ denote the simplicial object of $\mathrm{LMod}_T(\mathcal{C})$ given by the bar construction $\mathrm{Bar}_T(T, M)_\bullet$. Then $M \simeq |M_\bullet|$, and $M_\bullet$ is a G''-split by Example 6.2.1.5. Each M_n belongs to the essential image of F'' and therefore to $\mathfrak{X}_0$, so that $G(F'(M_\bullet)) \simeq G''(M_\bullet)$ is a split simplicial object of $\mathcal{C}$ and therefore $F'(M_\bullet)$ is a G-split simplicial object of $\mathcal{D}$. It follows from assumption (2) that $F'(M_\bullet)$ admits a colimit $D \in \mathcal{D}$, so that $M \in \mathfrak{X}$ and $F'(M) \simeq D$. The map u_M is given by the composition

$$G''(M) \simeq |G''(M_\bullet)| \simeq |G(F'(M_\bullet))| \to G|F'(M_\bullet)| \simeq GF'(M)$$

and is therefore an equivalence, since G preserves colimits of G-split simplicial objects. It follows that $\mathfrak{X}_0 = \mathrm{LMod}_T(\mathcal{C})$. In other words, the functor $G' : \mathcal{D} \to \mathrm{LMod}_T(\mathcal{C})$ admits a left adjoint F', and the unit map $\mathrm{id} \to G' \circ F'$ induces an equivalence of functors $u : G'' \to G'' \circ G' \circ F'$. Since the functor G'' is conservative (Corollary 4.2.3.2), we conclude that $G' \circ F' \simeq \mathrm{id}_{\mathrm{LMod}_T(\mathcal{C})}$, so that F' is fully faithful. $\qquad\square$

For later use, we record another consequence of the proof of Lemma 6.2.2.11:

Proposition 6.2.2.12. *Let* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *be a pair of adjoint functors between ∞-categories which satisfies conditions* (1) *and* (2) *of Theorem 6.2.2.5. For every object $D \in \mathcal{D}$, there exists a G-split simplicial object* $D_\bullet : \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathcal{D}$ *having colimit D, such that each D_n lies in the essential image of F.*

Remark 6.2.2.13. In the situation of Proposition 6.2.2.12, the simplicial object $D_\bullet$ can be chosen to depend functorially on D.

We now describe a few applications of Proposition 6.2.2.12:

Corollary 6.2.2.14. *Suppose we are given a commutative diagram of ∞-categories*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ \ U\ \ } & \mathcal{C}' \\ & {}_G\searrow \quad \swarrow{}_{G'} & \\ & \mathcal{D} & \end{array}$$
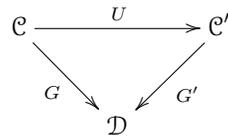
Assume that:

  (1) *The functors G and G' admit left adjoints F and F'.*

  (2) *Every G-split simplicial object of $\mathcal{C}$ admits a colimit in $\mathcal{C}$, which is preserved by G.*

  (3) *Every G'-split simplicial object of $\mathcal{C}'$ admits a colimit in $\mathcal{C}'$, which is preserved by G'.*

  (4) *The functor G' is conservative.*

  (5) *For each object $D \in \mathcal{D}$, the unit map unit map $D \to GF(D) \simeq G'(UF(D))$ induces an equivalence* $\alpha_D : F'(D) \to UF(D)$ *in $\mathcal{C}'$.*

Then U admits a fully faithful left adjoint. Moreover, U is an equivalence of ∞-categories if and only if G is conservative.

Remark 6.2.2.15. In the situation of Corollary 6.2.2.14, we can replace (5) by the following apparently weaker condition: for each $D \in \mathcal{D}$, the morphism $G'(\alpha_D) : G'F'(D) \to GF(D)$ is an equivalence in $\mathcal{D}$ (since the functor G' is conservative).

Proof. We begin by constructing a left adjoint T to the functor U. Let $j' : \mathcal{C}'^{op} \to \mathrm{Fun}(\mathcal{C}', \mathcal{S})$ be the Yoneda embedding, and let T_0 denote the composition $\mathcal{C}' \xrightarrow{j} \mathrm{Fun}(\mathcal{C}', \mathcal{S}) \xrightarrow{\circ U} \mathrm{Fun}(\mathcal{C}, \mathcal{S})$. Let $\mathrm{Fun}_0(\mathcal{C}, \mathcal{S})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$ spanned by the corepresentable functors (that is, the essential image of the Yoneda embedding $j : \mathcal{C}^{op} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S})$. To prove the existence of T, it will suffice to show that the essential image of T_0 is contained in $\mathrm{Fun}_0(\mathcal{C}, \mathcal{S})$; we can then write T as the composition $j^{-1} \circ T_0$, where j^{-1} is a homotopy inverse to $j : \mathcal{C}^{op} \simeq \mathrm{Fun}_0(\mathcal{C}, \mathcal{S})$.

Fix an object $C' \in \mathcal{C}'$; we wish to show that $T_0(C') \in \mathrm{Fun}_0(\mathcal{C}, \mathcal{S})$. Using Proposition 6.2.2.12, we can write C' as the geometric realization of a G'-split simplicial object $C'_\bullet$, such that each $C'_n \simeq F'(D_n)$ for some $D_n \in \mathcal{D}$. It follows that $T_0(C') \in \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ is the limit of of the cosimplicial object $T_0(C'_\bullet)$. Each of the functors $T_0(C'_n)$ is corepresentable by the object $F(D'_n)$, and therefore belongs to $\mathrm{Fun}_0(\mathcal{C}, \mathcal{S})$. It follows that we can write $T_0(C'_\bullet) = j(C_\bullet)$ for some simplicial object $C_\bullet$ in $\mathcal{C}$. By construction, there is an evident map $\alpha : C'_\bullet \to U(C_\bullet)$. Condition $(*)$ implies that $G'(\alpha)$ is an equivalence $G'(C'_\bullet) \to (G' \circ U)(C_\bullet) \simeq G(C_\bullet)$, so that $C_\bullet$ is G-split. It follows that $C_\bullet$ admits a colimit $C \in \mathcal{C}$, so that

$$T_0(C') \simeq \varprojlim T_0(C'_\bullet) \simeq \varprojlim j(C_\bullet) \simeq j(C)$$

is representable by C. This completes the proof of the existence of T.

We now prove that T is fully faithful. This is equivalent to showing that the unit map

$$u : \mathrm{id}_{\mathcal{C}'} \to U \circ T$$

induces an equivalence $C' \to (U \circ T)(C')$ for each object $C' \in \mathcal{C}$. Let $C'_\bullet$ and $C_\bullet$ be as above. We wish to prove that the canonical equivalence $C'_\bullet \simeq U(C_\bullet)$ induces an equivalence

$$C' \simeq \varinjlim C'_\bullet \simeq \varinjlim U(C_\bullet) \to U(\varinjlim C_\bullet).$$

In other words, we wish to prove that U preserves the colimit of the simplicial object $C_\bullet$. Since G' is conservative, it will suffice to show that $G = G' \circ U$ preserves the colimit of the simplicial object $C_\bullet$. This follows from our hypothesis on G, since $C_\bullet$ is G-split.

We conclude by observing that if U is an equivalence, then $G \simeq G' \circ U$ is conservative by virtue of (4). Conversely, if G is conservative, then U is conservative and therefore (since it admits a fully faithful left adjoint) is an equivalence of ∞-categories. $\qquad\qquad\square$

Our next result makes use of the notation and terminology of §T.5.5.8.

Corollary 6.2.2.16. *Suppose given a pair of adjoint functors between ∞-categories $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$. Assume that:*

(*i*) *The ∞-category $\mathcal{D}$ admits filtered colimits and geometric realizations, and G preserves filtered colimits and geometric realizations.*

(*ii*) *The ∞-category $\mathcal{C}$ is projectively generated (Definition T.5.5.8.23).*

(*iii*) *The functor G is conservative.*

Then:

(1) *The ∞-category $\mathcal{D}$ is projectively generated.*

(2) *An object $D \in \mathcal{D}$ is compact and projective if and only if there exists a compact projective object $C \in \mathcal{C}$ such that D is a retract of $F(C)$.*

(3) *The functor G preserves all sifted colimits.*

Proof. Let $\mathcal{C}^0$ denote the full subcategory of $\mathcal{C}$ spanned by the compact projective objects. Let $\mathcal{D}^0$ denote the essential image of $F| \, \mathcal{C}^0$. Using assumption (i), we deduce that $\mathcal{D}^0$ consists of compact projective objects of $\mathcal{D}$. Without loss of generality, we may assume that the ∞-category $\mathcal{D}$ is minimal, so that $\mathcal{D}^0$ is small. Moreover, since $\mathcal{C}^0$ is stable under finite coproducts in $\mathcal{C}$, and F preserves finite coproducts, we conclude that $\mathcal{D}^0$ admits finite coproducts (which are also finite coproducts in $\mathcal{D}$).

Let $\mathcal{D}' = \mathcal{P}_\Sigma(\mathcal{D}^0)$ (see Definition T.5.5.8.8). Using Proposition T.5.5.8.15, we deduce that the inclusion $\mathcal{D}^0 \subseteq \mathcal{D}$ is homotopic to a composition

$$\mathcal{D}^0 \xrightarrow{j} \mathcal{D}' \xrightarrow{f} \mathcal{D},$$

where the functor f preserves filtered colimits and geometric realizations. Combining Proposition T.5.5.8.22 with Proposition 6.2.2.12 and assumption (ii), we conclude that f is an equivalence of ∞-categories. This proves (1). Moreover, the proof shows that $\mathcal{D}^0$ is spanned by a set of compact projective generators for $\mathcal{D}$ (Definition T.5.5.8.23), so that assertion (2) follows from Proposition T.5.5.8.25. Assertion (3) now follows from Proposition T.5.5.8.15. $\square$

For later use, we record the following the following consequence of the Barr-Beck theorem:

Proposition 6.2.2.17. *Suppose given a commutative diagram of ∞-categories*

$$
\begin{array}{ccc}
& \mathcal{D} & \\
{\scriptstyle G'} \nearrow & & \searrow {\scriptstyle G''} \\
\mathcal{E} & \xrightarrow[\;\;G\;\;]{} & \mathcal{C}.
\end{array}
$$

Suppose that G exhibits $\mathcal{E}$ as monadic over $\mathcal{C}$, that G'' is conservative, and that G' admits a left adjoint. Then G' exhibits $\mathcal{E}$ as monadic over $\mathcal{D}$.

Warning 6.2.2.18. Monadicity is not transitive: in the situation of Proposition 6.2.2.17, if G'' exhibits $\mathcal{D}$ as monadic over $\mathcal{C}$ and G' exhibits $\mathcal{E}$ as monadic over $\mathcal{D}$, then G need not exhibit $\mathcal{E}$ as monadic over $\mathcal{C}$.

Proof. We will show that the functor G' satisfies the hypotheses of Theorem 6.2.2.5:

(1) The functor G' is conservative. For suppose that $\alpha : E \to E'$ is a morphism in $\mathcal{E}$ such that $G'(\alpha)$ is an equivalence. Then $G(\alpha)$ is an equivalence. Since G is conservative (by Theorem 6.2.2.5), we deduce that α is an equivalence.

(2) Let $E_\bullet$ be a simplicial object of $\mathcal{E}$, and suppose that $G'E_\bullet$ is a split simplicial object of $\mathcal{D}$. Then $GE_\bullet = G''G'E_\bullet$ is a split simplicial object of $\mathcal{C}$. Since G exhibits $\mathcal{E}$ as monadic over $\mathcal{C}$, Theorem 6.2.2.5 implies that $E_\bullet$ admits a colimit $|E_\bullet|$ in $\mathcal{E}$, and that the canonical map $\alpha : |GE_\bullet| \to G|E_\bullet|$ is an equivalence in $\mathcal{C}$. We wish to show that $\beta : |G'E_\bullet| \to G'|E_\bullet|$ is an equivalence in $\mathcal{D}$. Since G'' is conservative, it will suffice to show that $G''(\beta)$ is an equivalence in $\mathcal{C}$. Using the commutative diagram

$$
\begin{array}{ccc}
& G''|G'E_\bullet| & \\
{\scriptstyle \gamma} \nearrow & & \searrow {\scriptstyle G''(\beta)} \\
|GE_\bullet| & \xrightarrow[\;\;\alpha\;\;]{} & G|E_\bullet|,
\end{array}
$$

we are reduced to proving that γ is an equivalence. In other words, we must show that G'' preserves the colimit of the split simplicial object $G'E_\bullet$, which follows from Remark 6.2.1.4.

$\square$

6.2.3   BiCartesian Fibrations

Recall that a pair of functors

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

are adjoint if there exists a correspondence $q : \mathcal{M} \to \Delta^1$ with $q^{-1}\{0\} \simeq \mathcal{C}$ and $q^{-1}\{1\} \simeq \mathcal{D}$, such that q is both a coCartesian fibration associated to the functor $F : \mathcal{C} \to \mathcal{D}$ and a Cartesian fibration associated to $G : \mathcal{D} \to \mathcal{C}$ (see §T.5.2.2). In this case, F is determined up to equivalence by G and vice-versa. The formation of adjoint functors is compatible with composition up to homotopy: that is, we are are given a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $F' : \mathcal{D} \to \mathcal{E}$ having right adjoints G and G', respectively, then $G \circ G'$ is right adjoint to $F' \circ F$ (Proposition T.5.2.2.6). In some cases, it is necessary to be more precise: for example, we would like to know that an adjunction between F and G and an adjunction between F' and G' determines an adjunction between $G \circ G'$ and $F' \circ F$ up to a contractible space of choices, and that the resulting "composition law" for adjunctions is associative up to coherent homotopy. To formulate this idea more precisely, it is convenient to adopt the following definition:

Definition 6.2.3.1. We will say that a map of simplicial sets $q : X \to S$ is a *biCartesian fibration* if it is both a Cartesian fibration and a coCartesian fibration.

Fix a simplicial set S. We let $(\mathcal{S}et_\Delta^+)_{/S}$ denote the category of marked simplicial sets over S (see §T.3.1). An object of $(\mathcal{S}et_\Delta^+)_{/S}$ is given by a pair $(f : X \to S, M)$, where f is a map of simplicial sets and M is a collection of edges in X which contains all degenerate edges. The category $(\mathcal{S}et_\Delta^+)_{/S}$ can be endowed with the *Cartesian model structure* of Proposition T.3.1.3.7, whose fibrant objects are those pairs $(f : X \to S, M)$ where f is a Cartesian fibration and M is the collection of f-Cartesian edges of X. Similarly, $(\mathcal{S}et_\Delta^+)_{/S}$ can be endowed with the *coCartesian model structure* of Remark T.3.1.3.9, whose fibrant objects are pairs $(f : X \to S, M)$ where f is a coCartesian fibration and M is the collection of f-coCartesian edges of X. We now introduce a variant on these model structures, which is adapted to a discussion of biCartesian fibrations.

Definition 6.2.3.2. Let S be a simplicial set. We define a category $(\mathcal{S}et_\Delta^{++})_{/S}$ as follows:

(a) The objects of $(\mathcal{S}et_\Delta^{++})_{/S}$ are triples (X, E, E') where X is a simplicial set equipped with a map $X \to S$ and $E, E' \subseteq \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^1, X)$ are sets of edges which contain all degenerate edges of X.

(b) Given a pair of objects $(X, E, E'), (Y, F, F') \in (\mathcal{S}et_\Delta^{++})_{/S}$, a morphism from (X, E, E') to (Y, F, F') is a map of simplicial sets $f : X \to Y$ such that $f(E) \subseteq F$, $f(E') \subseteq F'$, and the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\quad f \quad} & Y \\
& \searrow \qquad \swarrow & \\
& S &
\end{array}
$$

is commutative.

We will say that an object $(X, E, E') \in (\mathcal{S}et_\Delta^{++})_{/S}$ is *S-bifibered* if the map $p : X \to S$ is a biCartesian fibration, E is the collection of p-Cartesian edges of X, and E' is the collection of p-coCartesian edges of X.

Remark 6.2.3.3. Fix a simplicial set S. For every object $(X, E, E') \in (\mathcal{S}et_\Delta^{++})_{/S}$ and every simplicial set K, we define a new object $K \otimes (X, E, E') \in (\mathcal{S}et_\Delta^{++})_{/S}$ by the formula

$$K \otimes (X, E, E') = (K \times X, \phi^{-1}(E), \phi^{-1}(E')),$$

where $\phi : K \times X \to X$ denotes the projection onto the second factor. Via this construction, we can regard $(\mathcal{S}et_\Delta^{++})_{/S}$ as enriched, tensored, and cotensored over the category of simplicial sets. In particular, for every pair of objects $\overline{X} = (X, E, E'), \overline{Y} = (Y, F, F') \in (\mathcal{S}et_\Delta^{++})_{/S}$, we have a mapping space $\mathrm{Map}_{(\mathcal{S}et_\Delta^{++})_{/S}}(\overline{X}, \overline{Y}) \subseteq \mathrm{Fun}_S(X, Y)$.

Remark 6.2.3.4. Fix a simplicial set S. Suppose we are given a morphism $\alpha : \overline{X} = (X, E, E') \to (Y, F, F') = \overline{Y}$ in $(\mathcal{S}et_\Delta^{++})_{/S}$, and that $\overline{Z} = (Z, G, G') \in (\mathcal{S}et_\Delta^{++})_{/S}$ is S-bifibered. Then α induces a categorical fibration of ∞-categories $\mathrm{Fun}_S(Y, Z) \to \mathrm{Fun}_S(X, Z)$. The simplicial sets $\mathrm{Map}_{(\mathcal{S}et_\Delta^{++})_{/S}}(\overline{X}, \overline{Z})$ can be described as the subcategory of $\mathrm{Fun}_S(X, Z)$ whose objects are maps $f : X \to Y$ such that $f(E) \subseteq G$ and $f(E') \subseteq G'$, and whose morphisms are equivalences. The simplicial set $\mathrm{Map}_{(\mathcal{S}et_\Delta^{++})_{/S}}(\overline{Y}, \overline{Z})$ has an analogous description as a subcategory of $\mathrm{Fun}_S(Y, Z)$. It follows that composition with α induces a Kan fibration of Kan complexes

$$\mathrm{Map}_{(\mathcal{S}et_\Delta^{++})_{/S}}(\overline{Y}, \overline{Z}) \to \mathrm{Map}_{(\mathcal{S}et_\Delta^{++})_{/S}}(\overline{X}, \overline{Z}).$$

Definition 6.2.3.5. Let S be a simplicial set and let $f : \overline{X} \to \overline{Y}$ be a morphism in $(\mathcal{S}et_\Delta^{++})_{/S}$. We will say that f is a *biCartesian equivalence* if, for every S-bifibered object $\overline{Z} \in (\mathcal{S}et_\Delta^{++})_{/S}$, composition with f induces a homotopy equivalence of Kan complexes

$$\mathrm{Map}_{(\mathcal{S}et_\Delta^{++})_{/S}}(\overline{Y}, \overline{Z}) \to \mathrm{Map}_{(\mathcal{S}et_\Delta^{++})_{/S}}(\overline{X}, \overline{Z}).$$

Example 6.2.3.6. Let $f : (X, E, E') \to (Y, F', F')$ be a morphism between S-bifibered objects of $(\mathcal{S}et_\Delta^{++})_{/S}$. Then f is a biCartesian equivalence if and only if it admits a homotopy inverse in the simplicial category $(\mathcal{S}et_\Delta^{++})_{/S}$. Using Proposition T.3.1.3.5, we see that this is equivalent to the requirement that for each vertex $s \in S$, the induced map $f_s : X_s \to Y_s$ is an equivalence of ∞-categories.

Remark 6.2.3.7. Fix a simplicial set S, and let $(\mathcal{S}et_\Delta^+)_{/S}$ denote the category of marked simplicial sets over S. There are evident forgetful functors $(\mathcal{S}et_\Delta^+)_{/S} \overset{\pi}{\leftarrow} (\mathcal{S}et_\Delta^{++})_{/S} \overset{\pi'}{\to} (\mathcal{S}et_\Delta^+)_{/S}$, given by the formulas

$$\pi(X, E, E') = (X, E) \qquad \pi'(X, E, E') = (X, E').$$

Note that $\overline{X} = (X, E, E')$ is S-bifibered if and only if $\pi(\overline{X})$ is fibrant with respect to the Cartesian model structure on $(\mathcal{S}et_\Delta^+)_{/S}$ and $\pi'(\overline{X})$ is fibrant with respect to the coCartesian model structure on $(\mathcal{S}et_\Delta^+)_{/S}$.

The functors π and π' admit left adjoints $\pi^*, \pi'^* : (\mathcal{S}et_\Delta^+)_{/S} \to (\mathcal{S}et_\Delta^{++})_{/S}$, given by the formulas

$$\pi^*(X, E) = (X, E, E_0) \qquad \pi'^*(X, E) = (X, E_0, E)$$

where E_0 denotes the collection of all degenerate edges in X. It follows immediately from the definitions that π^* carries Cartesian equivalences in $(\mathcal{S}et_\Delta^+)_{/S}$ to biCartesian equivalences in $(\mathcal{S}et_\Delta^{++})_{/S}$, and that π'^* carries coCartesian equivalences in $(\mathcal{S}et_\Delta^+)_{/S}$ to biCartesian equivalences in $(\mathcal{S}et_\Delta^{++})_{/S}$.

Note that an an object $\overline{X} \in (\mathcal{S}et_\Delta^{++})_{/S}$ is S-bifibered if and only if, for every marked anodyne morphism $\alpha : \overline{Y} \to \overline{Z}$ in $(\mathcal{S}et_\Delta^+)_{/S}$, $\overline{X}$ has the extension property with respect to $\pi^*(\alpha)$ and $\pi'^*(\alpha^{op})$. Arguing as in the proof of Proposition T.3.1.3.7, we deduce the following:

Lemma 6.2.3.8. *Let S be a simplicial set. There exists a functor $T : (\mathcal{S}et_\Delta^{++})_{/S} \to (\mathcal{S}et_\Delta^{++})_{/S}$ and a natural transformation $u : \mathrm{id} \to T$ with the following properties:*

(1) *For every object $\overline{X} \in (\mathcal{S}et_\Delta^{++})_{/S}$, the object $T\overline{X}$ is S-bifibered.*

(2) *For every object $\overline{X} \in (\mathcal{S}et_\Delta^{++})_{/S}$, the natural transformation u induces a biCartesian equivalence $\overline{X} \to T\overline{X}$.*

(3) *The functor T commutes with filtered colimits.*

If T is the functor of Lemma 6.2.3.8, then a map $\alpha : \overline{X} \to \overline{Y}$ is a biCartesian equivalence if and only if the induced map $T(\alpha) : T\overline{X} \to T\overline{Y}$ is a biCartesian equivalence. By Example 6.2.3.6, this is equivalent to the requirement that $T(\alpha)$ induces a categorical equivalence after passing to the fiber over each vertex $s \in S$. Since the collection of categorical equivalences in $\mathcal{S}et_\Delta$ is perfect (in the sense of Definition T.A.2.6.10), Corollary T.A.2.6.12 guarantees the following:

Lemma 6.2.3.9. *Let S be a simplicial set. The collection of biCartesian equivalences in $(\mathcal{S}et_{\Delta}^{+})_{/S}$ is perfect (in particular, it is stable under filtered colimits).*

Theorem 6.2.3.10. *Let S be a simplicial set. There exists a combinatorial left proper model structure on the category $(\mathcal{S}et_{\Delta}^{++})_{/S}$, which can be characterized as follows:*

(C) *A morphism $\alpha : (X, E, E') \to (Y, F, F')$ in $(\mathcal{S}et_{\Delta}^{++})_{/S}$ is a cofibration if and only if the underlying map of simplicial sets $X \to Y$ is a monomorphism.*

(W) *A morphism $\alpha : (X, E, E') \to (Y, F, F')$ in $(\mathcal{S}et_{\Delta}^{++})_{/S}$ is a weak equivalence if and only if it is a biCartesian equivalence.*

(F) *A morphism $\alpha : (X, E, E') \to (Y, F, F')$ in $(\mathcal{S}et_{\Delta}^{++})_{/S}$ is a fibration if and only it has the right lifting property with respect to every morphism β which is both a cofibration and a weak equivalence.*

We will refer to the model structure of Theorem 6.2.3.10 as the *biCartesian model structure on $(\mathcal{S}et_{\Delta}^{++})_{/S}$.*

Proof. It will suffice to show that the hypotheses of Proposition T.A.2.6.13 are satisfied:

(1) The collection of biCartesian equivalences is perfect (Lemma 6.2.3.9).

(2) The collection of biCartesian equivalences is closed under pushouts by cofibrations. This follows from Remark 6.2.3.4, together with the right properness of the usual model structure on the category of simplicial sets.

(3) Let $\alpha : (X, E, E') \to (Y, F, F')$ be a morphism which has the right lifting property with respect to every cofibration. Then the underlying map $X \to Y$ is a trivial Kan fibration of simplicial sets, and we have $E = \alpha^{-1}F$ and $E' = \alpha^{-1}F'$. It follows that α admits a section β, and the composition $\beta \circ \alpha$ is (fiberwise) homotopic to the identity map form (X, E, E') to itself; in particular, α is a biCartesian equivalence.

$\square$

Remark 6.2.3.11. For every simplicial set S, the biCartesian model structure on $(\mathcal{S}et_{\Delta}^{++})_{/S}$ is compatible with the simplicial structure of Remark 6.2.3.3.

Remark 6.2.3.12. Let S be a simplicial set. Let us regard $(\mathcal{S}et_{\Delta}^{+})_{/S}$ as endowed with the Cartesian model structure of §T.3.1.3, and let $(\mathcal{S}et_{\Delta}^{+})_{/S'}$ denote the same category $(\mathcal{S}et_{\Delta}^{+})_{/S}$ endowed with the coCartesian model structure. The functors $(\mathcal{S}et_{\Delta}^{+})_{/S} \xleftarrow{\pi} (\mathcal{S}et_{\Delta}^{++})_{/S} \xrightarrow{\pi'} (\mathcal{S}et_{\Delta}^{+})'_{/S}$ of Remark 6.2.3.7 are simplicial right Quillen functors, and induce limit-preserving functors between the underlying ∞-categories

$$\mathrm{N}((\mathcal{S}et_{\Delta}^{+})^{\circ}_{/S}) \leftarrow \mathrm{N}((\mathcal{S}et_{\Delta}^{++})^{\circ}_{/S}) \to \mathrm{N}((\mathcal{S}et_{\Delta}^{+})^{\circ}_{/S'}).$$

These functors allow us to identify $\mathrm{N}((\mathcal{S}et_{\Delta}^{++})^{\circ}_{/S})$ with a subcategory of either $\mathrm{N}((\mathcal{S}et_{\Delta}^{+})^{\circ}_{/S})$ or $\mathrm{N}((\mathcal{S}et_{\Delta}^{+})^{\circ}_{/S'})$.

Our next goal is to identify the image of $\mathrm{N}(((\mathcal{S}et_{\Delta}^{++})_{/S})^{\circ})$ under these equivalences. We begin by recalling a few definitions.

Definition 6.2.3.13. Suppose we are given a diagram of ∞-categories σ :

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{G} & \mathcal{D} \\
\downarrow{\scriptstyle U} & & \downarrow{\scriptstyle V} \\
\mathcal{C}' & \xrightarrow{G'} & \mathcal{D}'
\end{array}
$$

which commutes up to a specified equivalence $\alpha : V \circ G \simeq G' \circ U$. We say that this diagram σ is *left adjointable* if the functors G and G' admit left adjoints F and F', respectively, and if the composite transformation

$$F' \circ V \to F' \circ V \circ G \circ F \stackrel{\alpha}{\simeq} F' \circ G' \circ U \circ F \to U \circ F$$

is an equivalence. We say that σ is *right adjointable* if the functors G and G' admit right adjoints H and H', and the composite transformation

$$U \circ H \to H' \circ G' \circ U \circ H \stackrel{\alpha^{-1}}{\longrightarrow} H' \circ V \circ G \circ H \to H' \circ V$$

is an equivalence.

Remark 6.2.3.14. Suppose we are given a commutative diagram of ∞-categories σ :

$$
\begin{array}{ccc}
\mathcal{C} & \stackrel{G}{\longrightarrow} & \mathcal{D} \\
\downarrow{\scriptstyle U} & & \downarrow{\scriptstyle V} \\
\mathcal{C}' & \stackrel{G'}{\longrightarrow} & \mathcal{D}'
\end{array}
$$

where the functors G and G' admit left adjoints, and the functors U and V admit right adjoints. Then σ is left adjointable if and only if the transposed diagram

$$
\begin{array}{ccc}
\mathcal{C} & \stackrel{U}{\longrightarrow} & \mathcal{C}' \\
\downarrow{\scriptstyle G} & & \downarrow{\scriptstyle G'} \\
\mathcal{D} & \stackrel{V}{\longrightarrow} & \mathcal{D}'
\end{array}
$$

is right adjointable.

Remark 6.2.3.15. In classical category theory, the adjointability of a commutative square of categories is sometimes referred to as the *Beck-Chevalley condition*.

Definition 6.2.3.16. Let S be a simplicial set. We define subcategories

$$\mathrm{Fun}^{\mathrm{LAd}}(S, \mathcal{C}\mathrm{at}_\infty), \mathrm{Fun}^{\mathrm{RAd}}(S, \mathcal{C}\mathrm{at}_\infty) \subseteq \mathrm{Fun}(S, \mathcal{C}\mathrm{at}_\infty)$$

as follows:

(1) Let $F \in \mathrm{Fun}(S, \mathcal{C}\mathrm{at}_\infty)$. Then F belongs to $\mathrm{Fun}^{\mathrm{LAd}}(S, \mathcal{C}\mathrm{at}_\infty)$ ($\mathrm{Fun}^{\mathrm{RAd}}(S, \mathcal{C}\mathrm{at}_\infty)$) if and only if, for every edge $s \to s'$ in S, the induced functor $F(s) \to F(s')$ admits a left adjoint (right adjoint).

(2) Let $\alpha : F \to F'$ be a morphism in $\mathrm{Fun}(S, \mathcal{C}\mathrm{at}_\infty)$, where F and F' belong to $\mathrm{Fun}^{\mathrm{LAd}}(S, \mathcal{C}\mathrm{at}_\infty)$ ($\mathrm{Fun}^{\mathrm{RAd}}(S, \mathcal{C}\mathrm{at}_\infty)$). Then α is a morphism in $\mathrm{Fun}^{\mathrm{LAd}}(S, \mathcal{C}\mathrm{at}_\infty)$ ($\mathrm{Fun}^{\mathrm{RAd}}(S, \mathcal{C}\mathrm{at}_\infty)$) if and only if, for every edge $s \to s'$ in S, the diagram

$$
\begin{array}{ccc}
F(s) & \longrightarrow & F(s') \\
\downarrow & & \downarrow \\
F'(s) & \longrightarrow & F'(s')
\end{array}
$$

is left adjointable (right adjointable).

The following result is essentially an unwinding of definitions (see Remark T.7.3.1.3):

Proposition 6.2.3.17. *Let S be a simplicial set.*

(1) *Let $\phi : \mathrm{N}((\mathcal{S}et_\Delta^{++})^o_{/S}) \hookrightarrow \mathrm{N}((\mathcal{S}et_\Delta^+)^o_{/S})$ be the monomorphism by the right Quillen functor π of Remark 6.2.3.12, and let $\psi : \mathrm{N}((\mathcal{S}et_\Delta^+)^o_{/S}) \simeq \mathrm{Fun}(S^{op}, \mathrm{Cat}_\infty)$ be the equivalence furnished by Theorem T.3.2.0.1 and Proposition T.4.2.4.4. Then the composition $\psi \circ \phi$ induces an equivalence from $\mathrm{N}((\mathcal{S}et_\Delta^{++})^o_{/S})$ onto the subcategory $\mathrm{Fun}^{\mathrm{LAd}}(S^{op}, \mathrm{Cat}_\infty) \subseteq \mathrm{Fun}(S^{op}, \mathrm{Cat}_\infty)$.*

(2) *Let $\phi' : \mathrm{N}((\mathcal{S}et_\Delta^{++})^o_{/S}) \hookrightarrow \mathrm{N}((\mathcal{S}et_\Delta^+)^o_{/S'})$ be the monomorphism by the right Quillen functor π' of Remark 6.2.3.12, and let $\psi' : \mathrm{N}((\mathcal{S}et_\Delta^+)^o_{/S'}) \simeq \mathrm{Fun}(S, \mathrm{Cat}_\infty)$ be the equivalence furnished by Theorem T.3.2.0.1 and Proposition T.4.2.4.4. Then the composition $\psi' \circ \phi'$ induces an equivalence from $\mathrm{N}((\mathcal{S}et_\Delta^{++})^o_{/S})$ onto the subcategory $\mathrm{Fun}^{\mathrm{RAd}}(S, \mathrm{Cat}_\infty) \subseteq \mathrm{Fun}(S, \mathrm{Cat}_\infty)$.*

Corollary 6.2.3.18. *Let S be a simplicial set. Then:*

(1) *The ∞-categories $\mathrm{Fun}^{\mathrm{LAd}}(S, \mathrm{Cat}_\infty)$ and $\mathrm{Fun}^{\mathrm{RAd}}(S, \mathrm{Cat}_\infty)$ are presentable. In particular, they admit small limits.*

(2) *The inclusions $\mathrm{Fun}^{\mathrm{LAd}}(S, \mathrm{Cat}_\infty) \subseteq \mathrm{Fun}(S, \mathrm{Cat}_\infty)$ and $\mathrm{Fun}^{\mathrm{RAd}}(S, \mathrm{Cat}_\infty) \subseteq \mathrm{Fun}(S, \mathrm{Cat}_\infty)$ admit left adjoints; in particular, they preserve small limits.*

(3) *There is a canonical equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{LAd}}(S^{op}, \mathrm{Cat}_\infty) \simeq \mathrm{Fun}^{\mathrm{RAd}}(S, \mathrm{Cat}_\infty).$$

We conclude this section by sketching an application of the above ideas, which will be useful. Recall that Pr^{L} denotes the subcategory of $\widehat{\mathrm{Cat}}_\infty$ whose objects are presentable ∞-categories and whose morphisms are functors which preserve small colimits, and $\mathrm{Pr}^{\mathrm{R}} \subseteq \widehat{\mathrm{Cat}}_\infty$ the subcategory whose objects are presentable ∞-categories and whose morphisms are accessible functors which preserve small limits. The following result gives a useful criterion for commuting limits and colimits in Pr^{L} (or Pr^{R}).

Proposition 6.2.3.19. *Let S and T be simplicial sets, and suppose we are given a diagram $\chi : S \times T \to \mathrm{Pr}^{\mathrm{L}}$ with the following properties:*

($*$) *For every edge $s \to s'$ in S and every edge $t \to t'$ in T, the diagram of ∞-categories*

$$
\begin{array}{ccc}
\chi(s,t) & \longrightarrow & \chi(s',t) \\
\downarrow & & \downarrow \\
\chi(s,t') & \longrightarrow & \chi(s',t')
\end{array}
$$

   is right adjointable.

Then there exists a diagram $\overline{\chi} : S^{\triangleright} \times T^{\triangleleft} \to \mathrm{Pr}^{\mathrm{L}}$ with the following properties:

(a) *For each $t \in T^{\triangleleft}$, the map $\overline{\chi}_t : S^{\triangleright} \times \{t\} \to \mathrm{Pr}^{\mathrm{L}}$ is a colimit diagram.*

(b) *For each $s \in S^{\triangleright}$, the map $\overline{\chi}_s : \{s\} \times T^{\triangleleft} \to \mathrm{Pr}^{\mathrm{L}}$ is a limit diagram.*

(c) *For every edge $s \to s'$ in $S^{\triangleright}$ and every edge $t \to t'$ in $T^{\triangleleft}$, the diagram of ∞-categories*

$$
\begin{array}{ccc}
\chi(s,t) & \longrightarrow & \chi(s',t) \\
\downarrow & & \downarrow \\
\chi(s,t') & \longrightarrow & \chi(s',t')
\end{array}
$$

   is right adjointable.

More informally: the hypothesis $(*)$ of Proposition 6.2.3.19 guarantee that the canonical map

$$\varinjlim_{s \in S} \varprojlim_{t \in T} \chi(s, t) \to \varprojlim_{t \in T} \varinjlim_{s \in S} \chi(s, t)$$

is an equivalence in the ∞-category Pr^{L}.

Proof. Let $\mathrm{Fun}^{\mathrm{RAd}}(S, \widehat{\mathrm{Cat}}_\infty)$ be as in Definition 6.2.3.16, but in the setting of ∞-categories which are not necessarily small. Using assumption $(*)$, we can identify χ with a map $\chi : T \to \mathrm{Fun}^{\mathrm{RAd}}(S, \widehat{\mathrm{Cat}}_\infty)$. Let $\chi' : T \to \mathrm{Fun}^{\mathrm{LAd}}(S^{op}, \widehat{\mathrm{Cat}}_\infty)$ be the composition of χ with the equivalence of Corollary 6.2.3.18, and identify χ' with a functor $S^{op} \times T \to \widehat{\mathrm{Cat}}_\infty$.

Let s_0 denote the cone point of $S^\triangleleft$, and let $\widetilde{\chi}' : (S^\triangleleft)^{op} \times T \to \widehat{\mathrm{Cat}}_\infty$ be a right Kan extension of χ'. We claim that, for every vertex $s \in S$ and every edge $t \to t'$ in T, the diagram of ∞-categories

$$
\begin{array}{ccc}
\widetilde{\chi}'(s_0, t) & \longrightarrow & \widetilde{\chi}'(s, t) \\
\downarrow & & \downarrow \\
\widetilde{\chi}'(s_0, t') & \longrightarrow & \widetilde{\chi}'(s, t')
\end{array}
$$

is left adjointable. Using Corollary T.5.5.3.4 and Theorem T.5.5.3.18, we conclude that the horizontal functors are morphisms of Pr^{R} and therefore admit left adjoints. It will therefore suffice to show that the diagram σ :

$$
\begin{array}{ccc}
\widetilde{\chi}'(s_0, t) & \longrightarrow & \widetilde{\chi}'(s_0, t') \\
\downarrow & & \downarrow \\
\widetilde{\chi}'(s, t) & \longrightarrow & \widetilde{\chi}'(s, t')
\end{array}
$$

is right adjointable (Remark 6.2.3.14). This follows from Corollary 6.2.3.18, since for every edge $s \to s'$ in S the diagram

$$
\begin{array}{ccc}
\chi'(s', t) & \longrightarrow & \chi'(s', t') \\
\downarrow & & \downarrow \\
\chi'(s, t) & \longrightarrow & \chi'(s, t')
\end{array}
$$

is right adjointable (by Remark 6.2.3.14).

Let $\overline{\chi}' : (S^\triangleright)^{op} \times T^\triangleleft \to \widehat{\mathrm{Cat}}_\infty$ be a right Kan extension of $\widetilde{\chi}'$. It follows from Corollary 6.2.3.18 that, for every edge $s \to s'$ in $S^\triangleright$ and every edge $t \to t'$ in $T^\triangleleft$, the diagram

$$
\begin{array}{ccc}
\overline{\chi}'(s', t) & \longrightarrow & \overline{\chi}'(s, t) \\
\downarrow & & \downarrow \\
\overline{\chi}'(s', t') & \longrightarrow & \overline{\chi}'(s, t')
\end{array}
$$

is left adjointable. Consequently, we may identify $\overline{\chi}'$ with a map $T^\triangleleft \to \mathrm{Fun}^{\mathrm{LAd}}((S^\triangleright)^{op}, \widehat{\mathrm{Cat}}_\infty)$. Let $\overline{\chi} : T^\triangleleft \to \mathrm{Fun}^{\mathrm{RAd}}(S^\triangleright, \widehat{\mathrm{Cat}}_\infty)$ be the composition of $\overline{\chi}'$ with the equivalence of Corollary 6.2.3.18. We will identify $\overline{\chi}$ with a functor $S^\triangleright \times T^\triangleleft \to \widehat{\mathrm{Cat}}_\infty$. The restriction $\overline{\chi}|(S \times T)$ is equivalent to χ; replacing $\overline{\chi}$ by an equivalent diagram if necessary, we may assume that $\overline{\chi}$ is an extension of χ. By construction, the functor $\overline{\chi}$ satisfies (c).

We next claim that the diagram $\overline{\chi}$ takes values in $\mathrm{Pr}^{\mathrm{L}} \subseteq \widehat{\mathrm{Cat}}_\infty$. To prove this, we first show that for every vertex $(s, t) \in S^\triangleright \times T^\triangleleft$, the ∞-category $\overline{\chi}(s, t) \simeq \overline{\chi}'(s, t)$ is presentable. If $(s, t) \in S \times T$, then the result is

obvious. If $s = s_0$ and $t \in T$, then $\overline{\chi}'(s,t) \simeq \varinjlim \chi'_t$ is presentable by Theorem T.5.5.3.18. If $s \in S$ and t is the cone point of $T^{\triangleright}$, then $\overline{\chi}'(s,t) \simeq \varprojlim \chi'_s$ is presentable by Proposition T.5.5.3.13. The same argument will apply when $s = s_0$, provided that we can show that the diagram $\widetilde{\chi}'|(\{s_0\} \times T)$ takes values in $\mathcal{P}r^{L}$. In other words, we must show that for every edge $t \to t'$ in T, the induced functor $\widetilde{\chi}'(s_0, t) \to \widetilde{\chi}'(s_0, t')$ preserves small colimits. This is obvious if $S = \emptyset$ (in that case, both ∞-categories are contractible). Otherwise, we can choose a vertex $s \in S$ and the desired result follows from the left adjointability of the diagram σ considered above.

We next show that for every edge $\alpha : (s,t) \to (s',t')$ in $S \times T$, the induced functor $\overline{\chi}(\alpha) : \overline{\chi}(s,t) \to \overline{\chi}(s',t')$ preserves small colimits. We can factor $\overline{\chi}(\alpha)$ as a composition

$$\overline{\chi}(s,t) \xrightarrow{U} \overline{\chi}(s,t') \xrightarrow{V} \overline{\chi}(s',t').$$

The functor V admits a right adjoint, since $\overline{\chi}$ satisfies (c). It therefore suffices to show that U preserves small colimits. This was established above when $t \to t'$ is an edge of T. If t is the cone point of $T^{\triangleright}$, the desired result follows from Proposition T.5.5.3.13.

It remains to verify that $\overline{\chi}$ satisfies (a) and (b). Assertion (b) follows immediately from Proposition T.5.5.3.13. To prove (a), it will suffice (by Corollary T.5.5.3.4) to show that $\overline{\chi}'_t$ is a limit diagram in $\mathcal{P}r^{R}$. According to Theorem T.5.5.3.18, this is equivalent to the condition that $\overline{\chi}'_t$ is a limit diagram in $\widehat{\mathcal{C}at}_{\infty}$. This is true by construction for $t \in T$, and follows in general using Lemma T.5.5.2.3. $\square$

6.2.4   Descent and the Beck-Chevalley Condition

Let $f : A \to B$ be a homomorphism of commutative rings, and let M^0 be a B-module. The classical theory of *descent* is an attempt to answer the following question: under what circumstances can one find an A-module M and an isomorphism $\eta : M \otimes_A B \simeq M^0$? A choice of such an isomorphism η determines an isomorphism of $B \otimes_A B$-modules

$$\tau : B \otimes_A M^0 \simeq M^0 \otimes_A B$$

(since both sides are canonically isomorphic to $M \otimes_A (B \otimes_A B)$). Moreover, the map τ satisfies a "cocycle condition": namely, it renders the diagram

commutative, where τ' is given by τ on the outer factors and the identity id_B on the middle factor. A *descent datum* is a pair (M^0, τ), where M^0 is a B-module and $\tau : B \otimes_A M^0 \to M^0 \otimes_A B$ is an isomorphism of $B \otimes_A B$-modules making the above diagram commute. The collection of descent data can be organized into a category $\mathrm{Desc}(f)$, and the construction $M \mapsto M \otimes_A B$ determines a functor F from the category of A-modules into $\mathrm{Desc}(f)$. A classical theorem of Grothendieck asserts that this functor is an equivalence of categories when $f : A \to B$ is a faithfully flat morphism of commutative rings.

In this section, we will explain how to use the Barr-Beck theorem (Theorem 6.2.2.5) to set up an ∞-categorical version of the machinery of descent theory. We begin by recasting the problem. Let $f : A \to B$ be a map of commutative rings. Taking the Čech nerve of f (in the opposite of the category of commutative rings), we obtain a cosimplicial commutative ring $B^{\bullet}$, where B^n denotes the nth tensor power of B over A. The category $\mathrm{Desc}(f)$ can be identified with the category of cosimplicial modules $M^{\bullet}$ over $B^{\bullet}$ satisfying the requirement that for every map $[m] \to [n]$ in $\boldsymbol{\Delta}$, the induced map $M^m \otimes_{B^m} B^n \to M^n$ is an isomorphism. Put another way, $\mathrm{Desc}(f)^{op}$ can be identified with the homotopy limit of a cosimplicial category $\mathcal{C}^{\bullet}$, where each $\mathcal{C}^n$ is given by the opposite of the category of modules over the commutative ring B^n. In good cases (for example if f is faithfully flat), one can show that this homotopy limit is equivalent to the (opposite of the) category of A-modules. Our goal is to develop some general tools for proving results of this kind.

Very broadly, we can state our main problem as follows: given a cosimplicial ∞-category $\mathcal{C}^\bullet$, we would like to understand the totalization $\varprojlim \mathcal{C}^\bullet$. There is an evident forgetful functor $\varprojlim \mathcal{C}^\bullet \to \mathcal{C}^0$. By analogy with the situation considered above, it is natural to expect that an object of $\varprojlim \mathcal{C}^\bullet$ can be identified with an object of $\mathcal{C}^0$ together with some sort of "descent data." One way to articulate this idea precisely is to show that the forgetful functor $\varprojlim \mathcal{C}^\bullet \to \mathcal{C}^0$ is *monadic*: that is, it exhibits $\varprojlim \mathcal{C}^\bullet$ as the ∞-category $\mathrm{LMod}_T(\mathcal{C}^0)$ for some monad T on $\mathcal{C}^0$. This is true under very general assumptions (Proposition 6.2.4.1). Moreover, if we are willing to make some mild assumptions on $\mathcal{C}^\bullet$, then we can obtain an explicit description of the monad T (Theorem 6.2.4.2). Using this description, we can give a simple criterion for a functor $\mathcal{C}^{-1} \to \varprojlim \mathcal{C}^\bullet$ (encoded by an augmented cosimplicial ∞-category extending $\mathcal{C}^\bullet$) to be an equivalence (Corollary 6.2.4.3).

We begin with a very general result about the limit of a diagram of ∞-categories.

Proposition 6.2.4.1. *Let $\mathcal{J}$ be a small ∞-category and let $q : \mathcal{J} \to \mathcal{C}\mathrm{at}_\infty$ be a diagram having a limit $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$. Let $0 \in \mathcal{J}$ be an object and let $\mathcal{C}^0 = q(0)$. Suppose that the following conditions are satisfied:*

(1) *The functor $G : \mathcal{C} \to \mathcal{C}^0$ admits a left adjoint F.*

(2) *For every object $J \in \mathcal{J}$, there exists a morphism $0 \to J$ in $\mathcal{J}$.*

Then the adjoint functors $\mathcal{C}^0 \overset{F}{\underset{G}{\rightleftarrows}} \mathcal{C}$ satisfy the hypotheses of Theorem 6.2.2.5, so that $\mathcal{C}$ is equivalent to the ∞-category of modules $\mathrm{LMod}_T(\mathcal{C}^0)$ for the induced monad $T \simeq G \circ F$ on $\mathcal{C}^0$.

Proof. We first show that G is conservative. Let $\alpha : C \to C'$ be a morphism in $\mathcal{C}$ such that $G(\alpha)$ is an equivalence. Using condition (2), we deduce that the image of α in $q(J)$ is an equivalence for each $J \in \mathcal{J}$, so that α is an equivalence.

Now suppose that $X_\bullet$ is a G-split simplicial object of $\mathcal{C}$; we wish to prove that $X_\bullet$ admits a colimit in $\mathcal{C}$ which is preserved by G. The diagram q classifies a coCartesian fibration $p : \overline{\mathcal{J}} \to \mathcal{J}$. According to Proposition T.3.3.3.1, we can identify $\mathcal{C}$ with the full subcategory of $\mathrm{Fun}_\mathcal{J}(\mathcal{J}, \overline{\mathcal{J}})$ spanned by those sections of p which carry every morphism in $\mathcal{J}$ to a p-coCartesian morphism in $\overline{\mathcal{J}}$. In particular, we can identify $X_\bullet$ with a functor $X : \mathcal{J} \times \mathrm{N}(\mathbf{\Delta})^{op} \to \overline{\mathcal{J}}$. We next prove:

(∗) There exists a functor $\overline{X} \in \mathrm{Fun}_\mathcal{J}(\mathcal{J} \times \mathrm{N}(\mathbf{\Delta}_+)^{op}, \overline{\mathcal{J}})$ which is a p-left Kan extension of X.

In view of Lemma T.4.3.2.13, it will suffice to show that for each object $J \in \mathcal{J}$, the diagram

$$
\begin{array}{ccc}
\mathcal{J}_{/J} \times \mathrm{N}(\mathbf{\Delta})^{op} & \longrightarrow & \overline{\mathcal{J}} \\
\downarrow & \nearrow & \downarrow {\scriptstyle p} \\
(\mathcal{J}_{J/} \times \mathrm{N}(\mathbf{\Delta})^{op})^{\triangleright} & \longrightarrow & \mathcal{J}
\end{array}
$$

admits an extension as indicated which is a p-colimit diagram. Because the inclusion $\{J\} \times \mathrm{N}(\mathbf{\Delta})^{op} \hookrightarrow \mathcal{J}_{J/} \times \mathrm{N}(\mathbf{\Delta})^{op}$ is left cofinal, it suffices to construct a p-colimit diagram $\overline{X}^J : \mathrm{N}(\mathbf{\Delta}_+)^{op} \to \overline{\mathcal{J}}_J$ which extends to the composite map $X_\bullet^J : \mathrm{N}(\mathbf{\Delta})^{op} \overset{X_\bullet}{\to} \mathcal{C} \to q(J) \simeq \overline{\mathcal{J}}_J$. According to Proposition T.4.3.1.10, it suffices to show that the simplicial object $X_\bullet^J$ admits a colimit in $q(J) = \overline{\mathcal{J}}_J$, and that this colimit is preserved by the functor $q(J) \to q(J')$ associated to each morphism in $\alpha : J \to J'$ in $\mathcal{J}$. For this, it is sufficient to show that $X_\bullet^J$ extends to a split simplicial object of $q(J)$. Using assumption (2), we can reduce to the case $J = 0$. We conclude by invoking the assumption that $X_\bullet$ is G-split.

Let $\overline{X}$ be as in (∗), so we can identify $\overline{X}$ with an augmented simplicial object of $\mathrm{Fun}_\mathcal{J}(\mathcal{J}, \overline{\mathcal{J}})$. It follows from Lemma 3.2.2.9 that this augmented simplicial object is a colimit diagram, which is obviously preserved by G. To complete the proof, it will suffice to show that this augmented simplicial object belongs to the full subcategory $\mathcal{C} \subseteq \mathrm{Fun}_\mathcal{J}(\mathcal{J}, \overline{\mathcal{J}})$. In other words, we must show that for every morphism $\alpha : J \to J'$ in $\mathcal{J}$, the induced map $\overline{X}(J, [-1]) \to \overline{X}(J', [-1])$ is p-coCartesian. This is a translation of the condition that the functor $q(\alpha) : q(J) \to q(J')$ preserves the colimit of the simplicial object $X_\bullet^J$ (since $X_\bullet^J$ is split). $\qquad \square$

In order to apply Proposition 6.2.4.1 in practice, we would like to understand the monad $G \circ F$ on the ∞-category $\mathcal{C}^0$. For this, we specialize to the case where $\mathcal{J} = \mathrm{N}(\boldsymbol{\Delta})$ (so that $q : \mathcal{J} \to \mathcal{C}\mathrm{at}_\infty$ can be identified with a cosimplicial ∞-category $\mathcal{C}^\bullet$). For each $n \geq 0$, we let d^0 denote the coface map $\mathcal{C}^n \to \mathcal{C}^{n+1}$ associated to the inclusion $[n] \hookrightarrow [0] \star [n] \simeq [n+1]$.

The main result of this section can be stated as follows:

Theorem 6.2.4.2. *Let* $\mathcal{C}^\bullet : \mathrm{N}(\boldsymbol{\Delta}) \to \mathcal{C}\mathrm{at}_\infty$ *be a cosimplicial ∞-category which satisfies the following property:*

($*$) *For every map* $\alpha : [m] \to [n]$ *in* $\boldsymbol{\Delta}$*, the induced diagram*

$$
\begin{array}{ccc}
\mathcal{C}^m & \xrightarrow{\ d^0\ } & \mathcal{C}^{m+1} \\
\downarrow & & \downarrow \\
\mathcal{C}^n & \xrightarrow{\ d^0\ } & \mathcal{C}^{n+1}
\end{array}
$$

is left adjointable. In particular, each coface map $d^0 : \mathcal{C}^n \to \mathcal{C}^{n+1}$ *admits a left adjoint* $F(n) : \mathcal{C}^{n+1} \to \mathcal{C}^n$.

Let $\mathcal{C} = \varprojlim \mathcal{C}^\bullet$ *be a totalization of* $\mathcal{C}^\bullet$*. Then:*

(1) *The forgetful functor* $G : \mathcal{C} \to \mathcal{C}^0$ *admits a left adjoint* F.

(2) *The diagram*

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ G\ } & \mathcal{C}^0 \\
\downarrow{\scriptstyle G} & & \downarrow{\scriptstyle d^1} \\
\mathcal{C}^0 & \xrightarrow{\ d^0\ } & \mathcal{C}^1
\end{array}
$$

is left adjointable. That is, the canonical map $F(0) \circ d^1 \to G \circ F$ *is an equivalence of functors from* $\mathcal{C}^0$ *to itself.*

(3) *The adjoint functors* $\mathcal{C}^0 \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}$ *satisfy the conditions of Theorem 6.2.2.5, so that* $\mathcal{C}$ *is equivalent to the ∞-category* $\mathrm{LMod}_T(\mathcal{C}^0)$ *of algebras over the monad* $T \simeq G \circ F \simeq F(0) \circ d^1$.

Before giving the proof of Theorem 6.2.4.2, we describe our main application.

Corollary 6.2.4.3. *Let* $\mathcal{C}^\bullet : \mathrm{N}(\boldsymbol{\Delta}_+) \to \mathcal{C}\mathrm{at}_\infty$ *be an augmented cosimplicial ∞-category, and set* $\mathcal{C} = \mathcal{C}^{-1}$*. Let* $G : \mathcal{C} \to \mathcal{C}^0$ *be the evident functor. Assume that:*

(1) *The ∞-category* $\mathcal{C}^{-1}$ *admits geometric realizations of G-split simplicial objects, and those geometric realizations are preserved by* G.

(2) *For every morphism* $\alpha : [m] \to [n]$ *in* $\boldsymbol{\Delta}_+$*, the diagram*

$$
\begin{array}{ccc}
\mathcal{C}^m & \xrightarrow{\ d^0\ } & \mathcal{C}^{m+1} \\
\downarrow & & \downarrow \\
\mathcal{C}^n & \xrightarrow{\ d^0\ } & \mathcal{C}^{n+1}
\end{array}
$$

is left adjointable.

Then the canonical map $\theta : \mathcal{C} \to \varprojlim_{n \in \boldsymbol{\Delta}} \mathcal{C}^n$ *admits a fully faithful left adjoint. If G is conservative, then θ is an equivalence.*

Proof. Let $\mathcal{D}^{\bullet} : N(\boldsymbol{\Delta}_+) \to \mathrm{Cat}_{\infty}$ be a limit of the diagram $\mathcal{C}^{\bullet} \,|\, N(\boldsymbol{\Delta})$, so that we have a map of cosimplicial ∞-categories $\alpha : \mathcal{C}^{\bullet} \to \mathcal{D}^{\bullet}$ which induces the identity map from $\mathcal{C}^n = \mathcal{D}^n$ to itself for $n \geq 0$. Using (2) and Theorem 6.2.4.2, we conclude that the canonical map $G' : \mathcal{D}^{-1} \to \mathcal{D}^0 = \mathcal{C}^0$ admits a left adjoint F' which satisfies the hypotheses of Theorem 6.2.2.5. Applying Corollary 6.2.2.14 (and Remark 6.2.2.15) to the diagram

$$
\begin{array}{ccc}
\mathcal{C}^{-1} & \xrightarrow{\;\;U\;\;} & \mathcal{D}^{-1} \\
& {\scriptstyle G}\searrow \quad \swarrow {\scriptstyle G'} & \\
& \mathcal{C}^0, &
\end{array}
$$

we are reduced to showing that the canonical map $G'F' \to GF$ is an equivalence of functors from $\mathcal{C}^0$ to itself. This is clear: assumption (2) and Theorem 6.2.4.2 allow us to identify both functors with the composition

$$
\mathcal{C}^0 \xrightarrow{d^1} \mathcal{C}^1 \xrightarrow{T} \mathcal{C}^0,
$$

where T is a left adjoint to the face map $d^0 : \mathcal{C}^0 \to \mathcal{C}^1$. $\qquad \square$

Proof of Theorem 6.2.4.2. Let $\overline{\mathcal{C}}^{\bullet} : N(\boldsymbol{\Delta}_+) \to \mathrm{Cat}_{\infty}$ be a limit of the diagram $\mathcal{C}^{\bullet}$. Let $T : N(\boldsymbol{\Delta}_+) \times \Delta^1 \to N(\boldsymbol{\Delta}_+)$ be the functor given by the formula

$$
T([n], i) = \begin{cases} [n] & \text{if } i = 0 \\ [0] \star [n] \simeq [n+1] & \text{if } i = 1. \end{cases}
$$

Let us regard the composite functor

$$
N(\boldsymbol{\Delta}_+) \times \Delta^1 \xrightarrow{T} N(\boldsymbol{\Delta}_+) \xrightarrow{\overline{\mathcal{C}}^{\bullet}} \mathrm{Cat}_{\infty}
$$

as an augmented cosimplicial object $\overline{X}^{\bullet}$ of the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_{\infty})$. We claim that $\overline{X}^{\bullet}$ is a limit diagram in $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_{\infty})$. This is equivalent to the requirement that for $i \in \{0, 1\}$, the restriction $(\overline{\mathcal{C}}^{\bullet} \circ T)|(N(\boldsymbol{\Delta}_+) \times \{i\})$ is a limit diagram. For $i = 0$, this follows from our construction of $\overline{\mathcal{C}}^{\bullet}$, and for $i = 1$ it follows from the observation that the augmented cosimplicial ∞-category $(\overline{\mathcal{C}}^{\bullet} \circ T)|(N(\boldsymbol{\Delta}_+) \times \{1\})$ is split.

Let $X^{\bullet} = \overline{X}^{\bullet}$ be the underlying cosimplicial object of $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_{\infty})$. Condition $(*)$ is equivalent to the requirement that $X^{\bullet}$ is a cosimplicial object of $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \mathrm{Cat}_{\infty})$. It follows from Corollary 6.2.3.18 that $\overline{X}^{\bullet}$ is an augmented cosimplicial object of $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \mathrm{Cat}_{\infty})$. Since $\overline{X}^{-1}$ is an object of $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \mathrm{Cat}_{\infty})$, we deduce that the forgetful functor $G : \mathcal{C} \simeq \overline{\mathcal{C}}^{-1} \to \mathcal{C}^0$ admits a left adjoint. This proves (1). Assertion (2) is equivalent to the requirement that the map $X^{-1} \to X^0$ is a morphism of $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \mathrm{Cat}_{\infty})$, and assertion (3) follows immediately from Proposition 6.2.4.1. $\qquad \square$

6.3 Tensor Products of ∞-Categories

In the earlier chapters of this book, we developed a general theory of symmetric monoidal ∞-categories. Our goal in this section is to study some specific examples of symmetric monoidal ∞-categories which play an important role in higher category theory. We begin in §6.3.1 by showing that the ∞-category Pr^{L} of presentable ∞-categories admits a symmetric monoidal structure: if $\mathcal{C}$ and $\mathcal{D}$ are presentable ∞-categories, then the tensor product $\mathcal{C} \otimes \mathcal{D}$ is universal among presentable ∞-categories which receive a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ which preserves small colimits separately in each variable.

Let Sp denote the ∞-category of spectra introduced in §1.4.3. Then Sp is a presentable stable ∞-category, which we can regard as an object of Pr^{L}. In §6.3.2, we will show that Sp admits the structure of a

commutative algebra object of Pr^{L} (in an essentially unique way). In other words, the ∞-category of spectra admits a symmetric monoidal structure, where the tensor product

$$\otimes : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$$

preserves small colimits separately in each variable. We will refer to this structure as the *smash product symmetric monoidal structure.* The theory of commutative and associative algebras in Sp will be the subject of Chapter 8.

The symmetric monoidal structure on Pr^{L} has a number of other applications. Suppose that $\mathcal{C}$ is an associative algebra object of Pr^{L}: that is, $\mathcal{C}$ is a presentable monoidal ∞-category for which the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. Let A be an associative algebra object of $\mathcal{C}$. In §4.3.2, we saw that the ∞-category $\mathrm{RMod}_A(\mathcal{C})$ is naturally left-tensored over the monoidal ∞-category $\mathcal{C}$. In §6.3.3, we will show that the construction $A \mapsto \mathrm{RMod}_A(\mathcal{C})$ can be regarded as a functor $\mathrm{Alg}(\mathcal{C}) \to \mathrm{LMod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}})$. We will make a detailed study of this functor in §6.3.5. Our analysis relies on the fact that $\mathrm{RMod}_A(\mathcal{C})$ can be characterized (as an ∞-category left-tensored over $\mathcal{C}$) by two different universal properties, which we will verify in §6.3.4.

In the special case where $\mathcal{C}$ is a symmetric monoidal ∞-category, both $\mathrm{Alg}(\mathcal{C})$ and $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}})$ inherit symmetric monoidal structures, and the construction $A \mapsto \mathrm{RMod}_A(\mathcal{C})$ is a symmetric monoidal functor. Consequently, if A is an $\mathbb{E}_k$-algebra object of $\mathcal{C}$, then we can regard A as an $\mathbb{E}_{k-1}$ object of $\mathrm{Alg}(\mathcal{C})$ (see Theorem 5.1.2.2), so that $\mathrm{RMod}_A(\mathcal{C})$ is an $\mathbb{E}_{k-1}$-algebra object of $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}})$ and in particular inherits the structure of an $\mathbb{E}_{k-1}$-monoidal ∞-category. In §6.3.7, we will use this observation to construct a higher categorical analogue of the adjoint action of an associative algebra on itself (Theorem 6.3.7.5), which plays an important role in studying the deformation theory of $\mathbb{E}_k$-algebras (see §8.3.5). In the special case $k = \infty$, the $\mathbb{E}_{k-1}$-monoidal structure on $\mathrm{RMod}_A(\mathcal{C})$ can be viewed as a symmetric monoidal structure on $\mathrm{RMod}_A(\mathcal{C})$. In §6.3.6, we will show that this symmetric monoidal structure corresponds, under the equivalence $\mathrm{Mod}_A(\mathcal{C}) \to \mathrm{RMod}_A(\mathcal{C})$ studied in §4.4.1, to the symmetric monoidal structure on $\mathrm{Mod}_A(\mathcal{C})$ constructed in §3.3.

6.3.1   The Monoidal Structure on $\mathrm{Cat}_\infty(\mathcal{K})$

In §2.4.1, we saw that any ∞-category $\mathcal{C}$ which admits finite products can be regarded as a symmetric monoidal ∞-category, where the symmetric monoidal structure on $\mathcal{C}$ is given by the Cartesian product functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$. In particular, the ∞-category Cat_∞ of (small) ∞-categories is endowed with a symmetric monoidal structure. The commutative algebra objects of Cat_∞ can be thought of as symmetric monoidal ∞-categories (Remark 2.4.2.6). In practice, it is often convenient to work not with an arbitrary symmetric monoidal ∞-category $\mathcal{D}^\otimes$, but to introduce some restrictions: for example, we might want to require that $\mathcal{D}$ admits colimits indexed by certain diagrams, and that the tensor product on $\mathcal{D}$ is compatible with such colimits. In this case, $\mathcal{D}^\otimes$ can be identified with a commutative algebra object of a suitable subcategory of Cat_∞, which inherits a symmetric monoidal structure from the Cartesian symmetric monoidal structure on Cat_∞.

Definition 6.3.1.1. Let $\mathcal{K}$ be collection of simplicial sets. We let $\mathrm{Cat}_\infty(\mathcal{K})$ denote the subcategory of Cat_∞ spanned by those ∞-categories $\mathcal{C}$ which admit K-indexed colimits for each $K \in \mathcal{K}$, and those functors $f : \mathcal{C} \to \mathcal{D}$ which preserve K-indexed colimits for each $K \in \mathcal{K}$.

Our first goal in this section is to show that for every small collection of simplicial sets $\mathcal{K}$, the ∞-category $\mathrm{Cat}_\infty(\mathcal{K})$ inherits a symmetric monoidal structure from Cat_∞. To describe this symmetric monoidal structure, we need to introduce a bit of notation.

Notation 6.3.1.2. Let $\mathbf{A}$ be the category of marked simplicial sets, as defined in §T.3.1. Then $\mathbf{A}$ has the structure of a simplicial symmetric monoidal model category, where the symmetric monoidal structure is given by Cartesian product. We let $\mathrm{Cat}_\infty^\otimes$ denote the ∞-category $\mathrm{N}(\mathbf{A}^{\otimes,\mathrm{o}})$ appearing in the statement of Proposition 4.1.3.10. More concretely:

   (i) The objects of $\mathrm{Cat}_\infty^\otimes$ are finite sequences $[X_1, \ldots, X_n]$, where each X_i is an ∞-category.

(*ii*) Given a pair of objects $[X_1, \ldots, X_n], [Y_1, \ldots, Y_m] \in \mathrm{Cat}_\infty^\otimes$, a morphism from $[X_1, \ldots, X_n]$ to $[Y_1, \ldots, Y_m]$ consists of a map $\alpha : \langle n \rangle \to \langle m \rangle$ and a collection of functors $\eta_j : \prod_{\alpha(i)=j} X_i \to Y_j$.

Let P denote the collection of all sets of simplicial sets, partially ordered by inclusion. We define a subcategory $\mathcal{M}$ of $\mathrm{Cat}_\infty^\otimes \times \mathrm{N}(P)$ as follows:

(*iii*) An object $([X_1, \ldots, X_n], \mathcal{K})$ of $\mathrm{Cat}_\infty^\otimes \times \mathrm{N}(P)$ belongs to $\mathcal{M}$ if and only if each of the ∞-categories X_i admits $\mathcal{K}$-indexed colimits.

(*iv*) Let $f : ([X_1, \ldots, X_n], \mathcal{K}) \to ([Y_1, \ldots, Y_m], \mathcal{K}')$ be a morphism in $\mathrm{Cat}_\infty^\otimes \times \mathrm{N}(P)$, covering a map $\alpha : \langle n \rangle \to \langle m \rangle$ in $\mathcal{F}\mathrm{in}_*$. Then f belongs to $\mathcal{M}$ if and only if each of the associated functors

$$\prod_{\alpha(i)=j} X_i \to Y_j$$

preserves $\mathcal{K}$-indexed colimits separately in each variable.

If $\mathcal{K}$ is a set of simplicial sets, we let $\mathrm{Cat}_\infty(\mathcal{K})^\otimes$ denote the fiber product $\mathcal{M} \times_{\mathrm{N}(P)} \{\mathcal{K}\}$.

We will need the following technical result.

Proposition 6.3.1.3. *Let $\mathcal{M}$ be defined as in Notation 6.3.1.2. Then the forgetful functor $p : \mathcal{M} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \mathrm{N}(P)$ is a coCartesian fibration.*

Corollary 6.3.1.4. *Let $\mathcal{K}$ be a set of simplicial sets. Then the subcategory $\mathrm{Cat}_\infty(\mathcal{K})^\otimes$ is a symmetric monoidal ∞-category, and the inclusion $\mathrm{Cat}_\infty(\mathcal{K})^\otimes \hookrightarrow \mathrm{Cat}_\infty^\otimes$ is a lax symmetric monoidal functor.*

Proof. It follows from Proposition 6.3.1.3 that the forgetful functor $p : \mathrm{Cat}_\infty(\mathcal{K})^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a coCartesian fibration. Moreover, there is a canonical isomorphism of simplicial sets $\mathrm{Cat}_\infty(\mathcal{K})_{\langle n \rangle}^\otimes \simeq \mathrm{Cat}_\infty(\mathcal{K})^n$, which is induced by the functor p. $\square$

Remark 6.3.1.5. Suppose $\mathcal{K}$ is a collection of *sifted* simplicial sets. Then, for each $K \in \mathcal{K}$, a functor

$$\mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{D}$$

preserves K-indexed colimits if and only if it preserves K-indexed colimits separately in each variable. It follows that the symmetric monoidal structure on $\mathrm{Cat}_\infty(\mathcal{K})$ described in Corollary 6.3.1.4 is Cartesian.

Remark 6.3.1.6. Let $\mathcal{K}$ be a set of simplicial sets. Then the symmetric monoidal ∞-category $\mathrm{Cat}_\infty(\mathcal{K})$ of Corollary 6.3.1.4 is closed (see Definition 4.1.1.17): for every pair of objects $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_\infty(\mathcal{K})$, if we let $\mathcal{E}$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which preserve $\mathcal{K}$-indexed colimits, then the evaluation map

$$\mathcal{C} \times \mathcal{E} \hookrightarrow \mathcal{C} \times \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$$

preserves $\mathcal{K}$-indexed colimits separately in each variable and induces, for each object $\mathcal{T} \in \mathrm{Cat}_\infty(\mathcal{K})$, a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Cat}_\infty(\mathcal{K})}(\mathcal{T}, \mathcal{E}) \to \mathrm{Map}_{\mathrm{Cat}_\infty(\mathcal{K})}(\mathcal{C} \otimes \mathcal{T}, \mathcal{D}).$$

The proof of Proposition 6.3.1.3 will require a bit of notation.

Notation 6.3.1.7. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be ∞-categories. Given a pair of collections of diagrams $\mathcal{R}_1 = \{\overline{p}_\alpha : K_\alpha^\triangleright \to \mathcal{C}_1\}$ and $\mathcal{R}_2 = \{\overline{q}_\alpha : K_\beta^\triangleright \to \mathcal{C}_2\}$, we let $\mathcal{R}_1 \boxtimes \mathcal{R}_2$ denote the collection of all diagrams $\overline{r} : K^\triangleright \to \mathcal{C}_1 \times \mathcal{C}_2$ satisfying one of the following conditions:

(1) There exists an index α and an object $C_2 \in \mathcal{C}_2$ such that $K = K_\alpha$ and $\overline{r}$ is given by the composition

$$K^\triangleright \simeq K_\alpha^\triangleright \times \{C_2\} \xrightarrow{\overline{p}_\alpha} \mathcal{C}_1 \times \mathcal{C}_2.$$

(2) There exists an index β and an object $C_1 \in \mathcal{C}_1$ such that $K = K_\beta$ and $\bar{r}$ is given by the composition

$$K^\triangleright \simeq \{C_1\} \times K_\beta^\triangleright \xrightarrow{\bar{q}_\beta} \mathcal{C}_1 \times \mathcal{C}_2.$$

The operation $\boxtimes$ is coherently associative (and commutative). In other words, if for each $1 \leq i \leq n$ we are given an ∞-category $\mathcal{C}_i$ and a collection $\mathcal{R}_i$ of diagrams in $\mathcal{C}_i$, then we have a well-defined collection $\mathcal{R}_1 \boxtimes \ldots \boxtimes \mathcal{R}_n$ of diagrams in the product $\mathcal{C}_1 \times \ldots \times \mathcal{C}_n$. so there is no ambiguity in writing expressions such as $\mathcal{R}_1 \boxtimes \ldots \boxtimes \mathcal{R}_n$. In the case $n = 0$, we agree that this product coincides with the empty set of diagrams in the final ∞-category Δ^0.

Proof of Proposition 6.3.1.3. We first show that p is a locally coCartesian fibration. Suppose given an object $([\mathcal{C}_1, \ldots, \mathcal{C}_n], \mathcal{K})$ in $\mathcal{M}$, and a morphism $\alpha : (\langle n \rangle, \mathcal{K}) \to (\langle m \rangle, \mathcal{K}')$ in $N(\mathcal{F}in_*) \times N(P)$. We wish to show that α can be lifted to a locally p-coCartesian morphism in $\mathcal{M}$. Supplying a lift of α is tantamount to choosing a collection of functors

$$f_j : \prod_{\alpha(i)=j} \mathcal{C}_i \to \mathcal{D}_j$$

for $1 \leq j \leq m$, such that $\mathcal{D}_j$ admits $\mathcal{K}'$-indexed colimits, and f_j preserves $\mathcal{K}$-indexed colimits separately in each variable. For $1 \leq i \leq n$, let $\mathcal{R}_i$ denote the collection of all colimit diagrams in $\mathcal{C}_i$, indexed by simplicial sets belonging to $\mathcal{K}$. We now set $\mathcal{D}_j = \mathcal{P}_{\mathcal{R}}^{\mathcal{K}'}(\prod_{\alpha(i)=j} \mathcal{C}_i)$, where $\mathcal{R}$ denotes the $\boxtimes$-product of $\{\mathcal{R}_i\}_{\alpha(i)=j}$ (we refer the reader to §T.5.3.6 for an explanation of this notation). It follows from Proposition T.5.3.6.2 that the functors $\{f_j\}_{1 \leq j \leq m}$ assemble to a locally p-coCartesian morphism in $\mathcal{M}$.

To complete the proof that p is a coCartesian fibration, it will suffice to show that the locally p-coCartesian morphisms are closed under composition (Proposition T.2.4.2.8). In view of the construction of locally p-coCartesian morphisms given above, this follows immediately from Proposition T.5.3.6.11. $\square$

Remark 6.3.1.8. The coCartesian fibration p of Proposition 6.3.1.3 classifies a functor $N(\mathcal{F}in_*) \times N(P) \to \widehat{\mathcal{C}at}_\infty$, which we may identify with a functor from $N(P)$ to the ∞-category of commutative monoid objects of $\widehat{\mathcal{C}at}_\infty$. Consequently, we obtain a functor from P to the ∞-category of symmetric monoidal ∞-categories. In other words, if $\mathcal{K} \subseteq \mathcal{K}'$ are collections of simplicial sets, then we obtain a symmetric monoidal functor from $\widehat{\mathcal{C}at}_\infty(\mathcal{K})$ to $\widehat{\mathcal{C}at}_\infty(\mathcal{K}')$. It follows from the proof of Proposition 6.3.1.3 that this functor is given on objects by the formula $\mathcal{C} \mapsto \mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{C})$.

Remark 6.3.1.9. Let $\mathcal{O}^\otimes$ be an ∞-operad, so that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}at_\infty)$ can be identified with the ∞-category of $\mathcal{O}$-monoidal ∞-categories (Example 2.4.2.4 and Proposition 2.4.2.5). Unwinding the definitions, we see that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}at_\infty(\mathcal{K}))$ can be identified with the subcategory of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}at_\infty)$ spanned by the $\mathcal{O}$-monoidal ∞-categories which are compatible with $\mathcal{K}$-indexed colimits, and those $\mathcal{O}$-monoidal functors which preserve $\mathcal{K}$-indexed colimits.

Given an inclusion $\mathcal{K} \subseteq \mathcal{K}'$ of sets of simplicial sets, the induced inclusion

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}at_\infty(\mathcal{K}')) \subseteq \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}at_\infty(\mathcal{K}))$$

admits a left adjoint, given by composition with the symmetric monoidal functor $\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}$. Contemplating the unit of this adjunction, we arrive at the following result:

Proposition 6.3.1.10. *Let $\mathcal{K} \subseteq \mathcal{K}'$ be collections of simplicial sets and $q : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ a coCartesian fibration of ∞-operads. Assume that the $\mathcal{O}$-monoidal structure on $\mathcal{C}$ is compatible with $\mathcal{K}$-indexed colimits. Then there exists a $\mathcal{O}$-monoidal functor $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ with the following properties:*

(1) *The $\mathcal{O}$-monoidal structure on $\mathcal{D}^\otimes$ is compatible with $\mathcal{K}'$-indexed colimits.*

(2) *For each $X \in \mathcal{O}$, the underlying functor $f_X : \mathcal{C}_X \to \mathcal{D}_X$ preserves $\mathcal{K}$-indexed colimits.*

(3) *For every ∞-category $\mathcal{E}$ which admits $\mathcal{K}'$-indexed colimits and every object $X \in \mathcal{O}$, composition with f_X induces an equivalence of ∞-categories $\mathrm{Fun}_{\mathcal{K}'}(\mathcal{D}_X, \mathcal{E}) \to \mathrm{Fun}_{\mathcal{K}}(\mathcal{C}_X, \mathcal{E})$. In particular, f induces an identification $\mathcal{D}_X \simeq \mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{C}_X)$, and is therefore fully faithful (Proposition T.5.3.6.2).*

(4) *If $\mathcal{E}^{\otimes} \to \mathcal{O}^{\otimes}$ is a coCartesian fibration of ∞-operads which is compatible with $\mathcal{K}'$-indexed colimits, then the induced map $\mathrm{Alg}_{\mathcal{D}}(\mathcal{E}) \to \mathrm{Alg}_{\mathcal{C}}(\mathcal{E})$ induces an equivalence of ∞-categories from the full subcategory of $\mathrm{Alg}_{\mathcal{D}}(\mathcal{E})$ spanned by those algebra objects A such that each $A_X : \mathcal{D}_X \to \mathcal{E}_X$ preserves $\mathcal{K}'$-indexed colimits to the full subcategory of $\mathrm{Alg}_{\mathcal{C}}(\mathcal{E})$ spanned by those algebra objects B such that each $B_X : \mathcal{C}_X \to \mathcal{E}_X$ preserves $\mathcal{K}$-indexed colimits.*

Variant 6.3.1.11. In Notation 6.3.1.2, we can replace $\mathcal{C}\mathrm{at}_\infty$ by the (very large) ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$ of ∞-categories which are not necessarily small, and P by the collection $\widehat{P}$ of not necessarily small collections of simplicial sets. We then obtain a coCartesian fibration

$$\widehat{M} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \mathrm{N}(\widehat{P}).$$

We denote the fiber of this map over an object $\mathcal{K} \in \widehat{P}$ by $\widehat{\mathcal{C}\mathrm{at}}_\infty(\mathcal{K})^{\otimes}$.

Let $\mathcal{C}$ be a symmetric monoidal category. The category $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et})$ of presheaves of sets on $\mathcal{C}$ then inherits a symmetric monoidal structure, which is characterized by the following assertions:

(1) The Yoneda embedding $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et})$ is a monoidal functor.

(2) The tensor product functor $\otimes : \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et}) \times \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et})$ preserves colimits separately in each variable.

The bifunctor $\otimes : \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et}) \times \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et})$ is called the *Day convolution product*. More concretely, this operation is given by the composition

$$
\begin{aligned}
\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et}) \times \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et}) &\to \mathrm{Fun}(\mathcal{C}^{op} \times \mathcal{C}^{op}, \mathcal{S}\mathrm{et} \times \mathcal{S}\mathrm{et}) \\
&\xrightarrow{\phi} \mathrm{Fun}(\mathcal{C}^{op} \times \mathcal{C}^{op}, \mathcal{S}\mathrm{et}) \\
&\xrightarrow{\phi'} \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et})
\end{aligned}
$$

where ϕ is induced by the Cartesian product functor $\mathcal{S}\mathrm{et} \times \mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$, and ϕ' is given by left Kan extension along the functor $\mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathcal{C}^{op}$ determined by the symmetric monoidal structure on $\mathcal{C}$. This symmetric monoidal structure on $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et})$ is called the *Day convolution product*). We can use Proposition 6.3.1.10 to generalize the Day convolution product to the ∞-categorical setting. Taking $\mathcal{K}$ to be empty, $\mathcal{K}'$ to be the collection of all small simplicial sets, and $\mathcal{O}^{\otimes}$ is the commutative ∞-operad, we deduce the following:

Corollary 6.3.1.12. *Let $\mathcal{C}$ be a small symmetric monoidal ∞-category. Then there exists a symmetric monoidal structure on the ∞-category $\mathcal{P}(\mathcal{C})$ of presheaves on $\mathcal{C}$. It is characterized up to (symmetric monoidal) equivalence by the following properties:*

(1) *The Yoneda embedding $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ can be extended to a symmetric monoidal functor.*

(2) *The tensor product $\otimes : \mathcal{P}(\mathcal{C}) \times \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$ preserves small colimits separately in each variable.*

Proof. The existence of the desired symmetric monoidal structures on $\mathcal{P}(\mathcal{C})$ and j follows from Proposition 6.3.1.10. The uniqueness follows from the universal property given in assertion (3) of Proposition 6.3.1.10). $\square$

Applying Proposition 6.3.1.10 in the case where $\mathcal{K}$ is empty and $\mathcal{K}'$ is the class of all small *filtered* simplicial sets, we deduce the following:

Corollary 6.3.1.13. *Let $\mathcal{C}$ be a small symmetric monoidal ∞-category. Then there exists a symmetric monoidal structure on the ∞-category $\mathrm{Ind}(\mathcal{C})$ of Ind-objects on $\mathcal{C}$. It is characterized up to (symmetric monoidal) equivalence by the following properties:*

(1) *The Yoneda embedding $j : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ can be extended to a symmetric monoidal functor.*

(2) *The tensor product $\otimes : \mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$ preserves small filtered colimits separately in each variable.*

Moreover, if $\mathcal{C}$ admits finite colimits, and the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves finite colimits separately in each variable, then assertion (2) can be strengthened as follows:

(2′) *The tensor product functor $\otimes : \mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$ preserves small colimits separately in each variable.*

Moreover, if $\mathcal{D}^{\otimes}$ is any symmetric monoidal ∞-category such that $\mathcal{D}$ admits small filtered colimits and the tensor product $\mathcal{D} \times \mathcal{D} \to \mathcal{D}$ preserves small filtered colimits, then the restriction functor $\mathrm{Fun}^{\otimes}(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is an equivalence of ∞-categories.

Proof. Only assertion (2′) requires proof. For each $D \in \mathrm{Ind}(\mathcal{C})$, let e_D denote the functor $C \mapsto C \otimes D$. Let $\mathcal{D}$ denote the full subcategory spanned by those objects $D \in \mathrm{Ind}(\mathcal{C})$ such that the functor e_D preserves small colimits separately in each variable. We wish to prove that $\mathcal{D} = \mathrm{Ind}(\mathcal{C})$. Assertion (2) implies that the correspondence $D \mapsto e_D$ is given by a functor $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Fun}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C}))$ which preserves filtered colimits. Consequently, $\mathcal{D}$ is stable under filtered colimits in $\mathrm{Ind}(\mathcal{C})$. It will therefore suffice to show that $\mathcal{D}$ contains the essential image of j.

Let $C \in \mathcal{C}$. Since the Yoneda embedding j is a symmetric monoidal functor, we have a homotopy commutative diagram

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ e_C\ } & \mathcal{C} \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} \\
\mathrm{Ind}(\mathcal{C}) & \xrightarrow{\ e_{j(C)}\ } & \mathrm{Ind}(\mathcal{C}).
\end{array}
$$

The desired result now follows from Proposition T.5.5.1.9, since $e_{j(C)}$ preserves filtered colimits and $j \circ e_C$ preserves finite colimits. $\qquad\square$

Let $\mathcal{K}$ be the collection of all small simplicial sets. Let us now consider the symmetric monoidal ∞-category $\widehat{\mathrm{Cat}}_{\infty}(\mathcal{K})$, whose objects are ∞-categories which admit small colimits and whose morphisms are functors which preserve small colimits. We let Pr^{L} denote the full subcategory of $\widehat{\mathrm{Cat}}_{\infty}(\mathcal{K})$ spanned by the presentable ∞-categories.

Proposition 6.3.1.14. *Let $\mathcal{K}$ denote the collection of all small simplicial sets. The ∞-category Pr^{L} of presentable ∞-categories is closed under tensor products in $\widehat{\mathrm{Cat}}_{\infty}(\mathcal{K})$, and therefore inherits a symmetric monoidal structure (see Proposition 2.2.1.1).*

Proof. Corollary 6.3.1.12 implies that the unit object of $\widehat{\mathrm{Cat}}_{\infty}(\mathcal{K})$ is given by $\mathcal{P}(\Delta^0) \simeq \mathcal{S}$, which is a presentable ∞-category. It will therefore suffice to show that if $\mathcal{C}$ and $\mathcal{C}'$ are two presentable ∞-categories, then their tensor product $\mathcal{C} \otimes \mathcal{C}'$ in $\widehat{\mathrm{Cat}}_{\infty}(\mathcal{K})$ is also presentable. If $\mathcal{C} = \mathcal{P}(\mathcal{C}_0)$ and $\mathcal{C}' = \mathcal{P}(\mathcal{C}_0')$ for a pair of small ∞-categories $\mathcal{C}_0$ and $\mathcal{C}_0'$, then Corollary 6.3.1.12 yields an equivalence $\mathcal{C} \otimes \mathcal{C}' \simeq \mathcal{P}(\mathcal{C}_0 \times \mathcal{C}_0')$, so that $\mathcal{C} \otimes \mathcal{C}'$ is presentable as desired. In view of Theorem T.5.5.1.1, every presentable ∞-category is a localization of an ∞-category of presheaves. It will therefore suffice to prove the following:

(∗) Let $\mathcal{C}$ and $\mathcal{C}'$ be presentable ∞-categories, and assume that the tensor product $\mathcal{C} \otimes \mathcal{C}'$ is presentable. Let S be a (small) set of morphisms in $\mathcal{C}$. Then the tensor product $S^{-1}\mathcal{C} \otimes \mathcal{C}'$ is presentable.

To prove $(*)$, we choose a (small) set of objects $M = \{C'_\alpha\}$ which generates $\mathcal{C}'$ under colimits. Let $f : \mathcal{C} \times \mathcal{C}' \to \mathcal{C} \otimes \mathcal{C}'$ be the canonical map, and let T be the collection of all morphisms in $\mathcal{C} \otimes \mathcal{C}'$ having the form $f(s \times \mathrm{id}_{C'})$, where $s \in S$ and $C' \in M$. Consider now the composition

$$g : S^{-1}\,\mathcal{C} \times \mathcal{C}' \subseteq \mathcal{C} \times \mathcal{C}' \overset{f}{\to} \mathcal{C} \otimes \mathcal{C}' \overset{L}{\to} T^{-1}(\mathcal{C} \otimes \mathcal{C}'),$$

where L is a left adjoint to the inclusion $T^{-1}(\mathcal{C} \otimes \mathcal{C}') \subseteq \mathcal{C} \otimes \mathcal{C}'$. We claim that g exhibits $T^{-1}(\mathcal{C} \otimes \mathcal{C}')$ as a tensor product of $S^{-1}\,\mathcal{C}$ with $\mathcal{C}'$. In other words, we claim that if $\mathcal{D}$ is an arbitrary ∞-category which admits small colimits, then composition with g induces an equivalence

$$\mathrm{Fun}_{\mathcal{K}}(T^{-1}(\mathcal{C} \otimes \mathcal{C}'), \mathcal{D}) \to \mathrm{Fun}_{\mathcal{K} \boxtimes \mathcal{K}}(S^{-1}\,\mathcal{C} \times \mathcal{C}', \mathcal{D}).$$

The left hand side can be identified with the full subcategory of $\mathrm{Fun}_{\mathcal{K}}(\mathcal{C} \otimes \mathcal{C}', \mathcal{D})$ spanned by those functors which carry each morphism in T to an equivalence. Under the equivalence

$$\mathrm{Fun}_{\mathcal{K}}(\mathcal{C} \otimes \mathcal{C}', \mathcal{D}) \simeq \mathrm{Fun}_{\mathcal{K} \boxtimes \mathcal{K}}(\mathcal{C} \times \mathcal{C}', \mathcal{D}) \simeq \mathrm{Fun}_{\mathcal{K}}(\mathcal{C}, \mathrm{Fun}_{\mathcal{K}}(\mathcal{C}', \mathcal{D})),$$

this corresponds to the full subcategory spanned by those functors $F : \mathcal{C} \to \mathrm{Fun}_{\mathcal{K}}(\mathcal{C}', \mathcal{D})$ which carry each morphism in S to an equivalence. This ∞-category is equivalent to $\mathrm{Fun}_{\mathcal{K}}(S^{-1}\,\mathcal{C}, \mathrm{Fun}_{\mathcal{K}}(\mathcal{C}', \mathcal{D})) \simeq \mathrm{Fun}_{\mathcal{K} \boxtimes \mathcal{K}}(S^{-1}\,\mathcal{C} \times \mathcal{C}', \mathcal{D})$, as desired. $\qquad\square$

In §6.3.2, we will need a more explicit description of the tensor product on Pr^{L}.

Lemma 6.3.1.15. *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories. Then $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \mathcal{D})$ is a presentable ∞-category.*

Proof. Using Theorem T.5.5.1.1 and the results of §T.5.5.4, we can choose a small ∞-category $\mathcal{C}'$, a small collection S of morphisms in $\mathcal{P}(\mathcal{C}')$, and an equivalence $\mathcal{C} \simeq S^{-1}\,\mathcal{P}(\mathcal{C}')$. Then

$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{P}(\mathcal{C}')^{op}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{P}(\mathcal{C}'), \mathcal{D}^{op})^{op} \simeq \mathrm{Fun}(\mathcal{C}', \mathcal{D}^{op})^{op} \simeq \mathrm{Fun}(\mathcal{C}'^{op}, \mathcal{D})$$

is presentable by Proposition T.5.5.3.6, where the second equivalence is given by composition with the Yoneda embedding (Theorem T.5.1.5.6). For each morphism $\alpha \in S$, let $\mathcal{E}(\alpha)$ denote the full subcategory of $\mathrm{Fun}^{\mathrm{R}}(\mathcal{P}(\mathcal{C}')^{op}, \mathcal{D})$ spanned by those functors which carry α to an equivalence in $\mathcal{D}$. Then $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \mathcal{D})$ is equivalent to the intersection $\bigcap_{\alpha \in S} \mathcal{E}(\alpha)$. In view of Lemma T.5.5.4.18, it will suffice to show that each $\mathcal{E}(\alpha)$ is a localization of $\mathrm{Fun}^{\mathrm{R}}(\mathcal{P}(\mathcal{C}')^{op}, \mathcal{D})$. We now observe that $\mathcal{E}(\alpha)$ is given by a pullback diagram

$$
\begin{array}{ccc}
\mathcal{E}(\alpha) & \longrightarrow & \mathrm{Fun}^{\mathrm{R}}(\mathcal{P}(\mathcal{C}')^{op}, \mathcal{D}) \\
\downarrow & & \downarrow \\
\mathcal{E} & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{D}),
\end{array}
$$

where $\mathcal{E}$ denotes the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{D})$ spanned by the equivalences. According to Lemma T.5.5.4.17, it will suffice to show that $\mathcal{E}$ is an accessible localization of $\mathrm{Fun}(\Delta^1, \mathcal{D})$, which is clear. $\qquad\square$

Proposition 6.3.1.16. *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories. Then there is a canonical equivalence*

$$\mathcal{C} \otimes \mathcal{D} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \mathcal{D}).$$

Proof. Let $\mathcal{E}$ be an arbitrary presentable ∞-category, and let $\mathrm{Fun}'(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ be the full subcategory of $\mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ spanned by those functors which preserve small colimits separately in each variable. Then we have a canonical isomorphism $\mathrm{Fun}'(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathrm{Fun}^{\mathrm{L}}(\mathcal{D}, \mathcal{E}))$. Here the ∞-category $\mathrm{Fun}^{\mathrm{L}}(\mathcal{D}, \mathcal{E})$

is presentable (Proposition T.5.5.3.8). Using Corollary T.5.5.2.9 and Proposition T.5.2.6.3, we can identify $\mathrm{Fun}^{\mathrm{L}}(\mathcal{D}, \mathcal{E})$ with the full subcategory of $\mathrm{Fun}^{\mathrm{R}}(\mathcal{E}, \mathcal{D})^{op}$ spanned by those functors which are accessible. Consequently, we get a fully faithful embedding

$$
\begin{aligned}
\mathrm{Fun}'(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \quad &\rightarrow \quad \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathrm{Fun}^{\mathrm{R}}(\mathcal{E}, \mathcal{D})^{op}) \\
&\simeq \quad \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathrm{Fun}^{\mathrm{L}}(\mathcal{E}^{op}, \mathcal{D}^{op})) \\
&\simeq \quad \mathrm{Fun}^{\mathrm{L}}(\mathcal{E}^{op}, \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}^{op})) \\
&\simeq \quad \mathrm{Fun}^{\mathrm{R}}(\mathcal{E}, \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \mathcal{D}))^{op}
\end{aligned}
$$

whose essential image consists of the collection of accessible functors from $\mathcal{E}$ to $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \mathcal{D})$. We now apply Lemma 6.3.1.15 to conclude that $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \mathcal{C})$ is presentable, so that (using Corollary T.5.5.2.9 and Proposition T.5.2.6.3 again) $\mathrm{Fun}'(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ can be identified with $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \mathcal{D}), \mathcal{E})$. It follows that $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \mathcal{D})$ and $\mathcal{C} \otimes \mathcal{D}$ corepresent the same functor on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and are therefore canonically equivalent. $\square$

Remark 6.3.1.17. The symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ described in Proposition 6.3.1.14 is closed (see Definition 4.1.1.17): for every pair of presentable ∞-categories $\mathcal{C}$ and $\mathcal{D}$, the ∞-category $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ of colimit-preserving functors from $\mathcal{C}$ to $\mathcal{D}$ (which is presentable by Proposition T.5.5.3.8) is equipped with a map $\mathcal{C} \otimes \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$ which exhibits $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ as an exponential of $\mathcal{D}$ by $\mathcal{C}$.

Example 6.3.1.18. Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi. Then $\mathcal{X} \otimes \mathcal{Y}$ is an ∞-topos, and can be identified with the (Cartesian) product of $\mathcal{X}$ and $\mathcal{Y}$ in the ∞-category of ∞-topoi. For a proof of a slightly weaker assertion, we refer the reader to Theorem T.7.3.3.9. The general statement can be proven using a similar argument.

Example 6.3.1.19. Remark 6.3.1.8 implies that the ∞-category $\mathcal{S} = \mathcal{P}(\Delta^0)$ is a unit object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. In particular, for every presentable ∞-category $\mathcal{C}$ we have canonical equivalences $\mathcal{C} \simeq \mathcal{C} \otimes \mathcal{S} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \mathcal{S})$. The essential surjectivity of the composition is a restatement of the representability criterion of Proposition T.5.5.2.2.

Example 6.3.1.20. Recall that, for every ∞-category $\mathcal{C}$, the ∞-category $\mathcal{C}_*$ of *pointed objects* of $\mathcal{C}$ is defined to be the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by those functors $F : \Delta^1 \to \mathcal{C}$ for which $F(0)$ is a final object of $\mathcal{C}$. The canonical isomorphism of simplicial sets $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \mathcal{D}_*) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \mathcal{D})_*$ induces an equivalence $\mathcal{C} \otimes \mathcal{D}_* \simeq (\mathcal{C} \otimes \mathcal{D})_*$ for every pair of presentable ∞-categories $\mathcal{C}, \mathcal{D} \in \mathrm{L}\mathcal{P}\mathrm{r}$. In particular, we have a canonical equivalence $\mathcal{C} \otimes \mathcal{S}_* \simeq \mathcal{C}_*$.

Example 6.3.1.21. Let $n \geq -2$ be an integer, and let $\tau_{\leq n} \mathcal{S}$ denote the full subcategory of $\mathcal{S}$ spanned by the n-truncated spaces. For any presentable ∞-category $\mathcal{C}$, we can identify $\mathcal{C} \otimes \tau_{\leq n} \mathcal{S}$ with the ∞-category $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \tau_{\leq n} \mathcal{S}) \subseteq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \mathcal{S})$. Under the Yoneda equivalence $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \mathcal{S}) \simeq \mathcal{C}$, $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \tau_{\leq n} \mathcal{S})$ corresponds to the full subcategory of $\mathcal{C}$ spanned by those objects C such that $\mathrm{Map}_{\mathcal{C}}(C', C)$ is n-truncated, for each $C' \in \mathcal{C}$. In other words, we have a canonical equivalence $\mathcal{C} \otimes \tau_{\leq n} \mathcal{S} \simeq \tau_{\leq n} \mathcal{C}$.

Example 6.3.1.22. Let Sp denote the ∞-category of spectra. Then Sp can be identified with a homotopy limit of the tower

$$
\ldots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* .
$$

Consequently, for every presentable ∞-category $\mathcal{C}$, we have equivalences

$$
\mathcal{C} \otimes \mathrm{Sp} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \mathrm{Sp}) \simeq \mathrm{holim}\{\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \mathcal{S}_*)\} \simeq \mathrm{holim}\{\mathcal{C}_*\} \simeq \mathrm{Sp}(\mathcal{C}),
$$

where $\mathrm{Sp}(\mathcal{C})$ is defined as in §1.4.2.

Remark 6.3.1.23. Combining Proposition 6.3.1.16 with Theorem T.5.5.3.18, we conclude that the bifunctor $\otimes : \mathcal{P}\mathrm{r}^{\mathrm{L}} \times \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ preserves small colimits separately in each variable (remember that *colimits* in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ can also be computed as *limits* in $\mathcal{P}\mathrm{r}^{\mathrm{R}}$, which are computed by forming limits in $\mathcal{C}\mathrm{at}_{\infty}$ by Theorem T.5.5.3.18). Alternatively, one can observe that $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ is actually a *closed* monoidal category, with internal mapping objects given by $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ (see Proposition T.5.5.3.8).

6.3.2 The Smash Product Monoidal Structure

Let Sp denote the ∞-category of *spectra* (as defined in §1.4.3). The homotopy category hSp can be identified with the classical *stable homotopy category*. Given a pair of spectra $X, Y \in \mathrm{hSp}$, one can define a new spectrum called the *smash product* of X and Y. The smash product operation determines a monoidal structure on hSp. In this section, we will show that this monoidal structure is determined by a monoidal structure which exists on the ∞-category Sp itself. There are at least three ways to see this.

($S1$) Choose a simplicial model category **A** equipped with a compatible monoidal structure, whose underlying ∞-category is equivalent to Sp. For example, we can take **A** to be the category of *symmetric spectra* (see [73]). According to Proposition 4.1.3.10, the underlying ∞-category $\mathrm{N}(\mathbf{A}^{\mathrm{o}}) \simeq \mathrm{Sp}$ is endowed with a symmetric monoidal structure. The advantage of this perspective is that it permits us to easily compare the algebras and modules considered in this paper with more classical approaches to the theory of structured ring spectra. For example, Theorem 4.1.4.4 implies that $\mathrm{Alg}(\mathrm{Sp})$ is (equivalent to) the ∞-category underlying the model category of algebras in symmetric spectra (that is, strictly associative monoids in **A**); see Example 4.1.4.6.

The main disadvantage of this approach is that it seems to require auxiliary data (namely, a strictly associative model for the smash product functor), which could be supplied in many different ways. From a conceptual point of view, the existence of such a model ought to be irrelevant: the purpose of higher category theory is to provide a formalism which allows us to avoid assumptions like strict associativity.

($S2$) Let $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}, \mathrm{Sp})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})$ spanned by those functors from Sp to Sp which preserve small colimits. Corollary 1.4.4.6 asserts that evaluation on the sphere spectrum yields an equivalence of ∞-categories $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}, \mathrm{Sp}) \to \mathrm{Sp}$. On the other hand, since $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}, \mathrm{Sp})$ is stable under composition in $\mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})$, the composition monoidal structure on $\mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})$ induces a monoidal structure on $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}, \mathrm{Sp})$. This definition has the virtue of being very concrete (the smash product operation is simply given by composition of functors), and it allows us to identify the algebra objects of $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}, \mathrm{Sp})$: they are precisely the colimit-preserving monads on the ∞-category Sp (for an application of this last observation, see Theorem 8.1.2.1). The disadvantage of this definition is that it is very "associative" in nature, and therefore does not generalize easily to show that Sp is a *symmetric* monoidal ∞-category, as we will show below.

($S3$) Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ be the ∞-category whose objects are presentable ∞-categories and whose morphisms are colimit-preserving functors (see §T.5.5.3), and let $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ be the full subcategory of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ spanned by those presentable ∞-categories which are stable. In §6.3.1, we showed that there is a symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. We will show that the full subcategory $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ inherits a symmetric monoidal structure from that of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. The commutative algebra objects of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ can be identified with symmetric monoidal ∞-categories $\mathcal{C}$ which are stable, presentable, and have the property that the bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. We will see that Sp is the unit object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ (with respect to its tensor structure). It follows from Proposition 3.2.1.8 that Sp can be endowed with the structure of a commutative algebra object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$, and in fact is *initial* among such algebra objects. This establishes not only the existence of a symmetric monoidal structure on Sp, but also a universal property which can be used to prove its uniqueness (Corollary 6.3.2.19).

We will adopt approach ($S3$). We begin with some general remarks concerning idempotent objects of monoidal ∞-categories.

Definition 6.3.2.1. Let $\mathcal{C}$ be a monoidal ∞-category with unit object $\mathbf{1}$. We will say that a map $e : \mathbf{1} \to E$ is an *idempotent object* of $\mathcal{C}$ if the induced maps

$$E \simeq E \otimes \mathbf{1} \xrightarrow{\mathrm{id} \otimes e} E \otimes E \qquad E \simeq \mathbf{1} \otimes E \xrightarrow{e \otimes \mathrm{id}} E \otimes E$$

are equivalences in $\mathcal{C}$.

Remark 6.3.2.2. In the situation of Definition 6.3.2.1, we will often abuse terminology and simply say that E is an idempotent object of $\mathcal{C}$, or that $e : \mathbf{1} \to E$ *exhibits* E *as an idempotent object of* $\mathcal{C}$.

Example 6.3.2.3. Let $\mathcal{C}$ be an ∞-category. Then the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ is a strict monoidal ∞-category via composition (that is, composition endows $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ with the structure of a simplicial monoid: see Example 4.1.4.7). An idempotent object of $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ is a functor $L : \mathcal{C} \to \mathcal{C}$ equipped with a natural transformation $e : \mathrm{id}_{\mathcal{C}} \to L$ such that the induced maps

$$L = L \circ \mathrm{id}_{\mathcal{C}} \to L \circ L \qquad L = \mathrm{id}_{\mathcal{C}} \circ L \to L \circ L$$

are equivalences in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$. This is equivalent to the assertion that L is a localization functor from $\mathcal{C}$ to itself (see Proposition T.5.2.7.4).

Proposition 6.3.2.4. *Let $\mathcal{C}$ be a monoidal ∞-category and let $e : \mathbf{1} \to E$ be a morphism in $\mathcal{C}$. The following conditions are equivalent:*

(1) *The map e exhibits E as an idempotent object of $\mathcal{C}$.*

(2) *For every ∞-category $\mathcal{M}$, let $l_E : \mathcal{M} \to \mathcal{M}$ be the functor given by tensor product with $E \in \mathcal{C}$, so that e induces a natural transformation $\alpha_{\mathcal{M}} : \mathrm{id}_{\mathcal{M}} \to l_E$. Then each of the natural transformations α_M exhibits l_E as a localization functor on $\mathcal{M}$.*

(3) *Let $l_E : \mathcal{C} \to \mathcal{C}$ be the functor given by left tensor product with E. Then e induces a functor $\alpha : \mathrm{id}_{\mathcal{C}} \to l_E$ which exhibits l_E as a localization functor on $\mathcal{C}$.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Proposition T.5.2.7.4 and the implication (2) $\Rightarrow$ (3) is obvious. Assume that (3) is satisfied. Then Proposition T.5.2.7.4 implies that the transformation $\alpha : \mathrm{id}_{\mathcal{C}} \to l_E$ induces equivalences of functors $l_E \to l_E^2$. Evaluating these functors at the unit object of $\mathcal{C}$, we deduce that the two natural maps $E \to E \otimes E$ are equivalences. $\qquad\square$

Remark 6.3.2.5. Let $e : \mathbf{1} \to E$ be an idempotent object of a monoidal ∞-category $\mathcal{C}$ and let $L : \mathcal{C} \to \mathcal{C}$ be the localization functor given by tensor product with E. Suppose that e admits a left inverse. Then $\mathbf{1}$ is a retract of E. Since E is L-local, we conclude that $\mathbf{1}$ is L-local. Thus e is an L-equivalence between L-local objects, and therefore an equivalence.

In this section, we will be primarily interested in idempotent objects of *symmetric* monoidal ∞-categories.

Remark 6.3.2.6. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then a morphism $e : \mathbf{1} \to E$ in $\mathcal{C}$ exhibits E as an idempotent object of $\mathcal{C}$ if and only if the induced map $\mathrm{id}_E \otimes e : E \to E \otimes E$ is an equivalence; the symmetry of $\mathcal{C}$ guarantees that $e \otimes \mathrm{id}_E$ is homotopic to $\mathrm{id}_E \otimes e$.

Proposition 6.3.2.7. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $e : \mathbf{1} \to E$ be an idempotent object of $\mathcal{C}$. Let $L : \mathcal{C} \to \mathcal{C}$ be the localization functor given by $C \mapsto E \otimes C$ (see Proposition 6.3.2.4). Then L is compatible with the symmetric monoidal structure on $\mathcal{C}$, so that $L\mathcal{C}$ inherits a symmetric monoidal structure (see Proposition 2.2.1.9).*

Proof. Let $f : C \to C'$ be an L-equivalence and let $D \in \mathcal{C}$; we wish to show that the induced map $C \otimes D \to C' \otimes D$ is an L-equivalence. In other words, we wish to show that f induces an equivalence $E \otimes C \otimes D \to E \otimes C' \otimes D$. This is clear, since f induces an equivalence $E \otimes C \to E \otimes C'$. $\qquad\square$

In the situation of Proposition 6.3.2.7, the inclusion $(L\mathcal{C})^{\otimes} \hookrightarrow \mathcal{C}^{\otimes}$ is a fully faithful embedding of ∞-operads, which induces a fully faithful embedding $\mathrm{CAlg}(L\mathcal{C}) \hookrightarrow \mathrm{CAlg}(\mathcal{C})$ whose essential image is the collection of commutative algebra objects $A \in \mathrm{CAlg}(\mathcal{C})$ such that e induces an equivalence $A \to E \otimes E$. Note that the unit object of $L\mathcal{C}$ is given by $L\mathbf{1} \simeq E$. It follows that E has the structure of a commutative algebra object of $L\mathcal{C}$, and therefore a commutative algebra object of $\mathcal{C}$. To describe the situation more precisely, it is convenient to introduce a bit of terminology:

Definition 6.3.2.8. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We will say that a commutative algebra object $A \in \mathrm{CAlg}(\mathcal{C})$ is *idempotent* if the multiplication map $A \otimes A \to A$ is an equivalence. Let $\mathrm{CAlg}^{\mathrm{idem}}(\mathcal{C})$ denote the full subcategory of $\mathrm{CAlg}(\mathcal{C})$ spanned by the idempotent commutative algebra objects.

Note that if A is a commutative algebra object of $\mathcal{C}$ with unit map $e : \mathbf{1} \to A$, then the map $\mathrm{id}_A \otimes e : A \to A \otimes A$ is a right inverse to the multiplication map $A \otimes A \to A$. It follows that A is an idempotent commutative algebra object of $\mathcal{C}$ if and only if e exhibits A as an idempotent object of $\mathcal{C}$. In fact, we can be more precise:

Proposition 6.3.2.9. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category with unit object $\mathbf{1}$, which we regard as a trivial algebra object of $\mathcal{C}$. Then the functor*

$$\theta : \mathrm{CAlg}^{\mathrm{idem}}(\mathcal{C}) \subseteq \mathrm{CAlg}(\mathcal{C}) \simeq \mathrm{CAlg}(\mathcal{C})_{\mathbf{1}/} \to \mathcal{C}_{\mathbf{1}/}$$

is fully faithful, and its essential image is the collection of maps $e : \mathbf{1} \to E$ which exhibit E as an idempotent object of $\mathcal{C}$.

Proof. The arguments sketched above show that θ is essentially surjective. We prove that θ is fully faithful. Let $A, A' \in \mathrm{CAlg}(\mathcal{C})$ be idempotent algebras with units $e : \mathbf{1} \to A$ and $e' : \mathbf{1} \to A'$. It will suffice to show that if $\mathrm{Map}_{\mathcal{C}_{\mathbf{1}/}}(A, A')$ is nonempty, then $\mathrm{Map}_{\mathcal{C}_{\mathbf{1}/}}(A, A')$ and $\mathrm{Map}_{\mathrm{CAlg}(\mathcal{C})}(A, A')$ are both contractible. Assume therefore that $\mathrm{Map}_{\mathcal{C}_{\mathbf{1}/}}(A, A')$ is nonempty, so that e' factors as a composition $\mathbf{1} \xrightarrow{e} A \xrightarrow{f} A'$ for some morphism f in $\mathcal{C}$. Note that A and A' are idempotent objects of $\mathcal{C}$; let L and L' denote the associated localization functors. Then $L'(e)$ exhibits $L'A$ as an idempotent object of $L'\mathcal{C}$, and $L'(e)$ has a left inverse. It follows that $L'(e)$ is an equivalenSince A' is L'-local, we immediately deduce that $\mathrm{Map}_{\mathcal{C}_{\mathbf{1}/}}(A, A') \simeq \mathrm{Map}_{\mathcal{C}_{L'\mathbf{1}/}}(L'A, L'A')$ is contractible.

Note that A can be identified with an initial object of the full subcategory $\mathrm{CAlg}(L\mathcal{C}) \subseteq \mathrm{CAlg}(\mathcal{C})$. To show that $\mathrm{Map}_{\mathrm{CAlg}(\mathcal{C})}(A, A')$ is contractible, it suffices to show that $A' \in L\mathcal{C}$. We have a commutative diagram

Since the horizontal map is an equivalence, this diagram exhibits A' as a retract of $A \otimes A'$. Since $A \otimes A'$ is L-local, we conclude that A' is L-local. $\square$

Proposition 6.3.2.10. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let A be an idempotent object of $\mathrm{CAlg}(\mathcal{C})$. Let $L : \mathcal{C} \to \mathcal{C}$ be the associated localization functor (given by tensor product with A), so that $L\mathcal{C}$ inherits the structure of a symmetric monoidal ∞-category (Proposition 6.3.2.7). Then the forgetful functor $G : \mathrm{Mod}_A(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$ determines an equivalence of symmetric monoidal ∞-categories $\mathrm{Mod}_A(\mathcal{C})^{\otimes} \to (L\mathcal{C})^{\otimes}$.*

Proof. The forgetful functor G admits a symmetric monoidal left adjoint $F : \mathcal{C}^{\otimes} \to \mathrm{Mod}_A(\mathcal{C})^{\otimes}$, given by tensor product with F. Let $M \in \mathrm{Mod}_A(\mathcal{C})$; we will abuse notation by identifying M with its image in $\mathcal{C}$. The composition

$$M \simeq \mathbf{1} \otimes M \to A \otimes M \to M$$

is homotopic to the identity, so that M is a retract of $A \otimes M$ in $\mathcal{C}$ and therefore belongs to $L\mathcal{C}$. It follows that the counit map $F \circ G \to \mathrm{id}$ is an equivalence of functors for $\mathrm{Mod}_A(\mathcal{C})^{\otimes}$ to itself, so that G determines an equivalence onto the full subcategory of $\mathcal{C}^{\otimes}$ spanned by those objects X such that the unit map $X \to (G \circ F)(X)$ is an equivalence. Unwinding the definitions, we see that this is subcategory coincides with $(L\mathcal{C})^{\otimes}$. $\square$

We now specialize to the case where $\mathcal{C}$ is the ∞-category Pr^{L} of presentable ∞-categories, endowed with the symmetric monoidal structure described in §6.3.1. The unit object of Pr^{L} is the ∞-category of spaces. If $\mathcal{C}$ is presentable ∞-category with a distinguished object $C \in \mathcal{C}$, we will say that $(\mathcal{C}, C)$ is *idempotent* if there is a colimit-preserving functor $F : \mathcal{S} \to \mathcal{C}$ with $F(*) = C$ which exhibits $\mathcal{C}$ as an idempotent object of Pr^{L}. Note that in this case, the functor F is well-defined up to a contractible space of choices (Theorem T.5.1.5.6). It then follows from Proposition 6.3.2.9 that $\mathcal{C}$ has the structure of a commutative algebra object of Pr^{L}, and that $F : \mathcal{S} \to \mathcal{C}$ is the unit map for this algebra structure. In other words, there is a canonical symmetric monoidal structure on $\mathcal{C}$ for which $C \in \mathcal{C}$ is the unit object.

We now give some examples of idempotent objects of Pr^{L}.

Proposition 6.3.2.11. *Let $\mathcal{S}_*$ denote the ∞-category of pointed spaces, and let S^0 denote the 0-sphere (that is, a pointed space consisting of exactly two points). Then $(\mathcal{S}_*, S^0)$ is idempotent. Moreover, the forgetful functor $\mathrm{Mod}_{\mathcal{S}_*}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Pr}^{\mathrm{L}}$ determines a fully faithful embedding, whose essential image is the full subcategory $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Pt}} \subseteq \mathrm{Pr}^{\mathrm{L}}$ spanned by the pointed presentable ∞-categories.*

Lemma 6.3.2.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and assume that $\mathcal{C}$ has an initial object. Let $\mathrm{Fun}'(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which carry initial objects of $\mathcal{C}$ to final objects of $\mathcal{D}$, and define $\mathrm{Fun}'(\mathcal{C}, \mathcal{D}_*)$ similarly. Then the forgetful functor $\theta : \mathrm{Fun}'(\mathcal{C}, \mathcal{D}_*) \to \mathrm{Fun}'(\mathcal{C}, \mathcal{D})$ is a trivial fibration of simplicial sets.*

Remark 6.3.2.13. If the ∞-category $\mathcal{D}$ does not have a final object, then the conclusion of Lemma 6.3.2.12 is true but vacuous: both of the relevant ∞-categories of functors are empty.

Proof. We first observe that objects of $\mathrm{Fun}'(\mathcal{C}, \mathcal{D}_*)$ can be identified with maps $F : \mathcal{C} \times \Delta^1 \to \mathcal{D}$ with the following properties:

(a) For every initial object $C \in \mathcal{C}$, $F(C, 1)$ is a final object of $\mathcal{D}$.

(b) For every object $C \in \mathcal{C}$, $F(C, 0)$ is a final object of $\mathcal{D}$.

Assume for the moment that (a) is satisfied, and let $\mathcal{C}' \subseteq \mathcal{C} \times \Delta^1$ be the full subcategory spanned by those objects (C, i) for which either $i = 1$, or C is an initial object of $\mathcal{C}$. We observe that (b) is equivalent to the following pair of conditions:

(b') The functor $F | \mathcal{C}'$ is a right Kan extension of $F | \mathcal{C} \times \{1\}$.

(b'') The functor F is a left Kan extension of $F | \mathcal{C}'$.

Let $\mathcal{E}$ be the full subcategory of $\mathrm{Fun}(\mathcal{C} \times \Delta^1, \mathcal{D})$ spanned by those functors which satisfy conditions (b') and (b''). Using Proposition T.4.3.2.15, we deduce that the projection $\bar{\theta} : \mathcal{E} \to \mathrm{Fun}(\mathcal{C} \times \{1\}, \mathcal{D})$ is a trivial Kan fibration. Since θ is a pullback of $\bar{\theta}$, we conclude that θ is a trivial Kan fibration. $\square$

Proof of Proposition 6.3.2.11. The object $S^0 \in \mathcal{S}_*$ corresponds to the colimit-preserving functor $F : \mathcal{S} \to \mathcal{S}_*$ which assigns to each space $X \in \mathcal{S}$ the pointed space $X_+ = X \coprod \{*\}$ obtained by adjoining to X a disjoint base point. According to Propositions 6.3.2.4 and 6.3.2.10, it will suffice to show the following:

(∗) For every presentable ∞-category $\mathcal{C}$, the functor F induces a map $F_{\mathcal{C}} : \mathcal{C} \simeq \mathcal{C} \otimes \mathcal{S} \to \mathcal{C} \otimes \mathcal{S}_*$ which exhibits $\mathcal{C} \otimes \mathcal{S}_*$ as a $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Pt}}$-localization of $\mathcal{C}$.

In view of Example 6.3.1.20, it will suffice to show that if $\mathcal{C}$ and $\mathcal{D}$ are presentable ∞-categories such that $\mathcal{D}$ is pointed, then the canonical map $\mathrm{Fun}^{\mathrm{R}'}(\mathcal{D}, \mathcal{C}_*) \to \mathrm{Fun}^{\mathrm{R}'}(\mathcal{D}, \mathcal{C})$ is an equivalence of ∞-categories (here $\mathrm{Fun}^{\mathrm{R}'}(\mathcal{D}, \mathcal{C})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ spanned by those accessible functors which preserve small limits, and $\mathrm{Fun}^{\mathrm{R}'}(\mathcal{D}, \mathcal{C}_*)$ is defined similarly). This follows from Lemma 6.3.2.12, Proposition T.1.2.13.8, and Proposition T.4.4.2.9. $\square$

Remark 6.3.2.14. It follows from Proposition 6.3.2.11 that the ∞-category $\mathcal{S}_*$ of pointed spaces inherits a symmetric monoidal structure. This symmetric monoidal structure is uniquely determined by the requirements that the tensor product $\otimes : \mathcal{S}_* \times \mathcal{S}_* \to \mathcal{S}_*$ preserve colimits separately in each variable and that the unit object is given by $S^0 \in \mathcal{S}_*$ (Proposition 6.3.2.9). It follows that this tensor product is given by the classical *smash product* of pointed spaces: that is, it assigns to a pair of pointed spaces X and Y the space $X \wedge Y$ obtained from $X \times Y$ by collapsing the subspaces X and Y to the base point.

Proposition 6.3.2.15. *Let $n \geq -2$ be an integer, and let $\tau_{\leq n}\, \mathcal{S}$ denote the full subcategory of $\mathcal{S}$ spanned by the n-truncated spaces. Then the pair $(\tau_{\leq n}\, \mathcal{S}, *)$ is idempotent (here $* \in \mathcal{S}$ denotes the space consisting of a single point, which is automatically n-truncated). Moreover, the forgetful functor $\mathrm{Mod}_{\mathcal{S}_*}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Pr}^{\mathrm{L}}$ determines a fully faithful embedding, whose essential image is the full subcategory of $\mathfrak{X} \subseteq \mathrm{Pr}^{\mathrm{L}}$ spanned by the presentable ∞-categories which are equivalent to $(n+1)$-categories (see Definition T.2.3.4.1).*

Proof. The object $* \in \tau_{\leq n}\, \mathcal{S}$ corresponds to the colimit-preserving functor $\tau_{\leq n} : \mathcal{S} \to \tau_{\leq n}\, \mathcal{S}$ which assigns to each space $X \in \mathcal{S}$ its n-truncation. According to Propositions 6.3.2.4 and 6.3.2.10, it will suffice to show the following:

($*$) For every presentable ∞-category $\mathcal{C}$, the functor $\tau_{\leq n}$ induces a map $\theta : \mathcal{C} \simeq \mathcal{C} \otimes \mathcal{S} \to \mathcal{C} \otimes \tau_{\leq n}\, \mathcal{S}$ which exhibits $\mathcal{C} \otimes \tau_{\leq n}\, \mathcal{S}$ as a $\mathfrak{X}$-localization of $\mathcal{C}$.

Using Example 6.3.1.21, we are reduced to the following statement: if $\mathcal{C}$ and $\mathcal{D}$ are presentable ∞-categories such that $\mathcal{D}$ is equivalent to an $(n+1)$-category, then every $F \in \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ has a right adjoint $G : \mathcal{D} \to \mathcal{C}$ which factors through $\tau_{\leq n}\, \mathcal{C}$. This is immediate, since G is left exact and therefore carries objects of $\mathcal{D}$ (which are automatically n-truncated by virtue of our assumption on $\mathcal{D}$) to n-truncated objects of $\mathcal{C}$. $\quad\square$

Remark 6.3.2.16. Proposition 6.3.2.15 implies that each $\tau_{\leq n}\, \mathcal{S}$ is equipped with a symmetric monoidal structure, which is uniquely determined by the requirement that the tensor product $\tau_{\leq n}\, \mathcal{S} \times \tau_{\leq n}\, \mathcal{S} \to \tau_{\leq n}\, \mathcal{S}$ preserve colimits separately in each variable, and that the object $* \in \tau_{\leq n}\, \mathcal{S}$ be the unit object. It follows that this symmetric monoidal structure coincides with the Cartesian symmetric monoidal structure introduced in §2.4.1.

Remark 6.3.2.17. The proof of Proposition 6.3.2.15 shows that for any presentable ∞-category $\mathcal{C}$ and any $n \geq 0$, the tensor product $\mathcal{C} \otimes \tau_{\leq n}\, \mathcal{S}$ can be identified with the full subcategory $\tau_{\leq n}\, \mathcal{C} \subseteq \mathcal{C}$ spanned by the n-truncated objects of $\mathcal{C}$.

We now turn to the main case of interest.

Proposition 6.3.2.18. *Let Sp denote the ∞-category of spectra, and let $S \in \mathrm{Sp}$ be the sphere spectrum. Then (Sp, S) is idempotent. Moreover, the forgetful functor $\mathrm{Mod}_{\mathrm{Sp}}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Pr}^{\mathrm{L}}$ determines a fully faithful embedding whose essential image is the full subcategory $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}} \subseteq \mathrm{Pr}^{\mathrm{L}}$ spanned by the stable presentable ∞-categories.*

Proof. The object $S \in \mathrm{Sp}$ corresponds to the colimit-preserving functor $\Sigma^{\infty} : \mathcal{S} \to \mathrm{Sp}$. According to Propositions 6.3.2.4 and 6.3.2.10, it will suffice to show the following:

($*$) For every presentable ∞-category $\mathcal{C}$, the functor Σ^{∞} induces a map $F : \mathcal{C} \simeq \mathcal{C} \otimes \mathcal{S} \to \mathcal{C} \otimes \mathrm{Sp}$ which exhibits $\mathcal{C} \otimes \mathrm{Sp}$ as a $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$-localization of $\mathcal{C}$.

This follows immediately from Example 6.3.1.22 and Corollary 1.4.4.5. $\quad\square$

Corollary 6.3.2.19. *There exists a symmetric monoidal structure on the ∞-category Sp such that $S \in \mathrm{Sp}$ is the unit object and the tensor product $\otimes : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ preserves small colimits separately in each variable. Moreover, if $\mathcal{C}^{\otimes}$ is an arbitrary symmetric monoidal ∞-category such that $\mathcal{C}$ is stable and presentable and the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves colimits separately in each variable, then there exists a symmetric monoidal functor $\mathrm{Sp}^{\otimes} \to \mathcal{C}^{\otimes}$ such that the underlying functor $\mathrm{Sp} \to \mathcal{C}$ preserves small colimits. Moreover, the collection of such functors is parametrized by a contractible Kan complex.*

In other words, the ∞-category Sp admits a symmetric monoidal structure, which is uniquely characterized by the following properties:

(a) The bifunctor $\otimes : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ preserves small colimits separately in each variable.

(b) The unit object of Sp is the sphere spectrum S.

We will refer to this monoidal structure on Sp as the *smash product symmetric monoidal structure*.

Remark 6.3.2.20. It follows from Proposition 6.3.2.18 that every stable presentable ∞-category $\mathcal{C}$ is canonically tensored over the symmetric monoidal ∞-category Sp.

Warning 6.3.2.21. Given a pair of objects $X, Y \in \mathrm{hSp}$, the smash product of X and Y is usually denoted by $X \wedge Y$. We will depart from this convention by writing instead $X \otimes Y$ for the smash product.

Remark 6.3.2.22. For any ∞-operad $\mathcal{O}^{\otimes}$, the symmetric monoidal structure on Sp restricts to a $\mathcal{O}$-monoidal structure on spectra. If $\mathcal{O}^{\otimes}$ is unital, then the proof of Corollary 6.3.2.19 shows that this $\mathcal{O}$-monoidal structure is uniquely determined by the requirement that for each $X \in \mathcal{O}$, the corresponding unit object of Sp is identified with the sphere spectrum.

Let $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}, \mathrm{Sp})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})$ spanned by those functors which preserve small colimits. According to Corollary 1.4.4.6, evaluation at the sphere spectrum $S \in \mathrm{Sp}$ induces an equivalence of ∞-categories $\mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp}) \to \mathrm{Sp}$, so that the smash product monoidal structure on Sp determines a monoidal structure on $\mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})$. It follows from the above argument that this monoidal structure can be identified with the monoidal structure given by pointwise composition.

In Chapter 8, we will make an extensive study of the associative and commutative algebra objects of the ∞-category Sp.

6.3.3   Algebras and their Module Categories

Let $\mathcal{C}$ be a monoidal ∞-category and let $A \in \mathrm{Alg}(\mathcal{C})$ be an algebra object of $\mathcal{C}$. In §4.3.2, we saw that the ∞-category $\mathrm{RMod}_A(\mathcal{C})$ of right A-module objects of $\mathcal{C}$ is left-tensored over the ∞-category $\mathcal{C}$. In other words, we can identify $\mathrm{RMod}_A(\mathcal{C})$ with a *left* module over $\mathcal{C}$ in the ∞-category $\mathcal{C}\mathrm{at}_{\infty}$ (endowed with the Cartesian monoidal structure). Our goal in this section is to make a systematic study of this construction. We will proceed in several steps:

(i) We will begin by introducing an ∞-category $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Alg}}$ whose objects are pairs $(\mathcal{C}^{\otimes}, A)$, where $\mathcal{C}^{\otimes}$ is a monoidal ∞-category and A is an algebra object of $\mathcal{C}^{\otimes}$. Roughly speaking, a morphism from $(\mathcal{C}^{\otimes}, A)$ to $(\mathcal{D}^{\otimes}, B)$ in $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Alg}}$ consists of a monoidal functor $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ together with a map of algebras $F(A) \to B$. For a more precise discussion, see Definition 6.3.3.8 below.

(ii) We will define another ∞-category $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Mod}}$, whose objects can be viewed as pairs $(\mathcal{C}^{\otimes}, \mathcal{M})$ where $\mathcal{C}^{\otimes}$ is a monoidal ∞-category and $\mathcal{M}$ is an ∞-category left-tensored over $\mathcal{C}^{\otimes}$.

(iii) We will construct a functor Θ from a subcategory of $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Alg}}$ to $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Mod}}$. Informally, the functor Θ associates to every pair $(\mathcal{C}^{\otimes}, A)$ the ∞-category $\mathrm{RMod}_A(\mathcal{C})$ of right A-module objects of $\mathcal{C}^{\otimes}$.

The relevant constructions are straightforward but somewhat tedious. The reader who does not wish to become burdened by technicalities is invited to proceed directly to §6.3.4 and §6.3.5, where we will undertake a deeper study of the functor Θ.

We begin by recalling some terminology.

Definition 6.3.3.1. Let S be a simplicial set and let $\mathcal{O}^{\otimes}$ be an ∞-operad. A *coCartesian S-family of $\mathcal{O}$-monoidal ∞-categories* is a coCartesian fibration $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \times S$ with the following property: for every vertex $s \in S$, the induced map of fibers $\mathcal{C}_s^{\otimes} = \mathcal{C}^{\otimes} \times_S \{s\} \to \mathcal{O}^{\otimes}$ is a coCartesian fibration of ∞-operads.

In this case, we will say that q *exhibits* $\mathcal{C}^{\otimes}$ *as a coCartesian S-family of $\mathcal{O}$-monoidal ∞-categories*. In the special case $\mathcal{O}^{\otimes} = \mathcal{A}\mathrm{ss}^{\otimes}$, we will say that q is a *coCartesian S-family of monoidal ∞-categories*, or that q *exhibits* $\mathcal{C}^{\otimes}$ *as a coCartesian S-family of monoidal ∞-categories*.

Notation 6.3.3.2. If $q : \mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes} \times S$ is a coCartesian S-family of monoidal ∞-categories, we let $\mathcal{C}$ denote the fiber product $\mathcal{C}^{\otimes} \times_{\mathrm{N}(\boldsymbol{\Delta})^{op}} \{\langle 1 \rangle\}$, so that q induces a coCartesian fibration $\mathcal{C} \to S$.

Example 6.3.3.3. Let $\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty})$ denote the ∞-category of associative monoid objects of $\mathcal{C}\mathrm{at}_{\infty}$. There is a canonical map $\mathcal{A}\mathrm{ss}^{\otimes} \times \mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty}) \to \mathcal{C}\mathrm{at}_{\infty}$, which classifies a coCartesian fibration $q :$ $\widetilde{\mathrm{Mon}}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty}) \to \mathcal{A}\mathrm{ss}^{\otimes} \times \mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty})$. The coCartesian fibration q exhibits $\widetilde{\mathrm{Mon}}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty})$ as a co-Cartesian $\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty})$-family of monoidal ∞-categories. Moreover, this family of monoidal ∞-categories is universal in the following sense: for every simplicial set S, the construction $(\phi : S \to \mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty})) \mapsto$ $S \times_{\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty})} \widetilde{\mathrm{Mon}}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty})$ establishes a bijection between the collection of equivalence classes of diagrams $S \to \mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty})$ and the collection of equivalence classes of coCartesian S-families of monoidal ∞-categories $\mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes} \times S$ (with essentially small fibers).

Definition 6.3.3.4. Let K and S be simplicial sets. We will say that a coCartesian S-family of ∞-categories $q : \mathcal{C}^{\otimes} \to \mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty}) \times S$ is *compatible with K-indexed colimits* if the following conditions are satisfied:

(i) For each vertex $s \in S$, the fiber $\mathcal{C}_s$ admits K-indexed colimits.

(ii) For each vertex $s \in S$, the tensor product functor $\mathcal{C}_s \times \mathcal{C}_s \to \mathcal{C}_s$ preserves K-indexed colimits separately in each variable.

(iii) For every edge $s \to t$ in S, the induced functor $\mathcal{C}_s \to \mathcal{C}_t$ preserves K-indexed colimits.

If $\mathcal{K}$ is a collection of simplicial sets, we will say that q is *compatible with $\mathcal{K}$-indexed colimits* if it is compatible with K-indexed colimits for each $K \in \mathcal{K}$.

Notation 6.3.3.5. Let $\mathcal{K}$ be a collection of simplicial sets. Let $\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}^{\mathcal{K}}(\mathcal{C}\mathrm{at}_{\infty})$ denote the subcategory of $\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty})$ whose objects are monoidal ∞-categories $\mathcal{C}^{\otimes}$ which are compatible with $\mathcal{K}$-indexed colimits and whose morphisms are monoidal functors $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ such that the underlying functor $\mathcal{C} \to \mathcal{D}$ preserves $\mathcal{K}$-indexed colimits. Let $\widetilde{\mathrm{Mon}}_{\mathcal{A}\mathrm{ss}}^{\mathcal{K}}(\mathcal{C}\mathrm{at}_{\infty})$ denote the fiber product $\widetilde{\mathrm{Mon}}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty}) \times_{\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty})} \mathrm{Mon}_{\mathcal{A}\mathrm{ss}}^{\mathcal{K}}(\mathcal{C}\mathrm{at}_{\infty})$. The evident map $q : \widetilde{\mathrm{Mon}}_{\mathcal{A}\mathrm{ss}}^{\mathcal{K}}(\mathcal{C}\mathrm{at}_{\infty}) \to \mathcal{A}\mathrm{ss}^{\otimes} \times \mathrm{Mon}_{\mathcal{A}\mathrm{ss}}^{\mathcal{K}}(\mathcal{C}\mathrm{at}_{\infty})$ exhibits $\widetilde{\mathrm{Mon}}_{\mathcal{A}\mathrm{ss}}^{\mathcal{K}}(\mathcal{C}\mathrm{at}_{\infty})$ as a coCartesian $\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}^{\mathcal{K}}(\mathcal{C}\mathrm{at}_{\infty})$-family of monoidal ∞-categories which is compatible with $\mathcal{K}$-indexed colimits. Moreover, it is universal with respect to this property: for every simplicial set S, pullback along q induces a bijection from equivalence classes of diagrams $S \to \mathrm{Mon}_{\mathcal{A}\mathrm{ss}}^{\mathcal{K}}(\mathcal{C}\mathrm{at}_{\infty})$ and equivalence classes of coCartesian S-families of monoidal ∞-categories which are compatible with $\mathcal{K}$-indexed colimits.

Remark 6.3.3.6. Let $\mathcal{C}\mathrm{at}_{\infty}(\mathcal{K})$ be defined as in Definition 6.3.1.1, and endowed with the symmetric monoidal structure described in Corollary 6.3.1.4. For every set $\mathcal{K}$ of ∞-categories, the equivalence

$$\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty}) \simeq \mathrm{Alg}(\mathcal{C}\mathrm{at}_{\infty})$$

restricts to an equivalence $\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}^{\mathcal{K}}(\mathcal{C}\mathrm{at}_{\infty}) \simeq \mathrm{Alg}(\mathcal{C}\mathrm{at}_{\infty}(\mathcal{K}))$.

Definition 6.3.3.7. We let $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Alg}}$ denote the full subcategory of the fiber product

$$\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty}) \times_{\mathrm{Fun}_{\mathcal{A}\mathrm{ss}^{\otimes}}(\mathcal{A}\mathrm{ss}^{\otimes}, \mathcal{A}\mathrm{ss}^{\otimes} \times \mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty}))} \mathrm{Fun}_{\mathcal{A}\mathrm{ss}^{\otimes}}(\mathcal{A}\mathrm{ss}^{\otimes}, \widetilde{\mathrm{Mon}}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty}))$$

spanned by those pairs $(\mathcal{C}^{\otimes}, A)$, where $\mathcal{C}^{\otimes} \in \mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty})$ is a monoidal ∞-category and A is an algebra object of the monoidal ∞-category $\widetilde{\mathrm{Mon}}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty}) \times_{\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty})} \{\mathcal{C}^{\otimes}\} \simeq \mathcal{C}^{\otimes}$. If $\mathcal{K}$ is a collection of simplicial sets, we let $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Alg}}(\mathcal{K})$ denote the fiber product $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Alg}} \times_{\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathcal{C}\mathrm{at}_{\infty})} \mathrm{Mon}_{\mathcal{A}\mathrm{ss}}^{\mathcal{K}}(\mathcal{C}\mathrm{at}_{\infty})$.

Remark 6.3.3.8. The ∞-category $\mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathcal{K})$ is characterized up to equivalence by the following universal property: for any simplicial set S, there is a bijection between equivalence classes of diagrams $S \to \mathrm{Cat}_\infty^{\mathrm{Alg}}(\mathcal{K})$ and equivalence classes of diagrams

$$
\begin{array}{ccc}
 & & \mathcal{C}^\otimes \\
 & \nearrow^{A} & \downarrow{q} \\
\mathcal{A}\mathrm{ss}^\otimes \times S & \xrightarrow{\;\mathrm{id}\;} & \mathcal{A}\mathrm{ss}^\otimes \times S,
\end{array}
$$

where q exhibits $\mathcal{C}^\otimes$ as a coCartesian S-family of monoidal ∞-categories whose fibers are essentially small, q is compatible with $\mathcal{K}$-indexed colimits, and A is an S-family of associative algebra objects of $\mathcal{C}^\otimes$.

Definition 6.3.3.9. If $q : \mathcal{C}^\otimes \to \mathcal{A}\mathrm{ss}^\otimes \times S$ is a coCartesian S-family of monoidal ∞-categories, then we will say that a map $p : \mathcal{M}^\otimes \to \mathcal{LM}^\otimes \times S$ *exhibits $\mathcal{M}^\otimes$ as a coCartesian S-family of ∞-categories left-tensored over* $\mathcal{C}^\otimes$ if p is a coCartesian S-family of $\mathcal{LM}$-operads (in the sense of Definition 6.1.4.16) and we are given an isomorphism $\mathcal{C}^\otimes \simeq \mathcal{M}^\otimes \times_{\mathcal{LM}^\otimes} \mathcal{A}\mathrm{ss}^\otimes$.

Notation 6.3.3.10. If $\mathcal{M}^\otimes \to \mathcal{LM}^\otimes \times S$ is as in Definition 6.3.3.9, we let $\mathcal{M}$ denote the fiber product $\mathcal{M}^\otimes \times_{\mathcal{LM}^\otimes} \{\mathfrak{m}\}$.

Notation 6.3.3.11. Let $\mathcal{C}^\otimes \to \mathcal{A}\mathrm{ss}^\otimes \times S$ be a coCartesian S-family of monoidal ∞-categories. We define a simplicial set $\widetilde{\mathrm{Alg}}(\mathcal{C})$ equipped with a forgetful map $\widetilde{\mathrm{Alg}}(\mathcal{C}) \to S$ so that the following universal property is satisfied: for every map of simplicial sets $K \to S$, there is a canonical bijection

$$
\mathrm{Hom}_S(K, \widetilde{\mathrm{Alg}}(\mathcal{C})) \simeq \mathrm{Hom}_{\mathcal{A}\mathrm{ss}^\otimes \times S}(\mathcal{A}\mathrm{ss}^\otimes \times K, \mathcal{C}^\otimes).
$$

We let $\mathrm{Alg}(\mathcal{C})$ denote the full simplicial subset of $\widetilde{\mathrm{Alg}}(\mathcal{C})$ spanned by those vertices which correspond to algebra objects in the monoidal ∞-category $\mathcal{C}_s^\otimes$, for some vertex $s \in S$.

Suppose we are given a coCartesian S-family $\mathcal{M}^\otimes \to \mathcal{LM}^\otimes \times S$ of ∞-categories left-tensored over $\mathcal{C}^\otimes$. We let $\widetilde{\mathrm{LMod}}(\mathcal{M})$ denote a simplicial set with a map $\widetilde{\mathrm{LMod}}(\mathcal{M}) \to S$ hacing the following universal property: for every map of simplicial sets $K \to S$, there is a canonical bijection

$$
\mathrm{Hom}_S(K, \widetilde{\mathrm{LMod}}(\mathcal{M})) \simeq \mathrm{Hom}_{\mathcal{LM}^\otimes \times S}(\mathcal{LM}^\otimes \times K, \mathcal{M}^\otimes).
$$

We let $\mathrm{LMod}(\mathcal{M})$ denote the full simplicial subset of $\widetilde{\mathrm{LMod}}(\mathcal{M})$ whose vertices are left module objects of $\mathcal{M}_s^\otimes$, for some vertex $s \in S$.

Remark 6.3.3.12. In the special case where $S = \Delta^0$, the terminology of Notation 6.3.3.11 agrees with that of Definitions 4.1.1.6 and 4.2.1.13.

The following result is an easy consequence of Proposition T.3.1.2.1:

Lemma 6.3.3.13. *Let $q : \mathcal{C}^\otimes \to \mathcal{A}\mathrm{ss}^\otimes \times S$ be a coCartesian S-family of monoidal ∞-categories, let $p : \mathcal{M}^\otimes \to \mathcal{LM}^\otimes \times S$ be a coCartesian S-family of ∞-categories left-tensored over $\mathcal{C}^\otimes$. Then:*

(1) *The map $q' : \mathrm{Alg}(\mathcal{C}) \to S$ is a coCartesian fibration of simplicial sets.*

(2) *A morphism $A \to A'$ in $\mathrm{Alg}(\mathcal{C})$ is q'-coCartesian if and only if the underlying map $A(\langle 1 \rangle) \to A'(\langle 1 \rangle)$ is a q-coCartesian morphism in $\mathcal{C} \subseteq \mathcal{C}^\otimes$.*

(3) *The map $r : \mathrm{LMod}(\mathcal{M}) \to S$ is a coCartesian fibration of simplicial sets.*

(4) *A morphism $M \to M'$ in $\mathrm{LMod}(\mathcal{M})$ is r-coCartesian if and only if its image in $\mathrm{Alg}(\mathcal{C})$ is q'-coCartesian, and the induced map $M(\mathfrak{m}) \to M'(\mathfrak{m})$ is a p-coCartesian morphism in $\mathcal{M} \subseteq \mathcal{M}^\otimes$.*

Definition 6.3.3.14. Let $\mathcal{K}$ be a collection of simplicial sets, and let $\mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes op} \times S$ be a coCartesian S-family of monoidal ∞-categories which is compatible with $\mathcal{K}$-indexed colimits, and let $\mathcal{M}^{\otimes} \to \mathcal{L}\mathcal{M}^{\otimes} \times S$ be a coCartesian S-family of ∞-categories left-tensored over $\mathcal{C}^{\otimes}$. We will say that $\mathcal{M}^{\otimes}$ is *compatible with $\mathcal{K}$-indexed colimits* if the following conditions are satisfied:

(*i*) For every vertex $s \in S$ and each $K \in \mathcal{K}$, the ∞-category $\mathcal{M}_s$ admits K-indexed colimits.

(*ii*) For every vertex $s \in S$ and each $K \in \mathcal{K}$, the action map $\mathcal{C}_s \times \mathcal{M}_s \to \mathcal{M}_s$ preserves K-indexed colimits separately in each variable.

(*iii*) For every edge $s \to t$ in S and each $K \in \mathcal{K}$, the induced functor $\mathcal{M}_s \to \mathcal{M}_t$ preserves K-indexed colimits.

Lemma 6.3.3.15. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes} \times S$ be a coCartesian S-family of monoidal ∞-categories and let $q : \mathcal{M}^{\otimes} \to \mathcal{L}\mathcal{M}^{\otimes} \times S$ be a coCartesian S-family of ∞-categories left-tensored over $\mathcal{C}^{\otimes}$. Assume that both p and q are compatible with $\mathrm{N}(\boldsymbol{\Delta})^{op}$-indexed colimits. Then:*

(1) *The forgetful functor $r : \mathrm{LMod}(\mathcal{M}) \to \mathrm{Alg}(\mathcal{C})$ is a coCartesian fibration of simplicial sets.*

(2) *Let $f : M \to N$ be an edge of $\mathrm{LMod}(\mathcal{M})$, lying over an edge $f_0 : A \to B$ in $\mathrm{Alg}(\mathcal{C})$, which in turn lies over an edge $\alpha : s \to t$ in S. Then f is r-coCartesian if and only if it induces an equivalence $B \otimes_{\alpha_! A} M \to N$ in the ∞-category $\mathrm{LMod}(\mathcal{M})_t$, where $\alpha_! : \mathrm{Alg}(\mathcal{C})_s \to \mathrm{Alg}(\mathcal{C})_t$ denotes the functor induced by α.*

Proof. Choose a vertex $M \in \mathrm{LMod}(\mathcal{M})$ lying over $A \in \mathrm{Alg}(\mathcal{C})$, and let $f_0 : A \to B$ be an edge of $\mathrm{Alg}(\mathcal{C})$ lying over an edge $\alpha : s \to t$ in S. To prove (1), we will show that f_0 can be lifted to an r-coCartesian morphism of $\mathrm{LMod}(\mathcal{M})$; assertion (2) will be a consequence of our construction. Let $r' : \mathrm{Alg}(\mathcal{C}) \to S$ denote the canonical projection. Using Lemma 6.3.3.13, we can lift α to an $(r' \circ r)$-coCartesian morphism $f' : M \to M'$ in $\mathrm{LMod}(\mathcal{M})$; let $f_0' : A \to A'$ denote the image of f' in $\mathrm{Alg}(\mathcal{C})$. Lemma 6.3.3.13 guarantees that f_0' is r'-coCartesian, so we can identify A' with $\alpha_! A$; moreover, there exists a 2-simplex σ of $\mathrm{Alg}(\mathcal{C})$ corresponding to a diagram

$$
\begin{array}{ccc}
 & A' & \\
{\scriptstyle f_0'}\nearrow & & \searrow{\scriptstyle f_0''} \\
A & \xrightarrow{\ \ f_0\ \ } & B.
\end{array}
$$

We will prove that f_0'' can be lifted to an r-coCartesian morphism f'' of $\mathrm{LMod}(\mathcal{M})$. Using the fact that r is an inner fibration, it will follow that there is a composition $f = f'' \circ f'$ lifting f_0, which is also r-coCartesian by virtue of Proposition T.2.4.1.7. We may therefore replace f_0 by f_0'' and thereby reduce to the case where $s = t$ and the edge α is degenerate.

Proposition 4.3.7.14 shows that f_0 can be lifted to a locally r-coCartesian morphism $f : M \to B \otimes_A M$ in $\mathrm{LMod}(\mathcal{M})_s$. Since the projection $r_s : \mathrm{LMod}(\mathcal{M})_s \to \mathrm{Alg}(\mathcal{M})_s$ is a Cartesian fibration (Corollary 4.2.3.2), we deduce that f is r_s-coCartesian (Corollary T.5.2.2.4). To prove that f is r-coCartesian, it will suffice to show that for every edge $\beta : s \to t$ in S, the image $\beta_!(f)$ is an r_t-coCartesian morphism of the fiber $\mathrm{LMod}(\mathcal{M})_t$. Using the characterization of r_t-coCartesian morphisms supplied by Proposition 4.3.7.14, we see that this is equivalent to the requirement that the canonical map $\alpha_!(B) \otimes_{\alpha_! A} \alpha_! M \to \alpha_!(B \otimes_A M)$ is an equivalence in $\mathrm{LMod}(\mathcal{M})_t$. This is clear, since the functor $\alpha_!$ preserves tensor products and geometric realizations of simplicial objects. $\square$

Definition 6.3.3.16. Let $p : \mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes} \times S$ be a coCartesian S-family of monoidal ∞-categories. An *S-family of algebra objects of $\mathcal{C}^{\otimes}$* is a section of the projection map $\mathrm{Alg}(\mathcal{C}) \to S$.

If $q : \mathcal{M}^{\otimes} \to \mathcal{L}\mathcal{M}^{\otimes} \times S$ is a coCartesian S-family of ∞-categories left-tensored over $\mathcal{C}^{\otimes}$ and A is an S-family of algebra objects of $\mathcal{C}^{\otimes}$, then we let $\mathrm{LMod}_A(\mathcal{M})$ denote the fiber product $\mathrm{LMod}(\mathcal{M}) \times_{\mathrm{Alg}(\mathcal{C})} S$.

Remark 6.3.3.17. In the situation of Definition 6.3.3.16, if p and q are compatible with $\mathrm{N}(\mathbf{\Delta})^{op}$-indexed colimits, then Lemma 6.3.3.15 implies that the projection map $\mathrm{LMod}(\mathcal{M}) \to S$ is a coCartesian fibration of simplicial sets.

Variant 6.3.3.18. Let $p : \mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes} \times S$ be a coCartesian S-family of monoidal ∞-categories. We will say that an inner fibration $q : \mathcal{M}^{\otimes} \to \mathcal{LM}^{\otimes} \times S$ is a *locally coCartesian S-family of ∞-categories left-tensored over $\mathcal{C}^{\otimes}$* if we are given an isomorphism $\mathcal{C}^{\otimes} \simeq \mathcal{M}^{\otimes} \times_{\mathcal{LM}^{\otimes}} \mathcal{A}\mathrm{ss}^{\otimes}$ and, for every edge $\Delta^1 \to S$, the induced map $q_{\Delta^1} : \mathcal{M}^{\otimes} \times_S \Delta^1 \to \mathcal{LM}^{\otimes} \times \Delta^1$ is a coCartesian Δ^1-family of ∞-categories left-tensored over $\mathcal{C}^{\otimes} \times_S \Delta^1$. If K is a simplicial set, we will say that q is *compatible with K-indexed colimits* if each q_{Δ^1} is compatible with K-indexed colimits. If p and q are compatible with $\mathrm{N}(\mathbf{\Delta})^{op}$-indexed colimits and A is an S-family of algebra objects of $\mathcal{C}^{\otimes}$, then we define $\mathrm{LMod}(\mathcal{M})$ as in Notation 6.3.3.11 and $\mathrm{LMod}_A(\mathcal{M})$ as in Definition 6.3.3.16. It follows from Remark 6.3.3.17 that the map $\mathrm{LMod}_A(\mathcal{M}) \to S$ is a locally coCartesian fibration of simplicial sets.

Variant 6.3.3.19. In the situation of Definition 6.3.3.9, there is an evident dual notion of a *locally coCartesian S-family of ∞-categories $\mathcal{M}^{\otimes} \to \mathcal{RM}^{\otimes} \times S$ right-tensored over $\mathcal{C}^{\otimes}$*. Given an S-family of algebra objects A of $\mathcal{C}^{\otimes}$, we can then define a locally coCartesian fibration $\mathrm{RMod}_A(\mathcal{M}) \to S$ (provided that $\mathcal{C}^{\otimes}$ and $\mathcal{M}^{\otimes}$ are compatible with $\mathrm{N}(\mathbf{\Delta})^{op}$-indexed colimits), whose fiber over a vertex $s \in S$ is the ∞-category of right A-module objects of the fiber $\mathcal{M}_s$.

We let $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Mod}} = \mathrm{Mon}_{\mathcal{LM}}(\mathcal{C}\mathrm{at}_{\infty})$ denote the ∞-category of $\mathcal{LM}$-monoid objects of $\mathcal{C}\mathrm{at}_{\infty}$. We will informally describe the objects of $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Mod}}$ as pairs $(\mathcal{C}, \mathcal{M})$, where $\mathcal{C}$ is an ∞-category equipped with a monoidal structure and $\mathcal{M}$ is an ∞-category equipped with a left action of $\mathcal{C}$. If $\mathcal{K}$ is a collection of simplicial sets, we let $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Mod}}(\mathcal{K})$ denote the subcategory of $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Mod}}$ whose objects are diagrams where $\mathcal{C}$ and $\mathcal{M}$ admit $\mathcal{K}$-indexed colimits and the tensor product functors

$$\mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \mathcal{C} \times \mathcal{M} \to \mathcal{M}$$

preserve $\mathcal{K}$-indexed colimits separately in each variable, and whose morphisms are maps $(\mathcal{C}, \mathcal{M}) \to (\mathcal{C}', \mathcal{M}')$ such that the underlying functors $\mathcal{C} \to \mathcal{C}'$, $\mathcal{M} \to \mathcal{M}'$ preserve $\mathcal{K}$-indexed colimits.

Remark 6.3.3.20. The ∞-category $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Mod}}(\mathcal{K})$ is characterized by the following universal property: for every simplicial set S, there is a canonical bijection between equivalence classes of maps $S \to \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Mod}}(\mathcal{K})$ and equivalence classes of diagrams $\mathcal{M}^{\otimes} \to \mathcal{LM}^{\otimes} \times S$ which exhibit $\mathcal{M}^{\otimes}$ as a coCartesian S-family of ∞-categories left-tensored over some coCartesian S-family of monoidal ∞-categoires $\mathcal{C}^{\otimes} = \mathcal{M}^{\otimes} \times_{\mathcal{LM}^{\otimes}} \mathcal{A}\mathrm{ss}^{\otimes}$ such that $\mathcal{C}^{\otimes}$ and $\mathcal{M}^{\otimes}$ are compatible with $\mathcal{K}$-indexed colimits.

We now sketch the construction of the functor Θ.

Construction 6.3.3.21. Let $\mathbf{Pr} : \mathcal{LM}^{\otimes} \times \mathcal{RM}^{\otimes} \to \mathcal{BM}^{\otimes}$ be defined as in Construction 4.3.2.1, and let $\mathbf{Pr}_0$ denote the composition of $\mathbf{Pr}$ with the forgetful functor $\mathcal{BM}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$. If $\mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes} \times S$ is a coCartesian S-family of monoidal ∞-categories, we let $\overline{\mathcal{C}}^{\otimes}$ denote the pullback $\overline{\mathcal{C}}^{\otimes} \times_{\mathcal{A}\mathrm{ss}^{\otimes}} (\mathcal{LM}^{\otimes} \times \mathcal{RM}^{\otimes})$, which we regard as a coCartesian $(\mathcal{LM}^{\otimes} \times S)$-family of $\mathcal{RM}^{\otimes}$-monoidal ∞-categories; let $\overline{\mathcal{C}}_{\mathfrak{a}}^{\otimes} = \overline{\mathcal{C}}^{\otimes} \times_{\mathcal{RM}^{\otimes}} \mathcal{A}\mathrm{ss}^{\otimes}$ be the underlying $(\mathcal{LM}^{\otimes} \times S)$-family of monoidal ∞-categories.

Suppose that $A : S \to \mathrm{Alg}(\mathcal{C})$ is an S-family of algebra objects of $\mathcal{C}^{\otimes}$. Then A determines a $(\mathcal{LM}^{\otimes} \times S)$-family of algebra objects of $\overline{\mathcal{C}}_{\mathfrak{a}}^{\otimes}$, which we will denote by $\overline{A}$. We let $\mathrm{RMod}_A(\mathcal{C})^{\otimes}$ denote the simplicial set $\mathrm{RMod}_{\overline{A}}(\overline{\mathcal{C}})$ of Variant 6.3.3.19. If $\mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes} \times S$ is compatible with $\mathrm{N}(\mathbf{\Delta})^{op}$-indexed colimits, then Lemma 6.3.3.15 implies that the map $\mathrm{RMod}_A(\mathcal{C})^{\otimes} \to \mathcal{LM}^{\otimes} \times S$ is a coCartesian S-family of $\mathcal{LM}^{\otimes}$-monoidal ∞-categories. Let $\mathcal{C}'^{\otimes} = \mathrm{RMod}_A(\mathcal{C})^{\otimes} \times_{\mathcal{LM}^{\otimes}} \mathcal{A}\mathrm{ss}^{\otimes}$ be the underlying S-family of monoidal ∞-categories. Using Propositions 4.3.2.6 and T.3.3.1.5, we deduce that the inclusion $\{\mathfrak{m}\} \hookrightarrow \mathcal{RM}^{\otimes}$ induces a categorical equivalence $\mathcal{C}'^{\otimes} \to \mathcal{C}^{\otimes}$.

In the situation of Construction 6.3.3.21, if the original family $\mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes} \times S$ is compatible with $\mathcal{K}$-indexed colimits for some collection of simplicial sets $\mathcal{K}$ containing $\mathrm{N}(\mathbf{\Delta})^{op}$, then $\mathrm{RMod}_A(\mathcal{C}) \to \mathcal{LM}^{\otimes} \times S$ is also compatible with $\mathcal{K}$-indexed colimits. We can summarize the situation as follows:

Proposition 6.3.3.22. *Let $\mathcal{K}$ be a collection of simplicial sets which includes $\mathrm{N}(\mathbf{\Delta})^{op}$, let $q : \mathcal{C}^{\otimes} \to \mathcal{A}ss^{\otimes} \times S$ be a coCartesian S-family of monoidal ∞-categories which is compatible with $\mathcal{K}$-indexed colimits, and let A be an S-family of algebra objects of $\mathcal{C}^{\otimes}$. Then the forgetful functor $p : \mathrm{RMod}_A(\mathcal{C})^{\otimes} \to \mathcal{L}\mathcal{M}^{\otimes} \times S$ exhibits the coCartesian fibration $\mathrm{RMod}_A(\mathcal{C}) \to S$ of Variant 6.3.3.19 as left-tensored over a coCartesian S-family of monoidal ∞-categories $\mathcal{C}'^{\otimes}$ which is equivalent to $\mathcal{C}^{\otimes}$. Moreover, p is also compatible with $\mathcal{K}$-indexed colimits.*

Remark 6.3.3.23. In the special case $S = \Delta^0$, Proposition 6.3.3.22 amounts to the fact that there is a natural action of the monoidal ∞-category $\mathcal{C}$ on $\mathrm{RMod}_A(\mathcal{C})$, for each $A \in \mathrm{Alg}(\mathcal{C})$. This statement was established in §4.3.2 (and does not require any assumptions on $\mathcal{C}$).

Construction 6.3.3.24. Fix a collection of simplicial sets $\mathcal{K}$ which includes $\mathrm{N}(\mathbf{\Delta})^{op}$. Let $\mathcal{C}^{\otimes}$ denote the fiber product $\mathcal{C}at_{\infty}^{\mathrm{Alg}}(\mathcal{K}) \times_{\mathrm{Mon}_{\mathcal{A}ss}(\mathcal{C}at_{\infty})} \widetilde{\mathrm{Mon}}_{\mathcal{A}ss}(\mathcal{C}at_{\infty})$, so that we have a coCartesian $\mathcal{C}at_{\infty}^{\mathrm{Alg}}(\mathcal{K})$-family of monoidal ∞-categories $\mathcal{C}^{\otimes} \to \mathcal{A}ss^{\otimes} \times \mathcal{C}at_{\infty}^{\mathrm{Alg}}(\mathcal{K})$. By construction, there is a canonical $\mathcal{C}at_{\infty}^{\mathrm{Alg}}(\mathcal{K})$-family of algebra objects of $\mathcal{C}^{\otimes}$, which we will denote by A. Let $\mathrm{RMod}_A(\mathcal{C})^{\otimes} \to \mathcal{L}\mathcal{M}^{\otimes} \times \mathcal{C}at_{\infty}^{\mathrm{Alg}}(\mathcal{K})$ be defined as in Construction 6.3.3.21, so that $\mathrm{RMod}_A(\mathcal{C})^{\otimes}$ is a coCartesian $\mathcal{C}at_{\infty}^{\mathrm{Alg}}(\mathcal{K})$-family of $\mathcal{L}\mathcal{M}^{\otimes}$-monoidal ∞-categories which is compatible with $\mathcal{K}$-indexed colimits. Remark 6.3.3.20 implies that this family is classified by a functor $\Theta : \mathcal{C}at_{\infty}^{\mathrm{Alg}}(\mathcal{K}) \to \mathcal{C}at_{\infty}^{\mathrm{Mod}}(\mathcal{K})$. Note that the composite functor $\mathcal{C}at_{\infty}^{\mathrm{Alg}}(\mathcal{K}) \to \mathcal{C}at_{\infty}^{\mathrm{Mod}}(\mathcal{K}) \to \mathrm{Mon}_{\mathcal{A}ss}^{\mathcal{K}}(\mathcal{C}at_{\infty})$ classifies the coCartesian $\mathcal{C}at_{\infty}^{\mathrm{Alg}}(\mathcal{K})$-family of monoidal ∞-categories $\mathcal{C}^{\otimes}$, and is therefore equivalent to the evident forgetful functor $\mathcal{C}at_{\infty}^{\mathrm{Alg}}(\mathcal{K}) \to \mathrm{Mon}_{\mathcal{A}ss}^{\mathcal{K}}(\mathcal{C}at_{\infty})$. Replacing Θ by an equivalent functor if necessary, we will henceforth assume that the diagram

$$
\begin{array}{ccc}
\mathcal{C}at_{\infty}^{\mathrm{Alg}}(\mathcal{K}) & \xrightarrow{\quad\Theta\quad} & \mathcal{C}at_{\infty}^{\mathrm{Mod}}(\mathcal{K}) \\
& \searrow \qquad\qquad \swarrow & \\
& \mathrm{Mon}_{\mathcal{A}ss}^{\mathcal{K}}(\mathcal{C}at_{\infty}) &
\end{array}
$$

is commutative.

Remark 6.3.3.25. More informally, we can describe the functor $\Theta : \mathcal{C}at_{\infty}^{\mathrm{Alg}}(\mathcal{K}) \to \mathcal{C}at_{\infty}^{\mathrm{Mod}}(\mathcal{K})$ as follows: to every object $(\mathcal{C}^{\otimes}, A)$ of $\mathcal{C}at_{\infty}^{\mathrm{Alg}}(\mathcal{K})$ (given by a monoidal ∞-category $\mathcal{C}^{\otimes}$ and an algebra object $A \in \mathrm{Alg}(\mathcal{C})$), it associates the ∞-category $\mathrm{RMod}_A(\mathcal{C})$ of right A-module objects of $\mathcal{C}$, viewed as an ∞-category left-tensored over $\mathcal{C}$.

6.3.4 Properties of $\mathrm{RMod}_A(\mathcal{C})$

Let $\mathcal{C}$ be a monoidal ∞-category and let A be an algebra object of $\mathcal{C}$. The ∞-category $\mathrm{RMod}_A(\mathcal{C})$ of right A-module objects of $\mathcal{C}$ admits a left action of the ∞-category $\mathcal{C}$: informally speaking, if M is a right A-module and $C \in \mathcal{C}$, then $C \otimes M$ admits a right A-module structure given by the map

$$(C \otimes M) \otimes A \simeq C \otimes (M \otimes A) \to C \otimes M.$$

(for the formal construction, we refer the reader to §4.3.2). In this section, we will prove that (under some mild hypotheses) the ∞-category $\mathrm{RMod}_A(\mathcal{C})$ enjoys two important features (which will be formulated more precisely below):

(A) If $\mathcal{N}$ is an ∞-category left-tensored over $\mathcal{C}$, then the ∞-category of $\mathcal{C}$-linear functors from $\mathrm{RMod}_A(\mathcal{C})$ to $\mathcal{N}$ is equivalent to the ∞-category $\mathrm{LMod}_A(\mathcal{N})$ of left A-module objects of $\mathcal{N}$ (Theorem 6.3.4.1).

(B) If $\mathcal{M}$ is an ∞-category right-tensored over $\mathcal{C}$, then the tensor product $\mathcal{M} \otimes_{\mathcal{C}} \mathrm{RMod}_A(\mathcal{C})$ is equivalent to the ∞-category $\mathrm{RMod}_A(\mathcal{M})$ of right A-module objects of $\mathcal{M}$ (Theorem 6.3.4.6).

We begin with a precise formulation of assertion (A):

Theorem 6.3.4.1. *Let $\mathcal{K}$ be a collection of simplicial sets which includes $\mathrm{N}(\mathbf{\Delta})^{op}$, let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category, and $\mathcal{M}$ an ∞-category left-tensored over $\mathcal{C}$. Assume that $\mathcal{C}$ and $\mathcal{M}$ admit $\mathcal{K}$-indexed colimits, and that the tensor product functors*

$$\mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \mathcal{C} \times \mathcal{M} \to \mathcal{M}$$

preserve $\mathcal{K}$-indexed colimits separately in each variable. Let A be an algebra object of $\mathcal{C}$, and let θ denote the composition

$$
\begin{aligned}
\mathrm{LinFun}_{\mathcal{C}}^{\mathcal{K}}(\mathrm{RMod}_A(\mathcal{C}), \mathcal{M}) \quad &\subseteq \quad \mathrm{LinFun}_{\mathcal{C}}(\mathrm{RMod}_A(\mathcal{C}), \mathcal{M}) \\
&\xrightarrow{\theta'} \quad \mathrm{Fun}(\mathrm{LMod}_A(\mathrm{RMod}_A(\mathcal{C})), \mathrm{LMod}_A(\mathcal{M})) \\
&\xrightarrow{\theta''} \quad \mathrm{LMod}_A(\mathcal{M}),
\end{aligned}
$$

where θ' is the map described in Remark 4.3.7.8 and θ'' is given by evaluation at the A-bimodule given by A. Then θ is an equivalence of ∞-categories.

A precise formulation of (B) requires more effort. In order to make sense of the relative tensor product $\mathcal{M} \otimes_{\mathcal{C}} \mathrm{RMod}_A(\mathcal{C})$, we need to interpret each factor as an object of a relevant ∞-category. To this end, let us recall a bit of notation. Fix a collection of simplicial sets $\mathcal{K}$. We let $\mathcal{C}at_{\infty}(\mathcal{K})$ be the subcategory of $\mathcal{C}at_{\infty}$ whose objects are ∞-categories which admit $\mathcal{K}$-indexed colimits and whose morphisms are functors which preserve $\mathcal{K}$-indexed colimits, and regard $\mathcal{C}at_{\infty}(\mathcal{K})$ as endowed with the (symmetric) monoidal structure described in §6.3.1 Some basic features of $\mathcal{C}at_{\infty}(\mathcal{K})$ are summarized in the following result:

Lemma 6.3.4.2. *Let $\mathcal{K}$ be a small collection of simplicial sets. Then the ∞-category $\mathcal{C}at_{\infty}(\mathcal{K})$ is presentable, and the tensor product $\otimes : \mathcal{C}at_{\infty}(\mathcal{K}) \times \mathcal{C}at_{\infty}(\mathcal{K}) \to \mathcal{C}at_{\infty}(\mathcal{K})$ preserves small colimits separately in each variable.*

Proof. We first show that $\mathcal{C}at_{\infty}(\mathcal{K})$ admits small colimits. Let $\mathcal{J}$ be an ∞-category, and let $\chi : \mathcal{J} \to \mathcal{C}at_{\infty}(\mathcal{K})$ be a diagram. Let χ' denote the composition

$$\mathcal{J} \xrightarrow{\chi} \mathcal{C}at_{\infty}(\mathcal{K}) \subseteq \mathcal{C}at_{\infty},$$

and let $\mathcal{C}$ be a colimit of the diagram χ' in $\mathcal{C}at_{\infty}$. Let $\mathcal{R}$ denote the collection of all diagrams in $\mathcal{C}$ given by a composition

$$K^{\triangleright} \xrightarrow{p} \chi(J) \to \mathcal{C},$$

where $K \in \mathcal{K}$ and p is a colimit diagram. It follows from Proposition T.5.3.6.2 that there exists a functor $F : \mathcal{C} \to \mathcal{D}$ with the following properties:

(*i*) For every diagram $q : K^{\triangleright} \to \mathcal{C}$ belonging to $\mathcal{R}$, the composition $F \circ q$ is a colimit diagram.

(*ii*) The ∞-category $\mathcal{D}$ admits $\mathcal{K}$-indexed colimits.

(*iii*) For every ∞-category $\mathcal{E}$ which admits $\mathcal{K}$-indexed colimits, composition with F induces an equivalence from the full subcategory of $\mathrm{Fun}(\mathcal{D}, \mathcal{E})$ spanned by those functors which preserve $\mathcal{K}$-indexed colimits to the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$ spanned by those functors such that the composition $\chi(J) \to \mathcal{C} \to \mathcal{E}$ preserves $\mathcal{K}$-indexed colimits for each $J \in \mathcal{J}$.

The map F allows us to promote $\mathcal{D}$ to an object of $\overline{\mathcal{D}} = (\mathcal{C}at_{\infty})_{\chi'/}$. Using (*i*) and (*ii*), we deduce that $\overline{\mathcal{D}}$ lies in the subcategory $\mathcal{C}at_{\infty}(\mathcal{K})_{\chi/} \subseteq (\mathcal{C}at_{\infty})_{\chi'/}$, and (*iii*) that this lifting exhibits $\mathcal{D}$ as a colimit of the diagram χ.

We next show that the tensor product $\otimes : \mathcal{C}at_{\infty}(\mathcal{K}) \times \mathcal{C}at_{\infty}(\mathcal{K}) \to \mathcal{C}at_{\infty}(\mathcal{K})$ preserves small colimits separately in each variable. It will suffice to show that for every object $\mathcal{C} \in \mathcal{C}at_{\infty}(\mathcal{K})$, the operation $\mathcal{D} \mapsto \mathcal{C} \otimes \mathcal{D}$ admits a right adjoint. This right adjoint is given by the formula $\mathcal{E} \mapsto \mathrm{Fun}^{\mathcal{K}}(\mathcal{C}, \mathcal{E})$, where

$\mathrm{Fun}^{\mathcal{K}}(\mathcal{C}, \mathcal{E})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$ spanned by those functors which preserve $\mathcal{K}$-indexed colimits.

We now complete the proof by showing that $\mathcal{C}at_\infty(\mathcal{K})$ is presentable. Fix an uncountable regular cardinal κ so that $\mathcal{K}$ is κ-small and every simplicial set $K \in \mathcal{K}$ is κ-small. Choose another regular cardinal τ such that $\kappa < \tau$ and $\kappa \ll \tau$: that is, $\tau_0^{\kappa_0} < \tau$ whenever $\tau_0 < \tau$ and $\kappa_0 < \kappa$. Let $\mathcal{C}at_\infty^\tau(\mathcal{K})$ denote the full subcategory of $\mathcal{C}at_\infty^\tau(\mathcal{K})$ spanned by those ∞-categories $\mathcal{C}$ which are τ-small and admit $\mathcal{K}$-indexed colimits. Then $\mathcal{C}at_\infty^\tau(\mathcal{K})$ is an essentially small ∞-category; it will therefore suffice to prove that every object $\mathcal{C} \in \mathcal{C}at_\infty(\mathcal{K})$ is the colimit (in $\mathcal{C}at_\infty(\mathcal{K})$) of a diagram taking values in $\mathcal{C}at_\infty^\tau(\mathcal{K})$.

Let A be the collection of all simplicial subsets $\mathcal{C}_0 \subseteq \mathcal{C}$ with the following properties:

 (a) The simplicial set $\mathcal{C}_0$ is an ∞-category.

 (b) The ∞-category $\mathcal{C}_0$ admits $\mathcal{K}$-indexed colimits.

 (c) The inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ preserves $\mathcal{K}$-indexed colimits.

 (d) The simplicial set $\mathcal{C}_0$ is τ-small.

Our proof rests on the following claim:

 (∗) For every τ-small simplicial subset $\mathcal{C}_0 \subseteq \mathcal{C}$, there exists a τ-small simplicial subset $\mathcal{C}_0' \subseteq \mathcal{C}$ which contains $\mathcal{C}_0$ and belongs to A.

Let us regard the set A as partially ordered with respect to inclusions, and we have an evident functor $\rho : A \to \mathcal{S}et_\Delta$. From assertion (∗), it follows that A is filtered (in fact, τ-filtered) and that $\mathcal{C}$ is the colimit of the diagram ρ (in the ordinary category $\mathcal{S}et_\Delta$). Since the collection of categorical equivalences in $\mathcal{S}et_\Delta$ is stable under filtered colimits, we deduce that $\mathcal{C}$ is the homotopy colimit of the diagram ρ (with respect to the Joyal model structure), so that $\mathcal{C}$ is the colimit of the induced diagram $\mathrm{N}(\rho) : \mathrm{N}(A) \to \mathcal{C}at_\infty$ (Theorem T.4.2.4.1). Requirement (c) guarantees that every inclusion $\mathcal{C}_0 \subseteq \mathcal{C}_1$ between elements of A is a functor which preserves $\mathcal{K}$-indexed colimits, so that $\mathrm{N}(\rho)$ factors through $\mathcal{C}at_\infty(\mathcal{K})$. We claim that $\mathcal{C}$ is a colimit of the diagram $\mathrm{N}(\rho)$ in the ∞-category $\mathcal{C}at_\infty(\mathcal{K})$. Unwinding the definitions, this amounts to the following assertion: for every ∞-category $\mathcal{E}$ which admits $\mathcal{K}$-indexed colimits, a functor $F : \mathcal{C} \to \mathcal{E}$ preserves $\mathcal{K}$-indexed colimits if and only if $F|\,\mathcal{C}_0$ preserves $\mathcal{K}$-indexed colimits for each $\mathcal{C}_0 \in A$. The "only if" direction is obvious. To prove the converse, choose $K \in \mathcal{K}$ and a colimit diagram $p : K^\triangleright \to \mathcal{C}$. The image of p is τ-small, so that (∗) guarantees that p factors through $\mathcal{C}_0$ for some $\mathcal{C}_0 \in A$. Requirement (c) guarantees that p is also a colimit diagram in $\mathcal{C}_0$, so that $F \circ p$ is a colimit diagram provided that $F|\,\mathcal{C}_0$ preserves $\mathcal{K}$-indexed colimits.

It remains only to prove assertion (∗). Fix a τ-small subset $\mathcal{C}_0 \subseteq \mathcal{C}$. We define a transfinite sequence of τ-small simplicial subsets $\{\mathcal{C}_\alpha \subseteq \mathcal{C}\}_{\alpha < \kappa}$. If α is a nonzero limit ordinal, we take $\mathcal{C}_\alpha = \bigcup_{\beta < \alpha} \mathcal{C}_\beta$. If $\alpha = \beta + 1$, we define $\mathcal{C}_\alpha$ to be any τ-small simplicial subset of $\mathcal{C}$ with the following properties:

 • Every map $\Lambda_i^n \to \mathcal{C}_\beta$ for $0 < i < n$ extends to an n-simplex of $\mathcal{C}_\alpha$.

 • For each $K \in \mathcal{K}$ and each map $q : K \to \mathcal{C}_\beta$, there exists an extension $\overline{q} : K^\triangleright \to \mathcal{C}_\alpha$ which is a colimit diagram in $\mathcal{C}$.

 • Given $n > 0$ and a map $f : K \star \partial\,\Delta^n \to \mathcal{C}_\beta$ such that the restriction $f|K \star \{0\}$ is a colimit diagram in $\mathcal{C}$, there exists a map $\overline{f} : K \star \Delta^n \to \mathcal{C}_\alpha$ extending f.

Our assumption that $\kappa \ll \tau$ guarantees that we can satisfy these conditions by adjoining a τ-small set of simplices to $\mathcal{C}_\beta$. Let $\mathcal{C}_0' = \bigcup_{\alpha < \kappa} \mathcal{C}_\alpha$. Then $\mathcal{C}_0'$ contains $\mathcal{C}_0$, and belongs to A as desired. $\qquad\square$

Now suppose that $\mathcal{C}^\otimes$ is a monoidal ∞-category, that $\mathcal{M}$ is an ∞-category right-tensored over $\mathcal{C}$, and that $\mathcal{N}$ is an ∞-category left-tensored over $\mathcal{C}$. Fixing a collection of simplicial sets $\mathcal{K}$, we further assume that $\mathcal{C}$, $\mathcal{M}$, and $\mathcal{N}$ admit $\mathcal{K}$-indexed colimits, and that the tensor product functors

$$\mathcal{M} \times \mathcal{C} \to \mathcal{M} \qquad \mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \mathcal{C} \times \mathcal{N} \to \mathcal{N}$$

preserve $\mathcal{K}$-indexed colimits separately in each variable. We can identify $\mathcal{C}^{\otimes}$ with an associative algebra object of $\mathcal{C}\mathrm{at}_{\infty}(\mathcal{K})$, and the ∞-categories $\mathcal{M}$ and $\mathcal{N}$ with right and left modules over this associative algebra, respectively (Remark 2.4.2.6). Consequently, the relative tensor product $\mathcal{M} \otimes_{\mathcal{C}} \mathcal{N}$ can be defined using the constructions of §4.3.5. This tensor product is given as the geometric realization (in $\mathcal{C}\mathrm{at}_{\infty}(\mathcal{K})$) of the simplicial object $\mathrm{Bar}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})_{\bullet}$ of Construction 4.3.5.7. The bar construction $\mathrm{Bar}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})_{\bullet}$ is given informally by the formula $[n] \mapsto \mathcal{M} \otimes \mathcal{C}^{(\otimes n)} \otimes \mathcal{N}$, where the tensor product is formed in the ∞-category $\mathcal{C}\mathrm{at}_{\infty}(\mathcal{K})$. Consequently, this bar construction is dependent on the choice of the collection $\mathcal{K}$. To emphasize this dependence, we will denote $\mathrm{Bar}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})_{\bullet}$ by $\mathrm{Bar}_{\mathcal{C}}^{\mathcal{K}}(\mathcal{M}, \mathcal{N})_{\bullet}$. If $\mathcal{K}' \subseteq \mathcal{K}$, then we have a forgetful functor $\mathcal{C}\mathrm{at}_{\infty}(\mathcal{K}) \to \mathcal{C}\mathrm{at}_{\infty}(\mathcal{K}')$ which is lax symmetric monoidal, and induces a natural transformation $\mathrm{Bar}_{\mathcal{C}}^{\mathcal{K}'}(\mathcal{M}, \mathcal{N})_{\bullet} \to \mathrm{Bar}_{\mathcal{C}}^{\mathcal{K}}(\mathcal{M}, \mathcal{N})$. In particular, we have a map $\theta : \mathrm{Bar}_{\mathcal{C}}^{\emptyset}(\mathcal{M}, \mathcal{N})_{\bullet} \to \mathrm{Bar}_{\mathcal{C}}^{\mathcal{K}}(\mathcal{M}, \mathcal{N})$. The map θ is characterized by the following universal property:

($*$) For each $n \geq 0$ and every ∞-category $\mathcal{E}$ which admits $\mathcal{K}$-indexed colimits, composition with θ induces an equivalence from the full subcategory of $\mathrm{Fun}(\mathrm{Bar}_{\mathcal{C}}^{\mathcal{K}}(\mathcal{M}, \mathcal{N})_n, \mathcal{E})$ spanned by those functors which preserve $\mathcal{K}$-indexed colimits to the full subcategory of $\mathrm{Fun}(\mathrm{Bar}_{\mathcal{C}}^{\emptyset}(\mathcal{M}, \mathcal{N})_n, \mathcal{E}) \simeq \mathrm{Fun}(\mathcal{M} \times \mathcal{C}^n \times \mathcal{N}, \mathcal{E})$ spanned by those functors which preserve $\mathcal{K}$-indexed colimits separately in each variable.

We can identify $\mathrm{Bar}_{\mathcal{C}}^{\emptyset}(\mathcal{M}, \mathcal{N})_{\bullet}$ with a simplicial object $\chi : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{C}\mathrm{at}_{\infty}$. Let $\mathcal{C}^{\circledast}$, and $\mathcal{N}^{\circledast}$ be defined as in Notation 4.2.2.16 and define $\mathcal{M}^{\circledast}$ similarly. Unwinding the definitions, we see that χ classifies the coCartesian fibration $q : \mathcal{M}^{\circledast} \times_{\mathcal{C}^{\circledast}} \mathcal{N}^{\circledast} \to \mathrm{N}(\mathbf{\Delta})^{op}$. Proposition T.3.3.4.2 allows us to identify the geometric realization $|\mathrm{Bar}_{\mathcal{C}}^{\emptyset}(\mathcal{M}, \mathcal{N})_{\bullet}|$ with the ∞-category obtained from $\mathcal{M}^{\circledast} \times_{\mathcal{C}^{\circledast}} \mathcal{N}^{\circledast}$ obtained by inverting all of the q-coCartesian morphisms. Combining this observation with ($*$), we obtain the following concrete description of the relative tensor product in $\mathcal{C}\mathrm{at}_{\infty}(\mathcal{K})$:

Lemma 6.3.4.3. *Let $\mathcal{K}$ be a small collection of simplicial sets and let $\mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast} \leftarrow \mathcal{N}^{\circledast}$ be as above. For every ∞-category $\mathcal{E}$ which admits $\mathcal{K}$-indexed colimits, the natural transformation $\mathrm{Bar}_{\mathcal{C}}^{\emptyset}(\mathcal{M}, \mathcal{N})_{\bullet} \to \mathrm{Bar}_{\mathcal{C}}^{\mathcal{K}}(\mathcal{M}, \mathcal{N})_{\bullet}$ induces an equivalence of ∞-categories from the full subcategory of $\mathrm{Fun}(\mathcal{M} \otimes_{\mathcal{C}} \mathcal{N}, \mathcal{E})$ spanned by those functors which preserve $\mathcal{K}$-indexed colimits to the full subcategory of $\mathrm{Fun}(\mathcal{M}^{\circledast} \times_{\mathcal{C}^{\circledast}} \mathcal{N}^{\circledast}, \mathcal{E})$ spanned by those functors F with the following properties:*

(i) *The functor F carries q-coCartesian morphisms to equivalences in $\mathcal{E}$, where $q : \mathcal{M}^{\circledast} \times_{\mathcal{C}^{\circledast}} \mathcal{N}^{\circledast} \to \mathrm{N}(\mathbf{\Delta})^{op}$ denotes the canonical projection.*

(ii) *For each $n \geq 0$, the functor*

$$\mathcal{M} \times \mathcal{C}^n \times \mathcal{N} \simeq q^{-1}\{[n]\} \to \mathcal{E}$$

preserves $\mathcal{K}$-indexed colimits separately in each variable.

Our next goal is to apply Lemma 6.3.4.3 to construct a canonical map $\mathcal{M} \otimes_{\mathcal{C}} \mathrm{RMod}_A(\mathcal{C}) \to \mathrm{RMod}_A(\mathcal{M})$.

Construction 6.3.4.4. Let $\mathrm{Cut} : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{A}\mathrm{ss}^{\otimes} \subseteq \mathcal{RM}^{\otimes}$ be as in Construction 4.1.2.5, and let $\mathrm{RCut} : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{RM}^{\otimes}$ be the functor given by the the composition

$$\mathrm{N}(\mathbf{\Delta})^{op} \overset{r}{\to} \mathrm{N}(\mathbf{\Delta})^{op} \overset{\mathrm{LCut}}{\to} \mathcal{LM}^{\otimes} \overset{r'}{\to} \mathcal{RM}^{\otimes},$$

where r and r' are the isomorphisms given by order-reversal and LCut is the functor described in Construction 4.2.2.6.

We have a commutative diagram σ:

$$
\begin{array}{ccccc}
F_- & \longleftarrow & F & \longrightarrow & F_+ \\
\downarrow & & \downarrow & & \downarrow \\
F'_- & \longleftarrow & F' & \longrightarrow & F'_+
\end{array}
$$

in the ∞-category $\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{op} \times \mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{RM}^{\otimes})$, where

$$F_-([m],[n]) = \mathrm{RCut}([m]) \qquad F([m],[n]) = \mathrm{RCut}([m] \star [n]) \qquad F_+([m],[n]) = \mathrm{RCut}([n])$$

$$F'_-([m],[n]) = \mathrm{Cut}([m]) \qquad F'([m],[n]) = \mathrm{Cut}([m] \star [n]) \qquad F'_+([m],[n]) = \mathrm{Cut}([n]).$$

The vertical maps in this diagram are induced by the natural transformation $\mathrm{RCut} \to \mathrm{Cut}$ described in Remark 4.2.2.8, and the horizontal maps by the inclusions $[m] \hookrightarrow [m] \star [n] \hookleftarrow [n]$.

Let $p : \mathcal{C}^{\otimes} \to \mathcal{A}ss^{\otimes}$ be a monoidal ∞-category, and let $q : \mathcal{M}^{\otimes} \to \mathcal{RM}^{\otimes}$ be an ∞-category right-tensored over $\mathcal{C}^{\otimes}$. We let $\mathcal{C}'$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{C})$ spanned by those functors which carry each morphism in $\mathrm{N}(\boldsymbol{\Delta})^{op}$ to an equivalence, and let $\mathcal{M}' \subseteq \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{M})$ be defined similarly (since $\mathrm{N}(\boldsymbol{\Delta})^{op}$ is weakly contractible, the diagonal embeddings $\mathcal{C} \to \mathcal{C}'$ and $\mathcal{M} \to \mathcal{M}'$ are categorical equivalences). We regard $\mathcal{C}'$ as a monoidal ∞-category and $\mathcal{M}'$ as an ∞-category right-tensored over $\mathcal{C}'$ (see Remark 2.1.3.4). Let $\mathcal{C}'^{\circledast} = \mathcal{C}'^{\otimes} \times_{\mathcal{A}ss^{\otimes}} \mathrm{N}(\boldsymbol{\Delta})^{op}$, let $\mathcal{M}'^{\circledast}$ be defined as in Notation 4.2.2.16 (suitably modified for purposes of discussing right actions, rather than left actions).

Fix an algebra object $A \in \mathrm{Alg}(\mathcal{C})$ and regard $\mathrm{RMod}_A(\mathcal{C})$ as an ∞-category left-tensored over $\mathcal{C}'$ as in §4.3.2. Let $\mathrm{RMod}_A(\mathcal{C})^{\circledast} \to \mathcal{C}'^{\circledast}$ be defined as in Notation 4.2.2.16, and let $\mathcal{X} = \mathcal{M}'^{\circledast} \times_{\mathcal{C}'^{\circledast}} \mathrm{RMod}_A(\mathcal{C})^{\circledast}$. We will lift σ to a commutative diagram

$$
\begin{array}{ccccc}
\overline{F}_- & \longleftarrow & \overline{F} & \longrightarrow & \overline{F}_+ \\
\downarrow & & \downarrow & & \downarrow \\
\overline{F}'_- & \longleftarrow & \overline{F}' & \longrightarrow & \overline{F}'_+
\end{array}
$$

of functors from $\mathcal{X} \times \mathrm{N}(\boldsymbol{\Delta})^{op}$ to $\mathcal{M}^{\otimes}$ as follows:

(a) Let $\overline{F}'_-$ be the composition

$$\mathcal{X} \times \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathcal{C}'^{\circledast} \times \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathcal{C}^{\otimes} \subseteq \mathcal{M}^{\otimes}.$$

(b) Let α' denote the composition $\mathcal{M}'^{\circledast} \to \mathcal{C}'^{\circledast} \to \mathcal{C}'^{\otimes} \subseteq \mathcal{M}'^{\otimes}$. By construction, $\mathcal{M}^{\circledast}$ is equipped with another functor $\alpha : \mathcal{M}'^{\circledast} \to \mathcal{M}'^{\otimes}$ and a natural transformation $\alpha \to \alpha'$. Composing α with the forgetful functor $\mathcal{X} \times \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathcal{M}'^{\circledast} \times \mathrm{N}(\boldsymbol{\Delta})^{op}$ and the evaluation functor $\mathcal{M}'^{\otimes} \times \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathcal{M}^{\otimes}$ we obtain a functor $\overline{F}_-$ equipped with a natural transformation $\overline{F}_- \to \overline{F}'_-$.

(c) The functor $\overline{F}'$ is given by the composition

$$
\begin{array}{rcl}
\mathcal{X} \times \mathrm{N}(\boldsymbol{\Delta})^{op} & \longrightarrow & \mathrm{RMod}_A(\mathcal{C})^{\otimes} \times \mathrm{N}(\boldsymbol{\Delta})^{op} \\
& \longrightarrow & \mathrm{Fun}(\mathcal{RM}^{\otimes}, \mathcal{C}^{\otimes}) \times \mathrm{N}(\boldsymbol{\Delta})^{op} \\
& \overset{\mathrm{RCut}}{\longrightarrow} & \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{C}^{\otimes}) \times \mathrm{N}(\boldsymbol{\Delta})^{op} \\
& \longrightarrow & \mathcal{C}^{\otimes} \\
& \subseteq & \mathcal{M}^{\otimes}.
\end{array}
$$

Note that the inert natural transformation $\mathrm{RCut} \to \mathrm{Cut}$ induces a map of functors $\overline{F}' \to \overline{F}'_-$.

(d) Using the assumption that q is a fibration of ∞-operads, we can choose a functor $\overline{F}$ compatible with F, so that the diagram

$$
\begin{array}{ccc}
\overline{F}_- & \longleftarrow & \overline{F} \\
\downarrow & & \downarrow \\
\overline{F}'_- & \longleftarrow & \overline{F}'
\end{array}
$$

is a q-limit square (when evaluated at any object of $\mathcal{X} \times \mathrm{N}(\boldsymbol{\Delta})^{op}$).

(e) Let $\overline{F}'_+$ be the functor given by

$$\mathfrak{X} \times \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathrm{N}(\boldsymbol{\Delta})^{op} \overset{\mathrm{Cut}}{\to} \mathcal{Ass}^{\otimes} \overset{A}{\to} \mathcal{C}^{\otimes} \subseteq \mathcal{M}^{\otimes}.$$

The construction of $\mathrm{RMod}_A(\mathcal{M})^{\circledast}$ guarantees a natural transformation of functors $\overline{F}' \to \overline{F}'_+$.

(f) Since q is a coCartesian fibration, we can choose a q-coCartesian natural transformation $\beta : \overline{F} \to \overline{F}^+$ covering the natural transformation $F \to F^+$. Since β is q-coCartesian, there is an essentially unique extension to a commutative diagram

$$
\begin{array}{ccc}
\overline{F} & \longrightarrow & \overline{F}_+ \\
\downarrow & & \downarrow \\
\overline{F}' & \longrightarrow & \overline{F}'_+
\end{array}
$$

compatible with σ.

We can identify γ with a functor $\Psi : \mathfrak{X} \times \mathrm{N}(\boldsymbol{\Delta}) \times \Delta^1 \to \mathcal{M}^{\otimes}$. It is not difficult to see that this functor determines a map $\mathfrak{X} \to {}^{\Delta}\mathrm{RMod}_A(\mathcal{M})$.

Remark 6.3.4.5. In the situation of Construction 6.3.4.4, we can think of an object of $\mathfrak{X}$ lying over $[n] \in \mathrm{N}(\boldsymbol{\Delta})^{op}$ as a finite sequence $(M, C_1, \dots, C_n, N)$, where $M \in \mathcal{M}$, each $C_i \in \mathcal{C}$, and N is a right A-module object of $\mathcal{C}$. The functor Ψ is given informally by the formula $(M, C_1, \dots, C_n, N) \mapsto M \otimes C_1 \otimes \dots \otimes C_n \otimes N$. From this description, we see that Ψ satisfies the requirements of Lemma 6.3.4.3. Let $\mathcal{K}$ is a collection of simplicial sets such that $\mathcal{C}$ and $\mathcal{M}$ admit $\mathcal{K}$-indexed colimits, and that the tensor product functors

$$\mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \mathcal{M} \times \mathcal{C} \to \mathcal{M}$$

preserve $\mathcal{K}$-indexed colimits separately in each variable. Then Ψ determines a functor $\Phi : \mathcal{M} \otimes_{\mathcal{C}} \mathrm{RMod}_A(\mathcal{C}) \to \mathrm{RMod}_A(\mathcal{M})$, which is well-defined up to equivalence.

We can now formulate (B) as follows:

Theorem 6.3.4.6. *Let $\mathcal{K}$ be a small collection of simplicial sets which includes $\mathrm{N}(\boldsymbol{\Delta})^{op}$, let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category, let $\mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes}$ be an ∞-category right-tensored over $\mathcal{C}$, and let $A \in \mathrm{Alg}(\mathcal{C})$ be an algebra object of $\mathcal{C}$. Suppose that the ∞-categories $\mathcal{C}$ and $\mathcal{M}$ admit $\mathcal{K}$-indexed colimits, and the tensor product functors*

$$\mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \mathcal{M} \times \mathcal{C} \to \mathcal{M}$$

preserve $\mathcal{K}$-indexed colimits separately in each variable. Then the above construction yields an equivalence of ∞-categories $\mathcal{M} \otimes_{\mathcal{C}} \mathrm{RMod}_A(\mathcal{C}) \to \mathrm{RMod}_A(\mathcal{M})$, where the tensor product is taken in $\mathrm{Cat}_{\infty}(\mathcal{K})$.

Remark 6.3.4.7. In the formulation of Theorems 6.3.4.6 and 6.3.4.1, the ∞-categories $\mathrm{RMod}_A(\mathcal{M})$ and $\mathrm{LMod}_A(\mathcal{M})$ do not depend on the class of simplicial sets $\mathcal{K}$. It follows that the ∞-categories $\mathcal{M} \otimes_{\mathcal{C}} \mathrm{RMod}_A(\mathcal{C})$ and $\mathrm{LinFun}_{\mathcal{C}}^{\mathcal{K}}(\mathrm{RMod}_A(\mathcal{C}), \mathcal{M})$ do not depend on $\mathcal{K}$, provided that $\mathcal{K}$ contains $\mathrm{N}(\boldsymbol{\Delta})^{op}$.

Remark 6.3.4.8. Let $\mathcal{K}$ be a collection of simplicial sets which includes $\mathrm{N}(\boldsymbol{\Delta})^{op}$ and regard $\mathrm{Cat}_{\infty}(\mathcal{K})$ be the symmetric monoidal ∞-category as in Corollary 6.3.1.4. Let $\mathcal{C}$ be a monoidal ∞-category which admits $\mathcal{K}$-indexed colimits, such that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves $\mathcal{K}$-indexed colimits separately in each variable. Then we can regard $\mathcal{C}$ as an algebra object of $\mathrm{Cat}_{\infty}(\mathcal{K})$. If $A \in \mathrm{Alg}(\mathcal{C})$, then we regard $\mathrm{LMod}_A(\mathcal{C})$ and $\mathrm{RMod}_A(\mathcal{C})$ as right and left modules over $\mathcal{C}$ (in $\mathrm{Cat}_{\infty}(\mathcal{K})$), respectively. Using Theorem 6.3.4.6, we can identify the tensor product $\mathrm{LMod}_A(\mathcal{C}) \otimes_{\mathcal{C}} \mathrm{RMod}_A(\mathcal{C})$ with the ∞-category $\mathrm{RMod}_A(\mathrm{LMod}_A(\mathcal{C})) \simeq {}_A\mathrm{BMod}_A(\mathcal{C})$. Regarding A as a bimodule over itself, we obtain an object ${}_A A_A \in {}_A\mathrm{BMod}_A(\mathcal{C})$, which classifies a functor

$$\mathcal{S}(\mathcal{K}) \overset{c}{\to} \mathrm{LMod}_A(\mathcal{C}) \otimes_{\mathcal{C}} \mathrm{RMod}_A(\mathcal{C}).$$

Here $\mathcal{S}(\mathcal{K})$ denotes the unit object of $\mathcal{C}\mathrm{at}_\infty(\mathcal{K})$ (see Notation 6.3.5.2). We claim that c exhibits $\mathrm{LMod}_A(\mathcal{C}) \in \mathrm{RMod}_\mathcal{C}(\mathcal{C}\mathrm{at}_\infty(\mathcal{K}))$ as the left dual of $\mathrm{RMod}_A(\mathcal{C}) \in \mathrm{LMod}_\mathcal{C}(\mathcal{C}\mathrm{at}_\infty(\mathcal{K}))$. To prove this, it suffices to verify condition $(*)$ of Proposition 4.3.7.15: that is, we must show that if $\mathcal{D} \in \mathcal{C}\mathrm{at}_\infty(\mathcal{K})$ and $\mathcal{M} \in \mathrm{RMod}_\mathcal{C}(\mathcal{C}\mathrm{at}_\infty(\mathcal{K}))$, then c induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{RMod}_\mathcal{C}(\mathcal{C}\mathrm{at}_\infty(\mathcal{K}))}(\mathcal{D} \otimes \mathrm{LMod}_A(\mathcal{C}), \mathcal{M}) \to \mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty(\mathcal{K})}(\mathcal{D}, \mathcal{M} \otimes_\mathcal{C} \mathrm{RMod}_A(\mathcal{C})).$$

This follows from Theorems 6.3.4.1 and 6.3.4.6, which allow us to identify both sides with the the subcategory of $\mathrm{Fun}(\mathcal{D}, \mathrm{RMod}_A(\mathcal{M}))$ spanned by those functors which preserve $\mathcal{K}$-indexed colimits and equivalences between them.

The proofs of Theorem 6.3.4.6 and 6.3.4.1 are very similar, and rest on an analysis of the forgetful functor $\mathrm{RMod}_A(\mathcal{C}) \to \mathcal{C}$. We observe that this functor is $\mathcal{C}$-linear. More precisely, evaluation at the object $\mathfrak{m} \in \mathcal{RM}$ induces a $\mathcal{C}$-linear functor $\mathrm{RMod}_A(\mathcal{C}) \to \mathcal{C}$. We have the following fundamental observation:

Lemma 6.3.4.9. *Let $\mathcal{C}^\otimes$ be a monoidal ∞-category containing an algebra object A, and let $\mathcal{N} = \mathcal{C}$, regarded as an ∞-category left-tensored over $\mathcal{C}$. Consider the commutative diagram*

$$\mathcal{N}^\circledast \xleftarrow{\quad G^\otimes \quad} {}^\triangle\mathrm{RMod}_A(\mathcal{C})^\circledast$$

$$\searrow{\scriptstyle q} \qquad \swarrow{\scriptstyle p'}$$

$$\mathcal{C}^\circledast,$$

where G is determined by the $\mathcal{C}$-linear forgetful functor $\mathrm{RMod}_A(\mathcal{C}) \to \mathcal{C}$. Then there exists a functor $F : \mathcal{N}^\circledast \to \mathrm{RMod}_A(\mathcal{C})^\circledast$ and a natural transformation $u : \mathrm{id}_{\mathcal{N}^\circledast} \to G \circ F$ which exhibits F as a left adjoint to G relative to $\mathcal{C}^\circledast$ (see Definition 8.3.2.2 and Remark 8.3.2.3). Moreover, F determines a $\mathcal{C}$-linear functor from $\mathcal{C}$ to $\mathrm{RMod}_A(\mathcal{C})$.

Remark 6.3.4.10. More informally, Lemma 6.3.4.9 asserts that the forgetful functor $\mathrm{RMod}_A(\mathcal{C}) \to \mathcal{C}$ and its left adjoint $C \mapsto C \otimes A$ commute with the action of $\mathcal{C}$ by left multiplication.

Proof. We observe that p and p' are locally coCartesian fibrations (Lemma 4.2.2.19). Moreover, for each object $C \in \mathcal{C}^\otimes$, the induced map on fibers $\mathrm{RMod}_A(\mathcal{C})^\otimes_C \to \mathcal{N}^\otimes_C$ is equivalent to the forgetful functor $\theta : \mathrm{RMod}_A(\mathcal{C}) \to \mathcal{C}$, and therefore admits a left adjoint (Corollary 4.2.4.8). We complete the proof by observing that the functor G satisfies hypothesis (2) of Proposition 8.3.2.11. Unwinding the definitions, this results from the observation that the canonical maps $(C \otimes D) \otimes A \to C \otimes (D \otimes A)$ expressing the coherent associativity of the tensor product on $\mathcal{C}$ are equivalences in $\mathcal{C}$. $\square$

The following result provides a reformulation of Definition 4.3.7.6:

Lemma 6.3.4.11. *Let $\mathcal{C}$ be a monoidal ∞-category, let $\mathcal{M}$ and $\mathcal{N}$ be ∞-categories left-tensored over $\mathcal{C}$, and let $q : \mathcal{M}^\circledast \to \mathcal{C}^\circledast$ and $q' : \mathcal{N}^\circledast \to \mathcal{C}^\circledast$ be as in Notation 4.2.2.16. Then the evident functor $\theta : \mathrm{LinFun}_\mathcal{C}(\mathcal{M}, \mathcal{N}) \to \mathrm{Fun}_{\mathcal{C}^\circledast}(\mathcal{M}^\circledast, \mathcal{N}^\circledast)$ is fully faithful, and its essential image consists of those functors $\mathcal{M}^\circledast \to \mathcal{N}^\circledast$ which carry locally q-coCartesian morphisms to locally q'-coCartesian morphisms.*

Proof. Let $\mathcal{X}$ be the full subcategory of $\mathrm{Fun}_{\mathcal{C}^\circledast}(\mathcal{M}^\circledast, \mathcal{N}^\circledast)$ spanned by those functors which carry locally q-coCartesian morphisms to locally q'-coCartesian morphisms. We wish to show that θ induces an equivalence $\mathrm{LinFun}_\mathcal{C}(\mathcal{M}, \mathcal{N}) \to \mathcal{X}$. Equivalently, we wish to show that for every simplicial set K, the induced map $\mathrm{Fun}(K, \mathrm{LinFun}_\mathcal{C}(\mathcal{M}, \mathcal{N})) \to \mathrm{Fun}(K, \mathcal{X})$ induces a bijection between equivalence classes of objects. Replacing $\mathcal{N}$ by $\mathrm{Fun}(K, \mathcal{N})$, we can reduce to the case $K = \Delta^0$. We will show that θ induces a homotopy equivalence between the Kan complexes $\mathrm{LinFun}_\mathcal{C}(\mathcal{M}, \mathcal{N})^\simeq$ and $\mathcal{X}^\simeq$.

Let $C : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{C}\mathrm{at}_\infty$ classify the coCartesian fibration $\mathcal{C}^\circledast \to \mathrm{N}(\mathbf{\Delta})^{op}$. The coCartesian fibration $\mathcal{M}^\otimes \times_{\mathcal{L}\mathcal{M}^\otimes} (\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}) \to \Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}$ is classified by a functor $(\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{C}\mathrm{at}_\infty$ which we

can identify with a natural transformation $\alpha : M \to C$ in $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathfrak{Cat}_\infty)$. Similarly, the coCartesian fibration $\mathcal{N}^\otimes \times_{\mathcal{L}\mathcal{M}^\otimes} (\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}) \to \Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}$ is classified by a natural transformation $\beta : N \to C$ in $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathfrak{Cat}_\infty)$. Using Propositions 4.1.2.6 and 4.2.2.9, we conclude that $\mathrm{LinFun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})^\simeq$ can be identified with the homotopy fiber of the forgetful functor

$$\mathrm{Map}_{\mathrm{Fun}(\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{op}, \mathfrak{Cat}_\infty)}(\alpha, \beta) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathfrak{Cat}_\infty)}(C, C)$$

over the point corresponding to id_C. This can in turn be identified with the homotopy fiber product

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathfrak{Cat}_\infty)}(M, N) \times_{\mathrm{Map}_{\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathfrak{Cat}_\infty)}(M, C)} \{\alpha\}.$$

This homotopy fiber is given by $\mathfrak{X}'^\simeq$, where $\mathfrak{X}' \subseteq \mathrm{Fun}_{\mathcal{C}^\otimes}(\mathcal{M}^\otimes, \mathcal{N}^\otimes)$ is the full subcategory spanned by those functors F which carry $(p \circ q)$-coCartesian morphisms of $\mathcal{M}^\otimes$ to $(p \circ q')$-coCartesian morphisms of $\mathcal{N}^\otimes$. Using the functor θ, we see that $\mathfrak{X}' \subseteq \mathfrak{X}$; we will complete the proof by showing that $\mathfrak{X} \subseteq \mathfrak{X}'$. Note that a morphism f in $\mathcal{M}^\otimes$ is $(p \circ q)$-coCartesian if and only if $q(f)$ is p-coCartesian and f is locally q-coCartesian. It follows that if $F \in \mathfrak{X}$, then $F(f)$ is locally q'-coCartesian; since $q'(F(f)) \simeq q(f)$ is p-coCartesian we conclude that $F(f)$ is $(p \circ q)$-coCartesian. This proves that $\mathfrak{X} \subseteq \mathfrak{X}'$ as desired. $\square$

Lemma 6.3.4.12. *Let $\mathcal{C}$ be a monoidal ∞-category and let $\mathcal{M}$ and $\mathcal{N}$ be ∞-categories left-tensored over $\mathcal{C}$. Let K be a simplicial set such that $\mathcal{N}$ admits K-indexed colimits, and such that for each $C \in \mathcal{C}$, the tensor product functor $\{C\} \times \mathcal{N} \subseteq \mathcal{C} \times \mathcal{N} \to \mathcal{N}$ preserves K-indexed colimits. Then:*

(1) *The ∞-category $\mathrm{LinFun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ admits K-indexed colimits.*

(2) *A map $f : K^\triangleright \to \mathrm{LinFun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ is a colimit diagram if and only if, for each $M \in \mathcal{M}$, the induced map $K^\triangleright \to \mathcal{N}$ is a colimit diagram.*

Remark 6.3.4.13. In the situation of Lemma 6.3.4.12, suppose that $\mathcal{K}$ is a class of simplicial sets such that $\mathcal{C}$, $\mathcal{M}$, and $\mathcal{N}$ admit $\mathcal{K}$-indexed colimits, and the tensor product functors

$$\mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \mathcal{C} \times \mathcal{M} \to \mathcal{M} \qquad \mathcal{C} \times \mathcal{N} \to \mathcal{N}$$

preserve $\mathcal{K}$-indexed colimits separately in each variable. Then the full subcategory $\mathrm{LinFun}_{\mathcal{C}}^{\mathcal{K}}(\mathcal{M}, \mathcal{N}) \subseteq \mathrm{LinFun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ is stable under K-indexed colimits: this follows from the characterization of K-indexed colimits supplied by Lemma 6.3.4.12 together with Lemma T.5.5.2.3.

Proof. Assertion (1) follows from Lemma 6.3.4.11 and Proposition T.5.4.7.11. Proposition T.5.4.7.11 also implies that a diagram $f : K^\triangleright \to \mathrm{LinFun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ is a colimit if and only if, for every object $M \in \mathcal{M}^\otimes$ having image $C \in \mathcal{C}^\otimes$ and $[n] \in \mathrm{N}(\mathbf{\Delta})^{op}$, the induced map $f_M : K^\triangleright \to \mathcal{N}_C^\otimes$ is a colimit diagram. The necessity of condition (2) is now obvious. For the sufficiency, let $p : \mathcal{C}^\otimes \to \mathrm{N}(\mathbf{\Delta})^{op}$, $q : \mathcal{M}^\otimes \to \mathcal{C}^\otimes$, and $q' : \mathcal{N}^\otimes \to \mathcal{C}^\otimes$ be as in Notation 4.2.2.16. Choose a p-coCartesian morphism $\alpha : C \to C_0$ in $\mathcal{C}^\otimes$ covering the inclusion $[0] \simeq \{n\} \subseteq [n]$ in $\mathbf{\Delta}$ and a locally q-coCartesian morphism $\overline{\alpha} : M \to M_0$ lifting α, then we have a homotopy commutative diagram

$$
\begin{array}{ccc}
 & K^\triangleright & \\
 {}^{p_M}\swarrow & & \searrow^{p_{M_0}} \\
\mathcal{N}_C^\otimes & \xrightarrow{\quad \alpha_! \quad} & \mathcal{N}_{C_0}^\otimes
\end{array}
$$

where $\alpha_!$ is an equivalence of ∞-categories, so that f_M is a colimit if and only if f_{M_0} is a colimit diagram. $\square$

We now turn to the proofs of Theorems 6.3.4.1 and 6.3.4.6.

Proof of Theorem 6.3.4.1. Let $\mathcal{N} = \mathcal{C}$, and regard $\mathcal{N}$ as an ∞-category left-tensored over $\mathcal{C}$. Let $G : \mathrm{RMod}_A(\mathcal{C})^{\otimes} \to \mathcal{N}^{\otimes}$ and $F : \mathcal{N}^{\otimes} \to \mathrm{RMod}(\mathcal{C})^{\otimes}$ be as in Lemma 6.3.4.9. Then F and G induce adjoint functors

$$\mathrm{LinFun}_{\mathcal{C}}^{\mathcal{K}}(\mathcal{N}, \mathcal{M}) \underset{g}{\overset{f}{\rightleftarrows}} \mathrm{LinFun}_{\mathcal{C}}^{\mathcal{K}}(\mathrm{RMod}_A(\mathcal{C}), \mathcal{M})$$

We first claim that evaluation at the unit object $\mathbf{1} \in \mathcal{N} \simeq \mathcal{C}$ induces an equivalence of ∞-categories $\phi : \mathrm{LinFun}_{\mathcal{C}}^{\mathcal{K}}(\mathcal{N}, \mathcal{M}) \to \mathcal{M}$. It will suffice to show that for every simplicial set K, the induced map $\mathrm{Fun}(K, \mathrm{LinFun}_{\mathcal{C}}^{\mathcal{K}}(\mathcal{N}, \mathcal{M})) \to \mathrm{Fun}(K, \mathcal{M})$ induces a bijection on equivalence classes of objects. Replacing $\mathcal{M}$ by $\mathrm{Fun}(K, \mathcal{M})$, we are reduced to proving that ϕ induces a bijection on equivalence classes of objects. On the left hand side, the set of equivalence classes can be identified with $\pi_0 \mathrm{Map}_{\mathrm{LMod}_{\mathcal{C}}(\mathcal{C}\mathrm{at}_{\infty}(\mathcal{K}))}(\mathcal{C}, \mathcal{M})$. Using Corollary 4.2.4.8, we can identify this with the set $\pi_0 \mathrm{Map}_{\mathcal{C}\mathrm{at}_{\infty}(\mathcal{K})}(\mathcal{S}(\mathcal{K}), \mathcal{M}) \simeq \pi_0 \mathrm{Map}_{\mathcal{C}\mathrm{at}_{\infty}}(\Delta^0, \mathcal{M})$, which is the set of equivalence classes of objects of $\mathcal{M}$ as required.

Let $T : \mathrm{LinFun}_{\mathcal{C}}^{\mathcal{K}}(\mathrm{RMod}_A(\mathcal{C}), \mathcal{M}) \to \mathcal{M}$ denote the composition of the functor g with the equivalence ϕ. We have a homotopy commutative diagram of ∞-categories

$$\mathrm{LinFun}_{\mathcal{C}}^{\mathcal{K}}(\mathrm{RMod}_A(\mathcal{C}), \mathcal{M}) \xrightarrow{\quad\theta\quad} \mathrm{LMod}_A(\mathcal{M})$$
$$\overset{T}{\searrow} \qquad \overset{T'}{\swarrow}$$
$$\mathcal{M},$$

where T is the evident forgetful functor. We will prove that θ is an equivalence showing that this diagram satisfies the hypotheses of Corollary 6.2.2.14:

(a) The ∞-categories $\mathrm{LinFun}_{\mathcal{C}}^{\mathcal{K}}(\mathrm{RMod}_A(\mathcal{C}), \mathcal{M})$ and $\mathrm{LMod}_A(\mathcal{M})$ admit geometric realizations of simplicial objects. In the first case, this follows from Lemma 6.3.4.12 and Remark 6.3.4.13. In the second, it follows from Corollary 4.2.3.5.

(b) The functors T and T' admit left adjoints, which we will denote by U and U'. The left adjoint U is given by composing f with a homotopy inverse to the equivalence ϕ, and the left adjoint U' is supplied by Corollary 4.2.4.8.

(c) The functor T' is conservative and preserves geometric realizations of simplicial objects. The first assertion follows from Corollary 4.2.3.2 and the second from Corollary 4.2.3.5.

(d) The functor T is conservative and preserves geometric realizations of simplicial objects. The second assertion follows from Lemma 6.3.4.12. To prove the first, suppose that $\alpha : S \to S'$ is a natural transformation of $\mathcal{C}$-linear functors from $\mathrm{RMod}_A(\mathcal{C})$ to $\mathcal{M}$, each of which preserves $\mathcal{K}$-indexed colimits, and that $T(\alpha)$ is an equivalence. We wish to show that α is an equivalence. Let us abuse notation by identifying S and S' with the underlying maps $\mathrm{RMod}_A(\mathcal{C}) \to \mathcal{M}$, and let $\mathcal{X}$ be the full subcategory of $\mathrm{RMod}_A(\mathcal{C})$ spanned by those objects X for which α induces an equivalence $S(X) \to S'(X)$ in $\mathcal{M}$. We wish to show that $\mathcal{X} = \mathrm{RMod}_A(\mathcal{C})$.

Since ϕ is an equivalence of ∞-categories, we conclude that $g(\alpha)$ is an equivalence in $\mathrm{LinFun}_{\mathcal{C}}^{\mathcal{K}}(\mathcal{N}, \mathcal{M})$. In other words, the ∞-category $\mathcal{X}$ contains the essential image of the free module functor $\mathcal{C} \simeq \mathcal{N} \to \mathrm{RMod}_A(\mathcal{C})$. Since S and S' preserve $\mathcal{K}$-indexed colimits, $\mathcal{X}$ is stable under geometric realizations of simplicial objects. The equality $\mathcal{X} = \mathrm{RMod}_A(\mathcal{C})$ now follows from Proposition 6.2.2.12.

(e) The natural transformation $T' \circ U' \to T \circ U$ is an equivalence of functors from $\mathcal{M}$ to itself. Unwinding the definitions, we see that both of these functors are given by tensoring with the object $A \in \mathcal{C}$.

$\square$

Proof of Theorem 6.3.4.6. The forgetful functor $\mathrm{RMod}_A(\mathcal{C}) \to \mathcal{C}$ can be viewed as a map between left $\mathcal{C}$-module objects in $\mathrm{Cat}_\infty(\mathcal{K})$, and therefore induces a functor $G : \mathcal{M} \otimes_\mathcal{C} \mathrm{RMod}_A(\mathcal{C}) \to \mathcal{M} \otimes_\mathcal{C} \mathcal{C} \simeq \mathcal{M}$. Let $G' : \mathrm{RMod}_A(\mathcal{M}) \to \mathcal{M}$ be the evident forgetful functor. We have a diagram

$$
\begin{array}{ccc}
\mathcal{M} \otimes_\mathcal{C} \mathrm{RMod}_A(\mathcal{C}) & \xrightarrow{\ \Phi\ } & \mathrm{RMod}_A(\mathcal{M}) \\
& {\scriptstyle G}\searrow \quad \swarrow {\scriptstyle G'} & \\
& \mathcal{M}, &
\end{array}
$$

which commutes up to canonical homotopy, where Φ is the functor defined in Remark 6.3.4.5. To prove that Φ is an equivalence of ∞-categories, it will suffice to show that this diagram satisfies the hypotheses of Corollary 6.2.2.14:

(a) The ∞-categories $\mathcal{M} \otimes_\mathcal{C} \mathrm{RMod}_A(\mathcal{C})$ and $\mathrm{RMod}_A(\mathcal{M})$ admit geometric realizations of simplicial objects. In the first case, this follows from our assumption that $\mathrm{N}(\mathbf{\Delta})^{op} \in \mathcal{K}$; in the second case, it follows from Corollary 4.2.3.5 (since $\mathcal{C}$ admits geometric realizations and tensor product with A preserves geometric realizations).

(b) The functors G and G' admit left adjoints, which we will denote by F and F'. The existence of F' follows from Corollary 4.2.4.8 (which also shows that F' is given informally by the formula $M \mapsto M \otimes A$). Similar reasoning shows that the forgetful functor $\mathrm{RMod}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint F_0. This left adjoint can be promoted to a map of ∞-categories left-tensored over $\mathcal{C}$, so that it induces a functor $\mathrm{id} \otimes F_0 : \mathcal{M} \otimes_\mathcal{C} \mathcal{C} \to \mathcal{M} \otimes_\mathcal{C} \mathrm{RMod}(\mathcal{C})$ which is left adjoint to G.

(c) The functor G' is conservative and preserves geometric realizations of simplicial objects. The first assertion follows from Corollary 4.2.3.2 and the second from Corollary 4.2.3.5 (since $\mathrm{N}(\mathbf{\Delta})^{op} \in \mathcal{K}$).

(d) The functor G is conservative and preserves geometric realizations of simplicial objects. The second assertion is obvious (since G is a morphism in $\mathrm{Cat}_\infty(\mathcal{K})$ by construction). The proof that G is conservative is a bit more involved. Let $c : \mathrm{N}(\mathbf{\Delta})^{op} \times \mathrm{N}(\mathbf{\Delta}_+)^{op} \times \mathrm{N}(\mathbf{\Delta})^{op} \to \mathrm{N}(\mathbf{\Delta})^{op}$ be the concatenation functor, given by the formula $([l], [m], [n]) \mapsto [l] \star [m] \star [n] \simeq [l + m + n + 2]$, and let $c_0 : \mathrm{N}(\mathbf{\Delta})^{op} \times \mathrm{N}(\mathbf{\Delta}_+)^{op} \times \mathrm{N}(\mathbf{\Delta})^{op} \to \mathrm{N}(\mathbf{\Delta})^{op}$ be given by $([l], [m], [n]) \mapsto [l] \star [n]$. The canonical inclusions $[l] \star [n] \hookrightarrow [l] \star [m] \star [n]$ induce a natural transformation of functors $\alpha : c \to c_0$. Let $p : \mathcal{C}^\otimes \to \mathrm{N}(\mathbf{\Delta})^{op}$ be the canonical map, let $\pi : \mathcal{C}^\otimes \times_{\mathrm{N}(\mathbf{\Delta})^{op}}(\mathrm{N}(\mathbf{\Delta})^{op} \times \mathrm{N}(\mathbf{\Delta}_+)^{op} \times \mathrm{N}(\mathbf{\Delta})^{op})) \to \mathcal{C}^\otimes$ denote the projection, and choose a p-coCartesian natural transformation $\overline{\alpha} : \pi \to \pi'$ covering α. Adjusting π' by a homotopy if necessary, we can assume that π' induces a functor $T_\bullet : \mathrm{N}(\mathbf{\Delta}_+)^{op} \times \mathrm{RMod}_A(\mathcal{C})^\otimes \to \mathrm{RMod}_A(\mathcal{C})^\otimes$. We will view the functor $T_\bullet$ as an augmented simplicial object in the category of left $\mathcal{C}$-module functors from $\mathrm{RMod}_A(\mathcal{C})$ to itself, given informally by the formula $T_n(N) = N \otimes A^{\otimes(n+1)}$; in particular, the functor T_{-1} is equivalent to the identity functor. For each object $N \in \mathrm{RMod}_A(\mathcal{C})$, the canonical map $\epsilon : |T_\bullet N| \to T_{-1}N \simeq N$ is an equivalence: to prove this, it suffices to show that the image of ϵ under the forgetful functor $\theta : \mathrm{RMod}_A(\mathcal{C}) \to \mathcal{C}$ is an equivalence (Corollary 4.2.3.2): we note that $\theta|T_\bullet N|$ can be identified with the relative tensor product $N \otimes_A A$ and that $\theta(\epsilon)$ is the canonical equivalence $N \otimes_A A \simeq N$).

Using Lemma 6.3.4.3, we see that $T_\bullet$ determines an augmented simplicial object $U_\bullet : \mathrm{N}(\mathbf{\Delta}_+)^{op} \to \mathrm{Fun}(\mathcal{M} \otimes_\mathcal{C} \mathrm{Mod}_A^R(\mathcal{C}), \mathcal{M} \otimes_\mathcal{C} \mathrm{RMod}_A(\mathcal{C}))$. Note that each U_n preserves $\mathcal{K}$-indexed colimits. Let $\mathcal{X}$ be the full subcategory of $\mathcal{M} \otimes_\mathcal{C} \mathrm{RMod}_A(\mathcal{C})$ spanned by those objects for which the canonical map $|U_\bullet X| \to U_{-1}X \simeq X$ is an equivalence. Since each U_n preserves $\mathcal{K}$-indexed colimits, the full subcategory $\mathcal{X}$ is stable under $\mathcal{K}$-indexed colimits in $\mathcal{M} \otimes_\mathcal{C} \mathrm{RMod}(\mathcal{C}))$. Because $\mathcal{M} \otimes_\mathcal{C} \mathrm{RMod}_A(\mathcal{C}))$ is generated under $\mathcal{K}$-indexed colimits by the essential image of the tensor product functor $\otimes : \mathcal{M} \times \mathrm{RMod}_A(\mathcal{C}) \to \mathcal{M} \otimes_\mathcal{C} \mathrm{RMod}_A(\mathcal{C})$ (which obviously belongs to $\mathcal{X}$), we conclude that $\mathcal{X} = \mathcal{M} \otimes_\mathcal{C} \mathrm{RMod}_A(\mathcal{C})$.

Now suppose that $f : X \to Y$ is a morphism in $\mathcal{M} \otimes_\mathcal{C} \mathrm{RMod}_A(\mathcal{C}))$ such that $G(f)$ is an equivalence. We wish to prove that f is an equivalence. Note that f is equivalent to the geometric realization

$|U_\bullet f|$ (in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{M} \otimes_\mathcal{C} \mathrm{RMod}_A(\mathcal{C})))$; it therefore suffices to show that $U_n(f)$ is an equivalence for $n \geq 0$. We complete the argument by observing that U_n factors through G (since T_n factors through the forgetful functor $\mathrm{RMod}_A(\mathcal{C}) \to \mathcal{C}$).

(e) The canonical natural transformation $G' \circ F' \to G \circ F$ is an equivalence of functors from $\mathcal{M}$ to itself. This follows easily from the descriptions of F and F' given above: both functors are given by tensor product with A.

$\square$

6.3.5 Behavior of the Functor Θ

In §4.3.2, we saw that if $\mathcal{C}^\otimes \to \mathcal{A}ss^\otimes$ is a monoidal ∞-category and A is an algebra object of $\mathcal{C}$, then the ∞-category $\mathrm{RMod}_A(\mathcal{C})$ of right A-module objects of $\mathcal{C}$ is left-tensored over $\mathcal{C}$. Moreover, in §6.3.3 we showed that the construction $(\mathcal{C}^\otimes, A) \mapsto (\mathcal{C}^\otimes, \mathrm{RMod}_A(\mathcal{C}))$ determines a functor

$$\Theta : \mathcal{C}\mathrm{at}_\infty^{\mathrm{Alg}}(\mathcal{K}) \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{Mod}}(\mathcal{K})$$

for any collection of simplicial sets $\mathcal{K}$ which contains $\mathrm{N}(\boldsymbol{\Delta})^{op}$ (see Construction 6.3.3.24). In this section, we will apply the main results of §6.3.4 (Theorems 6.3.4.6 and 6.3.4.1) to establish some basic formal properties of Θ. We can describe our goals more specifically as follows:

(1) If A is an associative ring, we can *almost* recover A from the category RMod_A of right A-modules. More precisely, if we let M denote the ring A itself, regarded as a right A-module, then left multiplication by elements of A determines a canonical isomorphism $A \to \mathrm{Hom}_A(M, M)$. In other words, the data of the associative ring A is equivalent to the data of the category RMod_A of right A-modules together with its distinguished object M. An analogous result holds if we replace the category of abelian groups by a more general monoidal ∞-category $\mathcal{C}$: the functor Θ induces a fully faithful embedding Θ_* from the ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Alg}}(\mathcal{K})$ to the ∞-category of triples $(\mathcal{C}^\otimes, \mathcal{M}, M)$, where $\mathcal{C}^\otimes$ is a monoidal ∞-category, $\mathcal{M}$ is an ∞-category left-tensored over $\mathcal{C}$, and $M \in \mathcal{M}$ is a distinguished object. We refer the reader to Theorem 6.3.5.5 for a precise statement.

(2) If we work in the setting of *presentable* ∞-categories, then the functor Θ_* admits a right adjoint, which carries a triple $(\mathcal{C}^\otimes, \mathcal{M}, M)$ to the pair $(\mathcal{C}^\otimes, A)$, where $A \in \mathrm{Alg}(\mathcal{C})$ is the algebra of endomorphisms of M (Theorem 6.3.5.10).

(3) The ∞-categories $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Alg}}(\mathcal{K})$ and $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Mod}}(\mathcal{K})$ admit symmetric monoidal structures, and Θ can be promoted to a symmetric monoidal functor (Theorem 6.3.5.14).

We begin by addressing a small technical point regarding the behavior of the functor Θ with respect to base change:

Proposition 6.3.5.1. *Let $\mathcal{K}$ be a small collection of simplicial sets which includes $\mathrm{N}(\boldsymbol{\Delta})^{op}$, and consider the commutative diagram*

$$
\begin{array}{ccc}
\mathcal{C}\mathrm{at}_\infty^{\mathrm{Alg}}(\mathcal{K}) & \xrightarrow{\ \ \Theta\ \ } & \mathcal{C}\mathrm{at}_\infty^{\mathrm{Mod}}(\mathcal{K}) \\
& {\phi}\searrow \qquad \swarrow{\psi} & \\
& \mathrm{Mon}_{\mathcal{A}ss}^{\mathcal{K}}(\mathcal{C}\mathrm{at}_\infty). &
\end{array}
$$

The functors ϕ and ψ are coCartesian fibrations, and the functor Θ carries ϕ-coCartesian morphisms to ψ-coCartesian morphisms.

Proof. We first show that ϕ is a coCartesian fibration. Let $\widetilde{\mathrm{Mon}}_{\mathcal{A}ss}\mathcal{K}(\mathcal{C}at_\infty)$ be as defined in Notation 6.3.3.5, and set

$$\mathcal{X} = \mathrm{Fun}(\mathcal{A}ss^\otimes, \widetilde{\mathrm{Mon}}_{\mathcal{A}ss}\mathcal{K}(\mathcal{C}at_\infty)) \times_{\mathrm{Fun}(\mathcal{A}ss^\otimes, \mathcal{A}ss^\otimes \times \mathrm{Mon}^{\mathcal{K}}_{\mathcal{A}ss}(\mathcal{C}at_\infty))} \mathrm{Mon}^{\mathcal{K}}_{\mathcal{A}ss}(\mathcal{C}at_\infty)$$

Let us denote an object of $\mathcal{X}$ by a pair $(\mathcal{C}^\otimes, A)$, where $\mathcal{C}^\otimes$ is a monoidal ∞-category (compatible with $\mathcal{K}$-indexed colimits) and $A \in \mathrm{Fun}_{\mathcal{A}ss^\otimes}(\mathcal{A}ss^\otimes, \mathcal{C}^\otimes)$. It follows from Proposition T.3.1.2.1 that the projection map $\phi' : \mathcal{X} \to \mathrm{Mon}^{\mathcal{K}}_{\mathcal{A}ss}(\mathcal{C}at_\infty)$ is a coCartesian fibration; moreover, a morphism $(\mathcal{C}^\otimes, A) \to (\mathcal{D}^\otimes, B)$ in $\mathcal{X}$ is ϕ'-coCartesian if and only if the underlying map $F(A) \to B$ is an equivalence, where $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ denotes the underlying monoidal functor. In this case, if A is an algebra object of $\mathcal{C}^\otimes$, then $B \simeq F(A)$ is an algebra object of $\mathcal{D}^\otimes$. Note that $\mathcal{C}at_\infty^{\mathrm{Alg}}(\mathcal{K})$ can be identified with the full subcategory of $\mathcal{X}$ spanned by those pairs $(\mathcal{C}^\otimes, A)$ where A is an algebra object of $\mathcal{C}^\otimes$. It follows that if $f : X \to Y$ is a ϕ'-coCartesian morphism of $\mathcal{X}$ such that $X \in \mathcal{C}at_\infty^{\mathrm{Alg}}(\mathcal{K})$, then $Y \in \mathcal{C}at_\infty^{\mathrm{Alg}}(\mathcal{K})$. We conclude that $\phi = \phi'|\mathcal{C}at_\infty^{\mathrm{Alg}}(\mathcal{K})$ is again a coCartesian fibration, and that a morphism in $\mathcal{C}at_\infty^{\mathrm{Alg}}(\mathcal{K})$ is ϕ-coCartesian if and only if it is ϕ'-coCartesian.

We next prove that ψ is a coCartesian fibration. Note that $\mathrm{t}\mathcal{C}at_\infty^{\mathrm{Mod}}(\mathcal{K}) \simeq \mathrm{LMod}(\mathcal{C}at_\infty(\mathcal{K}))$ and the functor ψ can be identified with the forgetful functor $\mathrm{LMod}(\mathcal{C}at_\infty(\mathcal{K})) \to \mathrm{Alg}(\mathcal{C}at_\infty(\mathcal{K}))$, and is therefore a Cartesian fibration (Corollary 4.2.3.2). Consequently, to prove that ψ is a coCartesian fibration, it will suffice to show that for every morphism $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ in the ∞-category $\mathrm{Alg}(\mathcal{C}at_\infty(\mathcal{K})) \simeq \mathrm{Mon}^{\mathcal{K}}_{\mathcal{A}ss}(\mathcal{C}at_\infty)$, the forgetful functor $\mathrm{LMod}_{\mathcal{D}^\otimes}(\mathcal{C}at_\infty(\mathcal{K})) \to \mathrm{LMod}_{\mathcal{C}^\otimes}(\mathcal{C}at_\infty(\mathcal{K}))$ admits a left adjoint. This follows immediately from Lemma 6.3.4.2 and Proposition 4.3.7.14.

It remains only to prove that Θ carries ϕ-coCartesian morphisms to ψ-coCartesian morphisms. Unwinding the definitions, we must show that if $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ is a morphism in $\mathrm{Alg}(\mathcal{C}at_\infty(\mathcal{K}))$ and A is an algebra object of $\mathcal{C}^\otimes$, then the canonical map $\rho : \mathcal{D} \otimes_{\mathcal{C}} \mathrm{RMod}_A(\mathcal{C}) \to \mathrm{RMod}_{FA}(\mathcal{D})$ is an equivalence of ∞-categories (each of which is left-tensored over $\mathcal{D}$). Note that $\mathrm{RMod}_{FA}(\mathcal{D})$ can be identified with the ∞-category $\mathrm{RMod}_A(\mathcal{D})$, where we regard $\mathcal{D}$ as right-tensored over the ∞-category $\mathcal{C}$ via the monoidal functor F. Under this identification, the functor ρ is given by the equivalence of Theorem 6.3.4.6. $\square$

Let us now study the image of the initial object of $\mathcal{C}at_\infty^{\mathrm{Alg}}(\mathcal{K})$ under the functor Θ.

Notation 6.3.5.2. Fix a small collection of simplicial sets $\mathcal{K}$. We let $\mathcal{S}(\mathcal{K})$ denote the unit object of the monoidal ∞-category $\mathcal{C}at_\infty(\mathcal{K})$: it can be described concretely as the smallest full subcategory of $\mathcal{S}$ which contains the final object Δ^0 and is closed under $\mathcal{K}$-filtered colimits (Remark T.5.3.5.9). Since the formation of Cartesian products in $\mathcal{S}$ preserves small colimits in each variable, the full subcategory $\mathcal{S}(\mathcal{K}) \subseteq \mathcal{S}$ is stable under finite products. We may therefore regard $\mathcal{S}(\mathcal{K})$ as equipped with the Cartesian monoidal structure, which endows it with the structure of an algebra object of $\mathcal{C}at_\infty(\mathcal{K})$. We let $\mathfrak{M}$ denote the object of $\mathcal{C}at_\infty^{\mathrm{Mod}}(\mathcal{K})$ given by the left action of $\mathcal{S}(\mathcal{K})$ on itself.

Using Proposition 4.2.4.9, we can identify $\mathfrak{M}$ with $\Theta(\mathcal{S}(\mathcal{K})^\times, \mathbf{1})$, where $\mathbf{1}$ denotes the unit object $\Delta^0 \in \mathcal{S}(\mathcal{K})$, regarded as an algebra object of $\mathcal{S}(\mathcal{K})$.

Lemma 6.3.5.3. *Let $\mathcal{K}$ be a small collection of simplicial sets. Then the pair $(\mathcal{S}(\mathcal{K})^\times, \mathbf{1})$ is an initial object of $\mathcal{C}at_\infty^{\mathrm{Alg}}(\mathcal{K})$.*

Proof. Let $\phi : \mathcal{C}at_\infty^{\mathrm{Alg}}(\mathcal{K}) \to \mathrm{Mon}^{\mathcal{K}}_{\mathcal{A}ss}(\mathcal{C}at_\infty)$ denote the forgetful functor. Then $\phi(\mathcal{S}(\mathcal{K})^\times, \mathbf{1})$ is an initial object of $\mathrm{Mon}^{\mathcal{K}}_{\mathcal{A}ss}(\mathcal{C}at_\infty) \simeq \mathrm{Alg}(\mathcal{C}at_\infty(\mathcal{K}))$ (Proposition 3.2.1.8). It will therefore suffice to show that $(\mathcal{S}(\mathcal{K})^\times, \mathbf{1})$ is a ϕ-initial object of $\mathcal{C}at_\infty^{\mathrm{Alg}}(\mathcal{K})$ (Proposition T.4.3.1.5). Since ϕ is a coCartesian fibration (Proposition 6.3.5.1), this is equivalent to the requirement that for every ϕ-coCartesian morphism $\alpha : (\mathcal{S}(\mathcal{K})^\times, \mathbf{1}) \to (\mathcal{C}^\otimes, A)$, the object A is initial in the fiber $\phi^{-1}\{\mathcal{C}^\otimes\} \simeq \mathrm{Alg}(\mathcal{C})$ (Proposition T.4.3.1.10). This follows immediately from Proposition 3.2.1.8. $\square$

It follows from Lemma 6.3.5.3 that the forgetful functor $\theta : \mathcal{C}at_\infty^{\mathrm{Alg}}(\mathcal{K})_{(\mathcal{S}(\mathcal{K})^\times, \mathbf{1})/} \to \mathcal{C}at_\infty^{\mathrm{Alg}}(\mathcal{K})$ is a trivial Kan fibration. We let Θ_* denote the composition

$$\mathcal{C}at_\infty^{\mathrm{Alg}}(\mathcal{K}) \simeq \mathcal{C}at_\infty^{\mathrm{Alg}}(\mathcal{K})_{(\mathcal{S}(\mathcal{K})^\times, \mathbf{1})} \xrightarrow{\Theta} \mathcal{C}at_\infty^{\mathrm{Mod}}(\mathcal{K})_{\mathfrak{M}/},$$

where the first map is given by a section of θ.

Remark 6.3.5.4. An object of the ∞-category $\mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathcal{K})_{\mathfrak{M}/}$ is given by a morphism $(\mathcal{S}(\mathcal{K})^\times, \mathcal{S}(\mathcal{K})) \to (\mathcal{C}^\otimes, \mathcal{M})$ in $\mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathcal{K})$, given by a monoidal functor $\mathcal{S}(\mathcal{K})^\times \to \mathcal{C}^\otimes$ which preserves $\mathcal{K}$-indexed colimits (which is unique up to a contractible space of choices by Proposition 3.2.1.8) together with a functor $f : \mathcal{S}(\mathcal{K}) \to \mathcal{M}$ which preserves $\mathcal{K}$-indexed colimits. In view of Remark T.5.3.5.9, such a functor is determined uniquely up to equivalence by the object $f(\Delta^0) \in \mathcal{M}$. Consequently, we can informally regard $\mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathcal{K})_{\mathfrak{M}/}$ as an ∞-category whose objects are triples $(\mathcal{C}^\otimes, \mathcal{M}, M)$, where $(\mathcal{C}^\otimes, \mathcal{M}) \in \mathrm{Cat}^{\mathrm{Mod}}_\infty$ and $M \in \mathcal{M}$ is an object.

Theorem 6.3.5.5. *Let $\mathcal{K}$ be a small collection of simplicial sets which contains $\mathrm{N}(\mathbf{\Delta})^{op}$. Then the functor $\Theta_* : \mathrm{Cat}^{\mathrm{Alg}}_\infty(\mathcal{K}) \to \mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathcal{K})_{\mathfrak{M}/}$ is fully faithful.*

Lemma 6.3.5.6. *Let $\mathcal{K}$ be a small collection of simplicial sets which contains $\mathrm{N}(\mathbf{\Delta})^{op}$, let $(\mathcal{C}^\otimes, \mathcal{M}, M)$ be an object of $\mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathcal{K})_{\mathfrak{M}/}$, and suppose that there exists an algebra object $E \in \mathrm{Alg}(\mathcal{C})$ such that M can be promoted to an object $\overline{M} \in \mathrm{LMod}_E(\mathcal{M})$ where the action $E \otimes M \to M$ exhibits E as a morphism object $\mathrm{Mor}_\mathcal{M}(M, M)$ (see morphism object $\mathrm{Mor}_\mathcal{M}(M, M)$ (Definition 4.2.1.28). Then E represents the right fibration*

$$\mathrm{Cat}^{\mathrm{Alg}}_\infty(\mathcal{K}) \times_{\mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathcal{K})_{\mathfrak{M}/}} (\mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathcal{K})_{\mathfrak{M}/})_{/(\mathcal{C}^\otimes, \mathcal{M}, M)}.$$

Proof. Theorem 6.3.4.1 implies the existence of a functor $\phi : \mathrm{RMod}_E(\mathcal{C})^\otimes \to \mathcal{M}^\otimes$ of ∞-categories left-tensored over $\mathcal{C}^\otimes$ together with an identification $\alpha : M \simeq \phi(E)$ of left E-modules. The pair (ϕ, α) determines an object $\eta \in \mathrm{Cat}^{\mathrm{Alg}}_\infty(\mathcal{K}) \times_{\mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathcal{K})_{\mathfrak{M}/}} (\mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathcal{K})_{\mathfrak{M}/})_{/(\mathcal{C}^\otimes, \mathcal{M}, M)}$ lying over E. We claim that this object is final. To prove this, consider an arbitrary object $(\mathcal{D}^\otimes, A) \in \mathrm{Cat}^{\mathrm{Alg}}_\infty(\mathcal{K})$; we wish to show that the map

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{Alg}}_\infty(\mathcal{K})}((\mathcal{D}^\otimes, A), (\mathcal{C}^\otimes, E)) \to \mathrm{Map}_{\mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathcal{K})_{\mathfrak{M}/}}(\Theta_*(\mathcal{D}^\otimes, A), (\mathcal{C}^\otimes, \mathcal{M}, M))$$

is a homotopy equivalence. It will suffice to prove the result after passing to the homotopy fiber over a point of $\mathrm{Map}_{\mathrm{Mon}^{\mathcal{K}}_{\mathcal{A}ss}(\mathrm{Cat}_\infty)}(\mathcal{D}^\otimes, \mathcal{C}^\otimes)$, corresponding to a monoidal functor F. Using Propositions 6.3.5.1 and T.2.4.4.2 and replacing $\mathcal{D}^\otimes$ by $\mathcal{C}^\otimes$ (and A by $FA \in \mathrm{Alg}(\mathcal{C})$), we are reduced to proving that the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Alg}(\mathcal{C})}(A, E) & \longrightarrow & \mathrm{Map}_{\mathrm{LMod}_\mathcal{C}(\mathrm{Cat}_\infty(\mathcal{K}))}(\mathrm{RMod}_A(\mathcal{C}), \mathcal{M}) \\ \downarrow & & \downarrow \\ \{M\} & \longrightarrow & \mathcal{M} \end{array}$$

is a pullback square. Theorem 6.3.4.1 allows us to identify the upper right corner of this diagram with the ∞-category $\mathrm{LMod}_A(\mathcal{M})$, and the desired result follows from Corollary 6.1.2.41. $\qquad\square$

Proof of Theorem 6.3.5.5. Fix objects $(\mathcal{C}^\otimes, A)$, $(\mathcal{D}^\otimes, B) \in \mathrm{Cat}^{\mathrm{Alg}}_\infty(\mathcal{K})$. We wish to show that the canonical map $\theta : \mathrm{Map}_{\mathrm{Cat}^{\mathrm{Alg}}_\infty(\mathcal{K})}((\mathcal{C}^\otimes, A), (\mathcal{D}^\otimes, B)) \to \mathrm{Map}_{\mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathcal{K})_{\mathfrak{M}/}}(\Theta_*(\mathcal{C}^\otimes, A), \Theta_*(\mathcal{D}^\otimes, B))$ is a homotopy equivalence. Let $M \in \mathrm{RMod}_B(\mathcal{D})$ denote the right B-module given by the action of B on itself. In view of Lemma 6.3.5.6, it will suffice to show that the canonical map $m : B \otimes M \to M$ exhibits B as a morphism object $\mathrm{Mor}_{\mathrm{RMod}_B(\mathcal{D})}(M, M)$. In other words, we must show that for every object $D \in \mathcal{D}$, the multiplication map m induces a homotopy equivalence $\mathrm{Map}_\mathcal{D}(D, B) \to \mathrm{Map}_{\mathrm{RMod}_B(\mathcal{D})}(D \otimes M, M)$. This follows from Proposition 4.2.4.2. $\qquad\square$

It is not difficult to describe the essential image of the fully faithful embedding $\Theta_* : \mathrm{Cat}^{\mathrm{Alg}}_\infty(\mathcal{K}) \to \mathrm{Cat}^{\mathrm{Mod}}_\infty(\mathcal{K})_{\mathfrak{M}/}$ of Theorem 6.3.5.5:

Proposition 6.3.5.7. *Let $\mathcal{C}$ be a monoidal ∞-category. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and that the tensor product $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations. Let $\mathcal{M}$ be an ∞-category left-tensored over $\mathcal{C}$ and let $M \in \mathcal{M}$ be an object. There exists an algebra object $A \in \mathrm{Alg}(\mathcal{C})$ and an equivalence $\mathrm{RMod}_A(\mathcal{C}) \simeq \mathcal{M}$ of ∞-categories left-tensored over $\mathcal{C}$ which carries A to M if and only if the following conditions are satisfied:*

(1) *The ∞-category $\mathcal{M}$ admits geometric realizations of simplicial objects.*

(2) *The action map $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$ preserves geometric realizations of simplicial objects.*

(3) *The functor $F : \mathcal{C} \to \mathcal{M}$ given by $F(C) = C \otimes M$ admits a right adjoint G.*

(4) *The functor G preserves geometric realizations of simplicial objects.*

(5) *The functor G is conservative.*

(6) *For every object $N \in \mathcal{M}$ and every object $C \in \mathcal{C}$, the evident map*

$$F(C \otimes G(N)) \simeq C \otimes G(N) \otimes M \simeq C \otimes FG(N) \to C \otimes N$$

is adjoint to an equivalence $C \otimes G(N) \to G(C \otimes N)$.

Proof. We first prove the "only if" direction. Without loss of generality, we may assume that $\mathcal{M} = \mathrm{RMod}_A(\mathcal{C})$ and $M = A$. Assertion (3) is clear: the functor G can be identified with the forgetful functor $\mathrm{RMod}_A(\mathcal{C}) \to \mathcal{C}$ (see Corollary 4.2.4.8). Assertions (1), (2), and (4) follow from Corollary 4.2.3.5, and assertion (5) from Corollary 4.2.3.2. Assertion (6) follows from the observation that the forgetful functor $\mathrm{RMod}_A(\mathcal{C}) \to \mathcal{C}$ is conservative and commutes with the left action of $\mathcal{C}$.

Now suppose that conditions (1) through (6) are satisfied. Fix $A = G(M) \in \mathcal{C}$. We have a counit map $v : A \otimes M \to M$ with the following universal property: for every object $C \in \mathcal{C}$, composition with v induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}}(C, A) \to \mathrm{Map}_{\mathcal{M}}(C \otimes M, A \otimes M) \to \mathrm{Map}_{\mathcal{M}}(C \otimes M, M).$$

In other words, we can identify A with a morphism object $\mathrm{Mor}_{\mathcal{M}}(M, M)$. It is therefore a final object of the monoidal ∞-category $\mathcal{C}^+[M]$; it follows that A can be regarded as an algebra object of $\mathcal{C}$ and that M has the structure of a left module over A. Using Theorem 6.3.4.6, we deduce the existence of a $\mathcal{C}$-linear functor $U : \mathrm{RMod}_A(\mathcal{C}) \to \mathcal{M}$ carrying A to M. To complete the proof, it will suffice to show that U is an equivalence of ∞-categories.

Let $G' : \mathrm{RMod}_A(\mathcal{C}) \to \mathcal{C}$ be the forgetful functor and F' a left adjoint to G'. Proposition 4.2.4.2 allows us to identify F' with the free module functor $C \mapsto C \otimes A$, so that $F \simeq U \circ F'$. This equivalence is adjoint to a natural transformation $\gamma : G' \to G \circ U$. We claim that γ is an equivalence of functors. Using condition (4), we see that G', G, and U commute with geometric realizations of simplicial objects. Since every right A-module N can be written as a geometric realization of free right A-modules (Proposition 6.2.2.12), we see that it suffices to prove that γ induces an equivalence

$$C \otimes A \to (G \circ U)(C \otimes A) \simeq G(C \otimes M)$$

for $C \in \mathcal{C}$, which is a special case of (6).

We now deduce that U is an equivalence of ∞-categories by applying Corollary 6.2.2.14 to the (homotopy commutative) diagram

$$\begin{array}{ccc} \mathrm{RMod}_A(\mathcal{C}) & \xrightarrow{\quad U \quad} & \mathcal{M} \\ & {\scriptstyle G'} \searrow \quad \swarrow {\scriptstyle G} & \\ & \mathcal{C}\,. & \end{array}$$

$\square$

Remark 6.3.5.8. Fix a monoidal ∞-category $\mathcal{C}$ which admits geometric realizations of simplicial objects, and assume that the tensor product functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects. Theorem 6.3.5.5 implies that Θ_* determines a fully faithful embedding from $\mathrm{Alg}(\mathcal{C})$ to the ∞-category $\mathrm{LMod}_{\mathcal{C}}(\mathcal{Cat}_{\infty}(\mathcal{K}))_{\mathcal{C}/}$ of pointed $\mathcal{C}$-module objects of $\mathcal{Cat}_{\infty}(\mathcal{K})$, where $\mathcal{K} = \{\mathrm{N}(\mathbf{\Delta})^{op}\}$. In fact, something

slightly stronger is true: $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_\infty(\mathcal{K}))_{\mathcal{C}/}$ can be identified with the underlying ∞-category of an $(\infty, 2)$-category, and Θ_* is fully faithful at the level of $(\infty, 2)$-categories. More concretely: given algebra objects $A, B \in \mathrm{Alg}(\mathcal{C})$ and $\mathcal{C}$-linear functors $f, g : \mathrm{RMod}_A(\mathcal{C}) \to \mathrm{RMod}_B(\mathcal{C})$ satisfying $f(A) = g(A) = B$, any $\mathcal{C}$-linear natural transformation $\alpha : f \to g$ which induces an equivalence $\alpha_A : B \simeq f(A) \to g(A) \simeq B$ is itself an equivalence. To prove this, let $\mathfrak{X} \subseteq \mathrm{RMod}_A(\mathcal{C})$ be the full subcategory spanned by those objects M for which $\alpha_M : f(M) \to g(M)$ is an equivalence. We wish to prove that $\mathfrak{X} = \mathrm{RMod}_A(\mathcal{C})$. Since f and g preserve geometric realizations, $\mathfrak{X}$ is stable under the formation of geometric realizations. Since every right A-module is a geometric realization of free right A-modules (Proposition 6.2.2.12), it suffices to show that every free module $C \otimes A \in \mathrm{RMod}_A(\mathcal{C})$ belongs to $\mathfrak{X}$. Since α is $\mathcal{C}$-linear, $\mathfrak{X}$ is stable under the left action of $\mathcal{C}$; we are therefore reduced to showing that the right module A belongs to $\mathfrak{X}$, which is true by assumption.

Lemma 6.3.5.6 can also be used to show that the fully faithful embedding Θ_* admits a right adjoint, provided that we can guarantee the existence of endomorphism objects $\mathrm{Mor}_{\mathcal{M}}(M, M)$. For this, it is convenient to work a setting where we require all ∞-categories to be presentable. For this, we need to introduce a bit of terminology.

Notation 6.3.5.9. Let $\widehat{\mathrm{Cat}}_\infty$ denote the ∞-category of (not necessarily small) ∞-categories, which contains Cat_∞ as a full subcategory. Similarly, we define ∞-categories $\widehat{\mathrm{Cat}}_\infty^{\mathrm{Alg}} \supset \mathrm{Cat}_\infty^{\mathrm{Alg}}$ and $\widehat{\mathrm{Cat}}_\infty^{\mathrm{Mod}} \supset \mathrm{Cat}_\infty^{\mathrm{Mod}}$ by allowing monoidal ∞-categories and left-tensored ∞-categories which are not small. Let $\mathcal{K}$ denote the collection of *all* small simplicial sets, and let $\widehat{\mathrm{Cat}}_\infty^{\mathrm{Alg}}(\mathcal{K})$ and $\widehat{\mathrm{Cat}}_\infty^{\mathrm{Mod}}(\mathcal{K})$ be defined as in §6.3.3. Construction 6.3.3.24 generalizes immediately to give a functor $\widehat{\Theta} : \widehat{\mathrm{Cat}}_\infty^{\mathrm{Alg}}(\mathcal{K}) \to \widehat{\mathrm{Cat}}_\infty^{\mathrm{Mod}}(\mathcal{K})$. We let $\mathrm{Pr}^{\mathrm{Alg}}$ denote the full subcategory of $\widehat{\mathrm{Cat}}_\infty^{\mathrm{Alg}}(\mathcal{K})$ spanned by those pairs $(\mathcal{C}^\otimes, A)$ where the ∞-category $\mathcal{C}$ is presentable, and $\mathrm{Pr}^{\mathrm{Mod}}$ the full subcategory of $\widehat{\mathrm{Cat}}_\infty^{\mathrm{Mod}}(\mathcal{K})$ spanned by those pairs $(\mathcal{C}^\otimes, \mathcal{M})$ where $\mathcal{C}$ and $\mathcal{M}$ are both presentable. It follows from Corollary 4.2.3.7 that the functor $\widehat{\Theta} : \widehat{\mathrm{Cat}}_\infty^{\mathrm{Alg}}(\mathcal{K}) \to \widehat{\mathrm{Cat}}_\infty^{\mathrm{Mod}}(\mathcal{K})$ restricts to a functor $\mathrm{Pr}^{\mathrm{Alg}} \to \mathrm{Pr}^{\mathrm{Mod}}$, which we will also denote by $\widehat{\Theta}$. Similarly, if we let $\mathfrak{M}$ denote the object $\Theta(\mathcal{S}^\times, \Delta^0) \simeq (\mathcal{S}^\times, \mathcal{S}) \in \mathrm{Pr}^{\mathrm{Mod}}$, then we have a functor $\widehat{\Theta}_* : \mathrm{Pr}^{\mathrm{Alg}} \to \mathrm{Pr}^{\mathrm{Mod}}_{\mathfrak{M}/}$.

Theorem 6.3.5.10. *The functor* $\widehat{\Theta}_* : \mathrm{Pr}^{\mathrm{Alg}} \to \mathrm{Pr}^{\mathrm{Mod}}_{\mathfrak{M}/}$ *is fully faithful and admits a right adjoint.*

Proof. The first assertion follows by applying Theorem 6.3.5.5 in a larger universe. For the second, it will suffice to show that for every object $X = (\mathcal{C}^\otimes, \mathcal{M}, M) \in \mathrm{Pr}^{\mathrm{Mod}}_{\mathfrak{M}/}$, the right fibration $\mathrm{Pr}^{\mathrm{Alg}} \times_{\mathrm{Pr}^{\mathrm{Mod}}_{\mathfrak{M}/}} (\mathrm{Pr}^{\mathrm{Mod}}_{\mathfrak{M}/})_{/X}$ is representable (Proposition T.5.2.4.2). In view of Lemma 6.3.5.3, it will suffice to show that there exists an algebra object $E \in \mathrm{Alg}(\mathcal{C})$ such that M can be promoted to a module $\overline{M} \in \mathrm{LMod}_E(\mathcal{C})$ such that the action $E \otimes M \to M$ exhibits E as a morphism object $\mathrm{Mor}_{\mathcal{M}}(M, M)$. This is equivalent to requiring the existence of an algebra object $E \in {}^{\Delta}\mathrm{Alg}(\mathcal{C}^+[M])$ such that the underlying object in $\mathcal{C}^+[M]$ is final. According to Corollary 3.2.2.4, it will suffice to show that $\mathcal{C}^+[M]$ has a final object (which then admits an essentially unique algebra structure): that is, it will suffice to show that there exists a morphism object $\mathrm{Mor}_{\mathcal{M}}(M, M)$. This follows from Proposition 4.2.1.33. $\square$

Remark 6.3.5.11. More informally, the right adjoint to $\widehat{\Theta}_*$ carries an object $(\mathcal{C}^\otimes, \mathcal{M}, M) \in \mathrm{Pr}^{\mathrm{Mod}}_{\mathfrak{M}/}$ to the pair $(\mathcal{C}^\otimes, E) \in \mathrm{Pr}^{\mathrm{Alg}}$, where $E \in \mathrm{Alg}(\mathcal{C})$ is the algebra of endomorphisms of the object $M \in \mathcal{M}$.

We next investigate the behavior of the functor Θ with respect to tensor products of ∞-categories.

Notation 6.3.5.12. The ∞-category $\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathrm{Cat}_\infty)$ of monoidal ∞-categories admits finite products, and can therefore be regarded as endowed with Cartesian symmetric monoidal structure. Let $\mathcal{K}$ be a small collection of simplicial sets. We define a subcategory $\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}^{\mathcal{K}}(\mathrm{Cat}_\infty)^\otimes \subseteq \mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathrm{Cat}_\infty)^\times$ as follows:

(1) Let C be an object of $\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}(\mathrm{Cat}_\infty)^\times$, given by a sequence of monoidal ∞-categories $(\mathcal{C}_1^\otimes, \ldots, \mathcal{C}_n^\otimes)$. Then $C \in \mathrm{Mon}_{\mathcal{A}\mathrm{ss}}^{\mathcal{K}}(\mathrm{Cat}_\infty)^\otimes$ if and only if each of the underlying ∞-categories $\mathcal{C}_i$ admits $\mathcal{K}$-indexed colimits, and the tensor product functors $\mathcal{C}_i \times \mathcal{C}_i \to \mathcal{C}_i$ preserve $\mathcal{K}$-indexed colimits separately in each variable.

(2) Let $F : (\mathcal{C}_1^\otimes, \ldots, \mathcal{C}_m^\otimes) \to (\mathcal{D}_1^\otimes, \ldots, \mathcal{D}_n^\otimes)$ be a morphism in $\mathrm{Mon}_{\mathcal{A}ss}(\mathcal{C}at_\infty)^\times$ covering a map $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}in_*$, where the objects $(\mathcal{C}_1^\otimes, \ldots, \mathcal{C}_m^\otimes)$ and $(\mathcal{D}_1^\otimes, \ldots, \mathcal{D}_n^\otimes)$ belong to $\mathrm{Mon}_{\mathcal{A}ss}^{\mathcal{K}}(\mathcal{C}at_\infty)^\otimes$. Then F belongs to $\mathrm{Mon}_{\mathcal{A}ss}^{\mathcal{K}}(\mathcal{C}at_\infty)^\otimes$ if and only if the induced functor $\prod_{\alpha(i)=j} \mathcal{C}_i \to \mathcal{D}_j$ preserves $\mathcal{K}$-indexed colimits separately in each variable, for $1 \leq j \leq n$.

We let $\mathcal{C}at_\infty^{\mathrm{Mod}}(\mathcal{K})^\otimes$ denote the subcategory of $\mathcal{C}at_\infty^{\mathrm{Mod}\,\times}$ described as follows:

(1′) Let C be an object of $\mathcal{C}at_\infty^{\mathrm{Mod}\,\times}$, corresponding to a finite sequence $((\mathcal{C}_1^\otimes, \mathcal{M}_1), \ldots, (\mathcal{C}_n^\otimes, \mathcal{M}_n))$. Then $C \in \mathcal{C}at_\infty^{\mathrm{Mod}}(\mathcal{K})^\otimes$ if and only if each $\mathcal{C}_i$ and each $\mathcal{M}_i$ admit $\mathcal{K}$-indexed colimits, and the tensor product functors

$$\mathcal{C}_i \times \mathcal{C}_i \to \mathcal{C}_i \qquad \mathcal{C}_i \times \mathcal{M}_i \to \mathcal{M}_i$$

preserves $\mathcal{K}$-indexed colimits separately in each variable.

(2′) Let $F : ((\mathcal{C}_1^\otimes, \mathcal{M}_1), \ldots, (\mathcal{C}_m^\otimes, \mathcal{M}_m)) \to ((\mathcal{D}_1^\otimes, \mathcal{N}_1), \ldots, (\mathcal{D}_n^\otimes, \mathcal{N}_n))$ be a morphism in $\mathcal{C}at_\infty^{\mathrm{Mon},\times}$ covering a map $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}in_*$, where the objects $((\mathcal{C}_1^\otimes, \mathcal{M}_1), \ldots, (\mathcal{C}_m^\otimes, \mathcal{M}_m))$ and $((\mathcal{D}_1^\otimes, \mathcal{N}_1), \ldots, (\mathcal{D}_n^\otimes, \mathcal{N}_n))$ belong to $\mathcal{C}at_\infty^{\mathrm{Mod}}(\mathcal{K})^\otimes$. Then F belongs to $\mathcal{C}at_\infty^{\mathrm{Mod}}(\mathcal{K})^\otimes$ if and only if the induced functors

$$\prod_{\alpha(i)=j} \mathcal{C}_i \to \mathcal{D}_j \qquad \prod_{\alpha(i)=j} \mathcal{M}_i \to \mathcal{N}_j$$

preserves $\mathcal{K}$-indexed colimits separately in each variable, for $1 \leq j \leq n$.

We let $\mathcal{C}at_\infty^{\mathrm{Alg}}(\mathcal{K})^\otimes$ denote the fiber product $(\mathcal{C}at_\infty^{\mathrm{Alg}})^\times \times_{\mathrm{Mon}_{\mathcal{A}ss}(\mathcal{C}at_\infty)^\times} \mathrm{Mon}_{\mathcal{A}ss}^{\mathcal{K}}(\mathcal{C}at_\infty)^\otimes$.

Remark 6.3.5.13. Assume that $\mathcal{K}$ consists entirely of *sifted* simplicial sets (this is satisfied, for example, if $\mathcal{K} = \{\mathrm{N}(\mathbf{\Delta})^{op}\}$. Then we can identify $\mathcal{C}at_\infty^{\mathrm{Alg}}(\mathcal{K})^\otimes$, $\mathcal{C}at_\infty^{\mathrm{Mod}}(\mathcal{K})^\otimes$, and $\mathrm{Mon}_{\mathcal{A}ss}^{\mathcal{K}}(\mathcal{C}at_\infty)^\otimes$ with $\mathcal{C}at_\infty^{\mathrm{Alg}}(\mathcal{K})^\times$, $\mathcal{C}at_\infty^{\mathrm{Mod}}(\mathcal{K})^\times$, and $\mathrm{Mon}_{\mathcal{A}ss}^{\mathcal{K}}(\mathcal{C}at_\infty)^\times$, respectively.

Theorem 6.3.5.14. *Let $\mathcal{K}$ be a small collection of simplicial sets. Then:*

(1) *The map $\mathrm{Mon}_{\mathcal{A}ss}^{\mathcal{K}}(\mathcal{C}at_\infty)^\otimes \to \mathrm{N}(\mathcal{F}in_*)$ determines a symmetric monoidal structure on $\mathrm{Mon}_{\mathcal{A}ss}^{\mathcal{K}}(\mathcal{C}at_\infty)$, and the maps $\mathcal{C}at_\infty^{\mathrm{Alg}}(\mathcal{K})^\otimes \to \mathrm{Mon}_{\mathcal{A}ss}^{\mathcal{K}}(\mathcal{C}at_\infty)^\otimes \leftarrow \mathcal{C}at_\infty^{\mathrm{Mod}}(\mathcal{K})^\otimes$ are coCartesian fibrations of symmetric monoidal ∞-categories.*

(2) *The functor $\Theta : \mathcal{C}at_\infty^{\mathrm{Alg}}(\{\mathrm{N}(\mathbf{\Delta})^{op}\}) \to \mathcal{C}at_\infty^{\mathrm{Mod}}(\{\mathrm{N}(\mathbf{\Delta})^{op}\})$ preserves products, and therefore induces a symmetric monoidal functor $\Theta^\times : \mathcal{C}at_\infty^{\mathrm{Alg}}(\{\mathrm{N}(\mathbf{\Delta})^{op}\})^\times \to \mathcal{C}at_\infty^{\mathrm{Mod}}(\{\mathrm{N}(\mathbf{\Delta})^{op}\})^\times$.*

(3) *Assume that $\mathrm{N}(\mathbf{\Delta})^{op} \in \mathcal{K}$. Then the functor $\Theta^\times$ of (2) restricts to a functor $\Theta^\otimes : \mathcal{C}at_\infty^{\mathrm{Alg}}(\mathcal{K})^\otimes \to \mathcal{C}at_\infty^{\mathrm{Mod}}(\mathcal{K})^\otimes$ (see Remark 6.3.5.13).*

(4) *The functor $\Theta^\otimes$ is symmetric monoidal.*

Proof. We first prove (1). Recall that $\mathrm{Mon}_{\mathcal{A}ss}(\mathcal{C}at_\infty)$ can be identified with the ∞-category $\mathrm{Alg}(\mathcal{C}at_\infty)$ of associative algebra objects of $\mathcal{C}at_\infty$ (Proposition 2.4.2.5). Here we regard $\mathcal{C}at_\infty$ as endowed with the Cartesian symmetric monoidal structure. The ∞-category $\mathrm{Alg}(\mathcal{C}at_\infty)$ inherits a symmetric monoidal structure from that of $\mathcal{C}at_\infty$ (see Example 3.2.4.4), which is also Cartesian; we therefore obtain an identification $\mathrm{Alg}(\mathcal{C}at_\infty)^\otimes \simeq \mathrm{Mon}_{\mathcal{A}ss}(\mathcal{C}at_\infty)^\times$. Under this equivalence, the subcategory $\mathrm{Mon}_{\mathcal{A}ss}^{\mathcal{K}}(\mathcal{C}at_\infty)^\otimes$ corresponds to the subcategory $\mathrm{Alg}(\mathcal{C}at_\infty(\mathcal{K}))^\otimes$, which is again a symmetric monoidal ∞-category. A similar argument shows that $\mathcal{C}at_\infty^{\mathrm{Mod}}(\mathcal{K})^\otimes \to \mathrm{Alg}(\mathcal{C}at_\infty(\mathcal{K}))^\otimes$ is a coCartesian fibration of symmetric monoidal ∞-categories. Finally, we observe that the functor $\mathcal{C}at_\infty^{\mathrm{Alg}}(\mathcal{K})^\otimes \to \mathrm{Mon}_{\mathcal{A}ss}^{\mathcal{K}}(\mathcal{C}at_\infty)^\otimes$ is a pullback of $(\mathcal{C}at_\infty^{\mathrm{Alg}})^\times \to \mathrm{Mon}_{\mathcal{A}ss}(\mathcal{C}at_\infty)^\times$, which is easily seen to be a coCartesian fibration of ∞-operads. This proves (1).

Assertion (2) is obvious, and assertion (3) follows from Corollary 4.2.3.5. We will prove (4). It is easy to see that $\Theta^\otimes$ is a map of ∞-operads, and it follows from Proposition 4.2.4.9 that $\Theta^\otimes$ preserves unit objects.

Consequently, it will suffice to show that for every pair of objects $(\mathcal{C}^{\otimes}, A), (\mathcal{D}^{\otimes}, B) \in \mathcal{C}at_{\infty}^{\mathrm{Alg}}(\mathcal{K})$, the induced map

$$\Theta(\mathcal{C}^{\otimes}, A) \otimes \Theta(\mathcal{D}^{\otimes}, B) \to \Theta((\mathcal{C}^{\otimes}, A) \otimes (\mathcal{D}^{\otimes}, B))$$

is an equivalence in $\mathcal{C}at_{\infty}^{\mathrm{Mod}}(\mathcal{K})$. In other words, we wish to show that Θ induces an equivalence of ∞-categories

$$\theta : \mathrm{RMod}_A(\mathcal{C}) \otimes \mathrm{RMod}_B(\mathcal{D}) \to \mathrm{RMod}_{A \otimes B}(\mathcal{C} \otimes \mathcal{D})$$

(here the tensor products are taken in $\mathcal{C}at_{\infty}(\mathcal{K})$). We have a homotopy commutative diagram of ∞-categories

To prove that θ is a categorical equivalence, it will suffice to show that this diagram satisfies the hypotheses of Corollary 6.2.2.14:

(a) The ∞-categories $\mathrm{RMod}_A(\mathcal{C}) \otimes \mathrm{RMod}_B(\mathcal{D})$ and $\mathrm{RMod}_{A \otimes B}(\mathcal{C} \otimes \mathcal{D})$ admit geometric realizations of simplicial objects. This follows from our assumption that $\mathrm{N}(\boldsymbol{\Delta})^{op} \in \mathcal{K}$.

(b) The functors G and G' admit left adjoints, which we will denote by F and F'. The existence of F' is guaranteed by Proposition 4.2.4.2, and is given informally by the formula $X \mapsto X \otimes (A \otimes B)$. Similarly, Proposition 4.2.4.2 guarantees that the forgetful functors $\mathrm{RMod}_A(\mathcal{C}) \to \mathcal{C}$ and $\mathrm{RMod}_B(\mathcal{D}) \to \mathcal{D}$ admit left adjoints, given by tensoring on the right with A and B, respectively. The tensor product of these left adjoints is a left adjoint to G.

(c) The functor G' is conservative and preserves geometric realizations of simplicial objects. The first assertion follows from Corollary 4.2.3.2 and the second from Corollary 4.2.3.5.

(d) The functor G is conservative and preserves geometric realizations of simplicial objects. The second assertion is obvious: G is a tensor product of the forgetful functors $\mathrm{RMod}_A(\mathcal{C}) \to \mathcal{C}$ and $\mathrm{RMod}_B(\mathcal{D}) \to \mathcal{D}$, each of which preserves geometric realizations (and can therefore be interpreted as a morphism in $\mathcal{C}at_{\infty}(\mathcal{K})$) by Corollary 4.2.3.5. To prove that G is conservative, we factor G as a composition

$$\mathrm{RMod}_A(\mathcal{C}) \otimes \mathrm{RMod}_B(\mathcal{D}) \xrightarrow{G_0} \mathcal{C} \otimes \mathrm{RMod}_B(\mathcal{D}) \xrightarrow{G_1} \mathcal{C} \otimes \mathcal{D}.$$

We can identify G_1 with the forgetful functor

$$(\mathcal{C} \otimes \mathcal{D}) \otimes_{\mathcal{D}} \mathrm{RMod}_B(\mathcal{D}) \to (\mathcal{C} \otimes \mathcal{D}) \otimes_{\mathcal{D}} \mathcal{D} \simeq \mathcal{C} \otimes \mathcal{D}.$$

Theorem 6.3.4.6 allows us to identify the left hand side with the ∞-category $\mathrm{RMod}_B(\mathcal{C} \otimes \mathcal{D})$. Under this identification, G_1 corresponds to the forgetful functor $\mathrm{RMod}_B(\mathcal{C} \otimes \mathcal{D}) \to \mathcal{C} \otimes \mathcal{D}$, which is conservative by Corollary 4.2.3.2. A similar arguments shows that G_0 is conservative, so that $G \simeq G_1 \circ G_0$ is conservative as required.

(e) The canonical natural transformation $G' \circ F' \to G \circ F$ is an equivalence of functors from $\mathcal{C} \otimes \mathcal{D}$ to itself. This is clear from the descriptions of F and F' given above: both compositions are given by right multiplication by the object $A \otimes B \in \mathcal{C} \otimes \mathcal{D}$.

$\square$

Remark 6.3.5.15. Fix a small collection of simplicial sets $\mathcal{K}$ which contains $\mathrm{N}(\boldsymbol{\Delta})^{op}$. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Assume that $\mathcal{C}$ admits $\mathcal{K}$-indexed colimits and that the tensor product $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$

preserves $\mathcal{K}$-indexed colimits separately in each variable. Then we $\mathcal{C}^{\otimes}$ as a commutative algebra object in the (symmetric monoidal) ∞-category $\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}^{\mathcal{K}}(\mathcal{C}\mathrm{at}_{\infty})$. The fiber products

$$\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times_{\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}^{\mathcal{K}}(\mathcal{C}\mathrm{at}_{\infty})} \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Alg}}(\mathcal{C})^{\otimes} \qquad \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times_{\mathrm{Mon}_{\mathcal{A}\mathrm{ss}}^{\mathcal{K}}(\mathcal{C}\mathrm{at}_{\infty})^{\otimes}} \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Mod}}(\mathcal{C})^{\otimes}$$

can be identified with the symmetric monoidal ∞-categories $\mathrm{Alg}(\mathcal{C})^{\otimes}$ and $\mathrm{LMod}_{\mathcal{C}}(\mathcal{C}\mathrm{at}_{\infty}(\mathcal{K}))^{\otimes}$, respectively. It follows from Theorem 6.3.5.14 and Proposition 6.3.5.1 that Θ determines a symmetric monoidal functor $\Theta_{\mathcal{C}}^{\otimes} : \mathrm{Alg}(\mathcal{C})^{\otimes} \to \mathrm{LMod}_{\mathcal{C}}(\mathcal{C}\mathrm{at}_{\infty}(\mathcal{K}))^{\otimes}$.

Corollary 6.3.5.16. *Let $\mathcal{K}$ and $\mathcal{C}^{\otimes}$ be as in Remark 6.3.5.15, and let $\mathcal{O}^{\otimes}$ be a unital ∞-operad. Then the functor $\Theta_{\mathcal{C}}^{\otimes}$ induces a fully faithful functor*

$$\theta : \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}(\mathcal{C})) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{LMod}_{\mathcal{C}}(\mathcal{C}\mathrm{at}_{\infty}(\mathcal{K}))).$$

Proof. Let $\mathrm{Alg}(\mathcal{C})_*^{\otimes}$ and $\mathrm{LMod}_{\mathcal{C}}(\mathcal{C}\mathrm{at}_{\infty}(\mathcal{K}))_*^{\otimes}$ be unitalizations of $\mathrm{Alg}(\mathcal{C})^{\otimes}$ and $\mathrm{LMod}_{\mathcal{C}}(\mathcal{C}\mathrm{at}_{\infty}(\mathcal{K}))^{\otimes}$, respectively (see §2.3.1; note that $\mathrm{Alg}(\mathcal{C})^{\otimes}$ is already a unital ∞-operad, so that $\mathrm{Alg}(\mathcal{C})_*^{\otimes} \simeq \mathrm{Alg}(\mathcal{C})^{\otimes}$). The functor $\Theta_{\mathcal{C}}^{\otimes}$ induces a symmetric monoidal functor $\mathrm{Alg}(\mathcal{C})_*^{\otimes} \to \mathrm{LMod}_{\mathcal{C}}(\mathcal{C}\mathrm{at}_{\infty}(\mathcal{K}))_*^{\otimes}$, and Theorem 6.3.5.5 guarantees that this functor is fully faithful. We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}(\mathcal{C})_*) & \xrightarrow{\theta_*} & \mathrm{Alg}_{\mathcal{O}}(\mathrm{LMod}_{\mathcal{C}}(\mathcal{C}\mathrm{at}_{\infty}(\mathcal{K}))_*) \\
\downarrow & & \downarrow \\
\mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}(\mathcal{C})) & \xrightarrow{\theta} & \mathrm{Alg}_{\mathcal{O}}(\mathrm{LMod}_{\mathcal{C}}(\mathcal{C}\mathrm{at}_{\infty}(\mathcal{K})))
\end{array}
$$

where θ_* is fully faithful. Since $\mathcal{O}^{\otimes}$ is unital, the vertical maps are categorical equivalences, so that θ is fully faithful as well. $\qquad\square$

Corollary 6.3.5.17. *Let $\mathcal{K}$ and $\mathcal{C}^{\otimes}$ be as in Remark 6.3.5.15. Then for $n \geq 1$, we have a fully faithful functor $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathrm{LMod}_{\mathcal{C}}(\mathcal{C}\mathrm{at}_{\infty}(\mathcal{K})))$.*

Proof. Combine Corollary 6.3.5.16 and Theorem 5.1.2.2. $\qquad\square$

Corollary 6.3.5.17 furnishes a convenient way of understanding the notion of an $\mathbb{E}_n$-algebra: giving an $\mathbb{E}_n$-algebra object $A \in \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$ is equivalent to giving the underlying associative algebra object $A_0 \in \mathrm{Alg}(\mathcal{C})$, together with an $\mathbb{E}_{n-1}$-structure on the ∞-category $\mathrm{RMod}_{A_0}(\mathcal{C})$ of right A_0-modules (with unit object given by the module A_0 itself).

We also have a version of Corollary 6.3.5.17 when $n = \infty$:

Corollary 6.3.5.18. *Let $\mathcal{K}$ be a class of simplicial sets which includes $\mathrm{N}(\boldsymbol{\Delta})^{op}$. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category which admits $\mathcal{K}$-indexed colimits, for which the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves $\mathcal{K}$-indexed colimits separately in each variable. Then the construction $A \mapsto \mathrm{Mod}_A(\mathcal{C})^{\otimes}$ determines a fully faithful embedding*

$$\theta : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C}\mathrm{at}_{\infty}(\mathcal{K}))_{\mathcal{C}^{\otimes}/}.$$

Moreover, a symmetric monoidal functor $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ belongs to the essential image of θ if and only if the following conditions are satisfied:

  (a) *The ∞-category $\mathcal{D}$ admits $\mathcal{K}$-indexed colimits, the tensor product $\otimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ preserves $\mathcal{K}$-indexed colimits separately in each variable.*

  (b) *The functor F admits a right adjoint G (this condition guarantees that F preserves $\mathcal{K}$-indexed colimits).*

  (c) *The functor G preserves geometric realizations of simplicial objects.*

  (d) *The functor G is conservative.*

(e) *For every object $C \in \mathcal{C}$ and $D \in \mathcal{D}$, the canonical map*

$$C \otimes G(D) \to G(F(C) \otimes D)$$

is an equivalence.

Proof. Since $\mathcal{A}ss^{\otimes}$ is a unital ∞-operad, Propositions 3.2.4.10 and 2.4.3.9 imply that the unique bifunctor of ∞-operads $\mathcal{A}ss^{\otimes} \times \mathrm{Comm}^{\otimes} \to \mathrm{Comm}^{\otimes}$ exhibits $\mathrm{Comm}^{\otimes}$ as a tensor product of $\mathcal{A}ss^{\otimes}$ and $\mathrm{Comm}^{\otimes}$. Corollary 6.3.5.16 yields a fully faithful embedding $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathrm{LMod}_{\mathcal{C}}(\mathcal{C}at_{\infty}(\mathcal{K})))$. The functor θ is obtained by composing this embedding with the equivalence

$$\mathrm{CAlg}(\mathrm{LMod}_{\mathcal{C}}(\mathcal{C}at_{\infty}(\mathcal{K}))) \simeq \mathrm{CAlg}(\mathcal{C}at_{\infty}(\mathcal{K}))_{\mathcal{C}/}$$

furnished by Proposition 3.4.1.3. The description of the essential image of θ follows from Proposition 6.3.5.7. $\qquad\square$

Corollary 6.3.5.19. *Let $\mathcal{K}$ and $\mathcal{C}^{\otimes}$ be as in Remark 6.3.5.15. Then the functor $\Theta_{\mathcal{C}}^{\otimes}$ induces a fully faithful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{C}}(\mathcal{C}at_{\infty}(\mathcal{K})))$.*

6.3.6 Comparison of Tensor Products

Let R be a commutative ring, and let $\mathcal{C}$ denote the category of R-modules. Then $\mathcal{C}$ inherits a symmetric monoidal structure, which can be described in (at least) two different ways:

(a) Let M and N be R-modules. Then we can regard M as a right R-module and N as a left R-module, and consider the relative tensor product $M \otimes_R N$.

(b) The absolute tensor product $M \otimes N$ can be viewed as a module over the tensor product $R \otimes R$. The relative tensor product $M \otimes_R N$ is the R-module determined by base change of $M \otimes N$ along the ring homomorphism $m : R \otimes R \to R$ encoding the multiplication on R.

These two constructions agree: that is, there is a canonical isomorphism

$$M \otimes_R N \simeq (M \otimes N) \otimes_{R \otimes R} R.$$

Our goal in this section is to prove a generalization of this assertion, where the ordinary category of abelian groups is replaced by a symmetric monoidal ∞-category $\mathcal{C}$.

We begin by formulating a somewhat more general problem. Let $n \geq 2$ be an integer, and let A be an $\mathbb{E}_n$-algebra object of a symmetric monoidal ∞-category $\mathcal{C}$. We can then consider the ∞-category $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C})$ of $\mathbb{E}_n$-modules over A (see §3.3.3). In good cases, $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C})$ inherits the structure of an $\mathbb{E}_n$-monoidal ∞-category (Theorem 3.4.4.2). On the other hand, we can identify A with an object of $\mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})) \simeq \mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathrm{Alg}(\mathcal{C}))$, so that (under some mild assumptions) the ∞-category $\mathrm{RMod}_A(\mathcal{C})$ of right A-modules is endowed with the structure of an $\mathbb{E}_{n-1}$-monoidal ∞-category (see Corollary 6.3.5.17). The ∞-categories $\mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C})$ and $\mathrm{RMod}_A(\mathcal{C})$ are usually not equivalent. However, we will show that there is a forgetful functor $\theta : \mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C}) \to \mathrm{RMod}_A$. Our main result (Theorem 6.3.6.10) asserts that, under some mild assumptions, θ can be promoted to an $\mathbb{E}_{n-1}$-monoidal functor.

We begin by setting up a bit of notation.

Notation 6.3.6.1. Let $\mathrm{Cut} : \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathcal{A}ss^{\otimes}$ be as in Construction 4.1.2.5 and let $\mathbb{E}_1^{\otimes} \to \mathcal{A}ss^{\otimes}$ be the equivalence of Example 5.1.0.7. We let $\mathcal{A}ss^{\square}$ denote the fiber product $\mathrm{N}(\boldsymbol{\Delta})^{op} \times_{\mathcal{A}ss^{\otimes}} \mathbb{E}_1^{\otimes}$, so that the projection map $\mathcal{A}ss^{\square} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$ is a trivial Kan fibration.

Let $\mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1 \to \mathcal{R}\mathcal{M}^{\otimes}$ be defined as in Remark 4.2.2.8 and set

$$\mathcal{X} = (\mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1) \times_{\mathcal{R}\mathcal{M}^{\otimes}} (\mathcal{R}\mathcal{M}^{\otimes} \times_{\mathcal{A}ss^{\otimes}} \mathbb{E}_1^{\otimes}).$$

There is a unique vertex $x \in \mathfrak{X}$ lying over the initial object $([0], 0) \in \mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1$. We let $\mathcal{LM}^{\square}$ denote the ∞-category $\mathfrak{X}^{x/}$. Example 5.1.0.7 implies that the map $\mathfrak{X} \to \mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1$ is a trivial Kan fibration, so that $x \in \mathfrak{X}$ is an initial object and therefore the map $\mathcal{LM}^{\square} \to \mathrm{N}(\boldsymbol{\Delta})^{op} \times \Delta^1$ is also a trivial Kan fibration. Moreover, the fiber product $\mathcal{LM}^{\square} \times_{\Delta^1} \{1\}$ is isomorphic to $\mathcal{A}ss^{\square}$.

We will say that a morphism f in $\mathcal{A}ss^{\square}$ or $\mathcal{LM}^{\square}$ is *inert* if its image in $\mathcal{LM}^{\otimes}$ is inert.

Construction 6.3.6.2. Let S be a simplicial set, and let $\mathcal{C}^{\otimes} \to \mathbb{E}_1^{\otimes} \times S$ be an coCartesian S-family of $\mathbb{E}_1$-monoidal ∞-categories. We let $\mathrm{Alg}^{\square}(\mathcal{C})$ denote the full subcategory of

$$\mathrm{Fun}_{\mathbb{E}_1^{\otimes}}(\mathcal{A}ss^{\square}, \mathcal{C}^{\otimes}) \times_{\mathrm{Fun}(\mathcal{A}ss^{\square}, S)} S$$

spanned by those vertices which correspond to a vertex $s \in S$ and a map $\mathcal{A}ss^{\square} \to \mathcal{C}_s^{\otimes}$ which carries every inert morphism in $\mathcal{A}ss^{\square}$ to an inert morphism in $\mathcal{C}_s^{\otimes}$. We let $\mathrm{RMod}^{\square}(\mathcal{C})$ denote the full subcategory of

$$\mathrm{Fun}_{\mathbb{E}_1^{\otimes}}(\mathcal{LM}^{\square}, \mathcal{C}^{\otimes}) \times_{\mathrm{Fun}(\mathcal{LM}^{\square}, S)} S$$

spanned by those vertices which correspond to a vertex $s \in S$ and a map $\mathcal{LM}^{\square} \to \mathcal{C}_s^{\otimes}$ which carries inert morphisms in $\mathcal{LM}^{\square}$ to inert morphisms in $\mathcal{C}_s^{\otimes}$.

Remark 6.3.6.3. Let $\mathcal{C}^{\otimes} \to \mathbb{E}_1^{\otimes} \times S$ be a coCartesian S-family of $\mathbb{E}_1$-monoidal ∞-categories. It follows from Proposition T.3.1.2.1 that the projection maps $p : \mathrm{Alg}^{\square}(\mathcal{C}) \to S \qquad q : \mathrm{RMod}^{\square}(\mathcal{C}) \to S$ are coCartesian fibrations. Moreover, the restriction map $\mathrm{RMod}^{\square}(\mathcal{C}) \to \mathrm{Alg}^{\square}(\mathcal{C})$ carries q-coCartesian morphisms to p-coCartesian morphisms.

Example 6.3.6.4. Let $\mathcal{C}_0^{\otimes} \to \mathcal{A}ss^{\otimes} \times S$ be a coCartesian S-family of monoidal ∞-categories, let $\mathcal{C}^{\otimes} = \mathcal{C}_0^{\otimes} \times_{\mathcal{A}ss^{\otimes}} \mathbb{E}_1^{\otimes}$, and let $\mathcal{M}^{\otimes} = \mathcal{C}_0^{\otimes} \times_{\mathcal{A}ss^{\otimes}} \mathcal{RM}^{\otimes}$. Let $\mathrm{Alg}(\mathcal{C}_0)$ and $\mathrm{RMod}(\mathcal{M})$ be defined as in Notation 6.3.3.11. Composition with the forgetful functors $\mathcal{A}ss^{\square} \to \mathcal{A}ss^{\otimes}$ and $\mathcal{LM}^{\square} \to \mathcal{LM}^{\otimes}$ determines restriction maps that fit into a commutative diagram

$$\begin{array}{ccc} \mathrm{RMod}(\mathcal{M}) & \longrightarrow & \mathrm{RMod}^{\square}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{Alg}(\mathcal{C}_0) & \longrightarrow & \mathrm{Alg}^{\square}(\mathcal{C}). \end{array}$$

Using Propositions 4.1.2.15, 4.2.2.11, and T.3.3.1.5, we deduce that the horizontal maps are categorical equivalences.

Remark 6.3.6.5. Let $q : \mathcal{C}^{\otimes} \to \mathbb{E}_1^{\otimes} \times S$ be a coCartesian S-family of $\mathbb{E}_1$-monoidal ∞-categories. Using Example 5.1.0.7, we deduce that q is equivalent to $\mathcal{C}_0^{\otimes} \times_{\mathcal{A}ss^{\otimes}} \mathbb{E}_1^{\otimes}$, for some coCartesian S-family of monoidal ∞-categories $\mathcal{C}_0^{\otimes} \to \mathbb{E}_1^{\otimes} \times S$. It follows that the ∞-categories $\mathrm{RMod}^{\square}(\mathcal{C})$ and $\mathrm{Alg}^{\square}(\mathcal{C})$ can be understood in terms of the constructions described in §6.3.3, as explained in Example 6.3.6.4.

Construction 6.3.6.6. Let $n \geq 1$, let $\mathcal{C}^{\otimes} \to \mathbb{E}_n^{\otimes}$ be a coCartesian fibration of ∞-operads and let $\mathcal{D}^{\otimes} = \mathcal{C}^{\otimes} \times_{\mathbb{E}_n^{\otimes}} (\mathbb{E}_1^{\otimes} \times \mathbb{E}_{n-1}^{\otimes})$, so that $\mathcal{D}^{\otimes} \to \mathbb{E}_1^{\otimes} \times \mathbb{E}_{n-1}^{\otimes}$ is a $\mathbb{E}_{n-1}^{\otimes}$-family of $\mathbb{E}_1$-monoidal ∞-categories. Let $A \in \mathrm{Alg}_{/\mathbb{E}_n}(\mathcal{C})$, so that composition with A determines a section s of the coCartesian fibration $\mathrm{Alg}^{\square}(\mathcal{D}) \to \mathbb{E}_{n-1}^{\otimes}$. We let $\mathrm{RMod}_A^{\square}(\mathcal{C})^{\otimes}$ denote the fiber product $\mathbb{E}_{n-1}^{\otimes} \times_{\mathrm{Alg}^{\square}(\mathcal{D})} \mathrm{RMod}^{\square}(\mathcal{D})$.

Combining Remark 6.3.6.5, Example 6.3.6.4, and Lemma 6.3.3.15, we obtain the following:

Proposition 6.3.6.7. *Let $n \geq 1$ and let $\mathcal{C}^{\otimes} \to \mathbb{E}_n^{\otimes}$ be a coCartesian fibration of ∞-operads. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and that the tensor product $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects. Then the projection map $\mathrm{RMod}_A^{\square}(\mathcal{C})^{\otimes} \to \mathbb{E}_{n-1}^{\otimes}$ is a coCartesian fibration of ∞-operads.*

Remark 6.3.6.8. In the situation of Construction 6.3.6.6, the ∞-category $\mathcal{C}$ admits a monoidal structure and A determines an algebra object $A' \in \mathrm{Alg}(\mathcal{C})$. We have a canonical equivalence of ∞-categories $\mathrm{RMod}_A^\square(\mathcal{C}) \simeq \mathrm{RMod}_{A'}(\mathcal{C})$.

Construction 6.3.6.9. By construction, there is a canonical map $\mathcal{LM}^\square \times \Delta^1 \to \mathbb{E}_1^\otimes$, which determines a functor $\mathcal{LM}^\square \to \mathrm{Fun}(\Delta^1, \mathbb{E}_1^\otimes)$. Combining this with the bifunctor of ∞-operads $\mathbb{E}_1^\otimes \times \mathbb{E}_{n-1}^\otimes \to \mathbb{E}_n^\otimes$ of Construction 5.1.2.1, we obtain a functor

$$\gamma : \mathcal{LM}^\square \times \mathbb{E}_{n-1}^\otimes \to \mathcal{K}_{\mathbb{E}_n} \subseteq \mathrm{Fun}(\Delta^1, \mathbb{E}_n^\otimes),$$

where $\mathcal{K}_{\mathbb{E}_n}$ denotes the full subcategory of $\mathrm{Fun}(\Delta^1, \mathbb{E}_n^\otimes)$ spanned by the semi-inert morphisms (see Notation 3.3.2.1).

Suppose that $\mathcal{C}^\otimes \to \mathbb{E}_n^\otimes$ is a coCartesian fibration of ∞-operads and let $A \in \mathrm{Alg}_{/\mathbb{E}_n}(\mathcal{C})$. Then composition with γ determines a functor

$$\mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C})^\otimes \times_{\mathbb{E}_n^\otimes} \mathbb{E}_{n-1}^\otimes \to \mathrm{RMod}_A^\square(\mathcal{C})^\otimes.$$

We can now state our main result as follows:

Theorem 6.3.6.10. *Let $n \geq 1$, and let $q : \mathcal{C}^\otimes \to \mathbb{E}_n^\otimes$ be a coCartesian fibration of ∞-operads. Assume that the ∞-category $\mathcal{C}$ admits geometric realizations of simplicial objects, and that the tensor product on $\mathcal{C}$ preserves geometric realizations separately in each variable. Then for each $A \in \mathrm{Alg}_{/\mathbb{E}_n}(\mathcal{C})$, the functor*

$$\mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C})^\otimes \times_{\mathbb{E}_n^\otimes} \mathbb{E}_{n-1}^\otimes \to \mathrm{RMod}_A^\square(\mathcal{C})^\otimes$$

is an $\mathbb{E}_{n-1}$-monoidal functor.

Corollary 6.3.6.11. *Let $\mathcal{C}^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a symmetric monoidal ∞-category. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and that the tensor product on $\mathcal{C}$ preserves geometric realizations of simplicial objects. Let $A \in \mathrm{CAlg}(\mathcal{C})$ and let A' be the image of A in $\mathrm{CAlg}(\mathrm{Alg}(\mathcal{C}))$. Then there is a canonical equivalence of symmetric monoidal ∞-categories*

$$\mathrm{Mod}_A(\mathcal{C})^\otimes \simeq \mathrm{RMod}_{A'}(\mathcal{C})^\otimes,$$

where $\mathrm{Mod}_A(\mathcal{C})^\otimes$ is the symmetric monoidal ∞-category of Theorem 4.4.2.1.

Proof. Let $\mathcal{D}^\otimes = \mathcal{C}^\otimes \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathbb{E}_1^\otimes \times \mathrm{N}(\mathcal{F}\mathrm{in}_*))$, and regard $\mathcal{D}^\otimes$ as a coCartesian $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$-family of $\mathbb{E}_1$-monoidal ∞-categories. The commutative algebra A determines a section of the coCartesian fibration $\mathrm{Alg}^\square(\mathcal{D}) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, and set $\mathcal{E}^\otimes = \mathrm{RMod}^\square(\mathcal{D}) \times_{\mathrm{Alg}^\square(\mathcal{D})} \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Arguing as in Example 6.3.6.4, we see that the forgetful functor $\mathcal{LM}^\square \to \mathcal{RM}^\otimes$ induces an equivalence $\mathrm{RMod}_{A'}(\mathcal{C})^\otimes \to \mathcal{E}^\otimes$ of symmetric monoidal ∞-categories. Combining the bifunctor of ∞-operads $\mathbb{E}_1^\otimes \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ with the map map $\mathcal{LM}^\square \to \mathrm{Fun}(\Delta^1, \mathbb{E}_1^\otimes)$ of Construction 6.3.6.9, we obtain a map $\gamma : \mathcal{LM}^\square \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathcal{K}_{\mathrm{Comm}} \subseteq \mathrm{Fun}(\Delta^1, \mathrm{N}(\mathcal{F}\mathrm{in}_*))$. Composition with γ determines a functor $\theta : \mathrm{Mod}_A(\mathcal{C})^\otimes \to \mathcal{E}^\otimes$. We will complete the proof by showing that θ is an equivalence of symmetric monoidal ∞-categories. Using Proposition 4.4.1.4, we deduce that the underlying map $\mathrm{Mod}_A(\mathcal{C}) \to \mathcal{E}$ is an equivalence of ∞-categories. It will therefore suffice to show that the functor θ is symmetric monoidal. Since every morphism in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ can be lifted to a morphism in $\mathbb{E}_1^\otimes$, it suffices to show that the induced map

$$\phi : \mathrm{Mod}_A(\mathcal{C})^\otimes \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathbb{E}_1^\otimes \to \mathcal{E}^\otimes \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathbb{E}_1^\otimes$$

is symmetric monoidal. Let B be the image of A in $\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{C})$. Then ϕ factors as a composition

$$\mathrm{Mod}_A(\mathcal{C})^\otimes \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathbb{E}_1^\otimes \xrightarrow{\phi'} \mathrm{Mod}_B^{\mathbb{E}_2}(\mathcal{C}) \times_{\mathbb{E}_2^\otimes} \mathbb{E}_1^\otimes \xrightarrow{\phi''} \mathrm{RMod}_B^\square(\mathcal{C})^\otimes,$$

where Theorem 6.3.6.10 implies that ϕ'' is an $\mathbb{E}_1$-monoidal functor. The forgetful functor $\mathrm{Mod}_B^{\mathbb{E}_2}(\mathcal{C}) \to \mathrm{Mod}_B^{\mathbb{E}_1}(\mathcal{C})$ is conservative (Corollary 3.4.3.4) and $\mathbb{E}_1$-monoidal (Theorem 5.1.4.2). It will therefore suffice to show that the composite functor $\mathrm{Mod}_A(\mathcal{C})^\otimes \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathbb{E}_1^\otimes \to \mathrm{Mod}_B^{\mathbb{E}_1}(\mathcal{C})^\otimes$ is $\mathbb{E}_1$-monoidal; this follows from Theorem 4.4.2.1 and Example 5.1.0.7. $\square$

Corollary 6.3.6.12. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects. Let $A, B \in \mathrm{Alg}(\mathcal{C})$ be associative algebra objects of $\mathcal{C}$, and let $B^{\mathrm{rev}} \in \mathrm{Alg}(\mathcal{C})$ denote the the opposite algebra of B (see Remark 4.1.1.8). Then there is a canonical equivalence of ∞-categories ${}_A\mathrm{BMod}_B(\mathcal{C}) \simeq \mathrm{LMod}_{A \otimes B^{\mathrm{rev}}}(\mathcal{C})$.*

Proof. Using Theorems 6.3.4.6 and 4.3.2.7, we obtain an equivalence

$$ {}_A\mathrm{BMod}_B(\mathcal{C}) \simeq \mathrm{LMod}_A(\mathrm{RMod}_B(\mathcal{C})) \simeq \mathrm{LMod}_A(\mathcal{C}) \otimes_{\mathcal{C}} \mathrm{RMod}_B(\mathcal{C}) $$

in $\mathrm{Cat}_\infty(\mathcal{K})$. Since the monoidal structure on $\mathcal{C}$ is symmetric, we can identify $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_\infty(\mathcal{K}))$ with $\mathrm{RMod}_{\mathcal{C}}(\mathrm{Cat}_\infty(\mathcal{K}))$. Under this identification, $\mathrm{RMod}_B(\mathcal{C})$ is sent to $\mathrm{LMod}_{B^{\mathrm{rev}}}(\mathcal{C})$. Let

$$ \boxtimes_{\mathcal{C}} : \mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_\infty(\mathcal{K})) \times \mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_\infty(\mathcal{K})) \to \mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_\infty(\mathcal{K})) $$

be the tensor product functor induced by the monoidal functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Using Theorem 4.4.2.1 and Corollary 6.3.6.11, we obtain an equivalence of ∞-categories

$$ \mathrm{LMod}_A(\mathcal{C}) \otimes_{\mathcal{C}} \mathrm{RMod}_B(\mathcal{C}) \simeq \mathrm{LMod}_A(\mathcal{C}) \boxtimes_{\mathcal{C}} \mathrm{LMod}_{B^{\mathrm{rev}}}(\mathcal{C}). $$

Theorem 6.3.5.14 furnishes an equivalence

$$ \mathrm{LMod}_A(\mathcal{C}) \boxtimes_{\mathcal{C}} \mathrm{LMod}_{B^{\mathrm{rev}}}(\mathcal{C}) \simeq \mathrm{LMod}_{A \otimes B^{\mathrm{rev}}}(\mathcal{C}). $$

Composing these functors, we obtain the desired equivalence

$$ {}_A\mathrm{BMod}_B(\mathcal{C}) \simeq \mathrm{LMod}_{A \otimes B^{\mathrm{rev}}}(\mathcal{C}). $$

$\square$

Remark 6.3.6.13. The conclusion of Corollary 6.3.6.12 does not require the full strength of our assumption that $\mathcal{C}$ be symmetric monoidal. It is possible to formulate an analogous result assuming only that $\mathcal{C}$ is an $\mathbb{E}_2$-monoidal ∞-category.

The proof of Theorem 6.3.6.10 will require a simple combinatorial lemma.

Lemma 6.3.6.14. *Let $\star : \mathbf{\Delta} \times \mathbf{\Delta} \to \mathbf{\Delta}$ be the join functor, given by $([m], [n]) \mapsto [m] \star [n] \simeq [m + n + 1]$. Then the induced map*

$$ \mathrm{N}(\mathbf{\Delta}) \times \mathrm{N}(\mathbf{\Delta}) \to \mathrm{N}(\mathbf{\Delta}) $$

is right cofinal.

Proof. Using Theorem T.4.1.3.1, we are reduced to proving the following: let $[n] \in \mathbf{\Delta}$, and let $\mathcal{C} = (\mathbf{\Delta} \times \mathbf{\Delta}) \times_{\mathbf{\Delta}} (\mathbf{\Delta}_{[n]})$ be the category of triples $([m], [m'], \alpha : [m] \star [m'] \to [n])$. Then the simplicial set $\mathrm{N}(\mathcal{C})$ is weakly contractible. Let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ spanned by those triples $([m], [m'], \alpha)$ for which α induces a bijection from $[m]$ to $\{0, 1, \ldots, i\}$ and a bijection from $[m']$ to $\{j, \ldots, n\}$ for some $0 \le i \le j \le n$. The inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ admits a left adjoint, so that the map $\mathrm{N}(\mathcal{C}_0) \to \mathrm{N}(\mathcal{C})$ is a weak homotopy equivalence. We conclude by observing that $\mathcal{C}_0$ has an initial object, given by the triple $([0], [0], \alpha : [1] \to [n])$ with $\alpha(0) = 0$, $\alpha(1) = n$. $\square$

Proof of Theorem 6.3.6.10. It is easy to see that the functor

$$ \mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C})^{\otimes} \times_{\mathbb{E}_n^{\otimes}} \mathbb{E}_{n-1}^{\otimes} \to \mathrm{RMod}_A^{\square}(\mathcal{C})^{\otimes} $$

is a map of ∞-operads which preserves unit objects. If $n = 1$, this completes the proof. For $n \ge 2$, we note that every morphism in $\mathbb{E}_{n-1}^{\otimes}$ is equivalent to the image of a morphism in $\mathbb{E}_1^{\otimes}$. It therefore suffices to show

that the induced functor $\theta : \mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C})^{\otimes} \times_{\mathbb{E}_n^{\otimes}} \mathbb{E}_1^{\otimes} \to \mathrm{RMod}_A^{\square}(\mathcal{C})^{\otimes} \times_{\mathbb{E}_{n-1}^{\otimes}} \mathbb{E}_1^{\otimes}$ is $\mathbb{E}_1$-monoidal. Let A' denote the image of A in $\mathrm{Alg}_{\mathbb{E}_2/\mathbb{E}_n}(\mathcal{C})$. The functor θ factors as a composition

$$\mathrm{Mod}_A^{\mathbb{E}_n}(\mathcal{C})^{\otimes} \times_{\mathbb{E}_n^{\otimes}} \mathbb{E}_1^{\otimes} \xrightarrow{\theta'} \mathrm{Mod}_{A'}^{\mathbb{E}_2}(\mathcal{C})^{\otimes} \times_{\mathbb{E}_2^{\otimes}} \mathbb{E}_1^{\otimes} \xrightarrow{\theta''} \mathrm{RMod}_{A'}^{\square}(\mathcal{C})^{\otimes} \simeq \mathrm{RMod}_A^{\square}(\mathcal{C})^{\otimes} \times_{\mathbb{E}_{n-1}^{\otimes}} \mathbb{E}_1^{\otimes},$$

where θ' is an $\mathbb{E}_1$-monoidal functor by Theorem 5.1.4.2. It will therefore suffice to show that θ'' is $\mathbb{E}_1$-monoidal. We may therefore replace A by A' and thereby reduce to the case $n = 2$.

Let $p : \mathrm{Mod}_A^{\mathbb{E}_2}(\mathcal{C})^{\otimes} \to \mathbb{E}_2^{\otimes}$ and $p' : \mathrm{RMod}_A^{\square}(\mathcal{C})^{\otimes} \to \mathbb{E}_1^{\otimes}$ be the projection maps. We wish to show that if α is a p-coCartesian morphism in $\mathrm{Mod}_A^{\mathbb{E}_2}(\mathcal{C})^{\otimes}$ whose image in $\mathbb{E}_2^{\otimes}$ is contained in $\mathbb{E}_1^{\otimes}$, then $\theta(\alpha)$ is a p'-coCartesian morphism in $\mathrm{RMod}_A^{\square}(\mathcal{C})^{\otimes}$. Since it suffices to show that θ preserves tensor products of pairs of object, we may assume without loss of generality that α covers the map $\alpha_0 : \langle 2 \rangle \to \langle 1 \rangle$ in $\mathbb{E}_2^{\otimes}$ corresponding to the rectilinear embedding

$$\square^2 \times \langle 2 \rangle^{\circ} \simeq ((-1,0) \times (-1,1)) \coprod ((0,1) \times (-1,1)) \hookrightarrow \square^2.$$

Then α_0 determines a map $\Delta^1 \to \mathbb{E}_2^{\otimes}$. We let $\mathcal{K}$ denote the fiber product $\Delta^1 \times_{\mathbb{E}_2^{\otimes}} \mathcal{K}_{\mathbb{E}_2}$, so that α determines a functor $F : \mathcal{K} \to \mathcal{C}^{\otimes}$.

We now introduce an auxiliary construction. We define a topological category $\mathcal{E}$ as follows:

(a) The set of objects of $\mathcal{E}$ is given by $\{X, Z\} \cup \{Y_n\}_{n \geq 0}$.

(b) Morphism spaces in $\mathcal{E}$ are given by

$$\mathrm{Map}_{\mathcal{E}}(X, X) = \mathrm{Map}_{\mathcal{E}}(Z, Z) = * \qquad \mathrm{Map}_{\mathcal{E}}(Z, Y_n) \simeq \mathrm{Map}_{\mathcal{E}}(Z, X) \simeq \mathrm{Map}_{\mathcal{E}}(Y_n, X) = \emptyset$$

$$\mathrm{Map}_{\mathcal{E}}(X, Y_n) = \mathrm{Rect}(\square^2 \times \{-\infty\}, \square^2 \times \{-\infty\}) \times \mathrm{Rect}(\square^2 \times \{\infty\}, \square^2 \times \{\infty\})$$

$$\mathrm{Map}_{\mathcal{E}}(X, Z) = \mathrm{Rect}(\square^2 \times \{-\infty, \infty\}, \square^2)$$

$$\mathrm{Map}_{\mathcal{E}}(Y_n, Z) = \mathrm{Rect}(\square^2 \times \{-\infty, -n, -n+1, \ldots, n, \infty\}, \square^2).$$

If $m, n \geq 0$, then $\mathrm{Map}_{\mathcal{E}}(Y_m, Y_n)$ is the disjoint union, over all nondecreasing maps $\beta : [n] \to [m]$, of the product of the spaces

$$\mathrm{Rect}(\square^2 \times \{-\infty, -m, \ldots, -\beta(n) + 1\}, \square^2 \times \{-\infty\})$$

$$\prod_{1 \leq i \leq n} \mathrm{Rect}(\square^2 \times \{-\beta(i), \ldots, -\beta(i-1) + 1\}, \square^2 \times \{-i\})$$

$$\mathrm{Rect}(\square^2 \times \{-\beta(0), -\beta(0) + 1, \ldots, \beta(0)\}, \square^2 \times \{0\})$$

$$\prod_{1 \leq i \leq n} \mathrm{Rect}(\square^2 \times \{\beta(i-1) + 1, \ldots, \beta(i)\}, \square^2 \times \{i\})$$

$$\mathrm{Rect}(\square^2 \times \{\beta(n) + 1, \ldots, m, \infty\}, \square^2 \times \{\infty\}),$$

(which we can regard as a subspace of the collection of all maps from $\square^2 \times \{-\infty, -m, \ldots, m, \infty\}$ to $\square^2 \times \{-\infty, -n, \ldots, n, \infty\}$).

(c) Composition of morphisms is given by composition of rectilinear embeddings.

There is an evident forgetful functor $\psi : \mathcal{E} \to^t \mathbb{E}_2^{\otimes}$, and α_0 lifts uniquely to a morphism $\overline{\alpha}_0 : X \to Z$ in $\mathcal{E}$. We define a topological subcategory $\mathcal{E}^0 \subseteq \mathcal{E}$, having the same objects as $\mathcal{E}$, as follows:

- A rectilinear embedding $f : \square^2 \times \{-\infty, \infty\} \to \square^k$ belongs to $\operatorname{Map}_{\mathcal{E}^0}(X, Z)$ if and only if there is a rectilinear embedding $f_0 : (-1, 1) \to (-1, 1)$ such that

$$f(s, t, -\infty) = (\frac{s-1}{2}, f_0(t)) \qquad f(s, t, \infty) = (\frac{s+1}{2}, f_0(t)).$$

- A pair of rectilinear embeddings $f_- : \square^2 \times \{-\infty\} \to \square^2 \times \{-\infty\}$, $f_+ : \square^2 \times \{\infty\} \to \square^2 \times \{\infty\}$ belongs to $\operatorname{Map}_{\mathcal{E}^0}(X, Y_n)$ if and only if there is a rectilinear embedding $f_0 : (-1, 1) \to (-1, 1)$ such that $f_+(s, t) = (s, f_0(t)) = f_-(s, t)$.

- A rectilinear embedding $f : \square^2 \times \{-\infty, -n, \ldots, n, \infty\} \to \square^2$ belongs to $\operatorname{Map}_{\mathcal{E}^0}(Y_n, Z)$ if and only if there exists a rectilinear embedding $f_0 : \square^1 \times \{-\infty, -n, \ldots, 0\} \to \square^1$ such that $f_0(t, i) < f_0(t, j)$ for $i < j$, and

$$f(s, t, i) = \begin{cases} (\frac{s-1}{2}, f_0(t, i)) & \text{if } i < 0 \\ (s, f_0(t, i)) & \text{if } i = 0 \\ (\frac{s+1}{2}, f_0(t, -i) & \text{if } i > 0. \end{cases}$$

- If $f : \square^2 \times \{-\infty, -m, \ldots, m, \infty) \to \square^2 \times \{-\infty, -n, \ldots, n, \infty\}$ is a morphism from Y_m to Y_n in $\mathcal{E}$ covering a map of linearly ordered sets $\beta : [n] \to [m]$, then then f belongs to $\mathcal{E}^0$ if and only if there exists a rectilinear embedding $f_0 : \square^1 \times \{-\infty, -m, \ldots, 0) \to \square^1 \times \{-\infty, -n, \ldots, 0\}$ such that $f_0(t, i) < f_0(t, j)$ for $i < j$ and

$$f(s, t, i) = \begin{cases} (s, f_0(t, i)) & \text{if } i < -\beta(0) \\ (\frac{s-1}{2}, f_0(t, i)) & \text{if } -\beta(0) \le i < 0 \\ (s, f_0(t, i)) & \text{if } i = 0 \\ (\frac{s+1}{2}, f_0(t, -i) & \text{if } 0 < i \le \beta(0) \\ (s, f_0(t, -i)) & \text{if } \beta(0) < i. \end{cases}$$

Note that $\overline{\alpha}_0$ is a morphism from X to Z in $\mathcal{E}^0$. Let $\mathcal{X}^0$ be the full subcategory of $(\mathrm{N}(\mathcal{E}^0)_{/Z})^{X/}$ spanned by those diagrams of the form

$$\begin{array}{ccc} & Y_n & \\ & \nearrow \quad \searrow & \\ X & \xrightarrow{\ \overline{\alpha}_0\ } & Z. \end{array}$$

There is an evident forgetful functor $\mathcal{X}^0 \to \mathrm{N}(\mathbf{\Delta})^{op}$ which is an equivalence of ∞-categories. The functor $F : \mathcal{K} \to \mathcal{C}^\otimes$ induces a functor $\delta^0 : (\mathcal{X}^0)^{\triangleright} \to \mathcal{C}^\otimes$. Using Lemma 6.3.3.15, we see that $\theta(\alpha)$ is q-coCartesian if and only if δ^0 is an operadic q-colimit diagram.

We now define another topological subcategory $\mathcal{E}^v \subseteq \mathcal{E}$. Let $\sigma : \square^2 \to \square^2$ be the reflection given by $\sigma(s, t) = (-s, t)$.

- A rectilinear embedding $f : \square^2 \times \{-\infty, \infty\} \to \square^k$ belongs to $\operatorname{Map}_{\mathcal{E}^v}(X, Z)$ if and only if there exist rectilinear embeddings $f_0, f_1 : \square^1 \to \square^1$

$$f(s, t, -\infty) = (\frac{-f_0(-s) - 1}{2}, f_1(t)) \qquad f(s, t, \infty) = (\frac{f_0(s) + 1}{2}, f_1(t)).$$

- A pair of rectilinear embeddings $f_- : \square^2 \times \{-\infty\} \to \square^2 \times \{-\infty\}$, $f_+ : \square^2 \times \{\infty\} \to \square^2 \times \{\infty\}$ belongs to $\operatorname{Map}_{\mathcal{E}^v}(X, Y_n)$ if and only if $f_+ \circ \sigma = \sigma \circ f_-$.

- Let $\pi_0, \pi_1 : \square^2 \to \square^1$ be the projection onto the first and second factors, respectively. A rectilinear embedding $f : \square^2 \times \{-\infty, -n, \ldots, n, \infty\} \to \square^2$ belongs to $\operatorname{Map}_{\mathcal{E}^v}(Y_n, Z)$ if and only if $\sigma f(s, t, i) = f(-s, t, -i)$, $\pi_1 f(s, t, i) < \pi_1 f(s', t', i')$ when $i < i' \le 0$, and $\pi_0 f(s, t, i) < 0$ for $i < 0$ (so that $\pi_0 f(s, t, i) > 0$ for $i > 0$).

- If $f : \square^2 \times \{-\infty, -m, \ldots, m, \infty) \to \square^2 \times \{-\infty, -n, \ldots, n, \infty\}$ is a morphism from Y_m to Y_n in $\mathcal{E}$ covering a map of linearly ordered sets $\beta : [n] \to [m]$ given by a collection of maps $\{f_i : \square^2 \to \square^2\}_{i \in \{-\infty, -m, \ldots, m, \infty\}}$, then f belongs to $\mathcal{E}^v$ if and only if $\sigma \circ f_i = f_{-i} \circ \sigma$ and $\pi_1 f_i(s, t) < \pi_1 f_{i'}(s', t')$ whenever $i < i' \le 0$ and f carries $\square^2 \times \{i\}$ and $\square^2 \times \{i'\}$ to the same connected component of $\square^2 \times \{-\infty, -n, \ldots, n, \infty\}$.

Note that $\mathcal{E}^v$ contains $\mathcal{E}^0$, and that the inclusion $\mathcal{E}^0 \hookrightarrow \mathcal{E}^v$ is a weak equivalence of topological categories. Let $\mathfrak{X}^v$ be the full subcategory of $(\mathrm{N}(\mathcal{E}^v)_{/Z})^{X/}$ spanned by those diagrams of the form

$$
\begin{array}{ccc}
 & Y_n & \\
 \nearrow & & \searrow \\
X & \xrightarrow{\;\;\overline{\alpha}_0\;\;} & Z.
\end{array}
$$
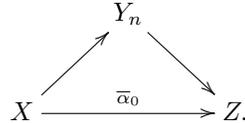
Then F determines a functor $\delta^v : (\mathfrak{X}^v)^{\triangleright} \to \mathcal{C}^{\otimes}$. Since the inclusion $\mathfrak{X}^0 \hookrightarrow \mathfrak{X}^v$ is a categorical equivalence, we are reduced to proving that δ^v is an operadic q-colimit diagram.

We define another subcategory $\mathcal{E}^h \subseteq \mathcal{E}$ as follows:

- A rectilinear embedding $f : \square^2 \times \{-\infty, \infty\} \to \square^k$ belongs to $\mathrm{Map}_{\mathcal{E}^h}(X, Z)$ if and only if it belongs to $\mathcal{E}^v$.

- A pair of rectilinear embeddings $f_- : \square^2 \times \{-\infty\} \to \square^2 \times \{-\infty\}$, $f_+ : \square^2 \times \{\infty\} \to \square^2 \times \{\infty\}$ belongs to $\mathrm{Map}_{\mathcal{E}^h}(X, Y_n)$ if and only if $f_+ \circ \sigma = \sigma \circ f_-$.

- A rectilinear embedding $f : \square^2 \times \{-\infty, -n, \ldots, n, \infty\} \to \square^2$ belongs to $\mathrm{Map}_{\mathcal{E}^h}(Y_n, Z)$ if and only if $\sigma f(s, t, i) = f(-s, t, -i)$ and $\pi_0 f(s, t, i) < \pi_0 f(s', t', i')$ when $i < i'$.

- If $f : \square^2 \times \{-\infty, -m, \ldots, m, \infty) \to \square^2 \times \{-\infty, -n, \ldots, n, \infty\}$ is a morphism from Y_m to Y_n in $\mathcal{E}$ covering a map of linearly ordered sets $\beta : [n] \to [m]$ given by a collection of maps $\{f_i : \square^2 \to \square^2\}_{i \in \{-\infty, -m, \ldots, m, \infty\}}$, then f belongs to $\mathcal{E}^v$ if and only if $\sigma \circ f_i = f_{-i} \circ \sigma$ and $\pi_0 f_i(s, t) < \pi_0 f_{i'}(s', t')$ whenever $i < i'$ and f carries $\square^2 \times \{i\}$ and $\square^2 \times \{i'\}$ to the same component of $\square^2 \times \{-\infty, -n, \ldots, n, \infty\}$.

Let $\mathcal{E}^{h,v} = \mathcal{E}^h \cap \mathcal{E}^v$. We observe that the inclusions

$$\mathcal{E}^v \hookleftarrow \mathcal{E}^{h,v} \hookrightarrow \mathcal{E}^v$$

are weak equivalences of topological categories. Let $\mathfrak{X}^h$ be the full subcategory of $(\mathrm{N}(\mathcal{E}^h)_{/Z})^{X/}$ spanned by those diagrams of the form

$$
\begin{array}{ccc}
 & Y_n & \\
 \nearrow & & \searrow \\
X & \xrightarrow{\;\;\overline{\alpha}_0\;\;} & Z,
\end{array}
$$
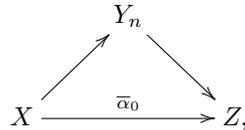
and let $\mathfrak{X}^{h,v}$ be defined similarly. Then F determines a functor $\delta^h : (\mathfrak{X}^h)^{\triangleright} \to \mathcal{C}^{\otimes}$. Since $\delta^h | (\mathfrak{X}^{h,v})^{\triangleright} = \delta^v | (\mathfrak{X}^{h,v})^{\triangleright}$ and the inclusions $\mathfrak{X}^h \hookleftarrow \mathfrak{X}^{h,v} \hookrightarrow \mathfrak{X}^v$ are categorical equivalences, we conclude that δ^v is an operadic q-colimit diagram if and only if δ^h is an operadic q-colimit diagram.

Let $\mathcal{E}^1$ be the topological subcategory of $\mathcal{E}^h$ whose morphisms are given by rectilinear embeddings which commute with projection onto the second coordinate. The inclusion $\mathcal{E}^1 \hookrightarrow \mathcal{E}^h$ is a weak equivalence of topological categories. Let $\mathfrak{X}^1$ be the full subcategory of $(\mathrm{N}(\mathcal{E}^1)_{/Z})^{X/}$ spanned by those diagrams of the form

$$
\begin{array}{ccc}
 & Y_n & \\
 \nearrow & & \searrow \\
X & \xrightarrow{\;\;\overline{\alpha}_0\;\;} & Z,
\end{array}
$$
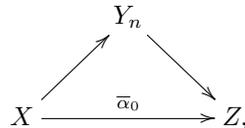
so that the inclusion $\mathfrak{X}^1 \to \mathfrak{X}^h$ is a categorical equivalence. We are therefore reduced to proving that $\delta^1 = \delta^h|(\mathfrak{X}^1)^{\triangleright}$ is an operadic q-colimit diagram.

Let $\mathcal{Y}$ denote the full subcategory of $((\mathbb{E}_1)_{\langle 1 \rangle})^{\langle 2 \rangle /}$ spanned by those diagrams

$$
\begin{array}{ccc}
 & \langle n \rangle & \\
\gamma \nearrow & & \searrow \gamma' \\
\langle 2 \rangle & \xrightarrow{\ \ \alpha_0\ \ } & \langle 1 \rangle
\end{array}
$$

such that γ is semi-inert, γ' is active, and image of γ contains the smallest and largest elements of $\langle n \rangle^{\circ}$ (with respect to the ordering induced by γ'). The map δ^1 factors as a composition

$$(\mathfrak{X}^1)^{\triangleright} \to \mathcal{Y}^{\triangleright} \xrightarrow{\delta'} \mathcal{C}^{\otimes}.$$

Since α is p-coCartesian, the proof of Theorem 5.1.4.2 shows that δ' is an operadic q-colimit diagram. It will therefore suffice to show that the map $\mathfrak{X}^1 \to \mathcal{Y}$ is left cofinal. We have a commutative diagram of ∞-categories

$$
\begin{array}{ccc}
\mathfrak{X}^1 & \longrightarrow & \mathcal{Y} \\
\downarrow & & \downarrow \\
\mathrm{N}(\boldsymbol{\Delta})^{op} & \xrightarrow{\ \ u\ \ } & \mathrm{N}(\boldsymbol{\Delta})^{op}
\end{array}
$$

where the vertical maps are categorical equivalences and u is the functor given by $[n] \mapsto [n] \star [n]^{op} \simeq [2n+1]$. The functor u factors as a composition

$$\mathrm{N}(\boldsymbol{\Delta})^{op} \xrightarrow{u'} \mathrm{N}(\boldsymbol{\Delta})^{op} \times \mathrm{N}(\boldsymbol{\Delta})^{op} \xrightarrow{u''} \mathrm{N}(\boldsymbol{\Delta})^{op} \times \mathrm{N}(\boldsymbol{\Delta})^{op} \xrightarrow{u'''} \mathrm{N}(\boldsymbol{\Delta})^{op}.$$

The map u' is left cofinal by Lemma T.5.5.8.4), the map u'' is an isomorphism (given by order-reversal on the second coordinate), and the map u''' is induced by the join operation which is a left cofinal functor (Lemma 6.3.6.14). It follows that u is left cofinal so that $\mathfrak{X}^1 \to \mathcal{Y}$ is cofinal, as desired. $\qquad\square$

6.3.7   The Adjoint Representation

Let A be an associative ring, and let $A^{\times}$ be the collection of units in A. Then $A^{\times}$ forms a group, which acts on A by conjugation. This action is given by a group homomorphism $\phi : A^{\times} \to \mathrm{Aut}(A)$ whose kernel is the subgroup of $A^{\times}$ consisting of units which belong to the center: this group can be identified with the group of units of the center $\mathfrak{Z}(A)$. In other words, we have an exact sequence of groups

$$0 \to \mathfrak{Z}(A)^{\times} \to A^{\times} \to \mathrm{Aut}(A).$$

Our goal in this section is to prove a result which generalizes this statement in the following ways:

(a) In place of a single associative ring A, we will consider instead a map of algebras $f : A \to B$. In this setting, we will replace the automorphism group $\mathrm{Aut}(A)$ by the set $\mathrm{Hom}(A, B)$ of algebra homomorphisms from A to B. This set is acted on (via conjugation) by the group $B^{\times}$ of units in B. Moreover, the stabilizer of the element $f \in \mathrm{Hom}(A, B)$ can be identified with the group of units $\mathfrak{Z}(f)^{\times}$ of the centralizer of the image of f. In particular, we have an exact sequence of pointed sets

$$\mathfrak{Z}(f)^{\times} \hookrightarrow B^{\times} \to \mathrm{Hom}(A, B).$$

(b) Rather than considering rings (which are associative algebra objects of the category of abelian groups), we will consider algebra objects in an arbitrary symmetric monoidal ∞-category $\mathcal{C}$. In this setting, we

need to determine appropriate analogues of the sets $\mathfrak{Z}(f)^\times$, $B^\times$, and $\text{Hom}(A, B)$ considered above. In the last case this is straightforward: the analogue of the set $\text{Hom}(A, B)$ of ring homomorphisms from A to B is the *space* $\text{Map}_{\text{Alg}(\mathcal{C})}(A, B)$ of morphisms in the ∞-category $\text{Alg}(\mathcal{C})$. Note that the spaces $\text{Map}_{\mathcal{C}}(\mathbf{1}, \mathfrak{Z}(f))$ and $\text{Map}_{\mathcal{C}}(\mathbf{1}, B)$ are equipped with (coherently) associative multiplications (see Definition 6.3.7.4), so that the sets $\pi_0 \text{Map}_{\mathcal{C}}(\mathbf{1}, \mathfrak{Z}(f))$ and $\pi_0 \text{Map}_{\mathcal{C}}(\mathbf{1}, B)$ have the structure of associative monoids. We let $\mathfrak{Z}(f)^\times \subseteq \text{Map}_{\mathcal{C}}(\mathbf{1}, \mathfrak{Z}(f))$ be the union of those connected components corresponding to invertible elements of $\pi_0 \text{Map}_{\mathcal{C}}(\mathbf{1}, \mathfrak{Z}(f))$, and let and $B^\times \subseteq \text{Map}_{\mathcal{C}}(\mathbf{1}, B)$ be defined similarly.

The collection of units in an associative ring R is equipped with the structure of group (with respect to multiplication). We will see that there is an analogous structure on the space of units $B^\times$ for an associative algebra object B of an arbitrary symmetric monoidal ∞-category $\mathcal{C}$: namely, $B^\times$ is a loop space. That is, there exists a pointed space $X(B)$ and a homotopy equivalence $B^\times \simeq \Omega X(B)$. There is an "action" of the loop space $B^\times$ on the mapping space $\text{Map}_{\text{Alg}(\mathcal{C})}(A, B)$. This action is encoded by a fibration $X(A, B) \to X(B)$, whose homotopy fiber (over the base point of $X(B)$) can be identified with $\text{Map}_{\text{Alg}(\mathcal{C})}(A, B)$. In particular, a morphism of associative algebra objects $f : A \to B$ determines a base point of $X(A, B)$, and we will see that the loop space $\Omega X(A, B)$ can be identified with the the space of units $\mathfrak{Z}(f)^\times$. In other words, we have a fiber sequence of spaces

$$\text{Map}_{\text{Alg}(\mathcal{C})}(A, B) \to X(A, B) \to X(B)$$

which, after looping the base and total space, yields a fiber sequence

$$\mathfrak{Z}(f)^\times \to B^\times \to \text{Map}_{\text{Alg}(\mathcal{C})}(A, B)$$

analogous to the exact sequence of sets described in (a).

(c) Instead of considering only associative algebras, we will consider algebras over an arbitrary little cubes operad $\mathbb{E}_k$ (according to Example 5.1.0.7, we can recover the case of associative algebras by setting $k = 1$). If B is an $\mathbb{E}_k$-algebra object of a symmetric monoidal ∞-category $\mathcal{C}$, then we can again define a space of units $B^\times \subseteq \text{Map}_{\mathcal{C}}(\mathbf{1}, B)$. The space $B^\times$ has the structure of a k-fold loop space: that is, one can define a pointed space $X(B)$ and a homotopy equivalence $B^\times \simeq \Omega^k X(B)$. If A is another $\mathbb{E}_k$-algebra object of $\mathcal{C}$, then there exists a fibration $X(A, B) \to X(B)$ whose fiber (over a well-chosen point of $X(B)$) can be identified with $\text{Map}_{\text{Alg}_{\mathbb{E}_k}(\mathcal{C})}(A, B)$. In particular, every $\mathbb{E}_k$-algebra map $f : A \to B$ determines a base point of the total space $X(A, B)$, and the k-fold loop space $\Omega^k X(A, B)$ can be identified with the space of units $\mathfrak{Z}_{\mathbb{E}_k}(f)^\times$ (see Definition 6.3.7.4 below). We therefore have a fiber sequence of spaces

$$\text{Map}_{\text{Alg}_{\mathbb{E}_k}(\mathcal{C})}(A, B) \to X(A, B) \to X(B)$$

which yields, after passing to loop spaces repeatedly, a fiber sequence

$$\mathfrak{Z}_{\mathbb{E}_k}(f)^\times \to B^\times \xrightarrow{\phi} \Omega^{k-1} \text{Map}_{\text{Alg}_{\mathbb{E}_k}(\mathcal{C})}(A, B).$$

We should regard the map ϕ as a k-dimensional analogue of the adjoint action of the unit group $B^\times$ of an associative ring B on the set of maps $\text{Hom}(A, B)$.

Our first step is to *define* the spaces of units appearing in the above discussion. This requires a bit of a digression.

Definition 6.3.7.1. Let $\mathcal{O}^\otimes$ be an ∞-operad, and let $\mathcal{O}^\otimes_* \subseteq \text{Fun}(\Delta^1, \mathcal{O}^\otimes)$ be the ∞-category of pointed objects of $\mathcal{O}^\otimes$. The forgetful functor $q : \mathcal{O}^\otimes_* \to \mathcal{O}^\otimes$ is a left fibration of simplicial sets. We let $\chi_{\mathcal{O}} : \mathcal{O}^\otimes \to \mathcal{S}$ denote a functor which classifies q.

Proposition 6.3.7.2. *Let* $q : \mathcal{O}^\otimes \to \text{N}(\mathcal{F}\text{in}_*)$ *be an* ∞-*operad and let* $\chi_{\mathcal{O}} : \mathcal{O}^\otimes \to \mathcal{S}$ *be as in Definition 6.3.7.1. Then* $\chi_{\mathcal{O}}$ *is a* $\mathcal{O}$-*monoid object of* $\mathcal{S}$.

Proof. We must show that if $X \in \mathcal{O}_{\langle n \rangle}^{\otimes}$, and if $\alpha_i : X \to X_i$ are a collection of inert morphisms in $\mathcal{O}^{\otimes}$ lifting the maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \leq i \leq n$, then the induced map $\chi_{\mathcal{O}}(X) \to \prod_{1 \leq i \leq n} \chi_{\mathcal{O}}(X_i)$ is a homotopy equivalence. Let 0 denote a final object of $\mathcal{O}^{\otimes}$; then the left hand side is homotopy equivalent to $\mathrm{Map}_{\mathcal{O}^{\otimes}}(0, X)$, while the right hand side is homotopy equivalent to $\prod_{1 \leq i \leq n} \mathrm{Map}_{\mathcal{O}^{\otimes}}(0, X_i)$. The desired result now follows from the observation that the maps α_i exhibit X as a q-product of the objects $\{X_i\}_{1 \leq i \leq n}$. $\square$

Remark 6.3.7.3. An ∞-operad $\mathcal{O}^{\otimes}$ is unital if and only if the functor $\chi_{\mathcal{O}} : \mathcal{O}^{\otimes} \to \mathcal{S}$ is equivalent to the constant functor taking the value Δ^0.

Definition 6.3.7.4. Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, where $\mathcal{O}^{\otimes}$ is unital, and let $\chi_{\mathcal{C}} : \mathcal{C}^{\otimes} \to \mathcal{S}$ be as in Definition 6.3.7.1. Composition with $\chi_{\mathcal{C}}$ determines a functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{S})$.

Suppose that $\mathcal{O}^{\otimes} = \mathbb{E}_k^{\otimes}$, where $k > 0$. Since the collection of grouplike $\mathbb{E}_k$-spaces is stable under colimits in $\mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$ (Remark 5.1.3.5) the inclusion $i : \mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}(\mathcal{S}) \subseteq \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$ preserves small colimits. It follows from Proposition 5.1.3.6 that $\mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}(\mathcal{S})$ is equivalent to $\mathcal{S}_*^{\geq k}$, and therefore presentable. Using Corollary T.5.5.2.9, we deduce that the inclusion functor i admits a right adjoint G. We let $\mathrm{GL}_1 : \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}) \to \mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}(\mathcal{S})$ denote the composite functor

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \overset{\chi_{\mathcal{C}} \circ}{\to} \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{C}) \overset{G}{\to} \mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}(\mathcal{C}).$$

If $A \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$, we will often write $A^{\times}$ in place of $\mathrm{GL}_1(A)$; we will refer to $A^{\times}$ as the $\mathbb{E}_k$-*space of units in* A.

In the special case $k = 0$, we let $\mathrm{GL}_1 : \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}) \to \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S}) \simeq \mathcal{S}_*$ be the functor defined by composition with $\chi_{\mathcal{C}}$; we will also denote this functor by $A \mapsto A^{\times}$.

We are now prepared to state our main result:

Theorem 6.3.7.5. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Assume that the underlying ∞-category $\mathcal{C}$ is presentable and that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. Fix an integer $k \geq 0$, and let $\mathrm{Map} : \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})^{op} \times \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}) \to \mathcal{S}$ be the adjoint of the Yoneda embedding $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}) \to \mathrm{Fun}(\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})^{op}, \mathcal{S})$. There exists another functor $X : \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})^{op} \times \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}) \to \mathcal{S}$ and a natural transformation $\alpha : \mathrm{Map} \to X$ with the following properties:*

(1) *For every object $B \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$ and every morphism $f : A' \to A$ in $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$, the diagram*

$$\begin{array}{ccc} \mathrm{Map}(A, B) & \longrightarrow & \mathrm{Map}(A', B) \\ \downarrow & & \downarrow \\ X(A, B) & \longrightarrow & X(A', B) \end{array}$$

   is a pullback square.

(2) *Let $f : A \to B$ be a morphism in $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$, so that the map f determines a base point of the space $X(A, B)$ (via α). Then there is a canonical homotopy equivalence $\Omega^k X(A, B) \simeq \mathfrak{Z}_{\mathbb{E}_k}(f)^{\times}$.*

Remark 6.3.7.6. In the situation of Theorem 6.3.7.5, it suffices to prove assertion (1) in the case where A' is the initial object $\mathbf{1} \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$. This follows by applying Lemma T.4.4.2.1 to the diagram

$$\begin{array}{ccccc} \mathrm{Map}(A, B) & \longrightarrow & \mathrm{Map}(A', B) & \longrightarrow & \mathrm{Map}(\mathbf{1}, B) \\ \downarrow & & \downarrow & & \downarrow \\ X(A, B) & \longrightarrow & X(A', B) & \longrightarrow & X(\mathbf{1}, B). \end{array}$$

Remark 6.3.7.7. In the special case where A' is the initial object $\mathbf{1} \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$, the space $\mathrm{Map}(A', B)$ is contractible, so that part (1) of Theorem 6.3.7.5 asserts the existence of a fiber sequence

$$\mathrm{Map}(A, B) \to X(A, B) \to X(\mathbf{1}, B).$$

Fixing a base point $(f : A \to B) \in \mathrm{Map}(A, B)$ and taking loop spaces repeatedly, we have a fiber sequence

$$\Omega^k X(A, B) \to \Omega^k X(\mathbf{1}, B) \to \Omega^{k-1} \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})}(A, B)$$

We observe that there is a canonical natural transformation $\beta : \mathbf{3}_{\mathbb{E}_k}(f_0) \to B$ of functors $\mathbb{E}_k^{\otimes} \to \mathcal{C}^{\otimes}$. The natural transformation β induces an equivalence of $\mathbb{E}_k$-spaces $\mathbf{3}_{\mathbb{E}_k}(f_0)^{\times} \to B^{\times}$. Invoking part (2) of Theorem 6.3.7.5, we obtain the fiber sequence

$$\mathbf{3}_{\mathbb{E}_k}(f)^{\times} \to B^{\times} \to \Omega^{k-1} \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})}(A, B)$$

described in (c).

An $\mathbb{E}_k$-algebra object A of a symmetric monoidal ∞-category $\mathcal{C}$ determines an (∞, k)-category $C(A)$ enriched over $\mathcal{C}$ (having a single j-morphism for each $j < k$). One approach to the proof of Theorem 6.3.7.5 would be to define $X(A, B)$ to be the space of functors from $C(A)$ into $C(B)$. Since we have not developed the theory of enriched (∞, k)-categories in this book, our proof will proceed along somewhat different lines. We will use an inductive approach, which iteratively replaces the ∞-category $\mathcal{C}$ by the ∞-category $\mathrm{LMod}_{\mathcal{C}}$ of ∞-categories left-tensored over $\mathcal{C}$. To guarantee that this replacement does not destroy our hypothesis that $\mathcal{C}$ is presentable, we need to introduce a few restrictions on the $\mathcal{C}$-modules that we allow.

Notation 6.3.7.8. Let κ be a regular cardinal. Recall that a presentable ∞-category $\mathcal{C}$ is *κ-compactly generated* if $\mathcal{C}$ is generated by its κ-compact objects under the formation of small, κ-filtered colimits (see §T.5.5.7). If $\mathcal{C}$ and $\mathcal{D}$ are κ-compactly generated ∞-categories, then we will say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *κ-good* if F preserves small colimits and carries κ-compact objects of $\mathcal{C}$ to κ-compact objects of $\mathcal{D}$. Equivalently, F is κ-good if F admits a right adjoint G which commutes with κ-filtered colimits (Proposition T.5.5.7.2).

Let Pr^{L} denote the ∞-category of presentable ∞-categories and colimit-preserving functors. We let $\mathrm{Pr}^{\mathrm{L}}_{\kappa}$ denote the subcategory of the ∞-category Pr^{L} whose objects are κ-compactly generated presentable ∞-categories and whose morphisms are κ-good functors.

Lemma 6.3.7.9. *Let κ be a regular cardinal. Then:*

(1) *Let $\mathcal{K}$ denote the collection of all κ-small simplicial sets together with the simplicial set* Idem *introduced in §T.4.4.5. Then the functor $\mathcal{C} \mapsto \mathrm{Ind}_{\kappa}(\mathcal{C})$ determines an equivalence of ∞-categories from $\mathrm{Cat}_{\infty}(\mathcal{K})$ to $\mathrm{Pr}^{\mathrm{L}}_{\kappa}$.*

(2) *The ∞-category $\mathrm{Pr}^{\mathrm{L}}_{\kappa}$ is presentable.*

(3) *The inclusion functor $\mathrm{Pr}^{\mathrm{L}}_{\kappa} \hookrightarrow \mathrm{Pr}^{\mathrm{L}}$ preserves small colimits.*

Remark 6.3.7.10. In the situation of Lemma 6.3.7.9, the objects of $\mathrm{Cat}_{\infty}(\mathcal{K})$ are idempotent complete ∞-categories which admit κ-small colimits, and the morphisms in $\mathrm{Cat}_{\infty}(\mathcal{K})$ are functors which preserve κ-small colimits. If κ is uncountable, then the requirement of idempotent completeness is automatically satisfied.

Proof. We will prove assertion (1); assertion (2) will then follow from (1) and Lemma 6.3.4.2, and assertion (3) from the observation that the functor $\mathrm{Ind}_{\kappa} : \mathrm{Cat}_{\infty}(\mathcal{K}) \to \mathrm{Pr}^{\mathrm{L}}$ preserves small colimits. It is clear that the functor $\mathrm{Ind}_{\kappa} : \mathrm{Cat}^0_{\infty}(\mathcal{K}) \to \mathrm{Pr}^{\mathrm{L}}_{\kappa}$ is essentially surjective. To prove that it is fully faithful, it will suffice to show that for every pair of ∞-categories $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}^0_{\infty}(\mathcal{K})$, the canonical map $\theta : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathrm{Ind}_{\kappa}(\mathcal{C}), \mathrm{Ind}_{\kappa}(\mathcal{D}))$ induces an equivalence of ∞-categories from the full subcategory $\mathrm{Fun}'(\mathcal{C}, \mathcal{D})$ of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the those functors which preserve $\mathcal{K}$-indexed colimits to the full subcategory $\mathrm{Fun}'(\mathrm{Ind}_{\kappa}(\mathcal{C}), \mathrm{Ind}_{\kappa}(\mathcal{D}))$ of $\mathrm{Fun}(\mathrm{Ind}_{\kappa}(\mathcal{C}), \mathrm{Ind}_{\kappa}(\mathcal{D}))$ spanned by the κ-good functors. Let $\mathrm{Fun}'(\mathcal{C}, \mathrm{Ind}_{\kappa}(\mathcal{D}))$

denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathrm{Ind}_\kappa(\mathcal{D}))$ consisting of those functors which preserve $\mathcal{K}$-indexed colimits and carry $\mathcal{C}$ into the full subcategory of $\mathrm{Ind}_\kappa(\mathcal{D})$ spanned by the κ-compact objects. We have a homotopy commutative diagram of ∞-categories

$$\mathrm{Fun}'(\mathcal{C}, \mathcal{D}) \xrightarrow{\quad\theta\quad} \mathrm{Fun}'(\mathrm{Ind}_\kappa(\mathcal{C}), \mathrm{Ind}_\kappa(\mathcal{D}))$$
$$\theta' \searrow \qquad \swarrow \theta''$$
$$\mathrm{Fun}'(\mathcal{C}, \mathrm{Ind}_\kappa(\mathcal{D})),$$

where θ' and θ'' are given by composing with the Yoneda embeddings for $\mathcal{D}$ and $\mathcal{C}$, respectively. To complete the proof, it will suffice to show that θ' and θ'' are categorical equivalences.

To show that θ' is a categorical equivalence, let $\mathcal{D}'$ denote the collection of all κ-compact objects of $\mathrm{Ind}_\kappa(\mathcal{D})$. Since $\mathcal{D}'$ is stable under κ-small colimits in $\mathrm{Ind}_\kappa(\mathcal{D})$, $\mathrm{Fun}'(\mathcal{C}, \mathrm{Ind}_\kappa(\mathcal{D}))$ is isomorphic to the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D}')$ spanned by those functors which preserve κ-small colimits. It will therefore suffice to show that the Yoneda embedding induces an equivalence $\mathcal{D} \to \mathcal{D}'$. This follows from Lemma T.5.4.2.4, by virtue of our assumption that $\mathcal{D}$ is idempotent-complete.

Repeating the previous argument with $\mathcal{C}$ in place of $\mathcal{D}$, we see that an object of $\mathrm{Ind}_\kappa(\mathcal{C})$ is κ-compact if and only if it lies in the image of the Yoneda embedding $j : \mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C})$. Consequently, to prove that θ'' is a categorical equivalence, it suffices to show that composition with j induces an equivalence from the full subcategory of $\mathrm{Fun}(\mathrm{Ind}_\kappa(\mathcal{C}), \mathrm{Ind}_\kappa(\mathcal{C}))$ spanned by those functors which preserve small colimits to the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathrm{Ind}_\kappa(\mathcal{C}))$ spanned by those functors which preserve κ-small colimits; this follows from Proposition T.5.5.1.9. $\qquad\square$

We now study the interaction between the subcategory $\mathrm{Pr}^{\mathrm{L}}_\kappa \subseteq \mathrm{Pr}^{\mathrm{L}}$ and the symmetric monoidal structure $\mathrm{Pr}^{\mathrm{L}\otimes}$ on Pr^{L} constructed in §6.3.1. Let $\mathrm{Pr}^{\mathrm{L}\otimes}_\kappa$ denote the subcategory of $\mathrm{Pr}^{\mathrm{L}\otimes}$ whose objects are finite sequences $(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ where each of the ∞-categories $\mathcal{C}_i$ is κ-compactly generated, and whose morphisms are given by maps $(\mathcal{C}_1, \ldots, \mathcal{C}_m) \to (\mathcal{D}_1, \ldots, \mathcal{D}_n)$ covering a map $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$ such that the functors $\prod_{\alpha(i)=j} \mathcal{C}_i \to \mathcal{D}_j$ preserve κ-compact objects for $1 \leq j \leq n$.

Lemma 6.3.7.11. *Let κ be a regular cardinal. Then:*

(1) *If $\mathcal{C}$ and $\mathcal{D}$ are κ-compactly generated presentable monoidal ∞-categories, then $\mathcal{C} \otimes \mathcal{D}$ is κ-compactly presented. Moreover, the collection of κ-compact objects of $\mathcal{C} \otimes \mathcal{D}$ is generated under κ-small colimits by tensor products of the form $C \otimes D$, where $C \in \mathcal{C}$ and $D \in \mathcal{D}$ are κ-compact.*

(2) *The composite map $\mathrm{Pr}^{\mathrm{L}\otimes}_\kappa \subseteq \mathrm{Pr}^{\mathrm{L}\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ exhibits $\mathrm{Pr}^{\mathrm{L}\otimes}_\kappa$ as a symmetric monoidal ∞-category, and the inclusion $\mathrm{Pr}^{\mathrm{L}\otimes}_\kappa \subseteq \mathrm{Pr}^{\mathrm{L}\otimes}$ is a symmetric monoidal functor.*

(3) *Let $\mathcal{K}$ be as in Lemma 6.3.7.9. The functor Ind_κ induces an equivalence of symmetric monoidal ∞-categories $\mathrm{Cat}_\infty(\mathcal{K})^\otimes \to \mathrm{Pr}^{\mathrm{L}\otimes}_\kappa$.*

(4) *The tensor product $\otimes : \mathrm{Pr}^{\mathrm{L}}_\kappa \times \mathrm{Pr}^{\mathrm{L}}_\kappa \to \mathrm{Pr}^{\mathrm{L}}_\kappa$ preserves small colimits separately in each variable.*

Proof. Remark 6.3.1.8 implies that the functor $\mathrm{Ind}_\kappa : \mathrm{Cat}_\infty(\mathcal{K}) \to \mathrm{Pr}^{\mathrm{L}}$ extends to a symmetric monoidal functor. To prove (1), we note that if $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{C}_0)$ and $\mathcal{D} \simeq \mathrm{Ind}_\kappa(\mathcal{D}_0)$, then $\mathcal{C} \otimes \mathcal{D} \simeq \mathrm{Ind}_\kappa(\mathcal{C}_0 \otimes \mathcal{D}_0)$ is a κ-compactly generated ∞-category. To prove the second assertion of (1), it suffices to show that $\mathcal{C}_0 \otimes \mathcal{D}_0$ is generated under κ-small colimits by the essential image of the functor $\mathcal{C}_0 \times \mathcal{D}_0 \to \mathcal{C}_0 \otimes \mathcal{D}_0$, which is clear. Assertion (2) follows immediately from (1). Assertion (3) follows from Lemma 6.3.7.9, and assertion (4) follows from (3) together with Lemma 6.3.4.2. $\qquad\square$

Lemma 6.3.7.12. *Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category. Assume that $\mathcal{C}$ is presentable and that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. Then there exists an uncountable regular cardinal κ with the following properties:*

(1) *The ∞-category $\mathcal{C}$ is κ-compactly generated.*

(2) *The tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves κ-compact objects, and the unit object $\mathbf{1} \in \mathcal{C}$ is κ-compact.*

(3) *For every algebra object $A \in \mathrm{Alg}(\mathcal{C})$, the ∞-category $\mathrm{RMod}_A(\mathcal{C})$ is κ-compactly generated.*

(4) *For every algebra object $A \in \mathrm{Alg}(\mathcal{C})$, the action functor $\otimes : \mathcal{C} \times \mathrm{RMod}_A(\mathcal{C}) \to \mathrm{RMod}_A(\mathcal{C})$ preserves κ-compact objects.*

Proof. Choose an regular cardinal κ_0 such that $\mathcal{C}$ is κ_0-compactly generated. Let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ spanned by the κ_0-compact objects, and let $\mathcal{C}_1$ denote the smallest full subcategory of $\mathcal{C}$ which contains $\mathcal{C}_0$, the unit object of $\mathcal{C}$, and the essential image of the tensor product functor $\otimes : \mathcal{C}_0 \times \mathcal{C}_0 \to \mathcal{C}$. Since $\mathcal{C}_1$ is essentially small, there exists a regular cardinal $\kappa > \kappa_0$ such that every object in $\mathcal{C}_1$ is κ-small. We claim that κ has the desired properties. It is clear that κ is uncountable and that (1) is satisfied.

To prove (2), choose κ-compact objects $C, D \in \mathcal{C}$. Then C and D can be written as κ-small colimits $\varinjlim(C_\alpha)$ and $\varinjlim(D_\beta)$, where the objects C_α and D_β are κ_0-compact. Then $C \otimes D \simeq \varinjlim(C_\alpha \otimes D_\beta)$ is a κ-small colimit of objects belonging to $\mathcal{C}_1$, and is therefore κ-compact.

We now prove (3). According to Corollary 4.2.3.5, the forgetful functor $G : \mathrm{RMod}_A(\mathcal{C}) \to \mathcal{C}$ preserves κ-filtered colimits (in fact, all small colimits). It follows from Proposition T.5.5.7.2 that the left adjoint F to G preserves κ-compact objects. Let $\mathcal{X}$ denote the full subcategory of $\mathrm{RMod}_A(\mathcal{C})$ generated under small colimits by objects of the form $F(C)$, where $C \in \mathcal{C}$ is κ-compact; we will show that $\mathcal{X} = \mathrm{RMod}_A(\mathcal{C})$. For each $M \in \mathrm{RMod}_A(\mathcal{C})$, we can write $M \simeq A \otimes_A M = |\mathrm{Bar}_A(A, M)_\bullet|$. Consequently, to show that $M \in \mathcal{X}$, it will suffice to show that X contains $F(A^{\otimes n-1} \otimes G(M))$ for each $n \geq 1$. We are therefore reduced to proving that $F(C) \in \mathcal{X}$ for each $C \in \mathcal{C}$, which is clear (the functor F preserves small colimits and C can be written as a colimit of κ-compact objects of $\mathcal{C}$ by (1)).

We now prove (4). Let $\mathcal{Y}$ denote the full subcategory of $\mathrm{RMod}_A(\mathcal{C})$ spanned by those objects M such that $C \otimes M \in \mathrm{RMod}_A(\mathcal{C})$ is κ-compact for every κ-compact object $C \in \mathcal{C}$. The ∞-category $\mathcal{Y}$ is evidently closed under κ-small colimits in $\mathrm{RMod}_A(\mathcal{C})$. Since $C \otimes F(D) \simeq F(C \otimes D)$, it follows from (2) that $\mathcal{Y}$ contains $F(D)$ for every κ-compact object $D \in \mathcal{C}$. Since every object of $\mathcal{Y}$ is κ-compact in $\mathrm{RMod}_A(\mathcal{C})$, we have a fully faithful embedding $f : \mathrm{Ind}_\kappa(\mathcal{Y}) \to \mathrm{Mod}_A^R(\mathcal{C})$, which preserves small colimits by Proposition T.5.5.1.9. The essential image $\mathcal{Y}'$ of f is stable under small colimits and contains $F(D)$ for every κ-compact object $D \in \mathcal{C}$, so that $\mathcal{X} \subseteq \mathcal{Y}'$. It follows that f is essentially surjective and therefore an equivalence of ∞-categories. Lemma T.5.4.2.4 now guarantees that the collection of κ-compact objects of $\mathrm{RMod}_A(\mathcal{C})$ is an idempotent completion of $\mathcal{Y}$. Since κ is uncountable, $\mathcal{Y}$ is stable under sequential colimits and therefore idempotent complete. It follows that $\mathcal{Y}$ contains every κ-compact object of $\mathrm{RMod}_A(\mathcal{C})$, as desired. $\qquad\square$

We now proceed with the proof of our main result.

Proof of Theorem 6.3.7.5. We proceed by induction on k. Assume first that $k = 0$. Let X denote the composite functor

$$\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})^{op} \times \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}) \to \mathcal{C}^{op} \times \mathcal{C} \xrightarrow{H'} \mathcal{S},$$

where H is the adjoint of the Yoneda embedding for $\mathcal{C}$ (given informally by $H(C, C') = \mathrm{Map}_{\mathcal{C}}(C, C')$). The forgetful functor $\theta : \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}) \to \mathcal{C}$ determines a natural transformation of functors $\mathrm{Map} \to X$. We claim that this functor satisfies conditions (1) and (2) of Theorem 6.3.7.5.

Suppose we are given a morphism $A' \to A$ in $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$ and an object $B \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$. Let $\mathbf{1}$ denote the unit object of $\mathcal{C}_\mathfrak{m}$. Proposition 2.1.3.9 implies that $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$ is equivalent to $(\mathcal{C})^{\mathbf{1}/}$. It follows that we have

a natural transformation of fiber sequences

$$
\begin{array}{ccc}
\operatorname{Map}(A,B) & \longrightarrow & \operatorname{Map}(A',B) \\
\downarrow & & \downarrow \\
X(A,B) & \longrightarrow & X(A',B) \\
\downarrow & & \downarrow \\
\operatorname{Map}_{\mathcal{C}}(\mathbf{1},\theta(B)) & \longrightarrow & \operatorname{Map}_{\mathcal{C}}(\mathbf{1},\theta(B)).
\end{array}
$$

Since the bottom horizontal map is a homotopy equivalence, the upper square is a homotopy pullback square. This proves (1). To prove (2), we invoke Corollary 6.1.4.28 to identify $\mathbf{3}_{\mathbb{E}_k}(f)^{\times} = \operatorname{Map}_{\mathcal{C}}(\mathbf{1}, \mathbf{3}_{\mathbb{E}_k}(f)^{\times})$ with the mapping space $\operatorname{Map}_{\mathcal{C}}(\theta(A),\theta(B)) = X(A,B)$.

We now treat the case where $k > 0$. Applying Corollary 6.3.5.17 (in the setting of ∞-categories which are not necessarily small, which admit small colimits) we obtain a fully faithful embedding $\psi : \operatorname{Alg}_{\mathbb{E}_k}(\mathcal{C}) \to \operatorname{Alg}_{\mathbb{E}_{k-1}}(\operatorname{LMod}_{\mathcal{C}}(\operatorname{Pr}^{\mathrm{L}}))$. Let κ be an uncountable regular cardinal satisfying the conditions of Lemma 6.3.7.12 and let $\mathcal{C}' = \operatorname{Mod}_{\mathcal{C}}(\operatorname{Pr}^{\mathrm{L}}_{\kappa})$. Using Corollary 4.2.3.7, Lemma 6.3.7.9, and Lemma 6.3.7.11, we deduce that $\mathcal{C}'$ is a presentable ∞-category equipped with a symmetric monoidal structure, such that the tensor product $\otimes : \mathcal{C}' \times \mathcal{C}' \to \mathcal{C}'$ preserves colimits separately in each variable. The functor ψ induces a fully faithful embedding $\operatorname{Alg}_{\mathbb{E}_k}(\mathcal{C}) \to \operatorname{Alg}_{\mathbb{E}_{k-1}}(\mathcal{C}')$, which we will also denote by ψ.

Let $\operatorname{Map}' : \operatorname{Alg}_{\mathbb{E}_{k-1}}(\mathcal{C}')^{op} \times \operatorname{Alg}_{\mathbb{E}_{k-1}}(\mathcal{C}') \to \mathcal{S}$ be the adjoint to the Yoneda embedding. Invoking the inductive hypothesis, we deduce that there exists another functor $X' : \operatorname{Alg}_{\mathbb{E}_{k-1}}(\mathcal{C}')^{op} \times \operatorname{Alg}_{\mathbb{E}_{k-1}}(\mathcal{C}') \to \mathcal{S}$ and a natural transformation $\alpha' : \operatorname{Map}' \to X'$ satisfying hypotheses (1) and (2) for the ∞-category $\mathcal{C}'$. Let X denote the composition

$$\operatorname{Alg}_{\mathbb{E}_k}(\mathcal{C})^{op} \times \operatorname{Alg}_{\mathbb{E}_k}(\mathcal{C}) \xrightarrow{\psi \times \psi} \operatorname{Alg}_{\mathbb{E}_{k-1}}(\mathcal{C}')^{op} \times \operatorname{Alg}_{\mathbb{E}_{k-1}}(\mathcal{C}') \xrightarrow{X'} \mathcal{S}.$$

Since ψ is fully faithful, the composition

$$\operatorname{Alg}_{\mathbb{E}_k}(\mathcal{C})^{op} \times \operatorname{Alg}_{\mathbb{E}_k}(\mathcal{C}) \xrightarrow{\psi \times \psi} \operatorname{Alg}_{\mathbb{E}_{k-1}}(\mathcal{C}')^{op} \times \operatorname{Alg}_{\mathbb{E}_{k-1}}(\mathcal{C}') \xrightarrow{\operatorname{Map}'} \mathcal{S}$$

is equivalent to Map, so that α' induces a natural transformation of functors $\alpha : \operatorname{Map} \to X$.

It is clear from the inductive hypothesis that the natural transformation α satisfies condition (1). We will prove that α satisfies (2). Let $f : A \to B$ be a morphism in $\operatorname{Alg}_{\mathbb{E}_k}(\mathcal{C})$, and let $\psi(f) : \mathcal{A} \to \mathcal{B}$ be the induced morphism in $\operatorname{Alg}_{\mathbb{E}_{k-1}}(\mathcal{C}')$. Let $Z = \mathbf{3}_{\mathbb{E}_k}(f)$, so that we have a commutative diagram

$$
\begin{array}{ccc}
 & Z \otimes A & \\
 \nearrow & & \searrow \\
 A & \xrightarrow{\quad f \quad} & B.
\end{array}
$$

Applying the (symmetric monoidal) functor ψ, we obtain a diagram

$$
\begin{array}{ccc}
 & \psi(Z) \otimes \mathcal{A} & \\
 \nearrow & & \searrow \\
 \mathcal{A} & \xrightarrow{\quad \psi(f) \quad} & \mathcal{B},
\end{array}
$$

which is classified by a map $\beta : \psi(Z) \to \mathbf{3}_{\mathbb{E}_{k-1}}(\psi(f))$. The inductive hypothesis guarantees a homotopy equivalence $\mathbf{3}_{\mathbb{E}_{k-1}}(\psi(f))^{\times} \simeq \Omega^{n-1} X'(\mathcal{A}, \mathcal{B}) \simeq \Omega^{n-1} X(A, B)$. Passing to loop spaces, we get an homotopy equivalence $\Omega \mathbf{3}_{\mathbb{E}_{k-1}}(\psi(f))^{\times} \simeq \Omega^n X(A, B)$. We will complete the proof by showing the following:

(a) There is a canonical homotopy equivalence $Z^\times \simeq \Omega\psi(Z)^\times$.

(b) The map β induces a homotopy equivalence $\Omega\psi(Z)^\times \to \Omega\mathbf{3}_{\mathbb{E}_{k-1}}(\psi(f))^\times$.

Assertion (a) is easy: the space $\Omega\psi(Z)^\times$ can be identified with the summand of the mapping space $\mathrm{Map}_{\mathrm{RMod}_Z(\mathcal{C})}(Z, Z)$ spanned by the equivalences from Z to itself. Corollary 4.2.4.7 furnishes an identification $\mathrm{Map}_{\mathrm{RMod}_Z(\mathcal{C})}(Z, Z) \simeq \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, Z)$, under which the summand $\Omega\psi(Z)^\times \subseteq \mathrm{Map}_{\mathrm{Mod}_Z^R(\mathcal{C})}(Z, Z)$ corresponds to the space of units $Z^\times$.

The proof of (b) is slightly more involved. We wish to show that β induces a homotopy equivalence

$$\phi : \Omega\,\mathrm{Map}_{\mathrm{Mod}_{\mathcal{C}}(\mathcal{P}\mathrm{r}_\kappa^L)}(\mathcal{C}, \mathrm{RMod}_Z(\mathcal{C})) \to \Omega\,\mathrm{Map}_{\mathrm{Mod}_{\mathcal{C}}(\mathcal{P}\mathrm{r}_\kappa^L)}(\mathcal{C}, \mathbf{3}_{\mathbb{E}_{k-1}}(\psi(f))).$$

Let $\mathcal{D}^\otimes$ be a unitalization of the symmetric monoidal ∞-category $\mathrm{Mod}_{\mathcal{C}}(\mathcal{P}\mathrm{r}^L)^\otimes$, so that the underlying ∞-category of $\mathcal{D}$ is equivalent to $\mathrm{Mod}_{\mathcal{C}}(\mathcal{P}\mathrm{r}^L)_{\mathcal{C}/}$. Since $\mathbb{E}_{k-1}$ is unital, we can regard $\mathrm{RMod}_Z(\mathcal{C})$ and $\mathbf{3}_{\mathbb{E}_{k-1}}(\psi(f))$ as $\mathbb{E}_{k-1}$-algebra objects of $\mathcal{D}$. Regard the ∞-category $\mathrm{Mod}_{\mathcal{C}}(\mathcal{P}\mathrm{r}^L)$ as tensored over spaces, and let $D = \mathcal{C} \otimes S^1$ (see §T.4.4.4), regarded as an object of $\mathcal{D}$ by choosing a base point $* \in S^1$. Then we can identify ϕ with the morphism $\mathrm{Map}_{\mathcal{D}}(D, \mathrm{RMod}_Z(\mathcal{C})) \to \mathrm{Map}_{\mathcal{D}}(D, \mathbf{3}_{\mathbb{E}_{k-1}}(\psi(f)))$.

Theorem 6.3.5.5 guarantees that the construction $C \mapsto \mathrm{RMod}_C(\mathcal{C})$ determines a fully faithful embedding of symmetric monoidal ∞-categories $F : \mathrm{Alg}(\mathcal{C})^\otimes \to \mathcal{D}^\otimes$. Theorem 6.3.5.10 guarantees that the underlying functor $f : \mathrm{Alg}(\mathcal{C}) \to \mathcal{D}$ admits a right adjoint g, so that f exhibits $\mathrm{Alg}(\mathcal{C})$ as a colocalization of $\mathcal{D}$ which is stable under tensor products in $\mathcal{D}$. Using Proposition 2.2.1.1, we see that g can be regarded as a lax symmetric monoidal functor, and induces a map $\gamma : \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_{k-1}}(\mathrm{Alg}(\mathcal{C})) \to \mathrm{Alg}_{\mathbb{E}_{k-1}}(\mathcal{D})$ which is right adjoint to the functor given by composition with F. Using the fact that ψ is a fully faithful symmetric monoidal functor, we deduce that $\gamma(\beta)$ is an equivalence in $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$. Consequently, to prove that ϕ induces an equivalence from $\mathrm{Map}_{\mathcal{D}}(D, \mathrm{Mod}_Z^R(\mathcal{C}))$ to $\mathrm{Map}_{\mathcal{D}}(D, \mathbf{3}_{\mathbb{E}_{k-1}}(\psi(f)))$, it will suffice to show that the object $D \in \mathcal{D}$ lies in the essential image of the functor f. In other words, we must show that there exists an algebra object $K \in \mathrm{Alg}(\mathcal{C})$ such that $\mathcal{C} \otimes S^1$ is equivalent to $\mathrm{RMod}_K(\mathcal{C})$ in the ∞-category $\mathrm{Mod}_{\mathcal{C}}(\mathcal{P}\mathrm{r}^L)_{\mathcal{C}/}$. Choosing a symmetric monoidal functor $\mathcal{S}^\times \to \mathcal{C}^\otimes$ (which is well-defined up to a contractible space of choices), we can reduce to the case where $\mathcal{C} = \mathcal{S}$, endowed with the Cartesian symmetric monoidal structure. In this case, $\mathrm{Mod}_{\mathcal{C}}(\mathcal{P}\mathrm{r}^L)$ is equivalent to the ∞-category $\mathcal{P}\mathrm{r}^L$ of symmetric monoidal ∞-categories, and the tensor product $\mathcal{C} \otimes S^1$ can be identified with the ∞-category $(\mathcal{S})_{/S^1}$ of spaces fibered over the circle. In this case, we can take $K = \mathbf{Z} \simeq \Omega(S^1) \in \mathrm{Mon}(\mathcal{S}) \simeq \mathrm{Alg}(\mathcal{S})$ to be the group of integers: the equivalence $\mathcal{S}_{/S^1} \simeq \mathrm{Alg}_K(\mathcal{S})$ is provided by Remark 5.1.3.10, and the free module functor $\mathcal{S} \to \mathrm{Alg}_K(\mathcal{S})$ corresponds to the map given by the base point on S^1 by virtue of Remark 5.1.3.11. $\qquad\qquad\square$

Warning 6.3.7.13. The spaces $X(A, B)$ constructed in the proof of Theorem 6.3.7.5 depend on the regular cardinals κ that are chosen at each stage of the induction. We can eliminate this dependence by replacing the functor X by the essential image of the natural transformation $\alpha : \mathrm{Map} \to X$ at each step.

Remark 6.3.7.14. With a bit more effort, one can show that the homotopy equivalence $\Omega^k X(A, B) \simeq \mathbf{3}_{\mathbb{E}_k}(f)^\times$ appearing in Theorem 6.3.7.5 is an equivalence of k-fold loop spaces, which depends functorially on A and B.

Chapter 7

The Calculus of Functors

Let $f : \mathbf{R} \to \mathbf{R}$ be a smooth function. Then, for each point $x_0 \in \mathbf{R}$, there exists a real number $s = f'(x_0)$ such that f is closely approximated by the linear function $x \mapsto f(x_0) + s(x - x_0)$ in a small neighborhood of x_0. For many purposes, this allows us to reduce questions about arbitrary smooth functions to questions about linear functions, which are usually much more tractable.

In this chapter, we will give an exposition of Goodwillie's *calculus of functors*, which attempts to exploit the same idea in a different context: rather than looking for approximations to a smooth function $f : \mathbf{R} \to \mathbf{R}$, we instead seek linear (or polynomial) approximations to a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$. Our investigation is loosely informed by the following table of analogies:

Differential Calculus	Calculus of Functors
Smooth manifold M	Compactly generated ∞-category $\mathcal{C}$
Smooth function $f : M \to N$	Functor $F : \mathcal{C} \to \mathcal{D}$ which preserves filtered colimits
Point $x \in M$	Object $C \in \mathcal{C}$
Real vector space	Stable ∞-category
Real numbers $\mathbf{R}$	∞-category Sp of spectra
Linear map of vector spaces	Exact functor between stable ∞-categories
Tangent space $T_{M,x}$ to M at x	∞-category of spectrum objects $\mathrm{Sp}(\mathcal{C}_{/C})$
Differential of a smooth function	Excisive approximation of a functor (see Theorem 7.1.1.10)

We will begin in §7.1 by reviewing the contents of Goodwillie's paper [60], which introduces and analyzes a sequence of *Taylor approximations* $P_n(F)$ to a functor $F : \mathcal{C} \to \mathcal{D}$ between compactly generated ∞-categories. Restricting to the case $n = 1$, we obtain a theory of first derivatives, which we will study in §7.2. One of our main results is the Klein-Rognes chain rule (Theorem 7.2.1.22), which asserts that (under some mild hypotheses) the first derivative of a composite functor $G \circ F$ is obtained by composing the derivative of G with the derivative of F. In §7.3, we discuss the chain rule of Arone-Ching, a more general statement which gives information about the higher derivatives of a composite functor. Using the theory of ∞-operads developed in §2, we formulate and prove a Koszul dual version of this chain rule.

7.1   The Calculus of Functors

Let $f : \mathbf{R} \to \mathbf{R}$ be a smooth (that is, infinitely differentiable) function. For each $n \geq 0$, Taylor's formula gives an identity

$$f(x) = c_0 + c_1 x + \cdots + c_n x^n + u(x) x^{n+1}$$

where $c_m = \frac{f^{(m)}(0)}{m!}$ and $u : \mathbf{R} \to \mathbf{R}$ is another infinitely differentiable function. We will refer to the polynomial $g(x) = c_0 + c_1 x + \cdots + c_n x^n$ as the *nth Taylor approximation* to f (at the point $0 \in \mathbf{R}$). It is uniquely characterized by the following properties:

(a) The function $g(x)$ is a polynomial of degree $\leq n$.

(b) The difference $f(x) - g(x)$ vanishes to order n at $0 \in \mathbf{R}$.

Our goal in this section is to give an exposition of Goodwillie's *calculus of functions*, which develops an analogous theory of Taylor approximations where we replace the real numbers $\mathbf{R}$ by the ∞-category Sp of spectra, and replace smooth functions $f : \mathbf{R} \to \mathbf{R}$ by functors $F : \mathrm{Sp} \to \mathrm{Sp}$ which commute with filtered colimits.

Question 7.1.0.1. Let $F : \mathrm{Sp} \to \mathrm{Sp}$ be a functor which commutes with filtered colimits. Can we find another functor $G : \mathrm{Sp} \to \mathrm{Sp}$ satisfying some analogues of conditions (a) and (b)?

To address Question 7.1.0.1, we first need to isolate a class of functors $G : \mathrm{Sp} \to \mathrm{Sp}$ which behave like polynomials of degree $\leq n$. Note that function $f : \mathbf{R} \to \mathbf{R}$ is a polynomial of degree $\leq n$ if and only if f can be written as an $\mathbf{R}$-linear combination of the functions $\{q_m : \mathbf{R} \to \mathbf{R}\}_{0 \leq m \leq n}$ given by $q_m(x) = x^m$. Each of these functions has an obvious analogue in the setting of functors from Sp to Sp: namely, the functor $Q_m : \mathrm{Sp} \to \mathrm{Sp}$ given by $Q_m(X) = X^{\otimes m}$ determined by the smash product monoidal structure on Sp (see §6.3.2). This motivates the following definition:

Definition 7.1.0.2. We let $\mathrm{Poly}^n(\mathrm{Sp}, \mathrm{Sp})$ denote the smallest full subcategory of $\mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})$ which is closed under translation, small colimits and contains the functors $Q_m : \mathrm{Sp} \to \mathrm{Sp}$ for $0 \leq m \leq n$. We will say that a functor $G : \mathrm{Sp} \to \mathrm{Sp}$ is *polynomial of degree* $\leq n$ if it belongs to $\mathrm{Poly}^n(\mathrm{Sp}, \mathrm{Sp})$.

Example 7.1.0.3. For every sequence of "coefficients" $C_0, C_1, \ldots, C_n \in \mathrm{Sp}$, the functor

$$X \mapsto \bigoplus_{0 \leq m \leq n} C_m \otimes X^{\otimes m}$$

from Sp to itself is polynomial of degree $\leq n$. However, not every polynomial functor has this form.

Definition 7.1.0.2 does a good job of capturing the intuitive notion of polynomial for a functor from Sp to Sp. However, for some purposes it is rather inconvenient:

(i) Given a functor $G : \mathrm{Sp} \to \mathrm{Sp}$, Definition 7.1.0.2 does not immediately suggest any method for testing whether or not G is a polynomial of degree $\leq n$.

(ii) Definition 7.1.0.2 relies on specific structural features of the ∞-category Sp (namely, the "monomial" functors $Q_m : \mathrm{Sp} \to \mathrm{Sp}$), and does not immediately generalize to other contexts.

For these reasons, it will be convenient to work with a more flexible definition. Suppose that $\mathcal{C}$ is an ∞-category which admits finite colimits and that $\mathcal{D}$ is an ∞-category which admits finite limits. In §7.1.1, we will introduce the notion of an *n-excisive functor* $F : \mathcal{C} \to \mathcal{D}$ (see Definition 7.1.1.3). This notion is completely intrinsic to F, and is applicable in a wide variety of situations. Moreover, it is closely related to Definition 7.1.0.2: we will eventually show that a functor $F : \mathrm{Sp} \to \mathrm{Sp}$ is polynomial of degree $\leq n$ if and only if it is n-excisive and commutes with filtered colimits (Corollary 7.1.4.15).

The collection of n-excisive functors from $\mathcal{C}$ to $\mathcal{D}$ span a full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, which we will denote by $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$. Our first main result is that, under some mild hypotheses (which we will suppress mention of for the moment), the inclusion functor $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ admits a left adjoint (Theorem 7.1.1.10). We will denote this left adjoint by $P_n : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$. Given an arbitrary functor $F : \mathcal{C} \to \mathcal{D}$, we can think of $P_n(F)$ as an nth Taylor approximation to F: it is, in a precise sense, a "best possible" approximation to F among n-excisive functors.

An easy consequence of the definition of n-excisive functors is that any n-excisive functor is also m-excisive for $m \leq n$ (Corollary 7.1.1.14). From this we deduce the existence of canonical maps $P_n(F) \to P_m(F)$, which we can arrange into a tower of natural transformations

$$\cdots \to P_n(F) \to P_{n-1}(F) \to \cdots \to P_0(F),$$

called the *Taylor tower* of F.

Remark 7.1.0.4. For any functor $F : \mathcal{C} \to \mathcal{D}$, there is a canonical map $F \to \varprojlim_n P_n(F)$. In many cases, one can show that this natural transformation is an equivalence, or at least an equivalence when restricted to a large subcategory of $\mathcal{C}$. However, this requires strong assumptions on F: it is analogous to the assertion that an infinitely differentiable function $f : \mathbf{R} \to \mathbf{R}$ can be recovered from its Taylor series

$$f(x) = \sum_{n \geq 0} \frac{f^{(n)}(0)}{n!} x^n.$$

We refer the reader to [59] for a treatment of these types of convergence questions.

In §7.1.2 we will show that the Taylor tower of any functor F is a tower of *principal fibrations*. That is, we can always recover $P_n(F)$ as the homotopy fiber of a map $P_{n-1}(F) \to R(F)$, where $R(F)$ is functor which is *homogeneous of degree n*: that is, $R(F)$ is n-excisive and $P_{n-1}(R(F))$ is trivial. In this sense, every n-excisive functor $F : \mathcal{C} \to \mathcal{D}$ can be "built from" m-homogeneous functors for $0 \leq m \leq n$. This should be regarded as an analogue of the assertion that every polynomial function $g : \mathbf{R} \to \mathbf{R}$ can be written (uniquely) as a sum of monomials which are homogeneous of degree m for $0 \leq m \leq n$.

In §7.1.4, we will review Goodwillie's classification of homogeneous functors. The main result is that every n-homogeneous functor $H : \mathcal{C} \to \mathcal{D}$ has a unique expression as

$$H(C) = \Omega_{\mathcal{D}}^\infty (h(C, C, \ldots, C)_{\Sigma_n}), \tag{7.1}$$

where $h : \mathcal{C}^n \to \mathrm{Sp}(\mathcal{D})$ is a functor which is 1-homogeneous in each variable and symmetric in its arguments, and $h(C, \ldots, C)_{\Sigma_n}$ denotes the coinvariants for the action of the symmetric group Σ_n on $h(C, \ldots, C)$ in the ∞-category $\mathrm{Sp}(\mathcal{D})$ (see Theorem 7.1.4.7). In the special case where $H = \mathrm{fib}(P_n(F) \to P_{n-1}(F))$ is the n-homogenous part of the n-excisive approximation to F, we can regard the functor h as an avatar of the nth derivative of F (evaluated at zero), so that 7.1 can be regarded as the analogue of the formula $c_m = \frac{f^{(m)}}{m!}$ for the coefficients appearing in the Taylor approximation $g(x) = c_0 + c_1 x + \ldots + c_n x^n$ for a smooth function $f : \mathbf{R} \to \mathbf{R}$. The proof of Theorem 4.2.1.26 requires an extension of the theory of n-excisiveness to the setting of functors of many variables, which we describe in §7.1.4.

The remainder of this section is devoted to studying the classification of n-excisive functors in general. This is quite a bit more difficult than the classification of n-homogeneous functors: in order to understand an n-excisive functor $F \simeq P_n(F)$, one must understand not only its homogeneous layers $\mathrm{fib}(P_m(F) \to P_{m-1}(F))$ for $0 \leq m \leq n$, but also the "k-invariants" which describe how these layers are connected to one another. In general, this is a difficult problem. However, there are special cases in which one say a great deal. In §7.1.5, we will show that an n-excisive functor $F : \mathcal{S} \to \mathrm{Sp}$ which commutes with filtered colimits is determined by its restriction to finite sets of cardinality $\leq n$, which may be prescribed arbitrarily (Theorem 7.1.5.1). In §7.1.6, we will study the classification of n-excisive functors between stable ∞-categories, where the relevant extension problems are controlled by a form of Tate cohomology (see Remark 7.1.6.29).

Remark 7.1.0.5. The calculus of functors was introduced by Tom Goodwillie, and most of the ideas presented in this section are due to him. In particular, our exposition in §7.1.1 through §7.1.4 can be regarded as a translation of Goodwillie's paper [60] to the language of ∞-categories (in fact, very little translation was necessary: the arguments given in [60] can be adapted to the present setting, without essential change).

7.1.1 n-**Excisive Functors**

Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and assume that $\mathcal{C}$ admits finite colimits. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *excisive* if it carries pushout squares in $\mathcal{C}$ to pullback squares in $\mathcal{D}$ (Definition 1.4.2.1). The condition that F be excisive can be regarded as an abstraction of the excision axiom in the definition of a homology theory (see Remark 1.4.3.3). However, there is another way of thinking about excisive functors. If $\mathcal{C}$ and $\mathcal{D}$ are stable ∞-categories, then a functor $F : \mathcal{C} \to \mathcal{D}$ is excisive if and only if it is the direct sum of a constant functor and an exact functor (see Remark 1.4.2.2). In the functor-function analogy, such functors correspond to maps between vector spaces which are *affine*: that is, which can be given by polynomials of degree at most 1. In this section, we will introduce the more general notion of an *n-excisive* functor $F : \mathcal{C} \to \mathcal{D}$, which can be viewed as the analogue of inhomogeneous polynomials of degrees $\leq n$ (Definition 7.1.1.3). The collection of n-excisive functors from $\mathcal{C}$ to $\mathcal{D}$ span a full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, which we will denote by $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$. Our main objective in this section is to show that that, under some mild assumptions on $\mathcal{C}$ and $\mathcal{D}$, the inclusion $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ admits a left adjoint $P_n : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$ (Theorem 7.1.1.10).

Notation 7.1.1.1. For every finite set S, we let $\mathbf{P}(S)$ denote the collection of subsets of S. We regard $\mathbf{P}(S)$ as a partially ordered set with respect to inclusion. Given an integer i, we let $\mathbf{P}_{\leq i}(S)$ denote the subset of $\mathbf{P}(S)$ consisting of subsets of S having cardinality at most i, and $\mathbf{P}_{>i}(S)$ the subset of $\mathbf{P}(S)$ consisting of those subsets of S having cardinality greater than i.

Definition 7.1.1.2. Let $\mathcal{C}$ be an ∞-category and S a finite set. An S-*cube* in $\mathcal{C}$ is a functor $\mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$. We let $\mathrm{Cb}_S(\mathcal{C}) = \mathrm{Fun}(\mathrm{N}(\mathbf{P}(S)), \mathcal{C})$ denote the ∞-category of S-cubes.

  We will say that an S-cube $X : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$ is *Cartesian* if it is a limit diagram: that is, if X induces an equivalence

$$X(\emptyset) \to \varprojlim_{\emptyset \neq S_0 \subseteq S} X(S_0).$$

We will say that an S-cube $X : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$ is *strongly coCartesian* if X is a left Kan extension of its restriction to $\mathbf{P}_{\leq 1}(S)$.

Definition 7.1.1.3. Let $\mathcal{C}$ be an ∞-category which admits finite colimits and $\mathcal{D}$ an ∞-category which admits finite limits. Let $n \geq 0$ be an integer and set $S = [n] = \{0, \ldots, n\}$. We will say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *n-excisive* if composition with F carries strongly coCartesian S-cubes in $\mathcal{C}$ to Cartesian S-cubes in $\mathcal{D}$. We let $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the n-excisive functors.

Example 7.1.1.4. Let $S = [0] = \{0\}$. Then an S-cube in an ∞-category $\mathcal{C}$ is just a morphism in $\mathcal{C}$. Every S-cube is strongly coCartesian, and an S-cube is Cartesian if and only if the corresponding morphism is an equivalence in $\mathcal{C}$. Consequently, a functor $F : \mathcal{C} \to \mathcal{D}$ is 0-excisive if and only if it factors through $\mathcal{D}^\simeq$: that is, if and only if it carries each morphism in $\mathcal{C}$ to an equivalence in $\mathcal{D}$.

Example 7.1.1.5. Let $S = [1] = \{0, 1\}$. Then an S-cube in an ∞-category $\mathcal{C}$ is just a commutative diagram

$$\begin{array}{ccc} C & \longrightarrow & C_0 \\ \downarrow & & \downarrow \\ C_1 & \longrightarrow & C_{01}. \end{array}$$

Such a diagram determines a Cartesian S-cube if and only if it is a pullback square, and a coCartesian S-cube if and only if it is a pushout square. Consequently, a functor $F : \mathcal{C} \to \mathcal{D}$ is 1-excisive if and only if it

is excisive, in the sense of Definition 1.4.2.1: that is, if and only if it carries pushout squares in $\mathcal{C}$ to pullback squares in $\mathcal{D}$.

Our main goal in this section is to construct a left adjoint to the inclusion functor $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. For this, we will need to introduce a mild assumption on $\mathcal{D}$.

Definition 7.1.1.6. Let $\mathcal{C}$ be an ∞-category. We will say that $\mathcal{C}$ is *differentiable* if it satisfies the following conditions:

(a) The ∞-category $\mathcal{C}$ admits finite limits.

(b) The ∞-category $\mathcal{C}$ admits sequential colimits: that is, every diagram $\mathrm{N}(\mathbf{Z}_{\geq 0}) \to \mathcal{C}$ admits a colimit in $\mathcal{C}$.

(c) The colimit functor $\varinjlim : \mathrm{Fun}(\mathrm{N}(\mathbf{Z}_{\geq 0}), \mathcal{C}) \to \mathcal{C}$ is left exact. More informally: the formation of sequential colimits in $\mathcal{C}$ commutes with finite limits.

Example 7.1.1.7. Let $\mathcal{C}$ be a stable ∞-category. Then $\mathcal{C}$ automatically satisfies condition (a) of Definition 7.1.1.6. Condition (b) is equivalent to the requirement that $\mathcal{C}$ admits countable coproducts (see Proposition 1.4.4.1). If this condition is satisfied, then (c) follows automatically.

Example 7.1.1.8. Every ∞-topos is differentiable (see Example T.7.3.4.7).

Example 7.1.1.9. Let $\mathcal{C}$ be a compactly generated ∞-category, and $\mathcal{C}_c$ the full subcategory of $\mathcal{C}$ spanned by the compact objects. Then $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}_c)$ can be identified with a full subcategory of $\mathrm{Fun}(\mathcal{C}_c^{op}, \mathcal{S})$ which is closed under filtered colimits. It follows that $\mathcal{C}$ is a presentable ∞-category and that filtered colimits in $\mathcal{C}$ are left exact, so that $\mathcal{C}$ is differentiable.

We can now state the main result of this section:

Theorem 7.1.1.10. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object, and let $\mathcal{D}$ be a differentiable ∞-category. Then:*

(1) *The inclusion $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ admits a left adjoint $P_n : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$.*

(2) *The functor P_n is left exact.*

Remark 7.1.1.11 (Rezk). Let $\mathcal{C}$ be a small ∞-category which admits finite colimits and has a final object, and let $\mathfrak{X}$ be an ∞-topos. Theorem 7.1.1.10 implies that $\mathrm{Exc}^n(\mathcal{C}, \mathfrak{X})$ is an accessible left-exact localization of the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathfrak{X})$, and therefore an ∞-topos (which is usually far from hypercomplete when $n > 0$).

We will give the proof of Theorem 7.1.1.10 at the end of this section. First, we record some basic facts about Cartesian and strongly coCartesian cubes.

Definition 7.1.1.12. Let S be a finite set, and suppose we are given a decomposition $S = T_- \amalg T \amalg T_+$. The construction $T_0 \mapsto T_- \amalg T_0$ determines an order-preserving map from $\mathbf{P}(T)$ to $\mathbf{P}(S)$. Given an S-cube $X : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$ in an ∞-category $\mathcal{C}$, the composition $\mathrm{N}(\mathbf{P}(T)) \to \mathrm{N}(\mathbf{P}(S)) \xrightarrow{X} \mathcal{C}$ is a T-cube in $\mathcal{C}$. We will refer to the T-cubes which arise in this way as T-*faces* of X.

Proposition 7.1.1.13. *Let S be a finite set and T a finite subset of S. Suppose we are given an S-cube $X : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$ in an ∞-category $\mathcal{C}$. Then:*

(1) *If X is strongly coCartesian, then every T-face of X is strongly coCartesian.*

(2) *If every T-face of X is Cartesian, then X is Cartesian.*

Proof. We first prove (1). Assume that X is strongly coCartesian, choose a decomposition $S = T_- \amalg T \amalg T_+$, and let $Y : \mathrm{N}(\mathbf{P}(T)) \to \mathcal{C}$ be the corresponding T-face of X. We wish to prove that Y is a left Kan extension of its restriction to $\mathbf{P}_{\leq 1}(T)$. Unwinding the definitions, we must show that for every subset $T_0 \subseteq T$, the functor X induces an equivalence

$$\varinjlim_{T_0'} X(T_- \coprod T_0') \to X(T_- \coprod T_0),$$

where T_0' ranges over all subsets of T_0 having cardinality at most 1. Let $\mathcal{J}$ denote the collection of all subsets $J \subseteq T_- \coprod T_0$ whose intersection with T_0 has cardinality at most 1. The construction $T_0' \mapsto T_- \coprod T_0'$ induces an injection $\mathbf{P}_{\leq 1}(T_0) \to \mathcal{J}$. This map admits a left adjoint and therefore induces a left cofinal map of simplicial sets $\mathrm{N}(\mathbf{P}_{\leq 1}(T_0)) \to \mathrm{N}(\mathcal{J})$. It will therefore suffice to show that X exhibits $X(T_- \coprod T_0)$ as a colimit of the diagram $X| \mathrm{N}(\mathcal{J})$. Note that $X| \mathrm{N}(\mathcal{J})$ is a left Kan extension of $X| \mathrm{N}(\mathbf{P}_{\leq 1}(T_- \coprod T_0))$. It will therefore suffice to show that $X(T_- \coprod T_0)$ is a colimit of the restriction $X| \mathrm{N}(\mathbf{P}_{\leq 1}(T_- \coprod T_0))$, which follows from our assumption that X is strongly coCartesian.

We now prove (2). Assume that every T-face of X is Cartesian; we will show that X is Cartesian. We wish to show that X exhibits $X(\emptyset)$ as a limit of $X| \mathrm{N}(\mathbf{P}_{>0}(S))$. Let $\mathcal{I}$ denote the subset of $\mathbf{P}(S)$ consisting of those subsets $S_0 \subseteq S$ which have nonempty intersection with T. Since the T-faces of X are Cartesian, $X| \mathrm{N}(\mathbf{P}_{>0}(S))$ is a right Kan extension of $X| \mathrm{N}(\mathcal{I})$. It will therefore suffice to show that X exhibits $X(\emptyset)$ as a limit of the diagram $X| \mathrm{N}(\mathcal{I})$. Note that the inclusion $\mathbf{P}_{>0}(T) \hookrightarrow \mathcal{I}$ admits a right adjoint, so that the map of ∞-categories $\mathrm{N}(\mathbf{P}_{>0}(T)) \to \mathrm{N}(\mathcal{I})$ is right cofinal. It will therefore suffice to show that X exhibits $X(\emptyset)$ as a limit of the diagram $X| \mathrm{N}(\mathbf{P}_{>0}(T))$, which follows from our assumption that the T-faces of X are Cartesian. $\qquad\square$

Corollary 7.1.1.14. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Assume that $\mathcal{C}$ admits finite colimits and $\mathcal{D}$ admits finite limits. If F is n-excisive, the F is m-excisive for each $m \geq n$.*

Proposition 7.1.1.15. *Let S be a finite set, let $\mathcal{C}$ be an ∞-category which admits finite colimits, and let $X : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$ be an S-cube. The following conditions are equivalent:*

(1) *The S-cube X is strongly coCartesian.*

(2) *For every pair of finite sets $T, T' \subseteq S$, the diagram*

$$\begin{array}{ccc} X(T \cap T') & \longrightarrow & X(T) \\ \downarrow & & \downarrow \\ X(T') & \longrightarrow & X(T \cup T') \end{array}$$

   is a pushout square in $\mathcal{C}$.

(3) *For every subset $T \subseteq S$ and every element $s \in S - T$, the diagram*

$$\begin{array}{ccc} X(\emptyset) & \longrightarrow & X(T) \\ \downarrow & & \downarrow \\ X(\{s\}) & \longrightarrow & X(T \cup \{s\}) \end{array}$$

   is a pushout square in $\mathcal{C}$.

Proof. We first show that $(1) \Rightarrow (2)$. Let $P \subseteq \mathbf{P}(S)$ denote the collection of those subsets S' such that $S' \subseteq T$ or $S' \subseteq T'$, let $P_0 = P \cap \mathbf{P}_{\leq 1}(S)$, and let $P_1 = \{T, T', T \cap T'\}$. We wish to prove that X exhibits $X(T \cup T')$ as a colimit of $X| \mathrm{N}(P_1)$. It follows from Theorem T.4.1.3.1 that the inclusion $\mathrm{N}(P_1) \subseteq \mathrm{N}(P_0)$

is left cofinal. It will therefore suffice to show that X exhibits $X(T \cup T')$ as a colimit of $X | \mathrm{N}(P)$. Our assumption that X is strongly coCartesian implies that $X | \mathrm{N}(P)$ is a left Kan extension of $X | \mathrm{N}(P_0)$. It will therefore suffice to show that X exhibits $X(T \cup T')$ as a colimit of $X | \mathrm{N}(P_0)$, which follows immediately from our assumption that X is strongly coCartesian.

The implication $(2) \Rightarrow (3)$ is obvious. We will complete the proof by showing that $(3) \Rightarrow (1)$. Let $X_0 = X | \mathrm{N}(\mathbf{P}_{\leq 1}(S))$. Since $\mathcal{C}$ admits finite colimits, we can extend X_0 to a strongly coCartesian S-cube $X' : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$. The identification $X' | \mathrm{N}(\mathbf{P}_{\leq 1}(S)) = X | \mathrm{N}(\mathbf{P}_{\leq 1}(S))$ extends to a natural transformation $\alpha : X' \to X$. To prove that X is strongly coCartesian, it will suffice to show that α is an equivalence. For each $T \subseteq S$, let $\alpha_T : X'(T) \to X(T)$ denote the induced map. We will prove that each of the maps α_T is an equivalence. We proceed by induction on the cardinality of T. The result is obvious if the cardinality of T is ≤ 1. Otherwise, choose an element $s \in T$ and let $T' = T - \{s\}$. Since X' and X both satisfy condition (3), we have a pushout diagram

$$
\begin{array}{ccc}
\alpha_\emptyset & \longrightarrow & \alpha_{T'} \\
\downarrow & & \downarrow \\
\alpha_{\{s\}} & \longrightarrow & \alpha_T
\end{array}
$$

in $\mathrm{Fun}(\Delta^1, \mathcal{C})$. Since $\alpha_\emptyset$, $\alpha_{T'}$, and $\alpha_{\{s\}}$ are equivalences by the inductive hypothesis, we conclude that α_T is an equivalence. $\qquad\square$

Corollary 7.1.1.16. *Let $\mathcal{C}$ be a stable ∞-category, let S be a finite set, and let $X : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$ be an S-cube in $\mathcal{C}$. Define $X' : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}^{op}$ by the formula $X'(T) = X(S - T)$. Then:*

(1) *The functor X is a strongly coCartesian S-cube in $\mathcal{C}$ if and only if X' is a strongly coCartesian S-cube in $\mathcal{C}^{op}$.*

(2) *The functor X is a Cartesian S-cube in $\mathcal{C}$ if and only if X' is a Cartesian S-cube in $\mathcal{C}^{op}$.*

Proof. Assertion (2) follows immediately from Proposition 1.2.4.13. Assertion (1) follows from Proposition 1.2.4.13 and the characterization of strongly coCartesian S-cubes given in Proposition 7.1.1.15. $\qquad\square$

Corollary 7.1.1.17. *Let $\mathcal{C}$ and $\mathcal{D}$ be stable ∞-categories, and let $n \geq 0$ be an integer. Then a functor $F : \mathcal{C} \to \mathcal{D}$ is n-excisive if and only if the induced map $\mathcal{C}^{op} \to \mathcal{D}^{op}$ is n-excisive.*

We now turn to the construction of the functor P_n appearing in the statement of Theorem 7.1.1.10. First, we need to introduce some notation.

Construction 7.1.1.18. Let $\mathcal{F}\mathrm{in}^{\mathrm{inj}}$ denote the category whose objects are finite sets and whose morphisms are injections, and let $\mathcal{F}\mathrm{in}^{\mathrm{inj}}_{\leq n}$ denote the full subcategory of $\mathcal{F}\mathrm{in}^{\mathrm{inj}}$ spanned by those finite sets having cardinality $\leq n$. Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object. Consider the following conditions on a functor $F : \mathrm{N}(\mathcal{F}\mathrm{in}^{\mathrm{inj}}) \to \mathcal{C}$:

(i) Whenever S is a set of cardinality exactly 1, the object $F(S) \in \mathcal{C}$ is final.

(ii) For every finite set S, F exhibits $F(S)$ as the colimit of the diagram $F | \mathrm{N}(\mathbf{P}_{\leq 1}(S))$.

Let $\overline{\mathcal{C}}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathcal{F}\mathrm{in}^{\mathrm{inj}}), \mathcal{C})$ spanned by those functors which satisfy (i) and (ii). Condition (i) is equivalent to the requirement that $F | \mathrm{N}(\mathcal{F}\mathrm{in}^{\mathrm{inj}}_{\leq 1})$ is a right Kan extension of $F | \mathrm{N}(\mathcal{F}\mathrm{in}^{\mathrm{inj}}_{\leq 0})$, and condition ($ii$) is equivalent to the requirement that F is a left Kan extension of $F | \mathrm{N}(\mathcal{F}\mathrm{in}^{\mathrm{inj}}_{\leq 1})$. Using Proposition T.4.3.2.15, we deduce that the evaluation functor $F \mapsto F(\emptyset)$ determines a trivial Kan fibration $\overline{\mathcal{C}} \to \mathcal{C}$. Choose a section of this Kan fibration. We can regard this section as determining a functor $\mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}^{\mathrm{inj}}) \to \mathcal{C}$, which we will denote by $(X, S) \mapsto C_S(X)$. We will refer to $C_S(X)$ as the *S-pointed cone on X*.

Example 7.1.1.19. Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object $*$. If $S = \emptyset$, we have $C_S(X) = X$ for each $X \in \mathcal{C}$. If S has a single element, then $C_S(X)$ is a final object of $\mathcal{C}$. If S has two elements, then we can identify $C_S : \mathcal{C} \to \mathcal{C}$ with the (unreduced) suspension functor $\Sigma_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, which carries an object $X \in \mathcal{C}$ to a pushout $* \amalg_X *$.

Remark 7.1.1.20. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories which admit finite colimits and final objects, and let

$$C^{\mathcal{C}} : \mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}^{\mathrm{inj}}) \to \mathcal{C} \qquad C^{\mathcal{D}} : \mathcal{D} \times \mathrm{N}(\mathcal{F}\mathrm{in}^{\mathrm{inj}}) \to \mathcal{D}$$

be as in Construction 7.1.1.18. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor which preserves final objects and pushouts. Then composition with F carries $\overline{\mathcal{C}}$ into $\overline{\mathcal{D}}$ (where $\overline{\mathcal{C}}$ and $\overline{\mathcal{D}}$ are defined as in Construction 7.1.1.18). It follows that the diagram

$$
\begin{array}{ccc}
\mathcal{C} \times \mathrm{N}(\mathcal{F}\mathrm{in}^{\mathrm{inj}}) & \xrightarrow{F \times \mathrm{id}} & \mathcal{D} \times \mathrm{N}(\mathcal{F}\mathrm{in}^{\mathrm{inj}}) \\
\downarrow{\scriptstyle C^{\mathcal{C}}} & & \downarrow{\scriptstyle C^{\mathcal{D}}} \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}
$$

commutes up to canonical homotopy.

Example 7.1.1.21. Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object. For every finite set S, the functor $X \mapsto C_S(X)$ preserves final objects and finite colimits. Using Remark 7.1.1.20, we conclude that for every pair of finite sets S and T, we have a canonical equivalence of functors $C_S \circ C_T \simeq C_T \circ C_S$. This equivalence depends functorially on the pair (S, T).

Construction 7.1.1.22. Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object, let $\mathcal{D}$ an ∞-category which admits finite limits, and let $F : \mathcal{C} \to \mathcal{D}$ a functor. For each integer $n \geq 0$, we define a new functor $T_n(F) : \mathcal{C} \to \mathcal{D}$ by the formula

$$(T_n F)(X) = \varprojlim_{\emptyset \neq S \subseteq [n]} F(C_S(X)).$$

The canonical map

$$F(X) = F(C_\emptyset(X)) \to \varprojlim_{\emptyset \neq S \subseteq [n]} F(C_S(X))$$

determines a natural transformation of functors $F \to T_n F$, which depends functorially on F.

Example 7.1.1.23. Let $F : \mathcal{C} \to \mathcal{D}$ be as in Construction 7.1.1.22, and let $*$ denote a final object of $\mathcal{C}$. Then $T_0 F$ is equivalent to the constant functor taking the value $F(*)$. If F is reduced (that is, $F(*)$ is a final object of $\mathcal{D}$), then $T_1(F)$ is given by the composition $\Omega_{\mathcal{D}} \circ F \circ \Sigma_{\mathcal{C}}$.

Remark 7.1.1.24. The construction $F \mapsto T_n F$ commutes with finite limits (in fact, it commutes with K-indexed limits, for any simplicial set K such that $\mathcal{D}$ admits K-indexed limits).

Remark 7.1.1.25. In the situation of Construction 7.1.1.22, suppose we are given another ∞-category $\mathcal{C}$ which admits finite colimits and a final object, and a functor $F' : \mathcal{C}' \to \mathcal{C}$ which preserves pushouts and final objects. Then we have a canonical equivalence of functors $T_n(F \circ F') \simeq (T_n F) \circ F'$.

Lemma 7.1.1.26. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and a final object, let $\mathcal{D}$ be an ∞-category which admits finite limits, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Let $n \geq 0$ and let $S = [n] = \{0, \ldots, n\}$. Suppose that $X : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$ is a strongly coCartesian S-cube. Then the canonical map of S-cubes $\theta_F : F(X) \to (T_n F)(X)$ factors through a Cartesian S-cube of $\mathcal{D}$.*

Proof (Rezk): Let $\theta : \mathcal{C} \to \mathrm{Cb}_S(\mathcal{C})$ be given by the formula $\theta(U)(I) = C_I(U)$. For every subset $I \subseteq S$, let $X_I : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$ denote the functor given by the formula $X_I(I') = X(I \cup I')$. Note that each X_I is a left Kan extension of its restriction to $\mathrm{N}(\mathbf{P}_{\leq 1}(S))$. Since $\theta(X(I)) | \mathrm{N}(\mathbf{P}_{\leq 1}(S))$ is a right Kan extension of its

restriction to $\mathrm{N}(\mathbf{P}_{\leq 0}(S))$, the identity map $X_I(\emptyset) \to \theta(X(I))(\emptyset)$ admits an essentially unique extension to a map of S-cubes $X_I \to \theta(X(I))$, depending functorially on I. Define

$$Y(I) = \varprojlim_{\emptyset \neq S' \subseteq S} F(X_I(S')).$$

The map $\theta : F(X) \to (T_n F)(X)$ factors canonically as a composition

$$F(X) \to Y \to (T_n F)(X).$$

To complete the proof, it will suffice to show that Y is a Cartesian S-cube. Since the collection of Cartesian S-cubes is stable under finite limits, it will suffice to show that for every nonempty set $S' \in S$, the functor $T \mapsto F(X_I(S')) = F(X(I \cup S'))$ is a Cartesian S-cube. Since every S'-face of this S-cube is constant and S' is nonempty, every S'-face is Cartesian; the desired result now follows from Proposition 7.1.1.13. $\qquad \square$

Construction 7.1.1.27. Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object. Suppose we are given a functor $F : \mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is differentiable. For each integer $n \geq 0$, we let $P_n F$ denote the colimit of the sequence of functors

$$F \xrightarrow{\theta_F} T_n F \xrightarrow{\theta_{T_n F}} T_n T_n F \to \cdots$$

We will refer to $P_n F$ as the *n-excisive approximation* to F.

Example 7.1.1.28. Let $F : \mathcal{C} \to \mathcal{D}$ be as in Construction 7.1.1.27. If F is reduced, then the 1-excisive approximation to F is given by $\varinjlim_m \Omega_{\mathcal{D}}^m \circ F \circ \Sigma_{\mathcal{C}}^m$ (see Example 7.1.1.23).

Remark 7.1.1.29. In the situation of Construction 7.1.1.27, the construction $F \mapsto P_n F$ commutes with finite limits. This follows from Remark 7.1.1.24, since the formation of finite limits in $\mathcal{D}$ commutes with sequential colimits.

Remark 7.1.1.30. In the situation of Construction 7.1.1.27, suppose we are given another ∞-category $\mathcal{C}'$ which admits finite colimits and a final object, and let $F' : \mathcal{C}' \to \mathcal{C}$ be a functor which preserves pushouts and final objects. Then we have a canonical equivalence of functors $P_n(F \circ F') \simeq (P_n F) \circ F'$.

Remark 7.1.1.31. In the situation of Construction 7.1.1.27, let K be a simplicial set such that $\mathcal{D}$ admits K-indexed colimits, and the formation of K-indexed colimits commutes with the formation of finite limits. It follows that the construction $F \mapsto T_n F$ commutes with K-indexed colimits, so that the construction $F \mapsto P_n F$ commutes with K-indexed colimits. In particular, the hypotheses of Construction 7.1.1.27 guarantee that the construction $F \mapsto P_n F$ commutes with sequential colimits.

Remark 7.1.1.32. In the situation of Construction 7.1.1.27, suppose we are given another differentiable ∞-category $\mathcal{D}'$ and let $G : \mathcal{D} \to \mathcal{D}'$ be a functor which preserves finite limits and sequential colimits. For any functor $F : \mathcal{C} \to \mathcal{D}$, we have a canonical equivalence $P_n(G \circ F) \simeq G \circ P_n(F)$.

We now wish to show that the functor $F \mapsto P_n F$ satisfies the conclusions of Theorem 7.1.1.10.

Lemma 7.1.1.33. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object. Suppose we are given a functor $F : \mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is differentiable. Then the functor $P_n F : \mathcal{C} \to \mathcal{D}$ is n-excisive.*

Proof. Let $S = [n] = \{0, \dots, n\}$, and let $X : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$ be a strongly coCartesian S-cube; we wish to show that $(P_n F)(X)$ is a Cartesian S-cube in $\mathcal{D}$. We can write $(P_n F)(X)$ as the colimit of a sequence of S-cubes

$$F(X) \to (T_n F)(X) \to (T_n^2 F)(X) \to \cdots$$

According to Lemma 7.1.1.26, each of the maps $(T_n^k F)(X) \to (T_n^{k+1} F)(X)$ factors through a Cartesian S-cube Y_k in $\mathcal{D}$. Then $(P_n F)(X)$ can be realized as the colimit of the sequence of S-cubes

$$Y_0 \to Y_1 \to Y_2 \to \cdots$$

Since each Y_i is Cartesian and finite limits in $\mathcal{D}$ commute with sequential colimits, we conclude that $(P_n F)(X) \simeq \varinjlim Y_i$ is Cartesian, as desired. $\qquad \square$

Lemma 7.1.1.34. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object. Suppose we are given a functor $F : \mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is a differentiable ∞-category. Let θ denote the canonical map from F to $T_n F$. Then θ induces an equivalence $P_n(F) \to P_n(T_n F)$.*

Proof. We have $T_n F = \varprojlim_{\emptyset \neq S \subseteq [n]} F \circ C_S$. Since P_n commutes with finite limits (Remark 7.1.1.29), the canonical map $P_n(T_n(F)) \to \varprojlim_{\emptyset \neq S \subseteq [n]} P_n(F \circ C_S)$ is an equivalence. Each of the functors C_S preserves pushouts and final objects, so that Remark 7.1.1.30 gives an equivalence $P_n(T_n F) \simeq \varprojlim_{\emptyset \neq S \subseteq [n]} (P_n F) \circ C_S$. It will therefore suffice to show that the canonical map $P_n F \to \varprojlim_{\emptyset \neq S \subseteq [n]} (P_n F) \circ C_S$ is an equivalence, which follows immediately from Lemma 7.1.1.33. $\square$

Lemma 7.1.1.35. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, where $\mathcal{C}$ admits finite colimits and a final object and $\mathcal{D}$ is differentiable. Let $\phi : F \to P_n F$ be the canonical natural transformation. Then $P_n(\phi) : P_n(F) \to P_n(P_n(F))$ is an equivalence.*

Proof. Using Remark 7.1.1.31, we can identify $P_n(\phi)$ with the colimit of the sequence of natural transformations $P_n(F) \to P_n(T_n^k(F))$, each of which factors as a composition of equivalences

$$P_n(F) \to P_n(T_n(F)) \to P_n(T_n^2(F)) \to \cdots \to P_n(T_n^k(F))$$

by virtue of Lemma 7.1.1.34. $\square$

Proof of Theorem 7.1.1.10. Let $\mathcal{C}$ be an ∞-category which admits finite colimits and a final object, let $\mathcal{D}$ be a differentiable ∞-category, and let $P_n : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be the functor given by Construction 7.1.1.27. We have already seen that P_n is left exact (Remark 7.1.1.29), and the essential image of P_n is contained in $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$ (Lemma 7.1.1.33). If $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is n-excisive, then it follows immediately from the definition that the canonical map $F \to T_n F$ is an equivalence. Applying this observation iteratively, we deduce that the canonical map $F \to P_n F$ is an equivalence, so that F belongs to the essential image of the functor P_n.

By construction, we have a natural transformation $\theta : \mathrm{id} \to P_n$ of functors from $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ to itself. We will complete the proof by showing that θ exhibits P_n as a localization functor. According to Proposition T.5.2.7.4, it will suffice to show that for every $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, the canonical maps

$$P_n(\theta_F), \theta_{P_n F} : P_n(F) \to P_n(P_n(F))$$

are equivalences. The map $\theta_{P_n(F)}$ is an equivalence by the argument given above, since $P_n(F)$ is n-excisive by Lemma 7.1.1.33. We conclude by applying Lemma 7.1.1.35 to deduce that $P_n(\theta_F)$ is also an equivalence. $\square$

7.1.2   The Taylor Tower

Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object, and let $\mathcal{D}$ be a differentiable ∞-category. For every integer $n \geq 0$, we let $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the n-excisive functors (Definition 7.1.1.3), and $P_n : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$ a left adjoint to the inclusion functor (see Theorem 7.1.1.10). According to Corollary 7.1.1.14, we have inclusions

$$\cdots \mathrm{Exc}^3(\mathcal{C}, \mathcal{D}) \supseteq \mathrm{Exc}^2(\mathcal{C}, \mathcal{D}) \supseteq \mathrm{Exc}^1(\mathcal{C}, \mathcal{D}) \supseteq \mathrm{Exc}^0(\mathcal{C}, \mathcal{D}),$$

so that the localization functors P_n form an inverse system

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0.$$

If $F : \mathcal{C} \to \mathcal{D}$ is a functor, then we obtain a diagram of functors

$$\cdots \to P_3 F \to P_2 F \to P_1 F \to P_0 F$$

is called the *Taylor tower* of F. We can think of the Taylor tower $\{P_n F\}_{n \geq 0}$ as a sequence of approximations to the functor F, which become more accurate as n grows large. Our goal in this section is to study the difference between successive Taylor approximations (as measured, for example, by taking fibers of the maps $P_n F \to P_{n-1} F$). Before we can state our main result, we need to introduce a bit of terminology.

Definition 7.1.2.1. Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object, and let $\mathcal{D}$ be a differentiable ∞-category. If n is a positive integer, we say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *n-reduced* if $P_{n-1} F$ is a final object of $\mathrm{Exc}^{n-1}(\mathcal{C}, \mathcal{D})$ (that is, if $(P_{n-1} F)(C)$ is a final object of $\mathcal{D}$, for each $C \in \mathcal{C}$). We will say that F is *n-homogeneous* if it is n-excisive and n-reduced. We let $\mathrm{Exc}_*^n(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which are n-excisive and 1-reduced, and $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$ the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which are n-homogeneous.

Remark 7.1.2.2. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, where $\mathcal{C}$ has finite colimits and a final object and $\mathcal{D}$ is differentiable. Then $P_0 F$ can be identified with the constant functor taking the value $F(*)$, where $*$ denotes a final object of $\mathcal{C}$. Consequently, the functor F is 1-reduced if and only if $F(*)$ is a final object of $\mathcal{D}$: that is, if and only if F is reduced, in the sense of Definition 1.4.2.1.

Remark 7.1.2.3. In the situation of Definition 7.1.2.1, the functors P_n and P_{n-1} commute with sequential colimits (Remark 7.1.1.31). It follows that the collections of n-excisive and n-reduced functors are closed under sequential colimits in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. In particular, the full subcategory $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is closed under sequential colimits.

We can now state the main result of this section.

Theorem 7.1.2.4 (Goodwillie). *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object, let $\mathcal{D}$ be a differentiable ∞-category, and let $n \geq 1$ be an integer. Then there exists a pullback diagram of functors*

$$
\begin{array}{ccc}
P_n & \longrightarrow & P_{n-1} \\
\downarrow & & \downarrow \\
K & \longrightarrow & R
\end{array}
$$

from $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ to itself having the following properties:

(1) *For every reduced functor $F : \mathcal{C} \to \mathcal{D}$, $K(F)$ carries every object of $\mathcal{C}$ to a final object of $\mathcal{D}$.*

(2) *For every reduced functor $F : \mathcal{C} \to \mathcal{D}$, the functor $R(F)$ is n-homogeneous.*

(3) *The functor $R : \mathrm{Fun}_*(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ is left exact.*

(4) *If $F \in \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ is $(n-1)$-excisive, then $R(F)$ carries each object of $\mathcal{C}$ to a final object of $\mathcal{D}$.*

Before proving Theorem 7.1.2.4, let us describe some of its consequences.

Theorem 7.1.2.5. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object, let $\mathcal{D}$ be a differentiable ∞-category, and let $n \geq 1$ be an integer. Let $\mathcal{E} \subseteq \mathrm{Fun}(\Lambda_2^2, \mathrm{Fun}_*(\mathcal{C}, \mathcal{D}))$ spanned by those diagrams of functors*

$$E \to H \leftarrow H_0$$

where E is reduced and $(n-1)$-excisive, H is n-homogeneous, and H_0 is a final object of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Then the construction

$$\varprojlim : \mathrm{Fun}(\Lambda_2^2, \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})) \to \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$$

induces an equivalence of ∞-categories $\mathcal{E} \to \mathrm{Exc}_*^n(\mathcal{C}, \mathcal{D})$.

In other words, every reduced n-excisive functor $F : \mathcal{C} \to \mathcal{D}$ can be written uniquely as a fiber of some natural transformation $\alpha : E \to H$, where E is a reduced $(n-1)$-excisive functor and H is n-homogeneous functor. Here the fiber is taken over a "base point" given by a natural transformation $\beta : H_0 \to H$, where H_0 is a final object of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. We can identify β with a lifting of H to an n-homogeneous functor from $\mathcal{C}$ to the ∞-category $\mathcal{D}_*$ of pointed objects of $\mathcal{D}$. The *existence* of the natural transformation α follows immediately from Theorem 7.1.2.4 (namely, we take α to be the natural transformation $P_{n-1}(F) \to R(F)$ appearing in the statement of Theorem 7.1.2.4). We will deduce the uniqueness from the following somewhat technical lemma:

Lemma 7.1.2.6. *Let $\mathcal{C}$ be an ∞-category which admits finite limits. Suppose we are given a diagram σ :*

$$
\begin{array}{ccccc}
X_{00} & \longrightarrow & X_{01} & \xleftarrow{\phi'} & X_{02} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\phi} \\
X_{10} & \longrightarrow & X_{11} & \longleftarrow & X_{12} \\
\uparrow{\scriptstyle\psi} & & \uparrow & & \uparrow \\
X_{20} & \xrightarrow{\psi'} & X_{21} & \longleftarrow & X_{22}
\end{array}
$$

For $i \in \{0, 1, 2\}$, let X_i^h denote the fiber product $X_{i0} \times_{X_{i1}} X_{i2}$, and let X_i^v denote the fiber product $X_{0i} \times_{X_{1i}} X_{2i}$. Assume that ϕ, ϕ', ψ, and ψ' are equivalences. Then the diagrams $\Lambda_2^2 \to \mathcal{C}$ given by

$$X_0^h \to X_1^h \leftarrow X_2^h$$

and

$$X_0^v \to X_1^v \leftarrow X_2^v$$

are equivalent. Moreover, the equivalence can be chosen to depend functorially on σ.

Proof. Let σ_- denote the diagram obtained from σ by omitting the lower right corner, σ_+ the diagram obtained from σ by omitting the upper left corner, and σ_0 the diagram obtained from σ by omitting both the upper left and lower right corners. We will prove that the diagram

$$X_0^h \to X_1^h \leftarrow X_2^h$$

is canonically equivalent to the diagram

$$\varprojlim \sigma_- \to \varprojlim \sigma_0 \leftarrow \varprojlim \sigma_+ \tag{7.2}$$

(via an equivalence which depends functorially in σ) By symmetry, it will follow that

$$X_0^v \to X_1^v \leftarrow X_2^v$$

is also equivalent to (7.2), and the proof will be complete.

Let σ_-' denote the diagram obtained from σ by omitting the lower row, σ_+' the diagram obtained from σ by omitting the upper row, and σ_0' the diagram obtained from σ by omitting both the upper and lower rows. Then

$$X_0^h \to X_1^h \leftarrow X_2^h$$

can be identified with the upper row of the commutative diagram

$$
\begin{array}{ccccc}
\varprojlim \sigma_-' & \longrightarrow & \varprojlim \sigma_0' & \longleftarrow & \varprojlim \sigma_+' \\
\uparrow & & \uparrow{\scriptstyle\theta} & & \uparrow \\
\varprojlim \sigma_- & \longrightarrow & \varprojlim \sigma_0 & \longleftarrow & \varprojlim \sigma_+.
\end{array}
$$

It will therefore suffice to show that the vertical maps in this diagram are equivalences. We will show that the map θ is an equivalence; the proofs for the other two maps are similar (but easier). Consider the diagram σ_0'' :

$$
\begin{array}{ccc}
 & & X_{02} \\
 & & \downarrow \phi \\
X_{10} \longrightarrow & X_{11} \longleftarrow & X_{12} \\
\uparrow \psi & & \\
X_{20}. & &
\end{array}
$$

The map θ factors as a composition

$$\varprojlim \sigma_0 \xrightarrow{\theta'} \varprojlim \sigma_0'' \xrightarrow{\theta''} \varprojlim \sigma_0'.$$

Since ϕ and ψ are equivalences, the diagram σ_0'' is a right Kan extension of σ_0', so that θ'' is an equivalence. The map θ' is an equivalence by a cofinality argument, so that θ is an equivalence as desired. $\qquad\square$

Proof of Theorem 7.1.2.5. Since the collection of reduced, n-excisive functors is closed under limits, it is clear that the formation of fiber products induces a functor $\phi : \mathcal{E} \to \mathrm{Exc}_*^n(\mathcal{C}, \mathcal{D})$. The construction which carries a functor F to the diagram

$$P_{n-1}(F) \to R(F) \leftarrow K(F)$$

(see Theorem 7.1.2.4) determines a functor $\psi : \mathrm{Exc}_*^n(\mathcal{C}, \mathcal{D}) \to \mathcal{E}$. It follows from Theorem 7.1.2.4 that the composition $\phi \circ \psi$ is equivalent to the identity on $\mathrm{Exc}_*^n(\mathcal{C}, \mathcal{D})$. We will complete the proof by showing that $\psi \circ \phi$ is equivalent to the identity functor. Consider an object $Y \in \mathcal{E}$, corresponding to a diagram of reduced functors

$$E \to H \leftarrow H_0$$

where E is $(n-1)$-excisive, H is n-homogeneous, and H_0 carries every object of $\mathcal{C}$ to an initial object of $\mathcal{D}$. Consider the diagram of functors σ :

$$
\begin{array}{ccc}
P_{n-1}(E) \longrightarrow & P_{n-1}(H) \longleftarrow & P_{n-1}(H_0) \\
\downarrow & \downarrow & \downarrow \\
R(E) \longrightarrow & R(H) \longleftarrow & R(H_0) \\
\uparrow & \uparrow & \uparrow \\
K(E) \longrightarrow & K(H) \longleftarrow & K(H_0).
\end{array}
$$

Assertion (1) of Theorem 7.1.2.4 implies that $K(E)$, $K(H)$, and $K(H_0)$ are final objects of $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$. Since P_{n-1} and R are left exact, the object $P_{n-1}(H_0)$ and $R(H_0)$ are final in $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$. The object $P_{n-1}(H) \in \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ is final since H is n-homogeneous, and the object $R(E) \in \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ is final by part (4) of Theorem 7.1.2.4. Since E, H, and H_0 are n-excisive, taking the limits along the columns of the diagram σ yields the diagram

$$E \to H \leftarrow H_0$$

given by Y. Since P_{n-1}, R, and K are left exact functors, taking the limits along the rows of the diagram σ gives the diagram $(\psi \circ \phi)(Y)$:

$$P_{n-1}(E \times_H H_0) \to R(E \times_H H_0) \leftarrow K(E \times_H H_0).$$

Invoking Lemma 7.1.2.6, we obtain an equivalence $Y \simeq (\psi \circ \phi)(Y)$, depending functorially on Y. $\qquad\square$

In the situation of Theorem 7.1.2.5, suppose that $F : \mathcal{C} \to \mathcal{D}$ is given by the fiber product $E \times_H H_0$ of a diagram in $\mathcal{E}$. Then $P_{n-1}F \simeq P_{n-1}E \times_{P_{n-1}H} P_{n-1}H_0 \simeq P_{n-1}E \simeq E$. In particular, F is n-homogeneous if and only if E is a final object of $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$. We therefore have the following specialization of Theorem 7.1.2.5:

Corollary 7.1.2.7. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object, let $\mathcal{D}$ be a differentiable ∞-category, and let $n \geq 1$ be an integer. Let $\mathcal{E}_0$ be the full subcategory of $\mathrm{Fun}(\Lambda_2^2, \mathrm{Fun}_*(\mathcal{C}, \mathcal{D}))$ spanned by those diagrams of functors*

$$H_1 \to H \leftarrow H_0$$

where H is n-homogeneous and the functors H_0 and H_1 carry each object of $\mathcal{C}$ to a final object of $\mathcal{D}$. Then the construction

$$\varprojlim : \mathrm{Fun}(\Lambda_2^2, \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})) \to \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$$

induces an equivalence $\mathcal{E}_0 \to \mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$.

An important special case of Corollary 7.1.2.7 occurs when the ∞-category $\mathcal{D}$ is *pointed*: that is, the final objects of $\mathcal{D}$ are also initial. In this case, the ∞-category $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ is also pointed. We may therefore identify the ∞-category $\mathcal{E}_0$ with $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$ (by means of the evaluation functor $(H_1 \to H \leftarrow H_0) \mapsto H$, which is a trivial Kan fibration by Proposition T.4.3.2.15). Corollary 7.1.2.7 now asserts that the loop functor $\Omega : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ restricts to an equivalence of ∞-categories $\Omega : \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \to \mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$. Combining this observation with Corollary 1.4.2.27, we obtain the following generalization of Proposition 1.4.2.16:

Corollary 7.1.2.8. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object, let $\mathcal{D}$ be a pointed differentiable ∞-category, and let $n \geq 1$ be an integer. Then the ∞-category $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$ is stable.*

In the situation of Corollary 7.1.2.8, let $\mathrm{Sp}(\mathcal{D})$ denote the ∞-category of spectrum objects of $\mathcal{D}$ (Definition 1.4.2.8), which we regard as a full subcategory of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{D})$. For every pointed finite space $K \in \mathcal{S}_*^{\mathrm{fin}}$, the evaluation $X \mapsto X(K)$ determines a functor $e_K : \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$. Note that $\mathrm{Sp}(\mathcal{D})$ is closed under finite limits and sequential colimits in $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{D})$. For every functor $F : \mathcal{C} \to \mathrm{Sp}(\mathcal{D})$, we have canonical equivalences

$$e_K \circ (P_n F) \simeq P_n(e_K \circ F) \qquad e_K \circ (P_{n-1}F) \simeq P_{n-1}(e_K \circ F)$$

(see Remark 7.1.1.32). It follows that F is n-excisive if and only if each of the functors $e_K \circ F$ is n-excisive, and n-reduced if and only if each of the functors $e_K \circ F$ is n-reduced. We therefore have canonical isomorphisms

$$\mathrm{Exc}_*^n(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \simeq \mathrm{Sp}(\mathrm{Exc}_*^n(\mathcal{C}, \mathcal{D})) \qquad \mathrm{Homog}^n(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \simeq \mathrm{Sp}(\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})).$$

Combining this second isomorphism, Proposition 1.4.2.21, and Corollary 7.1.2.8, we obtain the following result:

Corollary 7.1.2.9. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object, let $\mathcal{D}$ be a pointed differentiable ∞-category, and let $n \geq 1$ be an integer. Then composition with the functor $\Omega^\infty : \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$ induces an equivalence of ∞-categories*

$$\mathrm{Homog}^n(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \to \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}).$$

Remark 7.1.2.10. Let $\mathcal{C}$ be an ∞-category which admits finite colimits and a final object, and let $\mathcal{D}$ be a pointed differentiable ∞-category. Then the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is also pointed. For $n \geq 1$ and $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, let $U(F)$ denote the fiber of the map $F \to P_{n-1}(F)$. Since the functor P_{n-1} is left exact, $P_{n-1}U(F)$ is the fiber of the equivalence $P_{n-1}(F) \to P_{n-1}P_{n-1}(F)$. It follows that $U(F)$ is n-reduced. If $G : \mathcal{C} \to \mathcal{D}$ is any n-reduced functor, then $\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}, \mathcal{D})}(G, P_{n-1}F) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}, \mathcal{D})}(P_{n-1}G, P_{n-1}F)$ is contractible, so that the canonical map

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}, \mathcal{D})}(G, U(F)) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}, \mathcal{D})}(G, F)$$

is a homotopy equivalence. It follows that the construction $F \mapsto U(F)$ is a right adjoint to the inclusion from the ∞-category of n-reduced functors to the ∞-category of all functors from $\mathcal{C}$ to $\mathcal{D}$. Note that if F is n-excisive, then $U(F)$ is also n-excisive (and therefore n-homogeneous).

Let $\mathcal{C}$ be an ∞-category which admits finite colimits and a final object, and let $\mathcal{C}_*$ be the ∞-category of pointed objects of $\mathcal{C}$. The forgetful functor $\theta : \mathcal{C}_* \to \mathcal{C}$ preserves final objects and strongly coCartesian cubes. It follows that for any differentiable ∞-category $\mathcal{D}$, composition with θ induces a functor $\nu : \mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \to \mathrm{Exc}^n(\mathcal{C}_*, \mathcal{D})$. Using Remark 7.1.1.30, we obtain an equivalence equivalence $P_n(F \circ \theta) \simeq P_n(F) \circ \theta$ for every functor $F : \mathcal{C} \to \mathcal{D}$. In particular, if F is n-reduced, then $F \circ \theta$ is also n-reduced. Consequently, ν restricts to a map $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \to \mathrm{Homog}^n(\mathcal{C}_*, \mathcal{D})$.

Proposition 7.1.2.11. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and a final object, and let $\mathcal{D}$ be a pointed differentiable ∞-category. For each integer $n \geq 1$, composition with the forgetful functor $\mathcal{C}_* \to \mathcal{C}$ induces an equivalence of ∞-categories $\phi : \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \to \mathrm{Homog}^n(\mathcal{C}_*, \mathcal{D})$.*

Proof. We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Homog}^n(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) & \longrightarrow & \mathrm{Homog}^n(\mathcal{C}_*, \mathrm{Sp}(\mathcal{D})) \\
\downarrow & & \downarrow \\
\mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) & \longrightarrow & \mathrm{Homog}^n(\mathcal{C}_*, \mathcal{D})
\end{array}
$$

where the vertical maps are equivalences by Corollary 7.1.2.9. We may therefore replace $\mathcal{D}$ by $\mathrm{Sp}(\mathcal{D})$ and thereby reduce to the case where $\mathcal{D}$ is stable.

Let $*$ denote the final object of $\mathcal{C}$. Since $\mathcal{C}$ admits finite colimits, the forgetful functor $\mathcal{C}_* \to \mathcal{C}$ admits a left adjoint, given by $X \mapsto X_+ = X \amalg *$. Composition with this left adjoint determines a functor $\psi_0 : \mathrm{Fun}(\mathcal{C}_*, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Since the construction $X \mapsto X_+$ preserves finite colimits, ψ_0 restricts to a functor $\psi_1 : \mathrm{Exc}^n(\mathcal{C}_*, \mathcal{D}) \to \mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$, which is right adjoint to the forgetful functor $\mathrm{Ext}^{(n)}(\mathcal{C}, \mathcal{D}) \to \mathrm{Exc}^{(n)}(\mathcal{C}_*, \mathcal{D})$. Let U be as in Remark 7.1.2.10, and let ψ denote the composition

$$
\mathrm{Homog}^n(\mathcal{C}_*, \mathcal{D}) \subseteq \mathrm{Exc}^n(\mathcal{C}_*, \mathcal{D}) \xrightarrow{\psi_1} \mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \xrightarrow{U} \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}),
$$

so that ψ is right adjoint to ϕ.

We next prove that for every $F \in \mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$, the unit map $u : F \to \psi\phi F$ is an equivalence. We first prove that u is an equivalence. By assumption, F is n-reduced. Consequently, we can identify u with $U(u_0)$, where u_0 is the unit map $F \to \psi_0\phi F$. Since $\mathcal{D}$ is stable, to prove that u is an equivalence, it will suffice to show that $\mathrm{fib}(u_0)$ is $(n-1)$-excisive. Let $S = [n-1] = \{0, \ldots, n-1\}$ and let X be a strongly coCartesian S-cube in $\mathcal{C}$; we wish to show that $\mathrm{fib}(u_0)(X)$ is a Cartesian S-cube in $\mathcal{D}$. Let $S_+ = [n] = S \cup \{n\}$, and let Y be the strongly coCartesian S-cube given by

$$
Y(T) = \begin{cases} X(T) & \text{if } n \notin T \\ X(T)_+ & \text{if } n \in T. \end{cases}
$$

To prove that $\mathrm{fib}(u_0)(X)$ is a Cartesian S-cube, it suffices to show that $F(Y)$ is a Cartesian S_+. Since F is n-excisive, it suffices to show that Y is a strongly coCartesian S-cube, which is clear.

It follows from the above argument that the functor ϕ is fully faithful. To complete the proof, we will show that the functor ψ is conservative. Because ψ is an exact functor between stable ∞-categories, it will suffice to show that if $G \in \mathrm{Homog}^n(\mathcal{C}_*, \mathcal{D})$ satisfies $\psi G \simeq 0$, then G is a zero object of $\mathrm{Homog}^{(n)}(\mathcal{C}_*, \mathcal{D})$. The assumption that $\psi G \simeq 0$ implies that the canonical map $\psi_1 G \to P_{n-1}(\psi_1 G)$ is an equivalence: that is, $\psi_1 G$ is $(n-1)$-excisive. We will show that G is $(n-1)$-excisive: combined with our assumption that G is $(n-1)$-reduced, this will allow us to conclude that G is a zero object of $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$.

Let $S = [n-1] = \{0, \ldots, n-1\}$ and let $X : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}_*$ be a strongly coCartesian S-cube. Let $S_+ = [n] = S \cup \{n\}$ and let $Y : \mathrm{N}(\mathbf{P}(S_+)) \to \mathcal{C}_*$ be defined by the formula

$$Y(T) = \begin{cases} (\theta X(T))_+ & \text{if } n \notin T \\ X(T - \{n\}) & \text{if } n \in T. \end{cases}$$

The assumption that X is a strongly coCartesian S-cube implies that Y is a strongly coCartesian S_+-cube. Since G is n-excisive, we conclude that $G(Y)$ is a Cartesian S_+-cube in $\mathcal{D}$. It follows that the diagram

$$
\begin{array}{ccc}
G(Y(\emptyset)) & \longrightarrow & G(Y(\{n\})) \\
\downarrow & & \downarrow \\
\varprojlim_{\emptyset \neq T \subseteq S} G(Y(T)) & \longrightarrow & \varprojlim_{\emptyset \neq T \subseteq S} G(y(T \cup \{n\}))
\end{array}
$$

is a pullback square in $\mathcal{D}$. The assumption that $\psi_1 G$ is $(n-1)$-excisive implies that the left vertical map is an equivalence. Using the stability of $\mathcal{D}$, we conclude that the right vertical map is also an equivalence: that is, $G(X)$ is a Cartesian S-cube. $\square$

We now turn to the proof of Theorem 7.1.2.4. We will need a rather elaborate construction. Fix an integer $m \geq 0$. Let $\mathbf{P} = \mathbf{P}_{>0}([n])$ denote the partially ordered set of nonempty subsets of the set $[n] = \{0, \ldots, n\}$. We define a functor $\chi_m : \mathrm{N}(\mathbf{P})^m \times \mathrm{Fun}_*(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ by the formula

$$\chi_i(S_1, \ldots, S_m, F) = F \circ C_{S_1} \circ \cdots \circ C_{S_m},$$

where the functors $C_T : \mathcal{C} \to \mathcal{C}$ are defined as in Construction 7.1.1.18. For every subset $I \subseteq \mathbf{P}^m$, let $U_I : \mathrm{Fun}_*(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ be given by the formula

$$U_I(F) = \varprojlim \chi_m |(\mathrm{N}(I) \times \{F\}).$$

Remark 7.1.2.12. Suppose we are given subsets $I \subseteq \mathbf{P}^m$ and $I' \subseteq \mathbf{P}^{m'}$. Then we can identify $I \times I'$ with a subset of $\mathbf{P}^{m+m'}$. Moreover, we have a canonical equivalence of functors $U_{I \times I'} \simeq U_{I'} \circ U_I$ from $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ to itself.

Let $B \subseteq \mathbf{P}$ be the collection of those subsets $S \subseteq [n]$ having nonempty intersection with $[n-1] = \{0, \ldots, n-1\} \subseteq \{0, \ldots, n\} \subseteq [n]$. For each integer $m \geq 0$, let $A_m \subseteq \mathbf{P}^m$ denote the collection of tuples $(S_1, \ldots, S_m)$ such that at least one of the sets S_i contains $n \in [n]$. We have a commutative diagram of subsets of $\mathbf{P}^{m+1}$:

$$
\begin{array}{ccccc}
\mathbf{P}^{m+1} & \longleftarrow & B^m \times \mathbf{P} & \longleftarrow & B^{m+1} \\
\uparrow & & \uparrow & & \uparrow \\
A_{m+1} & \longleftarrow & A_{m+1} \cap (B^m \times \mathbf{P}) & \longleftarrow & A_{m+1} \cap B^{m+1} \\
\uparrow & & \uparrow & & \\
A_m \times \mathbf{P} & \longleftarrow & (A_m \cap B^m) \times \mathbf{P} & &
\end{array}
$$

which determines a commutative diagram of left exact functors τ_m:

$$
\begin{array}{ccccc}
U_{\mathbf{P}^{m+1}} & \longrightarrow & U_{B^m \times \mathbf{P}} & \longrightarrow & U_{B^{m+1}} \\
\downarrow & & \downarrow & & \downarrow \\
U_{A_{m+1}} & \longrightarrow & U_{A_{m+1} \cap (B^m \times \mathbf{P})} & \longrightarrow & U_{A_{m+1} \cap B^{m+1}} \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} & & \\
U_{A_m \times \mathbf{P}} & \longrightarrow & U_{(A_m \cap B^m) \times \mathbf{P}} & &
\end{array}
$$

from $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ to itself.

Lemma 7.1.2.13. *For each $m \geq 0$, the functor $U_{A_m} : \mathrm{Fun}_*(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ carries each $F \in \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ to a final object of $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$.*

Proof. Let $A'_m = \mathbf{P}^m - B^m$. We claim that the inclusion of simplicial sets $\mathrm{N}(A'_m) \to \mathrm{N}(A_m)$ is right cofinal. To prove this, consider a sequence $(S_1, \ldots, S_m) \in A_m$, and let $W = \{(S'_1, \ldots, S'_n) \in A'_m : (\forall 1 \leq i \leq n)[S'_i \subseteq S_i]\}$. According to Theorem T.4.1.3.1, it will suffice to show that the partially ordered set W has weakly contractible nerve. Let $I \subseteq \{1, \ldots, m\}$ be the set of indices for which $n \in S_i$. For every subset $J \subseteq I$, let W_J denote the subset of W consisting of those tuples $(S'_1, \ldots, S'_m)$ such that $S'_j = \{n\}$ for $j \in J$. Then $\mathrm{N}(W)$ is the homotopy limit of the diagram of simplicial sets $\{\mathrm{N}(W_J)\}_{\emptyset \neq J \subseteq I}$. Since I is nonempty, it will suffice to show that each of the simplicial set $\mathrm{N}(W_J)$ is contractible. This is clear, since W_J has a largest element (namely, the tuple $(S'_1, \ldots, S'_m)$ with $S'_i = \begin{cases} \{n\} & \text{if } i \in J \\ S_i & \text{otherwise.} \end{cases}$).

It follows from the above argument that the inclusion $A'_m \hookrightarrow A_m$ induces an equivalence $U_{A_m}(F) \to U_{A'_m}(F)$ for every $F \in \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$. Note that if $(S_1, \ldots, S_m) \in A'_m$, then at least one of the sets S_i is a singleton, so that the composite functor $C_{S_1} \circ \cdots \circ C_{S_m} : \mathcal{C} \to \mathcal{C}$ carries each object of $\mathcal{C}$ to a final object of $\mathcal{C}$. Since F is reduced, it follows that

$$U_{A'_m}(F) = \varprojlim_{(S_1, \ldots, S_m) \in A'_m} (F \circ C_{S_1} \circ \cdots \circ C_{S_m})$$

carries each object of $\mathcal{C}$ to a final object of $\mathcal{D}$, as desired. $\square$

Lemma 7.1.2.14. *In the above diagram, the maps ϕ and ϕ' are equivalences of functors from $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ to itself.*

Proof. We will prove that the map ϕ is an equivalence; the proof for ϕ' is similar. Observe that $\mathrm{N}(A_{m+1})$ is the union of the simplicial subsets $\mathrm{N}(A_m \times \mathbf{P})$ and $\mathrm{N}(B^m \times A_1)$, whose intersection is $\mathrm{N}((B^m \cap A_m) \times A_1)$. Using the results of §T.4.2.3, we deduce that the diagram of functors

$$\begin{array}{ccc} U_{A_{m+1}} & \longrightarrow & U_{B^m \times A_1} \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} \\ U_{A_m \times \mathbf{P}} & \longrightarrow & U_{(B^m \cap A_m) \times A_1} \end{array}$$

is a pullback square. It will therefore suffice to show that ψ is an equivalence. Note that $U_{B^m \times A_1} \simeq U_{A_1} \circ U_{B^m}$ (Remark 7.1.2.12), and therefore carries each $F \in \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ to a final object of $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ (Lemma 7.1.2.13). The same argument shows that $U_{(B^m \cap A_m) \times A_1}$ carries each $F \in \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ to a final object of $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$, so that ψ is an equivalence as desired. $\square$

Proof of Theorem 7.1.2.4. For each $m \geq 0$, let σ_m be the diagram

$$\begin{array}{ccc} U_{\mathbf{P}^m} & \longrightarrow & U_{B^m} \\ \downarrow & & \downarrow \\ U_{A_m} & \longrightarrow & U_{B^m \cap A^m} \end{array}$$

of functors from the ∞-category $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ to itself. Let $T_n : \mathrm{Fun}_*(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ be defined as in Construction 7.1.1.22, so that $T_n \simeq U_{\mathbf{P}}$. Using Remark 7.1.2.12 and Lemma 7.1.2.14, we can identify $T_n \sigma_m$ with the commutative diagram

$$\begin{array}{ccc} U_{\mathbf{P}^{m+1}} & \longrightarrow & U_{B^m \times \mathbf{P}} \\ \downarrow & & \downarrow \\ U_{A_{m+1}} & \longrightarrow & U_{A_{m+1} \cap (B^m \times \mathbf{P})}, \end{array}$$

so that the commutative diagram τ_m above induces a natural transformation $\alpha_m : T_n(\sigma_m) \to \sigma_{m+1}$. Let σ_∞ denote a colimit of the sequence

$$\sigma_0 \to T_n(\sigma_0) \xrightarrow{\alpha_0} \sigma_1 \to T_n(\sigma_1) \xrightarrow{\alpha_1} \sigma_2 \to \cdots$$

Then σ_∞ is a commutative diagram of functors

$$
\begin{array}{ccc}
P & \longrightarrow & P' \\
\downarrow & & \downarrow \\
K & \longrightarrow & R
\end{array}
$$

from $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ to itself. We claim that this commutative diagram has the desired properties.

Using the fact that $U_{\mathbf{P}} = T_n$ and Remark 7.1.2.12, we see that $U_{\mathbf{P}^m}$ can be identified with T_n^m. Unwinding the definitions, we deduce that P is the colimit of the sequence of functors

$$\mathrm{id} \to T_n \to T_n^2 \to \cdots,$$

and can therefore be identified with the functor P_n of Construction 7.1.1.27. There is an evident inclusion of partially ordered sets $\mathbf{P}_{>0}([n-1]) \hookrightarrow B$, which induces a right cofinal map of simplicial sets $\mathrm{N}(\mathbf{P}_{>0}([n-1])) \hookrightarrow \mathrm{N}(B)$. It follows that $U_B \simeq U_{\mathrm{N}(\mathbf{P}_{>0}([n-1]))^{op}}$ can be identified with the functor $T_{n-1} : \mathrm{Fun}_*(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$. Using Remark 7.1.2.12, we obtain an equivalence of functors $U_{B^m} \simeq T_{n-1}^m$. The functor P' can be identified with the colimit of the sequence

$$\mathrm{id} \to T_{n-1} \to T_{n-1}^2 \to T_{n-1}^3 \to \cdots,$$

which is the functor P_{n-1} of Construction 7.1.1.27. Lemma 7.1.2.13 implies that K is the colimit of a sequence of functors which carry every functor $F \in \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ to a final object of $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$. It follows that K carries every functor $F \in \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ to a final object of $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$. We can therefore identify σ_∞ with a commutative diagram

$$
\begin{array}{ccc}
P_n & \longrightarrow & P_{n-1} \\
\downarrow & & \downarrow \\
K & \longrightarrow & R.
\end{array}
$$

Note that for $I \subseteq \mathbf{P}^m$, the functor $U_I : \mathrm{Fun}_*(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ is a limit of functors which preserve finite limits, and is therefore itself preserve finite limits. Since finite limits in $\mathcal{D}$ commute with sequential colimits, finite limits in $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ commute with sequential colimits. In particular, the collection of functors from $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ to itself which commute with finite limits is closed under sequential colimits, so that R preserves finite limits.

We now prove that σ_∞ is a pullback square. Since finite limits in $\mathcal{D}$ commute with sequential colimits, the collection of pullback square in $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ is closed under sequential colimits. It will therefore suffice to show that each σ_m is a pullback square of functors from $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ to itself. This follows from the results of §T.4.2.3, since the simplicial set $\mathrm{N}(\mathbf{P}^m)$ is the union of the simplicial subsets $\mathrm{N}(B^m)$ and $\mathrm{N}(A_m)$ having intersection $\mathrm{N}(B^m \cap A_m)$.

We next prove that if $F : \mathcal{C} \to \mathcal{D}$ is reduced and $(n-1)$-excisive, then $R(F)$ is a final object of $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$. Since we can write $R(F)$ as the colimit of a sequence of functors $U_{B^m \cap A_m}(F)$, it will suffice to show that each $U_{B^m \cap A_m}(F)$ is a final object of $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$. There is an isomorphism of partially ordered sets

$$B^m \cap A_m \simeq \mathbf{P}_{>0}([n - 1]) \times \mathbf{P}_{>0}(\{1, \ldots, m\}),$$

given by

$$(S_1, \ldots, S_m) \mapsto (S_1 \cap [n - 1], \ldots, S_m \cap [n - 1], \{i : n \in S_i\})$$

We may therefore write $U_{B^m \cap A_m}(F)$ as a limit of a diagram

$$G : \mathrm{N}(\mathbf{P}_{>0}([n-1]))^m \times \mathrm{N}(\mathbf{P}_{>0}(\{1, \ldots, m\})) \to \mathrm{Fun}_*(\mathcal{C}, \mathcal{D}).$$

For every nonempty subset $T \subseteq \{1, \ldots, m\}$, let G_T denote the restriction of G to $\mathrm{N}(\mathbf{P}_{>0}([n-1])^m \times \{T\}$. Then $U_{B^m \cap A_m}(F)$ can be written as the limit $\varprojlim_{\emptyset \neq T \subseteq \{1, \ldots, m\}} \varprojlim G_T$. It will therefore suffice to show that each of the functors $\varprojlim G_T$ is a final object of $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$. Using our assumption that F is $(n-1)$-excisive, we obtain a canonical equivalence

$$\varprojlim G_T \simeq F \circ C_{S_1} \circ \cdots \circ C_{S_m},$$

where $S_i = \begin{cases} \{n\} & \text{if } i \in T \\ \emptyset & \text{if } i \notin T. \end{cases}$ Since T is nonempty, at least one of the functors C_{S_i} carries every object of $\mathcal{C}$ to the final object (see Example 7.1.1.23). Using our assumption that F is reduced, we conclude that $\varprojlim G_T$ carries every object of $\mathcal{C}$ to a final object in $\mathcal{D}$. This completes the proof that $R(F)$ is a final object of $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ whenever F is $(n-1)$-excisive.

We complete the proof by showing that for an arbitrary functor $F \in \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$, the functor $R(F)$ is n-homogeneous. Note that we have a canonical equivalence

$$P_{n-1}(R(F)) \simeq R(P_{n-1}F).$$

Since $P_{n-1}F$ is reduced and n-excisive, $R(P_{n-1}F)$ is a final object of $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$, so that $R(F)$ is n-reduced. It will therefore suffice to show that $R(F)$ is n-excisive. Let $S = [n]$ and let $X : \mathrm{N}\,\mathbf{P}(S) \to \mathcal{C}$ be a strongly coCartesian n-cube. Then $(R(F))(X)$ is given by the colimit of a sequence of S-cubes

$$U_{B^0 \cap A_0}(F)(X) \xrightarrow{\beta_0} T_n(U_{B^0 \cap A_0}(F))(X) \to U_{B^1 \cap A_1}(F)(X) \xrightarrow{\beta_1} T_n(U_{B^1 \cap A_1}(F))(X) \to \cdots$$

According to Lemma 7.1.1.26, each of the maps β_i factors through a Cartesian S-cube. It follows that $(R(F))(X)$ can be written as the colimit of a sequence of Cartesian S-cubes, and is therefore Cartesian (since finite colimits in $\mathcal{D}$ commute with sequential limits). $\qquad\square$

7.1.3 Functors of Many Variables

Suppose we are given a functor $F : \mathcal{C}_- \times \mathcal{C}_+ \to \mathcal{D}$ between ∞-categories. We can think about F in several different ways:

(a) We can understand F as a family of functors $\mathcal{C}_+ \to \mathcal{D}$, parametrized by the objects of $\mathcal{C}_-$.

(b) We can understand F as a family of functors $\mathcal{C}_- \to \mathcal{D}$, parametrized by the objects of $\mathcal{C}_+$.

(c) We can understand F as a single functor $\mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ denotes the product $\mathcal{C}_- \times \mathcal{C}_+$.

The ideas of §7.1.1 and 7.1.2 can be applied from each of these perspectives, to obtain several different notions of what it means for F to be reduced, excisive and homogeneous. Our goal in this section is to study the relationships between these notions. We begin by introducing some terminology.

Definition 7.1.3.1. Suppose we are given ∞-categories $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ which admit pushouts, and an ∞-category $\mathcal{D}$ which admits finite limits, and a sequence of nonnegative integers $\vec{n} = (n_1, \ldots, n_m)$. We will say that a functor $F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \to \mathcal{D}$ is $\vec{n}$-*excisive* if, for all $1 \leq i \leq m$ and every sequence of objects $\{X_j \in \mathcal{C}_j\}_{j \neq i}$, the induced functor

$$\mathcal{C}_i \hookrightarrow \mathcal{C}_i \times \prod_{j \neq i}\{X_j\} \hookrightarrow \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \xrightarrow{F} \mathcal{D}$$

is n_i-excisive. We let $\mathrm{Exc}^{\vec{n}}(\prod \mathcal{C}_i, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\prod \mathcal{C}_i, \mathcal{D})$ spanned by those functors which are $\vec{n}$-excisive. In the special case where $\vec{n} = (1, 1, \ldots, 1)$, we will denote $\mathrm{Exc}^{\vec{n}}(\prod \mathcal{C}_i, \mathcal{D})$ by $\mathrm{Exc}(\prod \mathcal{C}_i, \mathcal{D})$.

Warning 7.1.3.2. The notation of Definition 7.1.3.1 is potentially ambiguous. Suppose we are given a finite collection of ∞-categories $\{\mathcal{C}_s\}_{s \in S}$ which admit pushouts, and let $\mathcal{C} = \prod_{s \in S} \mathcal{C}_s$. Then $\mathrm{Exc}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$ and $\mathrm{Exc}(\mathcal{C}, \mathcal{D})$ denote two *different* ∞-categories: $\mathrm{Exc}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$ is the full subcategory of $\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$ spanned by those functors which are excisive separately in each variable, while $\mathrm{Exc}(\mathcal{C}, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$ spanned by those functors which are excisive when viewed as a functor of a single variable.

Remark 7.1.3.3. In the situation of Definition 7.1.3.1, suppose that $m > 0$ and set let $\vec{n}' = (n_2, \ldots, n_m)$. We then have a canonical isomorphism

$$\mathrm{Exc}^{\vec{n}}(\prod_{1 \leq i \leq m} \mathcal{C}_i, \mathcal{D}) \simeq \mathrm{Exc}^{n_1}(\mathcal{C}_1, \mathrm{Exc}^{\vec{n}'}(\prod_{2 \leq i \leq m} \mathcal{C}_i, \mathcal{D})).$$

Proposition 7.1.3.4. *Let $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be ∞-categories which admit finite colimits, $\mathcal{D}$ an ∞-category which admits finite limits, and $F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \to \mathcal{D}$ an $(n_1, \ldots, n_m)$-excisive functor. Then F is n-excisive when regarded as a functor of one variable (with values in $\mathcal{C} = \mathcal{C}_1 \times \ldots \times \mathcal{C}_m$), where $n = n_1 + \cdots + n_m$.*

Corollary 7.1.3.5. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits, $\delta : \mathcal{C} \to \mathcal{C}^m$ the diagonal map, $\mathcal{D}$ a functor which admits finite limits, and $F : \mathcal{C}^m \to \mathcal{D}$ an $(n_1, \ldots, n_m)$-excisive functor. Then $F \circ \delta$ is n-excisive, where $n = n_1 + \cdots + n_m$.*

Proof. Combine Proposition 7.1.3.4 with the observation that the diagonal map δ preserves strongly co-Cartesian cubes. $\square$

Proof of Proposition 7.1.3.4. Let $S = [n] = \{0, \ldots, n\}$ and let $X : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$ be a strongly coCartesian S-cube in $\mathcal{C}$, corresponding to a sequence of strongly coCartesian S-cubes $\{X_i : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}_i\}_{1 \leq i \leq m}$. Let $\delta_0 : \mathrm{N}(\mathbf{P}(S)) \to \mathrm{N}(\mathbf{P}(S))^m$ be the diagonal map. We wish to show that the composition

$$\mathrm{N}(\mathbf{P}(S)) \xrightarrow{\delta_0} \mathrm{N}(\mathbf{P}(S)^m) \xrightarrow{\prod X_i} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is a Cartesian S-cube in $\mathcal{D}$.

Let $A \subseteq \mathbf{P}(S)^m$ be the image of $\mathbf{P}_{>0}(S)$ under the map δ_0: that is, A is the collection of sequences $(S_1, \ldots, S_m)$ where $S_1 = S_2 = \cdots = S_m$ is a nonempty subset of S. Let $Y : \mathrm{N}(\mathbf{P}(S)^m) \to \mathcal{D}$ be the composition of F with $\prod X_i$. We wish to show that Y exhibits $Y(\emptyset, \ldots, \emptyset)$ as a limit of the diagram $Y | \mathrm{N}(A)$.

Let $B \subseteq \mathbf{P}(S)^m$ be the collection of sequences $(S_1, \ldots, S_m)$ for which the intersection $S_1 \cap \cdots \cap S_m$ is nonempty. Then $A \subseteq B$. Moreover, the inclusion $A \subseteq B$ admits a right adjoint, given by $(S_1, \ldots, S_m) \mapsto (T, \ldots, T)$ where $T = \bigcap S_i$. It follows that the canonical map $\varprojlim(Y | \mathrm{N}(B)) \to \varprojlim(Y | \mathrm{N}(A))$ is an equivalence. It will therefore suffice to show that Y exhibits $Y(\emptyset, \ldots, \emptyset)$ as a limit of the diagram $Y | \mathrm{N}(B)$. We will prove a stronger assertion: namely, that the functor Y is a right Kan extension of $Y | \mathrm{N}(B)$.

Choose a sequence of subsets

$$B = B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_k = \mathbf{P}(S)^m$$

so that the following conditions are satisfied:

(a) Each B_j is closed upwards in the partially ordered set $\mathbf{P}(S)^m$ (that is, if $(S_1, \ldots, S_m) \in B_j$ and $S_i \subseteq S_i' \subseteq S$, then $(S_1', \ldots, S_m') \in B_j$).

(b) For $0 < j \leq k$, the set B_j is obtained from B_{j-1} by adding a single element $(S_1, \ldots, S_m) \in \mathbf{P}(S)^m$.

To prove that Y is a right Kan extension of $Y | \mathrm{N}(B)$, it will suffice to show that $Y | \mathrm{N}(B_j)$ is a right Kan extension of $Y | \mathrm{N}(B_{j-1})$ for $0 < j \leq k$. Let us suppose that B_j is obtained from B_{j-1} by adjoining a single element $(S_1, \ldots, S_m) \in \mathbf{P}(S)^m$. Then $(S_1, \ldots, S_m) \notin B$, so we have $S = \bigcup_i (S - S_i)$. Since the cardinality of

S is larger than $n = n_1 + \cdots + n_m$, we conclude that there is an integer a with $1 \leq a \leq m$ such that $S - S_a$ has cardinality larger than n_a.

Let $T = \coprod_{1 \leq i \leq m}(S - S_i)$. Note that there is bijective correspondence between subsets of T and sequences $(S_1', \ldots, S_m') \in \mathbf{P}(S)^m$ such that $S_i \subseteq S_i'$ for all i. Consequently, Y determines a T-cube $Y_T : \mathrm{N}(\mathbf{P}(T)) \to \mathcal{D}$. Using (a), we see that $Y \,|\, \mathrm{N}(B_j)$ is a right Kan extension of $Y \,|\, \mathrm{N}(B_{j-1})$ if and only if Y_T is a Cartesian T-cube in $\mathcal{D}$. Let T_0 be a subset of $S - S_a$ having cardinality $n_a + 1$, and regard T_0 as a subset of T. According to Proposition 7.1.1.13, it will suffice to show that every T_0-face of Y_T is Cartesian. This follows immediately from our assumption that the functor F is n_a-excisive in the ath variable. $\qquad\square$

We have the following analogue of Theorem 7.1.1.10:

Proposition 7.1.3.6. *Let $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be ∞-categories which admit finite colimits and final objects, and let $\mathcal{D}$ be a pointed differentiable ∞-category. For every sequence of nonnegative integers $\vec{n} = (n_1, \ldots, n_m)$, the inclusion functor $\iota : \mathrm{Exc}^{\vec{n}}(\prod_{1 \leq i \leq m} \mathcal{C}_i, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\prod_{1 \leq i \leq m} \mathcal{C}_i, \mathcal{D})$ admits a left adjoint $P_{\vec{n}}$. Moreover, the functor $P_{\vec{n}}$ is left exact.*

Proof. We proceed by induction on m. If $m = 0$, then $\mathrm{Exc}^{\vec{n}}(\prod_{1 \leq i \leq m} \mathcal{C}_i, \mathcal{D}) = \mathrm{Fun}(\prod_{1 \leq i \leq m} \mathcal{C}_i, \mathcal{D})$ and there is nothing to prove. Assume therefore that $m > 0$, and set $\vec{n}' = (n_2, \ldots, n_m)$. Using Remark 7.1.3.3, we see that i is equivalent to the composition

$$\mathrm{Fun}(\mathcal{C}_1, \mathrm{Fun}(\prod_{2 \leq i \leq m} \mathcal{C}_i, \mathcal{D})) \xrightarrow{\iota'} \mathrm{Fun}(\mathcal{C}_1, \mathrm{Exc}^{\vec{n}'}(\prod_{2 \leq i \leq m} \mathcal{C}_i, \mathcal{D})) \xrightarrow{\iota''} \mathrm{Exc}^{n_1}(\mathcal{C}_1, \mathrm{Exc}^{\vec{n}'}(\prod_{2 \leq i \leq m} \mathcal{C}_i, \mathcal{D})).$$

The inductive hypothesis implies that the inclusion $\mathrm{Exc}^{\vec{n}'}(\prod_{2 \leq i \leq m} \mathcal{C}_i, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\prod_{2 \leq i \leq m} \mathcal{C}_i, \mathcal{D})$ admits a left exact left adjoint. It follows that ι' admits a left exact left adjoint. We now complete the proof by invoking Theorem 7.1.1.10, which guarantees that ι'' admits a left exact left adjoint. $\qquad\square$

We now turn our attention to reduced functors.

Definition 7.1.3.7. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ be ∞-category which admit finite colimits and final objects, let $\mathcal{D}$ be a differentiable ∞-category, and let $\vec{n} = (n_1, \ldots, n_m)$ be a sequence of positive integers. We will say that a functor $F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \to \mathcal{D}$ is *$\vec{n}$-reduced* if, for all $1 \leq i \leq m$ and every sequence of objects $\{X_j \in \mathcal{C}_j\}_{j \neq i}$, the induced functor

$$\mathcal{C}_i \hookrightarrow \mathcal{C}_i \times \prod_{j \neq i}\{X_j\} \hookrightarrow \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \xrightarrow{F} \mathcal{D}$$

is n_i-reduced. We will say that F is *$\vec{n}$-homogeneous* if F is $\vec{n}$-reduced and $\vec{n}$-excisive.

We will say that F is *reduced* if it is $(1, 1, \ldots, 1)$-reduced: that is, if F is reduced in each variable. We let $\mathrm{Fun}_*(\prod_{1 \leq i \leq m} \mathcal{C}_i, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\prod_{1 \leq i \leq m} \mathcal{C}_i, \mathcal{D})$ spanned by the reduced functors. We will say that F is *multilinear* if it is $(1, 1, \ldots, 1)$-homogeneous; we let

$$\mathrm{Exc}_*(\prod_{1 \leq i \leq m} \mathcal{C}_i, \mathcal{D}) = \mathrm{Exc}(\prod_{1 \leq i \leq m} \mathcal{C}_i, \mathcal{D}) \cap \mathrm{Fun}_*(\prod_{1 \leq i \leq m} \mathcal{C}_i, \mathcal{D})$$

denote the full subcategory of $\mathrm{Fun}(\prod_{1 \leq i \leq m} \mathcal{C}_i, \mathcal{D})$ spanned by the multilinear functors.

Warning 7.1.3.8. The notation of Definition 7.1.3.7 is potentially ambiguous. Given a finite collection of ∞-categories $\{\mathcal{C}_s\}_{s \in S}$ which admit finite colimits and final objects, set $\mathcal{C} = \prod_{s \in S} \mathcal{C}_s$. Then the full subcategories $\mathrm{Fun}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{D}), \mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{D}) \subseteq \mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$ are generally different from the full subcategories $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D}), \mathrm{Exc}_*(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ introduced in Definition 1.4.2.1. A functor $F : \mathcal{C} = \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ belongs to $\mathrm{Fun}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$ $(\mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{D}))$ if it is reduced (reduced and excisive) separately in each variable; it belongs to $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ $(\mathrm{Exc}_*(\mathcal{C}, \mathcal{D}))$ if it is reduced (reduced and excisive) when viewed as a functor of a single variable.

Remark 7.1.3.9. The requirement that a functor $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ be reduced can be stated more simply as follows: given a collection of objects $\{X_s \in \mathcal{C}_s\}$, if any X_s is a final object of $\mathcal{C}_s$, then $F(\{X_s\}_{s \in S})$ is a final object of $\mathcal{D}$.

Proposition 7.1.3.10. *Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ be ∞-category which admit finite colimits and final objects, let $\mathcal{D}$ be a differentiable ∞-category, and let $F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \to \mathcal{D}$ be a functor which is $(1, \ldots, 1)$-reduced. Let $\mathcal{C} = \prod_{1 \leq i \leq m} \mathcal{C}_i$. Then F is m-reduced when viewed as a functor from $\mathcal{C}$ to $\mathcal{D}$.*

Corollary 7.1.3.11. *In the situation of Proposition 7.1.3.10, suppose that F is $(1, \ldots, 1)$-homogeneous. Then F is m-homogeneous when regarded as a functor from $\mathcal{C}$ to $\mathcal{D}$.*

Proof. Combine Propositions 7.1.3.4 and 7.1.3.10. $\square$

Corollary 7.1.3.12. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and a final object, let $\mathcal{D}$ be a differentiable ∞-category, and let $F : \mathcal{C}^m \to \mathcal{D}$ be a functor which is $(1, \ldots, 1)$-reduced. Then the composite functor $f : \mathcal{C} \hookrightarrow \mathcal{C}^m \overset{F}{\to} \mathcal{D}$ is m-reduced. If F is multilinear, then f is m-homogeneous.*

Proof. The first assertion follows by combining Proposition 7.1.3.10 with Remark 7.1.1.30 (note that the diagonal map $\mathcal{C} \to \mathcal{C}^m$ preserves final objects and pushout squares), and the second assertion follows from the first and Proposition 7.1.3.4. $\square$

Proof of Proposition 7.1.3.10. We will prove the following:

($*$) Let $X \in \mathcal{C}$ and $G : \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \to \mathcal{D}$ be a $(1, \ldots, 1)$-reduced functor. Let G' be the same functor, regarded as a functor of one variable. Then the map $u : G(X) \to (T_{n-1}G')(X)$ factors through a final object of $\mathcal{D}$.

Assuming ($*$), we can complete the proof as follows. Let $X \in \mathcal{C}$ be an object and let F' be the functor F, regarded as a functor of one variable. We wish to show that $(P_{n-1}F')(X)$ is a final object of $\mathcal{D}$. We can write $(P_{n-1}F')(X)$ as the colimit of a sequence

$$F'(X) \overset{\alpha_0}{\to} (T_{n-1}F')(X) \overset{\alpha_1}{\to} (T_{n-1}^2 F')(X) \to \cdots .$$

Using ($*$), we deduce that each of the maps α_i factors through a final object of $\mathcal{D}$. Thus $(P_{n-1}F')(X)$ is the colimit of a sequence of final objects of $\mathcal{D}$, and is therefore itself final.

It remains to prove ($*$). The object $X \in \mathcal{C}$ corresponds to a sequence of objects $\{X_i \in \mathcal{C}_i\}_{1 \leq i \leq m}$. Let $S = \{1, \ldots, m\}$, and consider the functor $Y : \mathrm{N}(\mathbf{P}(S)^m) \to \mathcal{D}$ given by the formula

$$Y(S_1, \ldots, S_m) = G(C_{S_1}(X_1), \ldots, C_{S_m}(X_m)).$$

Let $A \subseteq \mathbf{P}(S)^m$ be the collection of all sequences of the form $(S_1, \ldots, S_m)$ where $S_1 = S_2 = \cdots = S_m$ and each S_i is nonempty. Unwinding the definitions, we can identify u with the restriction map

$$G(X) = Y(\emptyset, \ldots, \emptyset) \simeq \varprojlim(Y) \to \varprojlim(Y \,|\, \mathrm{N}(A)).$$

Let B denote the subset of $\mathbf{P}(S)^m$ consisting of those sequences $(S_1, \ldots, S_m)$ such that $i \in S_i$ for some $i \in \{1, \ldots, m\}$. Then $A \subseteq B$, so u factors as a composition

$$\varprojlim(Y) \to \varprojlim(Y \,|\, \mathrm{N}(B)) \to \varprojlim(Y \,|\, \mathrm{N}(A)).$$

We will complete the proof by showing that $\varprojlim(Y \,|\, \mathrm{N}(B))$ is a final object of $\mathcal{D}$.

Let $B_0 \subseteq B$ be the subset consisting of those sequences $(S_1, \ldots, S_m)$ such that $S_i = \{i\}$ for some $i \in \{1, \ldots, m\}$. We claim that the inclusion $\phi : \mathrm{N}(B_0) \hookrightarrow \mathrm{N}(B)$ is right cofinal. According to Theorem T.4.1.3.1, it will suffice to show that if $(S_1, \ldots, S_m) \in B$, then the partially ordered set

$$V = \{(S_1', \ldots, S_m') \in B_0 : S_i' \subseteq S_i\}$$

has weakly contractible nerve. Let V_0 be the subset of V consisting of those sequences $(S'_1, \ldots, S'_m)$ such that $S'_i \subseteq \{i\}$ for all i. The inclusion $V_0 \subseteq V$ has a right adjoint, given by the construction

$$(S'_1, \ldots, S'_m) \to (S'_1 \cap \{1\}, \ldots, S'_m \cap \{m\}).$$

It follows that the inclusion $N(V_0) \subseteq N(V)$ is a weak homotopy equivalence. It will therefore suffice to show that $N(V_0)$ is weakly contractible. This is clear, since V_0 has a final object.

The right cofinality of ϕ implies that the restriction map $\varprojlim(Y \,|\, N(B)) \to \varprojlim(Y \,|\, N(B_0))$ is an equivalence. It will therefore suffice to show that $\varprojlim(Y \,|\, N(B_0))$ is a final object of $\mathcal{D}$. In fact, we claim that $Y(S_1, \ldots, S_m) \in \mathcal{D}$ is final whenever $(S_1, \ldots, S_m) \in B_0$. For this, it suffices to observe that one of the sets S_i is a singleton, so that $C_{S_i}(X_i)$ is a final object of $\mathcal{C}$. Then

$$Y(S_1, \ldots, S_m) = G(C_{S_1}(X_1), \ldots, C_{S_m}(X_m))$$

is final by virtue of our assumption that G is $(1, \ldots, 1)$-reduced. $\qquad\square$

We next establish a partial converse to Proposition 7.1.3.4.

Proposition 7.1.3.13. *Let $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be ∞-categories which admit finite colimits and final objects, let $\mathcal{D}$ be an ∞-category which admits finite limits, and suppose that $F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \to \mathcal{D}$ is a functor which is reduced in each variable. Let $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_m$ and let $F' = F$, regarded as a functor (of one variable) from $\mathcal{C}$ to $\mathcal{D}$. If F' is m-excisive, then F is $(1, \ldots, 1)$-excisive.*

Proof. Without loss of generality, it will suffice to show that F is excisive in its first argument. Suppose we are given objects $\{X_i \in \mathcal{C}_i\}_{2 \leq i \leq n}$ and a pushout square σ :

$$
\begin{array}{ccc}
Y & \longrightarrow & Z \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Z'
\end{array}
$$

in $\mathcal{C}_1$. We wish to show that the diagram τ :

$$
\begin{array}{ccc}
F(Y, X_2, \ldots, X_m) & \longrightarrow & F(Z, X_2, \ldots, X_m) \\
\downarrow & & \downarrow \\
F(Y', X_2, \ldots, X_m) & \longrightarrow & F(Z', X_2, \ldots, X_m)
\end{array}
$$

is a pullback square in $\mathcal{D}$. Let $S = [m] = \{0, \ldots, m\}$. For $2 \leq i \leq m$, choose a morphism $X_i \to *_i$, where $*_i$ is a final object of $\mathcal{C}_i$. These morphisms determine maps of simplicial sets $\Delta^1 \to \mathcal{C}_i$. Taking the product of these maps with $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}_1$, we obtain a strongly coCartesian S-cube

$$U : N(\mathbf{P}(S)) \simeq (\Delta^1)^{m+1} \to \mathcal{C}_1 \times \cdots \times \mathcal{C}_m.$$

Since F' is m-excisive, we deduce that $F(U)$ is a Cartesian S-cube in $\mathcal{D}$. Our assumption that F is reduced in each variable implies that $F(U)(T)$ is a final object of $\mathcal{D}$ unless $T \subseteq \{0, 1\}$. It follows that $F(U)$ is a right Kan extension of its restriction to $N(\mathbf{P}(\{0, 1\}))$, so that $\tau = F(U) \,|\, N(\mathbf{P}(\{0, 1\}))$ is a pullback diagram as desired. $\qquad\square$

Corollary 7.1.3.14. *Let $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be ∞-categories which admit finite colimits and final objects, let $\mathcal{D}$ be a differentiable ∞-category, and let $F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \to \mathcal{D}$ be a functor which is reduced in each variable. Let $\mathcal{C} = \prod_{1 \leq i \leq m} \mathcal{C}_i$, and let F' denote the map F, regarded as a functor from $\mathcal{C}$ to $\mathcal{D}$. Then there is a canonical equivalence $P_m(F') \simeq P_{1, \ldots, 1}(F)$ (where $P_{1, \ldots, 1}(F)$ is defined as in Proposition 7.1.3.6).*

Proof. Since $P_{1,\ldots,1}(F)$ is $(1,\ldots,1)$-excisive, it is m-excisive when viewed as a functor of one variable (Proposition 7.1.3.4). It follows that the canonical map $F \to P_{1,\ldots,1}(F)$ factors as a composition

$$F \to P_m(F') \xrightarrow{\alpha} P_{1,\ldots,1}(F),$$

for some map α which is uniquely determined up to homotopy.

For $1 \le i \le m$, let $\mathcal{E}_i \subseteq \mathcal{C}$ be the full subcategory spanned by those sequences $(X_1,\ldots,X_m)$ where X_i is a final object of $\mathcal{C}_i$. The inclusion $\mathcal{E}_i \subseteq \mathcal{C}$ preserves pushout squares, so that $(P_m F')|\,\mathcal{E}_i = P_m(F'|\,\mathcal{E}_i)$. Since F is reduced in each variable, $F'|\,\mathcal{E}_i$ carries each object of $\mathcal{E}_i$ to a final object of $\mathcal{D}$. It follows that $P_m(F'|\,\mathcal{E}_i)$ has the same property, so that $P_m F'$ is reduced in each variable. Invoking Proposition 7.1.3.13, we deduce that $P_m F'$ is $(1,\ldots,1)$-excisive, so that the canonical map $F \to P_m F'$ admits a factorization

$$F \to P_{1,\ldots,1}(F) \xrightarrow{\beta} P_m(F').$$

It is easy to see that α and β are homotopy inverse to one another. $\square$

We now describe a procedure for replacing an arbitrary functor $F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \to \mathcal{D}$ by a reduced functor.

Construction 7.1.3.15. Let $\mathcal{C}_1,\ldots,\mathcal{C}_m$ be ∞-categories which admit final objects $\{*_i \in \mathcal{C}_i\}_{1 \le i \le m}$, and let $\mathcal{D}$ be a pointed ∞-category which admits finite limits. For $1 \le i \le m$, let $U_i : \mathcal{C}_i \to \mathcal{C}_i$ denote the constant functor taking the value $*_i$, and choose a natural transformation of functors $\alpha_i : \mathrm{id}_{\mathcal{C}_i} \to U_i$. Let $S = \{1,\ldots,m\}$. For each functor $F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \to \mathcal{D}$, consider the functor

$$\overline{F} : \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \times \mathrm{N}(\mathbf{P}(S)) \xrightarrow{\prod \alpha_i} \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \xrightarrow{F} \mathcal{D}$$

For each $T \subseteq S$, we let F^T denote the restriction of $\overline{F}$ to $\mathcal{C}_1 \times \cdots \times \mathcal{C}_m \times \{T\}$, so that F^T is given by the formula $F^T(X_1,\ldots,X_m) = F(X_1',\ldots,X_m')$ where $X_i' = \begin{cases} X_i & \text{if } i \notin T \\ *_i & \text{if } i \in T. \end{cases}$.

The functor $\overline{F}$ determines a natural transformation $\beta : F = F^\emptyset \to \varprojlim_{\emptyset \ne T \subseteq S} F^T$. We let $\mathrm{Red}(F)$ denote the fiber of β (in the pointed ∞-category $\mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \mathcal{D})$). We will refer to $\mathrm{Red}(F)$ as the *reduction* of F.

Example 7.1.3.16. In the situation of Construction 7.1.3.15, suppose that F is constant in its ith variable, for some $1 \le i \le m$. Then $\mathrm{Red}(F)$ carries each object of $\mathcal{C}_1 \times \ldots \times \mathcal{C}_m$ to a final object of $\mathcal{D}$.

Proposition 7.1.3.17. *Let $F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \to \mathcal{D}$ be a functor between ∞-categories. Assume that each $\mathcal{C}_i$ has a final object and that $\mathcal{D}$ is pointed and admits finite limits. Then:*

(a) *The functor $\mathrm{Red}(F) : \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \to \mathcal{D}$ is reduced.*

(b) *Let $G : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{D}$ be any reduced functor. Then the canonical map $\mathrm{Red}(F) \to F$ induces a homotopy equivalence*

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \mathcal{D})}(G, \mathrm{red}(F)) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \mathcal{D})}(G, F).$$

Proof. We first prove (a). For $T \subseteq S = \{1,\ldots,m\}$, let F^T be defined as in Construction 7.1.3.15. Suppose we are given objects $\{X_i \in \mathcal{C}_i\}_{1 \le i \le m}$ such that some X_j is a final object of $\mathcal{C}_j$. Then for $T \subseteq S$, the canonical map $F^T(X_1,\ldots,X_m) \to F^{T \cup \{j\}}(X_1,\ldots,X_m)$ is an equivalence. It follows that the diagram $\{F^T(X_1,\ldots,X_m)\}_{\emptyset \ne T \subseteq S}$ is a right Kan extension of $\{F^T(X_1,\ldots,X_m)\}_{\{j\} \subseteq T \subseteq S}$, so that the canonical map

$$\varprojlim_{\emptyset \ne T \subseteq S} F^T(X_1,\ldots,X_m) \to F^{\{j\}}(X_1,\ldots,X_m)$$

is an equivalence. Consequently, $\mathrm{Red}(F)(X_1, \ldots, X_m)$ is given by the fiber of the map

$$F^{\emptyset}(X_1, \ldots, X_m) \to F^{\{j\}}(X_1, \ldots, X_m).$$

Since this map is an equivalence, we deduce that $\mathrm{Red}(F)(X_1, \ldots, X_m)$ is a final object of $\mathcal{D}$.

We now prove (b). We have a fiber sequence of spaces

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \mathcal{D})}(G, \mathrm{Red}(F)) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \mathcal{D})}(G, F) \to \varprojlim_{\emptyset \neq T \subseteq S} \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \mathcal{D})}(G, F^T).$$

It will therefore suffice to show that the mapping space $\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \mathcal{D})}(G, F^T)$ is contractible for every nonempty subset $T \subseteq S$. Choose an element $j \in T$, and let $\mathcal{E} \subseteq \mathcal{C}_1 \times \cdots \times \mathcal{C}_m$ be the full subcategory spanned by those objects $(X_1, \ldots, X_m)$ for which X_j is a final object. Note that F^T is a right Kan extension of $F^T | \mathcal{E}$, so the restriction map $\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \mathcal{D})}(G, F^T) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{E}, \mathcal{D})}(G | \mathcal{E}, F^T | \mathcal{E})$ is a homotopy equivalence. It will therefore suffice to show that $\mathrm{Map}_{\mathrm{Fun}(\mathcal{E}, \mathcal{D})}(G | \mathcal{E}, F^T | \mathcal{E})$ is contractible. In fact, we claim that $G | \mathcal{E}$ is an initial object of $\mathrm{Fun}(\mathcal{E}, \mathcal{D})$. This follows immediately from our assumption that G is reduced (since the ∞-category $\mathcal{D}$ is assumed to be pointed). $\qquad\square$

Corollary 7.1.3.18. *Let $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be ∞-categories which admit final objects, let $\mathcal{D}$ be a pointed ∞-category which admits finite limits, and let $\mathrm{Fun}_*(\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n, \mathcal{D})$ spanned by the reduced functors. Then the inclusion*

$$\mathrm{Fun}_*(\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \mathcal{D})$$

admits a right adjoint, given by the construction $F \mapsto \mathrm{Red}(F)$.

Remark 7.1.3.19. Let $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be ∞-categories which admit finite colimits and final object, let $\mathcal{C} = \prod_{1 \leq i \leq m} \mathcal{C}_m$, and let $\mathcal{D}$ be a pointed differentiable ∞-category. Since the localization functors

$$P_n : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

are left exact, we have a canonical equivalence

$$P_n(\mathrm{Red}(F)) \simeq \mathrm{Red}(P_n F)$$

for every functor $F : \mathcal{C} \to \mathcal{D}$. In the case $n = m$, we can identify the left hand side with $P_{1,\ldots,1}(\mathrm{Red}(F))$ (Corollary 7.1.3.14) and thereby obtain an equivalence $P_{1,\ldots,1}(\mathrm{Red}(F)) \simeq \mathrm{Red}(P_n F)$.

Construction 7.1.3.20. Let $\mathcal{C}$ be an ∞-category which admits finite colimits and a final object, and let $\mathcal{D}$ be a pointed ∞-category which admits finite limits. Consider the functor $q : \mathcal{C}^n \to \mathcal{C}$, given by the formula

$$(X_1, \ldots, X_n) \to \coprod_{1 \leq i \leq n} X_i.$$

For every functor $F : \mathcal{C} \to \mathcal{D}$, we let $\mathrm{cr}_n(F) = \mathrm{Red}(F \circ q)$ denote the reduction of the functor

$$(X_1, \ldots, X_n) \mapsto F(X_1 \amalg \cdots \amalg X_n).$$

We will refer to $\mathrm{cr}_n(F)$ as the *nth cross effect* of F.

Variant 7.1.3.21. Suppose we are given a finite collection of ∞-categories $\{\mathcal{C}_s\}_{s \in S}$ which admit finite colimits and final objects and a collection of nonnegative integers $\vec{n} = \{n_s\}_{s \in S}$. Let $\mathcal{C} = \prod_{s \in S} \mathcal{C}_s$ and let $\mathcal{D}$ be a pointed ∞-category which admits finite limits. The product of the maps $q_s : \mathcal{C}_s^{n_s} \to \mathcal{C}_s$ of Construction 7.1.3.20 gives a map $q : \prod_{s \in S} \mathcal{C}_s^{n_s} \to \mathcal{C}$. We define $\mathrm{cr}_{\vec{n}}(F) = \mathrm{Red}(F \circ q)$. When S has a single element, this reduces to the cross effect appearing in Construction 7.1.3.20. When each of the integers n_s is equal to 1, it reproduces the reduction functor of Construction 7.1.3.15.

Proposition 7.1.3.22. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and a final object, let $\mathcal{D}$ be a pointed differentiable ∞-category, and let $F : \mathcal{C} \to \mathcal{D}$ be an n-excisive functor. For each $m \leq n + 1$, the cross-effect $\mathrm{cr}_m(F) : \mathcal{C}^m \to \mathcal{D}$ is $(n - m + 1, \dots, n - m + 1)$-excisive.*

Proof. The proof proceeds by induction on m. When $m = 0$ it is vacuous. When $m = 1$, $\mathrm{cr}_1(F)$ is given by the fiber of a natural transformation $F \to F_0$, where F_0 is a constant functor. Since F and F_0 are both n-excisive, we conclude that $\mathrm{cr}_1(F)$ is n-excisive. Let us therefore assume that $m \geq 2$. Fix objects $X_2, X_3, \dots, X_m \in \mathcal{C}$; we will show that the functor

$$X_1 \mapsto \mathrm{cr}_m(F)(X_1, \dots, X_m)$$

is $(n - m + 1)$-excisive. Let $*$ denote a final object of $\mathcal{C}$. Let $G, G'' : \mathcal{C} \to \mathcal{D}$ be defined by the formulas

$$G(X) = G(X \amalg X_m) \qquad G''(X) = F(X \amalg *).$$

Let G' be the fiber of the natural transformation $G \to G''$ induced by the map $X_m \to *$. Unwinding the definitions, we obtain an equivalence

$$\mathrm{cr}_m(F)(X_1, \dots, X_m) \simeq \mathrm{cr}_{m-1}(G')(X_1, \dots, X_{m-1}).$$

It will therefore suffice to show that $\mathrm{cr}_{m-1}(G')$ is $(n - m + 1, \dots, n - m + 1)$-excisive. Using the inductive hypothesis, we are reduced to proving that G' is $(n - 1)$-excisive. Let $S = [n - 1] = \{0, \dots, n - 1\}$ and let $Y : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$ be a strongly coCartesian S-cube. Let $S_+ = S \cup \{n\}$ and define a functor $Y_+ : \mathrm{N}(\mathbf{P}(S_+)) \to \mathcal{C}$ by the formula

$$Y_+(T) = \begin{cases} Y(T) \amalg X_m & \text{if } n \notin T \\ Y(T) \amalg * & \text{if } n \in T. \end{cases}$$

We observe that Y_+ is a strongly coCartesian S_+-cube in $\mathcal{C}$. Since F is n-excisive, $F(Y_+)$ is a Cartesian S_+-cube in $\mathcal{D}$. It follows that the diagram

$$\begin{array}{ccc} F(Y(\emptyset) \amalg X_m) & \longrightarrow & \varprojlim_{\emptyset \neq T \subseteq S} F(Y(T) \amalg X_m) \\ \downarrow & & \downarrow \\ F(Y(\emptyset) \amalg *) & \longrightarrow & \varprojlim_{\emptyset \neq T \subseteq S} F(Y(T) \amalg *) \end{array}$$

is a pullback square. Taking fibers in the vertical direction, we deduce that $G'(Y)$ is a Cartesian S-cube in $\mathcal{D}$. $\qquad\square$

Remark 7.1.3.23. Let $\mathcal{C}$ be an ∞-category which admits finite colimits and a final object, $\mathcal{D}$ a pointed differentiable ∞-category, and $F : \mathcal{C} \to \mathcal{D}$ a functor. Combining Remarks 7.1.3.19 and 7.1.1.30, we obtain a canonical equivalence

$$P_{1,\dots,1} \mathrm{cr}_n(F) \simeq \mathrm{cr}_n(P_n(F)).$$

It follows from Proposition 7.1.3.22 that the functor $\mathrm{cr}_n(P_{n-1}F)$ is a final object of $\mathrm{Fun}(\mathcal{C}^n, \mathcal{D})$. Since the functor cr_n is left exact, we obtain an equivalence $\mathrm{cr}_n(D_n(F)) \to \mathrm{cr}_n P_n(F)$, where $D_n(F) = \mathrm{fib}(P_n(F) \to P_{n-1}(F))$. We therefore obtain an equivalence of functors

$$\mathrm{cr}_n(D_n(F)) \simeq P_{1,\dots,1} \mathrm{cr}_n(F).$$

Proposition 7.1.3.24. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object, let $\mathcal{D}$ be a pointed differentiable ∞-category, let $n \geq 1$ be an integer, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is n-reduced if and only if it satisfies the following conditions:*

(1) *The functor F is 1-reduced: that is, it carries final objects of $\mathcal{C}$ to final objects of $\mathcal{D}$.*

(2) *For every positive integer $m < n$, the functor $P_{1,\ldots,1}\operatorname{cr}_m(F)$ carries each object of $\mathcal{C}^m$ to a final object of $\mathcal{D}$.*

Proof. If F is n-reduced, then assertion (1) is obvious and assertion (2) follows from Remark 7.1.3.23. Conversely, suppose that F satisfies (1) and (2). We will prove that F is k-reduced for $1 \leq k \leq n$. The proof proceeds by induction on k. When $k = 1$ the desired result follows from (1). To carry out the inductive step, it suffices to show that if F is m-reduced and $P_{1,\ldots,1}\operatorname{cr}_m F$ is trivial, then F is $(m+1)$-reduced. This follows immediately from Remark 7.1.3.23. $\qquad\square$

7.1.4 Symmetric Functors

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, where $\mathcal{C}$ is an ∞-category which admits finite colimits and a final object and $\mathcal{D}$ is a pointed differentiable ∞-category. In §7.1.2 we defined the the *Taylor tower*

$$\cdots \to P_3 F \to P_2 F \to P_1 F \to P_0 F.$$

In good cases, this affords a representation of F as a successive extension of homogeneous functors $D_n F = \operatorname{fib}(P_n F \to P_{n-1} F)$, each of which is an n-homogeneous functor from $\mathcal{C}$ to $\mathcal{D}$. In this section, we will continue the analysis by providing a classification of n-homogeneous functors.

Let us begin with a bit of motivation from linear algebra. Let V be a finite-dimensional vector space over the real numbers, and let $q : V \to \mathbf{R}$ be a quadratic form (that is, a map given by a homogeneous polynomial of degree 2). Then q determines a symmetric bilinear form $b : V \times V \to \mathbf{R}$, given by the formula

$$b(v, w) = q(v + w) - q(v) - q(w).$$

We will refer to b as the *polarization* of q. Conversely, any symmetric bilinear form $b : V \times V \to \mathbf{R}$ determines a quadratic form $q : V \to \mathbf{R}$, given by the formula $q(v) = \frac{1}{2}b(v, v)$. These two constructions are inverse to one another, and establish a bijective correspondence between symmetric bilinear forms on V and quadratic forms on V. In this section, we will establish an analogous correspondence in the setting of the calculus of functors. Suppose that $Q : \mathcal{C} \to \mathcal{D}$ is a 2-homogeneous functor. The analogue of the polarization in this context is the 2-fold cross-effect $\operatorname{cr}_2(F) : \mathcal{C} \times \mathcal{C} \to \mathcal{D}$ (see Construction 7.1.3.20). Using Proposition 7.1.3.22, we see that $\operatorname{cr}_2(F)$ is reduced and excisive in each variable: that is, it can be regarded as the analogue of a bilinear form. It is not difficult to see that $\operatorname{cr}_2(F)$ is symmetric in its two arguments. Our main goal is to show that F can be functorially recovered from $\operatorname{cr}_2(F)$. To carry out this recovery, we need to take into account the fact that $\operatorname{cr}_2(F)$ is a *symmetric* bifunctor: that is, we have a canonical equivalence $\operatorname{cr}_2(F)(X, Y) \simeq \operatorname{cr}_2(F)(Y, X)$ for $X, Y \in \mathcal{C}$. We begin by introducing some terminology for a more systematic treatment of symmetry.

Notation 7.1.4.1. For every group G, we let EG denote the simplicial set given by the Čech nerve of the map $G \to *$ (so that the set of m-simplices of EG is given by G^{m+1}, for each $m \geq 0$). Then EG is a contractible Kan complex with a free action of the group G. We let BG denote the quotient EG/G. We refer to BG as the *classifying space* of G.

Let $n \geq 0$ be an integer and let Σ_n be the symmetric group on n letters. For every simplicial set K, we let $K^{(n)}$ denote the quotient $(K^n \times E\Sigma_n)/\Sigma_n$. We refer to $K^{(n)}$ as the *nth extended power* of K. If K is an ∞-category, then $K^{(n)}$ is also an ∞-category, and is a model for the ∞-category $\operatorname{Sym}^n(K)$ given by the homotopy quotient for the action of Σ_n acting on K^n.

Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. A *symmetric n-ary functor* from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C}^{(n)} \to \mathcal{D}$. In this case, F determines a functor $\mathcal{C}^n \to \mathcal{D}$, which is invariant up to (coherent) homotopy under permutation of its arguments. We let $\operatorname{SymFun}^n(\mathcal{C}, \mathcal{D})$ denote the ∞-category $\operatorname{Fun}(\mathcal{C}^{(n)}, \mathcal{D})$ of symmetric n-ary functors from $\mathcal{C}$ to $\mathcal{D}$. If $\mathcal{C}$ and $\mathcal{D}$ admit final objects, we say that a symmetric n-ary functor $F : \mathcal{C}^{(n)} \to \mathcal{D}$ is *reduced* if the underlying functor $\mathcal{C}^n \to \mathcal{D}$ is reduced (in each variable). We let $\operatorname{SymFun}^n_*(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\operatorname{SymFun}^n(\mathcal{C}, \mathcal{D})$ spanned by the reduced symmetric n-ary functors.

Example 7.1.4.2. Let $\mathcal{C}$ be a symmetric monoidal ∞-category, which we can regard as a commutative monoid object of $\mathcal{C}at_\infty$. The symmetric monoidal structure determines a symmetric n-ary functor

$$\mathcal{C}^{(n)} \simeq \operatorname{Sym}^n(\mathcal{C}) \to \mathcal{C},$$

whose underlying map $\mathcal{C}^n \to \mathcal{C}$ is given by $(X_1, \ldots, X_n) \mapsto X_1 \otimes \cdots \otimes X_n$.

   If $\mathcal{C}$ is an ∞-category which admits finite coproducts, then we can regard $\mathcal{C}$ as endowed with the coCartesian symmetric monoidal structure of §2.4.3. We therefore obtain a symmetric n-ary functor $\coprod : \mathcal{C}^{(n)} \to \mathcal{C}$, whose underlying map $\mathcal{C}^n \to \mathcal{C}$ carries a sequence of objects $(X_1, \ldots, X_n)$ to the coproduct $\coprod_{1 \leq i \leq n} X_i$.

   Our first goal is to show that for any functor $F : \mathcal{C} \to \mathcal{D}$, the cross-effect $\operatorname{cr}_n(F)$ has the structure of a symmetric n-ary functor.

Proposition 7.1.4.3. *Let $\mathcal{C}$ be an ∞-category with a final object and let $\mathcal{D}$ be a pointed ∞-category which admits finite limits. Then the inclusion $i : \operatorname{SymFun}^n_*(\mathcal{C}, \mathcal{D}) \hookrightarrow \operatorname{SymFun}^n(\mathcal{C}, \mathcal{D})$ admits a right adjoint.*

Proof. Consider the inclusion map $j : \operatorname{Fun}_*(\mathcal{C}^n, \mathcal{D}) \to \operatorname{Fun}(\mathcal{C}^n, \mathcal{D})$ (here $\operatorname{Fun}_*(\mathcal{C}^n, \mathcal{D})$ denotes the full subcategory of $\operatorname{Fun}(\mathcal{C}^n, \mathcal{D})$ spanned by those functors which are reduced in each variable). The ∞-category $\operatorname{Fun}(\mathcal{C}^n, \mathcal{D})$ carries an action of the symmetric group Σ_n which preserves the image of j, and i is the map induced by j by taking homotopy invariants with respect to this action. Since the functor j has a right adjoint (Corollary 7.1.3.18), we conclude that i has a right adjoint. $\square$

Remark 7.1.4.4. The proof of Proposition 7.1.4.3 gives a bit more information: namely, it shows that the right adjoint $\operatorname{SymFun}^n(\mathcal{C}, \mathcal{D}) \to \operatorname{SymFun}^n_*(\mathcal{C}, \mathcal{D})$ to i fits into a commutative diagram of ∞-categories

$$
\begin{array}{ccc}
\operatorname{SymFun}^n(\mathcal{C}, \mathcal{D}) & \xrightarrow{\ \theta\ } & \operatorname{SymFun}^n_*(\mathcal{C}, \mathcal{D}) \\
\downarrow & & \downarrow \\
\operatorname{Fun}(\mathcal{C}^n, \mathcal{D}) & \xrightarrow{\ \operatorname{Red}\ } & \operatorname{Fun}_*(\mathcal{C}^n, \mathcal{D}),
\end{array}
$$

where the functor Red is as defined in Construction 7.1.3.15. We will abuse notation by denoting the induced functor $\operatorname{SymFun}^n(\mathcal{C}, \mathcal{D}) \to \operatorname{SymFun}^n_*(\mathcal{C}, \mathcal{D})$ also by Red. If $F : \mathcal{C}^n \to \mathcal{D}$ is a symmetric n-ary functor, we will refer to $\operatorname{Red}(F)$ as the *reduction* of F.

Construction 7.1.4.5. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Assume that $\mathcal{C}$ has finite coproducts and a final object, and that $\mathcal{D}$ is pointed and admits finite limits. Let $\coprod : \mathcal{C}^{(n)} \to \mathcal{C}$ be as in Example 7.1.4.2, so that the composition

$$\mathcal{C}^{(n)} \xrightarrow{\coprod} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is a symmetric n-ary functor from $\mathcal{C}$ to $\mathcal{D}$. We let $\operatorname{cr}_{(n)}(F) \in \operatorname{SymFun}^n_*(\mathcal{C}, \mathcal{D})$ denote the reduction of $F \circ \coprod$. We will refer to $\operatorname{cr}_{(n)}(F)$ as the *symmetric cross-effect of F*.

Remark 7.1.4.6. In the situation of Construction 7.1.4.5, the symmetric cross effect $\operatorname{cr}_{(n)}(F)$ induces a map $\mathcal{C}^n \to \mathcal{D}$, which can be identified with the cross effect $\operatorname{cr}_n(F)$ of Construction 7.1.3.20 (see Remark 7.1.4.4).

   We can now state the main result of this section.

Theorem 7.1.4.7. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits and let $\mathcal{D}$ be a pointed differentiable ∞-category. Then the formation of symmetric cross-effects induces a fully faithful embedding*

$$\operatorname{cr}_{(n)} : \operatorname{Homog}^n(\mathcal{C}, \mathcal{D}) \to \operatorname{SymFun}^n(\mathcal{C}, \mathcal{D}).$$

The essential image of $\operatorname{cr}_{(n)}$ is the full subcategory $\operatorname{SymFun}^n_{\operatorname{lin}}(\mathcal{C}, \mathcal{D}) \subseteq \operatorname{SymFun}^n(\mathcal{C}, \mathcal{D})$ spanned by those symmetric n-ary functors $E : \mathcal{C}^{(n)} \to \mathcal{D}$ such that the underlying functor $\mathcal{C}^n \to \mathcal{D}$ is multilinear.

We will give the proof of Theorem 7.1.4.7 at the end of this section. The main idea is to use Corollary 7.1.2.9 to reduce to the case where $\mathcal{D}$ is a stable ∞-category, in which case we can explicitly construct a homotopy inverse to the functor $\mathrm{cr}_{(n)}$. To carry out this strategy, we will need some preliminary results.

Lemma 7.1.4.8. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has final object, let $\mathcal{D}$ be an ∞-category which admits finite limits, and let $F : \mathcal{C} \to \mathcal{D}$ be an n-excisive functor for $n \geq 1$. The following conditions are equivalent:*

(1) *The functor F is $(n-1)$-excisive.*

(2) *Let $S = \{1, \dots, n\}$. For every strongly coCartesian S-cube $X : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$ such that $X(\emptyset)$ is a final object of $\mathcal{C}$, $F(X)$ is a Cartesian S-cube of $\mathcal{D}$.*

Proof. The implication $(1) \Rightarrow (2)$ is obvious. Suppose that (2) is satisfied. Let $S = \{1, \dots, n\}$ and let $X : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$ be a strongly coCartesian S-cube. Let $S_+ = S \cup \{0\}$ and choose a strongly coCartesian S_+-cube $X_+ : \mathrm{N}(\mathbf{P}(S_+)) \to \mathcal{C}$ extending S such that $X_+(\{0\})$ is a final object $* \in \mathcal{C}$. Since F is n-excisive, $F(X_+)$ is a Cartesian S_+ cube in $\mathcal{D}$ so we have a pullback square

$$
\begin{array}{ccc}
F(X(\emptyset)) & \longrightarrow & \varprojlim_{\emptyset \neq T \subseteq S} F(X(T)) \\
\downarrow & & \downarrow \\
F(*) & \longrightarrow & \varprojlim_{\emptyset \neq T \subseteq S} F(X(T \cup \{0\})).
\end{array}
$$

Using condition (2) and Proposition 7.1.1.13, we deduce that the lower horizontal map is an equivalence. It follows that the upper horizontal map is an equivalence, so that $F(X)$ is a Cartesian S-cube in $\mathcal{D}$. $\square$

Lemma 7.1.4.9. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object $*$, let $\mathcal{D}$ be a stable ∞-category, and let $F : \mathcal{C} \to \mathcal{D}$ be an n-excisive functor. Let $\coprod : \mathcal{C}^n \to \mathcal{C}$ be the functor given by $(C_1, \dots, C_n) \mapsto \coprod_{1 \leq i \leq n} C_i$. The following conditions are equivalent:*

(1) *The functor F is n-excisive.*

(2) *For every finite sequence of morphisms $\{* \to C_i\}_{1 \leq i \leq n}$ in $\mathcal{C}$ (given by maps $\alpha_i : \Delta^1 \to \mathcal{C}$), let $S = \{1, \dots, n\}$ and let $X : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$ be the S-cube given by the composition*

$$
\mathrm{N}(\mathbf{P}(S)) \simeq (\Delta^1)^n \xrightarrow{\prod_i \alpha_i} \mathcal{C}.
$$

Then $F(X)$ is a Cartesian S-cube in $\mathcal{D}$.

Proof. The implication $(1) \Rightarrow (2)$ is obvious. Assume that (2) is satisfied. We will show that F satisfies the criterion of Lemma 7.1.4.8. To this end, let $S = \{1, \dots, n\}$ and let $Y : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$ be a coCartesian S-cube in $\mathcal{C}$ with $Y(\emptyset) = *$. We wish to prove that $F(Y)$ is a Cartesian S-cube in $\mathcal{D}$. For $1 \leq i \leq n$, let $C_i = Y(\{i\})$, so that Y determines a map $* \to C_i$. Let X be the S-cube defined in (2). Then there is an evident natural transformation of S-cubes $\alpha : X \to Y$, which we can identify with a strongly coCartesian S_+-cube in $\mathcal{C}$ for $S_+ = \{0, \dots, n\}$. Since F is n-excisive, we have a pullback diagram

$$
\begin{array}{ccc}
F(X(\emptyset)) & \longrightarrow & \varprojlim_{\emptyset \neq T \subseteq S} F(X(T)) \\
\downarrow & & \downarrow \\
F(Y(\emptyset)) & \longrightarrow & \varprojlim_{\emptyset \neq T \subseteq S} F(Y(T)).
\end{array}
$$

Assumption (2) implies that the upper horizontal map is an equivalence. Since $\mathcal{D}$ is stable, this implies that the lower horizontal horizontal map is also an equivalence: that is, $F(Y)$ is a Cartesian S-cube in $\mathcal{D}$. $\square$

Proposition 7.1.4.10. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object, and let $\mathcal{D}$ be a stable ∞-category. Suppose that $F : \mathcal{C} \to \mathcal{D}$ is an n-excisive functor for some $n \geq 1$. The following conditions are equivalent:*

(a) *The functor F is $(n-1)$-excisive.*

(b) *The n-fold cross-effect $\mathrm{cr}_n(F)$ carries each object of $\mathcal{C}^n$ to a zero object of $\mathcal{D}$.*

Proof. If F is $(n-1)$-excisive, then Proposition 7.1.3.22 implies that $\mathrm{cr}_n(F)$ is $(0,0,\ldots,0)$-excisive: that is, constant. Since $\mathrm{cr}_n(F)$ is reduced (and $n > 0$) we conclude that $(a) \Rightarrow (b)$. Conversely, suppose that (b) is satisfied. We will prove that F is $(n-1)$-excisive by showing that it satisfies the second condition of Lemma 7.1.4.9. Let $*$ be a final object of $\mathcal{C}$ and suppose we are given a finite collection of morphisms $\{\alpha_i : * \to C_i\}_{1 \leq i \leq n}$. Let $S = \{1, \ldots, n\}$ and let $X : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$ be the strongly coCartesian S-cube defined in part (2) of Lemma 7.1.4.9. We wish to prove that $F(X)$ is a Cartesian S-cube in $\mathcal{D}$.

For $1 \leq i \leq n$, extend α_i to a 2-simplex σ_i :

$$
\begin{array}{ccc}
 & C_i & \\
{}^{\alpha_i}\nearrow & & \searrow \\
* \xrightarrow[\mathrm{id}]{} & & *
\end{array}
$$
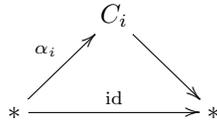
in $\mathcal{C}$. Let $\coprod : \mathcal{C}^S \to \mathcal{C}$ be the functor given by the formula

$$(K_1, \ldots, K_n) \mapsto \coprod_{1 \leq i \leq n} K_i,$$

and let $Y : (\Delta^2)^S \to \mathcal{C}$ denote the composition

$$(\Delta^2)^S \xrightarrow{\prod \sigma_i} \mathcal{C}^S \xrightarrow{\coprod} \mathcal{C}.$$

For $0 \leq i \leq n$, let $Y_i : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$ be defined by the formula $Y_i(T) = Y(a_1, \ldots, a_n)$, where

$$
a_j = \begin{cases}
0 & \text{if } j \geq i \text{ and } j \notin T \\
2 & \text{if } j < i \text{ and } j \in T \\
1 & \text{otherwise.}
\end{cases}
$$

Note that Y_n is equivalent to the S-cube X. Consequently, to complete the proof it will suffice to show that each $F(Y_i)$ is a Cartesian S-cube in $\mathcal{D}$. The proof proceeds by induction on i. When $i = 0$, we must show that the canonical map

$$u : F(Y_0(\emptyset)) \to \varprojlim_{\emptyset \neq T \subseteq S} F(Y_0(T))$$

is an equivalence. Unwinding the definitions, we see that the fiber of u is given by $\mathrm{cr}_n(F)(X_1, \ldots, X_n)$, which vanishes by (2).

Now suppose that $i > 0$, and let $S' = S - \{i\}$. We have a commutative diagram

$$
\begin{array}{ccc}
F(Y_i(\emptyset)) \longrightarrow & F(Y_{i-1}(\emptyset)) \longrightarrow & F(Y_{i-1}(\{i\})) \\
\downarrow & \downarrow & \downarrow \\
\varprojlim_{\emptyset \neq T \subseteq S'} F(Y_i(T)) \longrightarrow \varprojlim_{\emptyset \neq T \subseteq S'} F(Y_{i-1}(T)) \longrightarrow \varprojlim_{\emptyset \neq T \subseteq S'} F(Y_{i-1}(T \cup \{i\})).
\end{array}
$$

We wish to prove that the left square is a pullback diagram (note that $Y_{i-1}(T) = Y_i(T \cup \{i\})$ for $T \subseteq S'$). To prove this, we observe that the right square is a pullback diagram by the inductive hypothesis, and the outer rectangle is a pullback square because the horizontal compositions are equivalences. $\quad\square$

Corollary 7.1.4.11. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits, and let $\mathcal{D}$ be a stable ∞-category which admits countable colimits. Let $\alpha : F \to G$ be a natural transformation between n-homogeneous functors $F, G : \mathcal{C} \to \mathcal{D}$. Then α is an equivalence if and only if the induced map $\mathrm{cr}_n(F) \to \mathrm{cr}_n(G)$ is an equivalence.*

Proof. Suppose $\mathrm{cr}_n(\alpha)$ is an equivalence. Let H be the fiber of α. Note that H is n-homogeneous and that $\mathrm{cr}_n(H) \simeq \mathrm{fib}(\mathrm{cr}_n(\alpha))$ is a final object of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Using Proposition 7.1.4.10, we deduce that H is $(n-1)$-excisive. Since H is n-reduced, we conclude that H is a zero object of the stable ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, so that α is an equivalence. $\square$

Lemma 7.1.4.12. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits, let $\mathcal{D}$ be a stable ∞-category, and let $F : \mathcal{C} \to \mathcal{D}$ be a 1-excisive functor. Then F carries strongly coCartesian cubes in $\mathcal{C}$ to strongly coCartesian cubes in $\mathcal{D}$.*

Proof. This follows immediately from the characterization of strongly coCartesian cubes given in Proposition 7.1.1.15. $\square$

Proposition 7.1.4.13. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object, and let $\mathcal{D}$ be a stable ∞-category. Let $F : \mathcal{C}^n \to \mathcal{D}$ be a functor, and for every permutation σ in the symmetric group Σ_n, let $F^\sigma : \mathcal{C}^n \to \mathcal{D}$ be the composition of F with the isomorphism $\mathcal{C}^n \to \mathcal{C}^n$ obtained by applying the permutation σ. Let $\delta : \mathcal{C} \to \mathcal{C}^n$ be the diagonal map and let $f = F \circ \delta$, so that $f = F^\sigma \circ \delta$ for every permutation σ. Suppose that F is $(1, \ldots, 1)$-excisive. Then there is a canonical equivalence*

$$\mathrm{cr}_n(f) \simeq \bigoplus_{\sigma \in \Sigma_n} \mathrm{Red}(F^\sigma).$$

In particular, if F is $(1, \ldots, 1)$-homogeneous, then $\mathrm{cr}_n(f) \simeq \bigoplus_{\sigma \in \Sigma_n} F^\sigma$.

Proof. Let $S = \{1, \ldots, n\}$. Choose an initial object $\emptyset \in \mathcal{C}$. For every finite sequence of subsets $\vec{T} = (T_1, T_2, \ldots, T_n) \in \mathbf{P}(S)^n$, define $U_{\vec{T}} : \mathcal{C}^n \to \mathcal{C}^n$ by the formula

$$U_{\vec{T}}(X_1, \ldots, X_n) = (\coprod_{i \in T_1} X_i, \ldots, \coprod_{i \in T_n} X_i)$$

and let $F_{\vec{T}} = F \circ U_{\vec{T}}$. By construction, $\mathrm{cr}_n(f)$ is the reduction of the functor $F_{(S, \ldots, S)}$.

For any sequence of objects $X_1, \ldots, X_n \in \mathcal{C}$, our assumption that F is $(1, \ldots, 1)$-excisive implies that the construction $\vec{T} \mapsto F_{\vec{T}}(X_1, \ldots, X_n)$ is a strongly coCartesian separately in each variable (Lemma 7.1.4.12). It follows that the canonical map

$$\varinjlim_{\vec{T} \in \mathbf{P}_{\leq 1}(S)^n} F_{\vec{T}} \to F_{(S, \ldots, S)}$$

is an equivalence, so that $\mathrm{cr}_n(f)$ can be identified with the colimit of the diagram $Z : \mathrm{N}(\mathbf{P}_{\leq 1}(S)^n) \to \mathrm{Fun}(\mathcal{C}^n, \mathcal{D})$ given by $Z(\vec{T}) = \mathrm{Red}(F_{\vec{T}})$.

Let $P \subseteq \mathbf{P}_{\leq 1}(S)^n$ be the subset consisting of those sequences $\vec{T} = (T_1, \ldots, T_n)$ with $\bigcup_i T_i = S$. Note that if $\bigcup_i T_i \neq S$, then the functor $F_{\vec{T}}$ is independent of one of its arguments and therefore $\mathrm{Red}(F_{\vec{T}}) \simeq 0$ (Example 7.1.3.16). It follows that the diagram Z is a left Kan extension of $Z | \mathrm{N}(P)$. Moreover, $\mathrm{N}(P)$ is a discrete partially ordered set, whose elements can be identified with permutations of S. The desired result now follows from the observation that if $\vec{T}$ corresponds to a permutation $\sigma \in \Sigma_n$, then $F_{\vec{T}} \simeq F^\sigma$. $\square$

Proof of Theorem 7.1.4.7. We first show that the essential image of $\mathrm{cr}_{(n)}$ is contained in $\mathrm{SymFun}_{\mathrm{lin}}^n(\mathcal{C}, \mathcal{D})$. In view of Remark 7.1.4.6, it will suffice to show that if $F : \mathcal{C} \to \mathcal{D}$ is n-homogeneous, then $\mathrm{cr}_n(F)$ is $(1, \ldots, 1)$-homogeneous. The functor $\mathrm{cr}_n(F)$ is $(1, \ldots, 1)$-reduced by Proposition 7.1.3.17 and $(1, \ldots, 1)$-excisive by Proposition 7.1.3.22.

We have a commutative diagram of ∞-categories

$$
\begin{array}{ccc}
\mathrm{Homog}^n(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) & \xrightarrow{\mathrm{cr}_{(n)}} & \mathrm{SymFun}_{\mathrm{lin}}^n(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \\
\downarrow & & \downarrow{\psi} \\
\mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) & \xrightarrow{\mathrm{cr}_{(n)}} & \mathrm{SymFun}_{\mathrm{lin}}^n(\mathcal{C}, \mathcal{D}).
\end{array}
$$

The left vertical map is a categorical equivalence by Corollary 7.1.2.9. The functor ψ is obtained from the forgetful functor $\psi_0 : \mathrm{Exc}_*(\mathcal{C}^n, \mathrm{Sp}(\mathcal{D})) \to \mathrm{Exc}_*(\mathcal{C}^n, \mathcal{D})$ by taking homotopy invariants with respect to the action of Σ_n. Iterated application of Corollary 7.1.2.9 shows that ψ_0 is a categorical equivalence, so that ψ is a categorical equivalence. Consequently, to show that the lower horizontal map in the above diagram is a categorical equivalence, it will suffice to show that the upper horizontal map is a categorical equivalence. We may therefore replace $\mathcal{D}$ by $\mathrm{Sp}(\mathcal{D})$ and thereby reduce to the case where $\mathcal{D}$ is stable. In particular, $\mathcal{D}$ admits finite colimits. Since $\mathcal{D}$ admits sequential colimits, it admits countable filtered colimits, and therefore all countable colimits (Corollary T.4.2.3.11).

Let $\coprod : \mathcal{C}^{(n)} \to \mathcal{C}$ be as in Example 7.1.4.2. For every object $C \in \mathcal{C}$, the inclusion map $\{C\}^{(n)} \hookrightarrow \mathcal{C}^{(n)} \times_{\mathcal{C}} \mathcal{C}_{/C}$ is left cofinal. Note that $\{C\}^{(n)}$ is isomorphic to the classifying space $B\Sigma_n = E\Sigma_n/\Sigma_n$, which has countably many simplices. Consequently, every functor $\{C\}^{(n)} \to \mathcal{D}$ admits a colimit in $\mathcal{D}$. It follows that every functor $\mathcal{C}^{(n)} \to \mathcal{D}$ admits a left Kan extension along $\coprod$. The formation of left Kan extensions defines a functor $\psi : \mathrm{SymFun}^n(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ which may be described explicitly as follows: if $F : \mathcal{C}^{(n)} \to \mathcal{D}$ is a symmetric n-ary functor from $\mathcal{C}$ to $\mathcal{D}$ with underlying functor $f : \mathcal{C}^n \to \mathcal{D}$, then $(\psi F)(X)$ is given by extracting coinvariants with respect to the action of Σ_n on $f(X, \ldots, X) \in \mathcal{D}$. Note that the functor ψ is left adjoint to the forgetful functor $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{SymFun}^n(\mathcal{C}, \mathcal{D})$. It follows that $\psi | \mathrm{SymFun}_*^n(\mathcal{C}, \mathcal{D})$ is left adjoint to the symmetric cross effect construction $\mathrm{cr}_{(n)} : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{SymFun}_*^n(\mathcal{C}, \mathcal{D})$.

We now claim that ψ carries $\mathrm{SymFun}_{\mathrm{lin}}^n(\mathcal{C}, \mathcal{D})$ into $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$. Let $F : \mathcal{C}^{(n)} \to \mathcal{D}$ be a symmetric n-ary functor from $\mathcal{C}$ to $\mathcal{D}$ and let $f : \mathcal{C}^n \to \mathcal{D}$ be the underlying functor. Let $\delta : \mathcal{C} \to \mathcal{C}^n$ be the diagonal map. If f is 1-homogeneous in each variable, then $f \circ \delta$ is n-homogeneous (Corollaries 7.1.3.4 and 7.1.3.12). Note that F restricts to a map $\mathcal{C} \times B\Sigma_n \to \mathcal{D}$, which we can identify with a map $\chi : B\Sigma_n \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ carrying the base point to $f \circ \delta$. Moreover, $\psi(F)$ is given by the colimit of the diagram χ.

The collection of n-reduced functors from $\mathcal{C}$ to $\mathcal{D}$ is evidently stable under colimits. The functor $P_n : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is left exact by Remark 7.1.1.29. Since $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is stable, the functor P_n is also right exact. Since P_n preserves sequential colimits (Remark 7.1.1.31), it preserves countable filtered colimits and therefore all countable colimits (Corollary T.4.2.3.12). It follows that the collection of n-excisive functors from $\mathcal{C}$ to $\mathcal{D}$ is stable under countable colimits. Since the collection of n-reduced functors is evidently stable under countable colimits, we conclude that $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is stable under countable colimits. In particular, we deduce that $\psi(F) = \varinjlim(\chi)$ is n-homogeneous.

Let $\psi_0 = \psi | \mathrm{SymFun}_{\mathrm{lin}}^n(\mathcal{C}, \mathcal{D})$. The above arguments show that we have a pair of adjoint functors

$$
\mathrm{SymFun}_{\mathrm{lin}}^n(\mathcal{C}, \mathcal{D}) \underset{\mathrm{cr}_{(n)}}{\overset{\psi_0}{\rightleftarrows}} \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}).
$$

We claim that these adjoint functors are mutually inverse equivalences. It follows from Corollary 7.1.4.11 (and Remark 7.1.4.6) that the functor $\mathrm{cr}_{(n)}$ is conservative on $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$. It will therefore suffice to show that the unit map $u : \mathrm{id} \to \mathrm{cr}_{(n)} \circ \psi_0$ is an equivalence of functors from $\mathrm{SymFun}_{\mathrm{lin}}^n(\mathcal{C}, \mathcal{D})$ to itself.

Let $F : \mathcal{C}^{(n)} \to \mathcal{D}$ be a symmetric n-ary functor such that the underlying map $f : \mathcal{C}^n \to \mathcal{D}$ is 1-homogeneous in each variable. We wish to prove that u induces an equivalence of symmetric n-ary functors

$$
u_F : F \to \mathrm{cr}_{(n)}(\psi F).
$$

Since the forgetful functor $\mathrm{SymFun}^n(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}^n, \mathcal{D})$ is conservative, it will suffice to show that the induced map $v : f \to \mathrm{cr}_n(\psi F)$ is an equivalence. Because $\mathcal{D}$ is stable, the formation of colimits in $\mathcal{D}$ commutes

with finite limits. It follows that $\mathrm{cr}_n(\psi F)$ can be identified with the coinvariants for the permutation action of the symmetric group Σ_n on $\mathrm{cr}_n(f \circ \delta)$.

Proposition 7.1.4.13 gives a canonical equivalence $\mathrm{cr}_n(f \circ \delta) \simeq \bigoplus_{\sigma \in \Sigma_n} f_\sigma$, where the summands are permuted by the action of Σ_n. It follows that the Σ_n-coinvariants on $\mathrm{cr}_n(f \circ \delta)$ can be identified with the functor f. Unwinding the definitions, we see that this identification is given by the map v. $\qquad\square$

We will later need the following variant of Theorem 7.1.4.7:

Proposition 7.1.4.14. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and a final object, let $\mathcal{D}$ be a stable ∞-category which admits countable colimits, and let $\psi : \mathrm{SymFun}^n(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be defined as in the proof of Theorem 7.1.4.7 (so that if $F \in \mathrm{SymFun}^n(\mathcal{C}, \mathcal{D})$ has underlying functor $f : \mathcal{C}^n \to \mathcal{D}$, then $\psi(F)$ assigns to each object $X \in \mathcal{C}$ the coinvariants for the action of the symmetric group Σ_n on $f(X, \ldots, X)$). Then composition with ψ induces an equivalence of ∞-categories*

$$\mathrm{SymFun}^n_{\mathrm{lin}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}).$$

Proof. Let $F \in \mathrm{SymFun}^n_{\mathrm{lin}}(\mathcal{C}, \mathcal{D})$, and let $f : \mathcal{C}^n \to \mathcal{D}$ be the underlying functor. Using Corollary 7.1.3.11, we deduce that the functor $X \mapsto f(X, \ldots, X)$ is n-homogeneous. Note that $\mathcal{D}$ and therefore $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ admit countable colimits. The ∞-category of n-reduced functors from $\mathcal{C}$ to $\mathcal{D}$ is evidently closed under countable colimits. The functor $P_n : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is left exact by Remark 7.1.1.29, and therefore also right exact since $\mathcal{D}$ is stable. Since P_n commutes with sequential colimits (Remark 7.1.1.31), we conclude that the class of n-excisive functors is stable under finite colimits and sequential colimits and therefore under all countable colimits. Since $\psi(F)$ is a countable colimit of functors equivalent to f, we deduce that $\psi(F) \in \mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$.

Let $\mathcal{C}_*$ denote the ∞-category of pointed objects of $\mathcal{C}$, and let $\psi_* : \mathrm{SymFun}^n_{\mathrm{lin}}(\mathcal{C}_*, \mathcal{D}) \to \mathrm{Homog}^n(\mathcal{C}_*, \mathcal{D})$ be defined as above. We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{SymFun}^n_{\mathrm{lin}}(\mathcal{C}, \mathcal{D}) & \xrightarrow{\ \psi\ } & \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \\
\downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi'} \\
\mathrm{SymFun}^n_{\mathrm{lin}}(\mathcal{C}_*, \mathcal{D}) & \xrightarrow{\ \psi_*\ } & \mathrm{Homog}^n(\mathcal{C}_*, \mathcal{D}).
\end{array}
$$

The proof of Theorem 7.1.4.7 shows that ψ_* is left adjoint to an equivalence of ∞-categories, and is therefore itself an equivalence of ∞-categories. The map ϕ' is an equivalence of ∞-categories by Proposition 7.1.2.11. It will therefore suffice to show that ϕ is an equivalence of ∞-categories. Note that ϕ is obtained from a functor $\phi_0 : \mathrm{Exc}_*(\mathcal{C}^n_*, \mathcal{D}) \to \mathrm{Exc}_*(\mathcal{C}^n, \mathcal{D})$ by taking homotopy invariants with respect to the action of the symmetric group Σ_n. Here $\mathrm{Homog}^{(1, \ldots, 1)}(\mathcal{C}^n, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}^n, \mathcal{D})$ spanned by those functors which are $(1, \ldots, 1)$-homogeneous, and $\mathrm{Exc}_*(\mathcal{C}^n_*, \mathcal{D})$ is defined similarly. It will therefore suffice to show that ϕ_0 is an equivalence of ∞-categories. This follows from n applications of Proposition 7.1.2.11. $\qquad\square$

Corollary 7.1.4.15. *Let $F : \mathrm{Sp} \to \mathrm{Sp}$ be a functor, and let $n \geq 0$ be an integer. The following conditions are equivalent:*

(1) *The functor F is n-excisive and commutes with filtered colimits.*

(2) *The functor F is polynomial of degree $\leq n$, in the sense of Definition 7.1.0.2.*

Proof. Let $\mathrm{Poly}^n(\mathrm{Sp}, \mathrm{Sp}) \subseteq \mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})$ be as in Definition 7.1.0.2, and let $\mathcal{X} \subseteq \mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})$ be the full subcategory spanned by those functors which are n-excisive and commute with filtered colimits. We first show that $(2) \Rightarrow (1)$: that is, $\mathrm{Poly}^n(\mathrm{Sp}, \mathrm{Sp}) \subseteq \mathcal{X}$. For $0 \leq m \leq n$, the functor $X \mapsto X^{\otimes m}$ obviously commutes with filtered colimits, and is n-excisive by Corollary 7.1.3.5. It will therefore suffice to show that the subcategory $\mathcal{X} \subseteq \mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})$ is closed under translation and small colimits. The collection of functors $F : \mathrm{Sp} \to \mathrm{Sp}$ which commute with filtered colimits is closed under translations and small colimits by Lemma

T.5.5.2.3. We are therefore reduced to proving that $\mathrm{Exc}^n(\mathrm{Sp}, \mathrm{Sp}) \subseteq \mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})$ is closed under translation and small colimits. Since $\mathrm{Exc}^n(\mathrm{Sp}, \mathrm{Sp})$ is closed under filtered colimits, it suffices to show that it is a stable subcategory of $\mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})$. This is clear, since $\mathrm{Exc}^n(\mathrm{Sp}, \mathrm{Sp})$ is closed under translation and finite limits.

We now prove that $(1) \Rightarrow (2)$, using induction on n. Let $F : \mathrm{Sp} \to \mathrm{Sp}$ be an n-excisive functor which commutes with filtered colimits. The inductive hypothesis implies that $P_{n-1}(F) \in \mathrm{Poly}^{n-1}(\mathrm{Sp}, \mathrm{Sp}) \subseteq \mathrm{Poly}^n(\mathrm{Sp}, \mathrm{Sp})$. Using the fiber sequence

$$D_n(F) \to F \to P_{n-1}(F),$$

we are reduced to proving that $D_n(F) \in \mathrm{Poly}^n(\mathrm{Sp}, \mathrm{Sp})$. Using Proposition 7.1.4.14, we can write $D_n(F)$ as a colimit (indexed by the Kan complex $B\Sigma_n$) of functors of the form

$$\mathrm{Sp} \xrightarrow{\delta} \mathrm{Sp}^n \xrightarrow{G} \mathrm{Sp},$$

where $G \simeq \mathrm{cr}_n(F)$ is a functor which commutes with filtered colimits and is 1-homogeneous in each variable. It follows that G preserves small colimits separately in each variable. Let $S \in \mathrm{Sp}^n$ denote the sphere spectrum, let $E = G(S, \ldots, S)$, and let $G_E : \mathrm{Sp}^n \to \mathrm{Sp}$ be the functor given by $(X_1, \ldots, X_n) \mapsto E \otimes X_1 \otimes \cdots \otimes X_n$. Applying Corollary 1.4.4.6, we deduce that $G \simeq G_E$, so that $D_n(F)$ is a colimit of functors of the form G_E. Let $\mathcal{C} \subseteq \mathrm{Sp}$ denote the full subcategory spanned by those spectra E for which the functor G_E belongs to $\mathrm{Poly}^n(\mathrm{Sp}, \mathrm{Sp})$. Since $\mathrm{Poly}^n(\mathrm{Sp}, \mathrm{Sp})$ is a stable subcategory of $\mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})$ which is closed under small colimits, $\mathcal{C}$ is a stable subcategory of Sp which is closed under small colimits. By construction, we have $S \in \mathcal{C}$, so that $\mathcal{C} = \mathrm{Sp}$ and therefore $G_E \in \mathrm{Poly}^n(\mathrm{Sp}, \mathrm{Sp})$ as desired. $\qquad\square$

7.1.5   Functors from Spaces to Spectra

Let $\mathcal{C}$ be an ∞-category which admits finite colimits, and $\mathcal{D}$ an ∞-category which admits finite limits. In §7.1.1, we introduced the notion of an n-excisive functor from $\mathcal{C}$ to $\mathcal{D}$. If $\mathcal{C}$ is the ∞-category of spaces and $\mathcal{D}$ is the ∞-category of spectra, then n-excisive functors admit the following characterization:

Theorem 7.1.5.1. *Let $F : \mathcal{S} \to \mathrm{Sp}$ be a functor, and let $n \geq 0$ be an integer. The following conditions are equivalent:*

(1) *The functor F is a left Kan extension of $F| \mathrm{N}(\mathcal{F}\mathrm{in}^{\leq n})$. Here $\mathcal{F}\mathrm{in}^{\leq n}$ denotes the category of finite sets having cardinality $\leq n$.*

(2) *The functor F is n-excisive and commutes with small filtered colimits.*

Corollary 7.1.5.2. *Let $\mathrm{Exc}_c^n(\mathcal{S}, \mathrm{Sp})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{S}, \mathrm{Sp})$ spanned by those functors which are n-excisive and preserve small filtered colimits. Then the restriction functor $\mathrm{Exc}_c^n(\mathcal{S}, \mathrm{Sp}) \to \mathrm{Fun}(\mathrm{N}(\mathcal{F}\mathrm{in}^{\leq n}), \mathrm{Sp})$ is an equivalence of ∞-categories.*

Proof. Combine Theorem 7.1.5.1 with Proposition T.4.3.2.15. $\qquad\square$

Our goal in this section is to formulate and prove a slightly stronger form of Theorem 7.1.5.1. We begin by introducing some notation.

Notation 7.1.5.3. *Let $\mathcal{C}$ be an ∞-category which admits small colimits and $\mathcal{D}$ an ∞-category which admits finite limits and filtered colimits. We let $\mathrm{Exc}_c^n(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which are n-excisive and preserve small filtered colimits.*

Proposition 7.1.5.4. *Let $\mathcal{C}$ be a small ∞-category which admits finite colimits and let $\mathcal{D}$ be an ∞-category which admits finite limits and small filtered colimits. Assume that filtered colimits in $\mathcal{D}$ are left exact. Then composition with the Yoneda embedding $j : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ induces a fully faithful functor*

$$\theta : \mathrm{Exc}_c^n(\mathrm{Ind}(\mathcal{C}), \mathcal{D})) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$

whose essential image is the full subcategory $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the n-excisive functors.

Proof. Let $\mathrm{Fun}_c(\mathrm{Ind}(\mathcal{C}), \mathcal{D})$ be the full subcategory of $\mathrm{Fun}(\mathrm{Ind}(\mathcal{C}), \mathcal{D})$ spanned by those functors which preserve small filtered colimits. Proposition T.5.3.5.10 implies that the forgetful functor $\mathrm{Fun}_c(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is an equivalence of ∞-categories. It follows immediately that θ is fully faithful. The Yoneda embedding $j : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ preserves finite colimits (Proposition T.5.3.5.14) and therefore carries strongly coCartesian cubes to strongly coCartesian cubes. It follows that the essential image of θ is contained in $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$. Conversely, suppose that $f : \mathcal{C} \to \mathcal{D}$ is an n-excisive functor which factors as a composition

$$\mathcal{C} \xrightarrow{j} \mathrm{Ind}(\mathcal{C}) \xrightarrow{F} \mathcal{D}$$

where F preserves small filtered colimits. We wish to show that F is n-excisive. Let $S = [n] = \{0, \ldots, n\}$ and let $X : \mathrm{N}(\mathbf{P}(S)) \to \mathrm{Ind}(\mathcal{C})$ be a strongly coCartesian S-cube. We wish to show that $F(X)$ is a Cartesian S-cube in $\mathcal{D}$. Let $X_{\leq 1} = X | \mathrm{N}(\mathbf{P}_{\leq 1}(S))$. Using Proposition T.5.3.5.15, we can write $X_{\leq 1}$ as a small filtered colimits of diagrams $j \circ Y^\alpha$, where Y^α is a functor $\mathrm{N}(\mathbf{P}_{\leq 1}(S)) \to \mathcal{C}$. We can extend each Y^α to a strongly coCartesian S-cube $Z^\alpha : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}$, so that $X \simeq \varinjlim_\alpha (j \circ Z^\alpha)$. Since F commutes with small filtered colimits, we have $F(X) \simeq \varinjlim_\alpha f(Z^\alpha)$. Since filtered colimits in $\mathcal{D}$ commute with finite limits, it will suffice to show that $f(Z^\alpha)$ is a Cartesian S-cube in $\mathcal{D}$. This follows from our assumption that f is n-excisive. $\qquad\square$

Notation 7.1.5.5. Let $\mathcal{C}$ be a small ∞-category. We let $\mathcal{P}(\mathcal{C})$ denote the ∞-category $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ of presheaves of spaces on $\mathcal{C}$. We let $\mathcal{P}^c(\mathcal{C})$ denote the full subcategory of $\mathcal{P}(\mathcal{C})$ spanned by the compact objects, so that $\mathcal{P}^c(\mathcal{C})$ is an essentially small ∞-category with $\mathcal{P}(\mathcal{C}) \simeq \mathrm{Ind}(\mathcal{P}^c(\mathcal{C}))$. Let $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ be the Yoneda embedding.

For every integer $n \geq 0$, we let $\mathcal{P}^{\leq n}(\mathcal{C})$ denote the full subcategory of $\mathcal{C}$ spanned by the objects which can be written as a coproduct $\coprod_{1 \leq i \leq m} j(X_i)$ for some $m \leq n$ and some objects $X_i \in \mathcal{C}$.

We can now state the main result of this section.

Theorem 7.1.5.6. *Let $\mathcal{C}$ be a small ∞-category, let $\mathcal{D}$ be a presentable stable ∞-category, and let $n \geq 0$ be an integer. Let $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ be a functor. The following conditions are equivalent:*

(1) *The functor F is a left Kan extension of $F | \mathcal{P}^{\leq n}(\mathcal{C})$.*

(2) *The functor F is n-excisive and preserves small filtered colimits.*

Note that Theorem 7.1.5.6 immediately implies Theorem 7.1.5.1 (take $\mathcal{C} = \Delta^0$ and $\mathcal{D} = \mathrm{Sp}$). Together with Proposition 7.1.5.4, it implies the following version of Corollary 7.1.5.2.

Corollary 7.1.5.7. *Let $\mathcal{C}$ be a small ∞-category and let $\mathcal{D}$ be a presentable stable ∞-category. Then the restriction functors*
$$\mathrm{Exc}^n_c(\mathcal{P}(\mathcal{C}), \mathcal{D}) \to \mathrm{Exc}^n(\mathcal{P}^c(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{P}^{\leq n}(\mathcal{C}), \mathcal{D})$$
are equivalences of ∞-categories.

The proof of Theorem 7.1.5.6 will require some preliminary results.

Lemma 7.1.5.8. *Suppose we are given ∞-categories $\mathcal{C}_1, \ldots, \mathcal{C}_m$, and $\mathcal{C}$ which admit finite colimits, and a functor $F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \to \mathcal{C}$ which preserves finite colimits separately in each variable. Let S be a finite set and suppose we are given S-cubes $X_i : \mathrm{N}(\mathbf{P}(S)) \to \mathcal{C}_i$. Let X denote the S-cube in $\mathcal{C}$ given by the composition*

$$\mathrm{N}(\mathbf{P}(S)) \xrightarrow{\prod X_i} \prod \mathcal{C}_i \xrightarrow{F} \mathcal{C}.$$

Suppose we are given integers a_i such that each X_i is a left Kan extension of $X_i | \mathrm{N}(\mathbf{P}_{\leq a_i}(S))$. Then X is a left Kan extension of $X | \mathrm{N}(\mathbf{P}_{\leq a}(S))$, where $a = a_1 + \cdots + a_m$.

Proof. Define $Y : \mathrm{N}(\mathbf{P}(S))^m \to \mathcal{C}$ by the formula $Y(S_1, \ldots, S_m) = F(X_1(S_1), \ldots, X_m(S_m))$. Let $A \subseteq \mathbf{P}(S)^m$ be the subset consisting of sequences $(S_1, \ldots, S_m)$ where $S_1 = \cdots = S_m$ and each S_i has cardinality $\leq a$. We wish to show that Y exhibits $Y(S, \ldots, S)$ as a colimit of the diagram $Y \,|\, \mathrm{N}(A)$. Let $B_0 \subseteq \mathbf{P}(S)^m$ be the subset consisting of sequences $(S_1, \ldots, S_m)$ such that the union $\bigcup_i S_i$ has cardinality $\leq a$. Then $A \subseteq B$. The inclusion $\mathrm{N}(A) \to \mathrm{N}(B)$ admits a left adjoint and is therefore left cofinal. It will therefore suffice to show that Y exhibits $Y(S, \ldots, S)$ as a colimit of $Y \,|\, \mathrm{N}(B)$. We will prove the following stronger result: Y is a left Kan extension of $Y \,|\, \mathrm{N}(B)$.

Choose a sequence of downward-closed subsets

$$B = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_k = \mathbf{P}(S)^m,$$

where each B_j is obtained from B_{j-1} by adjoining a single element of $\mathbf{P}(S)^m$. To complete the proof, it will suffice to show that $Y \,|\, \mathrm{N}(B_j)$ is a left Kan extension of $Y \,|\, \mathrm{N}(B_{j-1})$ for $0 < j \leq k$. Suppose that B_j is obtained from B_{j-1} by adjoining the element $(S_1, \ldots, S_m) \in \mathbf{P}(S)^m$. Unwinding the definitions, we must show that $Y \,|\, \prod_{1 \leq i \leq m} \mathrm{N}(\mathbf{P}(S_i))$ is a colimit diagram in $\mathcal{C}$.

Since $(S_1, \ldots, S_m) \notin B_0$, the union $\bigcup S_i$ has cardinality larger than a. It follows that some S_i has cardinality larger than a_i. Without loss of generality, we may assume that S_1 has cardinality larger than a_1. Let $P = (\prod_i \mathbf{P}(S_i)) - \{(S_1, \ldots, S_m)\}$, and let $P_0 \subseteq P$ be the subset spanned by those sequences $(T_1, \ldots, T_m)$ such that $T_1 \neq S_1$, and let $P_1 \subseteq P_0$ be the subset consisting of those sequences having the form $(T_1, S_2, S_3, \ldots, S_m)$. Using the fact that X_1 is a left Kan extension of $X_1 \,|\, \mathrm{N}(\mathbf{P}_{\leq a_1}(S))$ and that F preserves finite colimits in the first variable, we deduce that $Y \,|\, \mathrm{N}(P)$ is a left Kan extension of the diagram $Y \,|\, \mathrm{N}(P_0)$. It will therefore suffice to show that Y exhibits $Y(S_1, \ldots, S_m)$ as a colimit of $Y \,|\, \mathrm{N}(P_0)$. Note that the inclusion $P_1 \subseteq P_0$ admits a left adjoint (given by $(T_1, \ldots, T_m) \mapsto (T_1, S_2, \ldots, S_m)$) and therefore induces a left cofinal map $\mathrm{N}(P_1) \to \mathrm{N}(P_0)$. We are therefore reduced to proving that Y exhibits $Y(S_1, \ldots, S_m)$ as a colimit of $Y \,|\, \mathrm{N}(P_1)$. Since F preserves finite colimits in the first variable, it suffices to show that $X_1 \,|\, \mathrm{N}(S_1)$ is a colimit diagram. Since S_1 has cardinality larger than a_1, this follows from our assumption that X_1 is a left Kan extension of $X_1 \,|\, \mathrm{N}(\mathbf{P}_{\leq a_1}(S))$. $\qquad\square$

Lemma 7.1.5.9. *Let $\mathcal{C}$ be a presentable ∞-category and let $\mathcal{D}$ be a presentable stable ∞-category. Let $F : \mathcal{C} \to \mathcal{D}$ be a 1-excisive functor such that the composition $F_+ : \mathcal{C}_* \to \mathcal{C} \xrightarrow{F} \mathcal{D}$ preserves small filtered colimits. Then F preserves small filtered colimits.*

Proof. Let $\emptyset$ and $*$ denote initial and final objects of $\mathcal{C}$, respectively, and let $G_\emptyset$ and G_* denote the constant functors $\mathcal{C} \to \mathcal{C}$ taking the values $\emptyset$ and $*$. Let U denote the pushout $\mathrm{id} \coprod_{G_\emptyset} G_*$, formed in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$. Since F is 1-excisive, we obtain a pullback diagram of functors

$$
\begin{array}{ccc}
F \circ G_\emptyset & \longrightarrow & F \circ G_* \\
\downarrow & & \downarrow \\
F & \longrightarrow & F \circ U.
\end{array}
$$

It follows that the fiber of the map $F \to F \circ U$ is equivalent to a constant functor from $\mathcal{C}$ to $\mathcal{D}$, and therefore commutes with filtered colimits. Consequently, to prove that F commutes with filtered colimits, it will suffice to show that $F \circ U$ commutes with filtered colimits. This is clear, since $F \circ U$ factors as a composition $\mathcal{C} \xrightarrow{T} \mathcal{C}_* \xrightarrow{F_+} \mathcal{D}$, where T is a left adjoint to the forgetful functor $\mathcal{C}_* \to \mathcal{C}$. $\qquad\square$

Remark 7.1.5.10. Let $\mathcal{C}$ be an ∞-category which admits finite colimits, and $\mathcal{D}$ a stable ∞-category which admits small colimits. The inclusion functor $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is a left exact functor between stable ∞-categories, and therefore right exact. It follows that the collection of n-excisive functors from $\mathcal{C}$ to $\mathcal{D}$ is closed under finite colimits. The collection of n-excisive functors from $\mathcal{C}$ to $\mathcal{D}$ is also closed under small filtered colimits (since filtered colimits in $\mathcal{D}$ are left exact), and therefore closed under all small colimits.

Proof of Theorem 7.1.5.6. We first show that $(1) \Rightarrow (2)$. Since the ∞-category $\mathcal{D}$ is presentable, we can regard it as tensored over the ∞-category $\mathcal{S}$ of spaces (see §6.3.1.14); we let $\otimes : \mathcal{S} \times \mathcal{D} \to \mathcal{D}$ denote the corresponding tensor product functor. Let $\mathcal{E} = \operatorname{Fun}(\mathcal{P}^{\leq n}(\mathcal{C}), \mathcal{D})$. For every object $X \in \mathcal{P}^{\leq n}(\mathcal{C})$, let $X^* : \mathcal{E} \to \mathcal{D}$ be the functor given by evaluation at X. This functor admits a left adjoint $X_!$, which is given on objects by the formula $(X_!(D))(Y) = \operatorname{Map}_{\mathcal{C}}(X, Y) \otimes D$.

Let $\mathcal{E}_0 \subseteq \mathcal{E}$ be the smallest full subcategory which contains the essential image of each of the functors $X_!$ and is closed under small colimits. We claim that $\mathcal{E}_0 = \mathcal{E}$. Note that both $\mathcal{E}_0$ and $\mathcal{E}$ are presentable, so Corollary T.5.5.2.9 implies that the inclusion $\mathcal{E}_0 \subseteq \mathcal{E}$ admits a right adjoint G. To prove that $\mathcal{E}_0 = \mathcal{E}$, it will suffice to show that G is conservative. This is clear: if $\alpha : E \to E'$ is a morphism in E such that $G(\alpha)$ is an equivalence, then $X^*(\alpha)$ is an equivalence for each $X \in \mathcal{P}^{\leq n}(\mathcal{C})$ and therefore α is an equivalence.

Let $\mathcal{E}_1$ denote the full subcategory of $\mathcal{E}$ spanned by those functors $E : \mathcal{P}^{\leq n}(\mathcal{C}) \to \mathcal{D}$ satisfying the following condition: if $\overline{E} : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ is a left Kan extension of E, then $\overline{E}$ is n-excisive and preserves filtered colimits. We wish to show that $\mathcal{E}_1 = \mathcal{E}$. In view of the preceding arguments, it will suffice to show that $\mathcal{E}_0 \subseteq \mathcal{E}_1$. Note that $\mathcal{E}_1$ is closed under small colimits in $\mathcal{E}$ by Remark 7.1.5.10. It will therefore suffice to show that $\mathcal{E}_1$ contains the functor $X_!(D)$, for each $X \in \mathcal{P}^{\leq n}(\mathcal{C})$ and each $D \in \mathcal{D}$. Since X is a compact object of $\mathcal{P}(\mathcal{C})$, the functor $X_!(D)$ preserves small filtered colimits. We must show that it is n-excisive.

Let $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ denote the Yoneda embedding, so we can write $X = \coprod_{1 \leq i \leq m} j(C_i)$ for some objects $C_i \in \mathcal{C}$, where $m \leq n$. We will show that $X_!(D)$ is m-excisive. Let $S = [m] = \{0, \ldots, m\}$ and choose a strongly coCartesian S-cube $Y : \operatorname{N}(\mathbf{P}(S)) \to \mathcal{P}(\mathcal{C})$. We wish to prove that $(X_!(D))(Y)$ is a Cartesian S-cube in $\mathcal{D}$. According to Proposition 1.2.4.13, it will suffice to show that $(X_!(D))(Y)$ is a colimit diagram in $\mathcal{D}$. Using the formula for $X_!(D)$ given above, we are reduced to proving that the S-cube of spaces $\{\operatorname{Map}_{\mathcal{P}(\mathcal{C})}(X, Y(T))\}_{T \subseteq S}$ is a colimit diagram: that is, it is a left Kan extension of its restriction to $\operatorname{N}(\mathbf{P}_{\leq m}(S))$. Note that this cube is equivalent to a product of the S-cubes of spaces given by $\{(Y(T))(C_i)\}_{T \subseteq S}$, each of which is strongly coCartesian (since we assumed that Y is strongly coCartesian). The desired result now follows from Lemma 7.1.5.8. This completes the proof that $(1) \Rightarrow (2)$.

We now prove that $(2) \Rightarrow (1)$. Let $\mathcal{E}' \subseteq \operatorname{Fun}(\mathcal{P}(\mathcal{C}), \mathcal{D})$ be the full subcategory of $\operatorname{Fun}(\mathcal{P}(\mathcal{C}), \mathcal{D})$ spanned by those functors which are left Kan extensions of their restrictions to $\mathcal{P}^{\leq n}(\mathcal{C})$. We wish to show that if $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ is an excisive functor which commutes with filtered colimits, then $F \in \mathcal{E}'$. The proof proceeds by induction on n. If $n = 0$, then F is constant functor and the result is obvious. Assume that $n > 0$. It follows from the inductive hypothesis (and the transitivity of left Kan extensions) that $\mathcal{E}'$ contains every $(n-1)$-excisive functor which commutes with filtered colimits. Applying Theorem 7.1.2.4, we obtain a fiber sequence of functors

$$F \to P_{n-1}F \to R$$

where R is n-homogeneous. Using the assumption that F commutes with filtered colimits and the construction of P_{n-1}, we conclude that $P_{n-1}F$ commutes with filtered colimits. It follows that $P_{n-1}F \in \mathcal{E}'$ and that R commutes with filtered colimits. Consequently, we may replace F by R and thereby reduce to the case where F is homogeneous.

Let $\psi : \operatorname{SymFun}_{\operatorname{lin}}^n(\mathcal{P}(\mathcal{C}), \mathcal{D}) \to \operatorname{Homog}^n(\mathcal{C}, \mathcal{D})$ be the equivalence of Proposition 7.1.4.14. We may assume that $F = \psi(\overline{H})$ for some $\overline{H} \in \operatorname{SymFun}_{\operatorname{lin}}^n(\mathcal{P}(\mathcal{C}), \mathcal{D})$. Let $H : \mathcal{P}(\mathcal{C})^n \to \mathcal{D}$ be the n-ary functor underlying $\overline{H}$, so that H is 1-homogeneous in each variable. The proof of Proposition 7.1.4.14 shows that the restriction $H | \mathcal{P}(\mathcal{C})^n_*$ is given by the cross effect $\operatorname{cr}_n(F | \mathcal{P}(\mathcal{C})_*)$, and therefore preserves filtered colimits separately in each variable. Using Lemma 7.1.5.9, we conclude that H preserves filtered colimits separately in each variable. Let $\delta : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})^n$ be the diagonal map, so that F is the colimit of a diagram $B\Sigma_n \to \operatorname{Fun}(\mathcal{P}(\mathcal{C}), \mathcal{D})$ carrying the vertex to $H \circ \delta$. Since $\mathcal{E}'$ is closed under small colimits, it will suffice to show that $H \circ \delta \in \mathcal{E}'$. Let $\mathcal{E}'' \subseteq \operatorname{Fun}(\mathcal{P}(\mathcal{C})^n, \mathcal{D})$ be the full subcategory spanned by those functors $G : \mathcal{P}(\mathcal{C})^n \to \mathcal{D}$ such that $G \circ \delta \in \mathcal{E}'$. To complete the proof, it will suffice to show the following:

(∗) Any functor $G : \mathcal{P}(\mathcal{C})^n \to \mathcal{D}$ which commutes with filtered colimits and is 1-excisive in each variable belongs to $\mathcal{E}''$.

The proof of (∗) proceeds by induction on n. Suppose that $G : \mathcal{P}(\mathcal{C})^n \to \mathcal{D}$ commutes with filtered colimits and is 1-excisive in each variable. Fix $1 \leq i \leq n$, let $\emptyset$ denote an initial object of $\mathcal{P}_{\mathcal{C}}$, and define

$G' : \mathcal{P}(\mathcal{C})^n \to \mathcal{D}$ by the formula $G'(X_1, \ldots, X_n) = G(X_1, \ldots, X_{i-1}, \emptyset, X_{i+1}, \ldots, X_n)$. Choose a fiber sequence of functors

$$G' \to G \to G''.$$

The inductive hypothesis implies that $G' \in \mathcal{E}''$. Consequently, to prove that $G \in \mathcal{E}''$ it will suffice to show that $G'' \in \mathcal{E}''$. As a functor of its ith argument, G'' is 1-excisive and preserves initial objects, and is therefore right exact. Since G'' preserves filtered colimits in the ith argument, we conclude that G'' preserves small colimits in the ith argument. Replacing G by G'', we can assume that G preserves small colimits in the ith argument. Applying this argument repeatedly, we can reduce to the case where G preserves small colimits separately in each variable. Let $j : \mathcal{C}^n \to \mathcal{P}(\mathcal{C})^n$ be the nth power of the Yoneda embedding. Using Lemma T.5.1.5.5 repeatedly, we deduce that G is a left Kan extension of $g = G \circ j$ along j. It will therefore suffice to prove the following:

($*'$) Let $g : \mathcal{C}^n \to \mathcal{D}$ be any functor, and let $G : \mathcal{P}(\mathcal{C})^n \to \mathcal{D}$ be a left Kan extension of g along $j : \mathcal{C}^n \to \mathcal{P}(\mathcal{C})^n$. Then $G \in \mathcal{E}''$.

Let $\mathcal{X}$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}^n, \mathcal{D})$ spanned by those functors g such that $G \in \mathcal{E}''$, where G is a left Kan extension of g along j. We wish to prove that $\mathcal{X} = \mathrm{Fun}(\mathcal{C}^n, \mathcal{D})$. For each object $\vec{C} = (C_1, \ldots, C_n) \in \mathcal{C}^n$, let $e_{\vec{X}} : \mathrm{Fun}(\mathcal{C}^n, \mathcal{D}) \to \mathcal{D}$ be the functor given by evaluation at $\vec{C}$, and let $\vec{C}_! : \mathcal{D} \to \mathrm{Fun}(\mathcal{C}^n, \mathcal{D})$ be a left adjoint to $e_{\vec{C}}$, so that $\vec{C}_!$ is given by the formula

$$(\vec{C}_!(D))(Y_1, \ldots, Y_n) = (\prod_i \mathrm{Map}_{\mathcal{C}}(C_i, Y_i)) \otimes D.$$

Let $\mathcal{X}'$ denote the smallest full subcategory of $\mathrm{Fun}(\mathcal{C}^n, \mathcal{D})$ which is closed under small colimits and contains all objects of the form $\vec{C}_!(D)$. Then $\mathcal{X}'$ is a presentable ∞-category which is closed under colimits in $\mathrm{Fun}(\mathcal{C}^n, \mathcal{D})$, so Corollary T.5.5.2.9 implies that the inclusion $\mathcal{X}' \to \mathrm{Fun}(\mathcal{C}^n, \mathcal{D})$ admits a right adjoint U. We claim that $\mathcal{X}' = \mathrm{Fun}(\mathcal{C}^n, \mathcal{D})$. To prove this, it suffices to show that U is conservative. This is clear: if $\alpha : g \to g'$ is a morphism in $\mathrm{Fun}(\mathcal{C}^n, \mathcal{D})$ such that $U(\alpha)$ is an equivalence, then $e_{\vec{C}}(\alpha)$ is an equivalence for each $\vec{C} \in \mathcal{C}^n$. Consequently, to prove that $\mathcal{X} = \mathrm{Fun}(\mathcal{C}^n, \mathcal{D})$, it will suffice to prove that $\mathcal{X}' \subseteq \mathcal{X}$. Because $\mathcal{X}$ is closed under small colimits in $\mathrm{Fun}(\mathcal{C}^n, \mathcal{D})$, it will suffice to show that $\mathcal{X}$ contains every object of the form $g = \vec{C}_!(D)$, where $\vec{C} = (C_1, \ldots, C_n) \in \mathcal{C}^n$ and $D \in \mathcal{D}$. Let $G : \mathcal{P}(\mathcal{C})^n \to \mathcal{D}$ be a left Kan extension of g along j. Unwinding the definitions, we see that G is given by the formula $G(X_1, \ldots, X_n) = (\prod_i X_i(C_i)) \otimes D$. In particular, $G \circ \delta$ is the functor given by

$$X \mapsto (\prod_i X(C_i)) \otimes D \simeq \mathrm{Map}_{\mathcal{P}(\mathcal{C})}(X_0, X) \otimes D,$$

where $X_0 \in \mathcal{P}^{\leq n}(\mathcal{C})$ denotes the coproduct of the functors represented by the objects $C_i \in \mathcal{C}$. It follows that $G \circ \delta$ is the left Kan extension of a constant functor (taking the value D) on the ∞-category $\{X_0\} \subseteq \mathcal{P}^{\leq n}(\mathcal{C})$, and therefore belongs to $\mathcal{E}'$ as desired. $\square$

7.1.6   Norm Maps

Let $\mathcal{C}$ and $\mathcal{D}$ be pointed ∞-categories, where $\mathcal{C}$ admits finite colimits and $\mathcal{D}$ is a differentiable ∞-category. According to Theorem 7.1.2.5, every reduced n-excisive functor from $\mathcal{C}$ to $\mathcal{D}$ can be described (in an essentially unique way) as the fiber of an "attaching map" $\nu : F \to G$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, where F is reduced and $(n-1)$-excisive and G is n-homogeneous. Consequently, the classification of all n-excisive functors from $\mathcal{C}$ to $\mathcal{D}$ can be broken into three problems:

(a) Classify all $(n-1)$-excisive functors from $\mathcal{C}$ to $\mathcal{D}$.

(b) Classify all n-homogeneous functors from $\mathcal{C}$ to $\mathcal{D}$.

(c) Given $F \in \mathrm{Exc}_*^{n-1}(\mathcal{C}, \mathcal{D})$ and $G \in \mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$, classify all natural transformations $\nu : F \to G$.

We can regard (a) as a simpler instance of the same problem (note that if $n = 0$, then $\mathrm{Exc}^n_*(\mathcal{C}, \mathcal{D})$ is a contractible Kan complex), and (b) is addressed in §7.1.4. In this section, we will discuss problem (c) in the special case where $\mathcal{C}$ and $\mathcal{D}$ are stable, and $\mathcal{D}$ admits countable limits. In this case, every n-homogeneous functor $G : \mathcal{C} \to \mathcal{D}$ can be described by the formula $C \mapsto g(C, C, \ldots, C)_{\Sigma_n}$, where $g : \mathcal{C}^{(n)} \to \mathcal{D}$ is an symmetric n-ary functor from $\mathcal{C}$ to $\mathcal{D}$ which is exact in each variable. In this situation, we will show that there is a universal example of an $(n-1)$-excisive functor F equipped with a natural transformation $F \to G$. We can describe F explicitly as the functor which carries an object $C \in \mathcal{C}$ to the fiber of the *norm map*

$$\mathrm{Nm} : g(C, C, \ldots, C)_{\Sigma_n} \to g(C, C, \ldots, C)^{\Sigma_n}.$$

We begin with a general discussion. Let M be an abelian group equipped with an action of a group G. We can associate to M the subgroup $M^G = \{x \in M : (\forall g \in G)[g(x) = x]\}$ consisting of G-invariant elements, as well as the quotient group $M_G = M/K$, where K is the subgroup of M generated by all elements of the form $g(x) - x$. When the group G is finite, there is a canonical norm map

$$\mathrm{Nm} : M_G \to M^G,$$

which is induced by the map from M to itself given by $x \mapsto \sum_{g \in G} g(x)$. Our first goal in this section is to describe an analogous construction in the ∞-categorical setting.

Notation 7.1.6.1. Let $\mathcal{C}$ be an ∞-category and X a Kan complex. We let $\mathcal{C}^X$ denote the ∞-category $\mathrm{Fun}(X, \mathcal{C})$ of all maps from X to $\mathcal{C}$. If $f : X \to Y$ is a map of Kan complexes, then composition with f induces a map $f^* : \mathcal{C}^Y \to \mathcal{C}^X$. Assume that $\mathcal{C}$ admits limits and colimits indexed by the simplicial sets $X \times_Y Y_{/y}$, for each $y \in Y$. Then f^* admits left and right adjoints, which we denote by f_* and $f_!$, respectively.

Example 7.1.6.2. Let G be a group and BG its classifying space (which we regard as a Kan complex). If $\mathcal{C}$ is an ∞-category, we define a *G-equivariant object* of $\mathcal{C}$ is an object of $\mathcal{C}^{BG}$. Let $f : BG \to \Delta^0$ be the projection map. If $\mathcal{C}$ admits small limits and colimits, then we have functors $f_*, f_! : \mathcal{C}^{BG} \to \mathcal{C}$. We will denote these functors by $M \mapsto M^G$ and $M \mapsto M_G$, respectively.

We can now formulate our problem more precisely. Let G be a finite group, $f : BG \to \Delta^0$ the projection map, and $\mathcal{C}$ be a sufficiently nice ∞-category. We wish to associate to the pair $(G, \mathcal{C})$ a natural transformation $\mathrm{Nm} : f_! \to f_*$. That is, we wish to construct a natural map $M_G \to M^G$ for each G-equivariant object $M \in \mathcal{C}$. It will be convenient to construct this natural transformation more generally for any map $f : X \to Y$ having reasonably simple homotopy fibers. We will proceed in several steps, each time allowing slightly more general homotopy fibers.

Lemma 7.1.6.3. *Suppose we are given a homotopy pullback diagram of Kan complexes σ :*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ X & \xrightarrow{f} & Y. \end{array}$$

Let $\mathcal{C}$ be an ∞-category, and assume that for each $y \in Y$ the ∞-category $\mathcal{C}$ admits limits indexed by the Kan complex $X \times_Y Y_{y/}$. Then the diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{C}^Y & \longrightarrow & \mathcal{C}^X \\ \downarrow & & \downarrow \\ \mathcal{C}^{Y'} & \longrightarrow & \mathcal{C}^{X'} \end{array}$$

is right adjointable.

Proof. Let $\mathcal{F} : X \to \mathcal{C}$ be a functor; we wish to show that the canonical map $g'^* f_* \mathcal{F} \to f'_* g^* \mathcal{F}$ is an equivalence in $\mathcal{C}^{Y'}$. Unwinding the definition, we must show that for each vertex $y' \in Y'$, the map

$$\varprojlim(\mathcal{F} | X \times_Y Y_{g'(y)/}) \to \varprojlim(\mathcal{F} | X' \times_{Y'} Y'_{y/})$$

is an equivalence in $\mathcal{C}$. This follows from the fact that the map $X' \times_{Y'} Y'_{y/} \to X \times_Y Y_{g'(y)/}$ is a homotopy equivalence, since we have assumed that σ is a homotopy pullback diagram. $\qquad\square$

Construction 7.1.6.4. Let $\mathcal{C}$ be an ∞-category which has both an initial object and a final object. It follows that for any map of Kan complexes $f : X \to Y$ with (-1)-truncated homotopy fibers, the pullback functor $f^* : \mathcal{C}^Y \to \mathcal{C}^X$ admits left and right adjoints $f_!$ and f_*, given by left and right Kan extension along f.

Let $X \times_Y X$ denote the homotopy fiber product of X with itself over Y and let $\delta : X \to X \times_Y X$ be the diagonal map. Since f is (-1)-truncated, δ is a homotopy equivalence. It follows that the Kan extension functors $\delta_!, \delta_* : \mathcal{C}^X \to \mathcal{C}^{X \times_Y X}$ are both homotopy inverse to δ^*, so there is a canonical equivalence $\delta_* \to \delta_!$. Let $p_0, p_1 : X \times_Y X \to X$ be the projection onto the first and second factor, respectively. We have a natural transformation of functors

$$p_0^* \to \delta_* \delta^* p_0^* \simeq \delta_* \simeq \delta_! \simeq \delta_! \delta^* p_1^* \to p_1^*$$

which is adjoint to a natural transformation $\beta : \mathrm{id}_{\mathcal{C}^X} \to (p_0)_* p_1^*$. Since we have a homotopy pullback diagram

$$
\begin{array}{ccc}
X \times_Y X & \xrightarrow{\ p_0\ } & X \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{\ f\ } & Y,
\end{array}
$$

Lemma 7.1.6.3 implies that the canonical map $f^* f_* \to (p_0)_* p_1^*$ is an equivalence, so that β determines a natural transformation $\mathrm{id}_{\mathcal{C}^X} \to f^* f_*$, which is in turn adjoint to a map $\mathrm{Nm}_f : f_! \to f_*$. We will refer to Nm_f as the *norm map* determined by f.

Example 7.1.6.5. Let $\mathcal{C}$ be an ∞-category with initial and final objects and let $f : X \to Y$ be a homotopy equivalence of Kan complexes. Then the natural transformation $\mathrm{Nm}_f : f_! \to f_*$ of Construction 7.1.6.4 is the equivalence determined by the observation that $f_!$ and f_* are both homotopy inverse to f^*: in other words, it is determined by the requirement that the induced map $f^* f_! \to f^* f_*$ is homotopy inverse to the composition of counit and unit maps

$$f^* f_* \to \mathrm{id}_{\mathcal{C}^X} \to f^* f_!.$$

Example 7.1.6.6. Let $Y = \Delta^0$ and let $\mathcal{C}$ be an ∞-category with initial and final objects. If $f : X \to Y$ is a (-1)-truncated map of Kan complexes, then X is either empty or contractible. If X is contractible, then the norm map Nm_f is the equivalence described in Example 7.1.6.5. If $X = \emptyset$, then $\mathcal{C}^X \simeq \Delta^0$ and the functors $f_!$ and f_* can be identified with initial and final objects of $\mathcal{C} \simeq \mathcal{C}^Y$, respectively. In this case, the norm map Nm_f is determined up to a contractible space of choices, since it is a map from an initial object of $\mathcal{C}$ to a final object of $\mathcal{C}$.

Proposition 7.1.6.7. *Let $\mathcal{C}$ be an ∞-category with an initial and final object. The following conditions are equivalent:*

(1) *For every map of Kan complexes $f : X \to Y$ with (-1)-truncated homotopy fibers, the norm map $\mathrm{Nm}_f : f_! \to f_*$ is an equivalence.*

(2) *Condition (1) holds whenever $Y = \Delta^0$.*

(3) *The ∞-category $\mathcal{C}$ is pointed.*

Proof. The equivalence of (1) and (2) is easy, and the equivalence of (2) and (3) follows from Example 7.1.6.6. ☐

When the hypotheses of Proposition 7.1.6.7 are satisfied, it is possible to perform a more elaborate version of Construction 7.1.6.4.

Construction 7.1.6.8. Let $\mathcal{C}$ be an ∞-category which admits finite products and coproducts. It follows that for any map of Kan complexes $f : X \to Y$ whose homotopy fibers are 0-truncated and have finitely many path components, the pullback functor $f^* : \mathcal{C}^Y \to \mathcal{C}^X$ admits left and right adjoints $f_!$ and f_*, given by left and right Kan extension along f.

Let $X \times_Y X$ denote the homotopy fiber product of X with itself over Y and let $\delta : X \to X \times_Y X$ be the diagonal map. Since f is 0-truncated, the map δ is (-1)-truncated, so that Construction 7.1.6.4 defines a norm map $\mathrm{Nm}_\delta : \delta_! \to \delta_*$. Assume that $\mathcal{C}$ is pointed. Proposition 7.1.6.7 implies that Nm_δ is an equivalence, and therefore admits a homotopy inverse $\mathrm{Nm}_\delta^{-1} : \delta_* \to \delta_!$.

Let $p_0, p_1 : X \times_Y X \to X$ be the projection onto the first and second factor, respectively. We have a natural transformation of functors

$$p_0^* \to \delta_* \delta^* p_0^* \xrightarrow{\mathrm{Nm}_\delta^{-1}} \delta_* \simeq \delta_! \simeq \delta_! \delta^* p_1^* \to p_1^*$$

which is adjoint to a natural transformation $\beta : \mathrm{id}_{\mathcal{C}^X} \to (p_0)_* p_1^*$. Since we have a homotopy pullback diagram

$$
\begin{array}{ccc}
X \times_Y X & \xrightarrow{\ p_0\ } & X \\
\ \downarrow{\scriptstyle p_1} & & \ \downarrow{\scriptstyle f} \\
X & \xrightarrow{\ f\ } & Y,
\end{array}
$$

Lemma 7.1.6.3 implies that the canonical map $f^* f_* \to (p_0)_* p_1^*$ is an equivalence, so that β determines a map $\mathrm{id}_{\mathcal{C}^X} \to f^* f_*$, which is adjoint to a natural transformation $\mathrm{Nm}_f : f_! \to f_*$. We will refer to Nm_f as the *norm map* determined by f.

Remark 7.1.6.9. In the situation of Construction 7.1.6.8, assume that f is (-1)-truncated. Then our definition of Nm_f is unambiguous: in other words, the natural transformations $\mathrm{Nm}_f : f_! \to f_*$ described in Constructions 7.1.6.4 and 7.1.6.8 agree. This follows immediately from Example 7.1.6.5.

Remark 7.1.6.10. Suppose we are given a homotopy pullback diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{\ f'\ } & Y' \\
\ \downarrow{\scriptstyle p'} & & \ \downarrow{\scriptstyle p} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

where the homotopy fibers of f are 0-truncated and have finitely many homotopy groups. Let $\mathcal{C}$ be a pointed ∞-category which admits finite products and coproducts. Using Lemma 7.1.6.3, it is not difficult to show that the natural transformation $f'_! \circ p'^* \to f'_* \circ p'^*$ determined by $\mathrm{Nm}_{f'}$ is homotopic to the composition

$$f'_! \circ p'^* \to p^* \circ f_! \xrightarrow{\mathrm{Nm}_f} p^* \circ f_* \to f'_* \circ p'^*.$$

Example 7.1.6.11. Let $\mathcal{C}$ be a pointed ∞-category which admits finite products and coproducts, let S be a finite set (regarded as a discrete simplicial set), and let $f : S \to \Delta^0$ be the canonical projection map. We can identify objects of $\mathcal{C}^S$ with tuples $C = (C_s \in \mathcal{C})_{s \in S}$. The norm map $f_!(C) \to f_*(C)$ can be identified with the map

$$\coprod_{s \in S} C_s \to \prod_{t \in S} C_t,$$

which classifies a collection of maps $\phi_{s,t} : C_s \to C_t$ in $\mathcal{C}$ where $\phi_{s,t} = \mathrm{id}$ if $s = t$ and the zero map otherwise.

Proposition 7.1.6.12. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite products and coproducts. The following conditions are equivalent:*

(1) *For every map of Kan complexes $f : X \to Y$ whose homotopy fibers are discrete and have finitely many connected components, the norm map $\mathrm{Nm}_f : f_! \to f_*$ is an equivalence.*

(2) *Condition (1) holds whenever $Y = \Delta^0$.*

(3) *For every finite collection of objects $\{C_s \in \mathcal{C}\}_{s \in S}$, the map*

$$\coprod_{s \in S} C_s \to \prod_{t \in S} C_t$$

described in Example 7.1.6.11 is an equivalence.

Proof. The implication (1) $\Rightarrow$ (2) is obvious and the converse follows from Remark 7.1.6.10. The equivalence of (2) and (3) follows from Example 7.1.6.11. $\square$

Definition 7.1.6.13. We will say that an ∞-category $\mathcal{C}$ is *semiadditive* if it satisfies the equivalent conditions of Proposition 7.1.6.12.

Remark 7.1.6.14. Let $\mathcal{C}$ be a semiadditive ∞-category. Suppose we are given a pair of objects $C, D \in \mathcal{C}$ and a finite collection of maps $\{\phi_s : C \to D\}_{s \in S}$. Then we can define a new map $\phi : C \to D$ by the composition

$$C \to \prod_{s \in S} C \xrightarrow{(\phi_s)_{s \in S}} \prod_{s \in S} D \simeq \coprod_{s \in S} D \to D,$$

where the first map is the diagonal of C and the last the codiagonal of D. This construction determines a map

$$\prod_{s \in S} \mathrm{Map}_{\mathcal{C}}(C, D) \to \mathrm{Map}_{\mathcal{C}}(C, D),$$

which endows $\mathrm{Map}_{\mathcal{C}}(C, D)$ with the structure of a commutative monoid up to homotopy. We will denote the image of a collection of morphisms $(\phi_s)_{s \in S}$ by $\sum_{s \in S} \phi_s$.

It is possible to make a much stronger assertion: the addition on $\mathrm{Map}_{\mathcal{C}}(C, D)$ is not only commutative and associative up to homotopy, but up to coherent homotopy. That is, each mapping space in $\mathcal{C}$ can be regarded as a commutative algebra object of $\mathcal{S}$, and the composition of morphisms in $\mathcal{C}$ is multilinear. Since we do not need this for the time being, we omit the proof.

Remark 7.1.6.15. Let $\mathcal{C}$ be an ∞-category which admits finite products and coproducts. Since products and coproducts in $\mathcal{C}$ are also products and coproducts in the homotopy category $h\mathcal{C}$, we see that $\mathcal{C}$ is semiadditive if and only if (the nerve of) the category $h\mathcal{C}$ is semiadditive.

Example 7.1.6.16. Let $\mathcal{A}$ be an additive category (see Definition 1.1.2.1). Then the ∞-category $N(\mathcal{A})$ is semiadditive.

Example 7.1.6.17. Let $\mathcal{C}$ be a stable ∞-category. Then the homotopy category $h\mathcal{C}$ is additive (Lemma 1.1.2.9). Combining this with Example 7.1.6.16 and Remark 7.1.6.15, we deduce that $\mathcal{C}$ is semiadditive.

Definition 7.1.6.18. Let X be a Kan complex. We will say that X is a *finite groupoid* if the following conditions are satisfied:

(1) The set of connected components $\pi_0 X$ is finite.

(2) For every point $x \in X$, the fundamental group $\pi_1(X, x)$ is finite.

(3) The homotopy groups $\pi_n(X, x)$ vanish for $n \geq 2$.

More generally, we say that a map of Kan complexes $f : X \to Y$ is a *relative finite groupoid* if the homotopy fibers of f are finite groupoids.

Construction 7.1.6.19. Let $\mathcal{C}$ be a semiadditive ∞-category which admits limits and colimits indexed by finite groupoids. It follows that for any map of Kan complexes $f : X \to Y$ which is a relative finite groupoid, the pullback functor $f^* : \mathcal{C}^Y \to \mathcal{C}^X$ admits left and right adjoints $f_!$ and f_*, given by left and right Kan extension along f.

Let $X \times_Y X$ denote the homotopy fiber product of X with itself over Y and let $\delta : X \to X \times_Y X$ be the diagonal map. Since f is a relative finite groupoid, the homotopy fibers of δ are homotopy equivalent to finite discrete spaces. Construction 7.1.6.8 defines a norm map $\mathrm{Nm}_\delta : \delta_! \to \delta_*$. Since $\mathcal{C}$ is semiadditive, the natural transformation Nm_δ is an equivalence and therefore admits a homotopy inverse $\mathrm{Nm}_\delta^{-1} : \delta_* \to \delta_!$.

Let $p_0, p_1 : X \times_Y X \to X$ be the projection onto the first and second factor, respectively. We have a natural transformation of functors

$$p_0^* \to \delta_* \delta^* p_0^* \xrightarrow{\mathrm{Nm}_\delta^{-1}} \delta_* \simeq \delta_! \simeq \delta_! \delta^* p_1^* \to p_1^*$$

which is adjoint to a natural transformation $\beta : \mathrm{id}_{\mathcal{C}^X} \to (p_0)_* p_1^*$. Since we have a homotopy pullback diagram

$$
\begin{array}{ccc}
X \times_Y X & \xrightarrow{p_0} & X \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{f} & Y,
\end{array}
$$

Lemma 7.1.6.3 implies that the canonical map $f^* f_* \to (p_0)_* p_1^*$ is an equivalence, so that β determines a map $\mathrm{id}_{\mathcal{C}^X} \to f^* f_*$, which is adjoint to a natural transformation $\mathrm{Nm}_f : f_! \to f_*$. We will refer to Nm_f as the *norm map* determined by f.

Remark 7.1.6.20. In the situation of Construction 7.1.6.19, assume that f is 0-truncated. Then the definition of Nm_f given in Construction 7.1.6.19 agrees with that given in Construction 7.1.6.8 (and, if f is (-1)-truncated, with that given in Construction 7.1.6.4): this follows easily from Remark 7.1.6.9.

Remark 7.1.6.21. In the situation of Construction 7.1.6.19, suppose we are given a homotopy pullback diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\
X & \xrightarrow{f} & Y.
\end{array}
$$

Using Lemma 7.1.6.3, we deduce that the natural transformation $f'_! \circ p'^* \to f'_* \circ p'^*$ determined by $\mathrm{Nm}_{f'}$ is homotopic to the composition

$$f'_! \circ p'^* \to p^* \circ f_! \xrightarrow{\mathrm{Nm}_f} p^* \circ f_* \to f'_* \circ p'^*.$$

Example 7.1.6.22. Let G be a finite group. Then the classifying space BG is a finite groupoid. Let $f : BG \to \Delta^0$ be the projection map. If $\mathcal{C}$ is semiadditive ∞-category which admits limits and colimits indexed by finite groupoids, then Construction 7.1.6.19 determines a natural transformation $\mathrm{Nm}_f : f_! \to f_*$. In particular, for every G-equivariant object $M \in \mathcal{C}$, we obtain a canonical map $\mathrm{Nm} : M_G \to M^G$.

Remark 7.1.6.23. Let $\mathcal{C}$ be a semiadditive ∞-category which admits limits and colimits indexed by finite groupoids, and let G be a finite group. Let M be a G-equivariant object of $\mathcal{C}$, and abuse notation by identifying M with its image in $\mathcal{C}$. We have canonical maps $e : M \to M_G$ and $e' : M^G \to M$. Unwinding the definitions, we see that the composition

$$M \xrightarrow{e} M_G \xrightarrow{\mathrm{Nm}} M^G \xrightarrow{e'} M$$

is given by $\sum_{g \in G} \phi_g$, where for each $g \in G$ we let $\phi_g : M \to M$ be the map given by evaluation on the 1-simplex of BG corresponding to g; here the sum is formed with respect to the addition described in Remark 7.1.6.14.

If $\mathcal{C}$ is equivalent to the nerve of an ordinary category, then the map $M \to M_G$ is a categorical epimorphism and the map $M^G \to M$ is a categorical monomorphism. It follows that the map $\mathrm{Nm} : M_G \to M^G$ is determined (up to homotopy) by the formula $e' \circ \mathrm{Nm} \circ e \simeq \sum_g \phi_g$. Moreover, the map Nm exists by virtue of the observation that the map $\sum_g \phi_g : M \to M$ is invariant under left and right composition with the maps ϕ_g.

Definition 7.1.6.24. Let $\mathcal{C}$ be a stable ∞-category which admits countable limits and colimits. Let G be a finite group, and let M be a G-equivariant object of $\mathcal{C}$. We will denote the cofiber of the norm map $\mathrm{Nm} : M_G \to M^G$ by M^{tG}. We refer to the formation $M \mapsto M^{tG}$ as the *Tate construction*.

Remark 7.1.6.25. Let $\mathcal{C}$ be a semiadditive ∞-category which admits limits and colimits indexed by finite groupoids. Assume that for every finite group G and every G-equivariant object M of $\mathcal{C}$, the norm map $\mathrm{Nm} : M_G \to M^G$ is an equivalence (if $\mathcal{C}$ is stable, this is equivalent to the requirement that the Tate construction M^{tG} vanish). It follows that for every relative finite groupoid $f : X \to Y$, the norm map $\mathrm{Nm}_f : f_! \to f_*$ is an equivalence of functors from $\mathcal{C}^X$ to $\mathcal{C}^Y$. We can then repeat Construction 7.1.6.19 to define a norm map $\mathrm{Nm}_f : f_! \to f_*$ for maps $f : X \to Y$ whose homotopy fibers are finite 2-groupoids. If $\mathcal{C}$ also admits limits and colimits indexed by finite 2-groupoids, then we can repeat Construction 7.1.6.19 to define a norm map $\mathrm{Nm}_f : f_! \to f_*$ whenever f is a relative finite 2-groupoid. This condition is satisfied, for example, if $\mathcal{C}$ is a $\mathbf{Q}$-linear ∞-category (here $\mathbf{Q}$ denotes the field of rational numbers), but is generally not satisfied for stable ∞-categories defined in positive or mixed characteristics. However, it is always satisfies in the setting of $K(n)$-local stable homotopy theory. We will study this construction in more detail in a future work.

Example 7.1.6.26. Let $\mathcal{C}$ be a semiadditive ∞-category which admits limits and colimits indexed by finite groupoids. Let G be a finite group, let $i : \Delta^0 \to BG$ be the inclusion of the base point and let $f : BG \to \Delta^0$ be the projection map. Let $M \in \mathcal{C} \simeq \mathcal{C}^{\Delta^0}$ and let $N = i_! M \in \mathcal{C}^{BG}$, so that $N = i_! M \simeq \coprod_{g \in G} M \simeq \prod_{g \in G} M \simeq i_* M$. Unwinding the definitions, we see that the norm map $f_!(N) \to f_*(N)$ is given by the composition

$$f_!(N) = f_! i_! M \simeq (\mathrm{id}_! M) \simeq (\mathrm{id}_* M) \simeq f_* i_*(M) \simeq f_*(N)$$

and is therefore an equivalence. If $\mathcal{C}$ is stable, we conclude that the Tate construction N^{tG} is a zero object of $\mathcal{C}$.

Let us now return to the calculus of functors. Suppose we are given a symmetric n-ary functor $F : \mathcal{C}^{(n)} \to \mathcal{D}$, where $\mathcal{D}$ is a stable ∞-category which admits countable limits and colimits. Restricting to the diagonal, F determines a diagram $(F\delta) : B\Sigma_n \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Taking the colimit and limit of this diagram, we obtain functors $(F\delta)^{\Sigma_n}, (F\delta)_{\Sigma_n} : \mathcal{C} \to \mathcal{D}$, and a natural transformation $\mathrm{Nm} : (F\delta)_{\Sigma_n} \to (F\delta)^{\Sigma_n}$. We will denote the cofiber of this natural transformation by $(F\delta)^{t\Sigma_n}$.

Proposition 7.1.6.27. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object, and let $\mathcal{D}$ be a stable ∞-category which admits finite limits and colimits. Let $F \in \mathrm{SymFun}_{\mathrm{lin}}^{(n)}(\mathcal{C}, \mathcal{D})$. Then the functor $(F\delta)^{t\Sigma_n}$ is $(n-1)$-excisive.*

Proof. The functor $(F\delta)_{\Sigma_n}$ is n-homogeneous by Proposition 7.1.5.4. Since the collection of n-excisive functors from $\mathcal{C}$ to $\mathcal{D}$ is stable under countable limits, Proposition 7.1.3.4 implies that $(F\delta)^{\Sigma_n}$ is n-excisive. It follows that $(F\delta)^{t\Sigma_n}$ is also n-excisive. Consequently, to prove that the cross effect $\mathrm{cr}_n (F\delta)^{t\Sigma_n}$ vanishes (Proposition 7.1.4.10). Let $(F\delta) : B\Sigma_n \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be as above, so that $\mathrm{cr}_n \circ (F\delta)$ determines a map $B\Sigma_n \to \mathrm{Fun}(\mathcal{C}^n, \mathcal{D})$. This diagram has both a colimit and a limit, which we will denote by $(\mathrm{cr}_n \circ (F\delta))_{\Sigma_n}$ and $(\mathrm{cr}_n \circ (F\delta))^{\Sigma_n}$, respectively. Moreover, we have a transfer map

$$\mathrm{Nm} : (\mathrm{cr}_n \circ (F\delta))_{\Sigma_n} \to (\mathrm{cr}_n \circ (F\delta))^{\Sigma_n}$$

whose cofiber is the cross-effect $\mathrm{cr}_n(F\delta)^{t\Sigma_n}$. It will therefore suffice to show that Nm is an equivalence. In view of Example 7.1.6.26, it will suffice to show that the diagram $\mathrm{cr}_n \circ (F\delta)$ is an induced representation of the symmetric group Σ_n: that is, that it is given by a left Kan extension along a map $\Delta^0 \to B\Sigma_n$. This follows immediately from Proposition 7.1.4.13. $\qquad\square$

Remark 7.1.6.28. Let $\mathcal{C}$ and $\mathcal{D}$ be stable ∞-categories, and assume that $\mathcal{D}$ admits countable limits and colimits. Suppose we are given a symmetric n-ary functor $F : \mathcal{C}^{(n)} \to \mathcal{D}$, and consider the fiber sequence

$$(F\delta)_{\Sigma_n} \overset{\mathrm{Nm}}{\to} (F\delta)^{\Sigma_n} \to (F\delta)^{t\Sigma_n}$$

constructed above. Proposition 7.1.4.14 implies that $(F\delta)_{\Sigma_n}$ is n-homogeneous: that is, it is n-excisive and $\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}((F\delta)_{\Sigma_n}, G)$ is contractible for every $(n-1)$-excisive functor $G : \mathcal{C} \to \mathcal{D}$. Replacing $\mathcal{C}$ and $\mathcal{D}$ by their opposite ∞-categories (which does not change the notion of k-excisive functor; see Corollary 7.1.1.17), the same argument shows that $(F\delta)^{\Sigma_n}$ is n-*cohomogeneous*: that is, it is n-excisive and the mapping space $\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(G, (F\delta)^{\Sigma_n}))$ is contractible whenever G is $(n-1)$-excisive. It follows that for any $(n-1)$-excisive functor G, the canonical map

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(G, (F\delta)^{t\Sigma_n}) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(G, (F\delta)_{\Sigma_n}[-1])$$

is a homotopy equivalence.

Remark 7.1.6.29. Let $\mathcal{C}$ and $\mathcal{D}$ be stable ∞-categories, let $n > 0$ be an integer, and assume that $\mathcal{D}$ admits countable limits and colimits. Combining Remark 7.1.6.28, Proposition 7.1.6.27, and Theorem 7.1.2.5, we deduce that giving a reduced n-excisive functor $F : \mathcal{C} \to \mathcal{D}$ is equivalent to giving the following data:

(a) A reduced $(n-1)$-excisive $E : \mathcal{C} \to \mathcal{D}$ (which will be given by $P_{n-1}(F)$).

(b) A functor $K \in \mathrm{SymFun}_{\mathrm{lin}}^{(n)}(\mathcal{C}, \mathcal{D})$ (which is a preimage of $\mathrm{fib}(F \to P_{n-1}F)$ under the equivalence of Proposition 7.1.4.14).

(c) A natural transformation of $(n-1)$-excisive functors $\alpha : E \to (K\delta)^{t\Sigma_n}$ (which is equivalent to the data of a map $E \to (K\delta)_{\Sigma_n}[-1]$, by Remark 7.1.6.28).

From the data of (a), (b), and (c), we can recover an n-excisive functor F by taking the fiber of the composite map $E \overset{\alpha}{\to} (K\delta)^{t\Sigma_n} \to (K\delta)_{\Sigma_n}[-1]$, which can also be described as the fiber product

$$E \times_{(K\delta)^{t\Sigma_n}} (K\delta)^{\Sigma_n}.$$

7.2   Differentiation

Let $\mathcal{C}$ be an ∞-category which admits finite limits. In §1.4.2, we introduced the stable ∞-category $\mathrm{Sp}(\mathcal{C})$ of spectrum objects of $\mathcal{C}$. In this section, we will discuss the extent to which the construction $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$ is functorial in $\mathcal{C}$. For example, suppose that $F : \mathcal{C} \to \mathcal{D}$ is a functor between ∞-categories which admits finite limits. Under what conditions does F determine a functor from $\mathrm{Sp}(\mathcal{C})$ to $\mathrm{Sp}(\mathcal{D})$? The most obvious case to consider is when the functor F is left exact. In this case, pointwise composition with F determines a $\partial F : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$. This functor f fits into a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Sp}(\mathcal{C}) & \overset{\partial F}{\longrightarrow} & \mathrm{Sp}(\mathcal{D}) \\
\downarrow{\scriptstyle \Omega^\infty} & & \downarrow{\scriptstyle \Omega^\infty} \\
\mathcal{C} & \underset{F}{\longrightarrow} & \mathcal{D} .
\end{array}
$$

There is a dual situation which is also important. Suppose $\mathcal{C}$ and $\mathcal{D}$ are presentable ∞-categories, and that the functor F preserves small colimits. Applying Corollary T.5.5.2.9, we deduce that F admits a

right adjoint G. Since G is left exact, we can apply the above reasoning to obtain an induced functor $\partial G : \mathrm{Sp}(\mathcal{D}) \to \mathrm{Sp}(\mathcal{C})$. We can use Corollary T.5.5.2.9 again to deduce that ∂G admits a left adjoint ∂F. This left adjoint fits into a commutative diagram of ∞-categories

$$
\begin{array}{ccc}
\mathrm{Sp}(\mathcal{C}) & \xrightarrow{\ \partial F\ } & \mathrm{Sp}(\mathcal{D}) \\
{\scriptstyle \Sigma^\infty_+}\big\uparrow & & \big\uparrow{\scriptstyle \Sigma^\infty_+} \\
\mathcal{C} & \xrightarrow{\ \ F\ \ } & \mathcal{D}\,.
\end{array}
$$

This raises a number of questions. For example, suppose that a functor $F : \mathcal{C} \to \mathcal{D}$ preserves small colimits *and* finite limits. In this case, we can apply either of the above constructions to produce a functor $\mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$: do the resulting functors coincide (up to homotopy)? On the other hand, suppose that F satisfies neither condition; can one still hope to find an exact functor $\partial F : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$ which is somehow related to F?

To address these questions, it is convenient to reformulate them in terms that do not mention spectrum objects at all. Let $\mathcal{C}$ and $\mathcal{D}$ be presentable pointed ∞-categories. Composition with the functors $\Sigma^\infty_{\mathcal{C}} : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$ and $\Omega^\infty_{\mathcal{D}} : \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$ determines a forgetful functor $\theta : \mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{D})) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, where $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{D}))$ denotes the ∞-category of colimit-preserving functors from $\mathrm{Sp}(\mathcal{C})$ to $\mathrm{Sp}(\mathcal{D})$ (that is, the ∞-category of exact functors which preserve filtered colimits). Under mild assumptions on $\mathcal{D}$, the functor θ is fully faithful, and its essential image is a full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those reduced, excisive functors which preserve filtered colimits. Consequently, we may rephrase our problem as follows: given a functor $F : \mathcal{C} \to \mathcal{D}$, can we choose a reduced, excisive functor $F' : \mathcal{C} \to \mathcal{D}$ which preserves filtered colimits and is, in some sense, a best approximation to the original functor F?

These questions can be readily addressed using ideas introduced in §7.1. Let us suppose that $F : \mathcal{C} \to \mathcal{D}$ is a reduced functor between compactly generated pointed ∞-categories, which preserves small filtered colimits. According to Theorem 7.1.1.10, there exists a natural transformation $\alpha : F \to P_1 F$, which is universal among natural transformations from F to an excisive functor. Corollary 7.1.2.9 implies that $P_1 F$ is given by the composition

$$
\mathcal{C} \xrightarrow{f} \mathrm{Sp}(\mathcal{D}) \xrightarrow{\Omega^\infty_{\mathcal{D}}} \mathcal{D},
$$

where f is reduced and excisive (and therefore right exact). The functor $P_1 F$ preserves small filtered colimits (Remark 7.1.1.31), so that f preserves small filtered colimits and therefore all small colimits. It follows from Corollary 1.4.4.5 that the functor f factors as a composition

$$
\mathcal{C} \xrightarrow{\Sigma^\infty_{\mathcal{C}}} \mathrm{Sp}(\mathcal{C}) \xrightarrow{\partial(F)} \mathrm{Sp}(\mathcal{D}),
$$

where $\partial(F)$ is an exact functor between stable ∞-categories. We can then regard α as a natural transformation from F to $\Omega^\infty_{\mathcal{D}} \circ \partial F \circ \Sigma^\infty_{\mathcal{C}}$. We will refer to the functor $\partial(F)$ as the *derivative* of F.

Our first objective in this section is to study the passage from F to $\partial(F)$. In §7.2.1, we will give a concrete description of $\partial(F)$, analogous to the formula $P_1 F \simeq \varinjlim \Omega^m_{\mathcal{D}} \circ F \circ \Sigma^m_{\mathcal{C}}$ of Example 7.1.1.28. As an application, we prove a version of the Klein-Rognes chain rule, which asserts that if $G : \mathcal{D} \to \mathcal{E}$ is another reduced functor between compactly generated pointed ∞-categories which commutes with filtered colimits, then there is canonical equivalence $\partial(G \circ F) \simeq \partial(G) \circ \partial(F)$ of functors from $\mathrm{Sp}(\mathcal{C})$ to $\mathrm{Sp}(\mathcal{E})$ (see Theorem 7.2.1.22 and Corollary 7.2.1.24). We can informally summarize the situation informally as follows: there is a functor of ∞-categories which carries each compactly generated pointed ∞-category $\mathcal{C}$ to the stable ∞-category $\mathrm{Sp}(\mathcal{C})$, and to each reduced functor $F : \mathcal{C} \to \mathcal{D}$ which commutes with filtered colimits its derivative $\partial(F)$. In §7.2.2, we will give a precise formulation and proof of this assertion (Theorem 7.2.2.1), using the a relative version of the stabilization construction $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$ (see Construction 7.2.2.2).

In §7.2.3, we will study the inclusion $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Under some mild hypotheses, we show that this functor admits a left adjoint, which carries a functor $F : \mathcal{C} \to \mathcal{D}$ to its *differential $DF : \mathcal{C} \to \mathcal{D}$*. If F is reduced, then DF can be identified with the 1-excisive approximation $P_1(F)$ introduced in §7.1.1.

In the general case, we have $DF \simeq P_1(\mathrm{cored}(F))$, where the *coreduction* of F is given by the formula $\mathrm{cored}(F)(C) = \mathrm{cofib}(F(*) \to F(C))$. Our main result is that there is a close connection between the derivative $\partial(F)$ and the differential DF for a large class of (possibly nonreduced) functors F (Corollary 7.2.3.24).

Throughout this section, we will study differentiation not only for one-variable functors $F : \mathcal{C} \to \mathcal{D}$, but also for multifunctors $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$. One motivation for this generality is that it gives a new perspective on the smash product monoidal structure on spectra constructed in §6.3.2: for each $n \geq 0$, the iterated smash product functor $\otimes : \mathrm{Sp}^n \to \mathrm{Sp}$ can be identified with the derivative of the iterated Cartesian product functor $\times : \mathcal{S}^n \to \mathcal{S}$. In §7.2.4, we will use this observation to obtain a generalization of the smash product to $\mathrm{Sp}(\mathcal{C})$, where $\mathcal{C}$ is an arbitrary compactly generated ∞-category. This generalization will play an important role in our discussion of the chain rule in §7.3. To construct it, we will need to generalize the stabilization construction of §7.2.2 to the case of multifunctors. Our treatment of this generalization is somewhat technical; we give the construction in §7.2.5, and characterize the result by a universal property in §7.2.6.

7.2.1 Derivatives of Functors

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories which admit finite limits. We will define the *derivative* of F to be an exact functor $\mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$ which is, in some sense, the best possible "linear" approximation to f. For later applications, we give a definition in the setting of functors of several variables.

Definition 7.2.1.1. Let $\{\mathcal{C}_s\}_{s \in S}$ be a finite collection of ∞-categories which admit finite limits, $\mathcal{D}$ an ∞-category which admits finite limits, and suppose we are given functors

$$F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D} \qquad f : \prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s) \to \mathrm{Sp}(\mathcal{D}).$$

We will say that a natural transformation

$$\alpha : F \circ \prod_{s \in S} \Omega_{\mathcal{C}_s}^\infty \to \Omega_{\mathcal{D}}^\infty \circ f$$

exhibits f as a derivative of F if the following conditions are satisfied:

(1) The functor f is multilinear (that is, it is exact in each variable).

(2) For every multilinear functor $g : \prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s) \to \mathrm{Sp}(\mathcal{D})$, composition with α induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s), \mathrm{Sp}(\mathcal{D}))}(f, g) \to \mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s), \mathcal{D})}(F \circ \prod_{s \in S} \Omega_{\mathcal{C}_s}^\infty, \Omega_{\mathcal{D}}^\infty \circ g).$$

Notation 7.2.1.2. Let $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ be as in Definition 7.2.1.1. If there exists a natural transformation $\alpha : F \circ \prod_{s \in S} \Omega_{\mathcal{C}_s}^\infty \to \Omega_{\mathcal{D}}^\infty \circ f$ which exhibits f as a derivative of F, then f is determined by F, up to canonical equivalence. We will emphasize the dependence of F on f by writing f as $\vec{\partial} F$. In the special case where S has a single element, we will denote f simply by ∂F.

Warning 7.2.1.3. The derivative $\vec{\partial}$ of a functor $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ depends not only on F, but also on the product decomposition of $\prod_{s \in S} \mathcal{C}_s$. For example, the derivative $\vec{\partial} F$ is generally quite different from the functor ∂F, obtained by viewing F as a functor of one variable from $\mathcal{C} = \prod_{s \in S} \mathcal{C}_s$ into $\mathcal{D}$.

Example 7.2.1.4. Let $F : \mathcal{C} \to \mathcal{D}$ be a left exact functor between ∞-categories which admit finite limits. Then composition with F induces a functor $f : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$, and we have an evident equivalence

$\alpha : F \circ \Omega^{\infty}_{\mathcal{C}} \simeq \Omega^{\infty}_{\mathcal{D}} \circ f$. Since F is left exact, the functor f is exact. Using Proposition 1.4.2.22, we deduce that α exhibits f as a derivative of F.

In particular, if $\mathcal{C}$ is an ∞-category which admits finite limits, then the identity functor $\mathrm{id}_{\mathrm{Sp}(\mathcal{C})}$ is a derivative of the identity functor $\mathrm{id}_{\mathcal{C}}$.

Example 7.2.1.5. In the situation of Definition 7.2.1.1, suppose that S is empty, so that we can identify a functor $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ with an object $D \in \mathcal{D}$ and a functor $f : \prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s) \to \mathrm{Sp}(\mathcal{D})$ with an object $X \in \mathrm{Sp}(\mathcal{D})$. A morphism $\alpha : D \to \Omega^{\infty}_{\mathcal{D}}(X)$ exhibits f as a derivative of F if and only if it has the following universal property: for every spectrum object $Y \in \mathrm{Sp}(\mathcal{D})$, composition with α induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Sp}(\mathcal{D})}(X, Y) \to \mathrm{Map}_{\mathcal{D}}(D, \Omega^{\infty}_{\mathcal{D}} Y).$$

The requirement that every functor $\prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ admits a derivative is then equivalent to the requirement that the functor $\Omega^{\infty}_{\mathcal{D}} : \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$ admits a left adjoint $\Sigma^{\infty}_{+} : \mathcal{D} \to \mathrm{Sp}(\mathcal{D})$.

Remark 7.2.1.6. Let $\{\mathcal{C}_s\}_{s \in S}$ be a finite collection of ∞-categories which admit finite limits, $\mathcal{D}$ an ∞-category which admits finite limits, and $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ a functor. For each $s \in S$, let $\mathcal{C}_{s,*}$ denote the ∞-category of pointed objects of $\mathcal{C}_s$, so that the forgetful functor $\mathcal{C}_{s,*} \to \mathcal{C}_s$ induces an equivalence of ∞-categories $u_s : \mathrm{Sp}(\mathcal{C}_{s,*}) \to \mathrm{Sp}(\mathcal{C}_s)$ (Remark 1.4.2.18). Let F_* denote the composite map

$$\prod_{s \in S} \mathcal{C}_{s,*} \to \prod_{s \in S} \mathcal{C}_s \xrightarrow{F} \mathcal{D},$$

and let u be the product of the functors u_s. Then a natural transformation $\alpha : F \circ \prod_{s \in S} \Omega^{\infty}_{\mathcal{C}_s} \to \Omega^{\infty}_{\mathcal{D}} \circ f$ exhibits f as a derivative of F if and only if the induced transformation

$$F_* \circ \prod_{s \in S} \Omega^{\infty}_{\mathcal{C}_{s,*}} \to \Omega^{\infty}_{\mathcal{D}} \circ f \circ u$$

exhibits $f \circ u$ as a derivative of F_*.

Remark 7.2.1.7. Let $\{\mathcal{C}_s\}_{s \in S}$ be finite collection of ∞-categories which admit finite limits, let $\mathcal{D}$ be an ∞-category which admits finite limits so that we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Sp}(\mathcal{D}_*) & \xrightarrow{\ u\ } & \mathrm{Sp}(\mathcal{D}) \\
\downarrow{\scriptstyle \Omega^{\infty}_{\mathcal{D}_*}} & & \downarrow{\scriptstyle \Omega^{\infty}_{\mathcal{D}}} \\
\mathcal{D}_* & \xrightarrow{\ v\ } & \mathcal{D} \, .
\end{array}$$

Suppose we are given functors

$$F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}_* \qquad f : \prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s) \to \mathrm{Sp}(\mathcal{D}_*).$$

For any functor $g : \prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s) \to \mathrm{Sp}(\mathcal{D}_*)$ which preserves final objects, the canonical maps

$$\mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s), \mathcal{D}_*)}(F \circ \prod_{s \in S} \Omega^{\infty}_{\mathcal{C}_s}, \Omega^{\infty}_{\mathcal{D}_*} \circ g) \to \mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s), \mathcal{D})}(v \circ F \circ \prod_{s \in S} \Omega^{\infty}_{\mathcal{C}_s}, \Omega^{\infty}_{\mathcal{D}} \circ u \circ g)$$

$$\mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s), \mathcal{D}_*)}(\Omega^{\infty}_{\mathcal{D}} \circ f, \Omega^{\infty}_{\mathcal{D}_*} \circ g) \to \mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s), \mathcal{D})}(\Omega^{\infty}_{\mathcal{D}} \circ u \circ f, \Omega^{\infty}_{\mathcal{D}} \circ u \circ g).$$

If S is nonempty, then any multilinear functor g preserves final objects. It follows that a natural transformation $\alpha : F \circ \prod_{s \in S} \Omega^{\infty}_{\mathcal{C}_s} \to \Omega^{\infty}_{\mathcal{D}} \circ f$ exhibits f as a derivative of F if and only if it exhibits $u \circ f$ as a derivative of $v \circ f$.

Remark 7.2.1.8. Let $\{\mathcal{C}_s\}_{s \in S}$ be a nonempty finite collection of pointed ∞-categories which admit finite limits, and let $\mathcal{D}$ be another ∞-category which admits finite limits. Repeatedly applying Proposition 1.4.2.22, we see that composition with $\Omega_{\mathcal{D}}^{\infty}$ induces an equivalence of ∞-categories

$$\mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathrm{Sp}(\mathcal{D})) \to \mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{D}).$$

Consequently, condition (2) of Definition 7.2.1.1 is equivalent to the following:

(2′) For every multilinear functor $G : \prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s) \to \mathcal{D}$, composition with α induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s), \mathcal{D})}(\Omega_{\mathcal{D}}^{\infty} \circ f, g) \to \mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s), \mathcal{D})}(F \circ \prod_{s \in S} \Omega_{\mathcal{C}_s}^{\infty}, G).$$

We have the following basic existence result for derivatives:

Proposition 7.2.1.9. *Let $\{\mathcal{C}_s\}_{s \in S}$ be a nonempty finite collection of ∞-categories which admit finite colimits, $\mathcal{D}$ a differentiable ∞-category, and $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ a functor which is reduced in each variable. Then F admits a derivative $\vec{\partial} F : \prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s) \to \mathrm{Sp}(\mathcal{D})$.*

Proof. Let $P_{\vec{1}} : \mathrm{Fun}_*(\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s), \mathcal{D}) \to \mathrm{Exc}_*(\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s), \mathcal{D})$ be a left adjoint to the inclusion (obtained by iterated application of Theorem 7.1.1.10), and let $F' = P_{\vec{1}}(F \circ \prod_{s \in S} \Omega_{\mathcal{C}_s}^{\infty})$. Then F' is multilinear, so Proposition 1.4.2.22 implies that $F' = \Omega_{\mathcal{D}}^{\infty} \circ f$ for some multilinear functor $f : \prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s) \to \mathrm{Sp}(\mathcal{D})$. It follows from Remark 7.2.1.8 that the canonical map

$$F \circ \prod_{s \in S} \Omega_{\mathcal{C}_s}^{\infty} \to F' \simeq \Omega_{\mathcal{D}}^{\infty} \circ f$$

exhibits f as a derivative of F. $\qquad\square$

Our next goal is to obtain a more explicit construction for the derivative $\vec{\partial} F$ of a reduced functor $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$. This will require a brief digression.

Notation 7.2.1.10. Let $\mathcal{S}_*$ denote the ∞-category of pointed spaces. According to Remark 6.3.2.14, the smash product functor $(X, Y) \mapsto X \wedge Y$ underlies a symmetric monoidal structure on $\mathcal{S}_*$, which is characterized up to equivalence by the requirement that $S^0 \in \mathcal{S}_*$ is the unit object and the smash product preserves small colimits separately in each variable. The full subcategory $\mathcal{S}_*^{\mathrm{fin}} \subseteq \mathcal{S}_*$ contains the unit object and is closed under smash products, and therefore inherits a symmetric monoidal structure from $\mathcal{S}_*$. In particular, for every finite set S, the iterated smash product determines a functor

$$\wedge : \prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}_*^{\mathrm{fin}}.$$

Proposition 7.2.1.11. *Let T be a nonempty finite set, and let $\mathcal{C}$ be an ∞-category which admits finite limits. Then composition with the smash product functor $\wedge : \prod_{t \in T} \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}_*^{\mathrm{fin}}$ induces an equivalence of ∞-categories*

$$\theta : \mathrm{Sp}(\mathcal{C}) = \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \to \mathrm{Exc}_*(\prod_{t \in T} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}).$$

Proof. Choose an element $s \in T$, and let $u : \mathcal{S}_*^{\mathrm{fin}} \to \prod_{t \in T} \mathcal{S}_*^{\mathrm{fin}}$ be the functor which is the identity on the sth component, and takes the value S^0 on all other components. Composition with u induces a functor $\theta' : \mathrm{Exc}_*(\prod_{t \in T} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \to \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) = \mathrm{Sp}(\mathcal{C})$. The composite functor

$$\mathcal{S}_*^{\mathrm{fin}} \xrightarrow{u} \prod_{t \in T} \mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\wedge} \mathcal{S}_*^{\mathrm{fin}}$$

is equivalent to the identity, so that $\theta' \circ \theta$ is an equivalence of ∞-categories. To prove that θ is an equivalence of ∞-categories, it will suffice to prove that θ' is an equivalence of ∞-categories. This follows by repeated application of Proposition 1.4.2.21. $\qquad\square$

Corollary 7.2.1.12. *Let T be a nonempty finite set and let $\mathcal{C}$ be a differentiable ∞-category. Then composition with the smash product functor $\wedge : \prod_{t \in T} \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}_*^{\mathrm{fin}}$ induces a fully faithful embedding $\mathrm{Sp}(\mathcal{C}) \to \mathrm{Fun}_*(\prod_{t \in T} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$, which admits a left adjoint $L_{\mathcal{C}}^T$.*

Proof. Proposition 7.2.1.11 implies that the functor $\mathrm{Sp}(\mathcal{C}) \to \mathrm{Fun}_*(\prod_{t \in T} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$ is a fully faithful embedding whose essential image is $\mathrm{Exc}_*(\prod_{t \in T} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$. It will therefore suffice to show that the inclusion

$$\mathrm{Exc}_*(\prod_{t \in T} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \hookrightarrow \mathrm{Fun}_*(\prod_{t \in T} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$$

admits a left adjoint. This left adjoint is given by the functor $P_{(1,\ldots,1)}$ of Proposition 7.1.3.6. $\square$

Remark 7.2.1.13. Let $\mathcal{C}$ be a differentiable ∞-category, let T be a nonempty set, and let

$$L_{\mathcal{C}}^T : \mathrm{Fun}_*(\prod_{t \in T} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \to \mathrm{Sp}(\mathcal{C})$$

be as in Corollary 7.2.1.12. Let $\mathbf{Z}_{\geq 0}^T$ be the collection of all finite sequences of natural numbers $\{n_s \in \mathbf{Z}_{\geq 0}\}_{t \in T}$, and for $\vec{n} \in \mathbf{Z}_{\geq 0}^T$ let $|\vec{n}| = \sum_{t \in T} n_t$. Using Example 7.1.1.28 and the proof of Proposition 7.2.1.11, we see that $\Omega^\infty \circ L_{\mathcal{C}}^T$ can be described explicitly by the formula

$$\Omega^\infty(L_{\mathcal{C}}^T F) = \varinjlim_{\vec{n} \in \mathbf{Z}_{\geq 0}^T} \Omega^{|\vec{n}|} F(\{S^{n_t}\}_{t \in T}) \in \mathcal{C}.$$

Construction 7.2.1.14. Let $q : S \to T$ be a surjection of nonempty finite sets. For each $t \in T$, we let $S_t = q^{-1}\{t\} \subseteq S$ denote the inverse image of t under the map q. Let $\{\mathcal{C}_t\}_{t \in T}$ be a collection of ∞-categories which admit finite limits and let $\mathcal{D}$ be a differentiable ∞-category. For every reduced functor $F : \prod_{t \in T} \mathcal{C} \to \mathcal{D}$, we let $F^+ : \prod_{t \in T} \mathrm{Fun}_*(\prod_{s \in S_t} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}_t) \to \mathrm{Fun}_*(\prod_{t \in T} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D})$ denote the functor which carries a collection of functors $\{X_t : \prod_{s \in S_t} \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}_t\}_{t \in T}$ to the composite functor

$$\prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}} \overset{\{X_t\}}{\to} \prod_{t \in T} \mathcal{C}_t \overset{F}{\to} \mathcal{D}.$$

We let $F' : \prod_{t \in T} \mathrm{Sp}(\mathcal{C}_t) \to \mathrm{Sp}(\mathcal{D})$ denote the functor given by the composition

$$\prod_{t \in T} \mathrm{Sp}(\mathcal{C}_t) \to \prod_{t \in T} \mathrm{Fun}_*(\prod_{s \in S_t} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}_t) \overset{F^+}{\to} \mathrm{Fun}_*(\prod_{t \in T} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D}) \overset{L_{\mathcal{D}}^T}{\to} \mathrm{Sp}(\mathcal{D}),$$

where $L_{\mathcal{D}}^T$ is defined as in Corollary 7.2.1.12.

Note that the notation of Construction 7.2.1.14 is somewhat abusive: the functor F^+ depends not only on the functor F, but also on a choice of surjective map $q : S \to T$. However, this ambiguity is mostly harmless: under some mild assumptions, we will show that the functor F' is a derivative of F (Proposition 7.2.1.19), and therefore canonically independent of q.

Remark 7.2.1.15. Let $\mathcal{J}$ be a filtered ∞-category with only countably many simplices. Then there exists a left cofinal map $\mathrm{N}(\mathbf{Z}_{\geq 0}) \to \mathcal{J}$. To prove this, we first invoke Proposition T.5.3.1.16 to choose a left cofinal map $\mathrm{N}(A) \to \mathcal{J}$, where A is a filtered partially ordered; note that the proof of Proposition T.5.3.1.16 produces a *countable* partially ordered set A in the case where $\mathcal{C}$ has only countably many simplices. Let $A = \{a_0, a_1, a_2, \ldots\}$. Let $b_0 = a_0$, and for each $n > 1$ choose an element $b_n \in A$ which is an upper bound for the set $\{b_{n-1}, a_n\}$. The sequence $b_0 \leq b_1 \leq b_2 \leq \ldots$ determines a map $\mathrm{N}(\mathbf{Z}_{\geq 0}) \to \mathrm{N}(A)$; Theorem T.4.1.3.1 implies that this map is left cofinal.

Remark 7.2.1.16. Using Remark 7.2.1.15, we deduce the following:

(1) Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ admits countable filtered colimits if and only if it admits sequential colimits. (In particular, if $\mathcal{C}$ also admits finite colimits and sequential colimits, then $\mathcal{C}$ admits all countable colimits.)

(2) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor where the ∞-category $\mathcal{C}$ satisfies the equivalent conditions of (1). Then F preserves countable filtered colimits if and only if F preserves sequential colimits.

Example 7.2.1.17. Let $\mathcal{C}$ be a differentiable ∞-category. Then $\mathcal{C}$ admits sequential colimits, so that $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$ also admits sequential colimits. Since sequential colimits in $\mathcal{C}$ are left exact, the full subcategory $\mathrm{Sp}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$ is closed under sequential colimits. It follows from Remark 7.2.1.16 that $\mathrm{Sp}(\mathcal{C})$ admits countable filtered colimits, and that the forgetful functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ preserves countable filtered colimits. Since $\mathrm{Sp}(\mathcal{C})$ is a stable ∞-category (Corollary 1.4.2.17), it admits finite colimits. It follows that $\mathrm{Sp}(\mathcal{C})$ admits all countable colimits (Proposition T.4.4.3.2). If $\mathcal{D}$ is another differentiable ∞-category, then a functor $F : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$ preserves countable colimits if and only if it is exact (that is, it preserves finite colimits) and preserves sequential colimits.

Proposition 7.2.1.18. *Let $\{\mathcal{C}_t\}_{t \in T}$ be a nonempty finite collection of differentiable ∞-categories, let $\mathcal{D}$ be a differentiable ∞-category, and let $F : \prod_{t \in T} \mathcal{C}_t \to \mathcal{D}$ be a functor which is reduced in each variable and preserves sequential colimits. For every surjection $q : S \to T$ of finite sets, the functor $F' : \prod_{t \in T} \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$ of Construction 7.2.1.14 preserves countable colimits separately in each variable. In particular, F' is multilinear.*

Proof. The functor F' is given by a composition

$$\prod_{t \in T} \mathrm{Sp}(\mathcal{C}_t) \xrightarrow{i} \prod_{t \in T} \mathrm{Fun}_*(\prod_{s \in S_t} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}_t) \xrightarrow{F^+} \mathrm{Fun}_*(\prod_{t \in T} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D}) \xrightarrow{L_{\mathcal{D}}^T} \mathrm{Sp}(\mathcal{D}).$$

The functor i preserves sequential colimits since $\mathrm{Exc}_*(\prod_{s \in S_t}, \mathcal{C}_t)$ is closed under sequential colimits in $\mathrm{Fun}_*(\prod_{s \in S_t} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}_t))$ (because sequential colimits in $\mathcal{C}$ are left exact). The functor F^+ preserves sequential colimits because F does, and the functor $L_{\mathcal{D}}^T$ preserves sequential colimits because it is a left adjoint. It follows that F' preserves sequential colimits.

To complete the proof, it will suffice to show that the functor F' is exact in each variable (Example 7.2.1.17). Fix an element $t_0 \in T$ and spectrum objects $\{Y_t \in \mathrm{Sp}(\mathcal{C}_t)\}_{t \in T - \{t_0\}}$, and let $G : \mathrm{Sp}(\mathcal{C}_s) \to \mathrm{Sp}(\mathcal{D})$ be the functor given by the formula $G(X) = F'(X, \{Y_t\}_{t \in T - \{t_0\}})$. We wish to show that G is exact. In view of Corollary 1.4.2.14, it will suffice to show that for every object $X \in \mathrm{Sp}(\mathcal{C}_s)$, the canonical map $\nu : \Sigma_{\mathrm{Sp}(\mathcal{D})} G(X) \to G(\Sigma_{\mathrm{Sp}(\mathcal{C})} X)$ is an equivalence in $\mathrm{Sp}(\mathcal{D})$. Note that $\Sigma_{\mathrm{Sp}(\mathcal{C})}(X) \simeq X \circ \Sigma_{\mathcal{S}_*^{\mathrm{fin}}}$. Choose an element $s_0 \in S$ such that $q(s) = t_0$, and let Let $U : \prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}} \to \prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}}$ be the functor given by the suspension on the s_0th coordinate, and the identity on the remaining coordinates. Then composition with U induces a functor from $\mathrm{Exc}_*(\prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D})$ to itself, which fits into a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Sp}(\mathcal{D}) & \xrightarrow{\Sigma_{\mathrm{Sp}(\mathcal{D})}} & \mathrm{Sp}(\mathcal{D}) \\
\downarrow & & \downarrow \\
\mathrm{Exc}_*(\prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D}) & \xrightarrow{\circ U} & \mathrm{Exc}_*(\prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D})
\end{array}
$$

where the vertical maps are given by the equivalence of Proposition 7.2.1.11. Let $Z : \prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{D}$ be the functor given by the composition

$$\prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}} \simeq \prod_{t \in T}(\prod_{s \in S_t} \mathcal{S}_*^{\mathrm{fin}}) \xrightarrow{\prod_{t \in T} \wedge} \prod_{t \in T} \mathcal{S}_*^{\mathrm{fin}} \xrightarrow{X, \{Y_t\}} \prod_{t \in T} \mathcal{C}_t \xrightarrow{F} \mathcal{D},$$

and let $P_{\overline{1}} : \mathrm{Fun}_*(\prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D}) \to \mathrm{Exc}_*(\prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D})$ be a left adjoint to the inclusion. Unwinding the definitions, we can identify ν with the canonical map $P_{\overline{1}}(Z) \circ U \to P_{\overline{1}}(Z \circ U)$, which is an equivalence by virtue of Remark 7.1.1.30. $\qquad\square$

Let $F : \prod_{t \in T} \mathcal{C}_t \to \mathcal{D}$ be as in Proposition 7.2.1.18 and let $q : S \to T$ be a surjection of finite sets. For $\{X_t \in \mathrm{Sp}(\mathcal{C}_t)\}_{t \in T}$, we have a canonical map

$$F(\{\Omega^\infty X_t\}_{t \in T}) = F^+(\{X_t\}_{t \in T})(S^0, S^0, \dots, S^0) \to \Omega^\infty F'(\{X_t\}).$$

This construction determines a natural transformation $\alpha : F \circ \prod_{t \in T} \Omega^\infty_{\mathcal{C}_t} \to \Omega^\infty_{\mathcal{D}} \circ F'$.

Proposition 7.2.1.19. *Let $\{\mathcal{C}_t\}_{t \in T}$ be a nonempty finite collection of differentiable ∞-categories, let $\mathcal{D}$ be a differentiable ∞-category, and let $F : \prod_{t \in T} \mathcal{C}_t \to \mathcal{D}$ be a reduced functor which preserves sequential colimits. For every surjection of finite sets $q : S \to T$, the natural transformation $\alpha : F \circ \prod_{t \in T} \Omega^\infty_{\mathcal{C}_t} \to \Omega^\infty_{\mathcal{D}} \circ F'$ defined above exhibits F' as a derivative of F.*

Proof. Let $P_{\vec{1}} : \mathrm{Fun}_*(\prod_{t \in T} \mathrm{Sp}(\mathcal{C}_t), \mathcal{D}) \to \mathrm{Exc}_*(\prod_{t \in T} \mathrm{Sp}(\mathcal{C}_t), \mathcal{D})$ be a left adjoint to the inclusion. Since the functor F' is multilinear, the natural transformation α factors as a composition

$$F \circ \prod_{t \in T} \Omega^\infty_{\mathcal{C}} \xrightarrow{\alpha'} P_{\vec{1}}(F \circ \prod_{t \in T} \Omega^\infty_{\mathcal{C}}) \xrightarrow{\alpha''} \Omega^\infty_{\mathcal{D}} \circ F'.$$

To prove that α exhibits F' as a derivative of F, it will suffice to show that α'' is an equivalence (Remark 7.2.1.8). Fix a collection of spectrum objects $\{X_t \in \mathrm{Sp}(\mathcal{C}_t)\}_{t \in T}$. For $\vec{n} \in \mathbf{Z}^S_{\geq 0}$ and $t \in T$, we let $\vec{n}_t$ denote the restriction of $\vec{n}$ to the subset $S_t \subseteq S$. Note that the construction $\vec{n} \mapsto \{|\vec{n}_t|\}_{t \in T}$ induces a cofinal map of partially ordered sets $\mathbf{Z}^S_{\geq 0} \to \mathbf{Z}^T_{\geq 0}$. Using Remark 7.2.1.13, we compute

$$
\begin{aligned}
(\Omega^\infty_{\mathcal{D}} \circ F')(\{X_t\}) &= (\Omega^\infty_{\mathcal{D}} \circ L_{\mathcal{D}})(F^+(\{X_t\})) \\
&\simeq \varinjlim_{\vec{n} \in \mathbf{Z}^S_{\geq 0}} \Omega^{|\vec{n}|}_{\mathcal{D}} F(\{X(S^{|\vec{n}_t|})\}) \\
&\simeq \varinjlim_{\vec{m} \in \mathbf{Z}^T_{\geq 0}} \Omega^{|\vec{m}|}_{\mathcal{D}} \circ (F \circ \prod_{t \in T} \Omega^\infty_{\mathcal{C}_t}) \circ \prod_{t \in T} \Sigma^{m_t}_{\mathrm{Sp}(\mathcal{C})}(X_t) \\
&\simeq P_{\vec{1}}(F \circ \Omega^\infty_{\mathcal{C}})(\{X_t\}).
\end{aligned}
$$

$\square$

When using Proposition 7.2.1.19 to compute derivatives of functors, the following result is often useful:

Proposition 7.2.1.20. *Let $q : S \to T$ be a surjective map of nonempty finite sets, let $\{\mathcal{C}_t\}_{t \in T}$ be a collection of differentiable ∞-categories, let $\mathcal{D}$ be a differentiable ∞-category, and let $F : \prod_{t \in T} \mathcal{C}_t \to \mathcal{D}$ be a reduced functor which preserves sequential colimits. For each $t \in T$, let $S_t = q^{-1}\{t\} \subseteq S$, and let*

$$F^+ : \prod_{t \in T} \mathrm{Fun}_*(\prod_{s \in S_t} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{C}_t) \to \mathrm{Fun}_*(\prod_{s \in S}, \mathcal{D})$$

denote the functor given by composition with F. Then F^+ carries $\prod_{t \in T} L^{S_t}_{\mathcal{C}_t}$-equivalences to $L^S_{\mathcal{D}}$-equivalences.

Proof. Let $P_{\vec{1}} : \mathrm{Fun}_*(\prod_{s \in S} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{D}) \to \mathrm{Exc}_*(\prod_{s \in S} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{D})$ be a left adjoint to the inclusion, and for each $t \in T$ left $P^t_{\vec{1}} : \mathrm{Fun}_*(\prod_{s \in S_t} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{C}_t) \to \mathrm{Exc}_*(\prod_{s \in S_t} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{C}_t)$ be a left adjoint to the inclusion. Suppose we are given a collection of morphisms $\{\alpha_t : X_t \to Y_t\}_{t \in T}$ in the ∞-categories $\mathrm{Fun}_*(\prod_{s \in S_t} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{C}_t)$ such that each $P^t_{\vec{1}}(\alpha_t)$ is an equivalence. We wish to show that the induced map $P_{\vec{1}} F^+(\{X_t\}) \to P_{\vec{1}} F^+(\{Y_t\})$ is an equivalence. Fix a finite collection of pointed spaces $\{K_s\}_{s \in S}$. Using the description of $P_{\vec{1}}$ supplied by Example 7.1.1.28, we are reduced to proving that the canonical map

$$\gamma : \varinjlim_{\vec{m} \in \mathbf{Z}^S_{\geq 0}} \Omega^{|\vec{m}|}_{\mathcal{D}} F(\{X_t(\{\Sigma^{m_s} K_s\}_{s \in S_t})\}_{t \in T}) \to \varinjlim_{\vec{n} \in \mathbf{Z}^S_{\geq 0}} \Omega^{|\vec{m}|}_{\mathcal{D}} F(\{Y_t(\{\Sigma^{m_s} K_s\}_{s \in S_t})\}_{t \in T})$$

is an equivalence in $\mathcal{D}$. For $\vec{n} \in \mathbf{Z}_{\geq 0}^S$, let $\vec{n}_t$ denote the restriction of $\vec{n}$ to the subset $S_t \subseteq S$. We have a commutative diagram

$$\varinjlim_{\vec{m} \in \mathbf{Z}_{\geq 0}^S} \Omega_{\mathcal{D}}^{|\vec{m}|} F(\{X_t(\{\Sigma^{m_s} K_s\}_{s \in S_t})\}_{t \in T}) \xrightarrow{\quad \gamma \quad} \varinjlim_{\vec{m} \in \mathbf{Z}_{\geq 0}^S} \Omega_{\mathcal{D}}^{|\vec{m}|} F(\{Y_t(\{\Sigma^{m_s} K_s\}_{s \in S_t})\}_{t \in T})$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\varinjlim_{\vec{m}, \vec{n} \in \mathbf{Z}_{\geq 0}^S} \Omega_{\mathcal{D}}^{|\vec{m}|} F(\{\Omega_{\mathcal{C}_t}^{|\vec{n}_t|} X_t\{\Sigma^{m_s+n_s} K_s\}_{s \in S_t}\}_{t \in T}) \xrightarrow{\quad \gamma' \quad} \varinjlim_{\vec{m}, \vec{n} \in \mathbf{Z}_{\geq 0}^S} \Omega_{\mathcal{D}}^{|\vec{m}|} F(\{\Omega_{\mathcal{C}_t}^{|\vec{n}_t|} Y_t\{\Sigma^{m_s+n_s} K_s\}_{s \in S_t}\}_{t \in T})$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\varinjlim_{\vec{m}, \vec{n} \in \mathbf{Z}_{\geq 0}^S} \Omega_{\mathcal{D}}^{|\vec{m}|+|\vec{n}|} F(\{X_t\{\Sigma^{m_s+n_s} K_s\}_{s \in S_t}\}_{t \in T}) \xrightarrow{\quad \gamma'' \quad} \varinjlim_{\vec{m}, \vec{n} \in \mathbf{Z}_{\geq 0}^S} \Omega_{\mathcal{D}}^{|\vec{m}|+|\vec{n}|} F(\{Y_t\{\Sigma^{m_s+n_s} K_s\}_{s \in S_t}\}_{t \in T}).$$

A simple cofinality argument shows that the vertical composite maps are equivalences, so that we can regard γ as a retract of γ' in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{D})$. It will therefore suffice to show that γ' is an equivalence. Since the functors $\Omega_{\mathcal{D}}$ and F commute with sequential colimits, it will suffice to show that each of the maps

$$\varinjlim_{\vec{n}_t \in \mathbf{Z}_{\geq 0}^{S_t}} \Omega_{\mathcal{C}_t}^{|\vec{n}_t|} X_t(\{\Sigma^{m_s+n_s} K_s\}_{s \in S_t}) \to \varinjlim_{\vec{n}_t \in \mathbf{Z}_{\geq 0}^{S_t}} \Omega_{\mathcal{C}_t}^{|\vec{n}_t|} Y_t(\{\Sigma^{m_s+n_s} K_s\}_{s \in S_t}),$$

which follows immediately from our assumption that $P_1^t(\alpha_t)$ is an equivalence. $\qquad\square$

We conclude this section by describing an application of Propositions 7.2.1.19 and 7.2.1.20.

Notation 7.2.1.21. Let $\{\mathcal{C}_s\}_{s \in S}$ and $\mathcal{D}$ be differentiable ∞-categories. We let $\mathrm{Fun}_\star(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$ spanned by those functors which are reduced in each variable and preserve sequential colimits. If each $\mathcal{C}_s$ admits finite colimits, we let $\mathrm{Exc}_\star(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$ spanned by those functors which are multilinear and preserve sequential colimits.

Theorem 7.2.1.22 (Chain Rule for First Derivatives). *Suppose we are given a surjective map of nonempty finite sets* $p : S \to T$, *differentiable ∞-categories* $\{\mathcal{C}_s\}_{s \in S}$, $\{\mathcal{D}_t\}_{t \in T}$, *and* $\mathcal{E}$, *and functors functors* $\{F_t \in \mathrm{Fun}_\star(\prod_{s \in S_t} \mathcal{C}_s, \mathcal{D}_t)\}_{t \in T}$ *and* $G \in \mathrm{Fun}_\star(\prod_{t \in T} \mathcal{D}_t, \mathcal{E})$. *Let* $\{f_t :\in \mathrm{Exc}_\star(\prod_{s \in S_t} \mathrm{Sp}(\mathcal{C}_s), \mathrm{Sp}(\mathcal{D}_t))\}_{t \in T}$ *and* $g \in \mathrm{Exc}_\star(\prod_{t \in T} \mathrm{Sp}(\mathcal{D}_t), \mathrm{Sp}(\mathcal{E}))$ *be functors equipped with natural transformations*

$$F_t \circ \prod_{s \in S_t} \Omega_{\mathcal{C}_s}^\infty \to \Omega_{\mathcal{D}_t}^\infty \circ f_t \qquad G \circ \prod_{t \in T} \Omega_{\mathcal{D}_t}^\infty \to \Omega_{\mathcal{E}}^\infty \circ g$$

which exhibit g as a derivative of G and each f_t as a derivative of F_t. Then the composite transformation

$$\gamma : G \circ \prod_{t \in T} F_t \circ \prod_{s \in S} \Omega_{\mathcal{C}_s}^\infty \quad \to \quad G \circ \prod_{t \in T} \Omega_{\mathcal{D}_t}^\infty \circ \prod_{t \in T} f_t$$

$$\to \quad \Omega_{\mathcal{E}}^\infty \circ g \circ \prod_{t \in T} f_t$$

exhibits the functor $g \circ \prod_{t \in T} f_t : \prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s) \to \mathrm{Sp}(\mathcal{E})$ *as a derivative of* $G \circ \prod_{t \in T} F_t$.

Remark 7.2.1.23. We can state Theorem 7.2.1.22 more informally as follows: given composable multifunctors $F_t : \prod_{s \in S_t} \mathcal{C}_s \to \mathcal{D}_t$ and $G : \prod_{t \in T} \mathcal{D}_t \to \mathcal{E}$ which are reduced in each variable and preserve sequential colimits, we have a canonical equivalence

$$\vec{\partial}(G \circ \prod_{t \in T} F_t) \simeq \vec{\partial}(G) \circ \prod_{t \in T} \vec{\partial} F_t.$$

Corollary 7.2.1.24 (Klein-Rognes). *Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be differentiable ∞-categories, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be reduced functors which preserve sequential colimits. Let $\alpha : F \circ \Omega_{\mathcal{C}}^{\infty} \to \Omega_{\mathcal{D}}^{\infty} \circ f$ be a natural transformation which exhibits a functor $f : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$ as a derivative of F, and let $\beta : G \circ \Omega_{\mathcal{D}}^{\infty} \to \Omega_{\mathcal{E}}^{\infty} \circ g$ be a natural transformation which exhibits $g : \mathrm{Sp}(\mathcal{D}) \to \mathrm{Sp}(\mathcal{E})$ as a derivative of G. Then the composite map*

$$\gamma : G \circ F \circ \Omega_{\mathcal{C}}^{\infty} \xrightarrow{\alpha} G \circ \Omega_{\mathcal{D}}^{\infty} \circ f \xrightarrow{\beta} \Omega_{\mathcal{E}}^{\infty} \circ g \circ f$$

exhibits $g \circ f$ as a derivative of $G \circ F$.

Remark 7.2.1.25. For a proof of Corollary 7.2.1.24 in the setting of classical homotopy theory, we refer the reader to [86].

Corollary 7.2.1.26. *Let $\{\mathcal{C}_s\}_{s \in S}$ be a nonempty finite collection of differentiable ∞-categories, let $\mathcal{D}$ and $\mathcal{E}$ be differentiable ∞-categories, and suppose we are given functors*

$$F \in \mathrm{Fun}_{\star}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D}) \qquad G \in \mathrm{Fun}_{\star}(\mathcal{D}, \mathcal{E}).$$

Assume that G is left exact, so that pointwise composition with G induces a functor $g : \mathrm{Sp}(\mathcal{D}) \to \mathrm{Sp}(\mathcal{E})$. Let $\alpha : F \circ \prod_{s \in S} \Omega_{\mathcal{C}_s}^{\infty} \to \Omega_{\mathcal{D}}^{\infty} \circ f$ be a natural transformation which exhibits f as a derivative of F. Then the induced map

$$G \circ F \circ \prod_{s \in S} \Omega_{\mathcal{C}_s}^{\infty} \xrightarrow{\alpha} G \circ \Omega_{\mathcal{D}}^{\infty} \circ f = \Omega_{\mathcal{E}}^{\infty} \circ g \circ f$$

exhibits $g \circ f$ as a derivative of $G \circ F$. In particular, we have a canonical equivalence

$$\vec{\partial}(G \circ F) \simeq g \circ \vec{\partial}(F).$$

Proof. Combine Theorem 7.2.1.22 with Example 7.2.1.4 (one can also deduce this result directly from the construction of the derivative supplied by the proof of Proposition 7.2.1.9). $\square$

Proof of Theorem 7.2.1.22. Let F_t^+ and G^+ be defined as in Construction 7.2.1.14. Using Proposition 7.2.1.19, we may assume that the functors f_t and g are given by the compositions

$$\prod_{s \in S_t} \mathrm{Sp}(\mathcal{C}_s) \xrightarrow{i_t} \prod_{s \in S_t} \mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}_s) \xrightarrow{F_t^+} \mathrm{Fun}_*(\prod_{s \in S_t} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D}_t) \xrightarrow{L_{\mathcal{D}_t}^{S_t}} \mathrm{Sp}(\mathcal{D}_t)$$

$$\prod_{t \in T} \mathrm{Sp}(\mathcal{D}_t) \xrightarrow{j} \prod_{t \in T} \mathrm{Fun}_*(\prod_{s \in S_t} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D}_t) \xrightarrow{G^+} \mathrm{Fun}_*(\prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{E}) \xrightarrow{L_{\mathcal{E}}^{S}} \mathrm{Sp}(\mathcal{E}).$$

The natural transformation γ can be written as a composition

$$G \circ \prod_{t \in T} F_t \circ \prod_{s \in S} \Omega_{\mathcal{C}_s}^{\infty} \quad \xrightarrow{\gamma'} \quad \Omega_{\mathcal{E}}^{\infty} \circ L_{\mathcal{E}}^{S} \circ G^+ \circ \prod_{t \in T} F_t^+ \circ \prod_{t \in T} i_t$$

$$\xrightarrow{\gamma''} \quad \Omega_{\mathcal{E}}^{\infty} \circ L_{\mathcal{E}}^{S} \circ G^+ \circ j \circ \prod_{t \in T} L_{\mathcal{D}_t}^{S_t} \circ \prod_{t \in T} F_t^+ \circ \prod_{t \in T} i_t.$$

Proposition 7.2.1.19 implies that γ' exhibits the functor $L_{\mathcal{E}}^{S} \circ G^+ \circ \prod_{t \in T} F_t^+ \circ \prod_{t \in T} i_t$ as a derivative of $G \circ \prod_{t \in T} F_t$. To complete the proof, it will suffice to show that γ'' is an equivalence. Note that γ'' is induced by a natural transformation

$$\beta : L_{\mathcal{E}}^{S} \circ G^+ \to L_{\mathcal{E}}^{S} \circ G^+ \circ j \circ \prod_{t \in T} L_{\mathcal{D}_t}^{S_t}$$

of functors from $\prod_{t \in T} \mathrm{Fun}(\prod_{s \in S_t} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D}_t)$ to $\mathrm{Sp}(\mathcal{E})$. Proposition 7.2.1.20 implies that β is an equivalence. $\square$

7.2.2 Stabilization of Differentiable Fibrations

Let $\mathcal{C}at_\infty^\star$ denote the subcategory of $\mathcal{C}at_\infty$ whose objects are differentiable ∞-categories and whose morphisms are reduced functors which preserve sequential colimits. Let $\mathcal{C}at_\infty^{\mathrm{Ex},\star}$ denote the subcategory of $\mathcal{C}at_\infty$ whose objects are stable ∞-categories which admit countable colimits and whose morphisms are functors which preserve countable colimits. We can define a functor

$$\Phi_0 : \mathrm{h}\mathcal{C}at_\infty^\star \to \mathrm{h}\mathcal{C}at_\infty^{\mathrm{Ex},\star},$$

given on objects by $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$ and on morphisms by $F \mapsto \partial F$. It follows from Example 7.2.1.4 and Corollary 7.2.1.24 that this construction preserves identity morphisms and composition of morphisms, up to homotopy. Our main goal in this section is to prove the following result:

Theorem 7.2.2.1. *The functor* $\Phi_0 : \mathrm{h}\mathcal{C}at_\infty^\star \to \mathrm{h}\mathcal{C}at_\infty^{\mathrm{Ex},\star}$ *lifts to a functor of ∞-categories* $\Phi : \mathcal{C}at_\infty^\star \to \mathcal{C}at_\infty^{\mathrm{Ex},\star}$.

Our basic strategy for proving Theorem 7.2.2.1 is to construct the coCartesian fibration $\mathcal{X} \to \mathcal{C}at_\infty^\star$ classified by the composite map

$$\mathcal{C}at_\infty^\star \xrightarrow{\Phi} \mathcal{C}at_\infty^{\mathrm{Ex},\star} \hookrightarrow \mathcal{C}at_\infty .$$

We can obtain $\mathcal{X}$ by *stabilizing* the fibration $\mathcal{Y} \to \mathcal{C}at_\infty^\star$ classified by the inclusion $\mathcal{C}at_\infty^\star \hookrightarrow \mathcal{C}at_\infty$.

Construction 7.2.2.2. Let $p : \mathcal{C} \to S$ be an inner fibration of simplicial sets. For each $s \in S$, we let $\mathcal{C}_s$ denote the ∞-category $\mathcal{C} \times_S \{s\}$. Assume that each of the ∞-categories $\mathcal{C}_s$ admits finite limits. We define simplicial sets

$$\mathrm{Stab}(p) \subseteq \mathrm{PStab}_*(p) \subseteq \mathrm{PStab}(p) \to S$$

as follows:

- For every map of simplicial sets $K \to S$, we have a canonical bijection

$$\mathrm{Hom}_S(K, \mathrm{PStab}(p)) \simeq \mathrm{Hom}_S(K \times \mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}).$$

  In particular, we can identify vertices of $\mathrm{PStab}(p)$ with pairs (s, X), where s is a vertex of S and $X : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}_s$ is a functor.

- We let $\mathrm{PStab}_*(p)$ denote the full simplicial subset of $\mathrm{PStab}(p)$ spanned by those pairs (s, X) for which X is a reduced functor.

- We let $\mathrm{Stab}(p)$ denote the full simplicial subset of $\mathrm{PStab}(p)$ spanned by those pairs (s, X) where X is a spectrum object of $\mathcal{C}_s$.

Remark 7.2.2.3. Suppose we are given a pullback diagram of simplicial sets

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \longrightarrow & S \end{array}$$

where the vertical maps are inner fibrations whose fibers admit finite limits. We then have canonical isomorphisms

$$\mathrm{PStab}(p') \simeq S' \times_S \mathrm{PStab}(p) \qquad \mathrm{PStab}_*(p') \simeq S' \times_S \mathrm{PStab}_*(p) \qquad \mathrm{Stab}(p') \simeq S' \times_S \mathrm{Stab}(p).$$

Remark 7.2.2.4. In the situation of Construction 7.2.2.2, we have a canonical isomorphism $\mathrm{Stab}(p)_s \simeq \mathrm{Sp}(\mathcal{C}_s)$ for each vertex $s \in S$. In other words, we can think of the construction $p \mapsto \mathrm{Stab}(p)$ as a relative version of the construction $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$.

Proposition 7.2.2.5. *Let $p : \mathcal{C} \to S$ be an inner fibration of simplicial sets, where each fiber $\mathcal{C}_s$ admits finite limits. Then the maps*

$$\mathrm{Stab}(p) \to \mathrm{PStab}_*(p) \to \mathrm{PStab}(p) \to S$$

are inner fibrations.

Proof. The map $\mathrm{PStab}(p) \to S$ is a pullback of the map $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, S)$, and therefore an inner fibration by Corollary T.2.3.2.5. The maps $\mathrm{Stab}(p) \hookrightarrow \mathrm{PStab}_*(p) \hookrightarrow \mathrm{PStab}(p)$ are inclusions of full simplicial subsets, and therefore automatically inner fibrations. $\square$

Definition 7.2.2.6. Let $p : \mathcal{C} \to S$ be a map of simplicial sets. We will say that p is a *locally differentiable fibration* if the following conditions are satisfied:

(a) The map p is a locally coCartesian fibration of simplicial sets.

(b) For each fiber $s \in S$, the ∞-category $\mathcal{C}_s$ is differentiable.

(c) For each edge $s \to s'$ in S, the induced functor $\mathcal{C}_s \to \mathcal{C}_{s'}$ preserves sequential colimits.

We will say that a locally differentiable fibration is *reduced* if it satisfies the following further condition:

(d) For each edge $s \to s'$ in S, the induced functor $\mathcal{C}_s \to \mathcal{C}_{s'}$ is reduced.

We will say that a locally differentiable fibration p is a *differentiable fibration* if it is locally differentiable and a coCartesian fibration.

Remark 7.2.2.7. Let $p : \mathcal{C} \to S$ be a coCartesian fibration of simplicial sets, classified by a map $\chi : S \to \mathcal{C}\mathrm{at}_\infty$. Then p is a reduced differentiable fibration if and only if χ factors through the subcategory $\mathcal{C}\mathrm{at}_\infty^\star \subseteq \mathcal{C}\mathrm{at}_\infty$.

Proposition 7.2.2.8. *Let $p : \mathcal{C} \to S$ be an inner fibration of simplicial sets, and assume that each fiber $\mathcal{C}_s$ of p admits finite limits.*

(1) *Assume that p is a coCartesian fibration. Then the induced map $q : \mathrm{PStab}(p) \to S$ is a coCartesian fibration. Moreover, an edge $e : (s, X) \to (s', X')$ in $\mathrm{PStab}(p)$ is q-coCartesian if and only if, for every finite pointed space $K \in \mathcal{S}_*^{\mathrm{fin}}$, the resulting edge $e(K) : X(K) \to X'(K)$ is a p-coCartesian edge of $\mathcal{C}$.*

(2) *Assume that p is a coCartesian fibration and that, for each edge $s \to s'$ in S, the induced functor $\mathcal{C}_s \to \mathcal{C}_{s'}$ is reduced. Then the map $q' : \mathrm{PStab}_*(p) \to S$ is a coCartesian fibration. Moreover, an edge e of $\mathrm{PStab}_*(p)$ is q'-coCartesian if and only if it is q-coCartesian, when regarded as an edge of $\mathrm{PStab}(p)$.*

(3) *If p is a reduced locally differentiable fibration, then the induced map $q'' : \mathrm{Stab}(p) \to S$ is a locally coCartesian fibration.*

(4) *If p is a reduced differentiable fibration, then $q'' : \mathrm{Stab}(p) \to S$ is a coCartesian fibration.*

Proof. Assertion (1) follows from Proposition T.3.1.2.1, and (2) follows immediately from (1). We now prove (3). Since q'' is an inner fibration by Proposition 7.2.2.5, we may reduce to the case where $S = \Delta^1$. Then the map $p : \mathcal{C} \to \Delta^1$ is a coCartesian fibration classifying a reduced functor $F : \mathcal{C}_0 \to \mathcal{C}_1$ which preserves sequential colimits. Let $X \in \mathrm{Sp}(\mathcal{C}_0) \subseteq \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}_0)$; we wish to show that there exists a morphism $\alpha : F \circ X \to Y$ in $\mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}_1)$, where $Y \in \mathrm{Sp}(\mathcal{C}_1)$ and composition with α induces a homotopy equivalence $\mathrm{Map}_{\mathrm{Sp}(\mathcal{C}_1)}(Y, Z) \to \mathrm{Map}_{\mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}_1)}(F \circ X, Z)$. To prove the existence of Y, it suffices to note that the inclusion $\mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}_1) \hookrightarrow \mathrm{Sp}(\mathcal{C}_1)$ admits a left adjoint (Corollary 7.2.1.12).

We now prove (4). Assume that p is a reduced differentiable fibration. Since $q'' : \mathrm{Stab}(p) \to S$ is a locally coCartesian fibration by (3), it will suffice to show that the collection of locally q''-coCartesian edges is closed under composition (Proposition T.2.4.2.8). Suppose we are given a 2-simplex

$$
\begin{array}{ccc}
 & Y & \\
\alpha \nearrow & & \searrow \beta \\
X & \xrightarrow{\quad\gamma\quad} & Z
\end{array}
$$

in $\mathrm{Stab}(p)$, where $X \in \mathrm{Sp}(\mathcal{C}_x)$, $Y \in \mathrm{Sp}(\mathcal{C}_y)$, and $Z \in \mathrm{Sp}(\mathcal{C}_z)$, and the morphisms α and β are locally q''-coCartesian. We wish to show that γ is locally q''-coCartesian. The images of α and β in S determine functors $F : \mathcal{C}_x \to \mathcal{C}_y$ and $G : \mathcal{C}_y \to \mathcal{C}_z$. Let $L_y : \mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}_y) \to \mathrm{Sp}(\mathcal{C}_y)$ and $L_z : \mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}_z) \to \mathrm{Sp}(\mathcal{C}_z)$ denote left adjoints to the inclusion functors. Using the first part of the proof, we may assume without loss of generality that $Y = L_y(F \circ X)$ and $Z = L_z(G \circ Y)$. Then α and β determine a natural transformation $\delta : G \circ F \circ X \to L_z(G \circ L_y(F \circ X))$. To prove that γ is locally q''-coCartesian, it will suffice to show that $L_z(\delta)$ is an equivalence. Equivalently, we wish to show that composition with G carries the map $F \circ Z \to L_y(F \circ X)$ to an L_z-equivalence, which follows from Proposition 7.2.1.20. $\qquad\square$

Notation 7.2.2.9. Let $p : \mathcal{C} \to S$ be an inner fibration whose fibers $\mathcal{C}_s$ admit finite limits. We let $\Omega_p^\infty : \mathrm{Stab}(p) \to \mathcal{C}$ denote the functor given by evaluation on the 0-sphere $S^0 \in \mathcal{S}_*^{\mathrm{fin}}$.

Proposition 7.2.2.10. *Let $p : \mathcal{C} \to S$ be a reduced locally differentiable fibration. Let $e : s \to s'$ be an edge of S, so that e induces functors*

$$
F : \mathcal{C}_s \to \mathcal{C}_{s'} \qquad f : \mathrm{Sp}(\mathcal{C}_s) \simeq \mathrm{Stab}(p)_s \to \mathrm{Stab}(p)_{s'} \simeq \mathrm{Sp}(\mathcal{C}_{s'}).
$$

Then the functor $\Omega_p^\infty : \mathrm{Stab}(p) \to \mathcal{C}$ induces a natural transformation

$$
F \circ \Omega_{\mathcal{C}_s}^\infty \to \Omega_{\mathcal{C}_{s'}}^\infty \circ f
$$

which exhibits f as a derivative of F.

Proof. Combine Proposition 7.2.1.19 with the proof of Proposition 7.2.2.8. $\qquad\square$

Corollary 7.2.2.11. *Let $p : \mathcal{C} \to S$ be a reduced locally differentiable fibration. Then for each edge $s \to s'$ in S the induced functor $\mathrm{Stab}(p)_s \to \mathrm{Stab}(p)_{s'}$ preserves countable colimits. In particular, it is an exact functor between stable ∞-categories.*

Proof. Combine Propositions 7.2.2.10 and 7.2.1.18. $\qquad\square$

Proof of Theorem 7.2.2.1. The inclusion functor $\mathcal{C}\mathrm{at}_\infty^\star \hookrightarrow \mathcal{C}\mathrm{at}_\infty$ classifies a reduced differentiable fibration $p : \mathcal{Y} \to \mathcal{C}\mathrm{at}_\infty^\star$ (Remark 7.2.2.7). Let $\mathcal{X} = \mathrm{Stab}(p)$. Then the projection map $q : \mathcal{X} \to \mathcal{C}\mathrm{at}_\infty^\star$ is a coCartesian fibration (Proposition 7.2.2.8), classified by a functor $\Phi : \mathcal{C}\mathrm{at}_\infty^\star \to \mathcal{C}\mathrm{at}_\infty$. Using Remark 7.2.2.4 and Proposition 7.2.2.10, we see that Φ is given on objects by $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$ and on morphisms by $F \mapsto \partial F$. It follows immediately that Φ factors through the subcategory $\mathcal{C}\mathrm{at}_\infty^{\star,\mathrm{Ex}} \subseteq \mathcal{C}\mathrm{at}_\infty$, and that Φ is a lift of the functor $\Phi_0 : \mathrm{h}\mathcal{C}\mathrm{at}_\infty^\star \to \mathrm{h}\mathcal{C}\mathrm{at}_\infty^{\star,\mathrm{Ex}}$ described in the introduction to this section. $\qquad\square$

Remark 7.2.2.12. Theorem 7.2.2.1 can be improved upon: if we take into account non-invertible natural transformations, we can regard $\mathcal{C}\mathrm{at}_\infty^\star$ and $\mathcal{C}\mathrm{at}_\infty^{\star,\mathrm{Ex}}$ as $(\infty, 2)$-categories, and the functor Φ can be extended to a functor of $(\infty, 2)$-categories. This can be deduced formally from the fact that the construction $p \mapsto \mathrm{Stab}(p)$ is defined on (reduced) locally differentiable fibrations, rather than merely on (reduced) differentiable fibrations.

Proposition 7.2.2.8 has a counterpart for Cartesian fibrations:

Proposition 7.2.2.13. *Let $p : \mathcal{C} \to S$ be a Cartesian fibration of simplicial sets, and assume that each fiber $\mathcal{C}_s$ of p admits finite limits.*

(1) *The induced map $q : \mathrm{PStab}(p) \to S$ is a coCartesian fibration. Moreover, an edge $e : (s, X) \to (s', X')$ in $\mathrm{PStab}(p)$ is q-Cartesian if and only if, for every finite pointed space $K \in \mathcal{S}_*^{\mathrm{fin}}$, the resulting edge $e(K) : X(K) \to X'(K)$ is a p-coCartesian edge of $\mathcal{C}$.*

(2) *Assume that, for each edge $s \to s'$ in S, the induced functor $\mathcal{C}_{s'} \to \mathcal{C}_s$ is reduced. Then the map $q' : \mathrm{PStab}_*(p) \to S$ is a Cartesian fibration. Moreover, an edge e of $\mathrm{PStab}_*(p)$ is q'-Cartesian if and only if it is q-coCartesian, when regarded as an edge of $\mathrm{PStab}(p)$.*

(3) *Assume that, for each edge $s \to s'$ in S, the induced functor $\mathcal{C}_{s'} \to \mathcal{C}_s$ is left exact. Then the map $q'' : \mathrm{Stab}(p) \to S$ is a Cartesian fibration. Moreover, an edge e of $\mathrm{Stab}(p)$ is q'-Cartesian if and only if it is q-Cartesian, when regarded as an edge of $\mathrm{PStab}(p)$.*

Proof. Assertion (1) follows from Proposition T.3.1.2.1, and assertions (2) and (3) follow immediately from (1). $\qquad\square$

Corollary 7.2.2.14. *Let $p : \mathcal{C} \to S$ be a locally Cartesian fibration of simplicial sets. Assume that each fiber $\mathcal{C}_s$ of p admits finite limits and that each edge $s \to s'$ in S induces a left exact functor $\mathcal{C}_{s'} \to \mathcal{C}_s$. Then:*

(1) *The induced map $q : \mathrm{Stab}(\mathcal{C}) \to S$ is a locally Cartesian fibration.*

(2) *The forgetful functor $\Omega_p^\infty : \mathrm{Stab}(p) \to \mathcal{C}$ carries locally q-Cartesian morphisms to locally p-Cartesian morphisms.*

(3) *Let $e : s \to s'$ be an edge of S, so that e induces functors*
$$F : \mathcal{C}_{s'} \to \mathcal{C}_s \qquad f : \mathrm{Sp}(\mathcal{C}_{s'}) \simeq \mathrm{Stab}(p)_{s'} \to \mathrm{Stab}(p)_s \simeq \mathrm{Sp}(\mathcal{C}_s).$$
Then Ω_p^∞ induces an equivalence
$$F \circ \Omega_{\mathcal{C}_{s'}}^\infty \simeq \Omega_{\mathcal{C}_s}^\infty \circ f,$$
which exhibits f as a derivative of F.

Proof. Assertion (1) follows from Propositions 7.2.2.13 and 7.2.2.5, assertion (2) follows from Proposition 7.2.2.13, and assertion (3) follows from (2) and Example 7.2.1.4. $\qquad\square$

It follows from the above analysis that differentiation preserves adjunctions:

Proposition 7.2.2.15. *Suppose we are given a pair of adjoint functors*
$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D},$$
where $\mathcal{C}$ and $\mathcal{D}$ are differentiable ∞-categories. Assume that F is reduced. Then:

(1) *The functor F preserves sequential colimits.*

(2) *The functor G preserves finite limits.*

(3) *The functors F and G admit derivatives ∂F, ∂G.*

(4) *The functors ∂F and ∂G are adjoint to one another.*

Proof. Assertions (1) and (2) follow from Proposition T.5.2.3.5, and assertion (3) follows from Example 7.2.1.4 and Proposition 7.2.1.9. To prove (4), we note that an adjunction between F and G determines a correspondence of ∞-categories $p : \mathcal{M} \to \Delta^1$, with $\mathcal{C} \simeq \mathcal{M} \times_{\Delta^1} \{0\}$ and $\mathcal{D} \simeq \mathcal{M} \times_{\Delta^1} \{1\}$. Then p is a reduced differentiable fibration, so that the induced map $q : \mathrm{Stab}(p) \to \Delta^1$ is a coCartesian fibration associated to the functor $\partial F : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$ (Proposition 7.2.2.10). Since p is also a Cartesian fibration, using (2) and Corollary 7.2.2.14 we conclude that q associated to the functor $\partial G : \mathrm{Sp}(\mathcal{D}) \to \mathrm{Sp}(\mathcal{C})$. It follows that the correspondence $\mathrm{Stab}(p) \to \Delta^1$ realizes an adjunction between the functors ∂F and ∂G. $\qquad\square$

Corollary 7.2.2.16. *Suppose given a pair of adjoint functors*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

between differentiable ∞-categories. Assume that F is reduced, and let $g : \operatorname{Sp}(\mathcal{D}) \to \operatorname{Sp}(\mathcal{C})$ be the functor given by pointwise application of G. Then:

(1) *The functor g admits a left adjoint $f : \operatorname{Sp}(\mathcal{C}) \to \operatorname{Sp}(\mathcal{D})$.*

(2) *If the functor G preserves sequential colimits, then the composite functor $g \circ f$ is equivalent to the derivative $\partial(G \circ F)$.*

(3) *If $\mathcal{D}$ is monadic over $\mathcal{C}$, then the adjunction $\operatorname{Sp}(\mathcal{C}) \underset{g}{\overset{f}{\rightleftarrows}} \operatorname{Sp}(\mathcal{D})$ exhibit $\operatorname{Sp}(\mathcal{D})$ as monadic over $\operatorname{Sp}(\mathcal{C})$.*

Proof. The functor g is a derivative of G (Example 7.2.1.4), so that assertion (1) follows from Proposition 7.2.2.15. Assertion (2) follows from Corollary 7.2.1.24, and assertion (3) follows from (1) and Example 6.2.2.9. $\qquad\square$

The following consequence of Corollary 7.2.2.16 will play an important role in §8.3:

Corollary 7.2.2.17. *Suppose given an adjunction $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ between differentiable ∞-categories. Assume that F is reduced, G preserves sequential colimits, and that G exhibits $\mathcal{D}$ as monadic over $\mathcal{C}$. If the unit map $\operatorname{id}_{\mathcal{C}} \to G \circ F$ induces an equivalence $\partial \operatorname{id}_{\mathcal{C}} \to \partial(G \circ F)$, then G induces an equivalence of ∞-categories $\operatorname{Sp}(\mathcal{D}) \to \operatorname{Sp}(\mathcal{C})$.*

We conclude this section by characterizing the construction $p \mapsto \operatorname{Stab}(p)$ by a universal property. As a first step, let us introduce a definition which summarizes the important properties of $\operatorname{Stab}(p)$:

Definition 7.2.2.18. Let $p : \mathcal{C} \to S$ be a reduced locally differentiable fibration of simplicial sets. We will say that a map of simplicial sets $U : \overline{\mathcal{C}} \to \mathcal{C}$ *exhibits $\overline{\mathcal{C}}$ as a stabilization of p* if the following conditions are satisfied:

(1) The composite map $q = p \circ U : \overline{\mathcal{C}} \to S$ is a locally coCartesian fibration.

(2) For each vertex $s \in S$, the ∞-category $\overline{\mathcal{C}}$ is stable.

(3) For each vertex $s \in S$, the functor $U_s : \overline{\mathcal{C}}_s \to \mathcal{C}_s$ is left exact. Consequently, U_s admits an essentially unique factorization $\overline{\mathcal{C}}_s \overset{V_s}{\to} \operatorname{Sp}(\mathcal{C}_s) \overset{\Omega^\infty_{\mathcal{C}_s}}{\to} \mathcal{C}_s$, where V_s is an exact functor (Corollary 1.4.2.23).

(4) For each vertex $s \in S$, the functor $V_s : \overline{\mathcal{C}}_s \to \operatorname{Sp}(\mathcal{C}_s)$ is an equivalence of ∞-categories, and therefore admits a homotopy inverse which we will denote by V_s^{-1}.

(5) Let $e : s \to s'$ be an edge of S, so that e induces functors

$$F : \mathcal{C}_s \to \mathcal{C}_{s'} \qquad \overline{F} : \overline{\mathcal{C}}_s \to \overline{\mathcal{C}}_{s'}.$$

Then the natural transformation

$$F \circ \Omega^\infty_{\mathcal{C}_s} \simeq F \circ U_s \circ V_s^{-1} \to U_{s'} \circ \overline{F} \circ V_s^{-1} \simeq \Omega^\infty_{\mathcal{C}_{s'}} \circ (V_{s'} \circ \overline{F} \circ V_s^{-1})$$

exhibits $V_{s'} \circ \overline{F} \circ V_s^{-1}$ as a derivative of F.

Example 7.2.2.19. Let $p : \mathcal{C} \to S$ be a reduced locally differentiable fibration of simplicial sets. Then the map $\Omega^\infty_p : \operatorname{Stab}(p) \to \mathcal{C}$ of Notation 7.2.2.9 exhibits $\operatorname{Stab}(p)$ as a stabilization of p.

Stabilizations of a reduced locally differentiable fibration enjoy the following universal property:

Theorem 7.2.2.20. *Let $p : \mathcal{C} \to S$ be a reduced locally differentiable fibration of simplicial sets and let $U : \overline{\mathcal{C}} \to \mathcal{C}$ be a map which exhibits $\overline{\mathcal{C}}$ as a stabilization of p. Let $q : \mathcal{D} \to S$ be a locally coCartesian fibration of simplicial sets. Suppose that each fiber $\mathcal{D}_s$ of q is a pointed ∞-category which admits finite colimits and each edge $s \to s'$ of S induces a functor $\mathcal{D}_s \to \mathcal{D}_{s'}$ which is reduced and right exact. Let $\mathfrak{X}$ denote the full subcategory of $\mathrm{Fun}_S(\mathcal{D}, \mathcal{C})$ spanned by those maps $F : \mathcal{D} \to \mathcal{C}$ which induce a reduced excisive functor $F_s : \mathcal{D}_s \to \mathcal{C}_s$ for each $s \in S$, and define $\overline{\mathfrak{X}} \subseteq \mathrm{Fun}_S(\mathcal{D}, \overline{\mathcal{C}})$ similarly. Then composition with U induces an equivalence of ∞-categories $\overline{\mathfrak{X}} \to \mathfrak{X}$.*

Remark 7.2.2.21. For most applications of Theorem 7.2.2.20, we will take q to be a locally coCartesian fibration whose fibers are stable ∞-categories.

Remark 7.2.2.22. We can regard Theorem 7.2.2.20 as a relative version of Proposition 1.4.2.22.

Corollary 7.2.2.23. *Let $p : \mathcal{C} \to S$ be a reduced locally differentiable fibration, and let $U : \overline{\mathcal{C}} \to \mathcal{C}$ be a map which exhibits $\overline{\mathcal{C}}$ as a stabilization of p. Then U factors as a composition*

$$\overline{\mathcal{C}} \xrightarrow{U'} \mathrm{Stab}(p) \xrightarrow{\Omega_p^\infty} \mathcal{C},$$

where U' is an equivalence of local coCartesian fibrations over S (that is, U' induces an equivalence of ∞-categories $\overline{\mathcal{C}}_s \to \mathrm{Stab}(p)_s \simeq \mathrm{Sp}(\mathcal{C}_s)$ for each $s \in S$, and carries locally $(p \circ U)$-coCartesian edges of $\overline{\mathcal{C}}$ to locally $(p \circ \Omega_p^\infty)$-coCartesian edges of $\mathrm{Stab}(p)$.

Remark 7.2.2.24. The conclusions of Corollary 7.2.2.23 guarantee that U' admits a homotopy inverse fiberwise over S: see Lemma B.2.4. In other words, $\mathrm{Stab}(p)$ is the unique stabilization of the reduced locally differentiable fibration $p : \mathcal{C} \to S$, up to fiberwise homotopy equivalence over S.

Proof of Corollary 7.2.2.23. Since U exhibits $\overline{\mathcal{C}}$ as a stabilization of p, each fiber $\overline{\mathcal{C}}_s$ is equivalent to $\mathrm{Stab}(\mathcal{C}_s)$ and is therefore a stable ∞-category which admits countable colimits. Moreover, every edge $s \to s'$ induces a functor $\overline{\mathcal{C}}_s \to \overline{\mathcal{C}}_{s'}$ which is equivalent to the derivative of the underlying functor $\mathcal{C}_s \to \mathcal{C}_{s'}$, and therefore preserves countable colimits (Proposition 7.2.1.18). Note that composition with Ω_p^∞ induces a categorical fibration of ∞-categories $\mathrm{Fun}_S(\overline{\mathcal{C}}, \mathrm{Stab}(p)) \to \mathrm{Fun}_S(\overline{\mathcal{C}}, \mathcal{C})$. Since Ω_p^∞ exhibits $\mathrm{Stab}(p)$ as a stabilization of p (Example 7.2.2.19), Theorem 7.2.2.20 implies that U factors as a composition $\Omega_p^\infty \circ U'$, where $U' : \overline{\mathcal{C}} \to \mathrm{Stab}(p)$ is a map which induces a left exact functor $U'_s : \overline{\mathcal{C}}_s \to \mathrm{Stab}(p)_s \simeq \mathrm{Sp}(\mathcal{C}_s)$ for each vertex $s \in S$. Since $\overline{\mathcal{C}}$ satisfies condition (4) of Definition 7.2.2.18, we deduce that each of the functors U'_s is an equivalence of ∞-categories. Condition (5) of Definition 7.2.2.18 guarantees that U' carries locally $(p \circ U)$-coCartesian edges to locally $(p \circ \Omega_p^\infty)$-coCartesian edges. $\square$

Our proof of Theorem 7.2.2.20 will require an analogue of Proposition T.3.2.2.7, which describes the structure of locally coCartesian fibrations over a simplex. To state this result, we need to introduce a bit of notation.

Notation 7.2.2.25. Fix an integer $n \geq 0$. We can identify objects of the simplicial category $\mathfrak{C}[\Delta^{n+1}]$ with elements of the linearly ordered set $[n + 1] = \{0, \ldots, n + 1\}$. For $0 \leq i \leq n$, we can identify vertices of $\mathrm{Map}_{\mathfrak{C}[\Delta^{n+1}]}(i, n + 1)$ with subsets $S \subseteq \{i, i + 1, \ldots, n\}$ which contain i. The construction which assigns to each subset S its largest element extends uniquely to a map of simplicial sets $\phi_i : \mathrm{Map}_{\mathfrak{C}[\Delta^{n+1}]}(i, n + 1) \to \Delta^n$.

Let $j_! : (\mathrm{Set}_\Delta)^{\mathfrak{C}[\Delta^n]} \to (\mathrm{Set}_\Delta)^{\mathfrak{C}[\Delta^{n+1}]}$ denote the functor given by left Kan extension along the inclusion $j : \mathfrak{C}[\Delta^n] \to \mathfrak{C}[\Delta^{n+1}]$, and let $\mathrm{M} : (\mathrm{Set}_\Delta)^{\mathfrak{C}[\Delta^n]} \to \mathrm{Set}_\Delta$ denote the composition of $i_!$ with the functor $(\mathrm{Set}_\Delta)^{\mathfrak{C}[\Delta^{n+1}]} \to \mathrm{Set}_\Delta$ given by evaluation at $n + 1$. For every $\mathcal{F} \in (\mathrm{Set}_\Delta)^\mathcal{C}$, we can identify $\mathrm{M}(\mathcal{F})$ with a quotient of the disjoint union $\coprod_{0 \leq i \leq n} \mathcal{F}(i) \times \mathrm{Map}_{\mathfrak{C}[\Delta^{n+1}]}(i, n + 1)$. The maps $\{\phi_i\}_{0 \leq i \leq n}$ determine a map of simplicial sets $\mathrm{M}(\mathcal{F}) \to \Delta^n$. This map depends functorially on $\mathcal{F}$; we may therefore view M as defining a functor from $(\mathrm{Set}_\Delta)^{\mathfrak{C}[\Delta^n]}$ to $(\mathrm{Set}_\Delta)_{/\Delta^n}$. We will abuse notation by denoting this functor also by M.

Remark 7.2.2.26. For every object $\mathcal{F} \in (\mathrm{Set}_\Delta)^{\mathfrak{C}[\Delta^n]}$ and every $0 \leq i \leq n$, there is a canonical isomorphism $\mathrm{M}(\mathcal{F}) \times_{\Delta^n} \{i\} \simeq \mathcal{F}(i)$. Moreover, the marking on this simplicial set provided by $\mathrm{M}(\mathcal{F})^+$ is trivial: only degenerate edges of $\mathcal{F}(i)$ are marked.

Proposition 7.2.2.27. *Let $\mathcal{F} \in \mathrm{Set}_\Delta{}^{\mathfrak{C}[\Delta^n]}$, and suppose we are given a commutative diagram*

$$
\begin{array}{ccc}
\mathrm{M}(\mathcal{F}) & \xrightarrow{\ f\ } & \mathcal{C} \\
& \searrow \quad \swarrow {\scriptstyle q} & \\
& \Delta^n &
\end{array}
$$

with the following properties:

(1) *The map q is a locally coCartesian fibration.*

(2) *Let v be a vertex of $\mathcal{F}(i)$, and let $S \subseteq S' \subseteq \{i, \ldots, n\}$ be subsets containing i, so that there is an edge e joining the vertex (v, S) to the vertex (v, S') in the simplicial set $\mathcal{F}(i) \times \mathrm{Map}_{\mathfrak{C}[\Delta^{n+1}]}(i, n+1)$. If $S' - S$ consists of a single element, then the image of e under the map*

$$ \mathcal{F}(i) \times \mathrm{Map}_{\mathfrak{C}[\Delta^{n+1}]}(i, n+1) \to \mathrm{M}(\mathcal{F}) \xrightarrow{f} \mathcal{C} $$

is a q-coCartesian morphism in $\mathcal{C}$.

(3) *For $0 \leq i \leq n$, the composite map*

$$ \mathcal{F}(i) \to \mathrm{M}(\mathcal{F}) \times_{\Delta^n} \{i\} \to \mathcal{C} \times_{\Delta^n} \{i\} $$

is a categorical equivalence.

Then f is a categorical equivalence.

Proof. The proof proceeds by induction on n. If $n = 0$ the result is obvious, so we may suppose $n > 0$. Let $\mathcal{F}_0 = \mathcal{F} \mid \mathfrak{C}[\Delta^{n-1}]$. Unwinding the definition, we have a canonical isomorphism of simplicial sets

$$ \alpha : \mathrm{M}(\mathcal{F}) \simeq (\mathrm{M}(\mathcal{F}') \times \Delta^1) \coprod_{\mathrm{M}(\mathcal{F}') \times \{1\}} \mathcal{F}(n) $$

Let q denote the composition $\mathcal{C} \to \Delta^n \xrightarrow{q_0} \Delta^1$, where $q_0^{-1}\{0\} = \Delta^{n-1} \subseteq \Delta^n$. The map q is a coCartesian fibration of simplicial sets. The desired result now follows by combining the inductive hypothesis with Proposition T.3.2.2.10. $\qquad\square$

Proposition 7.2.2.28. *Let $\mathcal{C} \to \Delta^n$ be a locally coCartesian fibration of ∞-categories. Then there exists a projectively cofibrant diagram $\mathcal{F} \in (\mathrm{Set}_\Delta)^{\mathfrak{C}[\Delta^n]}$ and a map $f : \mathrm{M}(\mathcal{F}) \to \mathcal{C}$ which satisfies the hypotheses of Proposition 7.2.2.27.*

Proof. The proof goes by induction on n. The result is obvious if $n = 0$, so assume $n > 0$. Let $\mathcal{C}_0 = \mathcal{C} \times_{\Delta^n} \Delta^{n-1}$. The inductive hypothesis guarantees the existence of a projectively cofibrant diagram $\mathcal{F}_0 \in (\mathrm{Set}_\Delta)^{\mathfrak{C}[\Delta^{n-1}]}$ and a map $f_0 : \mathrm{M}(\mathcal{F}_0) \to \mathcal{C}_0$ satisfying the hypotheses of Proposition 7.2.2.27. Let $q : \mathcal{C} \to \Delta^1$ be defined as in the proof of Proposition 7.2.2.27. Then q is a coCartesian fibration, and f_0 determines a map of simplicial sets $h_0 : \mathrm{M}(\mathcal{F}_0) \times \{0\} \to \mathcal{C} \times_{\Delta^1} \{0\}$. We can therefore choose a q-coCartesian extension of h_0 to a map $h : \mathrm{M}(\mathcal{F}_0) \times \Delta^1 \to \mathcal{C}$, where $h \mid \mathrm{M}(\mathcal{F}_0) \times \{1\}$ determines a map $h_1 : \mathrm{M}(\mathcal{F}_0) \to \mathcal{C} \times_{\Delta^n} \{n\}$. Choose a factorization of h_1 as a composition

$$ \mathrm{M}(\mathcal{F}_0) \xrightarrow{g'} X \xrightarrow{g''} \mathcal{C} \times_{\Delta^n} \{n\}, $$

where g' is a cofibration of simplicial sets and g'' is a categorical equivalence. The map g' determines an extension of $\mathcal{F}_0$ to a functor $\mathcal{F} \in (\mathrm{Set}_\Delta)^{\mathfrak{C}[\Delta^n]}$ with $\mathcal{F}(n) = X$, and the maps h and g'' can be amalgamated to a map of marked simplicial sets $\mathrm{M}(\mathcal{F}) \to \mathcal{C}$ with the desired properties. $\qquad\square$

Lemma 7.2.2.29. *Suppose given a commutative diagram of simplicial sets*

$$
\begin{array}{ccc}
\overline{\mathcal{C}} & \xrightarrow{\ U\ } & \mathcal{C} \\
& \searrow^{\overline{p}} \quad \swarrow_{p} & \\
& \Delta^n, &
\end{array}
$$

where p is a reduced locally differentiable fibration and U exhibits $\overline{\mathcal{C}}$ as a stabilization of p. Let $\mathcal{D}$ be a pointed ∞-category which admits finite colimits and a final object, let $V \subseteq \mathrm{Fun}(\mathcal{D}, \mathcal{C})$ denote the full subcategory spanned by those functors given by reduced, excisive maps $\mathcal{D} \to \mathcal{C}_i$ for some $0 \leq i \leq n$, and define $\overline{V} \subseteq \mathrm{Fun}(\mathcal{D}, \overline{\mathcal{C}}$ similarly. Then composition with U induces a categorical equivalence $\overline{V} \to V$.

Proof. Composition with the map p and evaluation at a final object of $\mathcal{D}$ determines a functor $q : V \to \Delta^n$. Let $\overline{q} = q \circ U$. Using Lemma B.2.4, we are reduced to proving the following:

  (*i*) The projection $\overline{q} : \overline{V} \to S$ is a locally coCartesian fibration.

 (*ii*) The projection $q : V \to S$ is a locally coCartesian fibration.

(*iii*) The map $\overline{V} \to V$ carries locally $\overline{q}$-coCartesian edges to q-coCartesian edges.

(*iv*) For every vertex $s \in S$, the induced map $\overline{V}_s \to V_s$ is an equivalence of ∞-categories.

The map q admits a factorization

$$
V \to \mathrm{Fun}(\mathcal{D}, \mathcal{C}) \times_{\mathrm{Fun}(\mathcal{D}, \Delta^n)} \Delta^n \to \Delta^n.
$$

The first map is the inclusion of a full subcategory, and therefore an inner fibration. The second map is a pullback of the rpojection $\mathrm{Fun}(\mathcal{D}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{D}, \Delta^n)$, and therefore an inner fibration by Corollary T.2.3.2.5. It follows that q is an inner fibration; likewise $\overline{q}$ is an inner fibration. To prove the remaining assertions, it suffices to treat the case $n = 1$. In particular, we may assume that p and $\overline{p}$ are coCartesian fibrations.

   We now prove (*i*). Proposition T.3.1.2.1 implies that the projection $\overline{q}' : \mathrm{Fun}(\mathcal{D}, \overline{\mathcal{C}}) \to \mathrm{Fun}(\mathcal{D}, \Delta^1)$ is a coCartesian fibration. Moreover, an edge $f \to g$ in the fiber product $\mathrm{Fun}(\mathcal{D}, \overline{\mathcal{C}})$ is $\overline{q}'$-coCartesian if and only if, for each $D \in \mathcal{D}$, the induced edge $f(D) \to g(D)$ is an $\overline{p}$-coCartesian edge of $\overline{\mathcal{C}}$. Since every edge $s \to s'$ induces an exact functor $\overline{\mathcal{C}}_s \to \overline{\mathcal{C}}_{s'}$, we conclude that if $f \in \overline{W}$, then $g \in \overline{W}$. This proves that $\overline{q} = \overline{q}'|\overline{W}$ is a coCartesian fibration, and that an edge of $\overline{W}$ is $\overline{q}$-coCartesian if and only if it is $\overline{q}'$-coCartesian.

   Assertion (*iv*) follows immediately from Proposition 1.4.2.22. To prove (*iii*), write $\mathcal{C}$ as the correspondence associated to a reduced functor $F : \mathcal{C}_0 \to \mathcal{C}_1$ which preserves sequential colimits. Since U exhibits $\overline{\mathcal{C}}$ as a stabilization of p, we can identify $\overline{\mathcal{C}}$ with the correspondence associated to the derivative $\partial F : \mathrm{Sp}(\mathcal{C}_0) \to \mathrm{Sp}(\mathcal{C}_1)$ of F. Let $G : \mathcal{D} \to \mathrm{Sp}(C_0)$ be a reduced, excisive functor and let $e : G \to \partial F \circ G$ be the corresponding $\overline{q}$-coCartesian edge of $\overline{V}$; we wish to show that the image of e in V is q-coCartesian. Unwinding the definitions, we are reduced to proving that the canonical natural transformation $F \circ \Omega^{\infty}_{\mathcal{C}_0} \circ G \to \Omega^{\infty}_{\mathcal{C}_1} \circ \partial F \circ G$ induces an equivalence $P_1(F \circ \Omega^{\infty}_{\mathcal{C}_0} \circ G) \simeq \Omega^{\infty}_{\mathcal{C}_1} \circ \partial F \circ G$. This follows from Remark 7.1.1.30.

   We now prove (*ii*). Suppose we are given an vertex $f : \mathcal{D} \to \mathcal{C}_0$ of V; we wish to show that there exists a q-coCartesian morphism $\alpha : f \to g$ in V, for some $g : \mathcal{D} \to \mathcal{C}_1$. Using (*iv*), we may assume without loss of generality that f can be lifted to a vertex $\overline{f} : \mathcal{D} \to \overline{\mathcal{C}}_0$ in $\overline{V}$. Using (*i*), we can choose an $\overline{q}$-coCartesian edge $\overline{f} \to \overline{g}$ for some $g : \mathcal{D} \to \overline{\mathcal{C}}_1$. We now take α to be the image of $\overline{\alpha}$ in V, which is q-coCartesian by virtue of (*iii*). $\square$

Proof of Theorem 7.2.2.20. According to Theorem B.0.19, there exists a model structure on the category of marked simplicial sets over S whose cofibrations are monomorphisms and whose fibrant objects are pairs (X, E), where $X \to S$ is a locally coCartesian fibration and E is the collection of locally coCartesian edges of X. Without loss of generality, we may assume that the map $U : \overline{\mathcal{C}} \to \mathcal{C}$ determines a fibration with respect to this model structure. We define a simplicial set Z by the following universal property: for every simplicial

set K, $\mathrm{Hom}_{\mathrm{Set}_\Delta}(K, Z)$ can be identified with the set of pairs (b, ϕ), where $b : K \to S$ is a map of simplicial sets and $\phi : K \times_S \mathcal{D} \to K \times_S \mathcal{C}$ is a map which is compatible with the projection to K, and induces a reduced, excisive functor $\mathcal{D}_{b(k)} \to \mathcal{C}_{b(k)}$ for each vertex k of K. Let $\overline{Z}$ be defined similarly, using $\overline{\mathcal{C}}$ in place of $\mathcal{C}$. The map $\overline{\mathcal{X}} \to \mathcal{X}$ is a pullback of the canonical map $\mathrm{Fun}(S, \overline{Z}) \to \mathrm{Fun}(S, Z)$. It will therefore suffice to show that the map $\overline{Z} \to Z$ is a trivial Kan fibration. In other words, we need only show that every lifting problem of the form

$$
\begin{array}{ccc}
\partial \Delta^n & \longrightarrow & \overline{Z} \\
\downarrow & & \downarrow \\
\Delta^n & \longrightarrow & Z
\end{array}
$$

admits a solution. Without loss of generality, we may replace S by Δ^n. Let $\mathcal{D}' = \mathcal{D} \times_{\Delta^n} \partial \Delta^n$. Unwinding the definitions, we are required to solve a lifting problem of the form

$$
\begin{array}{ccc}
\mathcal{D}' & \longrightarrow & \overline{\mathcal{C}} \\
\downarrow & \overset{\phi}{\nearrow} & \downarrow U \\
\mathcal{D} & \underset{\phi_0}{\longrightarrow} & \mathcal{C} .
\end{array}
$$

Moreover, if $n = 0$, we must further guarantee that the functor ϕ is left exact.

Let us first consider the case $n = 0$. By assumption, the map U is equivalent to the functor $\Omega_{\overline{\mathcal{C}}}^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$, and ϕ_0 is a left exact functor whose domain is stable. Invoking Proposition 1.4.2.22, we deduce that $\phi_0 \simeq U \circ \phi'$, where $\phi' : \mathcal{D} \to \overline{\mathcal{C}}$ is an exact functor. Since U is a fibration, any equivalence of $U \circ \phi'$ with ϕ_0 can be lifted to an equivalence of ϕ' with an exact functor $\phi : \mathcal{D} \to \overline{\mathcal{C}}$ satisfying $U \circ \phi = \phi_0$.

We now treat the case $n > 0$. Since q is a locally coCartesian fibration, Proposition 7.2.2.28 guarantees the existence of a simplicial functor $\mathcal{F} : \mathfrak{C}[\Delta^n] \to \mathrm{Set}_\Delta$ and a map $u : \mathrm{M}(\mathcal{F}) \to \mathcal{D}$ which induces categorical equivalences $\mathcal{F}(i) \to \mathcal{D} \times_{\Delta^n} \{i\}$ for $0 \leq i \leq n$. For every face $\sigma \subseteq \Delta^n$, let $W_\sigma = \mathrm{M}(\mathcal{F} \,|\, \mathfrak{C}[\sigma])$. Finally, for every simplicial subset $S' \subseteq S$, let $W_{S'}$ denote the colimit $\mathrm{colim}_{\sigma \in S'} W_\sigma$. For each $S' \subseteq S$, we have a canonical map $\psi_{S'} : W_{S'} \to Y \times_S S'$. Using Proposition 7.2.2.27, we deduce that $\psi_{S'}$ is a categorical equivalence whenever S' is a simplex. Since the domain and codomain of $\psi_{S'}$ both carry pushout squares of simplicial subsets of S to homotopy pushout squares of simplicial sets, we deduce that $\psi_{S'}$ is a categorical equivalence for all $S' \subseteq S$. Invoking Proposition T.A.2.3.1, we are reduced to solving the lifting problem depicted in the diagram

$$
\begin{array}{ccc}
W_{\partial \Delta^n} & \longrightarrow & \overline{\mathcal{C}} \\
\downarrow & \nearrow & \downarrow U \\
W_{\Delta^n} & \longrightarrow & \mathcal{C} .
\end{array}
$$

Let $C = (\Delta^1)^n$ denote an n-dimensional cube, and ∂C its boundary. Then the left vertical map is a pushout of the inclusion $(\partial C) \times \mathcal{F}(0) \subseteq C \times \mathcal{F}(0)$. Consequently, the above lifting problem is equivalent to providing a dotted arrow in the diagram

$$
\begin{array}{ccc}
(\partial C) \times \mathcal{F}(0) & \longrightarrow & \overline{\mathcal{C}} \\
\downarrow & \nearrow & \downarrow U \\
C \times \mathcal{F}(0) & \longrightarrow & \mathcal{C} .
\end{array}
$$

We may assume without loss of generality that the functor $\mathcal{F}$ is projectively fibrant (otherwise, we simply make a fibrant replacement for $\mathcal{F}$), so that $\mathcal{F}(0)$ is an ∞-category which is equivalent to the fiber $\mathcal{D} \times_{\Delta^n} \{0\}$. In particular, $\mathcal{F}(0)$ admits finite colimits and a final object. Let V denote the full simplicial subset of $\mathrm{Fun}(\mathcal{F}(0), \mathcal{C})$ spanned by those functors which belong to $\mathrm{Exc}_*(\mathcal{F}(0), \mathcal{C}_i)$ for some $0 \leq i \leq n$, and define $\overline{V}$

similarly. We can now rewrite our lifting problem yet again:

$$
\begin{array}{ccc}
\partial C & \longrightarrow & \overline{V} \\
\downarrow & & \downarrow{\scriptstyle U'} \\
C & \longrightarrow & V.
\end{array}
$$

To solve this lifting problem, it suffices to show that U' is a trivial Kan fibration. Our assumption on U guarantees that U' is a categorical fibration. We complete the proof by observing that Lemma 7.2.2.29 guarantees that U' is a categorical equivalence. □

7.2.3   Differentials of Functors

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories which admit finite limits. In §7.2.1, we defined the *derivative* ∂F of F, as a functor from $\mathrm{Sp}(\mathcal{C})$ to $\mathrm{Sp}(\mathcal{D})$. In this section, we introduce a closely related notion, which we call the *differential* of F.

Definition 7.2.3.1. Let $\{\mathcal{C}_s\}_{s \in S}$ be a finite collection of ∞-categories which admit finite colimits and final objects, let $\mathcal{D}$ be an ∞-category which admits finite limits, and suppose we are given functors $F, F' : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$. We will say that a natural transformation $\beta : F \to F'$ *exhibits F' as a differential of F* if the following conditions are satisfied:

(1) The functor F' is multilinear (that is, it is reduced and excisive in each variable).

(2) For every multilinear functor $G : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$, composition with β induces a homotopy equivalence

$$
\mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})}(F', G) \to \mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})}(F, G).
$$

Remark 7.2.3.2. In the situation of Definition 7.2.3.1, if $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ admits a differential F', then the F' is determined by F up to canonical equivalence. We will sometimes indicate the dependence of F' on F by writing $F' = DF$ or $F' = D(F)$.

Example 7.2.3.3. In the situation of Definition 7.2.3.1, suppose that S is empty. Then every functor $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ is multilinear, so that a natural transformation $\beta : F \to F'$ exhibits F' as a differential of F if and only β is an equivalence.

Remark 7.2.3.4. Let $\{\mathcal{C}_s\}_{s \in S}$ and $\mathcal{D}$ as in Definition 7.2.3.1, and suppose we are given functors

$$
F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D} \qquad f : \prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s) \to \mathrm{Sp}(\mathcal{D}).
$$

If S is nonempty, then a natural transformation $\alpha : F \circ \prod_{s \in S} \Omega^\infty_{\mathcal{C}_s} \to \Omega^\infty_{\mathcal{D}} \circ f$ exhibits f as a derivative of F if and only if it exhibits $\Omega^\infty_{\mathcal{D}} \circ f$ as a differential of $F \circ \prod_{s \in S} \Omega^\infty_{\mathcal{C}_s}$ (see Remark 7.2.1.8). This relationship between derivatives and differentials breaks down when S is empty: see Examples 7.2.3.3 and 7.2.1.5.

Example 7.2.3.5. Let $\{\mathcal{C}_s\}_{s \in S}$ be a finite collection of ∞-categories which admit finite colimits and final objects and let $\mathcal{D}$ be a differentiable ∞-category. Let $P_{\overline{1}} : \mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D}) \to \mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$ be a left adjoint to the inclusion (see Proposition 7.1.3.6). If $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ is a reduced functor, then $P_{\overline{1}}F$ is also reduced. It follows that the canonical map $F \to P_{\overline{1}}F$ exhibits $P_{\overline{1}}F$ as a differential of F. In particular, there exists a differential of F.

We can construct differentials (and derivatives) of more general functors by first passing to the case of reduced functors. For this, we introduce a dual version of Construction 7.1.3.15.

Construction 7.2.3.6. Let $\{\mathcal{C}_s\}_{s\in S}$ be ∞-categories which admit zero objects $*_s \in \mathcal{C}_s$, and let $\mathcal{D}$ be an ∞-category which admits finite colimits and a final object $*$. For $1 \le i \le n$, let $U_i : \mathcal{C}_i \to \mathcal{C}_i$ denote the constant functor taking the value $*_i$, and choose a natural transformation of functors $\alpha_i : U_i \to \mathrm{id}_{\mathcal{C}_i}$. For each functor $F : \prod_{s\in S} \mathcal{C}_s \to \mathcal{D}$, consider the functor

$$\overline{F} : \prod_{s\in S} \mathcal{C}_s \times \mathrm{N}(\mathbf{P}(S)) \xrightarrow{\prod \alpha_i} \prod_{s\in S} \mathcal{C}_s \xrightarrow{F} \mathcal{D}$$

For each $T \subseteq S$, we let F_T denote the restriction of $\overline{F}$ to $\prod_{s\in S} \mathcal{C}_s \times \{T\}$, so that F_T is given by the formula
$F_T(\{X_s\}) = F(\{X'_s\})$ where $X'_s = \begin{cases} X_s & \text{if } s \in T \\ *_s & \text{if } s \notin T. \end{cases}$. The functor $\overline{F}$ determines a natural transformation

$$\beta : \varinjlim_{T \subsetneq S} F_T \to F_S = F.$$

Let $\underline{*}$ denote the constant functor $\prod_{s\in S} \mathcal{C}_s \to \mathcal{D}$ taking the value $* \in \mathcal{D}$. We let $\mathrm{cored}(F)$ denote the pushout

$$\varinjlim_{T \subsetneq S} \underline{*} \coprod_{\varinjlim_{T \subsetneq S} F_T} F,$$

We will refer to $\mathrm{cored}(F)$ as the *coreduction* of F.

Remark 7.2.3.7. Let $F : \prod_{s\in S} \mathcal{C}_s \to \mathcal{D}$ be as in Construction 7.2.3.6. If S is empty, then the canonical map $F \to \mathrm{cored}(F)$ is an equivalence. Otherwise, the partially ordered set $\{T : T \subsetneq S\}$ is weakly contractible, so we can identify $\mathrm{cored}(F)$ with the cofiber of the canonical map $\varinjlim_{T \subsetneq S} F_T \to F$.

The coreduction of a functor F enjoys the following universal property:

Proposition 7.2.3.8. *Let $\{\mathcal{C}_s\}_{s\in S}$ be a finite collection of pointed ∞-categories, let $\mathcal{D}$ be an ∞-category which admits finite colimits and a final object, and let $F : \prod_{s\in S} \mathcal{C}_s \to \mathcal{D}$ be a functor. Then:*

(a) *The functor $\mathrm{cored}(F) : \prod_{s\in S} \mathcal{C}_S \to \mathcal{D}$ is reduced (in each variable).*

(b) *Let $G : \prod_{s\in S} \mathcal{C}_s \to \mathcal{D}$ be any reduced functor. Then the canonical map $F \to \mathrm{cored}(F)$ induces a homotopy equivalence*

$$\mathrm{Map}_{\mathrm{Fun}(\prod_{s\in S}\mathcal{C}_s,\mathcal{D})}(\mathrm{cored}(F), G) \to \mathrm{Map}_{\mathrm{Fun}(\prod_{s\in S}\mathcal{C}_s,\mathcal{D})}(F, G).$$

Proof. If S is empty, the result is obvious. Let us therefore assume that S is nonempty. If $\mathcal{D}$ is pointed, the desired result follows immediately from Proposition 7.1.3.17. The proof in the general case is similar. Suppose we are given a collection of objects $\vec{X} = \{X_s \in \mathcal{C}_s\}_{s\in S}$ such that some X_s is a zero object of $\mathcal{C}_s$. Then for $T \subseteq S$, the canonical map $F_T(\vec{X}) \to F_{T-\{s\}}(\vec{X})$ is an equivalence. It follows that the diagram $\{F_T(\vec{X})\}_{T \subsetneq S}$ is a left Kan extension of $\{F_T(\vec{X})\}_{j\notin T \subset S}$, so that the canonical map

$$F_{S-\{j\}}(\vec{X}) \to \varinjlim_{\emptyset \ne T \subseteq S} F_T(\vec{X})$$

is an equivalence. In particular, $\mathrm{cored}(F)(\vec{X})$ is given by the cofiber of the map $F_{S-\{j\}}(\vec{X}) \to F(\vec{X})$. Since this map is an equivalence, we conclude that $\mathrm{cored}(F)(\vec{X})$ is a final object of $\mathcal{D}$. This proves (a).

We now prove (b). Let $G : \prod_{s\in S} \mathcal{C}_s \to \mathcal{D}$ be reduced. We have a pullback diagram of mapping spaces

$$
\begin{array}{ccc}
\mathrm{Map}_{\mathrm{Fun}(\prod_{s\in S}\mathcal{C}_s,\mathcal{D})}(\mathrm{cored}(F), G) & \longrightarrow & \varprojlim_{T\subsetneq S} \mathrm{Map}_{\mathrm{Fun}(\prod_{s\in S}\mathcal{C}_s,\mathcal{D})}(\underline{*}, G) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathrm{Fun}(\prod_{s\in S}\mathcal{C}_s,\mathcal{D})}(F, G) & \longrightarrow & \varprojlim_{T\subsetneq S} \mathrm{Map}_{\mathrm{Fun}(\prod_{s\in S}\mathcal{C}_s,\mathcal{D})}(F_T, G),
\end{array}
$$

where $\underline{*}$ denotes the constant functor taking the value $* \in \mathcal{D}$. To prove that the left vertical map is a homotopy equivalence, it suffices to prove that the right vertical map is a homotopy equivalence. For this, we show that for every proper subset $T \subsetneq S$, the canonical map $\mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})}(\underline{*}, G) \, \mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})}(F_T, G)$ is a homotopy equivalence. Choose an element $t \in S - T$, and let $\mathcal{C}$ denote the full subcategory of $\prod_{s \in S} \mathcal{C}_s$ spanned by those objects whose tth coordinate is a zero object of $\mathcal{C}_t$. Then $\underline{*}$ and F_T are both left Kan extensions of their restrictions to $\mathcal{C}$. It will therefore suffice to show that the canonical map

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}, \mathcal{D})}(\underline{*}| \, \mathcal{C}, G| \, \mathcal{C}) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}, \mathcal{D})}(F_T| \, \mathcal{C}, G| \, \mathcal{C})$$

is a homotopy equivalence. This is clear, since $G| \, \mathcal{C}$ is a final object of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. $\qquad\square$

Corollary 7.2.3.9. *Let $\{\mathcal{C}_s\}_{s \in S}$ be a finite collection of pointed ∞-categories, let $\mathcal{D}$ an ∞-category which admits finite colimits and a final object, and let $\mathrm{Fun}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$ be the full subcategory of $\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$ spanned by the reduced functors. Then the inclusion*

$$\mathrm{Fun}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{D}) \to \mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$$

admits a left adjoint, given by the construction $F \mapsto \mathrm{cored}(F)$.

Example 7.2.3.10. Let $F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{D}$ be as in Construction 7.2.3.6. Suppose that there exist integers $1 \leq j < k \leq n$ such that, for every sequence of objects $\{X_i \in \mathcal{C}_i\}$, the diagram

$$F(X_1, \ldots, X_{j-1}, *_j, X_{j+1}, \ldots, X_{k-1}, *_k, X_{k+1}, \ldots, X_n)$$

$$F(X_1, \ldots, X_{j-1}, *_j, X_{j+1}, \ldots, X_n) \qquad\qquad F(X_1, \ldots, X_{k-1}, *_k, X_{k+1}, \ldots, X_n)$$

$$F(X_1, \ldots, X_n)$$

is a pushout square. Then the diagram of functors $\{F_T\}_{T \subsetneq S}$ is a left Kan extension of its restriction to $\mathrm{N}(\{T \in \mathbf{P}(S) : T \cap \{j, k\} \neq \{j, k\}\})$, which contains $\mathrm{N}(\{S - \{j\}, S - \{j, k\}, S - \{k\}\})$ as a left cofinal subset. It follows that the map $\beta : \varinjlim_{T \subsetneq S} F_T \to F_S = F$ is an equivalence, so that the coreduction $\mathrm{cored}(F)$ is trivial.

Proposition 7.2.3.11. *Let $\alpha : S \to T$ be a surjective map of finite sets, let $\{\mathcal{C}_s\}_{s \in S}$ and $\{\mathcal{D}_t\}_{t \in T}$ be pointed ∞-categories, and let $\mathcal{E}$ be an ∞-category which admits finite colimits and a final object. Suppose we are given functors*

$$G_t : \prod_{\alpha(s) = t} \mathcal{C}_s \to \mathcal{D}_t \qquad F : \prod_{t \in T} \mathcal{D}_t \to \mathcal{E}$$

where each G_t is reduced. Then we have a canonical equivalence $\mathrm{cored}(F \circ \prod_{t \in T} G_t) \simeq \mathrm{cored}(F) \circ \prod_{t \in T} G_t$.

Proof. Since each G_t is reduced, the functor $\mathrm{cored}(F) \circ \prod_{t \in T} G_t$. Consequently, the canonical map $F \circ \prod_{t \in T} G_t \to \mathrm{cored}(F) \circ \prod_{t \in T} G_t$ factors as a composition

$$F \circ \prod_{t \in T} G_t \to \mathrm{cored}(F \circ \prod_{t \in T} G_t) \xrightarrow{\beta} \mathrm{cored}(F) \circ \prod_{t \in T} G_t.$$

We wish to prove that β is an equivalence. For this, it will suffice to show that $\mathrm{cored}(F) \circ \prod_{t \in T} G_t$ satisfies the universal property of Proposition 7.2.3.8: that is, for every reduced functor $H : \prod_{s \in S} \mathcal{C}_s \to \mathcal{E}$ induces a homotopy equivalence

$$\theta : \mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{E})}(\mathrm{cored}(F) \circ \prod_{t \in T} G_t, H) \to \mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{E})}(F \circ \prod_{t \in T} G_t, H).$$

Let $\underline{*} : \prod_{t \in T} \mathcal{D}_t \to \mathcal{E}$ be the constant functor taking the value $* \in \mathcal{E}$, where $*$ is a final object of $\mathcal{E}$. Then θ is the pullback of a map

$$\varinjlim_{T' \subsetneq T} \mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{E})}(\underline{*} \circ \prod_{t \in T} G_t, H) \to \varprojlim_{T' \subsetneq T} \mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{E})}(F_{T'} \circ \prod_{t \in T} G_t, H).$$

It will therefore suffice to show that for every proper subset $T' \subsetneq T$, the map

$$\theta_{T'} : \mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{E})}(\underline{*} \circ \prod_{t \in T} G_t, H) \to \mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{E})}(F_{T'} \circ \prod_{t \in T} G_t, H)$$

is a homotopy equivalence. Choose an element $t \in T - T'$, choose an element $\bar{t} \in S$ with $\alpha(\bar{t}) = t$, and let $\mathcal{C} \subseteq \prod_{s \in S} \mathcal{C}_s$ be the full subcategory spanned by those objects whose $\bar{t}$th coordinate is a zero object of $\mathcal{C}_{\bar{t}}$. Then $F_{T'} \circ \prod_{t \in T} G_t$ and $\underline{*} \circ \prod_{t \in T} G_t$ are both left Kan extensions of their restrictions to $\mathcal{C}$. It will therefore suffice to show that the canonical map

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}, \mathcal{E})}((\underline{*} \circ \prod_{t \in T} G_t) | \mathcal{C}, H | \mathcal{C}) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}, \mathcal{E})}((F_{T'} \circ \prod_{t \in T} G_t) | \mathcal{C}, H | \mathcal{C})$$

is a homotopy equivalence. This is clear, since $H | \mathcal{C}$ is a final object of $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$. $\square$

Remark 7.2.3.12. Let $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ be as in Construction 7.2.3.6, and suppose we are given a collection of reduced functors $G_s : \mathcal{C}'_s \to \mathcal{C}_s$. Then we have a canonical equivalence

$$\mathrm{cored}(F) \circ \prod_{s \in S} G_s \simeq \mathrm{cored}(F \circ \prod_{s \in S} G_s).$$

Proposition 7.2.3.13. *Let $\{\mathcal{C}_s\}_{s \in S}$ be a finite collection of ∞-categories, let $\mathcal{D}$ be a differentiable ∞-category which admits finite colimits, and let $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ be a functor. Then:*

(1) *If each $\mathcal{C}_s$ admits finite colimits and a zero object, then there exists a differential of F.*

(2) *If each $\mathcal{C}_s$ admits finite limits, then there exists a derivative of F.*

Proof. We first prove (1). Set $G = \mathrm{cored}(F)$. Let $G = \mathrm{cored}(F)$. If $\beta : G \to G'$ exhibits G' as a differential of G, then the composite functor $F \to G \to G'$ exhibits G' as a differential of F. Consequently, assertion (1) follows from Example 7.2.3.5.

If S is empty, then assertion (2) follows from Example 7.2.1.5 and Proposition 7.2.3.16. Let us now prove (2) in the case where F is nonempty. Using Remark 7.2.1.6, we can replace each $\mathcal{C}_s$ by the ∞-category of pointed objects $\mathcal{C}_{s,*}$, and thereby reduce to the case where each $\mathcal{C}_s$ is pointed. Let $G = \mathrm{cored}(F)$ as above. If $\alpha : G \circ \prod_{s \in S} \Omega^\infty_{\mathcal{C}_s} \to \Omega^\infty_{\mathcal{D}} \circ g$ exhibits a functor $g : \prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s) \to \mathrm{Sp}(\mathcal{D})$ as a derivative of G, then the composite map

$$F \circ \prod_{s \in S} \Omega^\infty_{\mathcal{C}_s} \to G \circ \prod_{s \in S} \Omega^\infty_{\mathcal{C}_s} \xrightarrow{\alpha} \Omega^\infty_{\mathcal{D}} \circ g$$

exhibits g as a derivative of F (since $G \circ \prod_{s \in S} \Omega^\infty_{\mathcal{C}_s}$ is a coreduction of $F \circ \prod_{s \in S} \Omega^\infty_{\mathcal{C}_s}$, by Proposition 7.2.3.11). The desired result now follows from Proposition 7.2.1.9. $\square$

Lemma 7.2.3.14. *Let $\{\mathcal{C}_s\}_{s \in S}$ be a nonempty finite collection of ∞-categories which admit finite colimits and final objects, let $\mathcal{D}$ be an ∞-category which admits finite colimits and a final object, and let $\mathcal{E}$ be a differentiable ∞-category. Let*

$$P_1 : \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Exc}(\mathcal{D}, \mathcal{E}) \qquad P_{\bar{1}} : \mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{E}) \to \mathrm{Exc}(\prod_{s \in S} \mathcal{C}_s, \mathcal{E})$$

be left adjoints to the inclusion functors, and let $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ be a functor which is reduced and right exact in each variable. Then, for every reduced functor $G : \mathcal{D} \to \mathcal{E}$, we have a canonical equivalence $P_{\bar{1}}(G \circ F) \simeq P_1(G) \circ F$.

Proof. Since F is right exact in each variable and G is excisive, the functor $P_1(G) \circ F$ is excisive in each variable. Consequently, the canonical map $\alpha : G \circ F \to P_1(G) \circ F$ induces a natural transformation $\beta : P_{\overline{1}}(G \circ F) \to P_1(G) \circ F$. Let $H : \prod_{s \in S} \mathcal{C}_s \to \mathcal{E}$ be an arbitrary functor, so that composition with α induces a map

$$\theta : \mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{E})}(P_1(G) \circ F, H) \to \mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{E})}(G \circ F, H).$$

Fix an element $s \in S$, and regard F as defining a functor

$$f : \prod_{t \neq s} \mathcal{C}_t \to \mathrm{Fun}(\mathcal{C}_s, \mathcal{D}).$$

Applying Remark 7.1.1.30, we see that $P_1(G) \circ F$ classifies the functor

$$\prod_{t \neq s} \mathcal{C}_t \xrightarrow{f} \mathrm{Fun}(\mathcal{C}_s, \mathcal{D}) \xrightarrow{G \circ} \mathrm{Fun}(\mathcal{C}_s, \mathcal{E}) \xrightarrow{L} \mathrm{Exc}(\mathcal{C}_s, \mathcal{E}),$$

where $L : \mathrm{Fun}(\mathcal{C}_s, \mathcal{E}) \to \mathrm{Exc}(\mathcal{C}_s, \mathcal{E})$ denotes a left adjoint to the inclusion. It follows that the natural transformation θ is an homotopy equivalence whenever H is excisive in the sth variable. In particular, it is an equivalence when H is excisive in each variable, so that β is an equivalence. $\square$

Proposition 7.2.3.15. *Let $\alpha : S \to T$ be a surjective map of finite sets. Suppose we are given pointed ∞-categories $\{\mathcal{C}_s\}_{s \in S}$, $\{\mathcal{D}_t\}_{t \in T}$ which admit finite colimits, and a differentiable ∞-category $\mathcal{E}$ which is pointed and admits finite colimits. Let $\gamma : F \to F'$ be a natural transformation between functors $F, F' : \prod_{t \in T} \mathcal{D}_t \to \mathcal{E}$ which exhibits F' as a differential of F, and suppose we are given a collection of functors $G_t : \prod_{\alpha(s)=t} \mathcal{C}_s \to \mathcal{D}_t$ which are right exact in each variable. Then the induced natural transformation*

$$F \circ \prod_{t \in T} G_t \to F' \circ \prod_{t \in T} G_t$$

exhibits $F' \circ \prod_{t \in T} G_t$ as a differential of $F \circ \prod_{t \in T} G_t$.

Proof. Using Proposition 7.2.3.11 we can replace F by $\mathrm{cored}(F)$, and thereby reduce to the case where the functor F is reduced. In this case, the desired result follows from repeated application of Lemma 7.2.3.14. $\square$

Under some mild hypotheses, one can show that the derivative and differential of a functor $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ are interchangeable data. More precisely, we can recover the differential of F as the composition

$$\prod_{s \in S} \mathcal{C}_s \xrightarrow{\prod \Sigma_{\mathcal{C}_s}^\infty} \prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s) \xrightarrow{\vec{\partial} F} \mathrm{Sp}(\mathcal{D}) \xrightarrow{\Omega_{\mathcal{D}}^\infty} \mathcal{D},$$

where $\Sigma_{\mathcal{C}_s}^\infty$ denotes a left adjoint to the functor $\Omega_{\mathcal{C}_s}^\infty$. We begin by studying some existence criteria for the functors $\Sigma_{\mathcal{C}_s}^\infty$.

Proposition 7.2.3.16. *Let $\mathcal{C}$ be a differentiable ∞-category which admits finite colimits. Then the functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint.*

Definition 7.2.3.17. Let $\mathcal{C}$ be a differentiable ∞-category which admits finite colimits. We let $\Sigma_+^\infty : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$ denote a left adjoint to the functor $\Omega_{\mathcal{C}}^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$. We will refer to Σ_+^∞ as the *infinite suspension* functor. In the special case where the ∞-category $\mathcal{C}$ is pointed, we will denote the functor Σ_+^∞ simply by Σ^∞ (or by $\Sigma_{\mathcal{C}}^\infty$, if we wish to emphasize the dependence on $\mathcal{C}$).

Remark 7.2.3.18. The notation of Definition 7.2.3.17 is consistent with that of Proposition 1.4.4.4, in the special case where $\mathcal{C}$ is a presentable ∞-category.

The proof of Proposition 7.2.3.16 is based on the following lemma:

Lemma 7.2.3.19. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and a final object, and let $\theta :$ $\mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \to \mathcal{C}$ be the functor given by evaluation on the 0-sphere $S^0 \in \mathcal{S}_*^{\mathrm{fin}}$. Then θ admits a left adjoint $\overline{\Sigma}^\infty : \mathcal{C} \to \mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$.*

Proof. Let $\mathrm{Fun}^{\mathrm{Rex}}(\mathcal{S}^{\mathrm{fin}}, \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}, \mathcal{C})$ spanned by the right exact functors, so that evaluation on the one-point space $* \in \mathcal{S}^{\mathrm{fin}}$ induces a trivial Kan fibration $\mathrm{Fun}^{\mathrm{Rex}}(\mathcal{S}^{\mathrm{fin}}, \mathcal{C}) \to \mathcal{C}$ (see Remark 1.4.2.5). Choose a section of this trivial Kan fibration, which we will denote by $C \mapsto f_C$. For each $C \in \mathcal{C}$, we let $f_C^+ : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}$ denote the functor given by the formula

$$f_C^+(K) = f_C(K) \coprod_C *,$$

where $*$ denotes a final object of $\mathcal{C}$ and the map $C \to f_C(K)$ is determined by the base point of K. Let $\overline{\Sigma}^\infty : \mathcal{C} \to \mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$ be the functor given by $F(C) = f_C^+$. For each $C \in \mathcal{C}$, we have a canonical equivalence

$$\theta(F(C)) = f_C^+(S^0) \simeq C \amalg *,$$

which determines a natural transformation $u : \mathrm{id} \to \theta \circ \overline{\Sigma}^\infty$. We claim that u is the unit map for an adjunction between θ and F. To prove this, fix an object $C \in \mathcal{C}$ and a reduced functor $g : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}$; we wish to show that u induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})}(f_C^+, g) \to \mathrm{Map}_{\mathcal{C}}(f_C^+(S^0), g(S^0)) \to \mathrm{Map}_{\mathcal{C}}(C, g(S^0)).$$

Note that if $h : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}$ is a constant functor, then h is a left Kan extension of its restriction to the final object of $\mathcal{S}_*^{\mathrm{fin}}$. Since g is reduced, we deduce that $\mathrm{Map}_{\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})}(h, g)$ is contractible. Let $\phi : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}^{\mathrm{fin}}$ be the forgetful functor. We then have a pushout diagram of functors

$$
\begin{array}{ccc}
\underline{C} & \longrightarrow & \underline{*} \\
\downarrow & & \downarrow \\
f_C \circ \phi & \longrightarrow & f_C^+,
\end{array}
$$

where $\underline{C} : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}$ denotes the constant functor taking the value C and $\underline{*}$ is defined similarly. It follows that the restriction map $\mathrm{Map}_{\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})}(f_C^+, g) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})}(f_C \circ \phi, g)$ is a homotopy equivalence. Note that ϕ admits a left adjoint ψ, which carries each finite space K to the space obtained from K by adding a disjoint base point. Unwinding the definitions, we wish to prove that the canonical map

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})}(f_C \circ \phi, g) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}, \mathcal{C})}(f_C, g \circ \psi) \to \mathrm{Map}_{\mathcal{C}}(C, (g \circ \psi)(*))$$

is a homotopy equivalence. This follows from the fact that f_C is a left Kan extension of its restriction along the inclusion $\{*\} \hookrightarrow \mathcal{S}^{\mathrm{fin}}$. $\qquad \square$

Proof of Proposition 7.2.3.16. The functor Ω^∞ factors as a composition

$$\mathrm{Sp}(\mathcal{C}) \hookrightarrow \mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \xrightarrow{\theta} \mathcal{C},$$

where θ is given by evaluation at $S^0 \in \mathcal{S}_*^{\mathrm{fin}}$. The desired result now follows from Proposition 7.2.1.9 and Lemma 7.2.3.19. $\qquad \square$

Remark 7.2.3.20. Let S be a nonempty finite set and let $\mathcal{D}$ be a differentiable ∞-category which admits finite colimits, and let $L_{\mathcal{D}}^S : \mathrm{Fun}_*(\prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D}) \to \mathrm{Exc}_*(\prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D})$ be a left adjoint to the inclusion (see Construction 7.2.1.14). Then the inclusion $\mathrm{Fun}(\prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D}) \to \mathrm{Exc}_*(\prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D})$ admits a left adjoint, given by $X \mapsto L_{\mathcal{D}}^S \, \mathrm{cored}(X)$.

Now suppose we are given a collection of pointed differentiable ∞-categories $\{\mathcal{C}_s\}_{s \in S}$. For each functor $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$, let $F^+ : \prod_{s \in S} \mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \to \mathrm{Fun}(\prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D})$ be the functor given by composition with F. For every collection of reduced functors $X_s \in \mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}_s)$, Proposition 7.2.3.11 supplies an equivalence $\mathrm{cored}(F^+(\{X_s\})) \simeq \mathrm{cored}(F)^+(\{X_s\})$. Combining this observation with Proposition 7.2.1.19, we see that differential $\vec{\partial}F \simeq \vec{\partial}\,\mathrm{cored}(F)$ is given by the composition

$$\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s) \hookrightarrow \prod_{s \in S} \mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}_s) \xrightarrow{F^+} \mathrm{Fun}(\prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D}) \to \mathrm{Exc}_*(\prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D}),$$

where the last functor is a left adjoint to the inclusion.

The main result of this section is the following:

Theorem 7.2.3.21. *Let $\{\mathcal{C}_s\}_{s \in S}$ be a finite collection of pointed differentiable ∞-categories which admit finite colimits, and let $\mathcal{D}$ be a differentiable ∞-category. Then composition with the functors $\Sigma_{\mathcal{C}_s}^{\infty} : \mathcal{C}_s \to \mathrm{Sp}(\mathcal{C}_s)$ induces an equivalence of ∞-categories*

$$\mathrm{Exc}_\star(\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s), \mathcal{D}) \to \mathrm{Exc}_\star(\prod_{s \in S} \mathcal{C}_s, \mathcal{D}).$$

Before giving the proof of Theorem 7.2.3.21, let us describe some its consequences.

Corollary 7.2.3.22. *Let $\{\mathcal{C}_s\}_{s \in S}$ be a nonempty finite collection of pointed differentiable ∞-categories which admit finite colimits and let $\mathcal{D}$ be a differentiable ∞-category. Then the construction $f \mapsto \Omega_{\mathcal{D}}^{\infty} \circ f \circ \prod_{s \in S} \Sigma_{\mathcal{C}_s}^{\infty}$ induces an equivalence of ∞-categories $\phi : \mathrm{Exc}_\star(\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s), \mathrm{Sp}(\mathcal{D})) \to \mathrm{Exc}_\star(\prod_{s \in S} \mathcal{C}_s, \mathcal{D}).$*

Proof. Using Theorem 7.2.3.21, we are reduced to proving that composition with $\Omega_{\mathcal{D}}^{\infty}$ induces an equivalence of ∞-categories

$$\theta : \mathrm{Exc}_\star(\prod_{s \in S} \mathcal{C}_s, \mathrm{Sp}(\mathcal{D})) \to \mathrm{Exc}_\star(\prod_{s \in S}, \mathcal{D}).$$

This follows from Proposition 1.4.2.22 together with the following assertion:

($*$) Let $\mathcal{C}$ be an ∞-category which admits countable colimits and let $F : \mathcal{C} \to \mathrm{Sp}(\mathcal{D})$ be an excisive functor. Then F preserves sequential colimits if and only if the functor $\Omega_{\mathcal{D}}^{\infty} \circ F$ preserves sequential colimits.

The "only if" direction is obvious, since the functor $\Omega_{\mathcal{D}}^{\infty}$ preserves sequential colimits (see Example 7.2.1.17). Conversely, suppose that $\Omega_{\mathcal{D}}^{\infty} \circ F$ preserves sequential colimits; we wish to prove that F preserves sequential colimits. Using Remark 1.4.2.25, we are reduced to proving that for each $n \geq 0$, the functor $\Omega_{\mathcal{D}}^{\infty-n} \circ F \simeq \Omega_{\mathcal{D}}^{\infty}(\Sigma_{\mathrm{Sp}(\mathcal{D})}^n \circ F)$ commutes with sequential colimits. Since F is excisive, we can rewrite this functor as $\Omega_{\mathcal{D}}^{\infty} \circ F \circ \Sigma_{\mathcal{C}}^n$, which commutes with with sequential colimits by virtue of the fact that the functors $\Omega_{\mathcal{D}}^{\infty} \circ F$ and $\Sigma_{\mathcal{C}}^n$ commute with sequential colimits. $\square$

Remark 7.2.3.23. In the situation of Corollary 7.2.3.22, one can give a homotopy inverse to ϕ explicitly by the construction $F \mapsto \vec{\partial}F$. To see this, it suffices to show that for $f \in \mathrm{Exc}_\star(\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s), \mathrm{Sp}(\mathcal{D}))$, the canonical map

$$\alpha : \Omega_{\mathcal{D}}^{\infty} \circ f \circ \prod_{s \in S} \Sigma_{\mathcal{C}_s}^{\infty} \circ \prod_{s \in S} \Omega_{\mathcal{C}_s}^{\infty} \to \Omega_{\mathcal{D}}^{\infty} \circ f$$

exhibits f as a derivative of $F = \Omega_{\mathcal{D}}^{\infty} \circ f \circ \prod_{s \in S} \Sigma_{\mathcal{C}_s}^{\infty}$. Since f is multilinear, it suffices to show that for each multilinear functor $g : \prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s) \to \mathrm{Sp}(\mathcal{D})$, the map

$$
\begin{aligned}
\mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{D}))}(\Omega_{\mathcal{D}}^{\infty} \circ f, g) \quad &\to\quad \mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{D}))}(\Omega_{\mathcal{D}}^{\infty} \circ f \circ \prod_{s \in S} \Sigma_{\mathcal{C}_s}^{\infty} \circ \prod_{s \in S} \Omega_{\mathcal{C}_s}^{\infty}, g) \\
&\to\quad \mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathcal{C}, \mathrm{Sp}(\mathcal{D}))}(\Omega_{\mathcal{D}}^{\infty} \circ f \circ \prod_{s \in S} \Sigma_{\mathcal{C}_s}^{\infty}, g \circ \prod_{s \in S} \Sigma_{\mathcal{C}_s}^{\infty})
\end{aligned}
$$

is a homotopy equivalence. This follows from Theorem 7.2.3.21.

Corollary 7.2.3.24. *Let $\{\mathcal{C}_s\}_{s \in S}$ be a nonempty finite collection of pointed differentiable ∞-categories which admit finite colimits, let $\mathcal{D}$ be a differentiable ∞-category, and suppose we are reduced functors*

$$F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D} \qquad f : \prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s) \to \mathrm{Sp}(\mathcal{D})$$

where F preserves sequential colimits and f is exact in each variable. Let $\alpha : F \circ \prod_{s \in S} \Omega^\infty_{\mathcal{C}_s} \to \Omega^\infty_{\mathcal{D}} \circ f$ be a natural transformation. The following conditions are equivalent:

(a) *The natural transformation α exhibits f as a derivative of F.*

(b) *The composite transformation*

$$\beta : F \to F \circ \prod_{s \in S} \Omega^\infty_{\mathcal{C}_s} \circ \prod_{s \in S} \Sigma^\infty_{\mathcal{C}_s} \xrightarrow{\alpha} \Omega^\infty_{\mathcal{D}} \circ f \circ \prod_{s \in S} \Sigma^\infty_{\mathcal{C}_s}$$

exhibits $\Omega^\infty_{\mathcal{D}} \circ f \circ \prod_{s \in S} \Sigma^\infty_{\mathcal{C}_s}$ as a differential of f.

Remark 7.2.3.25. In the situation of Corollary 7.2.3.24, suppose that the ∞-category $\mathcal{D}$ admits finite colimits, so that the functor $\Omega^\infty_{\mathcal{D}} : \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$ admits a left adjoint $\Sigma^\infty_+ : \mathcal{D} \to \mathrm{Sp}(\mathcal{D})$ (Proposition 7.2.3.16). Using Proposition 1.4.2.22, we obtain the following additional variations on (a) and (b):

(c) The composite transformation

$$\gamma : \Sigma^\infty_+ \circ F \circ \prod_{s \in S} \Omega^\infty_{\mathcal{C}_s} \xrightarrow{\alpha} \Sigma^\infty_+ \circ \Omega^\infty_{\mathcal{D}} \circ f \to f$$

exhibits f as a differential of $\Sigma^\infty_+ \circ F \circ \prod_{s \in S} \Omega^\infty_{\mathcal{C}_s}$.

(d) The composite transformation

$$\delta : \Sigma^\infty_+ \circ F \to \Sigma^\infty_+ \circ F \circ \prod_{s \in S} \Omega^\infty_{\mathcal{C}_s} \circ \prod_{s \in S} \Sigma^\infty_{\mathcal{C}_s} \xrightarrow{\alpha} \Sigma^\infty_+ \circ \Omega^\infty_{\mathcal{D}} \circ f \circ \prod_{s \in S} \Sigma^\infty_{\mathcal{C}_s} \to f \circ \prod_{s \in S} \Sigma^\infty_{\mathcal{C}_s}$$

exhibits $f \circ \prod_{s \in S} \Sigma^\infty_{\mathcal{C}_s}$ as a differential of the functor $\Sigma^\infty_+ \circ F$.

In this situation, we will sometimes abuse terminology by saying that any of the natural transformations β, γ, and δ *exhibits f as a derivative of F.*

Theorem 7.2.3.21 is a close relative of Corollary 1.4.4.5. In fact, it is possible to deduce Theorem 7.2.3.21 from Corollary 1.4.4.5, by replacing the ∞-categories $\mathcal{C}_s$ by $\mathrm{Ind}_\kappa(\mathcal{C}_s)$, where κ denotes the least uncountable cardinal. However, we will give a more direct proof which is based on the following lemma:

Lemma 7.2.3.26. *Let $\mathcal{C}$ be a pointed differentiable ∞-category which admits finite colimits, and let $\mathcal{X}$ be a stable subcategory of $\mathrm{Sp}(\mathcal{C})$ which contains the essential image of the functor $\Sigma^\infty_{\mathcal{C}} : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$. If $\mathcal{X}$ is closed under sequential colimits, then $\mathcal{X} = \mathrm{Sp}(\mathcal{C})$.*

Proof. Example 7.2.1.4 implies that the identity functor $\mathrm{id}_{\mathrm{Sp}(\mathcal{C})}$ is a derivative of the identity functor $\mathrm{id}_{\mathcal{C}}$. It follows that the counit map $\Sigma^\infty_{\mathcal{C}} \circ \Omega^\infty_{\mathcal{C}} \to \mathrm{id}_{\mathrm{Sp}(\mathcal{C})}$ exhibits $\mathrm{id}_{\mathrm{Sp}(\mathcal{C})}$ as the 1-excisive approximation to $\Sigma^\infty_{\mathcal{C}} \circ \Omega^\infty_{\mathcal{C}}$: that is, the identity functor $\mathrm{id}_{\mathrm{Sp}(\mathcal{C})}$ is given by the colimit

$$\varinjlim_n \Omega^n_{\mathrm{Sp}(\mathcal{C})} \circ \Sigma^\infty_{\mathcal{C}} \circ \Omega^\infty_{\mathcal{C}} \circ \Sigma^n_{\mathrm{Sp}(\mathcal{C})}.$$

In particular, for every spectrum object $X \in \mathrm{Sp}(\mathcal{C})$, we have a canonical equivalence

$$X \simeq \varinjlim_n \Omega^n_{\mathrm{Sp}(\mathcal{C})}(\Sigma^\infty_{\mathcal{C}} X(S^n)).$$

Since $\mathcal{X}$ contains the essential image of $\Sigma^\infty_{\mathcal{C}}$ and is closed under desuspension, it contains each of the objects $\Omega^n_{\mathrm{Sp}(\mathcal{C})}(\Sigma^\infty_{\mathcal{C}} X(S^n))$. Using the fact that that $\mathcal{X}$ is closed under sequential colimits, we deduce that $X \in \mathcal{X}$. $\square$

Proof of Theorem 7.2.3.21. Working separately in each argument, we are reduced to proving the following assertion:

($*$) Let $\mathcal{C}$ be a pointed differentiable ∞-category which admits finite colimits and $\mathcal{D}$ an arbitrary differentiable ∞-category. Then composition with the functor $\Sigma^\infty_{\mathcal{C}}$ induces an equivalence of ∞-categories

$$\mathrm{Exc}_\star(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \to \mathrm{Exc}_\star(\mathcal{C}, \mathcal{D}).$$

Let $\phi : \mathrm{Exc}_\star(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ denote the functor given by precomposition with $\Sigma^\infty_{\mathcal{C}}$, and write ϕ as a composition of functors

$$\mathrm{Exc}_\star(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \hookrightarrow \mathrm{Fun}_\star(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \overset{\circ \Sigma^\infty_{\mathcal{C}}}{\to} \mathrm{Fun}_\star(\mathcal{C}, \mathcal{D}).$$

Each of these functors admits a left adjoint; it follows that ϕ admits a left adjoint ψ, given by the composition

$$\mathrm{Exc}_\star(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \overset{P_1}{\leftarrow} \mathrm{Fun}_\star(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \overset{\circ \Omega^\infty_{\mathcal{C}}}{\leftarrow} \mathrm{Fun}_\star(\mathcal{C}, \mathcal{D}),$$

which is given by $F \mapsto \Omega^\infty_{\mathcal{D}} \circ \partial F$. Since the functor $\Sigma^\infty_{\mathcal{C}}$ is right exact, the functor ϕ factors through $\mathrm{Exc}_\star(\mathcal{C}, \mathcal{D})$. We therefore obtain an adjunction

$$\mathrm{Exc}_\star(\mathcal{C}, \mathcal{D}) \underset{\phi_0}{\overset{\psi_0}{\rightleftarrows}} \mathrm{Exc}_\star(\mathrm{Sp}(\mathcal{C}), \mathcal{D})$$

We wish to show that these functors are mutually inverse equivalences of categories. We begin by showing that the functor ψ_0 is fully faithful. Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced, excisive functor which commutes with sequential colimits; we wish to show that the unit map

$$u_F : F \to \phi_0 \psi_0 F = \Omega^\infty_{\mathcal{D}} \circ \partial F \circ \Sigma^\infty_{\mathcal{C}}$$

is an equivalence. Fix an object $C \in \mathcal{C}$; we wish to show that u_F induces an equivalence $u_F(C) : F(C) \to \Omega^\infty_{\mathcal{D}} \circ \partial F \circ \Sigma^\infty_{\mathcal{C}}(C)$. Let $L_{\mathcal{C}} : \mathrm{Fun}_*(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C}) \to \mathrm{Sp}(\mathcal{C})$ and $L_{\mathcal{D}} : \mathrm{Fun}_*(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{D}) \to \mathrm{Sp}(\mathcal{D})$ be left adjoints to the inclusion, and let $F^+ : \mathrm{Fun}_*(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C}) \to \mathrm{Fun}_*(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{D})$ be the functor given by pointwise composition with F, and let $\overline{\Sigma}^\infty_{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}_*(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})$ be a left adjoint to the functor given by evaluation on $S^0 \in \mathcal{S}^{\mathrm{fin}}_*$. Then $\Sigma^\infty_{\mathcal{C}} \simeq L_{\mathcal{C}} \circ \overline{\Sigma}^\infty_{\mathcal{C}}$. Applying Propositions 7.2.1.19 and 7.2.1.20, we obtain equivalences

$$\partial F(\Sigma^\infty_{\mathcal{C}} C) \simeq L_{\mathcal{D}} F^+(\Sigma^\infty_{\mathcal{C}} C) \simeq L_{\mathcal{D}} F^+(\overline{\Sigma}^\infty_{\mathcal{C}} C).$$

Combining this with Remark 7.2.1.13, we obtain an equivalence

$$(\Omega^\infty_{\mathcal{D}} \circ \partial F \circ \Sigma^\infty_{\mathcal{C}}) \simeq (\Omega^\infty_{\mathcal{D}} \circ L_{\mathcal{D}}) F^+(\overline{\Sigma}^\infty_{\mathcal{C}} C) \simeq \varinjlim_n \Omega^n_{\mathcal{D}}(F^+(\overline{\Sigma}^\infty_{\mathcal{C}} C)(S^n)) \simeq \varinjlim_n \Omega^n_{\mathcal{D}} F(\Sigma^n_{\mathcal{C}} C) \simeq (P_1 F)(C).$$

Under this equivalence, the $u_F(C)$ corresponds to the canonical map from $F(C)$ to $(P_1 F)(C)$, which is an equivalence by virtue of our assumption that F is excisive.

To complete the proof that ϕ_0 and ψ_0 are mutually inverse equivalences of ∞-categories, it will suffice to show that the functor ϕ_0 is conservative. Let $\beta : f \to g$ be a morphism in $\mathrm{Exc}_\star(\mathrm{Sym}(\mathcal{C}), \mathcal{D})$ which induces an equivalence $f \circ \Sigma^\infty_{\mathcal{C}} \to g \circ \Sigma^\infty_{\mathcal{C}}$; we wish to show that β is an equivalence. Using Proposition 1.4.2.22, we can replace $\mathcal{D}$ by $\mathrm{Sp}(\mathcal{D})$ and thereby reduce to the case where $\mathcal{D}$ is stable, so that f and g are exact functors. Let $\mathcal{X} \subseteq \mathrm{Sp}(\mathcal{C})$ denote the full subcategory spanned by those objects X for β induces an equivalence $\beta_X : f(X) \to g(X)$ in $\mathcal{D}$. Since f and g are exact functors which commute with sequential colimits, $\mathcal{X}$ is a stable subcategory of $\mathrm{Sp}(\mathcal{C})$ which is closed under sequential colimits. Since $\mathcal{X}$ contains the essential image of the functor $\Sigma^\infty_{\mathcal{C}}$, we conclude from Lemma 7.2.3.26 that $\mathcal{X} = \mathrm{Sp}(\mathcal{C})$. $\square$

In Example 7.2.1.4, we saw that the derivative of a left exact functor has a simple description. We conclude this section by establishing an analogue for right exact functors:

Proposition 7.2.3.27. *Let $\{\mathcal{C}_s\}_{s \in S}$ and $\mathcal{D}$ be pointed differentiable ∞-categories which admit finite colimits, and suppose we are given functors*

$$F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D} \qquad f : \prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s) \to \mathrm{Sp}(\mathcal{D})$$

which preserve countable colimits in each variable. Then a natural transformation $\delta : \Sigma_{\mathcal{D}}^{\infty} \circ F \to f \circ \prod_{s \in S} \Sigma_{\mathcal{C}_s}^{\infty}$ exhibits f as a derivative of F (see Remark 7.2.3.25) if and only if δ is an equivalence.

Proof. Note that the hypotheses guarantee that F is reduced. Let

$$F^+ : \prod_{s \in S} \mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}_s) \to \mathrm{Fun}_*(\prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D})$$

and $L_{\mathcal{D}}^S : \mathrm{Fun}_*(\prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D}) \to \mathrm{Sp}(\mathcal{D})$ be defined as in Construction 7.2.1.14, let $L_{\mathcal{C}_s} : \mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}_s) \to \mathrm{Sp}(\mathcal{C}_s)$ be left adjoints to the inclusion functors, and let $\overline{\Sigma}_{\mathcal{C}_s}^{\infty}$ be defined as in Lemma 7.2.3.19. Repeated application of Lemma 7.2.3.19 shows that the functor $\mathrm{Fun}(\prod_{s \in S} \mathcal{S}_*^{\mathrm{fin}}, \mathcal{D}) \to \mathcal{D}$ given by evaluation at $(S^0, \ldots, S^0)$ admits a left adjoint U. Moreover, we have a canonical equivalence $U \circ F \simeq F^+ \circ \prod_{s \in S} \overline{\Sigma}_{\mathcal{C}}^{\infty}$. Using Propositions 7.2.1.20 and 7.2.1.19, we obtain equivalences

$$
\begin{aligned}
\Sigma_{\mathcal{D}}^{\infty} \circ F &\simeq & L_{\mathcal{D}}^S \circ U \circ F \\
&\simeq & L_{\mathcal{D}}^S \circ F^+ \circ \prod_{s \in S} \overline{\Sigma}_{\mathcal{C}_s}^{\infty} \\
&\simeq & L_{\mathcal{D}}^S \circ F^+ \circ \prod_{s \in S} L_{\mathcal{C}_s} \circ \prod_{s \in S} \overline{\Sigma}_{\mathcal{C}_s}^{\infty} \\
&\simeq & L_{\mathcal{D}}^S \circ F^+ \circ \prod_{s \in S} \Sigma_{\mathcal{C}_s}^{\infty} \\
&\simeq & \vec{\partial} F \circ \prod_{s \in S} \Sigma_{\mathcal{C}_s}^{\infty}.
\end{aligned}
$$

This proves the "if" direction of our assertion.

Conversely, suppose that δ is an equivalence. Let $\vec{\partial} F$ be a derivative of F. Since $f \circ \Sigma_{\mathcal{C}}^{\infty}$ is excisive, Corollary 7.2.3.24 implies that δ factors as a composition

$$\Sigma_{\mathcal{D}}^{\infty} \circ F \xrightarrow{\delta'} \vec{\partial} F \circ \prod_{s \in S} \Sigma_{\mathcal{C}_s}^{\infty} \xrightarrow{\delta''} f \circ \prod_{s \in S} \Sigma_{\mathcal{C}_s}^{\infty},$$

where δ' exhibits $\vec{\partial} F$ as a derivative of F. The first part of the proof shows that δ' is an equivalence, so that δ'' is an equivalence by the two-out-of-three property. It follows that δ exhibits f as a derivative of F. $\square$

Example 7.2.3.28. Let $F : \mathcal{S}_*^n \to \mathcal{S}_*$ be the functor given by

$$F(X_1, \ldots, X_n) = \prod X_i.$$

For $n \geq 1$, the functor $\mathrm{cored}(F)$ is given by the iterated smash product

$$\mathrm{cored}(F)(X_1, \ldots, X_n) = X_1 \wedge \cdots \wedge X_n.$$

Since the suspension spectrum functor $\Sigma^{\infty} : \mathcal{S}_* \to \mathrm{Sp}$ is symmetric monoidal, we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{S}_*^n & \xrightarrow{\Sigma^{\infty}} & \mathrm{Sp}^n \\
\downarrow{\scriptstyle \wedge} & & \downarrow{\scriptstyle \otimes} \\
\mathcal{S}_* & \xrightarrow{\Sigma^{\infty}} & \mathrm{Sp}.
\end{array}
$$

Applying Proposition 7.2.3.27, we can identify the derivative $\vec{\partial}(F) \simeq \vec{\partial}(\mathrm{cored}(F))$ with the iterated smash product functor $\otimes : \mathrm{Sp}^n \to \mathrm{Sp}$.

Variant 7.2.3.29. Let $F : \mathcal{S}^n \to \mathcal{S}$ be the functor given by the Cartesian product. If $n \geq 1$, then we can use Remark 7.2.1.6, Remark 7.2.1.7, and Example 7.2.3.28 to identify the derivative $\vec{\partial}(F)$ with the iterated smash product functor $\otimes : \mathrm{Sp}^n \to \mathrm{Sp}$. This description is also correct in the case $n = 0$: Example 7.2.1.5 allows us to identify $\vec{\partial}(F)$ with the sphere spectrum $S = \Sigma_+^\infty(*)$ (which is the unit object for the smash product monoidal structure on Sp).

7.2.4 Generalized Smash Products

Let $\mathrm{Sp} = \mathrm{Sp}(\mathcal{S})$ denote the ∞-category of spectra. In §6.3.2, we showed that Sp admits a symmetric monoidal structure, with tensor product $\otimes : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ given by the classical smash product of spectra. Our goal in this section is to address the following question:

Question 7.2.4.1. Let $\mathcal{C}$ be an ∞-category which admits finite limits, and let $\mathrm{Sp}(\mathcal{C})$ denote the ∞-category of spectrum objects of $\mathcal{C}$. Can we equip the ∞-category $\mathrm{Sp}(\mathcal{C})$ with some sort of smash product operation, generalizing the classical smash product of spectra?

We will attempt to answer Question 7.2.4.1 using the calculus of functors (more precisely, using the theory of derivatives developed in §7.2.1). According to Variant 7.2.3.29, the smash product functor $\otimes : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ can be identified with the derivative of the Cartesian product functor $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$. This suggests the following generalization:

Definition 7.2.4.2. Let $\mathcal{C}$ be a differentiable ∞-category which admits finite colimits. We define $\otimes : \mathrm{Sp}(\mathcal{C}) \times \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C})$ to be the derivative of the Cartesian product functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

Let us now study the properties of the tensor product functor $\otimes : \mathrm{Sp}(\mathcal{C}) \times \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C})$. By construction, it is exact in each variable (in fact, it even preserves countable colimits separately in each variable). Moreover, it is evidently symmetric: that is, we have canonical equivalences $X \otimes Y \simeq Y \otimes X$, depending functorially on $X, Y \in \mathrm{Sp}(\mathcal{C})$. The matter of associativity is more subtle. Consider the functors

$$F, G : \mathrm{Sp}(\mathcal{C}) \times \mathrm{Sp}(\mathcal{C}) \times \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C}),$$

given by $F(X, Y, Z) = X \otimes (Y \otimes Z)$, $G(X, Y, Z) = (X \otimes Y) \otimes Z$. By construction, for every pair of spectrum objects $X, Y \in \mathrm{Sp}(\mathcal{C})$, we have a canonical map

$$\alpha_{X,Y} : \Omega_{\mathcal{C}}^\infty(X) \times \Omega_{\mathcal{C}}^\infty(Y) \to \Omega_{\mathcal{C}}^\infty(X \otimes Y),$$

depending functorially on X and Y. We therefore obtain maps

$$\beta_{X,Y,Z} : \Omega_{\mathcal{C}}^\infty(X) \times \Omega_{\mathcal{C}}^\infty(Y) \times \Omega_{\mathcal{C}}^\infty(Z) \xrightarrow{\alpha_{Y,Z}} \Omega_{\mathcal{C}}^\infty(X) \times \Omega_{\mathcal{C}}^\infty(Y \otimes Z) \xrightarrow{\alpha_{X,Y \otimes Z}} \Omega_{\mathcal{C}}^\infty F(X,Y,Z)$$

$$\gamma_{X,Y,Z} : \Omega_{\mathcal{C}}^\infty(X) \times \Omega_{\mathcal{C}}^\infty(Y) \times \Omega_{\mathcal{C}}^\infty(Z) \xrightarrow{\alpha_{X,Y}} \Omega_{\mathcal{C}}^\infty(X \otimes Y) \times \Omega_{\mathcal{C}}^\infty(Z) \xrightarrow{\alpha_{X \otimes Y, Z}} \Omega_{\mathcal{C}}^\infty G(X,Y,Z).$$

Let $\otimes^3 : \mathrm{Sp}(\mathcal{C}) \times \mathrm{Sp}(\mathcal{C}) \times \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C})$ denote the derivative of the three-fold Cartesian product functor $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Since the functors F and G are exact in each variable, β and γ determine natural transformations $F \xleftarrow{\beta'} \otimes^3 \xrightarrow{\gamma'} G$. In particular, for every triple of spectrum objects $X, Y, Z \in \mathrm{Sp}(\mathcal{C})$, we have maps

$$X \otimes (Y \otimes Z) \leftarrow \otimes^3\{X, Y, Z\} \to (X \otimes Y) \otimes Z.$$

In the special case where $\mathcal{C} = \mathcal{S}$, these maps are equivalences, and determine an associativity constraint $X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$. However, this depends crucially on special properties of the ∞-category of spaces (specifically, the fact that the formation of Cartesian products commutes with colimits in each variable). In general, the tensor product functor of Definition 7.2.4.2 is not associative; however, we can regard the functor $\otimes^3$ and the natural transformations β' and γ' as providing a weak form of the associative law. To discuss this type of structure more systematically, it will be convenient to introduce the following definition:

Definition 7.2.4.3. Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad. We will say that $\mathcal{O}^{\otimes}$ is *corepresentable* if the map p is a locally coCartesian fibration.

Remark 7.2.4.4. Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a corepresentable ∞-operad. For $n \geq 0$, the unique active morphism $\langle n \rangle \to \langle 1 \rangle$ in $\mathcal{F}\mathrm{in}_*$ induces a functor

$$\mathcal{O}^n \simeq \mathcal{O}^{\otimes}_{\langle n \rangle} \to \mathcal{O}^{\otimes}_{\langle 1 \rangle} = \mathcal{O},$$

which we will denote by $\{X_i\}_{1 \leq i \leq n} \mapsto \otimes^n \{X_i\}$. Since this construction is Σ_n-equivariant, it can be described more invariantly: for every finite set I, we obtain a tensor product functor $\otimes^I : \mathcal{O}^I \to \mathcal{O}$.

Remark 7.2.4.5. Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad. Then p is corepresentable if and only if it satisfies the following conditions:

(∗) For every finite collection of objects $\{X_i\}_{i \in I}$ of $\mathcal{O}$, there exists a object $Y \in \mathcal{O}$ and an operation $\phi \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y)$ which is universal in the following sense: for every object $Z \in \mathcal{O}$, composition with ϕ induces a homotopy equivalence $\mathrm{Map}_{\mathcal{O}}(Y, Z) \to \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Z)$. Here $Y = \otimes \{X_i\}_{i \in I}$, where $\otimes$ denotes the functor of Remark 7.2.4.4.

Remark 7.2.4.6. Let $\mathcal{O}^{\otimes}$ be a corepresentable ∞-operad. The 0-fold tensor product $\otimes^0 : \Delta^0 \simeq \mathcal{O}^0 \to \mathcal{O}$ can be identified with an object $E \in \mathcal{O}$. Note that $\mathcal{O}^{\otimes}$ is a unital ∞-operad (in the sense of Definition 2.3.1.1) if and only if E is an initial object of $\mathcal{O}$.

Remark 7.2.4.7. Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a corepresentable ∞-operad. Then the underlying ∞-category $\mathcal{O}$ is equipped with a tensor product operation $\otimes : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$, given by the functor $\otimes^2$ of Remark 7.2.4.4. The tensor product $\otimes$ is commutative (up to canonical equivalence), but is generally not associative. To every triple of objects $X, Y, Z \in \mathcal{O}$, the locally coCartesian fibration $\mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ associates a 3-fold tensor product $\otimes^3 \{X, Y, Z\}$, which is equipped with canonical maps

$$X \otimes (Y \otimes Z) \leftarrow \otimes^3 \{X, Y, Z\} \to (X \otimes Y) \otimes Z.$$

These maps are equivalences if $\mathcal{O}^{\otimes}$ is a nonunital symmetric monoidal ∞-cateogry, in general need not be.

Remark 7.2.4.8. Let $\mathcal{O}^{\otimes}$ be a corepresentable ∞-operad, and suppose we are given a map of finite sets $\alpha : I \to J$. Let $\{X_i\}_{i \in I}$ be a collection of objects of $\mathcal{O}$. For $j \in J$, set $I_j = \alpha^{-1}\{j\} \subseteq I$, set $Y_j = \otimes^{I_j} \{X_i\}_{i \in I_j}$, and set $Z = \otimes^J \{Y_j\}_{j \in J}$. We have canonical operations

$$\phi_j \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I_j}, Y_j) \qquad \psi \in \mathrm{Mul}_{\mathcal{O}}(\{Y_j\}_{j \in J}, Z).$$

Composing these, we obtain a point of $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Z)$, which is classified by a map $\otimes^I \{X_i\}_{i \in I} \to Z$. This construction is functorial in each X_i, and therefore defines a natural transformation of functors

$$v_{\alpha} : \otimes^I \to \otimes^J \circ \prod_{j \in J} \otimes^{I_j}$$

from $\mathcal{O}^I$ to $\mathcal{O}$. In particular, for objects $X, Y, Z \in \mathcal{O}$, we have canonical maps

$$\otimes^2(X, \otimes^2(Y, Z)) \leftarrow \otimes^3(X, Y, Z) \to \otimes^2(\otimes^2(X, Y), Z).$$

Example 7.2.4.9. Every symmetric monoidal ∞-category is a corepresentable ∞-operad. Conversely, a corepresentable ∞-operad $\mathcal{O}^{\otimes}$ is a symmetric monoidal ∞-category if and only if, for every map of finite sets $\alpha : I \to J$, the natural transformation

$$v_{\alpha} : \otimes^I \to \otimes^J \circ \prod_{j \in J} \otimes^{I_j}$$

of Remark 7.2.4.8 is an equivalence of functors from $\mathcal{O}^I$ to $\mathcal{O}$.

We now discuss the process of *stabilizing* a corepresentable ∞-operad.

Definition 7.2.4.10. Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad. We will say that $\mathcal{O}^{\otimes}$ is *stable* if the following conditions are satisfied:

(1) The ∞-operad $\mathcal{O}^{\otimes}$ is corepresentable (that is, p is a locally coCartesian fibration).

(2) The underlying ∞-category $\mathcal{O}$ is stable.

(3) For every finite set I, the tensor product functor $\otimes^{I} : \mathcal{O}^{I} \to \mathcal{O}$ is exact in each variable.

Definition 7.2.4.11. Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be an ∞-operad. We will say that $\mathcal{O}^{\otimes}$ is *differentiable* if the following conditions are satisfied:

(1) The ∞-operad $\mathcal{O}^{\otimes}$ is corepresentable (that is, p is a locally coCartesian fibration).

(2) The underlying ∞-category $\mathcal{O}$ is differentiable and admits finite colimits.

(3) For every finite set I, the tensor product functors $\otimes^{I} : \mathcal{O}^{I} \to \mathcal{O}$ preserves sequential colimits.

Definition 7.2.4.12. Let $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a differentiable ∞-operad. We will say that a map of ∞-operads $q : \overline{\mathcal{O}}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *exhibits* $\overline{\mathcal{O}}^{\otimes}$ *as a stabilization of* $\mathcal{O}^{\otimes}$ if the following conditions are satisfied:

(1) The ∞-operad $\overline{\mathcal{O}}^{\otimes}$ is stable.

(2) The underlying functor $\overline{\mathcal{O}} \to \mathcal{O}$ is left exact.

It follows from (2) and Corollary 1.4.2.23 that the functor $\overline{\mathcal{O}} \to \mathcal{O}$ factors (in an essentially unique way) as a composition $\overline{\mathcal{O}} \xrightarrow{e} \mathrm{Sp}(\mathcal{O}) \to \mathcal{O}$, where the functor e is exact.

(3) The functor $e : \overline{\mathcal{O}} \to \mathrm{Sp}(\mathcal{O})$ is an equivalence of ∞-categories; we let e^{-1} denote a homotopy inverse to e.

(4) For every finite set I, let $\otimes^{I} : \mathcal{O}^{I} \to \mathcal{O}$ and $\overline{\otimes}^{I} : \overline{\mathcal{O}}^{I} \to \overline{\mathcal{O}}$ be defined as in Remark 7.2.4.4. Then q induces a natural transformation

$$\otimes^{I} \circ (\Omega_{\mathcal{O}}^{\infty})^{I} \to \Omega_{\mathcal{O}}^{\infty} \circ (e \circ \overline{\otimes}^{I} \circ (e^{-1})^{I})$$

of functors from $\mathrm{Sp}(\mathcal{O})^{I}$ into $\mathcal{O}$ which exhibits $e \circ \overline{\otimes}^{I} \circ (e^{-1})^{I}$ as a derivative of the functor $\otimes^{I}$.

Example 7.2.4.13. Let $\mathcal{S}^{\times}$ denote the Cartesian symmetric monoidal ∞-category whose underlying ∞-category is $\mathcal{S}$. Then $\mathcal{S}^{\times}$ is an initial object in the ∞-category of commutative algebra objects of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ (see Example 6.3.1.19). In particular, there is an essentially unique symmetric monoidal functor $\mathcal{S}^{\times} \to \mathrm{Sp}^{\otimes}$, where the underlying map of ∞-categories is given by the suspension spectrum functor $\Sigma_{+}^{\infty} : \mathcal{S} \to \mathrm{Sp}$. The functor Σ_{+}^{∞} is left adjoint to the functor $\Omega^{\infty} : \mathrm{Sp} \to \mathcal{S}$. Applying Corollary 8.3.2.7, we see that Ω^{∞} underlies a map of ∞-operads $q : \mathrm{Sp}^{\otimes} \to \mathcal{S}^{\times}$. Invoking the analysis of Example 7.2.3.28 and Variant 7.2.3.29, we deduce that q exhibits $\mathrm{Sp}^{\otimes}$ as the stabilization of the (differentiable) ∞-operad $\mathcal{S}^{\times}$.

The fundamental properties of Definition 7.2.4.12 are summarized in the following pair of results, which we will prove (in a more general form) in §7.2.5 and §7.2.6:

Proposition 7.2.4.14. *Let* $\mathcal{C}^{\otimes}$ *be a differentiable ∞-operad. Then there exists a stable ∞-operad* $\overline{\mathcal{C}}^{\otimes}$ *and a map of ∞-operads* $q : \overline{\mathcal{C}}^{\otimes} \to \mathcal{C}^{\otimes}$ *which exhibits* $\overline{\mathcal{C}}^{\otimes}$ *as a stabilization of* $\mathcal{C}^{\otimes}$.

Proposition 7.2.4.15. *Let* $q : \overline{\mathcal{C}}^{\otimes} \to \mathcal{C}^{\otimes}$ *be a map of ∞-operads which exhibits the stable ∞-operad* $\overline{\mathcal{C}}^{\otimes}$ *as a stabilization of the differentiable ∞-operad* $\mathcal{C}^{\otimes}$, *and let* $\mathcal{O}^{\otimes}$ *be an arbitrary stable ∞-operad. Let* $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{ex}}(\mathcal{C})$ *denote the full subcategory of* $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ *spanned by those ∞-operad maps* $\mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ *for which the underlying functor* $\mathcal{O} \to \mathcal{C}$ *is left exact, and define* $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{ex}}(\overline{\mathcal{C}})$ *similarly. Then composition with q induces an equivalence of ∞-categories* $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{ex}}(\overline{\mathcal{C}}) \to \mathrm{Alg}_{\mathcal{O}}^{\mathrm{ex}}(\mathcal{C})$.

Remark 7.2.4.16. It follows from Proposition 7.2.4.15 that the stabilization of a differentiable ∞-operad $\mathcal{C}^{\otimes}$ is determined up to equivalence by $\mathcal{C}^{\otimes}$.

Example 7.2.4.17. Let $\mathcal{S}_*$ denote the ∞-category of pointed spaces, which we regard as endowed with a symmetric monoidal structure via the smash product (see Remark 6.3.2.14). This symmetric monoidal structure is encoded by a coCartesian fibration $\mathcal{S}_*^{\wedge} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. The ∞-operad $\mathcal{S}_*^{\wedge}$ is differentiable, and therefore admits a stabilization $q : \mathcal{C}^{\otimes} \to \mathcal{S}_*^{\wedge}$ by Proposition 7.2.4.14. The underlying ∞-category $\mathcal{C}$ of $\mathcal{C}^{\otimes}$ can be identified with $\mathrm{Sp}(\mathcal{S}_*) \simeq \mathrm{Sp}$. It follows from the analysis of Example 7.2.3.28 that for every *nonempty* finite set I, the associated tensor product functor $\otimes^I : \mathcal{C}^I \to \mathcal{C}$ is given by the iterated smash product of spectra. However, when $I = \emptyset$ the tensor product functor $\otimes^I : \Delta^0 \simeq \mathcal{C}^I \to \mathcal{C}$ is given by a zero object of $\mathcal{C}$ (rather than the sphere spectrum). In particular, we see that $\mathcal{C}^{\otimes}$ is a unital corepresentable ∞-operad (Remark 7.2.4.6), which is not a symmetric monoidal ∞-category. Informally speaking, the corepresentable ∞-operad $\mathcal{C}^{\otimes}$ can be obtained from $\mathrm{Sp}^{\otimes}$ by "killing the unit object".

Using Example 7.2.3.28, we also see that $\mathcal{C}^{\otimes}$ can be identified with the stabilization of the differentiable ∞-operad $\mathcal{S}_*^{\times}$; the identification is induced by composing the map $q : \mathcal{C}^{\otimes} \to \mathcal{S}_*^{\wedge}$ with a map of ∞-operads $\mathcal{S}_*^{\wedge} \to \mathcal{S}_*^{\times}$.

7.2.5 Stabilization of ∞-Operads

Let $\mathcal{C}^{\otimes}$ be a differentiable ∞-operad (see Definition 7.2.4.11). In §7.2.4, we introduced the notion of a *stabilization* of $\mathcal{C}^{\otimes}$. Our goal in this section is to prove Proposition 7.2.4.14, which asserts that $\mathcal{C}^{\otimes}$ admits a stabilization. The proof will proceed by means of an explicit construction, which is closely related to the stabilization construction given in §7.2.2. However, the present case is considerably more complicated, because we must consider functors of several variables. For later use, it will be convenient to introduce one other complication: we will treat not only the case of a single differentiable ∞-operad, but a *family* of differentiable ∞-operads.

Definition 7.2.5.1. Let $p : \mathcal{O}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a map of simplicial sets. We will say that p is a *local S-family of ∞-operads* if, for every n-simplex of S, the induced map $\Delta^n \times_S \mathcal{O}^{\otimes} \to \Delta^n \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a Δ^n-family of ∞-operads, in the sense of Definition 2.3.2.10. We will say that p is a *corepresentable local S-family of ∞-operads* it is a local S-family of ∞-operads, and the map p is a locally coCartesian fibration.

Warning 7.2.5.2. Let S be an ∞-category. If $p : \mathcal{O}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is an S-family of ∞-operads (in the sense of Definition 2.3.2.10), then p is a local S-family of ∞-operads. The converse is generally not true: Definition 7.2.5.1 does not require that p be a categorical fibration of simplicial sets. However, if every every equivalence in S is a degenerate edge (for example, if S is the nerve of a partially ordered set), then every local S-family of ∞-operads is an S-family of ∞-operads. This follows from the characterization of categorical fibrations supplied by Corollary T.2.4.6.5.

Remark 7.2.5.3. Let $p : \mathcal{O}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a local S-family of ∞-operads. We will generally abuse terminology by referring to $\mathcal{O}^{\otimes}$ as a local S-family of ∞-operads, if the map p is clear in context. For each vertex $s \in S$, we let $\mathcal{O}_s^{\otimes}$ denote the ∞-operad given by the fiber product $\mathcal{O}^{\otimes} \times_S \{s\}$, and $\mathcal{O}_s$ its underlying ∞-category.

Remark 7.2.5.4. Let $q : \mathcal{O}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a corepresentable local S-family of ∞-operads, and let $e : X \to Y$ be an edge of $\mathcal{O}^{\otimes}$ lying over an edge $e_0 : s \to t$ in S. The following conditions are equivalent:

(1) The edge e is locally q-coCartesian.

(2) For every inert morphism $e' : Y \to Z$ in $\mathcal{O}_t^{\otimes}$ and every 2-simplex

$$
\begin{array}{ccc}
 & Y & \\
 {\scriptstyle e}\nearrow & & \searrow{\scriptstyle e'} \\
X & \xrightarrow{\quad f \quad} & Z
\end{array}
$$
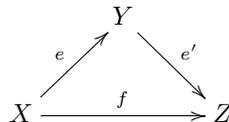
in $\mathcal{C}^{\otimes}$ lying over the degenerate 2-simplex

of S, f is a locally q-coCartesian edge of $\mathcal{O}^{\otimes}$.

(3) For every inert morphism $e' : Y \to Z$ in $\mathcal{O}_t^{\otimes}$ such that $Z \in \mathcal{O}_t$ and every 2-simplex
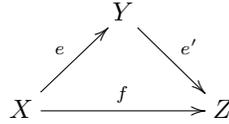
in $\mathcal{C}^{\otimes}$ lying over the degenerate 2-simplex
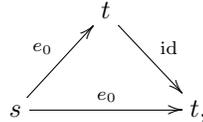
of S, f is a locally q-coCartesian edge of $\mathcal{O}^{\otimes}$.

Definition 7.2.5.5. Let $p : \mathcal{O}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a corepresentable local S-family of ∞-operads. We will say that p is *stable* (*differentiable*) if the following conditions are satisfied:

(a) For each vertex $s \in S$, the ∞-category $\mathcal{O}_s$ is stable (differentiable and admits finite colimits).

(b) For every edge $s \to t$ in S and every $n \geq 0$, the unique active morphism $\langle n \rangle \to \langle 1 \rangle$ induces a functor

$$\mathcal{O}_s^n \simeq \mathcal{O}_{(s,\langle n \rangle)}^{\otimes} \to \mathcal{O}_{(t,\langle 1 \rangle)}^{\otimes} \simeq \mathcal{O}_t$$

is exact in each variable (preserves sequential colimits).

Our goal is to describe a procedure for converting a differentiable local S-family of ∞-operads into a stable local S-family of ∞-operads. We first establish some terminology which will allow us to work *locally* on the simplicial set $S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

Definition 7.2.5.6. Let $\mathcal{C}$ be an ∞-category and let I be a finite set. An *I-decomposition* of $\mathcal{C}$ is a finite collection $\{W(i)\}_{i \in I}$, where each $W(i)$ is a collection of morphisms of $\mathcal{C}$, satisfying the following condition:

($*$) There exists an equivalence of ∞-categories $\mathcal{C} \to \prod_{i \in I} \mathcal{C}(i)$, where each $\mathcal{C}(i)$ is a weakly contractible ∞-category, such that for each $j \in I$, $W(j)$ is the collection of those morphisms f in $\mathcal{C}$ such that the image of f under the functor $\mathcal{C} \to \prod_{i \in I} \mathcal{C}(i) \to \mathcal{C}(j)$ is an equivalence in $\mathcal{C}(j)$.

Remark 7.2.5.7. Let $\mathcal{C}$ be an ∞-category equipped with an I-decomposition $\{W(i)\}_{i \in I}$. Then the equivalence $\mathcal{C} \to \prod_{i \in I} \mathcal{C}(i)$ appearing in condition ($*$) of Definition 7.2.5.6 is determined uniquely up to equivalence. For each $i \in I$, let $W'(i) = \bigcap_{j \neq i} W(j)$. Since $\mathcal{C}(j)$ is weakly contractible for $j \neq i$, we can identify $\mathcal{C}(i)$ with the ∞-category $\mathcal{C}[W'(i)^{-1}]$ obtained from $\mathcal{C}$ by formally inverting the morphisms belonging to $W'(i)$.

Definition 7.2.5.8. Let S be a simplicial set equipped with a map $\chi : S \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, and let $p : \mathcal{C} \to S$ be a locally coCartesian fibration. For each vertex $s \in S$, let $\chi(s) = I_*^s$, and let $\mathcal{C}_s$ denote the fiber of p over s. A *χ-decomposition* of $\mathcal{C}$ consists of an I^s-decomposition $\{W(i)^s\}_{i \in I^s}$ of each fiber $\mathcal{C}_s$, satisfying the following condition:

($*$) Let $e : s \to t$ be an edge of the simplicial set S, so that e induces a functor $e_! : \mathcal{C}_s \to \mathcal{C}_t$ be the induced functor and a map of pointed finite sets $\alpha : I_*^s \to I_*^t$. Then for each $j \in I^t$, the functor $e_!$ carries $\bigcap_{i \in \alpha^{-1}\{j\}} W(i)^s$ into $W(j)^t$.

Remark 7.2.5.9. Let $\chi : S \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a map of simplicial sets and let $p : \mathcal{C} \to S$ be a locally coCartesian fibration equipped with a χ-decomposition. For every vertex $s \in S$, the fiber $\mathcal{C}_s$ is equivalent to a product $\prod_{i \in I^s} \mathcal{C}_s(i)$, where $\chi(s) = I_*^s$. Moreover, every edge $e : s \to t$ determines a map of pointed finite sets $\alpha : I_*^s \to I_*^t$ and a functor $e_! : \prod_{i \in I^s} \mathcal{C}_s(i) \to \prod_{j \in I^t} \mathcal{C}_t(j)$, which factors as a composition

$$\prod_{i \in I^s} \mathcal{C}_s(i) \to \prod_{i \in \alpha^{-1} I^t} \mathcal{C}_s(i) \xrightarrow{\prod_{j \in I^t} F_j} \prod_{j \in I^t} \mathcal{C}_t(j),$$

for some functors $F_j : \prod_{i \in \alpha^{-1}\{j\}} \mathcal{C}_s(i) \to \mathcal{C}_t(j)$. Moreover, the functors F_j are uniquely determined up to equivalence.

Example 7.2.5.10. Let $p : \mathcal{O}^\otimes \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a corepresentable local S-family of ∞-operads, so that p induces a locally coCartesian fibration $q : \mathcal{O}^\otimes \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Let $\chi : S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be the projection onto the second factor. Assume that, for each vertex $s \in S$, the ∞-category $\mathcal{O}_s$ is weakly contractible. Then there is a canonical χ-decomposition on the locally coCartesian fibration q. For each vertex $(\langle n \rangle, s) \in \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times S$, we have a $\langle n \rangle^\circ$-decomposition $\{W(i)\}_{1 \leq i \leq n}$ of the fiber $\mathcal{O}_{\langle n \rangle, s}^\otimes$, where $W(i)$ is the collection of morphisms α in $\mathcal{O}_{\langle n \rangle, s}^\otimes$ such that $\phi(\alpha)$ is an equivalence, where $\phi : \mathcal{O}_{\langle n \rangle, s}^\otimes \to \mathcal{O}_s$ is the functor associated to the inert morphism $\rho^i : \langle n \rangle \to \langle 1 \rangle$ in the category $\mathcal{F}\mathrm{in}_*$.

Notation 7.2.5.11. Let $\chi : S \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a map of simplicial sets, which assigns to each vertex $s \in S$ a pointed finite set I_*^s. Suppose we are given locally coCartesian fibrations $p : \mathcal{C} \to S$ and $q : \mathcal{D} \to S$ equipped with χ-decompositions $\{W(i)_\mathcal{C}^s\}_{s \in S, i \in I^s}$ and $\{W(i)_\mathcal{D}^s\}_{s \in S, i \in I^s}$. We will say that a functor $U \in \mathrm{Fun}_S(\mathcal{C}, \mathcal{D})$ is *decomposition-compatible* if, for every vertex $s \in S$, the induced functor $U_s : \mathcal{C}_s \to \mathcal{D}_s$ carries $W(i)_\mathcal{C}^s$ into $W(i)_\mathcal{D}^s$, for each $i \in I^s$. We let $\mathrm{Fun}_\chi(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}_S(\mathcal{C}, \mathcal{D})$ spanned by those functors which are decomposition-compatible.

Remark 7.2.5.12. Let $\chi : S \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, $p : \mathcal{C} \to S$ and $q : \mathcal{D} \to S$ be as in Notation 7.2.5.11, so that the χ-decompositions of $\mathcal{C}$ and $\mathcal{D}$ determine equivalences

$$\mathcal{C}_s \simeq \prod_{i \in I^s} \mathcal{C}_s(i) \qquad \mathcal{D}_s \simeq \prod_{i \in I^s} \mathcal{D}_s(i)$$

for each vertex $s \in S$. If $U \in \mathrm{Fun}_S(\mathcal{C}, \mathcal{D})$ is decomposition-compatible, then the induced functor $U_s : \mathcal{C}_s \to \mathcal{D}_s$ can be identified with a product of functors $\{U_s(i) : \mathcal{C}_s(i) \to \mathcal{D}_s(i)\}_{i \in I^s}$ for $s \in S$.

Let $e : s \to t$ be an edge of S inducing a map of pointed finite sets $\alpha : I_*^s \to I_*^t$, so that the induced maps $\mathcal{C}_s \to \mathcal{C}_t$ and $\mathcal{D}_s \to \mathcal{D}_t$ determine functors

$$F_j : \prod_{i \in \alpha^{-1}\{j\}} \mathcal{C}_s(i) \to \mathcal{C}_t(j) \qquad G_j : \prod_{i \in \alpha^{-1}\{j\}} \mathcal{D}_s(i) \to \mathcal{D}_t(j)$$

for $j \in I^t$. For each $j \in I^t$, U determines a natural transformation

$$\beta_j : G_j \circ \prod_{i \in \alpha^{-1}\{j\}} U_s(i) \to U_t(j) \circ F_j$$

of functors from $\prod_{i \in \alpha^{-1}\{j\}} \mathcal{C}_s(i)$ to $\mathcal{D}_t(j)$.

Definition 7.2.5.13. Let $\chi : S \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a map of simplicial sets. Suppose we are given locally coCartesian fibrations $q : \overline{\mathcal{C}} \to S$ and $p : \mathcal{C} \to S$ equipped with χ-decompositions, and that each fiber $\mathcal{C}_s = \mathcal{C} \times_S \{s\}$ is an ∞-category which admits finite limits. We will say that a functor $U \in \mathrm{Fun}_S(\overline{\mathcal{C}}, \mathcal{C})$ *exhibits $\overline{\mathcal{C}}$ as a stabilization of p* if the following conditions are satisfied:

(a) The functor U is decomposition-compatible (Notation 7.2.5.11).

(b) For each $s \in S$, write $\chi(s) = I_*^s$ for some finite set I^s, so that the χ-decompositions of p and q determine equivalences

$$\mathcal{C}_s \simeq \prod_{i \in I^s} \mathcal{C}_s(i) \qquad \overline{\mathcal{C}}_s = \prod_{i \in I^s} \overline{\mathcal{C}}_s(i),$$

and U induces functors $U_s(i) :: \overline{\mathcal{C}}_s(i) \to \mathcal{C}_s(i)$. Then each $U_s(i)$ factors as a composition $\overline{\mathcal{C}}_s(i) \overset{V_s(i)}{\simeq}$ $\mathrm{Sp}(\mathcal{C}_s(i)) \overset{\Omega^\infty_{\overline{\mathcal{C}}_s(i)}}{\longrightarrow} \mathcal{C}_s(i)$. In particular, each $\overline{\mathcal{C}}_s(i)$ is a stable ∞-category, and each of the functors $U_s(i)$ is left exact.

(c) Let $e : s \to t$ be an edge of S, inducing a map of pointed finite sets $\alpha : I_*^s \to I_*^t$. Then e induces functors

$$F_j : \prod_{i \in \alpha^{-1}\{j\}} \mathcal{C}_s(i) \to \mathcal{C}_t(j) \qquad f_j : \prod_{i \in \alpha^{-1}\{j\}} \overline{\mathcal{C}}_s(i) \to \overline{\mathcal{C}}_t(j),$$

and U determines natural transformations

$$\beta_j : F_j \circ \prod_{i \in \alpha^{-1}\{j\}} U_s(i) \to U_t(j) \circ f_j$$

for $j \in I^t$. Then β_j exhibits the functors $V_t(j)^{-1} \circ f_j \circ \prod_{i \in \alpha^{-1}\{j\}} V_s(i)$ as derivatives of the functors F_j.

Remark 7.2.5.14. In the situation of Definition 7.2.5.13, suppose that $\chi : S \to \mathrm{N}(\mathfrak{Fin}_*)$ is the constant functor taking the value $\langle 1 \rangle$ and that $p : \mathcal{C} \to S$ is a locally differentiable fibration. Then a map $U : \overline{\mathcal{C}} \to \mathcal{C}$ exhibits $\overline{\mathcal{C}}$ as a stabilization of p in the sense of Definition 7.2.5.13 if and only if it exhibits $\overline{\mathcal{C}}$ as a stabilization of p in the sense of Definition 7.2.2.18.

Warning 7.2.5.15. Let S be a simplicial set equipped with a map $\chi : S \to \mathrm{N}(\mathfrak{Fin}_*)$, let $p : \mathcal{C} \to S$ be a locally differentiable fibration equipped with a χ-decomposition, and suppose that each fiber $\mathcal{C}_s$ admits finite colimits. Let $U : \overline{\mathcal{C}} \to \mathcal{C}$ be a map which exhibits $\overline{\mathcal{C}}$ as stabilization of p, and let $\overline{p} = p \circ U$. If p is a coCartesian fibration, it generally does not follow that $\overline{p}$ is a coCartesian fibration. To understand the issue, let us suppose for simplicity that $S = \Delta^2$, so that χ classifies a pair of maps $I_* \overset{\alpha}{\to} J_* \overset{\beta}{\to} K_*$ between pointed finite sets. Let us denote the fibers of p by $\mathcal{C}_0$, $\mathcal{C}_1$, and $\mathcal{C}_2$. Then the χ-decomposition of $\mathcal{C}$ determines equivalences

$$\mathcal{C}_0 \simeq \prod_{i \in I} \mathcal{C}(i) \qquad \mathcal{C}_1 \simeq \prod_{j \in J} \mathcal{C}(j) \qquad \mathcal{C}_2 \simeq \prod_{k \in K} \mathcal{C}(k),$$

together with functors

$$\left\{ F(j) : \prod_{\alpha(i)=j} \mathcal{C}(i) \to \mathcal{C}(j) \right\}_{j \in J} \qquad \left\{ G(k) : \prod_{\beta(j)=k} \mathcal{C}(j) \to \mathcal{C}(k) \right\}_{k \in K} \qquad \left\{ H(k) : \prod_{(\beta \circ \alpha)(i)=k} \mathcal{C}(i) \to \mathcal{C}(k) \right\}_{k \in K}$$

and natural transformations $\{\phi_k : H(k) \to G(k) \circ \prod_{\beta(j)=K} F(j)\}_{k \in K}$. The map p is a coCartesian fibration if and only if each of the natural transformations ϕ_k is an equivalence. However, the map $\overline{p}$ is a coCartesian fibration if and only if each of the composite maps

$$\vec{\partial} H(k) \to \vec{\partial}(G(k) \circ \prod_{\beta(j)=k} F(j)) \overset{\iota_k}{\to} \vec{\partial} G(k) \circ \prod_{\beta(j)=k} \vec{\partial} F(j).$$

Consequently, if p is a coCartesian fibration, then $\overline{p}$ is a coCartesian fibration if and only if each of the natural transformations ι_k is an equivalence. This is automatic in the following situations:

(a) The map α is inert, and each of the maps $F(j)$ is an equivalence.

(b) The map β is inert, and each of the maps $G(k)$ is an equivalence.

(c) The maps α and β are surjective, and the functors $F(j)$ and $G(k)$ are reduced in each variable (see Theorem 7.2.1.22).

Proposition 7.2.4.14 is an immediate consequence of the following pair of assertions:

Proposition 7.2.5.16. *Let $\chi : S \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a map of simplicial sets which carries each vertex $s \in S$ to a finite pointed set I^s_*. Let $p : \mathcal{C} \to S$ be a differentiable fibration equipped with a χ-decomposition, and suppose that each fiber $\mathcal{C}_s$ admits finite limits. Then there exists a locally coCartesian fibration $\overline{\mathcal{C}} \to S$ equipped with a χ-decomposition, and a map $U \in \mathrm{Fun}_S(\overline{\mathcal{C}}, \mathcal{C})$ which exhibits $\overline{\mathcal{C}}$ as a stabilization of $\mathcal{C}$.*

Proposition 7.2.5.17. *Let $p : \mathcal{O}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a differentiable S-family of ∞-operads. Let $\chi : S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be the projection onto the second factor, and regard $\mathcal{O}^{\otimes}$ as endowed with the χ-decomposition of Example 7.2.5.10. Suppose that $U : \overline{\mathcal{O}}^{\otimes} \to \mathcal{O}^{\otimes}$ exhibits $\overline{\mathcal{O}}^{\otimes}$ as a stabilization of $\mathcal{O}^{\otimes}$, in the sense of Definition 7.2.5.13. Then the underlying map $\overline{\mathcal{O}}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a stable local S-family of ∞-operads.*

We begin with the proof of Proposition 7.2.5.17, which is rather formal. We need the following general recognition principle:

Lemma 7.2.5.18. *Let $q : \mathcal{O}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a locally coCartesian fibration. Then q is a corepresentable local S-family of ∞-operads if and only if the following conditions are satisfied:*

(1) *Let $e : (s, \langle m \rangle) \to (t, \langle n \rangle)$ be an edge of $S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, suppose we are given an inert morphism $\alpha : \langle m' \rangle \to \langle m \rangle$, and let $\sigma : \Delta^2 \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be the 2-simplex corresponding to the diagram*

$$
\begin{array}{ccc}
 & (s, \langle m \rangle) & \\
{\scriptstyle (\mathrm{id}, \alpha)} \nearrow & & \searrow {\scriptstyle e} \\
(s, \langle m' \rangle) & \xrightarrow{\hspace{3cm}} & (t, \langle n \rangle).
\end{array}
$$
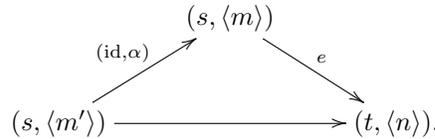
Then the projection map $\Delta^2 \times_{S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathcal{O}^{\otimes} \to \Delta^2$ is a coCartesian fibration.

(2) *Let $e : (s, \langle m \rangle) \to (t, \langle n \rangle)$ be an edge of $S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, let $\beta : \langle n \rangle \to \langle n' \rangle$ be an inert morphism, and let $\sigma : \Delta^2 \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be the 2-simplex corresponding to the diagram*

$$
\begin{array}{ccc}
 & (t, \langle n \rangle) & \\
{\scriptstyle e} \nearrow & & \searrow {\scriptstyle (\mathrm{id}, \beta)} \\
(s, \langle m \rangle) & \xrightarrow[\; e'_! \;]{} & (t, \langle n' \rangle).
\end{array}
$$
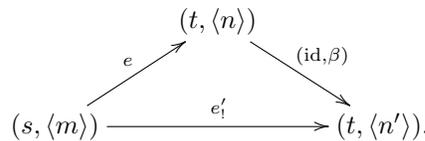
Then the projection map $\Delta^2 \times_{S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathcal{O}^{\otimes} \to \Delta^2$ is a coCartesian fibration.

(3) *For each vertex $s \in S$ and each $n \geq 0$, the collection of inert maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$ induce functors $\mathcal{O}^{\otimes}_{(s, \langle n \rangle)} \to \mathcal{O}^{\otimes}_{(s, \langle 1 \rangle)}$ which determine an equivalence $\mathcal{O}^{\otimes}_{(s, \langle n \rangle)} \simeq (\mathcal{O}^{\otimes}_{(s, \langle 1 \rangle)})^n$.*

Proof. We may assume without loss of generality that $S = \Delta^k$ for some integer k. Using Lemma T.2.4.2.7, we can reformulate (1) as follows:

(1') *For each vertex $s \in S$ and each locally q-coCartesian f in $\mathcal{O}^{\otimes}_s$ whose image in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is inert, f is q-coCartesian.*

Suppose that (3) is satisfied. We will show that (2) can be formulated as follows:

(2') Let t be a vertex of S, let $C \in \mathcal{O}_s^{\otimes}$ be an object, and suppose we are given a pair of locally q-coCartesian morphisms $C \to C'$ and $C \to C''$ in $\mathcal{O}_t^{\otimes}$ covering inert morphisms $\beta : \langle n \rangle \to \langle n' \rangle$ and $\gamma : \langle n \rangle \to \langle n'' \rangle$ which induce a bijection $\langle n' \rangle^{\circ} \coprod \langle n'' \rangle^{\circ} \to \langle n \rangle^{\circ}$. Then C is a q-product of C' with C''.

Note that $\mathcal{O}^{\otimes}$ is an S-family of ∞-operads if and only if it satisfies conditions (a), (b), and (c) of Definition 2.3.2.10. The desired result follows from the implications $(a) \Leftrightarrow (1')$, $(b) \Leftrightarrow (2')$, and $(a) + (b) + (c) \Rightarrow (3) \Rightarrow (c)$.

Assume now that (3) is satisfied; we will show that $(2) \Leftrightarrow (2')$. Fix a morphism $e : (s, \langle m \rangle) \to (t, \langle n \rangle)$ and an inert morphism $\beta : \langle n \rangle \to \langle n' \rangle$, and choose another inert morphism $\gamma : \langle n \rangle \to \langle n'' \rangle$ such that $\gamma^{-1} \langle n'' \rangle^{\circ}$ is the complement of $\beta^{-1} \langle n' \rangle^{\circ}$ in $\langle n \rangle^{\circ}$. Let $e' : (s, \langle m \rangle) \to (t, \langle n' \rangle)$ and $e'' : (s, \langle m \rangle) \to (t, \langle n'' \rangle)$ be the compositions of e with β and γ, respectively. Consider the induced functors

$$e_! : \mathcal{O}_{(s, \langle m \rangle)}^{\otimes} \to \mathcal{O}_{(t, \langle n \rangle)}^{\otimes} \qquad e'_! : \mathcal{O}_{(s, \langle m \rangle)}^{\otimes} \to \mathcal{O}_{(t, \langle n' \rangle)}^{\otimes} \qquad e''_! : \mathcal{O}_{(s, \langle m \rangle)}^{\otimes} \to \mathcal{O}_{(t, \langle n'' \rangle)}^{\otimes}$$

$$\beta_! : \mathcal{O}_{(t, \langle n \rangle)}^{\otimes} \to \mathcal{O}_{(t, \langle n' \rangle)}^{\otimes} \qquad \gamma_! : \mathcal{O}_{(t, \langle n \rangle)}^{\otimes} \to \mathcal{O}_{(t, \langle n'' \rangle)}^{\otimes},$$

so that we have natural transformations

$$e'_! \to \beta_! \circ e_! \qquad e''_! \to \gamma_! \circ e_!.$$

Condition (2) asserts that these natural transformations are equivalences. In other words, condition (2) is equivalent to the requirement that the induced map $e'_! \times e''_! \to (\beta_! \times \gamma_!) \circ e_!$ is an equivalence of functors from $\mathcal{O}_{(s, \langle m \rangle)}^{\otimes}$ to $\mathcal{O}_{(t, \langle n' \rangle)}^{\otimes} \times \mathcal{O}_{(t, \langle n'' \rangle)}^{\otimes}$. Condition (3) guarantees that $\beta_! \times \gamma_!$ is an equivalence of ∞-categories. Consequently, we can reformulate condition (2) as follows: for every pair (e, β) as above, for every object $X \in \mathcal{O}_{(s, \langle m \rangle)}^{\otimes}$ and for every object $Y \in \mathcal{O}_{(t, \langle n \rangle)}^{\otimes}$, the canonical map

$$\mathrm{Map}_{\mathcal{O}_{(t, \langle n \rangle}^{\otimes}} (e_! X, Y) \to \mathrm{Map}_{\mathcal{O}_{(t, \langle n' \rangle)}^{\otimes}} (e'_! X, \beta_! Y) \times \mathrm{Map}_{\mathcal{O}_{(t, \langle n'' \rangle)}^{\otimes}} (e''_! X, \gamma_! Y)$$

is a homotopy equivalence. If we regard Y and β as fixed, then this condition is satisfied for all pairs (e, X) if and only if the canonical maps $\beta_! Y \leftarrow Y \to \gamma_! Y$ exhibit Y as a q-product of $\beta_! Y$ with $\gamma_! Y$. It follows that $(2) \Leftrightarrow (2')$, as desired. $\qquad \square$

Proof of Proposition 7.2.5.17. Let $p : \mathcal{O}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a differentiable local S-family of ∞-operads, and let $\overline{\mathcal{O}}^{\otimes}$ be a stabilization of p (in the sense of Definition 7.2.5.13). We will show that $\overline{\mathcal{O}}^{\otimes}$ is a corepresentable local S-family of ∞-operads (the stability of $\overline{\mathcal{O}}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ will then follow immediately from the definition of a stabilization). For this, it will suffice to show that the underlying locally coCartesian fibration $q : \overline{\mathcal{O}}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ satisfies hypotheses (1), (2) and (3) of Lemma 7.2.5.18. Condition (3) follows immediately from Definition 7.2.5.13, and conditions (1) and (2) follow from the discussion in Warning 7.2.5.15. $\qquad \square$

We devote the remainder of this section to an explicit construction of the stabilizations whose existence is asserted by Proposition 7.2.5.16.

Notation 7.2.5.19. The ∞-category $\mathcal{S}_*^{\mathrm{fin}}$ of pointed finite spaces admits a symmetric monoidal structure given by the smash product of pointed spaces (see Notation 7.2.1.10), encoded by a coCartesian fibration $p : (\mathcal{S}_*^{\mathrm{fin}})^{\wedge} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. The coCartesian fibration p is equipped with a χ-decomposition, where χ denotes the identity map from $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ to itself (see Example 7.2.5.10).

Construction 7.2.5.20. Let $\chi : S \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a map of simplicial sets which carries each vertex $s \in S$ to a finite pointed set I_*^s. Let $p : \mathcal{C} \to S$ be a locally coCartesian fibration equipped with a χ-decomposition $\{W(i)^s\}_{s \in S, i \in I^s}$, and assume that each of the ∞-categories $\mathcal{C}_s$ admits finite limits.

We define a simplicial set $\mathrm{PStab}_{\chi}(p)$ equipped with a map $\mathrm{PStab}_{\chi}(p) \to S$ so that the following universal property is satisfied: for every map of simplicial sets $K \to S$, we have a bijection

$$\mathrm{Fun}_S(K, \mathrm{PStab}_{\chi}(p)) = \mathrm{Fun}_S(K \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} (\mathcal{S}_*^{\mathrm{fin}})^{\wedge}, \mathcal{C}).$$

For each vertex $s \in S$, we can identify the fiber $\mathrm{PStab}_\chi(p)_s$ with the ∞-category $\mathrm{Fun}((\mathcal{S}_*^{\mathrm{fin}})_{I_*^s}^\wedge, \mathcal{C}_s)$. If we are given an object of $\mathrm{PStab}_\chi(p)_s$ corresponding to a decomposition-compatible functor $F : (\mathcal{S}_*^{\mathrm{fin}})_{I_*^s}^\wedge \to \mathcal{C}_s$, then F factors as a composition

$$(\mathcal{S}_*^{\mathrm{fin}})_{I_*^s}^\wedge \simeq \prod_{i \in I^s} \mathcal{S}_*^{\mathrm{fin}} \xrightarrow{F_i} \prod_{i \in I^s} \mathcal{C}_s(i) \simeq \mathcal{C}_s$$

for some functors $F_i : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}_s(i)$ (here the ∞-categories $\mathcal{C}_s(i)$ are defined as in Remark 7.2.5.9). We let $\mathrm{Stab}_\chi(p)$ denote the full simplicial subset of $\mathrm{PStab}_\chi(p)$ spanned by those vertices which correspond to decomposition-compatible functors F for which each of the functors $F(i)$ is reduced and excisive.

Note that the unit $S^0 \in \mathcal{S}_*^{\mathrm{fin}}$ has the structure of a commutative algebra object of $\mathcal{S}_*^{\mathrm{fin}}$, and therefore determines a section of the coCartesian fibration $(\mathcal{S}_*^{\mathrm{fin}})^\wedge \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Composition with this section determines an evaluation map $\mathrm{PStab}_\chi(p) \to \mathcal{C}$, which restricts to a map $\Omega_\chi^\infty : \mathrm{Stab}_\chi(p) \to \mathcal{C}$.

Remark 7.2.5.21. In the special case where $\chi : S \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is the constant functor taking the value $\langle 1 \rangle$, Construction 7.2.5.20 reduces to Construction 7.2.2.2.

Let $\mathcal{C} \xrightarrow{p} S \xrightarrow{\chi} \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be as in Construction 7.2.5.20. Our next goal is to prove that, under some mild assumptions, the map $\Omega_\chi^\infty : \mathrm{Stab}_\chi(p) \to \mathcal{C}$ exhibits $\mathrm{Stab}_\chi(p)$ as a stabilization of $\mathcal{C}$. Here we will depart slightly from the exposition of §7.2.2: for the applications in §7.3, we need to treat the case of locally differentiable fibrations which are not reduced. In this case, the existence of the requisite derivatives requires some additional assumptions.

Remark 7.2.5.22. Let $\{\mathcal{C}_i\}_{i \in I}$ be a finite collection of weakly contractible ∞-categories. Then the product $\prod_{i \in I} \mathcal{C}_i$ is differentiable, pointed, or admits finite colimits if and only if each of the ∞-categories $\mathcal{C}_i$ has the same property. Consequently, if $\chi : S \to \mathcal{F}\mathrm{in}_*$ is a map of simplicial sets and $p : \mathcal{C} \to S$ is a locally coCartesian fibration equipped with a χ-decomposition, then p is a locally differentiable fibration if and only if the following conditions are satisfied:

(a) For each vertex $s \in S$, let $\chi(s) = I_*^s$ so that the χ-decomposition of p determines an equivalence $\mathcal{C}_s \simeq \prod_{i \in I^s} \mathcal{C}_s(i)$. Then each $\mathcal{C}_s(i)$ is a differentiable ∞-category.

(b) Let $s \to t$ be an edge of S inducing a map of pointed finite sets $\alpha : I^s \to I^t$. Then, for each $j \in I^t$, the induced functor

$$\prod_{i \in \alpha^{-1}\{j\}} \mathcal{C}_s(i) \to \mathcal{C}_t(j)$$

preserves sequential colimits separately in each variable.

Example 7.2.5.23. Let $S = \Delta^0$ and let $\chi : S \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be the map given by an object $I_* \in \mathcal{F}\mathrm{in}_*$, where I is some finite set. Any ∞-category $\mathcal{C}$ admits a unique locally coCartesian fibration $p : \mathcal{C} \to S$, and giving a χ-decomposition of $\mathcal{C}$ is equivalent to giving an I-decomposition of $\mathcal{C}$ in the sense of Definition 7.2.5.6. Such a decomposition determines an equivalence $\mathcal{C} \simeq \prod_{i \in I} \mathcal{C}(i)$. If $\mathcal{C}$ admits finite limits, we have a canonical equivalence of ∞-categories $\mathrm{Stab}_\chi(p) \simeq \prod_{i \in I} \mathrm{Sp}(\mathcal{C}(i)) \simeq \mathrm{Sp}(\mathcal{C})$.

Remark 7.2.5.24. Let $\chi : S \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a map of simplicial sets, and let $p : \mathcal{C} \to S$ be a locally coCartesian fibration equipped with a χ-decomposition. For every map of simplicial sets $\phi : T \to S$, let $p_T : \mathcal{C} \times_S T \to T$ be the induced locally coCartesian fibration, and observe that $\mathcal{C} \times_S T$ inherits a $(\chi \circ \phi)$-decomposition. Unwinding the definitions, we obtain a canonical isomorphism of simplicial sets $\mathrm{Stab}_{\chi \circ \phi}(p_T) \simeq \mathrm{Stab}_\chi(p) \times_S T$.

Proposition 7.2.5.16 is an immediate consequence of the following more precise assertion:

Theorem 7.2.5.25. *Let $\chi : S \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a map of simplicial sets which carries each vertex $s \in S$ to a finite pointed set I_*^s. Let $p : \mathcal{C} \to S$ be a differentiable fibration equipped with a χ-decomposition $\{W(i)^s\}_{s \in S, i \in I^s}$. Assume that, for each $s \in S$, the ∞-category $\mathcal{C}_s$ admits finite colimits. Then:*

(1) *The induced map $q : \mathrm{Stab}_\chi(p) \to S$ is a locally coCartesian fibration.*

(2) *For each $s \in S$ and $i \in I^s$, let $\overline{W}_s(i)$ be the collection of those morphisms $\alpha : F \to F'$ in $\mathrm{Stab}_\chi(p)_s$ such that, for each object $X \in (S_*^{\mathrm{fin}})^\wedge_{I_*^s}$, the induced map $F(X) \to F'(X)$ belongs to $W(i)^s$. Then the collection $\{\overline{W}_s(i)\}_{s \in S, i \in I^s}$ determines a χ-decomposition of $\mathrm{Stab}_\chi(p)$.*

(3) *Let $\Omega_\chi^\infty : \mathrm{Stab}_\chi(p) \to \mathcal{C}$ be as in Construction 7.2.5.20. Then Ω_χ^∞ exhibits $\mathrm{Stab}_\chi(p)$ as a stabilization of p, in the sense of Definition 7.2.5.13.*

Proof. We first prove that the map $\mathrm{Stab}_\chi(p) \to S$ is an inner fibration. Using Remark 7.2.5.24, we can reduce to the case where S is a simplex. In this case, Theorem T.2.4.6.1 implies that p is a categorical fibration. The projection map $(S_*^{\mathrm{fin}})^\wedge \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a coCartesian fibration and therefore a flat categorical fibration (Example B.3.11). Using Proposition B.4.5, we deduce that $\mathrm{PStab}_\chi(p) \to S$ is a categorical fibration, and in particular an inner fibration. Since $\mathrm{Stab}_\chi(q)$ is a full simplicial subset of $\mathrm{PStab}_\chi(p)$, we conclude that $\mathrm{Stab}_\chi(p) \to S$ is also an inner fibration.

We now complete the proof of (1) by showing that $q : \mathrm{Stab}_\chi(p) \to S$ is a locally coCartesian fibration. Fix an edge $e : s \to t$ in the simplicial set S and an object $F \in \mathrm{Stab}_\chi(p)_s$. We wish to show that there exists an object $G \in \mathrm{Stab}_\chi(p)_t$ and a locally q-coCartesian edge $\overline{e} : F \to G$ lifting e. Replacing $\mathcal{C}$ by the fiber product $\Delta^1 \times_S \mathcal{C}$, we may reduce to the case where $S = \Delta^1$. Evaluating χ on the edge e, we obtain a map of pointed finite sets $\alpha : I_*^s \to I_*^t$. The χ-decomposition of $\mathcal{C}$ determines equivalences

$$\mathcal{C}_s \simeq \prod_{i \in I^s} \mathcal{C}_s(i) \qquad \mathcal{C}_t \simeq \prod_{j \in I^t} \mathcal{C}_t(j).$$

Moreover, the edge e determines a functor $e_! : \mathcal{C}_s \to \mathcal{C}_t$, which is given as a product of functors $\{e_!(j) : \prod_{\alpha(i)=j} \mathcal{C}_s(i) \to \mathcal{C}_t(j)\}_{j \in I^t}$. Let us identify F with a sequence of reduced, excisive functors $\{F(i) : S_*^{\mathrm{fin}} \to \mathcal{C}_s(i)\}_{i \in I^s}$. Let H be any object of $\mathrm{Stab}_\chi(p)_t$, which we can identify with a sequence of reduced excisive functors $\{H(j) : S_*^{\mathrm{fin}} \to \mathcal{C}_t(j)\}_{j \in I^t}$. Unwinding the definitions, we can identify $\mathrm{Map}_{\mathrm{Stab}_\chi(p)}(F, H)$ with the product of the mapping spaces

$$\mathrm{Map}_{\mathrm{Fun}(\prod_{\alpha(i)=j} S_*^{\mathrm{fin}}, \mathcal{D}_t(j))}\left(e_!(j) \circ \prod_{\alpha(i)=j} F(i), H(j) \circ \wedge^{(j)}\right),$$

where j ranges over the set I^t and $\wedge^{(j)} : \prod_{\alpha(i)=j} S_*^{\mathrm{fin}} \to S_*^{\mathrm{fin}}$ denotes the iterated smash product functor.

To complete the proof of (1), we wish to show that there exists an object $G \in \mathrm{Stab}_\chi(p)_t$ and a morphism $\overline{e} : F \to G$ in $\mathrm{Stab}_\chi(p)$ such that composition with $\overline{e}$ induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Stab}_\chi(p)_t}(G, H) \to \mathrm{Map}_{\mathrm{Stab}_\chi(p)}(F, H)$$

for every object $H \in \mathrm{Stab}_\chi(p)_t$. To achieve this, we choose $\overline{e}$ to correspond to a sequence of natural transformations

$$\{\beta_j : e_!(j) \circ \prod_{\alpha(i)=j} F(i), G(j) \circ \wedge^{(j)}\}_{j \in I^t}$$

with the following properties:

(a) If $\alpha^{-1}\{j\}$ is nonempty, then β_j exhibits the functor $G(j) \circ \wedge^{(j)}$ as a differential of $e_!(j) \circ \prod_{\alpha(i)=j} F(i)$ (the existence of β_j follows from Propositions 7.2.3.13 and 7.2.1.11).

(b) If $\alpha^{-1}\{j\} = \emptyset$, then $e_!(j)$ determines an object $X \in \mathcal{C}_t(j)$. In this case, we choose $G(j)$ to correspond to the spectrum $\Sigma_+^\infty(X) \in \mathrm{Sp}(\mathcal{C}_t(j))$ (see Proposition 7.2.3.16) and β_j to correspond to the unit map $X \to \Omega_{\mathcal{C}_t}^\infty \Sigma_+^\infty X$.

Assertion (2) follows immediately from the construction, and assertion (3) follows from the construction together with the description of derivatives supplied by Remark 7.2.3.20 and Example 7.2.1.5. $\square$

7.2.6   Uniqueness of Stabilizations

Let $\mathcal{C}^{\otimes}$ be a differentiable ∞-operad. In the last section, we saw that there exists a map of ∞-operads $\overline{\mathcal{C}}^{\otimes} \to \mathcal{C}^{\otimes}$ which exhibits $\overline{\mathcal{C}}^{\otimes}$ as a stabilization of $\mathcal{C}^{\otimes}$. In this section, we will prove that $\overline{\mathcal{C}}^{\otimes}$ is uniquely determined up to equivalence. For this, it will suffice to show that $\overline{\mathcal{C}}^{\otimes}$ can be characterized by a universal property (Proposition 7.2.4.15). As in §7.2.5, it will be convenient to work in the more general setting of (local) ∞-operad families.

Definition 7.2.6.1. Let $p : \mathcal{O}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a local S-family of ∞-operads. We will say that $\mathcal{O}^{\otimes}$ is *right exact* if the following conditions are satisfied:

(a) The map p is a locally coCartesian fibration (that is, $\mathcal{O}^{\otimes}$ is a corepresentable local S-family of ∞-operads).

(b) For each vertex $s \in S$, the ∞-category $\mathcal{O}_s$ is pointed and admits finite colimits.

(c) Fpr each edge $s \to t$ in S and each $n \geq 1$, the unique active morphism $\langle n \rangle \to \langle 1 \rangle$ determines a functor

$$\mathcal{O}_s^n \simeq \mathcal{O}_{(s,\langle n \rangle)}^{\otimes} \to \mathcal{O}_{(t,\langle 1 \rangle)}^{\otimes} = \mathcal{O}_t$$

which is right exact in each variable.

Suppose that $\mathcal{O}^{\otimes}$ is a right exact local S-family of ∞-operads, and that $\mathcal{C}^{\otimes}$ is a differentiable local S-family of ∞-operads. We let $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{ex}}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}_{S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$ spanned by those maps $F : \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ with the following properties:

(i) For each $s \in S$, the restriction $F_s : \mathcal{O}_s^{\otimes} \to \mathcal{C}_s^{\otimes}$ of F is a map of ∞-operads.

(ii) For each $s \in S$, the underlying map of ∞-categories $\mathcal{O}_s \to \mathcal{C}_s$ is reduced and excisive.

Theorem 7.2.6.2. *Let $\mathcal{C}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a differentiable local S-family of ∞-operads, let $\overline{\mathcal{C}}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a stable local S-family of ∞-operads, and let $U : \overline{\mathcal{C}}^{\otimes} \to \mathcal{C}^{\otimes}$ exhibit $\overline{\mathcal{C}}^{\otimes}$ as a stabilization of $\mathcal{C}^{\otimes}$ (in the sense of Definition 7.2.5.13). Then, for every right exact local S-family of ∞-operads $\mathcal{O}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, composition with U induces an equivalence of ∞-categories $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{ex}}(\overline{\mathcal{C}}) \to \mathrm{Alg}_{\mathcal{O}}^{\mathrm{ex}}(\mathcal{C})$.*

Remark 7.2.6.3. Proposition 7.2.4.15 follows immediately from the implication (1) $\Rightarrow$ (3) of Theorem 7.2.6.2, applied in the case $S = \Delta^0$.

The proof of Theorem 7.2.6.2 involves a local analysis on each simplex of $S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. To carry out this analysis, it will be convenient to formulate a "local" version of Theorem 7.2.6.2.

Definition 7.2.6.4. Let $\chi : S \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a map of simplicial sets and let $p : \mathcal{C} \to S$ be a locally coCartesian fibration equipped with a χ-decomposition. We will say that the χ-decomposition of $\mathcal{C}$ is *right exact* if the following conditions are satisfied:

(a) Let $s \in S$, write $\chi(s) = I_*^s$, and let $\mathcal{C}_s \simeq \prod_{i \in I^s} \mathcal{C}_s(i)$ be the corresponding product decomposition of $\mathcal{C}_s$. Then each $\mathcal{C}_s(i)$ is a pointed ∞-category which admits finite colimits.

(b) Let $e : s \to t$ be an edge of S, so that the associated functor $e_! : \mathcal{C}_s \to \mathcal{C}_t$ is given by a product of functors

$$F_j : \prod_{\alpha(i)=j} \mathcal{C}_s(i) \to \mathcal{C}_t(j)$$

(see Remark 7.2.5.9). Then each of the functors F_j is right exact in each variable.

We will say that the χ-decomposition of $\mathcal{C}$ is *stable* if it satisfies (b) together with the following stronger version of (a):

(a') Let $s \in S$, write $\chi(s) = I_*^s$, and let $\mathcal{C}_s \simeq \prod_{i \in I^s} \mathcal{C}_s(i)$ be the corresponding product decomposition of $\mathcal{C}_s$. Then each factor $\mathcal{C}_s(i)$ is a stable ∞-category.

Notation 7.2.6.5. Let $\chi : S \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a map of simplicial sets, and suppose we are given locally coCartesian fibrations $p : \mathcal{C} \to S$ and $q : \mathcal{D} \to S$ equipped with χ-decompositions. For each $s \in S$, write $\chi(s) = I_*^s$. Assume that the χ-decomposition of $\mathcal{C}$ is right exact and that each fiber $\mathcal{D}_s$ of q admits finite limits. We let $\mathrm{Exc}_*^\chi(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}_S(\mathcal{C}, \mathcal{D})$ spanned by those functors $F : \mathcal{C} \to \mathcal{D}$ which are decomposition-compatible (Notation 7.2.5.11) and such that the induced map $\mathcal{C}_s(i) \to \mathcal{D}_s(i)$ is reduced and excisive for each $s \in S$ and each $i \in I^s$.

We can now formulate our main result.

Theorem 7.2.6.6. *Let $\chi : S \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a map of simplicial sets. Suppose we are given a locally differentiable fibration $p : \mathcal{C} \to S$ equipped with a χ-decomposition, and suppose that each fiber $\mathcal{C}_s$ of p admits finite colimits. Let $\overline{p} : \overline{\mathcal{C}} \to S$ be another locally coCartesian fibration equipped with a stable χ-decomposition, and let $U \in \mathrm{Exc}_*^\chi(\overline{\mathcal{C}}, \mathcal{C})$ (so that U induces a left exact functor $\overline{\mathcal{C}}_s \to \mathcal{C}_s$ for each $s \in S$). The following conditions are equivalent:*

(1) *The map U exhibits $\overline{\mathcal{C}}$ as a stabilization of p (in the sense of Definition 7.2.5.13).*

(2) *Let $q : \mathcal{D} \to S$ be a locally coCartesian fibration equipped with a right exact χ-decomposition. Then composition with U induces an equivalence of ∞-categories*

$$\mathrm{Exc}_*^\chi(\mathcal{D}, \overline{\mathcal{C}}) \to \mathrm{Exc}_*^\chi(\mathcal{D}, \mathcal{C}).$$

(3) *Let $p : \mathcal{D} \to S$ be a locally coCartesian fibration equipped with a stable χ-decomposition. Then composition with U induces an equivalence of ∞-categories*

$$\mathrm{Exc}_*^\chi(\mathcal{D}, \overline{\mathcal{C}}) \to \mathrm{Exc}_*^\chi(\mathcal{D}, \mathcal{C}).$$

We will give the proof of Theorem 7.2.6.6 at the end of this section.

Proof of Theorem 7.2.6.2. Let S be a simplicial set, let $\chi : S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ denote the projection onto the second factor. To deduce Theorem 7.2.6.2 from Theorem 7.2.6.6, it will suffice to verify the following:

($*$) Let $\mathcal{O}^\otimes \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a right exact local S-family of ∞-operads, let $\mathcal{C}^\otimes \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a differentiable local S-family of ∞-operads, and let $U : \overline{\mathcal{C}}^\otimes \to \mathcal{C}^\otimes$ exhibit $\overline{\mathcal{C}}^\otimes$ as a stabilization of $\mathcal{C}^\otimes$. Regard $\mathcal{O}^\otimes$, $\mathcal{C}^\otimes$, and $\overline{\mathcal{C}}^\otimes$ as endowed with the χ-decompositions described in Example 7.2.5.10. Then a map $F \in \mathrm{Exc}_*^\chi(\mathcal{O}^\otimes, \overline{\mathcal{C}}^\otimes)$ belongs to $\mathrm{Alg}_\mathcal{O}^{\mathrm{ex}}(\overline{\mathcal{C}})$ if and only if $U \circ F$ belongs to $\mathrm{Alg}_\mathcal{O}^{\mathrm{ex}}(\mathcal{C})$.

The "only if" direction is obvious. To prove the converse, assume that $U \circ F$ belongs to $\mathrm{Alg}_\mathcal{O}^{\mathrm{ex}}(\mathcal{C})$; we wish to show that $F \in \mathrm{Alg}_\mathcal{O}^{\mathrm{ex}}(\overline{\mathcal{C}})$. To prove this, it suffices to verify that for each vertex $s \in S$, the induced map $F_s : \mathcal{O}_s^\otimes \to \overline{\mathcal{C}}_s^\otimes$ is a map of ∞-operads. Let $\alpha : \langle m \rangle \to \langle n \rangle$ be a morphism in $\mathcal{F}\mathrm{in}_*$, so that α determines functors $\alpha_! : \mathcal{O}_{(s, \langle m \rangle)}^\otimes \to \mathcal{O}_{(s, \langle n \rangle)}^\otimes$ and $\alpha_!' : \overline{\mathcal{C}}_{(s, \langle m \rangle)}^\otimes \to \overline{\mathcal{C}}_{(s, \langle n \rangle)}^\otimes$, while F induces functors $F_m : \mathcal{O}_{(s, \langle m \rangle)}^\otimes \to \mathcal{C}_{(s, \langle m \rangle)}^\otimes$ and $F_n : \mathcal{O}_{(s, \langle n \rangle)}^\otimes \to \mathcal{C}_{(s, \langle n \rangle)}^\otimes$, together with a natural transformation $u_\alpha : \alpha_!' \circ F_m \to F_n \circ \alpha_!$. We wish to show that u_α is an equivalence whenever α is inert. By assumption, u_α induces an equivalence $U \circ \alpha_!' \circ F_m \to U \circ F_n \circ \alpha_!$ of functors from $\mathcal{O}_{(s, \langle m \rangle)}^\otimes$ to $\mathcal{C}_{(s, \langle n \rangle)}^\otimes$. Since U exhibits $\overline{\mathcal{C}}^\otimes$ as a stabilization of $\mathcal{C}^\otimes$, Proposition 1.4.2.22 implies that composition with U induces an equivalence of ∞-categories $\mathrm{Exc}_*(\mathcal{O}_{(s, \langle m \rangle)}^\otimes, \overline{\mathcal{C}}_{(s, \langle n \rangle)}^\otimes) \to \mathrm{Exc}_*(\mathcal{O}_{(s, \langle m \rangle)}^\otimes, \mathcal{C}_{(s, \langle n \rangle)}^\otimes)$. It will therefore suffice to show that the functors $\alpha_! \circ F_m$ and $F_n \circ \alpha_!$ are reduced and excisive (when viewed as functors of a single variable). This is clear: F_m and F_n are reduced and excisive (since $F \in \mathrm{Exc}_*^\chi(\mathcal{O}^\otimes, \overline{\mathcal{C}}^\otimes)$), the functor $\alpha_!$ is right exact (since α is inert), and the functor $\alpha_!'$ is left exact (again because α is inert). $\square$

The proof of Theorem 7.2.6.6 will require some preliminaries.

Lemma 7.2.6.7. *Let $\mathcal{C}$ be an ∞-category, let $n \geq 1$ be an integer, and suppose we are given a map of simplicial sets $f : \partial\Delta^n \to \mathcal{C}$ with $f(0) = X$ and $f(n) = Y$. Let $g : \mathcal{C} \to \mathcal{D}$ be a functor, and suppose that g induces a homotopy equivalence $\eta : \mathrm{Map}_{\mathcal{C}}(X,Y) \to \mathrm{Map}_{\mathcal{D}}(X,Y)$. Then g induces a homotopy equivalence of Kan complexes*

$$\theta : \mathrm{Fun}(\Delta^n, \mathcal{C}) \times_{\mathrm{Fun}(\partial\Delta^n, \mathcal{C})} \{f\} \to \mathrm{Fun}(\Delta^n, \mathcal{D}) \times_{\mathrm{Fun}(\partial\Delta^n, \mathcal{D})} \{g \circ f\}.$$

Proof. We proceed by induction on n. In the case $n = 1$, we can identify η with θ (see Corollary T.4.2.1.8) so there is nothing to prove. If $n > 1$, we can choose an integer $0 < i < n$. Let $f_0 = f|\Lambda_i^n$. We have a diagram of fiber sequences

$$
\begin{array}{ccc}
\mathrm{Fun}(\Delta^n, \mathcal{C}) \times_{\mathrm{Fun}(\partial\Delta^n, \mathcal{C})} \{f\} & \xrightarrow{\ \theta\ } & \mathrm{Fun}(\Delta^n, \mathcal{D}) \times_{\mathrm{Fun}(\partial\Delta^n, \mathcal{D})} \{g \circ f\} \\
\downarrow & & \downarrow \\
\mathrm{Fun}(\Delta^n, \mathcal{C}) \times_{\mathrm{Fun}(\Lambda_i^n, \mathcal{C})} \{f_0\} & \xrightarrow{\ \theta'\ } & \mathrm{Fun}(\Delta^n, \mathcal{D}) \times_{\mathrm{Fun}(\Lambda_i^n, \mathcal{D})} \{g \circ f_0\} \\
\downarrow & & \downarrow \\
\mathrm{Fun}(\partial\Delta^n, \mathcal{C}) \times_{\mathrm{Fun}(\Lambda_i^n, \mathcal{C})} \{f_0\} & \xrightarrow{\ \theta''\ } & \mathrm{Fun}(\partial\Delta^n, \mathcal{D}) \times_{\mathrm{Fun}(\partial\Delta^n, \mathcal{D})} \{g \circ f_0\}.
\end{array}
$$

Since the inclusion Λ_i^n is inner anodyne, the domain and codomain of θ' are contractible. We are therefore reduced to proving that θ'' is a homotopy equivalence. Let f_1 denote the restriction of f to the face of Δ^n opposite the ith vertex. Then θ'' is a (homotopy) pullback of the map

$$\theta : \mathrm{Fun}(\Delta^{n-1}, \mathcal{C}) \times_{\mathrm{Fun}(\partial\Delta^{n-1}, \mathcal{C})} \{f_1\} \to \mathrm{Fun}(\Delta^{n-1}, \mathcal{D}) \times_{\mathrm{Fun}(\partial\Delta^{n-1}, \mathcal{D})} \{g \circ f_1\},$$

and therefore a homotopy equivalence by the inductive hypothesis. $\qquad\square$

Lemma 7.2.6.8. *Let $n \geq 0$ be an integer, and let Y be the simplicial subset of $\Delta^n \times \Delta^1$ given by the union of $\partial\Delta^n \times \Delta^1$ and $\Delta^n \times \partial\Delta^1$. Then the inclusion $Y \hookrightarrow \Delta^n \times \Delta^1$ factors as a composition $Y \overset{i}{\hookrightarrow} X \overset{j}{\hookrightarrow} \Delta^n \times \Delta^1$, where i is inner anodyne and j fits into a pushout diagram*

$$
\begin{array}{ccc}
\partial\Delta^{n+1} & \longrightarrow & \Delta^{n+1} \\
\downarrow & & \downarrow{\scriptstyle \sigma} \\
X & \xrightarrow{\ j\ } & \Delta^n \times \Delta^1
\end{array}
$$

with $\sigma(0) = (0,0)$ and $\sigma(n+1) = (n,1)$.

Proof. Use the filtration described in the proof of Proposition T.2.1.2.6. $\qquad\square$

Lemma 7.2.6.9. *Let $C = (\Delta^1)^n$ denote a cube of dimension n, let $v = (0, \ldots, 0)$ be the initial vertex of C, and let $w = (1, \ldots, 1)$ be the final vertex of C. Then the inclusion $\partial C \hookrightarrow C$ factors as a composition*

$$\partial C \overset{i}{\hookrightarrow} X \overset{j}{\hookrightarrow} C,$$

where i is inner anodyne and j fits into a pushout diagram

$$
\begin{array}{ccc}
\partial\Delta^n & \longrightarrow & \Delta^n \\
\downarrow & & \downarrow{\scriptstyle \sigma} \\
X & \xrightarrow{\ j\ } & C
\end{array}
$$

with $\sigma(0) = v$ and $\sigma(n) = w$.

Proof. We proceed by induction on n, the case $n = 0$ being obvious. If $n > 0$, set $C' = (\Delta^1)^{n-1}$ and use the inductive hypothesis to factor the inclusion $\partial\, C' \hookrightarrow C'$ as a composition $\partial\, C' \xhookrightarrow{i'} X' \xhookrightarrow{j'} C'$. Write $C = C' \times \Delta^1$, and let Y be the simplicial subset of C given by the union of $X' \times \Delta^1$ and $C' \times (\partial\, \Delta^1)$. Then the inclusion $\partial\, C \hookrightarrow Y$ is inner anodyne (Corollary T.2.3.2.4). The inclusion $Y \hookrightarrow C$ is a pushout of the inclusion

$$(\partial\, \Delta^{n-1} \times \Delta^1) \amalg_{\partial\, \Delta^{n-1} \times \partial\, \Delta^1} (\Delta^{n-1} \times \partial\, \Delta^1) \hookrightarrow \Delta^{n-1} \times \Delta^1.$$

We now conclude by applying Lemma 7.2.6.8. $\square$

Lemma 7.2.6.10. *Let $C = (\Delta^1)^n$ denote a cube of dimension $n > 0$, let $\mathcal{C}$ be an ∞-category, and let $f : \partial\, C \to \mathcal{C}$ be a functor carrying the initial vertex of C to an object $X \in \mathcal{C}$ and the final vertex of C to an object $Y \in \mathcal{C}$. Suppose that $g : \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories which induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathcal{D}}(g(X), g(Y))$. Then g induces a homotopy equivalence of Kan complexes*

$$\mathrm{Fun}(C, \mathcal{C}) \times_{\mathrm{Fun}(\partial\, C, \mathcal{C})} \{f\} \to \mathrm{Fun}(C, \mathcal{D}) \times_{\mathrm{Fun}(\partial\, C, \mathcal{D})} \{g \circ f\}.$$

Proof. Combine Lemmas 7.2.6.7 and 7.2.6.9. $\square$

Lemma 7.2.6.11. *Let I be a finite set. Suppose we are given differentiable ∞-categories $\{\mathcal{D}_i\}_{i \in I}$ and $\mathcal{E}$ which admit finite colimits, a pair of functors*

$$F : \prod_{i \in I} \mathcal{D}_i \to \mathcal{E} \qquad f : \prod_{i \in I} \mathrm{Sp}(\mathcal{D}_i) \to \mathrm{Sp}(\mathcal{E}),$$

and a natural transformation $\epsilon : F \circ \prod_{i \in I} \Omega^{\infty}_{\mathcal{D}_i} \to \Omega^{\infty}_{\mathcal{E}} \circ f$ which exhibits f as a derivative of F. Suppose further that we are given a collection of pointed ∞-categories $\{\mathcal{C}_i\}_{i \in I}$ which admit finite colimits, together with functors

$$\{G_i : \mathcal{C}_i \to \mathrm{Sp}(\mathcal{D}_i)\}_{i \in I} \qquad H : \prod_{i \in I} \mathcal{C}_i \to \mathrm{Sp}(\mathcal{E})$$

where each G_i is right exact and H is right exact in each variable. Then composition with ϵ induces a homotopy equivalence

$$\theta : \mathrm{Map}_{\mathrm{Fun}(\prod_{i \in I} \mathcal{C}_i, \mathrm{Sp}(\mathcal{E}))}(f \circ \prod_{i \in I} G_j, H) \to \mathrm{Map}_{\mathrm{Fun}(\prod_{i \in I} \mathcal{C}_i, \mathcal{E})}(F \circ \prod_{i \in I}(\Omega^{\infty}_{\mathcal{D}_i} \circ G_j), \Omega^{\infty}_{\mathcal{E}} \circ H).$$

Proof. We will assume that the set I is empty (otherwise the statement is a tautology). Since the functors H and $f \circ \prod_{i \in I} G_i$ are multilinear, Proposition 1.4.2.22 implies that composition with $\Omega^{\infty}_{\mathcal{E}}$ induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\prod_{i \in I} \mathcal{C}_i, \mathrm{Sp}(\mathcal{E}))}(f \circ \prod_{i \in I} G_j, H) \to \mathrm{Map}_{\mathrm{Fun}(\prod_{i \in I} \mathcal{C}_i, \mathcal{E})}(\Omega^{\infty}_{\mathcal{E}} \circ f \circ \prod_{i \in I} G_i, \Omega^{\infty}_{\mathcal{E}} \circ H).$$

It will therefore suffice to show that ϵ induces a natural transformation $\epsilon' : F \circ \prod_{i \in I} \Omega^{\infty}_{\mathcal{D}_i} \circ \prod_{i \in I} G_i \to \Omega^{\infty}_{\mathcal{E}} \circ f \circ \prod_{i \in I} G_j$ which exhibits $\Omega^{\infty}_{\mathcal{E}} \circ f \circ \prod_{i \in I} G_j$ as a differential of $F \circ \prod_{i \in I} \Omega^{\infty}_{\mathcal{D}_i} \circ \prod_{i \in I} G_j$. This follows from Remark 7.2.3.4 together with Proposition 7.2.3.15. $\square$

Proof of Theorem 7.2.6.6. The implication $(2) \Rightarrow (3)$ is obvious. Assume that $(1) \Rightarrow (2)$ for the moment; we will show that $(3) \Rightarrow (1)$. Let us regard $p : \mathcal{C} \to S$ as fixed. It is clear that if there exists a diagram

$$\begin{array}{ccc} \overline{\mathcal{C}} & \xrightarrow{\ \ U\ \ } & \mathcal{C} \\ & \searrow^{\overline{p}} \quad \swarrow_{p} & \\ & S & \end{array}$$

where $\overline{q}$ is a locally coCartesian fibration equipped with a stable χ-decomposition satisfying condition (3), then $\overline{\mathcal{C}}$ is well-defined up to equivalence fiberwise over S. Consequently, to prove that $(3) \Rightarrow (1)$, it will suffice to exhibit such a diagram which having the additional property that U exhibits $\overline{\mathcal{C}}$ as a stabilization of $\mathcal{C}$. Since $(1) \Rightarrow (2) \Rightarrow (3)$, it suffices to show that p admits a stabilization, which follows from Theorem 7.2.5.25.

It remains to prove that $(1) \Rightarrow (2)$. Assume that $U : \overline{\mathcal{C}} \to \mathcal{C}$ exhibits $\overline{\mathcal{C}}$ as a stabilization of p, and let $q : \mathcal{D} \to S$ be a locally coCartesian fibration equipped with a right exact χ-decomposition. We wish to prove that composition with U induces an equivalence of ∞-categories

$$\mathrm{Exc}_*^\chi(\mathcal{D}, \overline{\mathcal{C}}) \to \mathrm{Exc}_*^\chi(\mathcal{D}, \mathcal{C}).$$

For this, it suffices to show that for every simplicial set K, the induced map $\mathrm{Fun}(K, \mathrm{Exc}_*^\chi(\mathcal{D}, \overline{\mathcal{C}})) \to \mathrm{Fun}(K, \mathrm{Exc}_*^\chi(\mathcal{D}, \mathcal{C}))$ restricts to a homotopy equivalence $\mathrm{Fun}(K, \mathrm{Exc}_*^\chi(\mathcal{D}, \overline{\mathcal{C}}))^\simeq \to \mathrm{Fun}(K, \mathrm{Exc}_*^\chi(\mathcal{D}, \mathcal{C}))^\simeq$ between the underlying Kan complexes. Replacing $\mathcal{C}$ by $\mathrm{Fun}(K, \mathcal{C}) \times_{\mathrm{Fun}(K,S)} S$ and $\overline{\mathcal{C}}$ by $\mathrm{Fun}(K, \overline{\mathcal{C}}) \times_{\mathrm{Fun}(K,S)} S$, we are reduced to proving that the map $\theta : \mathrm{Exc}_*^\chi(\mathcal{D}, \overline{\mathcal{C}})^\simeq \to \mathrm{Exc}_*^\chi(\mathcal{D}, \mathcal{C})^\simeq$ is a homotopy equivalence of Kan complexes.

For every map of simplicial sets $T \to S$, let $\mathcal{D}_T = T \times_S \mathcal{D}$, $\mathcal{C}_T = T \times_S \mathcal{C}$, and $\overline{\mathcal{C}}_T = T \times_S \overline{\mathcal{C}}$. Set

$$X_T = \mathrm{Exc}_*^{\chi|T}(\mathcal{D}_T, \overline{\mathcal{C}}_T)^\simeq \qquad Y_T = \mathrm{Exc}_*^{\chi|T}(\mathcal{D}_T, \mathcal{C}_T)^\simeq.$$

Composition with the map U induces a map of Kan complexes $\theta_T : X_T \to Y_T$. We will prove that θ_T is a homotopy equivalence of Kan complexes, for every map $T \to S$. Write T as a union of its skeleta

$$\emptyset = \mathrm{sk}^{-1} T \subseteq \mathrm{sk}^0 T \subseteq \mathrm{sk}^1 T \subseteq \cdots.$$

Then θ_T is a homotopy limit of the tower of maps $\{\theta_{\mathrm{sk}^i T}\}$. It will therefore suffice to prove that each of the morphisms $\theta_{\mathrm{sk}^n T}$ is a homotopy equivalence. We may therefore assume without loss of generality that T has dimension $\leq n$, for some integer n. We proceed by induction on n, the case $n = -1$ being trivial. Assume $n \geq 0$ and let A denote the set of n-simplices of T, so that we have a pushout diagram of simplicial sets

$$
\begin{array}{ccc}
A \times \partial\,\Delta^n & \longrightarrow & A \times \Delta^n \\
\downarrow & & \downarrow \\
\mathrm{sk}^{n-1} T & \longrightarrow & T,
\end{array}
$$

which determines a homotopy pullback diagram

$$
\begin{array}{ccc}
\theta_{A \times \partial\,\Delta^n} & \longleftarrow & \theta_{A \times \Delta^n} \\
\uparrow & & \uparrow \\
\theta_{\mathrm{sk}^{n-1} T} & \longleftarrow & \theta_T.
\end{array}
$$

Using the inductive hypothesis, we deduce that $\theta_{A \times \partial\,\Delta^n}$ and $\theta_{\mathrm{sk}^{n-1} T}$ are homotopy equivalences. Consequently, to prove that θ_T is a homotopy equivalence, it will suffice to show that $\theta_{A \times \Delta^n}$ is a homotopy equivalence. Note that $\theta_{A \times \Delta^n}$ is a product of the functors $\theta_{\{a\} \times \Delta^n}$ (where the product is taken over the elements $a \in A$). It will therefore suffice to show that each of the morphisms $\theta_{\{a\} \times \Delta^n}$ is a homotopy equivalence. Replacing S by $\{a\} \times \Delta^n$, we may reduce to the case where S is an n-simplex for some $n \geq 0$.

We first treat the case where $n = 0$, so that S consists of a single vertex. The functor χ carries this vertex to a pointed finite set I_*, and the χ-decompositions of $\mathcal{C}$, $\overline{\mathcal{C}}$, and $\mathcal{D}$ give equivalences

$$\mathcal{C} \simeq \prod_{i \in I} \mathcal{C}(i) \qquad \overline{\mathcal{C}} = \prod_{i \in I} \overline{\mathcal{C}}(i) \qquad \mathcal{D} \simeq \prod_{i \in I} \mathcal{D}(i)$$

Unwinding the definitions, we can identify θ_S with the product of the maps

$$\mathrm{Exc}_*(\mathcal{D}(i), \overline{\mathcal{C}}(i))^{\simeq} \to \mathrm{Exc}_*(\mathcal{D}(i), \mathcal{C}(i))^{\simeq}.$$

Since U satisfies condition (1), each of these functors is a homotopy equivalence by Proposition 1.4.2.22.

We now treat the case $n > 0$. For $0 \leq i \leq n$, write $\chi(i) = J_*^i$ for some finite set J^i. Since $p : \mathcal{D} \to \Delta^n$ is a locally coCartesian fibration, Proposition 7.2.2.28 guarantees the existence of a simplicial functor $\mathcal{F} : \mathfrak{C}[\Delta^n] \to \mathrm{Set}_\Delta$ and a map $u : M(\mathcal{F}) \to \mathcal{D}$ which induces categorical equivalences $\mathcal{F}(i) \to \mathcal{D} \times_{\Delta^n} \{i\}$ for $0 \leq i \leq n$. We may assume without loss of generality that $\mathcal{F}$ is a fibrant diagram, so that each $\mathcal{F}(i)$ is an ∞-category. Let $\mathcal{D}' = M(\mathcal{F})$. For every simplicial subset $T \subseteq \Delta^n$, let $\mathcal{D}'_T = \mathcal{D}' \times_{\Delta^n} T$, so that u induces a map $\mathcal{D}'_T \to \mathcal{D} \times_{\Delta^n} T$ (which is a categorical equivalence of ∞-categories if T is a simplex). Let $X'(T)$ denote the essential image of $X(T)$ in $\mathrm{Fun}_T(\mathcal{D}'(T), \overline{\mathcal{C}}_T)^{\simeq}$ and let $Y'(T)$ denote the essential image of $Y(T)$ in $\mathrm{Fun}_T(\mathcal{D}'(T), \mathcal{C}_T)^{\simeq}$. The evident maps $X(T) \to X'(T)$ and $Y(T) \to Y'(T)$ are homotopy equivalences. It follows from the inductive hypothesis that the canonical map $X'(T) \to Y'(T)$ is a homotopy equivalence for every proper subset $T \subseteq \Delta^n$, and we wish to show that $X'(T) \to Y'(T)$ is an equivalence when $T = \Delta^n$. For this, it suffices to show that the diagram of spaces σ :

$$
\begin{array}{ccc}
X'(\Delta^n) & \longrightarrow & Y'(\Delta^n) \\
\downarrow & & \downarrow \\
X'(\partial \Delta^n) & \longrightarrow & Y'(\partial \Delta^n)
\end{array}
$$

is a homotopy pullback square.

Let $\mathcal{X}$ denote the full subcategory of $\mathrm{Fun}(\mathcal{F}(0), \mathcal{C})$ spanned by those functors F which satisfy the following condition:

($*$) There exists a vertex $i \in \Delta^n$ such that F factors through $\mathcal{C}_i$ (this condition is actually automatic, since the ∞-category $\mathcal{C}_i$ is pointed). Let $\alpha : J_*^0 \to J_*^i$ denote the map of pointed finite sets determined by χ. The χ-decompositions of $\overline{\mathcal{C}}$ and $\mathcal{D}$ determine equivalences of ∞-categories

$$\mathcal{D}_0 \simeq \prod_{j \in J^0} \mathcal{D}_0(j) \qquad \overline{\mathcal{C}}_i = \prod_{j' \in J^i} \overline{\mathcal{C}}_i(j').$$

Then F is equivalent to a composition

$$\mathcal{F}(0) \simeq \prod_{j \in J^0} \mathcal{D}_0(j) \xrightarrow{\prod_{j' \in J^i} F_{j'}} \prod_{j' \in J^i} \overline{\mathcal{C}}_i(k) \simeq \overline{\mathcal{C}}_i$$

where each of the functors $F_k : \prod_{\alpha(j)=k} \mathcal{D}_0(j) \to \overline{\mathcal{C}}_i$ is reduced and excisive in each variable.

Define $\overline{\mathcal{X}} \subseteq \mathrm{Fun}(\mathcal{F}(0), \overline{\mathcal{C}})$ similarly. Let $C = (\Delta^1)^n$ denote a cube of dimension n. Unwinding the definitions, we obtain a pushout diagram of simplicial sets

$$
\begin{array}{ccc}
\partial C \times \mathcal{F}(0) & \longrightarrow & C \times \mathcal{F}(0) \\
\downarrow & & \downarrow \\
\mathcal{D}'_{\partial \Delta^n} & \longrightarrow & \mathcal{D}'_{\Delta^n},
\end{array}
$$

compatible with a projection map $\pi : C \to \Delta^n$. Set

$$X'' = \mathrm{Fun}(C, \overline{\mathcal{X}})^{\simeq} \times_{\mathrm{Fun}(C, \Delta^n)^{\simeq}} \{\pi\} \qquad X_0'' = \mathrm{Fun}(\partial C, \overline{\mathcal{X}})^{\simeq} \times_{\mathrm{Fun}(\partial C, \Delta^n)^{\simeq}} \{\pi | \partial C\}$$

$$Y'' = \mathrm{Fun}(C, \mathcal{X})^{\simeq} \times_{\mathrm{Fun}(C, \Delta^n)^{\simeq}} \{\pi\} \qquad Y_0'' = \mathrm{Fun}(\partial C, \mathcal{X})^{\simeq} \times_{\mathrm{Fun}(\partial C, \Delta^n)^{\simeq}} \{\pi | \partial C\}$$

We have a commutative diagram

$$
\begin{array}{ccccc}
X'(\Delta^n) & \longrightarrow & Y'(\Delta^n) & \longrightarrow & Y'' \\
\downarrow & & \downarrow & & \downarrow \\
X'(\partial\,\Delta^n) & \longrightarrow & Y'(\partial\,\Delta^n) & \longrightarrow & Y_0''
\end{array}
$$

where right square is a homotopy pullback. Consequently, to prove that σ is a homotopy pullback square, it will suffice to show that the outer rectangle is a homotopy pullback square. This outer rectangle fits into a commutative diagram

$$
\begin{array}{ccccc}
X'(\Delta^n) & \longrightarrow & X'' & \longrightarrow & Y'' \\
\downarrow & & \downarrow & & \downarrow \\
X'(\partial\,\Delta^n) & \longrightarrow & X_0'' & \longrightarrow & Y_0''
\end{array}
$$

where the left square is a homotopy pullback diagram. We are therefore reduced to showing that the right square in this diagram is a homotopy pullback. For this, it suffices to show that for every point $x \in X_0''$, the induced map $X'' \times_{X_0''} \{x\} \to Y'' \times_{Y_0''} \{x\}$ is a homotopy equivalence of Kan complexes. Let us identify x with a map $f : \partial\, C \to \overline{\mathfrak{X}}$; we wish to show that the projection map $g : \overline{\mathfrak{X}} \to \mathfrak{X}$ induces a homotopy equivalence

$$
\mathrm{Fun}(C,\overline{\mathfrak{X}}) \times_{\mathrm{Fun}(\partial\, C,\overline{\mathfrak{X}})} \{f\} \to \mathrm{Fun}(C,\mathfrak{X}) \times_{\mathrm{Fun}(\partial\, C,\mathfrak{X})} \{g \circ f\}.
$$

Let $F, F' \in \overline{\mathfrak{X}}$ be the images under f of the initial and final vertices of C, respectively. Using Lemma 7.2.6.9, we are reduced to proving that g induces a homotopy equivalence $\mathrm{Map}_{\overline{\mathfrak{X}}}(F, F') \to \mathrm{Map}_{\mathfrak{X}}(g(F), g(F'))$. This follows from Lemma 7.2.6.11 (after an unpacking of definitions). $\qquad\square$

7.3 The Chain Rule

Let $\mathcal{C}$ and $\mathcal{D}$ be compactly generated pointed ∞-categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor which preserves filtered colimits. In §7.1.2, we saw how to associate to F a tower of approximations

$$
\cdots \to P_3(F) \to P_2(F) \to P_1(F) \to P_0(F) \simeq *,
$$

where each $P_n F$ is n-excisive. Roughly speaking, we can think of this tower as a providing a filtration of F whose "successive quotients" $D_n(F) = \mathrm{fib}(P_n(F) \to P_{n-1}(F))$ are n-homogeneous. According to Theorem 7.1.4.7, each functor D_n is determined by its symmetric cross-effect $\mathrm{cr}_{(n)}\, D_n(F) \in \mathrm{SymFun}_{\mathrm{lin}}^n(\mathcal{C}, \mathcal{D})$. Corollary 7.2.3.22 supplies an equivalence of ∞-categories

$$
\theta : \mathrm{Exc}_\star(\mathrm{Sp}(\mathcal{C})^n, \mathrm{Sp}(\mathcal{D})) \to \mathrm{Exc}_\star(\mathcal{C}^n, \mathcal{D}).
$$

Consequently, the functor $\mathrm{cr}_n\, D_n(F) : \mathcal{C}^n \to \mathcal{D}$ is given by the composition

$$
\mathcal{C}^n \xrightarrow{\Sigma_\mathcal{C}^\infty} \mathrm{Sp}(\mathcal{C})^n \xrightarrow{\partial_n(F)} \mathrm{Sp}(\mathcal{D}) \xrightarrow{\Omega_\mathcal{D}^\infty} \mathcal{D}
$$

for some functor $\partial_n(F) : \mathrm{Sp}(\mathcal{C})^n \to \mathrm{Sp}(\mathcal{D})$ which is excisive in each variable. We will refer to $\partial_n(F)$ as the *nth derivative* of the functor F.

In this section, we will be concerned with the following:

Question 7.3.0.1. Suppose that $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are reduced functors between pointed compactly generated ∞-categories which preserve filtered colimits. Can one compute the derivatives of the composite functor $G \circ F$ in terms of the derivatives of G and F?

To address Question 7.3.0.1, it will be convenient to introduce some terminology. Note that the equivalence of ∞-categories θ is equivariant with respect to the action of the symmetric group Σ^n. It follows that each of the functors $\partial_n(F) : \mathrm{Sp}(\mathcal{C})^n \to \mathrm{Sp}(\mathcal{D})$ is invariant under permutations of its arguments. More precisely, $\partial_n(F)$ underlies a symmetric multilinear functor $\partial_{(n)}(F) \in \mathrm{SymFun}_{\mathrm{lin}}^n(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{D}))$.

Definition 7.3.0.2. Let $\mathcal{C}$ and $\mathcal{D}$ be stable ∞-categories. A *symmetric sequence* of functors from $\mathcal{C}$ to $\mathcal{D}$ is a collection of symmetric multilinear functors $\{F_{(n)} : \mathrm{SymFun}_{\mathrm{lin}}^n(\mathcal{C}, \mathcal{D})\}_{n \geq 1}$. We let $\mathrm{SSeq}(\mathcal{C}, \mathcal{D})$ denote the ∞-category $\prod_{n \geq 1} \mathrm{SymFun}_{\mathrm{lin}}^n(\mathcal{C}, \mathcal{D})$ of symmetric sequences from $\mathcal{C}$ to $\mathcal{D}$.

Remark 7.3.0.3. We will generally denote a symmetric sequence of functors $\{F_{(n)} \in \mathrm{SymFun}_{\mathrm{lin}}^n(\mathcal{C}, \mathcal{D})\}_{n \geq 1}$ simply by $F_* \in \mathrm{SSeq}(\mathcal{C}, \mathcal{D})$.

Remark 7.3.0.4. Let $\mathcal{C}$ and $\mathcal{D}$ be stable ∞-categories. We can think of a symmetric sequence $F_* \in \mathrm{SSeq}(\mathcal{C}, \mathcal{D})$ as a collection of functors $F_I : \mathcal{C}^I \to \mathcal{D}$ which are exact in each variable, defined for every nonempty finite set I and depending functorially on I.

Example 7.3.0.5. Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor between compactly generated pointed ∞-categories, and assume that F commutes with filtered colimits. Then the collection of derivates $\{\partial_{(n)}(F)\}_{n \geq 1}$ is a symmetric sequence from $\mathrm{Sp}(\mathcal{C})$ to $\mathrm{Sp}(\mathcal{D})$, which we will denote by $\partial_*(F) \in \mathrm{SSeq}(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{D}))$.

Suppose we are given a triple of stable ∞-categories $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$. There is a composition product

$$\circ : \mathrm{SSeq}(\mathcal{D}, \mathcal{E}) \times \mathrm{SSeq}(\mathcal{C}, \mathcal{D}) \to \mathrm{SSeq}(\mathcal{C}, \mathcal{E}),$$

which is given informally by $G_* \circ F_* = H_*$, with

$$H_I = \bigoplus_E G_{I/E} \circ \{F_J\}_{J \in I/E}$$

where the sum is taken over all equivalence relations E on the nonempty finite set I. This composition product is coherently associative. In particular, if $\mathcal{C}$ is a compactly generated stable ∞-category, then $\mathrm{SSeq}(\mathcal{C}, \mathcal{C})$ can be regarded as a monoidal ∞-category. If $\mathcal{D}$ is another compactly generated stable ∞-category, then $\mathrm{SSeq}(\mathcal{C}, \mathcal{D})$ is left tensored over $\mathrm{SSeq}(\mathcal{D}, \mathcal{D})$ and right tensored over $\mathrm{SSeq}(\mathcal{C}, \mathcal{C})$.

Remark 7.3.0.6. We will not give a precise definition for the composition product of symmetric sequences in this book. The reader can regard the above discussion as heuristic (though, with some effort, it can be made precise). Note that the associativity of the composition product described above depends crucially on the fact that we are working with stable ∞-categories, and that all functors are assumed to be exact.

A complete answer to Question 7.3.0.1 can be given as follows:

Conjecture 7.3.0.7 (Chain Rule).   (1) Let $\mathcal{C}$ be a compactly generated pointed ∞-category, and let $\mathrm{id}_{\mathcal{C}} :$ $\mathcal{C} \to \mathcal{C}$ denote the identity functor. Then the symmetric sequence $\partial_*(\mathrm{id}_{\mathcal{C}})$ is equipped with a coherently associative multiplication: that is, it can be regarded as an algebra object of the monoidal ∞-category $\mathrm{SSeq}(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{C}))$.

   (2) Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor between compactly generated pointed ∞-categories which commutes with filtered colimits. Then $\partial_*(F)$ can be regarded as an $\partial_*(\mathrm{id}_{\mathcal{D}})$-$\partial_*(\mathrm{id}_{\mathcal{C}})$ bimodule object of $\mathrm{SSeq}(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{D}))$.

   (3) Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be reduced functors between compactly generated pointed ∞-categories which commute with filtered colimits. Then there is a canonical equivalence

$$\partial_*(G \circ F) \simeq \partial_*(G) \otimes_{\partial_*(\mathrm{id}_{\mathcal{D}})} \partial_*(F)$$

in the ∞-category of $\partial_*(\mathrm{id}_{\mathcal{E}})$-$\partial_*(\mathrm{id}_{\mathcal{C}})$ bimodule objects of $\mathrm{SSeq}(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{E}))$.

Remark 7.3.0.8. In the case where $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \{\mathrm{Sp}, \mathcal{S}_*\}$, Conjecture 7.3.0.7 was proven (in a different setting) by Arone and Ching. We refer the reader to [2] for more details.

Remark 7.3.0.9. If we restrict our attention to first derivatives, Conjecture 7.3.0.7 reduces to (the single variable version of) Theorem 7.2.1.22.

To arrive at a more precise formulation of Conjecture 7.3.0.7, it is convenient to use the language of corepresentable ∞-operads developed in §7.2.4. Suppose that $\mathcal{O}^{\otimes}$ is a stable ∞-operad (see Definition 7.2.4.10). For each $n > 0$, Remark 7.2.4.4 provides a functor $\otimes^n : \mathcal{O}^n \to \mathcal{O}$, which is exact in each variable. We can regard the collection of functors $\{\otimes^n\}_{n \geq 1}$ as a symmetric sequence from $\mathcal{O}$ to itself, which we will denote by $\otimes^* \in \mathrm{SSeq}(\mathcal{O}, \mathcal{O})$. Let I be a finite set and let $\mathrm{Equiv}(I)$ denote the set of all equivalence relations on I. For every equivalence relation E on I, Remark 7.2.4.8 supplies a natural transformation of functors

$$\otimes^I \to \otimes^{I/E} \circ \prod_{J \in I/E} \otimes^J.$$

Taking the direct sum of these natural transformations over all E, we obtain a map $\otimes^I \to (\otimes^* \circ \otimes^*)^I$ Allowing I to vary, we get a map of symmetric sequences $\delta : \otimes^* \to \otimes^* \circ \otimes^*$.

Let us regard the stable ∞-operad $\mathcal{O}^{\otimes}$ as encoded by a locally coCartesian fibration $p : \mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. The underlying ∞-category $\mathcal{O}$ can be recovered by studying the fibers of p over vertices of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ (in fact, over the single vertex $\langle 1 \rangle \in \mathrm{N}(\mathcal{F}\mathrm{in}_*)$), the symmetric sequence $\otimes^*$ can be recovered by studying the restriction of p to edges of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, and the comultiplication map δ can be recovered by studying the restriction of p to 2-simplices of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. We can regard the entire locally coCartesian fibration p as witnessing the fact that the comultiplication δ is associative up coherent homotopy. In other words, a corepresentable ∞-operad $\mathcal{O}^{\otimes}$ determines a stable ∞-category $\mathcal{O}$ together with a coalgebra object of the monoidal ∞-category $\mathrm{SSeq}(\mathcal{O}, \mathcal{O})$.

Remark 7.3.0.10. Let $\mathcal{O}$ be a stable ∞-category. With some effort, one can show that the construction sketched above underlies an equivalence between the following two types of data:

(*a*) Coalgebra objects F^* of $\mathrm{SSeq}(\mathcal{O}, \mathcal{O})$ for which the counit map restricts to an equivalence $F^1 \simeq \mathrm{id}_{\mathcal{O}}$.

(*b*) Stable unital ∞-operads $\mathcal{O}^{\otimes}$ having underlying ∞-category $\mathcal{O}$.

Since we will not need this fact, we will not give a precise formulation here.

Example 7.3.0.11. Let $\mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a stable ∞-operad. Suppose that the underlying ∞-category $\mathcal{O}$ is the ∞-category of spectra, and that each of the functors $\otimes^n : \mathcal{O}^n \to \mathcal{O}$ preserves filtered colimits. Then each of the functors $\otimes^n : \mathcal{O}^n \to \mathcal{O}$ preserves small colimits separately in each variable, and is therefore given by the formula

$$(X_1, X_2, \dots, X_n) \mapsto E_n \otimes X_1 \otimes \cdots \otimes X_n$$

for some spectrum E_n (given concretely by $E_n = \otimes^n\{S\}_{1 \leq i \leq n}$). Then the collection $\{E_n\}_{n \geq 1}$ can be regarded as a *cooperad* in ∞-category of spectra. That is, each E_n is equipped with an action of the symmetric group Σ_n, and we have coproduct maps

$$E_{n_1 + \cdots + n_k} \to E_k \otimes \bigotimes_{1 \leq i \leq k} E_{n_i}$$

which satisfy an associative law up to coherent homotopy.

Warning 7.3.0.12. The dictionary provided by Example 7.3.0.11 poses some danger of creating confusion. To every *cooperad* $\{E_n\}_{n \geq 1}$ in spectra (in the sense of classical homotopy theory), we can reverse engineer the construction of Example 7.3.0.11 to produce a unital stable ∞-*operad* $\mathcal{O}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ (in the sense of Definition 7.2.4.10). In other words, the same mathematical structure has two incarnations: first, as an cooperad (enriched in spectra) and second, as an operad (with several colors, encriched in spaces). The first perspective is useful for comparing the constructions given here with the existing literature ([2]). However, we will avoid it in what follows, to avoid conflict with the terminology established earlier in this book.

Using the above dictionary and the canonical equivalence $\mathrm{SSeq}(\mathcal{O}, \mathcal{O})^{op} \simeq \mathrm{SSeq}(\mathcal{O}^{op}, \mathcal{O}^{op})$, we can now give a precise formulation of the first assertion of Conjecture 7.3.0.7:

Conjecture 7.3.0.13. Let $\mathcal{C}$ be a compactly generated pointed ∞-category. Then there exists a unital stable ∞-operad $\mathcal{O}^{\otimes}$ with the following properties:

(a) The underlying ∞-category of $\mathcal{O}^{\otimes}$ is given by $\mathcal{O} \simeq \mathrm{Sp}(\mathcal{C})^{op}$.

(b) For each $n \geq 1$, the tensor product functor $\otimes^n : \mathcal{O}^n \to \mathcal{O}$ is equivalent to (the opposite of) the functor $\partial_n(\mathrm{id}_{\mathcal{C}}) : \mathrm{Sp}(\mathcal{C})^n \to \mathrm{Sp}(\mathcal{C})$.

Using the ideas developed in §7.2, we can immediately deduce a close relative of Conjecture 7.3.0.13. Let $\mathcal{C}$ be a compactly generated pointed ∞-category, and let $\mathcal{C}^{\times}$ be the associated Cartesian symmetric monoidal ∞-category. Then $\mathcal{C}^{\times}$ is a differentiable ∞-operad, so it admits a stabilization $\mathcal{O}'^{\otimes}$ (Proposition 7.2.4.14). Note that since $\mathcal{C}$ is pointed, the ∞-operad $\mathcal{O}'^{\otimes}$ is unital. This proves the following:

Proposition 7.3.0.14. *Let $\mathcal{C}$ be a compactly generated pointed ∞-category. Then there exists a unital stable ∞-operad $\mathcal{O}'^{\otimes}$ with the following properties:*

(a) *The underlying ∞-category of $\mathrm{Sp}(\mathcal{C})^{\otimes}$ with underlying ∞-category $\mathrm{Sp}(\mathcal{C})$.*

(b) *For each $n \geq 1$, the tensor product functor $\otimes^n : \mathrm{Sp}(\mathcal{C})^n \to \mathrm{Sp}(\mathcal{C})$ is given by the derivative of the Cartesian product functor $\mathcal{C}^n \to \mathcal{C}$.*

To bring out the analogy between Conjecture 7.3.0.13 Proposition 7.3.0.14, let us recall how to describe explicitly the nth derivative $\partial_n(F)$ of a functor $F : \mathcal{C} \to \mathcal{D}$. Combining Remark 7.1.3.23 with Proposition 7.2.3.15, we see that $\partial_n(F)$ can be identified with the (multivariate) derivative $\vec{\partial} \mathrm{cr}_n(F)$, where $\mathrm{cr}_n(F)$ denotes the nth cross effect of F. In other words, we have a canonical equivalence

$$\partial_n(F) \simeq \vec{\partial} \mathrm{red}(F^{\mathrm{II}}),$$

where $F^{\mathrm{II}} : \mathcal{C}^n \to \mathcal{D}$ is the functor given by $F^{\mathrm{II}}(C_1, \ldots, C_n) = F(C_1 \amalg \cdots \amalg C_n)$. For our purposes, it will be easier to study a dual construction, where we replace coproducts by products and reduction by coreduction.

Definition 7.3.0.15. Let $\mathcal{C}$ and $\mathcal{D}$ be compactly generated pointed ∞-categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor which commutes with filtered colimits. For each integer $n \geq 1$, let $\partial^n(F) : \mathrm{Sp}(\mathcal{C})^n \to \mathrm{Sp}(\mathcal{D})$ denote the functor $\vec{\partial} \mathrm{cored}(F^{\times})$, where $F^{\times} : \mathcal{C}^n \to \mathcal{D}$ is given by $F^{\times}(C_1, \ldots, C_n) = F(C_1 \times \cdots \times C_n)$. We will refer to $\partial^n(F)$ as the *nth coderivative of F*.

Remark 7.3.0.16. Our description of $\partial^n(F)$ as $\vec{\partial} \mathrm{cored}(F^{\times})$ was made to emphasize the analogy with the description of the usual derivative $\partial_n(F)$ as $\vec{\partial} \mathrm{red}(F^{\mathrm{II}})$. However, Definition 7.3.0.15 can be simplified: since the canonical map $F^{\times} \to \mathrm{cored}(F^{\times})$ induces an equivalence of derivatives, we have $\partial^n(F) = \vec{\partial}(F^{\times})$. This observation renders the study of coderivatives much more tractable than the formally dual theory of derivatives.

We can summarize Proposition 7.3.0.14 informally as follows: for every compactly generated ∞-category $\mathcal{C}$, the symmetric sequence of coderivatives $\{\partial^n(\mathrm{id}_{\mathcal{C}})\}_{n \geq 1}$ can be regarded as a coalgebra object of $\mathrm{SSeq}(\mathcal{O}, \mathcal{O})$. In other words, we can regard Proposition 7.3.0.14 as a dual version of the first part of Conjecture 7.3.0.7. Our goal in this section is to formulate and prove dual versions of the remaining assertions of Conjecture 7.3.0.7:

(2′) Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor between compactly generated pointed ∞-categories which commutes with filtered colimits. Then $\partial^*(F)$ is equipped with compatible left and right (co)actions of the coalgebras $\partial^*(\mathrm{id}_{\mathcal{D}})$ and $\partial^*(\mathrm{id}_{\mathcal{C}})$.

(3′) Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be reduced functors between compactly generated pointed ∞-categories which commute with filtered colimits. Then there is a canonical equivalence

$$\partial^*(G \circ F) \simeq \partial^*(G) \otimes^{\partial^*(\mathrm{id}_{\mathcal{D}})} \partial^*(F),$$

where the superscript indicates that the relative tensor product is formed in the opposite ∞-category: that is, the right hand side is given by the totalization of a cosimplicial object

$$\partial^*(G) \circ \partial^*(F) \rightrightarrows \partial^*(G) \circ \partial^*(\mathrm{id}_{\mathcal{D}}) \circ \partial^*(F) \Rrightarrow \cdots$$

in the ∞-category $\mathrm{SSeq}(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{E}))$.

The proof of (2′) is essentially already contained in §7.2: in §7.3.1, we will construct the relevant structures on $\partial^*(F)$ can be obtained by applying the stabilization construction of §7.2.5 to a suitable correspondence between ∞-operads. In §7.3.2 we will give a precise formulation of (3′) by constructing a map

$$\partial^*(G \circ F) \to \partial^*(G) \otimes^{\partial^*(\mathrm{id}_{\mathcal{D}})} \partial^*(F).$$

The hard part is to show that this map is an equivalence, which we prove in §7.3.6 (see Theorem 7.3.2.1). Our strategy is to use the formula $\partial^n(F) \simeq \vec{\partial}(F^\times)$ of Remark 7.3.0.16 to reduce to the chain rule for first derivatives (Theorem 7.2.1.22). The main obstacle is that Theorem 7.2.1.22 applies only to functors which are reduced in each variable. In §7.3.5, we will explain how to circumvent this difficulty by introducing suitable "correction terms" into the formula for the chain rule. The main ingredient is a certain technical result concerning the commutation of differentiation with limits (Theorem 7.3.3.14), which we prove in §7.3.4. This result can be regarded as a relative version of Arone-Mahowald calculation of the derivatives of the identity functor on the ∞-category of pointed spaces, which we review in §7.3.3.

Remark 7.3.0.17. Let $\mathcal{C}$ be a compactly generated pointed ∞-category. The relationship between the symmetric sequences $\partial^*(\mathrm{id}_{\mathcal{C}})$ and $\partial_*(\mathrm{id}_{\mathcal{C}})$ is more than an analogy: they are *Koszul dual* to one another. Given a suitable theory of Koszul duality, one can deduce Conjecture 7.3.0.13 from Proposition 7.3.0.14, and the remaining assertions of Conjecture 7.3.0.7 from the versions of (2′) and (3′) that we prove in this section. We plan to return to the subject in a future work.

7.3.1 Cartesian Structures

Let $\mathcal{C}$ be a compactly generated pointed ∞-category. According to Proposition 7.3.0.14, the symmetric sequence of coderivatives $\{\partial^n(\mathrm{id}_{\mathcal{C}})\}_{n \geq 1}$ is equipped with a coherently associative comultiplication, encoded by a (unital) stable ∞-operad $\mathrm{Sp}(\mathcal{C})^\otimes$ with underlying ∞-category $\mathrm{Sp}(\mathcal{C})$. The ∞-operad $\mathrm{Sp}(\mathcal{C})^\otimes$ was constructed in §7.2.4 by stabilizing the Cartesian symmetric monoidal ∞-category $\mathcal{C}^\times$.

Now suppose that $F : \mathcal{C} \to \mathcal{D}$ is a reduced functor between compactly generated pointed ∞-categories. Our goal in this section is to show that the symmetric sequence $\{\partial^n(F)\}_{n \geq 1}$ is equipped with (compatible) left and right (co)actions of $\{\partial^n(\mathrm{id}_{\mathcal{D}})\}_{n \geq 1}$ and $\{\partial^n(\mathrm{id}_{\mathcal{C}})\}_{n \geq 1}$, respectively. We first explain how to encode these actions using the language of (families of) ∞-operads.

Definition 7.3.1.1. A *correspondence of ∞-operads* is Δ^1-family of ∞-operads $p : \mathcal{O}^\otimes \to \Delta^1 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. We will say that $\mathcal{O}^\otimes$ is a *correspondence* from the ∞-operad $\mathcal{O}_0^\otimes = \{0\} \times_{\Delta^1} \mathcal{O}^\otimes$ to the ∞-operad $\mathcal{O}_1^\otimes = \{1\} \times_{\Delta^1} \mathcal{O}^\otimes$. We will say that a correspondence of ∞-operads p is *corepresentable* (*stable*, *differentiable*) it is corepresentable (stable, differentiable) when regarded as a (local) Δ^1-family of ∞-operads.

Let $p : \mathcal{O}^\otimes \times \Delta^1 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a corepresentable correspondence from an ∞-operad $\mathcal{O}_0^\otimes$ to an ∞-operad $\mathcal{O}_1^\otimes$. For each $n \geq 0$, the unique active morphism $\langle n \rangle \to \langle 1 \rangle$ determines a map from $(0, \langle n \rangle)$ to $(1, \langle 1 \rangle)$ in $\Delta^1 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, to which we can associate a functor

$$F^n : \mathcal{O}_0^n \simeq \mathcal{O}_{(0,\langle n \rangle)}^\otimes \to \mathcal{O}_{(1,\langle 1 \rangle)}^\otimes \simeq \mathcal{O}_1.$$

Each F^n is equivariant with respect to the action of the symmetric group Σ_n on $\mathcal{O}_0^n$; we may therefore think of this construction as giving a family of functors $F^I : \mathcal{O}_0^I \to \mathcal{O}_1$ for every nonempty finite set I. If $\mathcal{O}^\otimes$ is a stable correspondence of ∞-operads, then each of the functors F^I is exact in each variable, so that we can view $\{F^n\}_{n \geq 1}$ as a symmetric sequence of functors from $\mathcal{O}_0$ to $\mathcal{O}_1$. We will denote this symmetric sequence by F^*.

Since $\mathcal{O}_0^\otimes$ and $\mathcal{O}_1^\otimes$ are corepresentable ∞-operads, Remark 7.2.4.4 supplies tensor product functors

$$\otimes_0^I : \mathcal{O}_0^I \to \mathcal{O}_0 \qquad \otimes_1^I : \mathcal{O}_1^I \to \mathcal{O}_1$$

for every nonempty finite set I. If E is an equivalence relation on I, we have canonical natural transformations

$$\otimes_1^{I/E} \circ \prod_{J \in I/E} F^J \leftarrow F^I \to F^{I/E} \circ \prod_{J \in I/E} \otimes_0^J.$$

If we suppose that p is a stable correspondence of ∞-operads, then we can combine these maps as E varies over $\mathrm{Equiv}(I)$, to obtain maps

$$\bigoplus_{E \in \mathrm{Equiv}(I)} \otimes_1^{I/E} \circ \prod_{J \in I/E} F^J \leftarrow F^I \to \bigoplus_{E \in \mathrm{Equiv}(I)} F^{I/E} \circ \prod_{J \in I/E} \otimes_0^J.$$

Allowing I to vary, we obtain maps of symmetric sequences

$$F^* \to F^* \circ \otimes_0^* \qquad F^* \to \otimes_1^* \circ F^*.$$

Remark 7.3.1.2. Let $\mathcal{O}_0^\otimes$ and $\mathcal{O}_1^\otimes$ be reduced stable ∞-operads, corresponding (under the dictionary of Remark 7.3.0.10) to associative coalgebra objects $\otimes_0^*$ and $\otimes_1^*$ of $\mathrm{SSeq}(\mathcal{O}_0, \mathcal{O}_0)$ and $\mathrm{SSeq}(\mathcal{O}_1, \mathcal{O}_1)$, respectively.

(a) Symmetric sequences $F^* \in \mathrm{SSeq}(\mathcal{O}_0, \mathcal{O}_1)$ which are equipped with (compatible) left coactions of $\otimes_1^*$ and right coactions of $\otimes_0^*$.

(b) Stable correspondences $\mathcal{O}^\otimes$ from $\mathcal{O}_0^\otimes$ to $\mathcal{O}_1^\otimes$ which are *unital* in the sense that the initial object of $\mathcal{O}_0^\otimes$ is also an initial object of $\mathcal{O}^\otimes$.

As in Remark 7.3.0.10, we will be content to view this as a heuristic principle: we will not attempt to give a proof (or even a precise formulation) in this book.

Motivated by this analysis, we formulate the following dual version of assertion (2) of Conjecture 7.3.0.7:

Proposition 7.3.1.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be compactly generated pointed ∞-categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor which preserves filtered colimits. Then there exists a stable correspondence of ∞-operads $q : \mathcal{O}^\otimes \to \Delta^1 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ with the following properties:*

(1) *The underlying stable ∞-operads $\mathcal{O}_0^\otimes$ and $\mathcal{O}_1^\otimes$ are given by $\mathrm{Sp}(\mathcal{C})^\otimes$ and $\mathrm{Sp}(\mathcal{D})^\otimes$, respectively (where $\mathrm{Sp}(\mathcal{C})^\otimes$ and $\mathrm{Sp}(\mathcal{D})^\otimes$ are as in Proposition 7.3.0.14).*

(2) *For each $n \geq 1$, the functor $\mathrm{Sp}(\mathcal{C})^n \to \mathrm{Sp}(\mathcal{D})$ determined by q is equivalent to the coderivative $\partial^n(F)$.*

Our construction of the correspondence $\mathcal{O}^\otimes \to \Delta^1 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ of Proposition 7.3.1.3 will proceed in three steps:

(a) First, take $p : \mathcal{M} \to \Delta^1$ be a correspondence of ∞-categories associated to the functor F. That is, p is a coCartesian fibration with fibers $\mathcal{C} \simeq \mathcal{M}_0 = \mathcal{M} \times_{\Delta^1} \{0\}$ and $\mathcal{D} \simeq \mathcal{M}_1 = \mathcal{M} \times_{\Delta^1} \{1\}$, and the induced map from $\mathcal{M}_0$ to $\mathcal{M}_1$ is given by F.

(b) We will show that the correspondence $\mathcal{M}$ from $\mathcal{C}$ to $\mathcal{D}$ determines a correspondence of ∞-operads $\mathcal{M}^\times$ from $\mathcal{C}^\times$ to $\mathcal{D}^\times$, using a relative version of Construction 2.4.1.4.

(c) By applying the stabilization construction described in §7.2.5 to $\mathcal{M}^{\times}$, we will obtain a stable correspondence from $\mathrm{Sp}(\mathcal{C})^{\otimes}$ to $\mathrm{Sp}(\mathcal{D})^{\otimes}$.

We will devote most of this section to carrying out step (b). We begin with a relative version of Definition 2.4.0.1.

Definition 7.3.1.4. Let S be a simplicial set, and let $q : \mathcal{O}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a map of simplicial sets). We will say that q is a *Cartesian local S-family of ∞-operads* if it satisfies the following conditions:

(1) The map q is a corepresentable local S-family of ∞-operads.

(2) For each $s \in S$, the induced map $q_s : \mathcal{C}_s^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a coCartesian fibration which determines a Cartesian symmetric monoidal structure on the ∞-category $\mathcal{C}_s$ (see Definition 2.4.0.1).

(3) For every $s \in S$, the inclusion $\mathcal{C}_s^{\otimes} \hookrightarrow \mathcal{C}^{\otimes}$ carries q_s-coCartesian edges to q-coCartesian edges.

Remark 7.3.1.5. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories which admit finite products, and let $q : \mathcal{O}^{\otimes} \to \Delta^1 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a corepresentable correspondence between the symmetric monoidal ∞-categories $\mathcal{O}_0^{\otimes} = \mathcal{C}^{\times}$ and $\mathcal{O}_1^{\otimes} = \mathcal{D}^{\times}$. For each $n \geq 0$, restricting q to the active morphism $\alpha : (0, \langle n \rangle) \to (1, \langle 1 \rangle)$ of $\Delta^1 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ gives a functor $F^n : \mathcal{C}^n \to \mathcal{D}$. Note that α factors as a composition $(0, \langle n \rangle) \to (0, \langle 1 \rangle) \to (1, \langle 1 \rangle)$. For every n-tuple of objects $C_1, \ldots, C_n \in \mathcal{C}$, this factorization determines a map

$$F^n(C_1, \ldots, C_n) \to F^1(C_1 \times \cdots \times C_n).$$

Condition (3) of Definition 7.3.1.4 guarantees that these map is an equivalences: that is, we can recover each of functors $F^n : \mathcal{C}^n \to \mathcal{D}$ by composing the single functor $F^1 : \mathcal{C} \to \mathcal{D}$ with the Cartesian product on $\mathcal{C}$.

The analysis given in Remark 7.3.1.5 suggests that Cartesian local S-family of ∞-operads should be determined by the underlying locally coCartesian fibration

$$\mathcal{O} \simeq \mathcal{O}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \{\langle 1 \rangle\} \to S.$$

We can formulate this more precisely as follows:

Theorem 7.3.1.6. *Let $\mathcal{C} \to S$ be a locally coCartesian fibration of simplicial sets. Then there exists a Cartesian local S-family of ∞-operads $\mathcal{C}^{\times} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ with $\mathcal{C} \simeq \mathcal{C}^{\times} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \{\langle 1 \rangle\}$. Moreover, $\mathcal{C}^{\times}$ is determined uniquely up to equivalence.*

Proof of Proposition 7.3.1.3. Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor between compactly generated pointed ∞-categories which preserves filtered limits, and let $\mathcal{M} \to \Delta^1$ be a correspondence from $\mathcal{C}$ to $\mathcal{D}$ associated to F. According to Theorem 7.3.1.6, we can extend $\mathcal{M}$ to a Cartesian Δ^1-family of ∞-operads $q : \mathcal{M}^{\times} \to \Delta^1 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Then q is a differentiable local S-family of ∞-operads; let $\mathcal{O}^{\otimes} \to \Delta^1 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ denote the stable Δ^1-family of ∞-operads obtained by applying Construction 7.2.5.20. Then $\mathcal{O}^{\otimes}$ is a stable correspondence from $\mathrm{Sp}(\mathcal{C})^{\otimes}$ to $\mathrm{Sp}(\mathcal{D})^{\otimes}$. The identification of the induced functors $\mathrm{Sp}(\mathcal{C})^n \to \mathrm{Sp}(\mathcal{D})$ with $\partial^n(F)$ follows from Remark 7.3.1.5. $\square$

Theorem 7.3.1.6 is an immediate consequence of a more precise result (Theorem 7.3.1.15), which we will prove at the end of this section. First, we need to formulate a generalization of Definition 2.4.1.1.

Definition 7.3.1.7. Let $q : \mathcal{C}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a corepresentable local S-family of ∞-operads, and let $p : \mathcal{D} \to S$ be a locally coCartesian fibration. We will say that a functor $\pi \in \mathrm{Fun}_S(\mathcal{C}^{\otimes}, \mathcal{D})$ is a *weak Cartesian structure* on $\mathcal{C}^{\otimes}$ if it satisfies the following conditions:

(1) For each $s \in S$, the fiber $\mathcal{C}_s^{\otimes}$ is a symmetric monoidal ∞-category and the induced map $\pi_s : \mathcal{C}_s^{\otimes} \to \mathcal{D}_s$ is a weak Cartesian structure on the symmetric monoidal ∞-category $\mathcal{C}_s^{\otimes}$ (see Definition 2.4.1.1).

(2) Let e be a locally q-coCartesian edge of $\mathcal{C}^{\otimes}$ lying over the unique active morphism $\langle n \rangle \to \langle 1 \rangle$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Then $p(e)$ is a locally p-coCartesain edge of $\mathcal{D}$.

We say that π is a *Cartesian structure* if it is a weak Cartesian structure which induces an equivalence of ∞-categories $\mathcal{C}_s \to \mathcal{D}_s$ for each vertex $s \in S$.

Lemma 7.3.1.8. *Let S be a simplicial set and let $q : \mathcal{C}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a corepresentable local S-family of ∞-operads. If q admits a Cartesian structure, then it is a Cartesian S-family of ∞-operads.*

Proof. It suffices to prove that each pullback $\mathcal{C}^{\otimes} \times_S \Delta^n$ is a corepresentable Δ^n-family of ∞-operads for every map $\Delta^n \to S$. We may therefore assume without loss of generality that S is an ∞-category. The only nontrivial point is to verify condition (3) of Definition 7.3.1.4. Let $f : X \to Y$ be a q_s-coCartesian morphism in $\mathcal{C}^{\otimes}$; we wish to show that f is q-coCartesian. According to Lemma T.2.4.2.7, it will suffice to show that for every locally q-coCartesian morphism $g : Y \to Z$, the composition $g \circ f$ is locally q-coCartesian. Using Remark 7.2.5.4, we can reduce to the case where $Z \in \mathcal{C} = \mathcal{C}^{\otimes}_{\langle 1 \rangle}$. The map g factors as a composition $Y \xrightarrow{g'} Y' \xrightarrow{g''} Z$, where g' is inert (and therefore q-coCartesian) and g'' is active. Since g is locally q-coCartesian, Lemma T.2.4.2.7 implies that g'' is locally q-coCartesian. We may replace Y by Y' and thereby reduce to the case where the image of g in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is active. Factor f as a composition $f'' \circ f'$, where f' is inert and f'' is active. Lemma T.2.4.2.7 implies that $g \circ f$ is locally q-coCartesian if and only if $g \circ f''$ is locally q-coCartesian. We may therefore replace f by f'' and thereby reduce to the case where f is active. Let s denote the image of Z in S, and factor $g \circ f$ as a composition

$$X \xrightarrow{h'} Z' \xrightarrow{h''} Z$$

where h' is locally q-coCartesian and h'' is a morphism in $\mathcal{C}_s$. We wish to prove that h'' is an equivalence.

Let $p : \mathcal{D} \to S$ be a locally coCartesian fibration and let $\pi : \mathcal{C}^{\otimes} \to \mathcal{D}$ be a Cartesian structure on $\mathcal{C}^{\otimes}$. Let $t \in S$ denote the image of Z. Then π induces an equivalence $\mathcal{C}_t \to \mathcal{D}_t$ (by condition (1) of Definition 7.3.1.7). It will therefore suffice to show that $\pi(h'')$ is an equivalence in $\mathcal{D}_t$. Since $\pi(h')$ is locally p-coCartesian (condition (2) of Definition 7.3.1.7), we are reduced to proving that $\pi(h) = \pi(g) \circ \pi(f)$ is locally q-coCartesian. Because f is an active q_s-coCartesian morphism in $\mathcal{C}^{\otimes}_s$, the map $\pi(f)$ is an equivalence in $\mathcal{C}_s$ (by condition (1) of Definition 7.3.1.7). We are therefore reduced to proving that $\pi(g)$ is locally q-coCartesian, which follows from immediately from condition (2) of Definition 7.3.1.7. $\square$

Proposition 7.3.1.9. *Let $p : \mathcal{C}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a Cartesian local S-family of ∞-operads, let $q : \mathcal{D} \to S$ be a locally coCartesian fibration such that each fiber $\mathcal{D}_s$ admit finite products. Let $\mathrm{Fun}^{\times}_S(\mathcal{C}^{\otimes}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}_S(\mathcal{C}^{\otimes}, \mathcal{D})$ spanned by the weak Cartesian structures and let $\mathrm{Fun}^{\times}_S(\mathcal{C}, \mathcal{D})$ be the full subcategory of $\mathrm{Fun}_S(\mathcal{C}, \mathcal{D})$ spanned by those maps $F : \mathcal{C} \to \mathcal{D}$ satisfying the following conditions:*

(a) *The functor F carries locally p-coCartesian edges to locally q-coCartesian edges.*

(b) *For each vertex $s \in S$, the induced map $\mathcal{C}_s \to \mathcal{D}_s$ preserves finite products.*

Then the restriction map $\mathrm{Fun}^{\times}_S(\mathcal{C}^{\otimes}, \mathcal{D}) \to \mathrm{Fun}^{\times}_S(\mathcal{C}, \mathcal{D})$ is an equivalence of ∞-categories.

Proof. For every map of simplicial sets $T \to S$, let $\mathcal{C}^{\otimes}_T = T \times_S \mathcal{C}^{\otimes}$ and $\mathcal{C}_T = T \times_S \mathcal{C}$. Let $\mathrm{Fun}^{\times}_S(\mathcal{C}^{\otimes}_T, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}_S(\mathcal{C}^{\otimes}_T, \mathcal{D})$ spanned by those functors which determine weak Cartesian structures $\mathcal{C}^{\otimes}_T \to T \times_S \mathcal{D}$, and define $\mathrm{Fun}^{\times}_S(\mathcal{C}_T, \mathcal{D}) \subseteq \mathrm{Fun}_S(\mathcal{C}_T, \mathcal{D})$ similarly. We will prove that each restriction map $\theta_T : \mathrm{Fun}^{\times}_S(\mathcal{C}^{\otimes}_T, \mathcal{D}) \to \mathrm{Fun}^{\times}_S(\mathcal{C}_T, \mathcal{D})$ is an equivalence of ∞-categories. The construction $T \mapsto \theta_T$ carries homotopy colimits to homotopy limits. We may therefore reduce to the case where T is a simplex. Replacing S by T, we may assume that $S = \Delta^k$ for some integer $k \geq 0$.

We define a subcategory $\mathcal{I} \subseteq S \times \mathcal{F}\mathrm{in}_* \times [1]$ as follows:

(a) *Every object of $S \times \mathcal{F}\mathrm{in}_* \times [1]$ belongs to $\mathcal{I}$.*

(b) A morphism $(s, \langle n \rangle, i) \to (s', \langle n' \rangle, i')$ in $\mathcal{F}\mathrm{in}_* \times [1]$ belongs to $\mathcal{I}$ if and only if either $i' = 1$ or the induced map $\alpha : \langle n \rangle \to \langle n' \rangle$ is active.

Let $\mathcal{C}'$ denote the fiber product $\mathcal{C}^{\otimes} \times_{S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\mathcal{I})$, which we regard as a subcategory of $\mathcal{C}^{\otimes} \times \Delta^1$, and let $p' : \mathcal{C}' \to \mathrm{N}(\mathcal{I})$ denote the projection. Let $\mathcal{C}'_0$ and $\mathcal{C}'_1$ denote the intersections of $\mathcal{C}'$ with $\mathcal{C}^{\otimes} \times \{0\}$ and $\mathcal{C}^{\otimes} \times \{1\}$, respectively. We note that there is a canonical isomorphism $\mathcal{C}'_1 \simeq \mathcal{C}^{\otimes}$.

Let $\mathcal{E}$ denote the full subcategory of $\mathrm{Fun}_S(\mathcal{C}', \mathcal{D})$ spanned by those functors F which satisfy the following conditions:

(i) For every object $C \in \mathcal{C}^{\otimes}$, the induced map $F(C, 0) \to F(C, 1)$ is an equivalence in $\mathcal{D}$.

(ii) The restriction $F | \mathcal{C}'_1$ is a weak Cartesian structure on $\mathcal{C}^{\otimes}$.

It is clear that if (i) and (ii) are satisfied, then the restriction $F_0 = F | \mathcal{C}'_0$ satisfies the following additional conditions:

(iii) For each $s \in S$, the restriction $F_0 | \mathcal{C}_s$ is a functor from $\mathcal{C}_s$ to $\mathcal{D}_s$ which preserves finite products.

(iv) Let $s \in S$, and let α be an active locally p_s-coCartesian morphism in $\mathcal{C}_s^{\otimes}$. Then $F_0(\alpha)$ is an equivalence in $\mathcal{D}$.

(v) Let α be a locally p-coCartesian morphism in $\mathcal{C}$. Then $F_0(\alpha)$ is a locally q-coCartesian morphism in $\mathcal{D}$.

Condition (i) is equivalent to the assertion that F is a right Kan extension of $F | \mathcal{C}'_1$. Proposition T.4.3.2.15 implies that the restriction map $r : \mathcal{E} \to \mathrm{Fun}_S^{\times}(\mathcal{C}^{\otimes}, \mathcal{D})$ induces a trivial Kan fibration onto its essential image. The map r has a section s, given by composition with the projection map $\mathcal{C}' \to \mathcal{C}^{\otimes}$. The restriction map $\mathrm{Fun}_S^{\times}(\mathcal{C}^{\otimes}, \mathcal{D}) \to \mathrm{Fun}_S^{\times}(\mathcal{C}, \mathcal{D})$ factors as a composition

$$\mathrm{Fun}_S^{\times}(\mathcal{C}^{\otimes}, \mathcal{D}) \xrightarrow{s} \mathcal{E} \xrightarrow{e} \mathrm{Fun}_S^{\times}(\mathcal{C}, \mathcal{D}),$$

where e is induced by composition with the inclusion $\mathcal{C} \subseteq \mathcal{C}'_0 \subseteq \mathcal{C}'$. Consequently, it will suffice to prove that e is an equivalence of ∞-categories.

Let $\mathcal{E}_0 \subseteq \mathrm{Fun}(\mathcal{C}'_0, \mathcal{D})$ be the full subcategory spanned by those functors which satisfy conditions (iii), (iv), and (v). The map e factors as a composition

$$\mathcal{E} \xrightarrow{e'} \mathcal{E}_0 \xrightarrow{e''} \mathrm{Fun}^{\times}(\mathcal{C}, \mathcal{D}).$$

We will complete the proof by showing that that e' and e'' are trivial Kan fibrations.

Let $f : \mathcal{C}'_0 \to \mathcal{D}$ be an arbitrary functor, and let $C \in \mathcal{C}_{j, \langle n \rangle}^{\otimes} \subseteq \mathcal{C}'_0$. There exists a unique map $\alpha : (j, \langle n \rangle, 0) \to (j, \langle 1 \rangle, 0)$ in $\mathcal{I}$; choose a locally p'-coCartesian morphism $\overline{\alpha} : C \to C'$ lifting α. Since $\mathcal{C}^{\otimes}$ is Cartesian S-family of ∞-operads, the morphism $\overline{\alpha}$ is p'-coCartesian. It follows that $\overline{\alpha}$ exhibits C' as an initial object of $\mathcal{C} \times_{(\mathcal{C}'_0)/C'} \times_{\mathcal{C}'_0} \mathcal{C}$. Consequently, f is a right Kan extension of $f | \mathcal{C}$ at C if and only if $f(\overline{\alpha})$ is an equivalence. It follows that f satisfies (iv) if and only if f is a right Kan extension of $f | \mathcal{C}$. The same argument (and Lemma T.4.3.2.7) shows that every functor $f_0 : \mathcal{C} \to \mathcal{D}$ admits a right Kan extension to $\mathcal{C}'_0$. Applying Proposition T.4.3.2.15, we deduce that e'' is a trivial Kan fibration.

It remains to show that e' is a trivial Kan fibration. In view of Proposition T.4.3.2.15, it will suffice to prove the following pair of assertions, for every functor $f \in \mathcal{E}_0$:

(1) There exist a functor $F : \mathcal{C}' \to \mathcal{D}$ which is a left Kan extension of $f = F | \mathcal{C}'_0$.

(2) Let $F : \mathcal{C}' \to \mathcal{D}$ be an arbitrary extension of f. Then F is a left Kan extension of f if and only if F belongs to $\mathcal{E}$.

For every finite linearly ordered set J, let J^+ denote the disjoint union $J \coprod \{\infty\}$, where ∞ is a new element larger than every element of J. Let $(C, 1) \in \mathcal{C}^\otimes_{J_*} \times \{1\} \subseteq \mathcal{C}'$ lying over a vertex $s \in S$. Since there exists a final object $1_s \in \mathcal{C}_s$, the ∞-category $\mathcal{C}'_0 \times_{\mathcal{C}'} \mathcal{C}'_{/C}$ also has a final object, given by the map $\alpha : (C', 0) \to (C, 1)$, where $C' \in \mathcal{C}^\otimes_{J^+_*}$ corresponds, under the equivalence

$$\mathcal{C}^\otimes_{s, J^+_*} \simeq \mathcal{C}_s \times \mathcal{C}^\otimes_{s, J_*},$$

to the pair $(1_s, C)$. We now apply Lemma T.4.3.2.13 to deduce (1), together with the following analogue of (2):

(2′) An arbitrary functor $F : \mathcal{C}' \to \mathcal{D}$ which extends f is a left Kan extension of f if and only if, for every morphism $\alpha : (C', 0) \to (C, 1)$ as above, the induced map $F(C', 0) \to F(C, 1)$ is an equivalence in $\mathcal{D}$.

To complete the proof, it will suffice to show that F satisfies the conditions stated in (2′) if and only if $F \in \mathcal{E}$. We first prove the "if" direction. Suppose that $F \in \mathcal{E}$ and let $\alpha : (C', 0) \to (C, 1)$ be as above; we wish to prove that $F(\alpha) : F(C', 0) \to F(C, 1)$ is an equivalence in $\mathcal{D}$. The map α factors as a composition

$$(C', 0) \xrightarrow{\alpha'} (C', 1) \xrightarrow{\alpha''} (C, 1).$$

Condition (i) guarantees that $F(\alpha')$ is an equivalence. Condition (ii) guarantees that $F(C', 1)$ is equivalent (as an object of $\mathcal{D}_s$) to a product to a product $F(1_s, 1) \times F(C, 1)$, and that $F(\alpha'')$ can be identified with the projection onto the second factor. Moreover, since 1_s is a final object of $\mathcal{C}_s$, condition (ii) also guarantees that $F(1_s, 1)$ is a final object of $\mathcal{D}_s$. It follows that $F(\alpha'')$ is an equivalence, so that $F(\alpha)$ is an equivalence as desired.

Now let us suppose that F satisfies the condition of (2′). We wish to prove that $F \in \mathcal{E}$. We begin by verifying condition (i). Let $C \in \mathcal{C}^\otimes_{J_*}$ for some finite linearly ordered set J, lying over a vertex $s \in S$. Let $\alpha : (C', 0) \to (C, 1)$ be defined as above. Let $\beta : (J_*, 0) \to (J^+_*, 0)$ be the morphism in $\mathcal{I}$ induced by the inclusion $J \subseteq J^+$. Choose a locally p'-coCartesian morphism $\overline{\beta} : (C, 0) \to (C'', 0)$ lifting β. Since the symmetric monoidal structure on $\mathcal{C}_s$ is Cartesian, the final object $1_s \in \mathcal{C}_s$ is also the unit object of $\mathcal{C}$ and we can identify C'' with C'. The composition $(C, 0) \xrightarrow{\overline{\beta}} (C', 1) \xrightarrow{\alpha} (C, 1)$ is homotopic to the canonical map $\gamma : (C, 0) \to (C, 1)$ appearing in the statement of (i). Condition (iv) guarantees that $F(\overline{\beta})$ is an equivalence, and (2′) guarantees that $F(\alpha)$ is an equivalence. Using the two-out-of-three property, we deduce that $F(\gamma)$ is an equivalence, so that F satisfies (i).

Let $F_1 = F| \mathcal{C}'_1$, so that we can regard F_1 as a functor $\mathcal{C}^\otimes \to \mathcal{D}$. To prove that F satisfies (ii), we must verify three conditions:

(ii₀) If $s \in S$ and β is an active p_s-coCartesian morphism of $\mathcal{C}^\otimes_s$, then $F_1(\beta)$ is an equivalence.

(ii₁) Let $C \in \mathcal{C}^\otimes_{s, \langle n \rangle}$ and choose inert morphisms $\gamma_i : \mathcal{C} \to \mathcal{C}_i$ in $\mathcal{C}^\otimes_s$ covering the maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \leq i \leq n$. Then the morphisms γ_i exhibit $F_1(C)$ as a product $\prod_{1 \leq i \leq n} F_1(C_i)$ in the ∞-category $\mathcal{D}_s$.

(ii₂) Let β be a locally p-coCartesian morphism in $\mathcal{C}^\otimes$ covering the unique active morphism $\langle n \rangle \to \langle 1 \rangle$ in $\mathcal{F}\text{in}_*$. Then $F_1(\beta)$ is a locally q-coCartesian morphism in $\mathcal{C}^\otimes$.

Condition (ii₀) follows immediately from (i) and (iv). To prove (ii₁), we consider the maps $\alpha : (C', 0) \to (C, 1)$ and $\alpha_i : (C'_i, 0) \to (C_i, 1)$ which appear in the statement of (2′). For each $1 \leq i \leq n$, we have a commutative diagram

$$
\begin{array}{ccc}
(C', 0) & \xrightarrow{\ \alpha\ } & (C, 1) \\
\Big\downarrow{\gamma'_j} & & \Big\downarrow{\gamma_j} \\
(C'_i, 0) & \xrightarrow{\ \alpha_j\ } & (C_i, 1).
\end{array}
$$

Condition $(2')$ guarantees that the maps $F(\alpha)$ and $F(\alpha_i)$ are equivalences in $\mathcal{D}$. Consequently, it will suffice to show that the maps $f(\gamma'_i)$ exhibit $f(C', 0)$ as a product $\prod_{j \in J} f(C'_j, 0)$ in $\mathcal{D}$. Let $f_0 = f|\mathcal{C}$. Using condition (iv), we obtain canonical equivalences

$$f(C', 0) \simeq f_0(1_s \otimes \otimes_{1 \le i \le n} C_i) \qquad f(C'_i, 0) \simeq f_0(1_s \otimes C_i)$$

Since condition (iii) guarantees that f_0 preserves products, it will suffice to show that the canonical map

$$1_s \otimes (\otimes_{1 \le i \le n} C_i) \to \otimes_{1 \le i \le n} (1_s \otimes C_j)$$

is an equivalence in the ∞-category $\mathcal{C}_s$. This follows easily from our assumption that the symmetric monoidal structure on $\mathcal{C}$ is Cartesian.

It remains to prove (ii_2). Let $C \in \mathcal{C}^{\otimes}_{s, \langle n \rangle}$ and let $\beta : C \to D$ be a locally p-coCartesian morphism in $\mathcal{C}^{\otimes}$ covering the unique active map $\beta_0 : \langle n \rangle \to \langle 1 \rangle$. Choose a p_s-coCartesian morphism $\beta' : C \to C'$ in $\mathcal{C}^{\otimes}_s$ lying over β_0. Since $\mathcal{C}^{\otimes}$ is a Cartesian S-family of ∞-operads, the morphism β' is p-coCartesian. We can therefore factor β as a composition

$$C \xrightarrow{\beta'} C' \xrightarrow{\beta''} D,$$

where β'' is a morphism in $\mathcal{C}$. Since β is locally p-coCartesian and β' is p-coCartesian, the morphism β'' is also locally p-coCartesian. Since $F_1(\beta')$ is an equivalence (by (ii_0)), it will suffice to show that $F_1(\beta'')$ is locally q-coCartesian. We have a commutative diagram

$$
\begin{array}{ccc}
F(C', 0) & \longrightarrow & F(C', 1) \\
\downarrow{\scriptstyle f(\beta'')} & & \downarrow{\scriptstyle F_1(\beta'')} \\
F(D, 0) & \longrightarrow & F(D, 1).
\end{array}
$$

Since F satisfies (i), the horizontal maps in this diagram are equivalences. We are therefore reduced to proving that $f(\beta'')$ is locally q-coCartesian, which follows from (v). $\square$

It follows from Proposition 7.3.1.12 that if $\mathcal{O}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a Cartesian local S-family of ∞-operads which admits a Cartesian structure, then $\mathcal{O}^{\otimes}$ is determined up to equivalence by the underlying locally Cartesian fibration $\mathcal{O} \to S$. We next prove a converse: given a locally coCartesian fibration $\mathcal{O} \to S$ where each fiber $\mathcal{O}_s$ admits finite products, we construct a local S-family of ∞-operads $\mathcal{O}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and a Cartesian structure $\mathcal{O}^{\otimes} \to \mathcal{O}$. For this, we need a relative version of Construction 2.4.1.4.

Construction 7.3.1.10. Let $\Gamma^{\times}$ be the category introduced in Notation 2.4.1.2 (so that the objects of $\Gamma^{\times}$ are pairs $(\langle n \rangle, K)$, where $K \subseteq \langle n \rangle^{\circ}$. Let $p : \mathcal{C} \to S$ be a locally coCartesian fibration of simplicial sets. We define a simplicial set $\widetilde{\mathcal{C}}^{\times}$ equipped with a map $\widetilde{\mathcal{C}}^{\times} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ by the following universal property: for every map of simplicial sets $K \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, we have an isomorphism

$$\mathrm{Fun}_{S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)}(K, \widetilde{\mathcal{C}}^{\times}) \simeq \mathrm{Fun}_S(K \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\Gamma^{\times}), \mathcal{C}).$$

Note that for each $s \in S$, we have a canonical isomorphism $\widetilde{\mathcal{C}}^{\times} \times_S \{s\} \simeq \widetilde{\mathcal{C}}^{\times}_s$, where $\widetilde{\mathcal{C}}^{\times}_s$ is obtained by applying Construction 2.4.1.4 to the ∞-category $\mathcal{C}_s$. We let $\mathcal{C}^{\times}$ denote the full simplicial subset of $\widetilde{\mathcal{C}}^{\times}$ spanned by those vertices which belong to $\mathcal{C}^{\times}_s \subseteq \widetilde{\mathcal{C}}^{\times}_s$, for some vertex $s \in S$.

Remark 7.3.1.11. Let $p : \mathcal{C} \to S$ be a locally coCartesian fibration of simplicial sets. We will identify vertices of $\widetilde{\mathcal{C}}^{\times}$ with triples $(\langle n \rangle, s, \lambda)$, where $\langle n \rangle$ is an object of $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, s is a vertex of S, and λ is a map from the nerve of the partially ordered set of subsets of $\langle n \rangle^{\circ} \simeq \{1, \dots, n\}$ (ordered by reverse inclusion) to the ∞-category $\mathcal{C}_s$.

The fundamental properties of Construction 7.3.1.10 are summarized in the following result:

Proposition 7.3.1.12. *Let $p : \mathcal{C} \to S$ be a locally coCartesian fibration of simplicial sets. Then:*

(1) *The projection $\widetilde{q} : \widetilde{\mathcal{C}}^{\times} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a locally coCartesian fibration.*

(2) *Let $\overline{\alpha} : (\langle n \rangle, s, \lambda) \to (\langle n' \rangle, t, \lambda')$ be an edge of $\widetilde{\mathcal{C}}^{\times}$, so that $\overline{\alpha}$ determines a map $\alpha : \langle n \rangle \to \langle n' \rangle$ in $\mathcal{F}\mathrm{in}_*$ and an edge $e : s \to s'$ in S. Then $\overline{\alpha}$ is locally $\widetilde{q}$-coCartesian if and only if, for every $K \subseteq \langle n' \rangle^{\circ}$, the induced map $\lambda(\alpha^{-1} K) \to \lambda'(K)$ is locally p-coCartesian.*

(3) *Suppose that each of the ∞-categories $\mathcal{C}_s$ admits finite products. Then $q : \mathcal{C}^{\times} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a locally coCartesian fibration. Moreover, an edge $\overline{\alpha} : (\langle n \rangle, s, \lambda) \to (\langle n' \rangle, t, \lambda')$ is locally q-coCartesian if and only if, for every element $j \in \langle n' \rangle^{\circ}$, the induced map $\lambda(\alpha^{-1}\{j\}) \to \lambda'(\{j\})$ is locally p-coCartesian.*

(4) *Suppose that each of the ∞-categories $\mathcal{C}_s$ admits finite products. Then q exhibits $\mathcal{C}^{\times}$ as a corepresentable local S-family of ∞-operads.*

(5) *Suppose that each $\mathcal{C}_s$ admits finite products, and let $\pi : \mathcal{C}^{\times} \to \mathcal{C}$ be the map given by composition with the section $s : \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathrm{N}(\Gamma^{\times})$ defined in Remark 2.4.1.3 (given by $\langle n \rangle \mapsto (\langle n \rangle, \langle n \rangle^{\circ})$). Then π is a Cartesian structure on $\mathcal{C}^{\times}$.*

Proof. The forgetful functor $\mathrm{N}(\Gamma^{\times}) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a Cartesian fibration, and therefore a flat categorical fibration (Example B.3.11). It follows from Proposition B.3.14 that q is a categorical fibration. To prove that q is a locally coCartesian fibration, it will suffice to show that for every edge $\Delta^1 \to S$ the induced map $\Delta^1 \times_S \widetilde{\mathcal{C}}^{\times} \to \Delta^1 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a locally coCartesian fibration. We may therefore replace S by Δ^1 and thereby reduce to the case where p is a coCartesian fibration. Assertions (1) and (2) now follow from Corollary T.3.2.2.13.

We now prove (3). Fix a vertex $(s, \langle n \rangle)$ of $S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, so that we can identify objects of $\widetilde{\mathcal{C}}^{\times}_{s, \langle n \rangle}$ with functors $\mathrm{N}(\mathbf{P}(\langle n \rangle^{\circ})) \to \mathcal{C}_s$. The full subcategory $\mathcal{C}^{\times}_{s, \langle n \rangle}$ is spanned by those functors which are right Kan extensions of their restriction to the full subcategory $\mathrm{N}(\mathbf{P}_1(\langle n \rangle^{\circ})) \subseteq \mathrm{N}(\mathbf{P}(\langle n \rangle^{\circ}))$, where $\mathbf{P}_1(\langle n \rangle^{\circ})$ consists of subsets having cardinality 1. Using our assumption that $\mathcal{C}_s$ admits finite products, we deduce that the inclusion $\mathcal{C}^{\times}_{s, \langle n \rangle} \hookrightarrow \widetilde{\mathcal{C}}^{\times}_{s, \langle n \rangle}$ admits a left adjoint $L_{s, \langle n \rangle}$. Moreover, a morphism $\lambda \to \lambda'$ in $\widetilde{\mathcal{C}}^{\times}_{s, \langle n \rangle}$ is an $L_{s, \langle n \rangle}$ equivalence if and only if, for every element $j \in \langle n \rangle^{\circ}$, the induced map $\lambda(\{j\}) \to \lambda'(\{j\})$ is an equivalence.

We now argue that q is a locally coCartesian fibration. Suppose we are given a vertex $(\langle n \rangle, s, \lambda)$ in $\mathcal{C}^{\otimes}$ and an edge $\alpha : (\langle n \rangle, s) \to (\langle n' \rangle, t)$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times S$. Let $\overline{\alpha}' : (\langle n \rangle, s, \lambda) \to (\langle n' \rangle, t, \lambda')$ be a locally $\widetilde{q}$-coCartesian morphism in $\widetilde{\mathcal{C}}^{\otimes}$ lifting α. Let $\overline{\alpha}'' : \lambda' \to \lambda''$ be a morphism in the ∞-category $\widetilde{\mathcal{C}}^{\times}_{t, \langle n' \rangle}$ which exhibits λ'' as a $\mathcal{C}^{\times}_{t, \langle n' \rangle}$-localization of λ'. Since $\widetilde{q}$ is an inner fibration, we can choose a 2-simplex

$$
\begin{array}{ccc}
 & (\langle n' \rangle, t, \lambda') & \\
\overset{\overline{\alpha}'}{\nearrow} & & \overset{\overline{\alpha}''}{\searrow} \\
(\langle n \rangle, s, \lambda) & \xrightarrow{\quad \overline{\alpha} \quad} & (\langle n' \rangle, t, \lambda'')
\end{array}
$$

lifting the degenerate 2-simplex

$$
\begin{array}{ccc}
 & (\langle n' \rangle, t) & \\
\overset{\alpha}{\nearrow} & & \overset{\mathrm{id}}{\searrow} \\
(\langle n \rangle, s) & \xrightarrow{\quad \alpha \quad} & (\langle n' \rangle, t)
\end{array}
$$

of $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times S$. We note that $\overline{\alpha}$ is locally q-coCartesian. Moreover, condition (2) implies that for every subset $K \subseteq \langle n' \rangle^{\circ}$, $\overline{\alpha}'$ induces a locally p-coCartesian edge $\lambda(\alpha^{-1} K) \to \lambda'(K)$. Since $\overline{\alpha}'$ is an $L_{t, \langle n' \rangle}$-equivalence, it

induces an equivalence $\lambda'(\{j\}) \to \lambda''(\{j\})$ for each $j \in \langle n'\rangle^\circ$. It follows by transitivity that $\lambda(\alpha^{-1}\{j\}) \to \lambda''(\{j\})$ is locally p-coCartesian for each $j \in \langle n'\rangle^\circ$. This proves the "only if" direction of the final assertion of (3). To prove the converse, suppose we are given an arbitrary edge of $\overline{\beta} : (\langle n\rangle, s, \lambda) \to (\langle n'\rangle, t, \mu)$ in $\mathcal{C}^\times$ which lifts α and induces locally p-coCartesian edges $\lambda(\alpha^{-1}\{j\}) \to \mu(\{j\})$ for each $j \in \langle n\rangle^\circ$. Since $\overline{\alpha}$ is locally q-coCartesian, we can choose a 2-simplex

$$
\begin{array}{ccc}
 & (\langle n'\rangle, t, \lambda'') & \\
 \overline{\alpha}\nearrow & & \searrow \gamma \\
 (\langle n\rangle, s, \lambda) & \xrightarrow{\;\;\overline{\beta}\;\;} & (\langle n'\rangle, t, \mu)
\end{array}
$$

where γ is a morphism in $\mathcal{C}^\times_{t,\langle n'\rangle}$. Then γ induces an equivalence $\lambda''(\{j\}) \to \mu(\{j\})$ for each $j \in \langle n'\rangle^\circ$. It follows that γ is an equivalence, so that $\overline{\beta}$ is also locally q-coCartesian.

We now prove (4). We may assume without loss of generality that $S = \Delta^n$. We will show that it is a corepresentable local S-family of ∞-categories by verifying the hypotheses of Lemma 7.2.5.18. The third hypothesis is clear (it follows from Proposition 2.4.1.5 that $\mathcal{C}^\times_s \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a symmetric monoidal ∞-category for each vertex $s \in S$). We will check the first hypothesis; the proof of the second is similar. Suppose we are given a 2-simplex

$$
\begin{array}{ccc}
 & (\langle m\rangle, s, \lambda') & \\
 \alpha\nearrow & & \searrow \beta \\
 (\langle m'\rangle, s, \lambda) & \xrightarrow{\;\;\gamma\;\;} & (\langle n\rangle, t, \lambda'')
\end{array}
$$

in $\mathcal{C}^\times$ where α is inert and β is locally q-coCartesian. Let $\gamma_0 : \langle m'\rangle \to \langle n\rangle$ be the image of γ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, and define α_0 and β_0 similarly. We wish to show that γ is locally q-coCartesian. Unwinding the definitions, we must show that for $j \in \langle m\rangle^\circ$, then the induced edge $\lambda(\gamma_0^{-1}\{j\}) \to \lambda''(\{j\})$ is locally p-coCartesian. Since β is locally q-coCartesian, the edge $\lambda'(\beta_0^{-1}\{j\}) \to \lambda''(\{j\})$ is locally p-coCartesian. It will therefore suffice to show that $\lambda(\gamma_0^{-1}\{j\}) \to \lambda'(\beta_0^{-1}\{j\})$ is an equivalence, which follows from the inertness of α.

To prove (5), we node that the first condition of Definition 7.3.1.7 follows from Proposition 2.4.1.5, and the second follows the description of the class of locally q-coCartesian morphisms given by (2) and (3). $\square$

Notation 7.3.1.13. Let S be a simplicial set and let $p : \mathcal{C}^\otimes \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and $q : \mathcal{D}^\otimes \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be corepresentable local S-families of ∞-operads. We let $\mathrm{Fun}^\otimes_S(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$ denote the full subcategory of $\mathrm{Fun}_{S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$ spanned by those maps F which carry locally p-coCartesian edges of $\mathcal{C}^\otimes$ to locally q-coCartesian edges of $\mathcal{D}^\otimes$.

Proposition 7.3.1.14. *Let* $p : \mathcal{C}^\otimes \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *be a corepresentable local S-family of ∞-operads, let* $q : \mathcal{D} \to S$ *be a locally coCartesian fibration, and assume that each of the ∞-categories $\mathcal{D}_s$ admits finite products. Let* $\pi : \mathcal{D}^\times \to \mathcal{D}$ *be the Cartesian structure of Proposition 2.4.1.5. Then composition with π induces a trivial Kan fibration*

$$\mathrm{Fun}^\otimes_S(\mathcal{C}^\otimes, \mathcal{D}^\times) \to \mathrm{Fun}^\times_S(\mathcal{C}^\otimes, \mathcal{D}),$$

where $\mathrm{Fun}^\times_S(\mathcal{C}^\otimes, \mathcal{D})$ *is defined as in Proposition 7.3.1.9.*

Proof. Arguing as in the proof of Proposition 7.3.1.9, we can reduce to the case where $S = \Delta^k$ is a simplex (and, in particular, an ∞-category). Unwinding the definitions, we can identify $\mathrm{Fun}^\otimes_S(\mathcal{C}^\otimes, \mathcal{D}^\times)$ with the full subcategory of $\mathrm{Fun}_S(\mathcal{C}^\otimes \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\Gamma^\times), \mathcal{D})$ spanned by those functors F which satisfy the following condition:

(1) For every vertex $s \in S$ and every object $C \in \mathcal{C}^\otimes_{s,\langle n\rangle}$ and every subset $J \subseteq \langle n\rangle^\circ$, the functor F induces an equivalence

$$F(C, J) \to \prod_{j \in J} F(C, \{j\})$$

in the ∞-category $\mathcal{D}_s$.

(2) For every locally p-coCartesian morphism $\alpha : C \to C'$ covering a map $\alpha_0 : \langle n \rangle \to \langle n' \rangle$ in $\mathfrak{Fin}_*$, and every element $j \in \langle n' \rangle^\circ$, the induced map $F(C, \alpha^{-1}\{j\}) \to F(C', \{j\})$ is a locally q-coCartesian morphism of $\mathcal{D}$.

The functor $F' = \pi \circ F$ can be described by the formula $F'(C) = F(C, \langle n \rangle^\circ)$, for each $C \in \mathcal{C}^\otimes_{\langle n \rangle}$. In other words, F' can be identified with the restriction of F to the full subcategory of $\mathcal{C}' \subseteq \mathcal{C}^\otimes \times_{\mathrm{N}(\mathfrak{Fin}_*)} \mathrm{N}(\Gamma^\times)$ spanned by objects of the form $(C, \langle n \rangle^\circ)$ (note that $\mathcal{C}'$ is canonically isomorphic to $\mathcal{C}^\otimes$).

Let $X = (C, J)$ be an object of the fiber product $\mathcal{C}^\otimes \times_{\mathrm{N}(\mathfrak{Fin}_*)} \mathrm{N}(\Gamma^\times)$. Here $C \in \mathcal{C}^\otimes_{\langle n \rangle}$ and $J \subseteq \langle n \rangle^\circ$. We claim that the ∞-category $\mathcal{C}'_{X/}$ has an initial object. More precisely, if we choose an inert morphism $\overline{\alpha} : C \to C'$ covering the map $\alpha : \langle n \rangle \to J_*$ given by the formula

$$\alpha(j) = \begin{cases} j & \text{if } j \in J \\ * & \text{otherwise,} \end{cases}$$

then the induced map $\widetilde{\alpha} : (C, J) \to (C', J)$ is an initial object of $\mathcal{C}'_{X/}$. It follows that every functor $F' : \mathcal{C}' \to \mathcal{D}$ admits a right Kan extension to $\mathcal{C}^\otimes \times_{\mathrm{N}(\mathfrak{Fin}_*)} \mathrm{N}(\Gamma^\times)$, and that an arbitrary functor $F : \mathcal{C}^\otimes \times_{\mathrm{N}(\mathfrak{Fin}_*)} \mathrm{N}(\Gamma^\times) \to \mathcal{D}$ is a right Kan extension of $F | \mathcal{C}'$ if and only if $F(\widetilde{\alpha})$ is an equivalence, for every $\widetilde{\alpha}$ defined as above.

Let $\mathcal{E}$ be the full subcategory of $\mathrm{Fun}(\mathcal{C}^\otimes \times_{\mathrm{N}(\mathfrak{Fin}_*)} \mathrm{N}(\Gamma^\times), \mathcal{D})$ spanned by those functors F which satisfy the following conditions:

(1') The restriction $F' = F | \mathcal{C}'$ is a weak Cartesian structure on $\mathcal{C}' \simeq \mathcal{C}^\otimes$.

(2') The functor F is a right Kan extension of F'.

Using Proposition T.4.3.2.15, we conclude that the restriction map $\mathcal{E} \to \mathrm{Fun}^\times_S(\mathcal{C}^\otimes, \mathcal{D})$ is a trivial fibration of simplicial sets. To prove that θ is a trivial Kan fibration, it will suffice to show that conditions (1) and (2) are equivalent to conditions (1') and (2').

Suppose first F is a functor satisfying conditions (1') and (2'). We first verify condition (1). Let $C \in \mathcal{C}^\otimes_{s, \langle n \rangle}$, let $J \subseteq \langle n \rangle^\circ$. We wish to show that the canonical map $F(C, J) \to \prod_{j \in J} F(C, \{j\})$ is an equivalence in $\mathcal{D}_s$. Let $\widetilde{\alpha} : (C, J) \to (C', J)$ and $\{\widetilde{\alpha}_j : (C, \{j\}) \to (C'_j, \{j\})\}_{j \in J}$ be defined as above. We have a commutative diagram

$$
\begin{array}{ccc}
F(C, J) & \longrightarrow & \prod_{j \in J} F(C, \{j\}) \\
\downarrow & & \downarrow \\
F(C', J) & \longrightarrow & \prod_{j \in J} F(C'_j, \{j\})
\end{array}
$$

in the ∞-category $\mathcal{D}_s$. Using condition (2'), we deduce that the vertical maps are equivalences. It will therefore suffice to show that the lower horizontal map is an equivalence, which follows immediately from (1').

We now verify condition (2). Choose a locally p-coCartesian morphism $\widetilde{\beta} : C \to C'$ covering a map $\beta : \langle n \rangle \to \langle n' \rangle$ in $\mathfrak{Fin}_*$ and let $j \in \langle n' \rangle^\circ$. We wish to prove that the induced map $F(C, \beta^{-1}\{j\}) \to F(C', \{j\})$ is locally q-coCartesian. Lift the morphism $\rho^j : \langle n \rangle \to \langle 1 \rangle$ to an inert morphism $C' \to C''$ in $\mathcal{C}^\otimes$. Condition (2') implies the induced map $F(C', \{j\}) \to F(C'', \{1\})$ is an equivalence. We may therefore replace C' by C'' and thereby reduce to the case where $n' = 1$. The map $\widetilde{\beta}$ factors as a composition

$$C \xrightarrow{\widetilde{\beta}'} C_0 \xrightarrow{\widetilde{\beta}''} C'$$

where $\widetilde{\beta}'$ is inert and $\widetilde{\beta}''$ is active. Using (2') again, we deduce that $F(\widetilde{\beta}')$ is an equivalence. We may therefore replace C by C_0 and thereby reduce to the case where $\widetilde{\beta}$ is active, so that $\beta^{-1}\{1\} = \langle n \rangle^\circ$. We are therefore reduced to proving that $F(\widetilde{\beta})$ is locally p-coCartesian, which follows from (1').

Now suppose that F satisfies (1) and (2). We first verify that F satisfies (2′). Fix an object $C \in \mathcal{C}^{\otimes}_{\langle n \rangle}$ and a subset $J \subseteq \langle n \rangle^{\circ}$, and choose an inert morphism $C \to C'$ covering the canonical map $\langle n \rangle \to J_*$. We wish to prove that the induced map $F(C, J) \to F(C', J)$ is an equivalence in $\mathcal{D}$. Using condition (1), we are reduced to proving that the induced map $F(C, \{j\}) \to F(C', \{j\})$ is an equivalence for each $j \in J$, which is a special case of (2).

We now verify condition (1′). We first show that for each vertex $s \in S$, F' induces a weak Cartesian structure on the ∞-operad $\mathcal{C}^{\otimes}_s$. Suppose that $C \in \mathcal{C}^{\otimes}_{s, \langle n \rangle}$, and choose inert morphisms $C \to C_i$ lying over $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \leq i \leq n$. We wish to show that the induced maps $F'(C) \to F'(C_i)$ exhibit $F'(C)$ as a product of the objects $F(C_i)$ in the ∞-category $\mathcal{D}_s$. It follows from condition (1) that $F'(C)$ is a product of the objects $F(C, \{i\})$ for $1 \leq i \leq n$. It therefore suffices to show that each of the maps $F(C, \{i\}) \to F(C_i, \{1\})$ is an equivalence, which follows from (2). To complete the proof that F' is a weak Cartesian structure on $\mathcal{C}^{\otimes}$, it suffices to show that if $e : C \to C'$ is a locally p-coCartesian morphism in $\mathcal{C}^{\otimes}$ covering an active map $\langle n \rangle \to \langle 1 \rangle$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, then $F'(e)$ is locally q-coCartesian. This is also a special case of assumption (2). $\square$

Theorem 7.3.1.15. *Fix a simplicial set S.*

(1) *Let $p : \mathcal{C} \to S$ be a locally coCartesian fibration, and suppose that for each vertex $s \in S$, the ∞-category $\mathcal{C}_s$ admits finite products. Then there exists a Cartesian local S-family of ∞-operads $p : \mathcal{O}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and a Cartesian structure $\pi : \mathcal{O}^{\otimes} \to \mathcal{C}$. In particular, there is an equivalence $\mathcal{C} \simeq \mathcal{O}$ (of locally coCartesian fibrations over S).*

(2) *Let $p : \mathcal{C}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and $q : \mathcal{D}^{\otimes} \to S \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be Cartesian S-families of ∞-operads, let $\mathrm{Fun}^{\otimes}_S(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ be defined as in Notation 7.3.1.13, and let $\mathrm{Fun}^{\times}_S(\mathcal{C}, \mathcal{D})$ be defined as in Proposition 7.3.1.9. Then the restriction map $\theta : \mathrm{Fun}^{\otimes}_S(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) \to \mathrm{Fun}^{\times}_S(\mathcal{C}, \mathcal{D})$ is an equivalence of ∞-categories.*

Proof. Assertion (1) follows immediately from Proposition 2.4.1.5. To prove (2), we first define $\mathcal{D}^{\times}$ as in the proof of Proposition 2.4.1.5. Since $\mathcal{D}^{\otimes}$ is a Cartesian S-family of ∞-operads, Proposition 7.3.1.9 implies that there exists a Cartesian structure $\pi : \mathcal{D}^{\otimes} \to \mathcal{D}$. Using Proposition 7.3.1.14, we can assume that π factors as a composition

$$\mathcal{D}^{\otimes} \xrightarrow{\phi} \mathcal{D}^{\times} \xrightarrow{\pi'} \mathcal{D},$$

where π' is the Cartesian structure of Proposition 2.4.1.5 and $\phi \in \mathrm{Fun}^{\otimes}_S(\mathcal{D}^{\otimes}, \mathcal{D}^{\times})$. For every vertex $s \in S$ and each $\langle n \rangle \in \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, the induced map

$$\mathcal{D}^{\otimes}_{s, \langle n \rangle} \to \mathcal{D}^{\times}_{s, \langle n \rangle}$$

is an equivalence of ∞-categories (since both sides can be identified with the nth power of the ∞-category $\mathcal{D}_s$). It follows that ϕ is a categorical equivalence. We may therefore replace $\mathcal{D}^{\otimes}$ by $\mathcal{D}^{\times}$. In this case, the functor θ factors as a composition

$$\mathrm{Fun}^{\otimes}_S(\mathcal{C}^{\otimes}, \mathcal{D}^{\times}) \xrightarrow{\theta'} \mathrm{Fun}^{\times}_S(\mathcal{C}^{\otimes}, \mathcal{D}) \xrightarrow{\theta''} \mathrm{Fun}^{\times}_S(\mathcal{C}, \mathcal{D}),$$

where θ' is an equivalence of ∞-categories by Proposition 7.3.1.14 and θ'' is an equivalence of ∞-categories by Proposition 7.3.1.9. $\square$

7.3.2 Composition of Correspondences

Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor between compactly generated pointed ∞-categories which commutes with filtered colimits. Propositions 7.3.0.14 and 7.3.1.3 can be summarized informally as follows: the symmetric sequences $\partial^*(\mathrm{id}_{\mathcal{C}})$ and $\partial^*(\mathrm{id}_{\mathcal{D}})$ are equipped with coherently associative comultiplications

$$\partial^*(\mathrm{id}_{\mathcal{C}}) \to \partial^*(\mathrm{id}_{\mathcal{C}}) \circ \partial^*(\mathrm{id}_{\mathcal{C}}) \qquad \partial^*(\mathrm{id}_{\mathcal{D}}) \to \partial^*(\mathrm{id}_{\mathcal{D}}) \circ \partial^*(\mathrm{id}_{\mathcal{D}}),$$

and the symmetric sequence $\partial^*(F)$ is equipped with commuting coactions

$$\partial^*(F) \to \partial^*(\mathrm{id}_{\mathcal{D}}) \circ \partial^*(F) \qquad \partial^*(F) \to \partial^*(F) \circ \partial^*(\mathrm{id}_{\mathcal{C}}).$$

We can regard these assertions as a dual version of parts (1) and (2) of Conjecture 7.3.0.7. In this section, we will study the analogue of the third part of Conjecture 7.3.0.7. That is, we wish to show that if $G : \mathcal{D} \to \mathcal{E}$ is another reduced functor between compactly generated pointed ∞-categories which commutes with filtered colimits, then the symmetric sequence $\partial^*(G \circ F)$ can be regarded as a kind of tensor product of $\partial^*(G)$ with $\partial^*(F)$ over $\partial^*(\mathrm{id}_{\mathcal{D}})$. We begin by formalizing this idea more precisely using the language of stable families of ∞-operads.

Suppose that $p : \mathcal{O}^{\otimes} \to \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a corepresentable Δ^2-family of ∞-operads. Taking the fibers of the map $\mathcal{O}^{\otimes} \to \Delta^2$, we obtain corepresentable ∞-operads $\mathcal{O}_0^{\otimes}$, $\mathcal{O}_1^{\otimes}$, and $\mathcal{O}_2^{\otimes}$, and in particular we obtain tensor product functors

$$\otimes_0^I : \mathcal{O}_0^I \to \mathcal{O}_0 \qquad \otimes_1^I : \mathcal{O}_1^I \to \mathcal{O}_1 \qquad \otimes_2^I : \mathcal{O}_2^I \to \mathcal{O}_2$$

(see Remark 7.2.4.4). Taking the inverse image of edges in Δ^2, we obtain corepresentable correspondences from $\mathcal{O}_i^{\otimes}$ to $\mathcal{O}_j^{\otimes}$ for $i < j$. In particular, for every finite set I, we obtain functors

$$F^I : \mathcal{O}_0^I \to \mathcal{O}_1 \qquad G^I : \mathcal{O}_1^I \to \mathcal{O}_2 \qquad H^I : \mathcal{O}_0^I \to \mathcal{O}_2$$

(see the discussion following Definition 7.3.1.1). Every equivalence relation E on a finite set I determines a 2-simplex $(0, I_*) \to (1, (I/E)_*) \to (2, \langle 1 \rangle)$ in $\Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, to which the locally coCartesian fibration p associates a natural transformation

$$H^I \to G^{I/E} \circ \prod_{J \in I/E} F^J$$

of functors from $\mathcal{O}_0^I$ to $\mathcal{O}_2$. If $\mathcal{O}^{\otimes}$ is a stable Δ^2-family of ∞-operads, then we obtain a map of symmetric sequences $\delta : H^* \to G^* \circ F^*$ in the ∞-category $\mathrm{SSeq}(\mathcal{O}_0, \mathcal{O}_2)$. By studying the restriction of p to 3-simplices of $\Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, it is not difficult to verify that the diagram of symmetric sequences

$$
\begin{array}{ccc}
H^* & \xrightarrow{\;\;\delta\;\;} & G^* \circ F^* \\
{\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \times \phi} \\
G^* \circ F^* & \xrightarrow{\;\psi\circ\mathrm{id}\;} & G^* \circ \otimes_1^* \circ F^*
\end{array}
$$

commutes up to homotopy, where $\phi : F^* \to \otimes_1^* \circ F^*$ and $\psi : G^* \to G^* \circ \otimes_1^*$ are given by the coactions of $\otimes_1^*$ on F^* and G^*, respectively. By studying the inverse image under p of higher-dimensional simplices of $\Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, we obtain certain higher coherence conditions. We may summarize the situation informally as follows:

(∗) A stable Δ^2-family of ∞-operads $\mathcal{O}^{\otimes} \to \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ determines coalgebras $\otimes_i^* \in \mathrm{SSeq}(\mathcal{O}_i, \mathcal{O}_i)$, an $\otimes_1^*$-$\otimes_0^*$ bimodule $F^* \in \mathrm{SSeq}(\mathcal{O}_0, \mathcal{O}_1)$, an $\otimes_2^*$-$\otimes_1^*$ bimodule $G^* \in \mathrm{SSeq}(\mathcal{O}_1, \mathcal{O}_2)$, and a map of $\otimes_2^*$-$\otimes_0^*$ bimodules $H^* \to G^* \circ^{\otimes_1^*} F^*$ in $\mathrm{SSeq}(\mathcal{O}_0, \mathcal{O}_2)$.

Now suppose we are given compactly generated pointed ∞-categories $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$, together with reduced functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$, and $H : \mathcal{C} \to \mathcal{E}$ commuting with filtered colimits, and a natural transformation $\alpha : H \to G \circ F$. We can encode this data by locally coCartesian fibration $\mathcal{M} \to \Delta^2$ with $\mathcal{M}_0 \simeq \mathcal{C}$, $\mathcal{M}_1 \simeq \mathcal{D}$, and $\mathcal{M}_2 \simeq \mathcal{D}$ Let $p : \mathcal{M}^{\times} \to \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be as in Construction 7.3.1.10. Then p is a differentiable Δ^2-family of ∞-operads, so it admits a stabilization $q : \mathcal{O}^{\otimes} \to \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. According to (∗), the stable Δ^2-family of ∞-operads q determines a map of symmetric sequences

$$\alpha^+ : \partial^*(H) \to \partial^*(G) \circ^{\partial^*(\mathrm{id}_{\mathcal{D}})} \partial^*(F).$$

We would like to say that if α is an equivalence (that is, if the map $\mathcal{M} \to \Delta^2$ is actually a coCartesian fibration), then the map α^+ is also an equivalence. What we will actually prove is a reformulation of this assertion, which does not require us to define (relative) composition products of symmetric sequences or to justify the heuristic $(*)$ given above. Our reformulation is based on the notion of a *thin* Δ^2-family of ∞-operads (Definition 7.3.2.12). Informally speaking, a stable Δ^2-family of ∞-operads $\mathcal{O}^\otimes \to \Delta^2 \times N(\mathcal{F}in_*)$ is *thin* if it determines a morphism $H^* \to G^* \circ_1^{\otimes^*} F^*$ which is an equivalence in $\mathrm{SSeq}(\mathcal{O}_0, \mathcal{O}_2)$. However, the precise definition of thinness does not directly reference the theory of symmetric sequences, and makes sense also in unstable situations. Granting the notion of flatness, we can formulate our main result as follows:

Theorem 7.3.2.1. *Let $p : \mathcal{C} \to \Delta^2$ be a coCartesian fibration. Assume that the fibers of p are pointed and compactly generated, and that for $i < j$ the induced functor $\mathcal{C}_i \to \mathcal{C}_j$ is reduced and preserves filtered colimits. Let $q : \overline{\mathcal{C}}^\otimes \to \Delta^2 \times N(\mathcal{F}in_*)$ be a stabilization of the differentiable Δ^2-family of ∞-operads $\mathcal{C}^\times \to \Delta^2 \to N(\mathcal{F}in_*)$. Then $\overline{\mathcal{C}}^\otimes$ is a thin Δ^2-family of ∞-operads.*

Theorem 7.3.2.1 is an immediate consequence of the following pair of assertions:

Theorem 7.3.2.2. *Let $p : \mathcal{C}^\otimes \to \Delta^2 \times N(\mathcal{F}in_*)$ be a Cartesian Δ^2-family of ∞-operads, and suppose that the underlying locally coCartesian fibration $p_0 : \mathcal{C} \to \Delta^2$ is a coCartesian fibration. Then $\mathcal{C}^\otimes$ is a thin Δ^2-family of ∞-operads.*

Theorem 7.3.2.3. *Let $q : \mathcal{C}^\otimes \to \Delta^2 \times N(\mathcal{F}in_*)$ be a corepresentable Δ^2-family of ∞-operads. Assume that:*

(1) *Each of the ∞-categories $\mathcal{C}_i$ is pointed and compactly generated.*

(2) *For every pair of induces $0 \le i \le j \le 2$ and every finite set I, the induced functor $\mathcal{C}_i^I \to \mathcal{C}_j$ preserves filtered colimits and zero objects.*

(3) *Let $0 \le i \le 2$, let $\alpha : \langle m \rangle \to \langle n \rangle$ be an injective map of pointed finite sets, and let $\overline{\alpha}$ be a morphism in $\mathcal{C}^\otimes$ lifting the induced map $(i, \langle m \rangle) \to (i, \langle n \rangle)$. If α is locally q-coCartesian, then α is q-coCartesian.*

Let $\overline{\mathcal{C}}^\otimes \to \Delta^2 \times N(\mathcal{F}in_*)$ be a stabilization of $\mathcal{C}^\otimes$. If $\mathcal{C}^\otimes$ is a thin Δ^2-family of ∞-operads, then $\overline{\mathcal{C}}^\otimes$ is also a thin Δ^2-family of ∞-operads.

Remark 7.3.2.4. Let $q : \mathcal{C}^\otimes \to \Delta^2 \times N(\mathcal{F}in_*)$ be a corepresentable Δ^2-family of ∞-operads satisfying conditions (1) and (2). Then each of the ∞-categories $\mathcal{C}_i$ admits a zero object $*_i$. If S is a finite set and $0 \le i \le j \le 2$, then q determines a functor $F_{i,j}^S : \mathcal{C}_i^S \to \mathcal{C}_j$ satisfying $F_{i,j}^S(*_i, *_i, \ldots, *_i) \simeq *_j$. Moreover, for every finite set T, we have canonical maps

$$F_{i,j}^S(\{X_s\}_{s \in S}) \to F_{i,j}^{S \cup T}(\{X_s\}_{s \in S}, \{*_i\}_{t \in T}).$$

Condition (3) of Theorem 7.3.2.1 asserts that each of these maps is an equivalence. Note that this condition is automatically satisfied if q is a Cartesian Δ^2-family of ∞-operads.

We will prove Theorem 7.3.2.2 at the end of this section. The proof of Theorem 7.3.2.3 is quite a bit more difficult, and will be given in §7.3.6.

Our first step is to define the notion of a thin Δ^2-family of ∞-operads. This will require a brief digression. Suppose that $q : X \to S$ is a locally coCartesian fibration of simplicial sets. Then every edge $e : s \to t$ in S induces a functor of ∞-categories $X_s \to X_t$. More generally, we can associate to every n-simplex $\sigma : \Delta^n \to S$ a functor from $X_{\sigma(0)}$ to $X_{\sigma(n)}$, given by the composition of the functors

$$\theta(\sigma) : X_{\sigma(0)} \to X_{\sigma(1)} \to \cdots \to X_{\sigma(n)}$$

associated to the edges belonging to the "spine" of Δ^n. If q is a coCartesian fibration, then the functor $\theta(\sigma)$ depends only on the edge $\sigma(0) \to \sigma(n)$ of S determined by σ. However, if we assume only that q is a locally coCartesian fibration, then $\theta(\sigma)$ depends on the entire simplex σ. We will need to precisely articulate the sense in which the functor $\theta(\sigma)$ depends on σ.

Notation 7.3.2.5. Let S be a simplicial set. We let $\mathbf{\Delta}_S$ denote the *category of simplices* of S. The objects of $\mathbf{\Delta}_S$ are maps $\sigma : \Delta^m \to S$. Given a pair of objects $\sigma : \Delta^m \to S$ and $\tau : \Delta^n \to S$, we let $\mathrm{Hom}_{\mathbf{\Delta}_S}(\sigma, \tau)$ denote the collection of commutative diagrams

$$
\begin{array}{ccc}
\Delta^m & \longrightarrow & \Delta^n \\
 & \searrow{\scriptstyle \sigma} \quad \swarrow{\scriptstyle \tau} & \\
 & S. &
\end{array}
$$

Suppose we are given a pair of vertices $s, t \in S$. We define a subcategory $\mathbf{\Delta}_S^{s,t}$ as follows:

(i) An object $\sigma : \Delta^m \to S$ belongs to $\mathbf{\Delta}_S^{s,t}$ if $n \geq 1$, $\sigma(0) = s$, and $\sigma(n) = t$.

(ii) Given objects $\sigma, \tau \in \mathbf{\Delta}_S^{s,t}$, a commutative diagram

$$
\begin{array}{ccc}
\Delta^m & \overset{\theta}{\longrightarrow} & \Delta^n \\
 & \searrow{\scriptstyle \sigma} \quad \swarrow{\scriptstyle \tau} & \\
 & S. &
\end{array}
$$

determines a morphism in $\mathbf{\Delta}_S^{s,t}$ if and only if $\theta(0) = 0$ and $\theta(m) = n$.

Construction 7.3.2.6. Let $q : X \to S$ be a locally coCartesian fibration of simplicial sets, and suppose we are given vertices $s, t \in S$. We define a functor $\phi : (\mathbf{\Delta}_S^{s,t})^{op} \to \mathrm{Set}_{\Delta}$ by the formula $\phi(\sigma : \Delta^m \to S) = \mathrm{Fun}_S(\Delta^m, X)^{op}$. Let $\mathcal{Z}$ denote the opposite of the relative nerve $\mathrm{N}_\phi((\mathbf{\Delta}_S^{s,t}))$ (see Definition T.3.2.5.2). Then $\mathcal{Z}$ is an ∞-category equipped with a Cartesian fibration $\mathcal{Z} \to \mathrm{N}(\mathbf{\Delta}_S^{s,t})$. Unwinding the definitions, we can identify the objects of $\mathcal{Z}$ with simplices $\sigma : \Delta^n \to X$ such that $n \geq 1$, $\sigma(0) \in X_s$, and $\sigma(n) \in X_t$. Note that the evaluation maps $\sigma \mapsto \sigma(0)$ and $\tau \mapsto \tau(0)$ determine functors $e_s : \mathcal{Z}_q \to X_s$ and $e_t : \mathcal{Z}_q \to X_t$.

Let $\mathcal{Z}^0$ denote the full subcategory of $\mathcal{Z}$ spanned by those simplices $\sigma : \Delta^n \to X$ such that, for each $1 \leq i \leq n$, the map $\sigma(i-1) \to \sigma(i)$ is locally q-coCartesian.

Lemma 7.3.2.7. *Let $q : X \to S$ be a locally coCartesian fibration of simplicial sets, let $s, t \in S$ be vertices, and let $\mathcal{Z}^0 \subseteq \mathcal{Z}$ be defined as in Construction 7.3.2.6. For each $\sigma \in \mathbf{\Delta}_S^{s,t}$, let $\mathcal{Z}_\sigma$ denote the fiber of $\mathcal{Z} \to \mathrm{N}(\mathbf{\Delta}_S^{s,t})^{op}$ over σ, and define $\mathcal{Z}_\sigma^0$ similarly. Then:*

(a) *For each $\sigma \in \mathbf{\Delta}_S^{s,t}$, the map e_s induces a trivial Kan fibration $\mathcal{Z}_\sigma^0 \to X_s$.*

(b) *For every object $\sigma \in \mathbf{\Delta}_S^{s,t}$, the inclusion $\mathcal{Z}_\sigma^0 \hookrightarrow \mathcal{Z}_\sigma$ admits a right adjoint L_σ. Moreover, if α is a morphism in $\mathcal{Z}_\sigma$, then $L_\sigma(\alpha)$ is an equivalence if and only if $e_0(\alpha)$ is an equivalence, where $e_0 : \mathcal{Z}_\sigma \to X_s$ is defined as in Construction 7.3.2.6.*

(c) *The forgetful functor $f : \mathcal{Z} \to \mathrm{N}(\mathbf{\Delta}_S^{s,t})^{op}$ restricts to a Cartesian fibration $f^0 : \mathcal{Z}^0 \to \mathrm{N}(\mathbf{\Delta}_S^{s,t})$.*

(d) *The map e_s induces an equivalence of ∞-categories*

$$
\mathcal{Z}_\sigma^0 \to X_s \times \mathrm{N}(\mathbf{\Delta}_S^{s,t}).
$$

Proof. We first prove (a). Fix an object $\sigma : \Delta^n \to S$ of $\mathbf{\Delta}_S^{s,t}$. For $0 \leq j \leq n$, let

$$
\mathcal{C}_j = \mathrm{Fun}_S(\Delta^{\{0,1\}} \coprod_{\{1\}} \Delta^{\{1,2\}} \coprod \cdots \coprod_{\{j-1\}} \Delta^{\{j-1,j\}}, X),
$$

and let $\mathcal{C}_j'$ denote the full subcategory of $\mathcal{C}_j$ spanned by those diagrams which carry each edge $\Delta^{\{i-1,i\}}$ to a q-coCartesian edge of X for $1 \leq i \leq j$. Each of the restriction maps $\mathcal{C}_j' \to \mathcal{C}_{j-1}'$ is a pullback of the restriction map

$$
r_j : \mathrm{Fun}_S'(\Delta^{\{j-1,j\}}, X) \to \mathrm{Fun}_S(\{j-1\}, X),
$$

where $\mathrm{Fun}'_S(\Delta^{\{j-1,j\}}, X)$ denotes the full subcategory of $\mathrm{Fun}_S(\Delta^{\{j-1,j\}}, X)$ spanned by the locally q-coCartesian edges. Proposition T.4.3.2.15 (and our assumption that q is a locally coCartesian fibration) imply that each r_j is a trivial Kan fibration. We have a pullback diagram

$$
\begin{array}{ccc}
\mathcal{Z}^0_\sigma & \longrightarrow & \mathcal{C}'_n \\
\downarrow & & \downarrow \\
\mathcal{Z}_\sigma & \longrightarrow & \mathcal{C}_n\,.
\end{array}
$$

The lower horizontal map is a trivial Kan fibration because the inclusion

$$
\Delta^{\{0,1\}} \coprod_{\{1\}} \Delta^{\{1,2\}} \coprod \cdots \coprod_{\{n-1\}} \Delta^{\{n-1,n\}} \hookrightarrow \Delta^n
$$

is inner anodyne. It follows that the restriction map $\mathcal{Z}^0_\sigma \to \mathcal{C}'_n$ is a trivial Kan fibration, so that the composite map

$$
\theta : \mathcal{Z}^0_\sigma \to \mathcal{C}'_n \to \mathcal{C}'_{n-1} \to \cdots \to \mathcal{C}'_1 \to \mathcal{C}'_0 \simeq X_s
$$

is a trivial Kan fibration, as desired.

We now prove (b). Let σ be as above, and let $s : X_s \to \mathcal{Z}^0_\sigma$ be a section of the trivial Kan fibration $\theta_0 : \mathcal{Z}^0_\sigma$. We will show that the identity map $\mathrm{id}_{X_s} \to \theta_0 \circ s$ exhibits s as a left adjoint to θ_0; it will then follow from (a) that the composition $s \circ \theta_0$ is a right adjoint to the inclusion. Fix an object $x \in X_s$ and an object $z \in \mathcal{Z}_\sigma$; we wish to show that the canonical map

$$
\mathrm{Map}_{\mathcal{Z}_\sigma}(s(x), z) \to \mathrm{Map}_{X_s}(x, \theta_0(z))
$$

is a homotopy equivalence. For each $0 \leq j \leq n$, let $\theta_j : \mathcal{Z}_\sigma \to \mathcal{C}_j$ be the restriction functor. We will prove that each of the maps

$$
\mathrm{Map}_{\mathcal{Z}_\sigma}(s(x), z) \to \mathrm{Map}_{\mathcal{C}_j}(\theta_j s(x), \theta_j(z))
$$

is a homotopy equivalence using descending induction on j. When $j = n$, the desired result is obvious (since θ_n is a trivial Kan fibration), and when $j = 0$ it will imply the desired result. To carry out the inductive step, it will suffice to show that the map

$$
u : \mathrm{Map}_{\mathcal{C}_j}(\theta_j s(x), \theta_j(z)) \to \mathrm{Map}_{\mathcal{C}_{j-1}}(\theta_{j-1} s(x), \theta_{j-1}(z))
$$

is a homotopy equivalence. Let x' and z' be the images of $s(x)$ and z in $\mathrm{Fun}_S(\Delta^{\{j-1,j\}}, X)$ and let x'_0 and z'_0 be their images in $X_{\sigma(j-1)}$. Then u is a pullback of the restriction map

$$
\mathrm{Map}_{\mathrm{Fun}_S(\Delta^{\{j-1,j\}}, X)}(x', z') \to \mathrm{Map}_{X_{\sigma(j-1)}}(x'_0, z'_0),
$$

which is a homotopy equivalence by virtue of the fact that x' is a locally q-coCartesian edge of X.

Assertion (c) follows from (b) and Lemma 2.2.1.11, and assertion (d) follows from Corollary T.2.4.4.4 together with (a) and (c). $\qquad\square$

Definition 7.3.2.8. Let $q : X \to S$ be a locally coCartesian fibration of simplicial sets. Let $s, t \in S$ be vertices, let $\mathcal{Z}^0 \subseteq \mathcal{Z}$ be defined as in Construction 7.3.2.6, and let $h : X_s \times \mathrm{N}(\boldsymbol{\Delta}^{s,t}_S) \to \mathcal{Z}^0$ be a homotopy inverse to the equivalence of ∞-categories $\mathcal{Z}^0 \to X_s \times \mathrm{N}(\boldsymbol{\Delta}^{s,t}_S)$ of Lemma 7.3.2.7. The composite map

$$
X_s \times \mathrm{N}(\boldsymbol{\Delta}^{s,t}_S) \overset{h}{\to} \mathcal{Z}^0 \subseteq \mathcal{Z} \overset{e_t}{\to} X_t
$$

determines a functor $\theta : \mathrm{N}(\boldsymbol{\Delta}^{s,t}_S) \to \mathrm{Fun}(X_s, X_t)$, which is well-defined up to homotopy. We will refer to θ as the *spray* associated to q.

Remark 7.3.2.9. In the situation of Definition 7.3.2.8, suppose that $\sigma : \Delta^n \to S$ is an object of $\mathbf{\Delta}_S^{s,t}$, so that σ determines a sequence of edges

$$s = s_0 \overset{f(1)}{\to} s_1 \overset{f(2)}{\to} \cdots \overset{f(n)}{\to} s_n = t.$$

Since q is a coCartesian fibration, each of the edges $f(i)$ determines a functor $f(i)_! : X_{\sigma(i-1)} \to X_{\sigma(i)}$. If $\theta : \mathrm{N}(\mathbf{\Delta}_S^{s,t}) \to \mathrm{Fun}(X_s, X_t)$ is the spray associated to q, then $\theta(\sigma)$ is given (up to homotopy) by the composition $f(n)_! \circ f(n-1)_! \circ \cdots \circ f(1)_!$.

Remark 7.3.2.10. In the situation of Definition 7.3.2.8, suppose that the map $q : X \to S$ is a coCartesian fibration of simplicial sets. Then the spray $\theta : \mathrm{N}(\mathbf{\Delta}_S^{s,t}) \to \mathrm{Fun}(X_s, X_t)$ is locally constant: that is, it carries each morphism in $\mathbf{\Delta}_S^{s,t}$ to an equivalence in $\mathrm{Fun}(X_s, X_t)$.

Notation 7.3.2.11. Let S be a finite set, and let $\mathrm{Equiv}(S)$ be the partially ordered set of equivalence relations on S (see Construction 7.3.3.12). We let $\mathrm{Part}(S)$ denote the partially ordered set consisting of linearly ordered subsets of $\mathrm{Equiv}(S)$. We regard $\mathrm{Part}(S)$ as partially ordered with respect to inclusions. Let $\mathrm{Part}^0(S) \subseteq \mathrm{Part}(S)$ denote the subset consisting of *nonempty* linearly ordered subsets of $\mathrm{Equiv}(S)$.

Suppose that $S = \{1, \ldots, n\}$. The construction

$$(E \in \mathrm{Equiv}(S)) \mapsto (S/E)_*$$

determines a functor $\mathrm{Equiv}(S) \to \mathcal{F}\mathrm{in}_*$. Let $s = (0, \langle n \rangle) \in \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and $t = (2, \langle 1 \rangle) \in \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. We define a functor

$$\chi : \mathrm{Part}(S) \to \mathbf{\Delta}_{\Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)}^{s,t}$$

as follows: to every chain of equivalence relations $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_k$ in $\mathrm{Equiv}(S)$, χ associates the $(k+1)$-simplex of $\Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ given by the sequence of active morphisms

$$(0, \langle n \rangle) \to (1, (S/E_1)_*) \to (1, (S/E_2)_*) \to \cdots \to (1, (S/E_k)_*) \to (2, \langle 1 \rangle).$$

Definition 7.3.2.12. Let $q : \mathcal{O}^{\otimes} \to \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a Δ^2-family of ∞-operads. We will say that $\mathcal{O}^{\otimes}$ is *thin* if is corepresentable, the ∞-category $\mathcal{O}_2$ admits finite limits and the following condition is satisfied for each $n \geq 1$:

(∗) Let $S = \{1, \ldots, n\}$, let $T = \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, let $\chi : \mathrm{Part}(S) \to \mathbf{\Delta}_T^{s,t}$ be as in Notation 7.3.2.11 and let $\theta : \mathrm{N}(\mathbf{\Delta}_T^{s,t}) \to \mathrm{Fun}(\mathcal{O}_{\langle n \rangle}^{\otimes}, \mathcal{O})$ be the spray associated to q (see Definition 7.3.2.8). Then the composite map

$$\mathrm{N}(\mathrm{Part}^0(S))^{\lhd} \simeq \mathrm{N}(\mathrm{Part}(S)) \overset{\chi}{\to} \mathrm{N}(\mathbf{\Delta}_T^{s,t}) \overset{\theta}{\to} \mathrm{Fun}(\mathcal{O}_{(0,\langle n \rangle)}^{\otimes}, \mathcal{O}_{(2,\langle 1 \rangle)}^{\otimes}) \simeq \mathrm{Fun}(\mathcal{O}_0^S, \mathcal{O}_2)$$

is a limit diagram in the ∞-category $\mathrm{Fun}(\mathcal{O}_0^S, \mathcal{O}_2)$.

Remark 7.3.2.13. Let $q : \mathcal{O}^{\otimes} \to \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a corepresentable Δ^2-family of ∞-operads, so that q determines functors

$$\otimes_0^m : \mathcal{O}_0^m \to \mathcal{O}_0 \qquad \otimes_1^m : \mathcal{O}_1^m \to \mathcal{O}_1 \qquad \otimes_2^m : \mathcal{O}_2^m \to \mathcal{O}_2$$
$$F^m : \mathcal{O}_0^m \to \mathcal{O}_1 \qquad G^m : \mathcal{O}_1^m \to \mathcal{O}_2 \qquad H^m : \mathcal{O}_0^m \to \mathcal{O}_2$$

for every integer $m \geq 0$. When $n = 1$, condition (∗) of Definition 7.3.2.12 asserts that for every object $X \in \mathcal{O}_0$, the canonical map $H^1(X) \to G^1(F^1(X))$ is an equivalence. When $n = 2$, condition (∗) of Definition 7.3.2.12 guarantees that for every pair of objects $X, Y \in \mathcal{O}_0$, the diagram

$$
\begin{array}{ccc}
H^2(X,Y) & \longrightarrow & G^1(F^2(X,Y)) \\
\downarrow & & \downarrow \\
G^2(F^1(X), F^1(Y)) & \longrightarrow & G^1(F^1(X) \otimes_1^2 F^1(Y)).
\end{array}
$$

For larger values of n, condition (∗) guarantees that the functor $H^n : \mathcal{O}_0^n \to \mathcal{O}_2$ can be recovered as the limit of a finite diagram of functors obtained by composing the functors F^p, G^q, and $\otimes_1^r$ for $p, q, r > 0$.

Remark 7.3.2.14. To place Definition 7.3.2.12 in context, we remark that there exists an $(\infty, 2)$-category SSeq which may be described roughly as follows:

- The objects of SSeq are pairs $(\mathcal{O}, U^*)$, where $\mathcal{O}$ is a stable ∞-category and U^* is a coalgebra object of SSeq$(\mathcal{O}, \mathcal{O})$ whose counit induces an equivalence $U^1 \to \mathrm{id}_{\mathcal{O}}$ (equivalently, we can define the objects of SSeq to be unital stable ∞-operads: see Remark 7.3.0.10).

- Given a pair of objects $(\mathcal{O}, U^*), (\mathcal{O}', U'^*) \in$ SSeq, the ∞-category of morphisms from $(\mathcal{O}, U^*)$ to $(\mathcal{O}', U'^*)$ in SSeq can be identified with the collection of U'^*-U^* comodule objects of SSeq$(\mathcal{O}, \mathcal{O}')$ (which we can think of as stable correspondences between the corresponding stable ∞-operads: see Remark 7.3.1.2).

Every stable Δ^2-family of ∞-operads $q : \mathcal{O}^{\otimes} \to \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ determines 1-morphisms

$$f : (\mathcal{O}_0, \otimes_0^*) \to (\mathcal{O}_1, \otimes_1^*) \qquad g : (\mathcal{O}_1, \otimes_1^*) \to (\mathcal{O}_2, \otimes_2^*) \qquad h : (\mathcal{O}_2, \otimes_0^*) \to (\mathcal{O}_2, \otimes_2^*)$$

together with a 2-morphism $\alpha : h \to g \circ f$ in the $(\infty, 2)$-category SSeq. The thinness of q is equivalent to the invertibility of the 2-morphism α.

We warn the reader that this $(\infty, 2)$-categorical interpretation of Definition 7.3.2.12 is specific to the case of stable families of ∞-operads, and can lead to misleading intuitions in the unstable case.

Proof of Theorem 7.3.2.2. Let $\mathcal{C}_0$, $\mathcal{C}_1$, and $\mathcal{C}_2$ denote the fibers of the coCartesian fibration $p_0 : \mathcal{C} \to \Delta^2$, so that p_0 determines functors $F : \mathcal{C}_0 \to \mathcal{C}_1$ and $G : \mathcal{C}_1 \to \mathcal{C}_2$. Fix a nonempty finite set I. The construction of Notation 7.3.2.11 determines a functor $\theta : \mathrm{N}(\mathrm{Part}(I)) \to \mathrm{Fun}(\mathcal{C}_0^I, \mathcal{C}_2)$. We wish to show that for every sequence of objects $\vec{C} = \{C_i \in \mathcal{C}_0\}_{i \in I}$, the induced map

$$\theta(\emptyset)(\vec{C}) \to \varinjlim_{P \in \mathrm{Part}^0(I)} \theta(P)(\vec{C})$$

is an equivalence in $\mathcal{C}_2$. Unwinding the definitions, we see that θ carries a nonempty chain of equivalence relations $P = (E_1 \subset E_2 \subset \cdots \subset E_k)$ to the functor $\theta(P)$ given by the formula

$$\theta(P)(\{C_i\}) = G(\prod_{J \in I/E_1} F(\prod_{j \in J} C_j)).$$

In particular, $\theta^0 = \theta | \mathrm{N}(\mathrm{Part}^0(I))$ factors as a composition

$$\mathrm{N}(\mathrm{Part}^0(I)) \xrightarrow{\phi} \mathrm{N}(\mathrm{Equiv}(I))^{op} \xrightarrow{\psi} \mathrm{Fun}(\mathcal{C}_0^I, \mathcal{C}_2),$$

where ϕ carries a chain of equivalence relations $(E_1 \subset E_2 \subset \cdots \subset E_k)$ to the equivalence relation E_1. We claim that ϕ is right cofinal. Since ϕ is a coCartesian fibration, it suffices to show that the fibers of ϕ are weakly contractible (Lemma T.4.1.3.2). This is clear, since each of these fibers has both an initial object. We are therefore reduced to proving that the canonical map

$$\theta(\emptyset)(\vec{C}) \to \varprojlim_{E \in \mathrm{Equiv}(I)} \psi(E)(\{C_i\})$$

is an equivalence. Note that $\mathrm{N}(\mathrm{Equiv}(I))$ has a final object, given by the indiscrete equivalence relation $E_{\top}$ (such that $iE_{\top}j$ for all $i, j \in I$). We are therefore reduced to proving that the natural transformation $\theta(\emptyset) \to \psi(E_{\top})$ is an equivalence of functors from $\mathcal{C}_0^I$ to $\mathcal{C}_2$, which follows immediately from the definitions. $\square$

7.3.3 Derivatives of the Identity Functor

Let $\mathcal{C}$ be a compactly generated pointed ∞-category. Conjecture 7.3.0.7 asserts in particular that the derivatives $\{\partial_n(\mathrm{id}_{\mathcal{C}})\}_{n \geq 1}$ can be regarded as an algebra object in the ∞-category of symmetric sequences

$\mathrm{SSeq}(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{C}))$. Let us consider the special case where $\mathcal{C} = \mathcal{S}_*$ is the ∞-category of pointed spaces, so that $\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Sp}$ is the ∞-category of spectra. Let $S \in \mathrm{Sp}(\mathcal{S}_*) \simeq \mathrm{Sp}$ denote the sphere spectrum. Repeatedly applying Corollary 1.4.4.6, we see that each of the functors $\partial_n(\mathrm{id}_{\mathcal{C}})$ is determined by the single object $(\partial_n F)(S, S, \ldots, S) \in \mathrm{Sp}(\mathcal{C})$, which we will denote by $\partial_n(F)$. In the special case where $\mathcal{C} = \mathcal{S}_*$, we can identify each $\partial_n(F)$ with a spectrum, so that $\{\partial_n(F)\}_{n \geq 1}$ can be viewed as a symmetric sequence of spectra.

Let $\mathrm{id} : \mathcal{S}_* \to \mathcal{S}_*$ be the identity functor. Conjecture 7.3.0.7 implies in particular that $\{\partial_n(\mathrm{id})\}_{n \geq 1}$ can be regarded as an algebra with respect to the composition product on symmetric sequences of spectra. In other words, it implies that can regard $\{\partial_n(\mathrm{id})\}_{n \geq 1}$ as an *operad* in the category of spectra, in that it is equipped with composition maps

$$\partial_m(\mathrm{id}) \otimes \partial_{n_1}(\mathrm{id}) \otimes \cdots \otimes \partial_{n_m}(\mathrm{id}) \to \partial_{n_1 + \cdots + n_m}(\mathrm{id})$$

satisfying suitable associative laws. A structure of this type was constructed by Ching, who showed that $\{\partial_n(\mathrm{id})\}_{n \geq 1}$ can be regarded as a homotopy-theoretic analogue of the Lie operad (see [29]). Ching's work built upon earlier results of Arone and Mahowald, who gave an explicit description of the spectra $\partial_n(\mathrm{id})$ as the Spanier-Whitehead dual of a certain partition complex. To recall their result, we need to introduce a bit of notation.

Definition 7.3.3.1. Let I be a nonempty finite set. We let $\mathrm{Equiv}(I)$ denote the collection of all equivalence relations on I. We will regard $\mathrm{Equiv}(I)$ as a partially ordered set, where $E \leq E'$ if xEy implies $xE'y$.

We let $E_\top$ denote the trivial equivalence relation on I (so that $xE_\top y$ for all $x, y \in S$) and $E_\perp$ the discrete equivalence relation on I (so that $xE_\perp y$ if and only if $x = y$). Then $E_\top$ and $E_\perp$ are the greatest and smallest elements of $\mathrm{Equiv}(I)$, respectively. We set

$$\mathrm{Equiv}^+(I) = \mathrm{Equiv}(I) - \{E_\top\} \qquad \mathrm{Equiv}^-(I) = \mathrm{Equiv}(I) - \{E_\perp\}$$

$$\mathrm{Equiv}^\pm(I) = \mathrm{Equiv}^+(I) \cap \mathrm{Equiv}^-(I) = \mathrm{Equiv}(I) - \{E_\top, E_\perp\}.$$

Notation 7.3.3.2. Let $S \in \mathrm{Sp}$ denote the sphere spectrum. Since the ∞-category Sp^{op} admits small colimits, Theorem T.5.1.5.6 implies that there is an essentially unique functor $F : \mathcal{S} \to \mathrm{Sp}^{op}$ which preserves small colimits and satisfies $F(*) = S$. If X is a space, we will denote the spectrum $F(X)$ by S^X, and refer to it as the *Spanier-Whitehead dual* of X.

More generally, if K is any simplicial set, we let S^K denote the Spanier-Whitehead dual of a fibrant replacement for K.

Theorem 7.3.3.3 (Arone-Mahowald)**.** *Let $n \geq 1$ be a positive integer and set $I = \{1, \ldots, n\}$. Then there is a fiber sequence of spectra*

$$\partial_n(\mathrm{id}) \to S^{\mathrm{N}(\mathrm{Equiv}(I))} \to S^{\mathrm{N}(\mathrm{Equiv}^+(I))} \times_{S^{\mathrm{N}(\mathrm{Equiv}^\pm(I))}} S^{\mathrm{N}(\mathrm{Equiv}^-(I))}.$$

Remark 7.3.3.4. The fiber sequence of Theorem 7.3.3.3 is equivariant with respect to the action of the symmetric group Σ_n.

Remark 7.3.3.5. If $n = 1$, then the partially ordered sets $\mathrm{Equiv}^+(I)$, $\mathrm{Equiv}^-(I)$, and $\mathrm{Equiv}^\pm(I)$ are empty. In this case, Theorem 7.3.3.3 asserts the existence of an equivalence $\partial_1(\mathrm{id}) \simeq S$, which can be deduced directly from Example 7.2.1.4. If $n > 1$, then the simplicial sets $\mathrm{N}(\mathrm{Equiv}(I))$, $\mathrm{N}(\mathrm{Equiv}^+(I))$, and $\mathrm{N}(\mathrm{Equiv}^-(I))$ are weakly contractible. In this case, Theorem 7.3.3.3 asserts the existence of an equivalence

$$\partial_n(\mathrm{id}) \to \Omega \, \mathrm{fib}(S \to S^{\mathrm{N}(\mathrm{Equiv}^\pm(I))}).$$

For the original proof of Theorem 7.3.3.3, we refer the reader to [3]. In this section, we will give a rather different proof which is based on the exactness properties of the construction $F \mapsto \vec{\partial}(F)$, where $F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{D}$ is a functor of several variables. Recall that the differential of F is given by $P_{\vec{1}} \mathrm{cored}(F)$, where $\mathrm{cored}(F)$ denotes the coreduction of F (Construction 7.2.3.6) and $\vec{1} = (1, \ldots, 1)$ (see

Example 7.2.3.5). The construction $F \mapsto P_{\bar{1}}F$ is preserves finite limits (Theorem 7.1.1.10), but the formation of coreductions generally does not. The key to our proof of Theorem 7.3.3.3 will be to show that, nevertheless, the construction $F \mapsto \partial F$ commutes with certain very special finite limits. Before we can formulate a precise result, we need to introduce some notation.

Notation 7.3.3.6. Let $\mathcal{C}$ be a pointed ∞-category which admits finite limits, let K be a finite simplicial set, and let $U : K^{\triangleleft} \to \mathcal{C}$ be a diagram. Let v denote the cone point of $K^{\triangleleft}$. We let $\mathrm{tfib}(U)$ denote the fiber of the induced map

$$U(v) \to \varprojlim(U|K).$$

We will refer to $\mathrm{tfib}(U)$ as the *total fiber* of the diagram U.

Remark 7.3.3.7. In the situation of Notation 7.3.3.6, if U is a limit diagram, then $\mathrm{tfib}(U)$ is a final object of $\mathcal{C}$. The converse holds if the ∞-category $\mathcal{C}$ is stable.

Remark 7.3.3.8. Let $\mathcal{C}$ and $\mathcal{D}$ be pointed ∞-categories which admit finite limits, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor which preserves zero objects. For every finite diagram $U : K^{\triangleleft} \to \mathcal{C}$, there is a canonical map $F(\mathrm{tfib}(U)) \to \mathrm{tfib}(F \circ U)$. If F is left exact, then this map is an equivalence.

Remark 7.3.3.9. Let $\{\mathcal{C}_s\}_{s \in S}$ be a finite collection of pointed ∞-categories which admit finite limits, let $\{K_s\}_{s \in S}$ be a finite collection of finite simplicial sets, and suppose we are given diagrams $U_s : K_s^{\triangleleft} \to \mathcal{C}_s$. Let $\mathcal{D}$ be a pointed ∞-category which admits finite limits, and let $G : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ be a functor which is left exact in each variable. Then $G(\{\mathrm{tfib}(U_s)\}_{s \in S})$ can be identified with the total fiber of the diagram

$$\prod_{s \in S} K_s^{\triangleleft} \xrightarrow{\prod U_s} \prod_{s \in S} \mathcal{C}_s \xrightarrow{G} \mathcal{D}.$$

Example 7.3.3.10. In the situation of Notation 7.3.3.6, suppose that $K = \Delta^0$, so that a map $U : K^{\triangleleft} \to \mathcal{C}$ can be identified with a morphism f in $\mathcal{C}$. We then have a canonical equivalence $\mathrm{tfib}(U) \simeq \mathrm{fib}(f)$.

Example 7.3.3.11. Let $\{\mathcal{C}_s\}_{s \in S}$ be a finite collection of ∞-categories which admit final objects, let $\mathcal{D}$ be a pointed ∞-category which admits finite limits, and let $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ be a functor. Then the reduction $\mathrm{Red}(F)$ introduced in Construction 7.1.3.15 is given by the total fiber of the diagram of functors $\{F^T\}_{T \subseteq S}$.

Our next goal is to construct a variant of Example 7.3.3.11. More precisely, we will show that if $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ is a functor which carries final objects of $\prod_{s \in S} \mathcal{C}_s$ to final objects of $\mathcal{D}$, then the reduction $\mathrm{Red}(F)$ has a different description as the total fiber of a diagram of functors (which is better behaved with respect to differentiation).

Construction 7.3.3.12. Let $\{\mathcal{C}_s\}_{s \in S}$ be a nonempty finite collection of ∞-categories which admit final objects $*_s \in \mathcal{C}_s$. Let $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ be a functor, where $\mathcal{D}$ is an ∞-category which admits finite products. For every equivalence relation E on the set S, we let $F^E : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ denote the functor given by the formula

$$F^E(\{X_s\}_{s \in S}) = \prod_{T \in S/E} F(\{X_s^T\}_{s \in S})$$

where

$$X_s^T = \begin{cases} X_s & \text{if } s \in T \\ *_s & \text{if } s \notin T. \end{cases}$$

It will be convenient to have a more formal construction of the collection of functors $\{F^E\}_{E \in \mathrm{Equiv}(S)}$. For this, we define a partially ordered set $\overline{\mathrm{Equiv}(S)}$ as follows:

(a) The objects of $\overline{\mathrm{Equiv}}(S)$ are pairs (T, E), where $T \subseteq S$ and E is an equivalence relation on T.

(b) We have $(T, E) \leq (T', E')$ if and only if $T \subseteq T'$ and xEy implies $xE'y$ for $x, y \in T$.

Let us identify $\mathrm{Equiv}(S)$ with the partially ordered subset of $\overline{\mathrm{Equiv}}(S)$ consisting of those pairs (T, E) where $T = S$, and let $\overline{\mathrm{Equiv}}_0(S)$ denote the subset of $\overline{\mathrm{Equiv}}(S)$ consisting of those pairs (T, E) where E is a trivial equivalence relation on T.

Now let $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ be as above, and for $T \subseteq S$ let F^T be the functor defined in Construction 7.1.3.15. The construction $(T, E) \mapsto F^{S-T}$ determines a functor $\mathrm{N}(\overline{\mathrm{Equiv}}_0(S))^{op} \to \mathrm{Fun}(\prod_s \mathcal{C}_s, \mathcal{D})$. This functor admits a right Kan extension $\mathrm{N}(\overline{\mathrm{Equiv}}(S))^{op} \to \mathrm{Fun}(\prod_s \mathcal{C}_s, \mathcal{D})$, which we will denote by $(T, E) \mapsto F^{(T,E)}$. When $T = S$, we will denote $F^{(T,E)}$ simply by F^E. Note that $F^E \simeq \prod_{T \in S/E} F^T$ agrees with the functor defined informally above.

Proposition 7.3.3.13. *Let S be a nonempty finite set, let $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ be a functor between ∞-categories. Assume that each $\mathcal{C}_s$ has a final object $*_s$, that $\mathcal{D}$ is pointed and admits finite limits, and that $F(\{*_s\})$ is a final object of $\mathcal{D}$. Let H denote the total fiber of the diagram $\{F^E\}_{E \in \mathrm{Equiv}(S)}$. Then:*

(a) *The functor $H : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ is reduced.*

(b) *Let $G : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ be any reduced functor. Then the canonical map $H \to F^{E_\top} \simeq F$ induces a homotopy equivalence*

$$\mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})}(G, H) \to \mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})}(G, F).$$

Consequently, we have a canonical equivalence $H \simeq \mathrm{Red}(F)$.

Proof. We first prove (a). We will assume that S has more than one element (otherwise $H = F$ and the result is obvious). Fix an element $t \in S$, and choose a sequence of objects $\{X_s \in \mathcal{C}_s\}$ such that $X_t = *_t$. We will prove that the canonical map $F(\{X_s\}) \to \varprojlim_{E \in \mathrm{Equiv}^+(S)} F^E(\{X_s\})$ is an equivalence. Let $U \subseteq \mathrm{Equiv}(S)$ denote the subset consisting of those equivalence relations E on S such that, if sEt, then $s = t$ (that is, the set $\{t\}$ is an equivalence class with respect to E). Note that the inclusion $\mathrm{N}(U) \to \mathrm{N}(\mathrm{Equiv}(S))$ admits a right adjoint, which we will denote by $E \mapsto E'$. Using our assumption that $X_t = *_t$ and that $F(\{*_s\})$ is a final object of $\mathcal{D}$, we deduce that the canonical map $F^E(\{X_s\}) \to F^{E'}(\{X_s\})$ is an equivalence for every equivalence relation $E \in \mathrm{Equiv}(S)$. Let $U^+ = U \cap \mathrm{Equiv}^+(S)$. The preceding argument shows that the diagram $\{F^E(\{X_s\})\}_{E \in \mathrm{Equiv}^+(S)}$ is a right Kan extension of its restriction to $\{F^E(\{X_s\})\}_{E \in U^+}$. Note that U^+ has a largest element (given by the equivalence E_0 relation corresponding to the partition $S = \{t\} \cup (S - \{t\})$), so that $\varprojlim_{E \in \mathrm{Equiv}^+(S)} F^E(\{X_s\})$ is given by $F^{E_0}(\{X_s\})$. It now suffices to observe that the map $F(\{X_s\}) \to F^{E_0}(\{X_s\})$ is an equivalence.

To prove (b), it will suffice to show that the space

$$\mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})}(G, \varprojlim_{E \in \mathrm{Equiv}^+(S)} F^E)$$

is contractible. In fact, we claim that the mapping space

$$\mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})}(G, F^E)$$

is contractible for every nontrivial equivalence relation E on S. Let $S_1, \dots, S_k$ denote the equivalence classes with respect to $\sim$, so that $F^E = \prod_{1 \leq i \leq k} F^{S_i}$ (where F^{S_i} is defined as in Construction 7.1.3.15). It will therefore suffice to show that each of the mapping spaces $\mathrm{Map}_{\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})}(G, F^{S_i})$ is contractible. Let $\mathcal{E} \subseteq \prod_{s \in S} \mathcal{C}_s$ be the full subcategory spanned by those sequences $\{Y_s\}_{s \in S}$ such that Y_s is a final object of $\mathcal{C}_s$ for $s \notin S_i$. Then F^{S_i} is a right Kan extension of its restriction to $\mathcal{E}$. It will therefore suffice to show that $\mathrm{Map}_{\mathrm{Fun}(\mathcal{E}, \mathcal{D})}(G|\mathcal{E}, F^{S_i}|\mathcal{E})$ is contractible. This is clear, since G carries each object of $\mathcal{E}$ to a zero object of $\mathcal{D}$. $\qquad\square$

The main ingredient in our proof of Theorem 7.3.3.3 is the following assertion:

Theorem 7.3.3.14. *Let $\{\mathcal{C}_s\}_{s \in S}$ be a nonempty finite collection of pointed ∞-categories which admit finite colimits. For each functor $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{S}_*$, let DF denote a differential of F (see Definition 7.2.3.1 and Proposition 7.2.3.13). Suppose that $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{S}_*$ is a functor such that $F(\{*_s\})$ is contractible, so that Proposition 7.3.3.13 furnishes an equivalence $\theta : \mathrm{Red}(F) \simeq \mathrm{tfib}\{F^E\}_{E \in \mathrm{Equiv}(S)}$. Then θ induces an equivalence $D\,\mathrm{Red}(F) \simeq \mathrm{tfib}\{DF^E\}_{E \in \mathrm{Equiv}(S)}$.*

Example 7.3.3.15. In the situation of Theorem 7.3.3.14, suppose that $S = \{0, 1\}$ and let $*$ denote the zero object of both $\mathcal{C}_0$ and $\mathcal{C}_1$. Theorem 7.3.3.14 asserts that, if $F : \mathcal{C}_0 \times \mathcal{C}_1 \to \mathcal{S}_*$ is a functor such that $F(*, *)$ is contractible and we define $G : \mathcal{C}_0 \times \mathcal{C}_1 \to \mathcal{S}_*$ by the formula $G(C, D) = F(C, *) \times F(*, D)$, then the fiber sequence

$$\mathrm{Red}(F) \to F \to G$$

induces a fiber sequence of differentials

$$D\,\mathrm{Red}(F) \to DF \to DG.$$

Combining Theorem 7.3.3.14 with Corollary 7.2.3.22, we obtain the following result:

Corollary 7.3.3.16. *Let $\{\mathcal{C}_s\}_{s \in S}$ be a nonempty finite collection of pointed differentiable ∞-categories which admit finite colimits, and let $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{S}_*$ be a functor which preserves final objects and sequential colimits. Then the equivalence $\mathrm{Red}(F) \simeq \mathrm{tfib}\{F^E\}_{E \in \mathrm{Equiv}(S)}$ induces an equivalence $\vec{\partial}\,\mathrm{Red}(F) \simeq \mathrm{tfib}\{\vec{\partial}F^E\}_{E \in \mathrm{Equiv}(S)}$ of functors from $\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s)$ to $\mathrm{Sp}(\mathcal{D})$.*

The proof of Theorem 7.3.3.14 is rather elaborate, and will be given in §7.3.4. We conclude this section by showing how Theorem 7.3.3.3 can be deduced from Theorem 7.3.3.14.

Proof of Theorem 7.3.3.3. Let $\mathrm{id} : \mathcal{S}_* \to \mathcal{S}_*$ be the identity functor; we wish to describe the spectrum $\partial_n(\mathrm{id})$ or equivalently the functor $\partial_n(\mathrm{id}) : \mathrm{Sp}^n \to \mathrm{Sp}$. Let $F : \mathcal{S}_*^n \to \mathcal{S}_*$ be the functor given by $F(X_1, \ldots, X_n) = X_1 \amalg \cdots \amalg X_n$, so that the functor $\partial_n(\mathrm{id})$ is given by the derivative of $\mathrm{Red}(F)$. Set $T = \{1, \ldots, n\}$. Using Corollary 7.3.3.16, we obtain an equivalence of functors $\alpha : \partial_n(\mathrm{id}) \simeq \mathrm{tfib}\{\vec{\partial}(F^E)\}_{E \in \mathrm{Equiv}(T)}$. Since the product functor on $\mathcal{S}_*$ preserves pushouts in each variable, it follows from Example 7.2.3.10 that $\mathrm{cored}(F^E)$ is trivial for $E \neq E_\perp$. Moreover, $F^{E_\perp}$ is given by the formula $F^{E_\perp}(X_1, \ldots, X_n) = X_1 \times \cdots \times X_n$, so that $\mathrm{cored}(F^{E_\perp})$ is the iterated smash product functor $\wedge : \mathcal{S}_*^n \to \mathcal{S}_*$ and therefore $\vec{\partial}(F^{E_\perp}) : \mathrm{Sp}^n \to \mathrm{Sp}$ is also given by the iterated smash product (see Example 7.2.3.28). In particular, we have $\vec{\partial}(F^E)(S, S, \ldots, S) \simeq G'(E)$, where $G' : \mathrm{N}(\mathrm{Equiv}(T))^{op} \to \mathrm{Sp}$ denote the functor given by the formula

$$G'(E) = \begin{cases} S & \text{if } E = E_\perp \\ 0 & \text{otherwise.} \end{cases}$$

Evaluating α on the sphere spectrum, we obtain an equivalence of spectra $\partial_n(\mathrm{id}) \simeq \mathrm{tfib}(G')$.

Let $G : \mathrm{N}(\mathrm{Equiv}(T))^{op} \to \mathrm{Sp}$ be the constant functor taking the value S, and let $G'' : \mathrm{N}(\mathrm{Equiv}(T))^{op} \to \mathrm{Sp}$ be given by $G''(E) = \begin{cases} 0 & \text{if } E = E_\perp \\ S & \text{otherwise.} \end{cases}$ so that we have a fiber sequence of functors

$$G' \to G \to G''$$

and therefore a fiber sequence of spectra

$$\partial_n(\mathrm{id}) \to \mathrm{tfib}(G) \to \mathrm{tfib}(G'').$$

Unwinding the definitions, we have equivalences

$$\mathrm{tfib}(G) = \mathrm{fib}(\varprojlim(G) \to \varprojlim(G \,|\, \mathrm{N}(\mathrm{Equiv}^+(T)))) \qquad \mathrm{tfib}(G'') = \mathrm{fib}(\varprojlim(G'') \to \varprojlim(G'' \,|\, \mathrm{N}(\mathrm{Equiv}^+(T)))).$$

Note that G'' is a right Kan extension of $G| \operatorname{N}(\operatorname{Equiv}^-(T))$, so that

$$\operatorname{tfib}(G'') \simeq \operatorname{fib}(\varprojlim(G| \operatorname{N}(\operatorname{Equiv}^-(T))) \to \varprojlim(G| \operatorname{N}(\operatorname{Equiv}^\pm(T)))).$$

We may therefore identify $\partial_n(\operatorname{id})$ with the total fiber of the diagram of spectra

$$
\begin{array}{ccc}
\varprojlim(G) & \longrightarrow & \varprojlim(G| \operatorname{N}(\operatorname{Equiv}^+(T))) \\
\downarrow & & \downarrow \\
\varprojlim(G| \operatorname{N}(\operatorname{Equiv}^-(T))) & \longrightarrow & \varprojlim(G| \operatorname{N}(\operatorname{Equiv}^\pm(T))),
\end{array}
$$

which is the fiber of the canonical map

$$S^{\operatorname{N}(\operatorname{Equiv}(T))} \to S^{\operatorname{N}(\operatorname{Equiv}^+(T))} \times_{S^{\operatorname{N}(\operatorname{Equiv}^\pm(T))}} S^{\operatorname{N}(\operatorname{Equiv}^-(T))}.$$

$\square$

7.3.4   Differentiation and Reduction

Let $\{\mathcal{C}_s\}_{s \in S}$ be a nonempty finite collection of pointed ∞-categories which admit finite colimits, and let $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{S}_*$ be a functor which preserves final objects. Our goal in this section is to prove Theorem 7.3.3.14, which asserts that the canonical map

$$d \operatorname{Red}(F) \to \operatorname{tfib}\{dF^E\}_{E \in \operatorname{Equiv}(S)}$$

is an equivalence of functors from $\prod_{s \in S} \mathcal{C}_s \to \mathcal{S}_*$ (here dG denotes the differential of a functor G). We can outline our strategy as follows:

(a) Let $\mathcal{C} = \prod_{s \in S} \mathcal{C}_s$. We will show that any functor $F : \mathcal{C} \to \mathcal{S}_*$ can be "approximated" by products of homogeneous functors (Proposition 7.3.4.4).

(b) We show that any homogeneous functor from $\mathcal{C}$ to $\mathcal{D}$ is a product of functors which are homogeneous in each variable (Proposition 7.3.4.11).

(c) Using (a) and (b), we are reduced to proving Theorem 7.3.3.14 for (products of) functors which are homogeneous in each variable. In this case, we will obtain the desired result by combining the classification of homogeneous functors given in §7.1.4 with an analysis of the partially ordered set $\operatorname{Equiv}(S)$ of equivalence relations on S.

To carry out step (a), we need to introduce some terminology.

Definition 7.3.4.1. Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object and let $\mathcal{D}$ be a differentiable ∞-category. We will say that a natural transformation $\alpha : F \to G$ of functors $F, G : \mathcal{C} \to \mathcal{D}$ is a *jet equivalence* if α induces an equivalence $P_n F \to P_n G$ for every integer n.

Remark 7.3.4.2. Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object, let $\mathcal{D}$ be a differentiable ∞-category, and suppose we are given a pullback diagram

$$
\begin{array}{ccc}
F & \xrightarrow{\ \alpha\ } & F' \\
\downarrow & & \downarrow \\
G & \xrightarrow{\ \beta\ } & G'
\end{array}
$$

in the ∞-category $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$. If β is a jet equivalence, then α is also a jet equivalence. This follows from the fact that each of the functors $P_n : \operatorname{Fun}(\mathcal{C}, \mathcal{D}) \to \operatorname{Fun}(\mathcal{C}, \mathcal{D})$ is left exact (Remark 7.1.1.29).

Remark 7.3.4.3. Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object and let $\mathcal{D}$ be a differentiable ∞-category. Assume that $\mathcal{D}$ admits K-indexed colimits, for some simplicial set K. Then the collection of jet equivalences is closed under K-indexed colimits (when regarded as a full subcategory of $\operatorname{Fun}(\Delta^1, \operatorname{Fun}(\mathcal{C}, \mathcal{D})))$. This follows immediately from the observation that $P_n : \operatorname{Fun}(\mathcal{C}, \mathcal{D}) \to \operatorname{Fun}(\mathcal{C}, \mathcal{D})$ is a localization functor.

The main ingredient in our proof of Theorem 7.3.3.14 is the following general approximation result:

Proposition 7.3.4.4. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object, let $F : \mathcal{C} \to \mathcal{S}_*$ be a reduced functor, and let $n \geq 0$ be an integer. Then there exists a simplicial object $F_\bullet$ of $\operatorname{Fun}(\mathcal{C}, \mathcal{S}_*)$ and a jet equivalence $\alpha : |F_\bullet| \to P_n F$ satisfying the following condition:*

$(*_n)$ *For each integer $p \geq 0$, the functor F_p can be written as a finite product $F_p \simeq \prod_\alpha F_{p,\alpha}$, each each $F_{p,\alpha} : \mathcal{C} \to \mathcal{S}_*$ is k_α-homogeneous for some $1 \leq k_\alpha \leq n$.*

The proof of Proposition 7.3.4.4 will require some preliminaries.

Lemma 7.3.4.5. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object, let $\mathcal{D}$ be a pointed differentiable ∞-category, and let $\alpha : F \to G$ be a natural transformation between functors $F, G : \mathcal{C} \to \mathcal{D}$. Assume that:*

(i) *The map α induces an equivalence $F(*) \to G(*)$, where $*$ denotes the final object of $\mathcal{C}$.*

(ii) *There exists an integer k such that α induces an equivalence of functors $\Omega^k(F) \to \Omega^k(G)$.*

Then α is a jet equivalence.

Proof. We will prove that $P_n(\alpha)$ is an equivalence using induction on n. If $n = 0$, the desired result follows from (i). If $n > 0$, we use Theorem 7.1.2.4 to construct a map of fiber sequences

$$
\begin{array}{ccccc}
P_n F & \longrightarrow & P_{n-1} F & \longrightarrow & RF \\
\downarrow {\scriptstyle P_n(\alpha)} & & \downarrow {\scriptstyle P_{n-1}(\alpha)} & & \downarrow {\scriptstyle R(\alpha)} \\
P_n G & \longrightarrow & P_{n-1} G & \longrightarrow & RG.
\end{array}
$$

Since $P_{n-1}(\alpha)$ is an equivalence by the inductive hypothesis, it will suffice to show that $R(\alpha)$ is an equivalence. Because the functor R is left exact, condition (ii) implies that $R(\alpha)$ induces an equivalence $\Omega^k RF \to \Omega^k RG$ for some integer k. Since the ∞-category $\operatorname{Homog}^n(\mathcal{C}, \mathcal{D})$ is stable (Corollary 7.1.2.8), this implies that $R(\alpha)$ is itself an equivalence. $\qquad\qquad\square$

Lemma 7.3.4.6. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits and has a final object. Let $n \geq 0$, and let $U : \operatorname{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \operatorname{Fun}_*(\mathcal{C}, \mathcal{S}_*)) \to \operatorname{Fun}_*(\mathcal{C}, \mathcal{S}_*)$ be the functor given by the formula $U(F_\bullet) = P_n |F_\bullet|$. Then U preserves finite limits.*

Proof. Since it is clear that U preserves final objects, it will suffice to show that U preserves pullback squares (Corollary T.4.4.2.5). Let $\tau_{\geq 1} : \mathcal{S}_* \to \mathcal{S}_*$ be the functor which assigns to each pointed space X the connected component of its base point (so that we have a fiber sequence $\tau_{\geq 1} X \to X \to \pi_0 X$). For every functor $F : \mathcal{C} \to \mathcal{S}_*$, let F^o denote the composite functor

$$
\mathcal{C} \xrightarrow{F} \mathcal{S}_* \xrightarrow{\tau_{\geq 1}} \mathcal{S}_* .
$$

There is an evident natural transformation $F^o \to F$. According to Lemma 7.3.4.5, this natural transformation is a jet equivalence whenever F is reduced (or, more generally, whenever F carries the final object of $\mathcal{C}$ to a connected space). If we are given a simplicial object $F_\bullet$ in $\operatorname{Fun}_*(\mathcal{C}, \mathcal{D})$, then we obtain a new simplicial object $F_\bullet^o$, and Remark 7.3.4.3 implies that the induced map $|F_\bullet^o| \to |F_\bullet|$ is also a jet equivalence, and therefore induces an equivalence $U(F_\bullet^o) \to U(F_\bullet)$.

Suppose now that we are given a pullback diagram σ :

$$
\begin{array}{ccc}
F_\bullet & \longrightarrow & F'_\bullet \\
\downarrow & & \downarrow \\
G_\bullet & \longrightarrow & G'_\bullet
\end{array}
$$

of simplicial objects of $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$. We would like to show that $U(\sigma)$ is also a pullback diagram. In view of the above arguments, it will suffice to show that U carries the diagram σ':

$$
\begin{array}{ccc}
F_\bullet^{\mathrm{o}} & \longrightarrow & F'^{\mathrm{o}}_\bullet \\
\downarrow & & \downarrow \\
G_\bullet^{\mathrm{o}} & \longrightarrow & G'^{\mathrm{o}}_\bullet.
\end{array}
$$

to a pullback diagram in $\mathrm{Fun}_*(\mathcal{C}, \mathcal{S}_*)$. Let $H_\bullet$ be the simplicial functor given by the fiber product $F'^{\mathrm{o}}_\bullet \times_{G'^{\mathrm{o}}_\bullet} G_\bullet^{\mathrm{o}}$. For every integer $k \geq 0$ and every object $X \in \mathcal{C}$, we have a pullback diagram

$$
\begin{array}{ccc}
F_k^{\mathrm{o}}(X) & \longrightarrow & H_k(X) \\
\downarrow & & \downarrow \\
\pi_0 F_k(X) & \longrightarrow & \pi_0 F'_k(X) \times_{\pi_0 G'_k(X)} \pi_0 G_k(X)
\end{array} \qquad ,
$$

so that the canonical map $F_k^{\mathrm{o}}(X) \to H_k(X)$ has nonempty, discrete homotopy fibers (that is, it is a covering map). Using Lemma 7.3.4.5, we deduce that the map $F_k^{\mathrm{o}} \to H_k$ is a jet equivalence, so that Remark 7.3.4.3 implies that $U(F_\bullet^{\mathrm{o}}) \to U(H_\bullet)$ is an equivalence. It will therefore suffice to show that the pullback diagram

$$
\begin{array}{ccc}
H_\bullet & \longrightarrow & F'^{\mathrm{o}}_\bullet \\
\downarrow & & \downarrow \\
G_\bullet^{\mathrm{o}} & \longrightarrow & G'^{\mathrm{o}}_\bullet
\end{array}
$$

remains a pullback diagram after applying the functor U. Because P_n is left exact (Remark 7.1.1.29), we are reduced to proving that the diagram

$$
\begin{array}{ccc}
|H_\bullet| & \longrightarrow & |F'^{\mathrm{o}}_\bullet| \\
\downarrow & & \downarrow \\
|G_\bullet^{\mathrm{o}}| & \longrightarrow & |G'^{\mathrm{o}}_\bullet|
\end{array}
$$

is a pullback square of functors. This is equivalent to the assertion that for every object $X \in \mathcal{C}$, the diagram

$$
\begin{array}{ccc}
|H_\bullet(X)| & \longrightarrow & |F'^{\mathrm{o}}_\bullet(X)| \\
\downarrow & & \downarrow \\
|G_\bullet^{\mathrm{o}}(X)| & \longrightarrow & |G'^{\mathrm{o}}_\bullet(X)|
\end{array}
$$

is a pullback square of spaces. This follows from Lemma 5.3.6.17 (note that each $G'^{\mathrm{o}}_k(X)$ is connected, by construction). $\square$

Proof of Proposition 7.3.4.4. Let $G : \mathcal{C} \to \mathcal{S}_*$ denote the constant functor taking the value Δ^0, and let $G_\bullet$ be the constant simplicial object of $\mathrm{Fun}(\mathcal{C}, \mathcal{S}_*)$ taking the value G. We proceed by induction on n. When $n = 0$, we can take $F_\bullet = G_\bullet$. Let us therefore assume that $n > 0$. By the inductive hypothesis, we can choose a jet equivalence $\alpha : |F_\bullet| \to P_{n-1}F$ satisfying condition $(*_{n-1}$. Choose a natural transformation $u : G_\bullet \to F_\bullet$, and let $F_{\bullet,\bullet}$ be the Čech nerve of u. For every integer $p \geq 0$, we have an augmentation map

$$v_p : \varinjlim_{[q]} F_{p,q} \to F_p$$

which exhibits $\varinjlim_{[q]} F_{p,q}(X)$ as the base point component of $F_p(X)$, for every object $X \in \mathcal{C}$. Using Lemma 7.3.4.5, we deduce that v_p is a jet equivalence. It follows that the composite map

$$P_m(\varinjlim_{[p],[q]} F_{p,q}) \to P_m|F_\bullet| \simeq P_{n-1}F$$

is also a jet equivalence. Define a simplicial object $F'_\bullet$ of $\mathrm{Fun}(\mathcal{C}, \mathcal{S}_*)$ by the formula $F'_p = F_{p,p}$. Since the ∞-category $\mathrm{N}(\mathbf{\Delta})^{op}$ is sifted, we conclude that the map $|F'_\bullet| \to P_{n-1}F$ is a jet equivalence.

Theorem 7.1.2.4 supplies a fiber sequence of functors

$$P_n F \to P_{n-1}F \to RF,$$

where R is n-homogeneous, and let $F''_\bullet$ denote the simplicial object of $\mathrm{Fun}(\mathcal{C}, \mathcal{S}_*)$ whose pth term is given by the fiber of the composite map

$$F'_p \to P_{n-1}F \to RF.$$

We claim that the evident map $\beta : |F''_\bullet| \to P_n F$ satisfies our requirements. Since colimits in $\mathcal{S}$ are universal, we have a pullback diagram

$$
\begin{array}{ccc}
|F''_\bullet| & \longrightarrow & |F'_\bullet| \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\
P_n F & \longrightarrow & P_{n-1}F.
\end{array}
$$

For every integer $p \geq 0$, the map $F'_p \to P_{n-1}F \to R$ factors through $F'_0 \simeq G$ and is therefore nullhomotopic. We therefore have an equivalence

$$F''_p \simeq \Omega(RF) \times F'_p \simeq \Omega(RF) \times F_{p,p} \simeq \Omega(RF) \times \Omega(F_p)^p.$$

Because the simplicial object $F_\bullet$ satisfies $(*_{n-1})$ and R is n-homogeneous, we deduce that $F''_\bullet$ satisfies $(*_n)$. $\square$

To apply Proposition 7.3.4.4 in our context, we need to know something about the classification of homogeneous functors with domain $\prod_{s \in S} \mathcal{C}_s$. For this, we need to generalize some of the results of §7.1.4 to the multivariate case. We begin by proving a generalization of Proposition 7.1.3.10.

Proposition 7.3.4.7. *Let $\{\mathcal{C}_s\}_{s \in S}$ be a nonempty finite collection of pointed ∞-categories which admit finite colimits, let $\mathcal{C} = \prod_{s \in S} \mathcal{C}_s$, and let $\mathcal{D}$ be a pointed differentiable ∞-category. Let $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ be a functor which is n_s-reduced in the sth argument. Then F is n-reduced (when regarded as a functor from $\mathcal{C}$ to $\mathcal{D}$.*

Lemma 7.3.4.8. *Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be ∞-categories which admit finite colimits and zero objects $*_0$ and $*_1$, let $\mathcal{C} = \mathcal{C}_0 \times \mathcal{C}_1$, and let $\mathcal{D}$ be an ∞-category which admits finite limits. Let ϕ_0 denote the composite functor*

$$\mathcal{C}_0 \simeq \mathcal{C}_0 \times \{*_1\} \hookrightarrow \mathcal{C}_0 \times \mathcal{C}_1 = \mathcal{C},$$

and define $\phi_1 : \mathcal{C}_1 \to \mathcal{C}$ similarly. Then composition with ϕ_0 and ϕ_1 induces an equivalence of ∞-categories

$$\theta : \mathrm{Exc}_*(\mathcal{C}, \mathcal{D}) \to \mathrm{Exc}_*(\mathcal{C}_0, \mathcal{D}) \times \mathrm{Exc}_*(\mathcal{C}_1, \mathcal{D}).$$

Proof. Let $\xi : \mathrm{Exc}_*(\mathcal{C}_0, \mathcal{D}) \times \mathrm{Exc}_*(\mathcal{C}_1, \mathcal{D}) \to \mathrm{Exc}_*(\mathcal{C}_0 \times \mathcal{C}_1, \mathcal{D})$ be the functor given by the formula

$$\xi(F_0, F_1)(X, Y) = F_0(X) \times F_1(Y).$$

It is easy to see that ξ is a homotopy inverse to θ. $\square$

Remark 7.3.4.9. Let $\{\mathcal{C}_s\}_{s \in S}$ be a nonempty finite collection of pointed ∞-categories which admit finite colimits, let $\mathcal{C} = \prod_{s \in S} \mathcal{C}_s$, let $\mathcal{D}$ be a pointed ∞-category which admits finite products. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $\mathrm{cr}_n(F) : \mathcal{C}^n \to \mathcal{D}$ be its nth cross effect. Let $d\,\mathrm{cr}_n(F)$ denote the differential of $\mathrm{cr}_n(F)$ (where we regard $\mathrm{cr}_n(F)$ as a functor of n variables). The proof of Lemma 7.3.4.8 shows that $d\,\mathrm{cr}_n(F)$ can be written as a product of functors given by the composition

$$\mathcal{C}^n \to \prod_{s \in S} \mathcal{C}_s^{n_s} \xrightarrow{d\,\mathrm{cr}_{\vec{n}}(F)} \mathcal{D}$$

where $\vec{n}$ varies over those tuples $\{n_s\}_{s \in S}$ having sum n and the cross effects $\mathrm{cr}_{\vec{n}}$ are defined as in Variant 7.1.3.21.

Proof of Proposition 7.3.4.7. If each $n_s = 0$, there is nothing to prove. Assume therefore that $n_s > 0$ for some $s \in S$. Then F is 1-reduced as a functor of its sth argument, and therefore 1-reduced when regarded as a functor from $\mathcal{C}$ to $\mathcal{D}$. According to Proposition 7.1.3.24, it will suffice to show that for $m < n$, the differential $d\,\mathrm{cr}_m(F)$ vanishes. Using Remark 7.3.4.9, we are reduced to proving that the functor

$$d\,\mathrm{cr}_{\vec{m}}(F) : \prod_{s \in S} \mathcal{C}_s^{m_s} \to \mathcal{D}$$

is trivial for every sequence $\{m_s\}_{s \in S}$ satisfying $\sum_{s \in S} m_s < n$. For such a sequence, we must have $m_s < n_s$ for some s. The desired result now follows by applying Remark 7.1.3.23 to F, regarded as a functor from $\mathcal{C}_s$ to $\mathrm{Fun}(\prod_{t \neq s} \mathcal{C}_t, \mathcal{D})$. $\square$

Corollary 7.3.4.10. *Let $\{\mathcal{C}_s\}_{s \in S}$ be a nonempty finite collection of ∞-categories which admit finite colimits and final objects, and let $\mathcal{D}$ be a pointed differentiable ∞-category. For every collection of nonnegative integers $\vec{n} = \{n_s\}_{s \in S}$, let $\mathrm{Homog}^{\vec{n}}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$ spanned by those functors F which are n_s-homogeneous when regarded as a functor from $\mathcal{C}_s$ to $\mathrm{Fun}(\prod_{t \neq s} \mathcal{C}_t, \mathcal{D})$. Then $\mathrm{Homog}^{\vec{n}}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D}) \subseteq \mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$, where $\mathcal{C} = \prod_{s \in S} \mathcal{C}_s$ and $n = \sum_{s \in S} n_s$.*

Proof. Combine Proposition 7.3.4.7 with Proposition 7.1.3.4. $\square$

In the situation of Corollary 7.3.4.10, the ∞-category $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$ can be reconstructed from the ∞-categories $\mathrm{Homog}^{\vec{n}}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$ as follows:

Proposition 7.3.4.11. *Let $\{\mathcal{C}_s\}_{s \in S}$ be a nonempty finite collection of pointed ∞-categories which admit finite colimits, let $\mathcal{C} = \prod_{s \in S} \mathcal{C}_s$, let $\mathcal{D}$ be a pointed differentiable ∞-category, and let $n \geq 0$ be an integer. For every collection of nonnegative integers $\vec{n} = \{n_s\}_{s \in S}$, let $|\vec{n}| = \sum_{s \in S} n_s$. Consider the functor*

$$\Phi : \prod_{|\vec{n}| = n} \mathrm{Homog}^{(\vec{n})}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D}) \to \mathrm{Homog}^{(n)}(\mathcal{C}, \mathcal{D})$$

given by

$$\Phi(\{F_{\vec{n}}\}) = \prod_{|\vec{n}| = n} F_{\vec{n}}.$$

Then Φ is an equivalence of ∞-categories.

Lemma 7.3.4.12. *Let S be a finite set. Suppose we are given a finite collection of pointed ∞-categories $\{\mathcal{C}_s\}_{s \in S}$ which admit finite colimits, and let $\mathcal{D}$ be an ∞-category which admits finite limits. Let $\mathcal{C} = \prod_{s \in S} \mathcal{C}_s$, and for each $s \in S$, let $\phi_s : \mathcal{C}_s \to \mathcal{C}$ be the functor given by the product of the identity map $\mathrm{id} : \mathcal{C}_s \to \mathcal{C}_s$ with constant functor $\mathcal{C}_s \to \mathcal{C}_t$ carrying $\mathcal{C}_s$ to a zero object of $\mathcal{C}_t$ for $t \neq s$.*

Fix an integer $n \geq 0$. For every collection of nonnegative integers $\vec{n} = \{n_s\}_{s \in S}$, set

$$\mathcal{C}^{\vec{n}} = \prod_{s \in S} \mathcal{C}_s^{n_s} \qquad \mathcal{C}^{(\vec{n})} = \prod_{s \in S} \mathcal{C}_s^{(n_s)},$$

where $\mathcal{C}_s^{(n_s)}$ denotes the n_sth extended power of $\mathcal{C}_s$ (see Notation 7.1.4.1). Let $\mathrm{SymFun}^{\vec{n}}(\mathcal{C}, \mathcal{D})$ denote the ∞-category $\mathrm{Fun}(\mathcal{C}^{(\vec{n})}, \mathcal{D})$, and let $\mathrm{SymFun}_{\mathrm{lin}}^{(\vec{n})}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{SymFun}^{\vec{n}}(\mathcal{C}, \mathcal{D})$ spanned by those functors for which the underlying functor $\prod_{s \in S} \mathcal{C}_s^{n_s} \to \mathcal{D}$ is 1-homogeneous in each variable. Let $U_{\vec{n}}$ denote the composite map

$$U_{\vec{n}} : \mathcal{C}^{\vec{n}} \to \prod_{s \in S} \mathcal{C}^{(n_s)} \to \mathcal{C}^{(n)}.$$

Then the maps $U_{\vec{n}}$ induce an equivalence of ∞-categories

$$\Psi : \mathrm{SymFun}_{\mathrm{lin}}^n(\mathcal{C}, \mathcal{D}) \to \prod_{|\vec{n}| = n} \mathrm{SymFun}_{\mathrm{lin}}^{\vec{n}}(\mathcal{C}, \mathcal{D}).$$

Proof. Using Lemma 7.3.4.8 repeatedly, we obtain an equivalence of ∞-categories

$$\mathrm{Exc}_*(\mathcal{C}^n, \mathcal{D}) \to \prod_{(s_1, \ldots, s_n) \in S^n} \mathrm{Exc}_*(\prod_{1 \leq i \leq n} \mathcal{C}_{s_i}, \mathcal{D}).$$

The desired result is now obtained by extracting homotopy fixed points with respect to the action of the symmetric group Σ_n on each side. $\qquad\square$

Remark 7.3.4.13. Let $\mathcal{C} = \prod_{s \in S} \mathcal{C}_s$ and $\mathcal{D}$ be as in Lemma 7.3.4.12, and assume that $\mathcal{D}$ is pointed. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. For every $\vec{n} = \{n_s\}_{s \in S}$ with $\sum_{s \in S} n_s = n$, we let $\mathrm{cr}_{(\vec{n})}(F) \in \mathrm{SymFun}^{(\vec{n})}(\mathcal{C}, \mathcal{D})$ be the composition of $\mathrm{cr}_{(n)}(F) \in \mathrm{SymFun}^{(n)}(\mathcal{C}, \mathcal{D})$ with the functor $U_{\vec{n}} : \mathcal{C}^{(\vec{n})} \to \mathcal{C}^{(n)}$ in the statement of Lemma 7.3.4.12. Note that the underlying n-ary functor of $\mathrm{cr}_{(\vec{n})}(F)$ is given by the functor $\mathrm{cr}_{\vec{n}}(F) : \prod_{s \in S} \mathcal{C}_s^{n_s} \to \mathcal{D}$ appearing in Variant 7.1.3.21.

Proof of Proposition 7.3.4.11. Suppose we are given two sequences $\vec{m}$ and $\vec{n}$ with $|\vec{m}| = |\vec{n}| = n$. Consider the composite functor

$$U_{\vec{m}, \vec{n}} : \mathrm{Homog}^{\vec{m}}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D}) \subseteq \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \overset{\mathrm{cr}_{(n)}}{\to} \mathrm{SymFun}_{\mathrm{lin}}^n(\mathcal{C}, \mathcal{D}) \to \mathrm{SymFun}_{\mathrm{lin}}^{\vec{n}}(\mathcal{C}, \mathcal{D}).$$

If $\vec{m} = \vec{n}$, then iterated application of Theorem 7.1.4.7 shows that $U_{\vec{m}, \vec{n}}$ is an equivalence of ∞-categories. It $\vec{m} \neq \vec{n}$, we claim that $U_{\vec{m}, \vec{n}}$ carries each object $F \in \mathrm{Homog}^{\vec{m}}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$ to a zero object of the ∞-category $\mathrm{SymFun}_{\mathrm{lin}}^{\vec{n}}(\mathcal{C}, \mathcal{D})$. Using Remark 7.3.4.13, we are reduced to proving that $\mathrm{cr}_{\vec{m}}(F)$ carries each object of $\prod_{s \in S} \mathcal{C}_s^{m_s}$ to a zero object of $\mathcal{D}$. Since $\sum_{s \in S} m_s = \sum_{s \in S} n_s$, we have $m_s > n_s$ for some $s \in S$, in which case the desired result follows by applying Proposition 7.1.3.22 to the functor $\mathcal{C}_s \to \mathrm{Fun}(\prod_{t \neq s} \mathcal{C}_t, \mathcal{D})$ determined by F.

It follows from the above argument that the composite functor

$$\prod_{|\vec{m}| = n} \mathrm{Homog}^{\vec{m}}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D}) \overset{\Phi}{\to} \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \overset{\mathrm{cr}_{(n)}}{\to} \mathrm{SymFun}_{\mathrm{lin}}^n(\mathcal{C}, \mathcal{D}) \overset{\Psi}{\to} \prod_{|\vec{n}| = n} \mathrm{SymFun}_{\mathrm{lin}}^{\vec{n}}(\mathcal{C}, \mathcal{D})$$

is an equivalence of ∞-categories, where Ψ is the equivalence of ∞-categories appearing in Lemma 7.3.4.12. Since $\mathrm{cr}_{(n)}$ is an equivalence of ∞-categories by Theorem 7.1.4.7, we conclude that Φ is an equivalence of ∞-categories. $\qquad\square$

We now turn to the proof of Theorem 7.3.3.14.

Remark 7.3.4.14. Let $\{\mathcal{C}_s\}$ be a finite collection of pointed ∞-categories which admit finite colimits. Suppose we are given functors $F, F' : \prod_{s \in S} \mathcal{C}_s \to \mathcal{S}_*$ and an element $s \in S$ such that F' is 2-reduced when regarded as a functor from $\mathcal{C}_s$ to $\mathrm{Fun}(\prod_{t \neq s} \mathcal{C}_t, \mathcal{S}_*)$. Form fiber sequences

$$H \to F \to \varprojlim_{E \in \mathrm{Equiv}^+(S)} F^E.$$

$$H' \to F' \to \varprojlim_{E \in \mathrm{Equiv}^+(S)} F'^E.$$

Using Remark 7.1.1.29, we deduce that H' and each of the functors F'^E is also 2-reduced when regarded as a functor $\mathcal{C}_s \to \mathrm{Fun}(\prod_{t \neq s} \mathcal{C}_t, \mathcal{S}_*)$. It follows that the projection maps

$$H \times H' \to H \qquad (F \times F')^E \simeq F^E \times F'^E \to F^E$$

induce equivalences after applying the functor $P_{\vec{1}}$ for $\vec{1} = (1, \ldots, 1)$, and therefore equivalences of differentials. Consequently, the conclusion of Theorem 7.3.3.14 is valid for F if and only if it is valid for the product $F \times F'$.

Proof of Theorem 7.3.3.14. Let n be the cardinality of the set S, let $\mathcal{C} = \prod_{s \in S} \mathcal{C}_s$, and let $\mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{S}_*)$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S}_*)$ spanned by those functors which are reduced and excisive in each variable. Note that any functor $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$ which is 1-homogeneous in each variable is n-homogeneous when regarded as a functor from $\mathcal{C}$ to $\mathcal{D}$ (Corollary 7.1.3.11). It follows that any natural transformation $F' \to F$ which induces an equivalence $P_n F' \to P_n F$ also induces an equivalence of differentials $DF' \to DF$.

Let $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{S}_*$ be a functor which preserves final objects. For every subset $T \subseteq S$, let $F^T : \mathcal{C} \to \mathcal{S}_*$ be defined as in Construction 7.1.3.15, and note that $(P_n F^T) \simeq P_n(F^T)$. Since P_n is left exact, we conclude that $P_n \mathrm{Red}(F) \simeq \mathrm{Red}(P_n F)$. In particular, we have $D \mathrm{Red}(F) \simeq D \mathrm{Red}(P_n F)$ and $DF^E \simeq DP_n(F^E) \simeq D(P_n F)^E$ for each equivalence relation E on S. We may therefore replace F by $P_n F$ and thereby reduce to the case where F is n-excisive.

Using Proposition 7.3.4.4, we can choose a jet equivalence $|F_\bullet| \to F$ such that each F_k is a finite product of homogeneous functors from $\mathcal{C}$ to $\mathcal{D}$. Let F' denote the functor $|F_\bullet|$. Note that for every subset $T \subseteq S$, the functor F'^T is given by $|F_\bullet^T|$ (using the notation of Construction 7.1.3.15). Since geometric realizations in $\mathcal{S}$ commute with products, we deduce that for every equivalence relation E on S, the canonical map $|F_\bullet^E| \to F'^E$ is an equivalence. Since $F' \to F$ is a jet equivalence, the left exactness of the functors P_m implies that $F'^E \to F^E$ is a jet equivalence for every $E \in \mathrm{Equiv}(S)$, so that we have a jet equivalence $|F_\bullet^E| \to F^E$. In particular, we obtain an equivalence of differentials $D|F_\bullet^E| \simeq DF^E$, so that DF^E is the geometric realization of the simplicial object $DF_\bullet^E$ in the ∞-category $\mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{S}_*)$. Since the ∞-category $\mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{S}_*)$ is stable (see Corollary 7.1.2.8), we deduce that the canonical map

$$|\mathrm{tfib}\{DF_\bullet^E\}_{E \in \mathrm{Equiv}(S)}| \to \mathrm{tfib}\{|DF_\bullet^E|\}_{E \in \mathrm{Equiv}(S)}$$

is an equivalence in $\mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{S}_*)$.

Let $H_\bullet$ be the simplicial functor given by $\mathrm{tfib}\{F_\bullet^E\}_{E \in \mathrm{Equiv}(S)}$, and let $H = \mathrm{tfib}\{F^E\}_{E \in \mathrm{Equiv}(S)}$. It follows from Lemma 7.3.4.6 that the canonical map $|H_\bullet| \to H$ is a jet equivalence, so that $DH \simeq D|H_\bullet|$ is the geometric realization of the simplicial object of the simplicial object $dH_\bullet$ in the ∞-category $\mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{S}_*)$. It follows that the map

$$DH \to \mathrm{tfib}\{DF^E)\}_{E \in \mathrm{Equiv}(S)}$$

can be realized as a colimit of maps $DH_m \to \mathrm{tfib}\{F_m^E\}_{E \in \mathrm{Equiv}(S)}$. We may therefore replace F by F_m and thereby reduce to the case where $F : \mathcal{C} \to \mathcal{S}_*$ is a finite product of reduced homogeneous functors.

Using Proposition 7.3.4.11, we can write F as a finite product $\prod_{\alpha \in A} F_\alpha$, where each $F(\alpha)$ belongs to $\mathrm{Homog}^{\vec{n}_\alpha}(\prod_{s \in S} \mathcal{C}_s, \mathcal{S}_*)$, where $\vec{n}_\alpha = \{n_{\alpha,s}\}_{s \in S}$ is a collection of nonnegative integers with $|\vec{n}_\alpha| = \sum_{s \in S} n_{\alpha,s} > 0$. Let $A_0 \subseteq A$ be the subset consisting of those indices α such that $n_{\alpha,s} \leq 1$ for all $s \in S$. Let

$\overline{F} = \prod_{\alpha \in A_0} F(\alpha)$. Applying Remark 7.3.4.14 repeatedly, we can replace F by $\overline{F}$ and thereby reduce to the case where $A_0 = A$. For every subset $T \subseteq S$, let F_T denote the product of those functors F_α for which $\alpha \in A$ satisfies $n_{\alpha,s} = \begin{cases} 1 & \text{if } s \in T \\ 0 & \text{if } s \notin T. \end{cases}$ Then $F \simeq \prod_{\emptyset \neq T \subseteq S} F_T$, where each F_T can be written as a composition

$$\prod_{s \in S} \mathcal{C}_s \to \prod_{s \in T} \mathcal{C}_s \xrightarrow{\widetilde{F_T}} \mathcal{S}_*$$

for some functor $\widetilde{F_T}$ which is reduced and excisive in each variable.

Let $\mathcal{U}$ be a collection of nonempty subsets of S, and set $F_{\mathcal{U}} = \prod_{T \in \mathcal{U}} F_T$. Note that for $E \in \mathrm{Equiv}(S)$, we have $F_{\mathcal{U}}^E = \prod_T F_T$, where the product is taken over all subsets $T \in \mathcal{U}$ which are contained in a single equivalence class of the equivalence relation E. If $T \neq S$, then the collection of equivalence relations $E \in \mathrm{Equiv}^+(S)$ satisfying this condition has a smallest element, and therefore a weakly contractible nerve. It follows that $\varprojlim_{E \in \mathrm{Equiv}^+(S)} F_{\mathcal{U}}^E$ is given by the product $\prod_{T \in \mathcal{U}, T \neq S} F_T$. Form a fiber sequence

$$H_{\mathcal{U}} \to F_{\mathcal{U}} \to \varprojlim_{E \in \mathrm{Equiv}^+(S)} F_{\mathcal{U}}^E,$$

so that $H_{\mathcal{U}} \simeq \begin{cases} F_S & \text{if } S \in \mathcal{U} \\ * & \text{otherwise,} \end{cases}$ where $*$ denotes the constant functor $\mathcal{C} \to \mathcal{S}_*$ carrying every object to a single point.

We will prove that for every collection $\mathcal{U}$ of nonempty subsets of S, the canonical map

$$\theta_{\mathcal{U}} : DH_{\mathcal{U}} \to \mathrm{tfib}\{DF_{\mathcal{U}}^E\}_{E \in \mathrm{Equiv}(S)}$$

is an equivalence. Taking $\mathcal{U} = \mathbf{P}(S)$, we obtain a proof of the desired result. Our proof will proceed by induction on the cardinality of $\mathcal{U}$. Let us therefore assume that $\theta_{\mathcal{U}'}$ is an an equivalence for every proper subset $\mathcal{U}' \subsetneq \mathcal{U}$. Since the ∞-category $\mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{S}_*)$ is stable (Corollary 7.1.2.8) and the differentiation functor $d : \mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{S}_*) \to \mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{S}_*)$ preserves colimits, we deduce that the map

$$\theta' : D(\varinjlim_{\mathcal{U}' \subsetneq \mathcal{U}} H_{\mathcal{U}'}) \to \mathrm{tfib}\{D(\varinjlim_{\mathcal{U}' \subsetneq \mathcal{U}} F_{\mathcal{U}'}^E\}_{E \in \mathrm{Equiv}(S)}$$

For each equivalence relation $E \in \mathrm{Equiv}(S)$, let $G(E)$ denote the cofiber of the canonical map

$$\varinjlim_{\mathcal{U}' \subsetneq \mathcal{U}} F_{\mathcal{U}'}^E \to F_{\mathcal{U}}^E.$$

When $E = E_\perp$ is the discrete equivalence relation on S (that is, $sE_\perp t$ for all $s, t \in S$), we denote $G(E)$ by G. Let H_0 denote the cofiber of the map $\varprojlim_{\mathcal{U}' \subsetneq \mathcal{U}} H_{\mathcal{U}'} \to H_{\mathcal{U}}$. Using the description of the functor $H_{\mathcal{U}}$ given above, we see that

$$H_0 = \begin{cases} F_S & \text{if } \mathcal{U} = \{S\} \\ 0 & \text{otherwise.} \end{cases}$$

We have a fiber sequence

$$\theta' \to \theta_{\mathcal{U}} \to \theta''$$

in the stable ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{S}_*))$, where θ'' denotes the canonical map

$$dH_0 \to \mathrm{tfib}\{DG(E)\}_{E \in \mathrm{Equiv}(S)}.$$

Consequently, to prove that $\theta_{\mathcal{U}}$ is an equivalence, it will suffice to show that θ'' is an equivalence. If $\mathcal{U} = \{S\}$, then $H_0 \simeq G \simeq F_S$ and $G(E)$ is trivial for $E \in \mathrm{Equiv}^+(S)$. Let us therefore assume that $\mathcal{U} \neq \{S\}$. Then H_0 is trivial, so we are reduced to proving that the canonical map

$$\phi : DG \to \mathrm{tfib}\{DG(E)\}_{E \in \mathrm{Equiv}(S)}$$

is an equivalence.

Let $\wedge : \mathcal{S}_* \times \mathcal{S}_* \to \mathcal{S}_*$ denote the smash product functor on pointed spaces (see Example 7.2.3.28). Unwinding the definitions, we see that for each equivalence relation $E \in \mathrm{Equiv}(S)$, the functor $G(E)$ is given by the smash product $\bigwedge_{T \in \mathcal{U}} F_T$ if each element $T \in \mathcal{U}$ is contained in an equivalence class of E, and is trivial otherwise. There are several cases to consider:

(a) Suppose there is some element $s \in S$ which does not belong to any element of $\mathcal{U}$. Then each of the functors $G(E)$ is constant when regarded a functor $\mathcal{C}_s \to \mathrm{Fun}(\prod_{t \neq s} \mathcal{C}_t, \mathcal{S}_*)$, so that the differential $DG(E)$ is a zero object of $\mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{S}_*)$. In this case, ϕ is a map between zero objects of $\mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{S}_*)$, and therefore an equivalence.

(b) Suppose that some element $s \in S$ belongs to $T \cap T'$ for some pair of distinct elements $T, T' \in \mathcal{U}$. Using Proposition 7.1.3.10, we deduce that each of the functors $G(E)$ is 2-reduced when regarded as a functor from $\mathcal{C}_s$ to $\mathrm{Fun}(\prod_{t \neq s} \mathcal{C}_t, \mathcal{S}_*)$, so that the differential $DG(E)$ vanishes (Remark 7.3.4.14). We again see that ϕ is a map between zero objects of $\mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{S}_*)$, and therefore an equivalence.

(c) Suppose that $\mathcal{U} = \{T_1, \ldots, T_k\}$ for some collection of disjoint nonempty subsets $T_1, \ldots, T_k \subseteq S$ satisfying $S = \bigcup T_i$. Let $\mathrm{Equiv}^{++}(S) \subseteq \mathrm{Equiv}^+(S)$ be the subset consisting of those equivalence relations such that each T_i is contained in an equivalence class. Our analysis above shows that the functors $G(E)$ are given by the formula

$$G(E) = \begin{cases} G & \text{if } E \in \mathrm{Equiv}^{++}(S) \\ * & \text{otherwise,} \end{cases}$$

so that the diagram $(E \in \mathrm{Equiv}^+(S)) \mapsto G(E)$ is a right Kan extension of its restriction to the ∞-category $\mathrm{N}(\mathrm{Equiv}^{++}(S))^{op}$. Consequently, to prove that ϕ is an equivalence, it will suffice to show that the simplicial set $\mathrm{N}(\mathrm{Equiv}^{++}(S))$ is weakly contractible. In fact, the partially ordered set $\mathrm{Equiv}^{++}(S)$ has a smallest element: namely, the equivalence relation whose equivalence classes are precisely the sets T_i (note that this equivalence relation belongs to $\mathrm{Equiv}^+(S)$ by virtue of our assumption that $\mathcal{U} \neq \{S\}$).

$\square$

7.3.5   Consequences of Theorem 7.3.3.14

Let $\{\mathcal{C}_s\}_{s \in S}$ be a nonempty finite collection of pointed ∞-categories which admit finite colimits, let $\mathcal{D}$ be a pointed differentiable ∞-category, and let

$$D : \mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D}) \to \mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$$

be a left adjoint to the inclusion (given by differentiation). When restricted to the full subcategory of $\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathcal{D})$ spanned by those functors which are reduced in each variable, the functor D coincides with $P_{\vec{1}}$ of Proposition 7.1.3.6 (with $\vec{1} = (1, 1, \ldots, 1)$), and is therefore left exact. In general D is not left exact. Nevertheless, Theorem 7.3.3.14 implies that D commutes with a very special type of limits. In this section, we will apply Theorem 7.3.3.14 to show that differentiation commutes with a larger class of limits (Theorem 7.3.5.5). We will also obtain a generalization of the chain rule of Theorem 7.2.1.22 to the case of nonreduced functors (Theorem 7.3.5.6).

We begin by formulating a more general version of Theorem 7.3.3.14.

Notation 7.3.5.1. Let $p : S \to T$ be a map of finite sets. For each $t \in T$, we let S_t denote the fiber of $p^{-1}\{t\} \subseteq S$. We let $\mathrm{Equiv}_p(S)$ denote the subset of $\mathrm{Equiv}(S)$ consisting of those equivalence relations E on S such that p is constant on each equivalence class: that is, equivalence relations for which xEy implies

$p(x) = p(y)$. If $E \in \mathrm{Equiv}(S)$ and $t \in T$, we let E_t denote the restriction of E to an equivalence relation on $S_t \subseteq S$. Note that the construction $E \mapsto \{E_t\}_{t \in T}$ determines an isomorphism of partially ordered sets $\mathrm{Equiv}_p(S) \simeq \prod_{t \in T} \mathrm{Equiv}(S_t)$. Note that $\mathrm{Equiv}_p(S)$ has a largest element E_p, where xE_py if and only if $p(x) = p(y)$. We let $\mathrm{Equiv}_p^+(S)$ denote the partially ordered set $\mathrm{Equiv}_p(S) - \{E_p\}$.

Proposition 7.3.5.2. *Let $p : S \to T$ be a surjective map of nonempty finite sets. Suppose we are given a collection of pointed ∞-categories $\{\mathcal{C}_s\}_{s \in S}$ which admit finite colimits, a collection of pointed ∞-categories $\{\mathcal{D}_t\}_{t \in T}$, and a pointed presentable differentiable ∞-category $\mathcal{E}$. Let $\{F_t : \prod_{p(s)=t} \mathcal{C}_s \to \mathcal{D}_t\}$ and $G : \prod_{t \in T} \mathcal{D}_t \to \mathcal{E}$ be functors which preserve zero objects. For each $E \in \mathrm{Equiv}_p(S)$, let F^E denote the functor*

$$\prod_{s \in S} \mathcal{C}_s \xrightarrow{\prod_{t \in T} F_t^{E_t}} \prod_{t \in T} \mathcal{D}_t .$$

Then the canonical map

$$D(G \circ \mathrm{tfib}\{F^E\}_{E \in \mathrm{Equiv}_p(S)}) \to \mathrm{tfib}\{D(G \circ F^E)\}_{E \in \mathrm{Equiv}_p(S)}$$

is an equivalence in the ∞-category $\mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{E})$.

Remark 7.3.5.3. In the special case where T has a single element, $\mathcal{E} = \mathcal{S}_*$, and the functor G is an equivalence, Proposition 7.3.5.2 reduces to Theorem 7.3.3.14.

Before giving the proof of Proposition 7.3.5.2, let us describe some of its consequences.

Notation 7.3.5.4. If $\mathcal{C}$ and $\mathcal{D}$ are compactly generated ∞-categories, we let $\mathrm{Fun}^c(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which preserve filtered colimits.

Theorem 7.3.5.5. *Let S be a nonempty finite set. Suppose we are given compactly generated pointed ∞-categories $\{\mathcal{C}_s\}_{s \in S}$, and $\mathcal{D}$. Let K be a finite simplicial set and suppose we are given a diagram*

$$\alpha : K^\triangleleft \to \mathrm{Fun}^c(\prod_{s \in S} \mathcal{C}_s, \mathcal{D}),$$

carrying each vertex $v \in K^\triangleleft$ to a functor $F_v : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$. For every nonempty subset $S' \subseteq S$, let $F_{v,S'}$ denote the restriction of F_v to

$$\prod_{s \in S'} \mathcal{C}_s \simeq \prod_{s \in S'} \mathcal{C}_s \times \prod_{s \in S-S'} \{*_s\} \subseteq \prod_{s \in S} \mathcal{C}_s;$$

here $*_s$ denotes a zero object of $\mathcal{C}_s$ for $s \in S$. Assume that:

(1) *Each of the functors F_v carries zero objects of $\prod_{s \in S} \mathcal{C}_s$ to zero objects of $\mathcal{D}$.*

(2) *For each equivalence relation $E \in \mathrm{Equiv}(S)$, there exists a map of finite simplicial sets $u : K \to \prod_{S' \in S/E} K_{S'}$ satisfying the following conditions:*

    (a) *The map of simplicial sets u is right cofinal.*

    (b) *For each $S' \in S/E$, there exists a limit diagram $\beta_{S'} : K_{S'}^\triangleleft \to \mathrm{Fun}^c(\prod_{s \in S'} \mathcal{C}_s, \mathcal{D})$ such that the composite map*

$$K^\triangleleft \xrightarrow{u} \prod_{S'' \in S/E} K_{S''}^\triangleleft \to K_{S'}^\triangleleft \xrightarrow{\beta_{S'}} \mathrm{Fun}^c(\prod_{s \in S'} \mathcal{C}_s, \mathcal{D})$$

    is given by $v \mapsto F_{v,S'}$.

Let v_0 denote the cone point of $K^\triangleleft$. Then the canonical map

$$\phi : \vec{\partial}(F_{v_0}) \to \varprojlim_{v \in K} \vec{\partial}(F_v)$$

is an equivalence in $\mathrm{Exc}_*(\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s), \mathrm{Sp}(\mathcal{D}))$.

Proof of Theorem 7.3.5.5. For every compact object $D \in \mathcal{D}$, let $J_D : \mathcal{D} \to \mathcal{S}_*$ denote the functor corepresented by D. Then J_D is left exact, so that its derivative $j_D = \partial(J_D) : \mathrm{Sp}(\mathcal{D}) \to \mathrm{Sp}$ is given by pointwise composition with J_D (Example 7.2.1.4). Since $\mathcal{D}$ is compactly generated, the functors j_D are jointly conservative. Consequently, it will suffice to show that each of the induced maps

$$\alpha_G : j_D \circ \vec{\partial}(F_{v_0}) \to \varprojlim j_D \circ \vec{\partial}(F_v$$

is an equivalence. Using Theorem 7.2.1.22, we can identify α with the map

$$\vec{\partial}(J_D \circ \prod_{s \in S} F_s) \to \varprojlim_{v \in K} \vec{\partial}(J_D \circ F_v).$$

We may therefore replace F by $J_D \circ F$ and thereby reduce to the case where $\mathcal{D} = \mathcal{S}_*$.

For each equivalence relation $E \in \mathrm{Equiv}(S)$, let ϕ_E denote the canonical map $\vec{\partial}(F_{v_0}^E) \to \varprojlim_{v \in K} \vec{\partial}(F_v^E)$. We will prove that each of the maps ϕ_E is an equivalence. The proof will proceed by induction. For every equivalence relation E, let $\mathrm{Equiv}_E(S)$ denote the collection of equivalence relations on S that refine E (that is, $\mathrm{Equiv}_E(S) = \{E' \in \mathrm{Equiv}(S) : E' \leq E\}$) and let $\mathrm{Equiv}_E^+(S) = \mathrm{Equiv}_E(S) - \{E\}$. To complete the proof, it will suffice to show that if $E \in \mathrm{Equiv}(S)$ has the property that $\phi_{E'}$ is an equivalence for each $E' \in \mathrm{Equiv}_E^+(S)$, then ϕ_E is an equivalence.

Fix an equivalence relation $E \in \mathrm{Equiv}(S)$ as above, and let $T = S/E$. Let $G : \prod_{t \in T} \mathcal{S}_* \to \mathcal{S}_*$ denote the functor given by iterated Cartesian product. Then ϕ_E can be identified with the upper horizontal map appearing in the diagram σ :

$$
\begin{array}{ccc}
\vec{\partial}(G \circ \prod_{S' \in T} F_{v_0, S'}) & \longrightarrow & \varprojlim_{v \in K} \vec{\partial}(G \circ \prod_{S' \in T} F_{v, S'}) \\
\downarrow & & \downarrow \\
\varprojlim_{E' \in \mathrm{Equiv}_E^+(S)} \mathrm{Diff}(G \circ \prod_{S' \in T} F_{v_0, S'}^{E'}) & \longrightarrow & \varprojlim_{v \in K} \varprojlim_{E' \in \mathrm{Equiv}_E^+(S)} (\vec{\partial}(G \circ \prod_{S' \in T} F_{v, S'}^{E'})).
\end{array}
$$

Here the lower horizontal map is an equivalence by the inductive hypothesis. Since $\mathrm{Exc}_*(\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s), \mathrm{Sp})$ is a stable ∞-category, it will suffice to show that σ induces an equivalence after passing to the fibers of the vertical maps. Using Proposition 7.3.5.2, we are reduced to providing that the map

$$\vec{\partial}(G \circ \prod_{S' \in T} \mathrm{Red}(F_{v_0, S'})) \to \varprojlim_{v \in K} \vec{\partial}(G \circ \prod_{S' \in T} \mathrm{Red}(F_{v, S'}))$$

is an equivalence.

Let $G' : \prod_{t \in T} \mathcal{S}_* \to \mathcal{S}_*$ denote the iterated smash product functor. For each vertex $v \in K$, the canonical map

$$G \circ \prod_{S' \in T} \mathrm{Red}(F_{v, S'}) \to G' \circ \prod_{S' \in T} \mathrm{Red}(F_{v, S'})$$

induces an equivalence of derivatives. It will therefore suffice to show that the canonical map

$$\theta : \vec{\partial}(G' \circ \prod_{S' \in T} \mathrm{Red}(F_{v_0, S'})) \to \varprojlim_{v \in K} \vec{\partial}(G' \circ \prod_{S' \in T} \mathrm{Red}(F_{v, S'}))$$

is an equivalence. Using Theorem 7.2.1.22 and Example 7.2.3.28, we can identify θ with the canonical map

$$\theta' : \otimes_{S' \in T} \vec{\partial}(\mathrm{Red}(F_{v_0, S'})) \to \varprojlim_{v \in K} \otimes_{S' \in T} \vec{\partial}(\mathrm{Red}(F_{v, S'})).$$

Choose a map $u : K \to \prod_{S' \in S/E} K_{S'}$ and maps $\beta_{S'} : K_{S'}^{\triangleleft} \to \mathrm{Fun}^c(\prod_{s \in S'} \mathcal{C}_s, \mathcal{D})$ as in (2). For $S' \in T$, let $v_{0, S'}$ denote the cone point of $K_{S'}^{\triangleleft}$. Using the right cofinality of u, we can identify θ with the canonical map

$$\otimes_{S' \in T} \vec{\partial} \mathrm{Red}(\beta_{S'}(v_{0, S'})) \to \varprojlim_{\{v_{S'} \in K_{S'}\}} \otimes_{S' \in T} \vec{\partial} \mathrm{Red}(\beta_{S'}(v_{S'})) \simeq \otimes_{S' \in T} \varprojlim_{v_{S'} \in K_{S'}} \vec{\partial} \mathrm{Red}(\beta_{S'}(v_{S'})).$$

We are therefore reduced to proving that for each $S' \in T$, the canonical map

$$\vec{\partial} \mathrm{Red}(\beta_{S'}(v_{0, S'})) \to \varprojlim_{v_{S'} \in K_{S'}} \vec{\partial} \mathrm{Red}(\beta_{S'}(v_{S'}))$$

is an equivalence. Since differentiation is left exact when restricted to functors which are reduced in each variable, so it suffices to show that $\mathrm{Red}(\beta_{S'}(v_{0, S'})) \simeq \varprojlim_{v_{S'} \in K_{S'}} \mathrm{Red}\, \beta_{S'}(v_{S'})$ is an equivalence. This follows from our assumption that $\beta_{S'}$ is a limit diagram, since the reduction functor Red is left exact. $\square$

Theorem 7.3.5.6. *Let $p : S \to T$ be a surjective map of nonempty finite sets. Suppose we are given compactly generated pointed ∞-categories $\{\mathcal{C}_s\}_{s \in S}$, $\{\mathcal{D}_t\}_{t \in T}$, and $\mathcal{E}$. Let $G \in \mathrm{Fun}^c(\prod_{t \in T} \mathcal{D}_t, \mathcal{E})$. Let K be a finite simplicial set, and suppose that for each $t \in T$ we are given a diagram*

$$K^{\triangleleft} \to \mathrm{Fun}^c(\prod_{s \in S_t} \mathcal{C}_s, \mathcal{D}_t),$$

carrying each vertex $v \in K^{\triangleleft}$ to a functor $F_{t,v} : \prod_{s \in S_t} \mathcal{C}_s \to \mathcal{D}_t$. Assume that:

(1) *Each of the functors $F_{t,v}$ carries zero objects of $\prod_{s \in S_t} \mathcal{C}_s$ to zero objects of $\mathcal{D}_t$, and G carries zero objects of $\prod_{t \in T} \mathcal{D}_t$ to zero objects of $\mathcal{E}$.*

(2) *For each $E \in \mathrm{Equiv}_p^+(S)$, there exists a map of finite simplicial sets $u : K^{\triangleleft} \to K_E^{\triangleleft}$ satisfying the following conditions:*

  (a) *The underlying map $K \to K_E^{\triangleleft}$ is right cofinal.*
  (b) *For each $t \in T$, the functor $K^{\triangleleft} \to \mathrm{Fun}^c(\prod_{s \in S_t} \mathcal{C}_s, \mathcal{D}_t)$ given by $v \mapsto F_{t,v}^{E_t}$ factors through u.*
  (c) *The map u carries the cone point of $K^{\triangleleft}$ to the cone point of $K_E^{\triangleleft}$.*

Let v_0 denote the cone point of $K^{\triangleleft}$. Then the diagram σ:

$$
\begin{array}{ccc}
\vec{\partial}(G \circ \prod_{t \in T} F_{t,v_0}) & \longrightarrow & \varprojlim_{v \in K} \vec{\partial}(G \circ \prod_{t \in T} F_{t,v}) \\
\downarrow & & \downarrow \\
\vec{\partial}(G) \circ \prod_{t \in T} \vec{\partial}(F_{t,v_0}) & \longrightarrow & \varprojlim_{v \in K} \vec{\partial}(G) \circ \prod_{t \in T} \vec{\partial}(F_{t,v})
\end{array}
$$

is a pullback square in $\mathrm{Fun}(\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s), \mathrm{Sp}(\mathcal{E}))$.

Proof. For each $E \in \mathrm{Equiv}_p^+(S)$, let $\sigma(E)$ denote the diagram

$$
\begin{array}{ccc}
\vec{\partial}(G \circ \prod_{t \in T} F_{t,v_0}^{E_t}) & \longrightarrow & \varprojlim_{v \in K} \vec{\partial}(G \circ \prod_{t \in T} F_{t,v}^{E_t}) \\
\downarrow & & \downarrow \\
\vec{\partial}(G) \circ \prod_{t \in T} \vec{\partial}(F_{t,v_0}^{E_t}) & \longrightarrow & \varprojlim_{v \in K} \vec{\partial}(G) \circ \prod_{t \in T} \vec{\partial}(F_{t,v}^{E_t}).
\end{array}
$$

Form a fiber sequence

$$\tau \to \sigma \to \varprojlim_{E \in \mathrm{Equiv}_p^+(S)} \sigma(E).$$

in the stable ∞-category $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{Exc}_*(\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s), \mathrm{Sp}(\mathcal{E})))$. To prove that σ is a pullback diagram, it will suffice to show that τ is a pullback diagram and that $\sigma(E)$ is a pullback diagram, for each $E \in \mathrm{Equiv}_p^+(S)$.

We first show that τ is a pullback diagram. Unwinding the definitions, we can write τ as a commutative diagram

$$
\begin{array}{ccc}
\mathrm{tfib}\{\vec{\partial}(G \circ \prod_t F_{t,v_0}^{E_t})\}_{E \in \mathrm{Equiv}_p(S)} & \longrightarrow & \varprojlim_{v \in K} \mathrm{tfib}\{\vec{\partial}(G \circ \prod_t F_{t,v}^{E_t})\}_{E \in \mathrm{Equiv}_p(S)} \\
\downarrow & & \downarrow \\
\mathrm{tfib}\{\vec{\partial}(G) \circ \prod_t \vec{\partial}(F_{t,v_0}^{E_t})\}_{E \in \mathrm{Equiv}_p(S)} & \longrightarrow & \varprojlim_{v \in K} \mathrm{tfib}\{\vec{\partial}(G) \circ \prod_t \vec{\partial}(F_{t,v}^{E_t})\}_{E \in \mathrm{Equiv}_p(S)}.
\end{array}
$$

To prove that this diagram is a pullback square, it will suffice to show that the vertical maps are equivalences. Fix a vertex $v \in K^{\triangleleft}$; we will show that the map

$$\theta : \mathrm{tfib}\{\vec{\partial}(G \circ \prod_{t \in T} F_{t,v}^{E_t})\}_{E \in \mathrm{Equiv}_p(S)} \to \mathrm{tfib}\{\vec{\partial}(G) \circ \prod_{t \in T} \vec{\partial}(F_{t,v}^{E_t})\}_{E \in \mathrm{Equiv}_p(S)}$$

is an equivalence. Since the functor $\vec{\partial}(G)$ is left exact in each variable, the right hand side is given by

$$\vec{\partial}(G) \circ \prod_{t \in T} \mathrm{tfib}\{\vec{\partial}(F_{t,v}^{E_t})\}_{E_t \in \mathrm{Equiv}(S_t)}.$$

We have a commutative diagram

$$
\begin{array}{ccc}
\vec{\partial}(G \circ \prod_{t \in T} \mathrm{tfib}\{F_{t,v}^{E_t}\}_{E_t \in \mathrm{Equiv}(S_t)}) & \longrightarrow & \mathrm{tfib}\{G \circ \prod_{t \in T} F_{t,v}^{E_t}\}_{E \in \mathrm{Equiv}_p(S)} \\
\downarrow \scriptstyle{\theta'} & & \downarrow \scriptstyle{\theta} \\
\vec{\partial}(G) \circ \prod_{t \in T} \vec{\partial}(\mathrm{tfib}\{F_{t,v}^{E_t}\}_{E_t \in \mathrm{Equiv}(S_t)}) & \longrightarrow & \vec{\partial}(G) \circ \prod_{t \in T} \mathrm{tfib}\{\vec{\partial}(F_{t,v}^{E_t})\}_{E_t \in \mathrm{Equiv}(S_t)}.
\end{array}
$$

The horizontal maps in this diagram are equivalences by Proposition 7.3.5.2. We are therefore reduced to proving that θ' is an equivalence. This follows from Theorem 7.2.1.22, since each of the functors $\mathrm{tfib}\{F_{t,v}^{E_t}\}$ is reduced in each variable. This completes the proof that τ is a pullback square.

Now suppose that $E \in \mathrm{Equiv}_p^+(S)$; we will show that the diagram $\sigma(E)$ is a pullback square. Choose a map $u : K^{\triangleleft} \to K_E^{\triangleleft}$ satisfying the requirements of hypothesis (2). Condition (b) implies that for each $t \in T$, we can choose a diagram $u_t' : K_{t,E}^{\triangleleft} \to \mathrm{Fun}^c(\prod_{s \in S_t} \mathcal{C}_s, \mathcal{D}_t)$ such that the composition of $u_t' \circ u_t$ is given by the formula $v \mapsto F_{t,v}^{E_t}$. Let $K_E^{\triangleleft}$ denote the product $\prod_{t \in T} K_{t,E}^{\triangleleft}$. For each vertex $w \in \{w_t\}_{t \in T}$, let $H_{t,w} : \prod_{s \in S_t} \mathcal{C}_s \to \mathcal{D}_t$ denote the image of w_t under the diagram u_t'. Then we rewrite $\sigma(E)$ as a diagram

$$
\begin{array}{ccc}
\vec{\partial}(G \circ \prod_{t \in T} H_{t,u_t(v_0)}) & \longrightarrow & \varprojlim_{v \in K} \vec{\partial}(G \circ \prod_{t \in T} H_{t,u_t(v)}) \\
\downarrow & & \downarrow \\
\vec{\partial}(G) \circ \prod_{t \in T} \vec{\partial}(H_{t,u_t(v_0)}) & \longrightarrow & \varprojlim_{v \in K} \vec{\partial}(G) \circ \prod_{t \in T} \vec{\partial}(H_{t,u_t(v)}).
\end{array}
$$

Let w_0 denote the cone point of $K_E^{\triangleleft}$. Using assumptions (a) and (c), we can rewrite the diagram $\sigma(E)$ again:

$$
\begin{array}{ccc}
\vec{\partial}(G \circ \prod_{t \in T} H_{t,w_0}) & \longrightarrow & \varprojlim_{w \in K_E^{\triangleleft}} \vec{\partial}(G \circ \prod_{t \in T} H_{t,w}) \\
\downarrow & & \downarrow \\
\vec{\partial}(G) \circ \prod_{t \in T} \vec{\partial}(H_{t,w_0}) & \longrightarrow & \varprojlim_{w \in K_E^{\triangleleft}} \vec{\partial}(G) \circ \prod_{t \in T} \vec{\partial}(H_{t,w}).
\end{array}
$$

Since $K_E^\triangleleft$ contains w_0 as an initial object, the horizontal maps in this diagram are equivalences. $\qquad\square$

The proof of Proposition 7.3.5.2 will require a bit of terminology.

Notation 7.3.5.7. Let $\{\mathcal{C}_s\}_{s \in S}$ be a nonempty finite collection of pointed ∞-categories which admit finite colimits, and let $\mathcal{D}$ be a differentiable ∞-category. Suppose we are given functors

$$F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D} \qquad f : \prod_{s \in S} \mathcal{C}_s \to \mathrm{Sp}(\mathcal{D}).$$

We will say that a natural transformation $\alpha : F \to \Omega_\mathcal{D}^\infty \circ f$ *exhibits f as a predifferential of F* if it exhibits $\Omega_\mathcal{D}^\infty \circ f$ as a differential of f, in the sense of Definition 7.2.3.1.

Using Proposition 1.4.2.22, we see that $\Omega_\mathcal{D}^\infty$ induces a trivial Kan fibration

$$\mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathrm{Sp}(\mathcal{D})) \to \mathrm{Exc}_*(\prod_{s \in S} \mathcal{C}_s, \mathcal{D}).$$

It follows from Proposition 7.2.3.13 that for every functor $F : \prod_{s \in S} \mathcal{C}_s \to \mathcal{D}$, there exists a functor $f : \prod_{s \in S} \mathcal{C}_s \to \mathrm{Sp}(\mathcal{D})$ and a natural transformation $\alpha : F \to \Omega_\mathcal{D}^\infty \circ f$. The functor f is determined by f up to equivalence; we will denote it by $d(F)$.

Example 7.3.5.8. In the situation of Notation 7.3.5.7, let $\Sigma_\mathcal{D}^\infty : \mathcal{D} \to \mathrm{Sp}(\mathcal{D})$ be a left adjoint to $\Omega_\mathcal{D}^\infty$ (Proposition 7.2.3.16). Then a natural transformation $\alpha : F \to \Omega_\mathcal{D}^\infty \circ f$ exhibits f as a predifferential of F if and only if the adjoint map $\Sigma_\mathcal{D}^\infty \circ F \to f$ exhibits f as a differential of $\Sigma_\mathcal{D}^\infty \circ F$.

Lemma 7.3.5.9. *Let $p : S \to T$ be a surjective map between nonempty finite sets, and let $\{\mathcal{C}_s\}_{s \in S}$ be a collection of pointed ∞-categories which admit finite colimits. Suppose we are given a collection of functors $\{F_t : \prod_{p(s)=t} \mathcal{C}_s \to \mathcal{S}_*\}_{t \in T}$ such that each $F_t(\{*_s\}_{p(s)=t})$ is contractible. Let F denote the composite functor*

$$\prod_{s \in S} \mathcal{C}_s \overset{\prod F_t}{\to} \prod_{t \in T} \mathcal{S}_* \overset{\times}{\to} \mathcal{S}_* .$$

For each $E \in \mathrm{Equiv}_p(S)$, let $F_\times^E$ denote the composition

$$\prod_{s \in S} \mathcal{C}_s \overset{\prod F_t^{E_t}}{\longrightarrow} \prod_{t \in T} \mathcal{S}_* \overset{\times}{\to} \mathcal{S}_* .$$

Then the canonical map

$$d\, \mathrm{tfib}\{F_\times^E\}_{E \in \mathrm{Equiv}_p(S)} \to \mathrm{tfib}\{dF_\times^E\}_{E \in \mathrm{Equiv}_p(S)}$$

is an equivalence.

Proof. Let $E \in \mathrm{Equiv}_p(S)$. Since the formation of products in $\mathcal{S}$ preserves colimits separately in each variable, the product functor on $\mathcal{S}_*$ preserves contractible colimits seperately in each variable. It follows that the coreduction of $F_\times^E$ is given by the composition

$$\prod_{s \in S} \mathcal{C}_s \overset{\prod_{t \in T} \mathrm{cored}(F_t^{E_t})}{\longrightarrow} \prod_{t \in T} \mathcal{S}_* \overset{\wedge}{\to} \mathcal{S}_* .$$

Consequently, the functor $\Sigma^\infty \circ \mathrm{cored}(F_\times^E) : \prod_{s \in S} \mathcal{C}_s \to \mathrm{Sp}$ is given by the composition

$$\prod_{s \in S} \mathcal{C}_s \overset{\prod_{t \in T} \mathrm{cored}(F_t^{E_t})}{\longrightarrow} \prod_{t \in T} \mathcal{S}_* \overset{\Sigma^\infty}{\to} \prod_{t \in T} \mathrm{Sp} \overset{\otimes}{\to} \mathrm{Sp},$$

where $\otimes$ denotes the functor given by iterated smash product of spectra. Since the functor $\otimes$ is left exact and preserves sequential colimits in each variable, we have canonical equivalences

$$
\begin{aligned}
d(F_\times^E) &\simeq d(\mathrm{cored}(F_\times^E)) \\
&\simeq D(\Sigma^\infty \circ \mathrm{cored}(F_\times^E)) \\
&\simeq \otimes_{t \in T} D(\Sigma^\infty \circ \mathrm{cored}(F_t^{E_t})) \\
&\simeq \otimes_{t \in T} d(\mathrm{cored}(F_t^{E_t})) \\
&\simeq \otimes_{t \in T} d(F_t^{E_t}).
\end{aligned}
$$

It follows that the differential $d(F_\times^E)$ is given by the composition

$$
\prod_{s \in S} \mathcal{C}_s \xrightarrow{d(F_t^{E_t})} \prod_{t \in T} \mathrm{Sp} \xrightarrow{\otimes} \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathcal{S}_* .
$$

Since the formation of smash products of spectra is left exact in each variable, Remark 7.3.3.9 implies that the total fiber $\mathrm{tfib}\{D(F_\times^E)\}_{E \in \mathrm{Equiv}_p(S)}$ is given by the composition

$$
\prod_{s \in S} \mathcal{C}_s \xrightarrow{\mathrm{tfib}\{d(F_t^{E_t})\}} \prod_{t \in T} \mathrm{Sp} \xrightarrow{\otimes} \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathcal{S}_* .
$$

Invoking Theorem 7.3.3.14, we can identify this composition with the functor

$$
\prod_{s \in S} \mathcal{C}_s \xrightarrow{\prod_{t \in T} d(\mathrm{Red}(F_t))} \prod_{t \in T} \mathrm{Sp} \xrightarrow{\otimes} \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathcal{S}_*,
$$

which is the differential of the functo

$$
\prod_{s \in S} \mathcal{C}_s \xrightarrow{\prod_t \mathrm{Red}(F_t)} \prod_{t \in T} \mathcal{S}_* \xrightarrow{\times} \mathcal{S}_* .
$$

Using Remark 7.3.3.9 again, we can identify this composition with $\mathrm{tfib}\{F_\times^E\}_{E \in \mathrm{Equiv}_p(S)}$. $\square$

Proof of Proposition 7.3.5.2. It will suffice to show that the natural transformation

$$
d(G \circ \mathrm{tfib}\{F^E\}_{E \in \mathrm{Equiv}_p(S)}) \to \mathrm{tfib}\{d(G \circ F^E)\}_{E \in \mathrm{Equiv}_p(S)}
$$

is an equivalence in $\mathrm{Fun}(\prod_{s \in S} \mathcal{C}_s, \mathrm{Sp}(\mathcal{E}))$ (we can then deduce the analogous result for differentials by composing with the left exact functor $\Omega_{\mathcal{E}}^\infty$). Replacing G by $\Sigma_{\mathcal{E}}^\infty \circ G$ (and using Example 7.3.5.8), we are reduced to proving Proposition 7.3.3.14 in the special case where the ∞-category $\mathcal{E}$ is stable.

We may assume without loss of generality that the ∞-categories $\mathcal{C}_s$ and $\mathcal{D}_t$ are small. Let $\mathcal{X}$ denote the full subcategory of $\mathrm{Fun}(\prod_{t \in T} \mathcal{D}_t, \mathcal{E})$ spanned by those functors which preserve zero objects. Let us regard the functors $F_t : \prod_{p(s)=t} \mathcal{C}_s \to \mathcal{D}_t$ as fixed, and allow G to vary over objects of $\mathcal{X}$. For each $G \in \mathcal{X}$, let θ_G denote the canonical map

$$
D(G \circ \mathrm{tfib}\{F^E\}_{E \in \mathrm{Equiv}_p(S)})) \to \mathrm{tfib}\{D(G \circ F^E)\}_{E \in \mathrm{Equiv}_p(S)}.
$$

Let $\mathcal{X}_0 \subseteq \mathcal{X}$ denote the full subcategory spanned by those functors $G \in \mathcal{X}$ such that θ_G is an equivalence. We wish to show that $\mathcal{X}_0 = \mathcal{X}$.

For every collection of objects $\vec{D} = \{D_t \in \mathcal{D}_t\}_{t \in T}$ and every object $E \in \mathcal{E}$, let $G_{\vec{D}, E} \in \mathcal{X}$ be the functor given by the formula

$$
G_{\vec{D}, E}(\{D_t'\}_{t \in T}) = (\prod_{t \in T} \mathrm{Map}_{\mathcal{D}_t}(D_t, D_t')) \wedge E
$$

(here $\wedge : \mathcal{S}_* \times \mathcal{E} \to \mathcal{E}$ indicates the action of the ∞-category of pointed spaces on $\mathcal{E}$; see Proposition 6.3.2.11). Let $\mathfrak{X}_1$ denote the smallest full subcategory of $\mathfrak{X}$ which is closed under small colimits and contains each $G_{\vec{D},E}$. Using the presentability of $\mathcal{E}$, we deduce that $\mathfrak{X}_1$ is generated by a small collection of objects under small colimits, and is therefore a presentable ∞-category.

Using Corollary T.5.5.2.9, we deduce that the inclusion $\mathfrak{X}_1 \to \mathfrak{X}$ admits a right adjoint V. We claim that V is conservative (so that $\mathfrak{X}_1 = \mathfrak{X}$). To prove this, consider a morphism $\alpha : G \to G'$ in $\mathfrak{X}$ such that $V(\alpha)$ is an equivalence. Then the composite map

$$\mathrm{Map}_{\mathcal{E}}(E, G(\vec{D})) \simeq \mathrm{Map}_{\mathfrak{X}}(G_{\vec{D},E}, G) \to \mathrm{Map}_{\mathfrak{X}}(G_{\vec{D},E}, G') \simeq \mathrm{Map}_{\mathcal{E}}(E, G'(\vec{D}))$$

is a homotopy equivalence for every pair $(\vec{D}, E)$, from which it follows that α is an equivalence.

To prove that $\mathfrak{X}_0 = \mathfrak{X}$, it will suffice to show that $\mathfrak{X}_1 \subseteq \mathfrak{X}_0$. Using the stability of $\mathcal{E}$, we deduce that the construction $G \mapsto \theta_G$ preserves small colimits. It follows that $\mathfrak{X}_0$ is closed under small colimits in $\mathfrak{X}$. It will therefore suffice to show that $G_{\vec{D},E} \in \mathfrak{X}_1$ for every pair $(\vec{D}, E)$. In other words, we can reduce to the case where the functor G is given by a composition

$$\prod_{t\in T} \mathcal{D}_t \xrightarrow{e_{\vec{D}}} \mathcal{S}_* \xrightarrow{\Sigma^\infty} \mathrm{Sp} \xrightarrow{\otimes E} \mathcal{E} \, .$$

where $e_{\vec{D}}$ denotes the functor corepresented by $\vec{D} \in \prod_{t\in T} \mathcal{D}_t$. Since the last of these functors commutes with small colimits and finite limits, we can replace G by the composition $\Sigma^\infty \circ e_{\vec{D}}$ and $\mathcal{E}$ by the ∞-category of spectra. Using Example 7.3.5.8, we are reduced to proving that the canonical map

$$\psi : d(e_{\vec{D}} \circ \mathrm{tfib}\{F^E\}_{E\in\mathrm{Equiv}_p(S)}) \to \mathrm{tfib}\{d(e_{\vec{D}} \circ F^E)\}_{E\in\mathrm{Equiv}_p(S)}.$$

For each $t \in T$, let $F'_t : \prod_{p(s)=t} \mathcal{C}_s \to \mathcal{S}_*$ denote the composite functor

$$\prod_{p(s)=t} \mathcal{C}_S \xrightarrow{F_t} \mathcal{D}_t \xrightarrow{\mathrm{Map}_{\mathcal{D}}(D_t,\bullet)} \mathcal{S}_* \, .$$

Since corepresentable functors preserve limits, we can identify ψ with the canonical map

$$D(\mathrm{tfib}\{F'^E\}_{E\in\mathrm{Equiv}_p(S)}) \to \mathrm{tfib}\{d(F'^E)\}_{E\in\mathrm{Equiv}_p(S)}.$$

The desired result now follows from Lemma 7.3.5.9. $\qquad\square$

7.3.6 The Dual Chain Rule

Recall that Theorem 7.3.2.3 asserts that if $q : \mathcal{C}^{\otimes} \to \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a thin Δ^2-family of ∞-operads satisfying some mild hypotheses, then the stabilization of q is also a thin Δ^2-family of ∞-operads. We will devote the entirety of this section to the proof of Theorem 7.3.2.3. We begin with a simple combinatorial lemma.

Lemma 7.3.6.1. *Let K be a finite product of simplices $\prod_{1\le i\le n} \Delta^{a_i}$, and let v denote the final vertex of K. Let P denote the partially ordered set of nondegenerate simplices of K (ordered by inclusion) and let $P^0 = P - \{v\}$. If at least one of the integers a_i is positive, then $\mathrm{N}(P^0)$ is weakly contractible.*

Proof. By omitting those factors Δ^{a_i} where $a_i = 0$, we may assume that each a_i is positive. For $1 \le i \le n$, let $P_i \subseteq P$ denote the subset consisting of those simplices $\sigma \subseteq K$ such that the projection map $K \to \Delta^{a_i}$ does not carry σ to the final vertex of Δ^{a_i}. For $I \subseteq \{1, \ldots, n\}$, we let $P_I = \bigcap_{i\in I} P_i$. We will identify each $\mathrm{N}(P_I)$ with a simplicial subset of $\mathrm{N}(P)$. Note that $\mathrm{N}(P_I) = \bigcap_{i\in I} \mathrm{N}(P_i)$ and that $\mathrm{N}(P^0) = \bigcup_{1\le i\le n} \mathrm{N}(P_i)$. It follows that $\mathrm{N}(P^0)$ can be identified with the homotopy colimit of the diagram $I \mapsto \mathrm{N}(P_I)$, where I ranges over the collection of nonempty subsets of $\{1, \ldots, n\}$. To complete the proof, it will suffice to show that each $\mathrm{N}(P_I)$ is weakly contractible.

For every nonempty subset $I \subseteq \{1, \ldots, n\}$, let K_I denote the product

$$\prod_{i \in I} \Delta^{a_i - 1} \times \prod_{i \notin I} \Delta^{a_i} \subseteq K.$$

Let Q_I denote the partially ordered collection of nondegenerate simplices of K_I. Then $\mathrm{N}(Q_I)$ is the barycentric subdivision of K_I, and therefore weakly contractible (since K_I is a product of simplices). The inclusion $\mathrm{N}(Q_I) \to \mathrm{N}(P_I)$ admits a right adjoint, which carries a simplex σ to its intersection with the simplicial subset $K_I \subseteq K$. It follows that $\mathrm{N}(P_I)$ is also weakly contractible, as desired. $\qquad\square$

Lemma 7.3.6.2. *Let $S \to T$ be a surjective map of nonempty finite sets, let $\{\mathcal{C}_s\}_{s \in S}$ be a collection of differentiable pointed ∞-categories, $\{\mathcal{D}_t\}$ a collection of differentiable stable ∞-categories, and $\mathcal{E}$ a differentiable stable ∞-category. Suppose that we are given a collection of functors*

$$\{F_t : \prod_{s \in S_t} \mathcal{C}_s \to \mathcal{D}_t\}_{t \in T} \qquad G : \prod_{t \in T} \mathcal{D}_t \to \mathcal{E}.$$

Assume that each F_t commutes with sequential colimits, and that G commutes with sequential colimits and is exact in each variable. For each $t \in T$, let $\alpha_t : F_t \to D(F_t)$ be a map which exhibits $D(F_t)$ as a differential of F_t. Then the induced map

$$\alpha : G \circ \prod_{t \in T} F_t \to G \circ \prod_{t \in T} D(F_t)$$

exhibits $G \circ \prod_{t \in T} D(F_t)$ as a differential of $G \circ \prod_{t \in T} F_t$.

Proof. Since the functor G is reduced and right exact in each variable, we the canonical map $G \circ \prod_{t \in T} F_t \to G \circ \prod_{t \in T} \mathrm{cored}(F_t)$ exhibits $G \circ \prod_{t \in T} \mathrm{cored}(F_t)$ as a coreduction of $G \circ \prod_{t \in T} F_t$. We may therefore replace each F_t by $\mathrm{cored}(F_t)$ and thereby reduce to the case the functors F_t are reduced in each variable. Using Corollary 7.2.3.22, we can write each of the functors $D(F_t)$ as a composition

$$\prod_{s \in S_t} \mathcal{C}_s \xrightarrow{\Sigma^\infty} \prod_{s \in S_t} \mathrm{Sp}(\mathcal{C}_s) \xrightarrow{\vec{\partial}(F_t)} \mathrm{Sp}(\mathcal{D}_t) \simeq \mathcal{D}_t$$

where $\vec{\partial}(F_t)$ is a multilinear functor, and the map α_t exhibits $\vec{\partial}(F_t)$ as a derivative of F_t. Let $\vec{\partial}G : \prod_{t \in T} \mathrm{Sp}(\mathcal{D}_t) \to \mathrm{Sp}(\mathcal{E})$ denote a derivative of G. Since the ∞-categories $\mathcal{D}_t$ and $\mathcal{E}$ are stable and the functor G is exact in each variable, the diagram

$$
\begin{array}{ccc}
\prod_{t \in T} \mathrm{Sp}(\mathcal{D}_t) & \xrightarrow{\vec{\partial}G} & \mathrm{Sp}(\mathcal{E}) \\
\downarrow{\scriptstyle \Omega^\infty} & & \downarrow{\scriptstyle \Omega^\infty} \\
\prod_{t \in T} \mathcal{D}_t & \longrightarrow & \mathcal{E}
\end{array}
$$

commutes (up to canonical homotopy). Using Theorem 7.2.1.22, we conclude that α exhibits the composite map

$$\prod_{s \in S} \mathrm{Sp}(\mathcal{C}_s) \xrightarrow{\prod \vec{\partial}(F_t)} \prod_{t \in T} \mathrm{Sp}(\mathcal{D}_t) \xrightarrow{\vec{\partial}G} \mathrm{Sp}(\mathcal{E})$$

as a derivative of $G \circ \prod_{t \in T} F_t$. It follows from Corollary 7.2.3.24 that α also exhibits $G \circ \prod_{t \in T} D(F_t)$ as a differential of $G \circ \prod_{t \in T} F_t$. $\qquad\square$

Proof of Theorem 7.3.2.3. Let $q : \mathcal{C}^\otimes \to \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a thin Δ^2-family of ∞-operads satisfying the following conditions:

(1) Each of the ∞-categories $\mathcal{C}_i$ is pointed and compactly generated.

(2) For every $i \leq j$, each of the induced functors $\mathcal{C}_i^n \to \mathcal{C}_j$ preserves final objects and filtered colimits.

(3) Let $i \in \Delta^2$ be a vertex and $\alpha : \langle m \rangle \to \langle n \rangle$ be an injective map of pointed finite sets. If $\overline{\alpha}$ is a locally q-coCartesian morphism in $\mathcal{C}^{\otimes}$ lifting the map $(i, \langle m \rangle) \to (i, \langle n \rangle)$ in $\Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, then $\overline{\alpha}$ is q-coCartesian.

Let $U : \mathrm{Sp}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$ be a map which exhibits $\mathrm{Sp}(\mathcal{C})^{\otimes}$ as a stabilization of $\mathcal{C}^{\otimes}$. We wish to show that $\mathrm{Sp}(\mathcal{C})^{\otimes}$ is also a thin Δ^2-family of ∞-operads.

Let us regard the canonical map $U^{op} : (\mathrm{Sp}(\mathcal{C})^{\otimes})^{op} \to (\mathcal{C}^{\otimes})^{op}$ as giving a diagram $\upsilon : [1] \to \mathcal{S}\mathrm{et}_{\Delta}$, and let $\mathcal{M}$ denote the relative nerve $\mathrm{N}_{\upsilon}(\Delta^1)^{op}$ (see Definition T.3.2.5.2). We have a canonical map $u_0 : \mathcal{M} \to \Delta^1$, whose fibers are given by $\mathcal{M}_0 = \mathcal{C}^{\otimes}$, $\mathcal{M}_1 = \mathrm{Sp}(\mathcal{C})^{\otimes}$. Moreover, the map Let $\upsilon_0 : [1] \to \mathcal{S}\mathrm{et}_{\Delta}$ be the constant functor taking the value $\Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Since the maps

$$\mathcal{C}^{\otimes} \to \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \qquad \mathrm{Sp}(\mathcal{C})^{\otimes} \to \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$$

are categorical fibrations, the map u_0 lifts to a categorical fibration

$$u : \mathcal{M} = \mathrm{N}_{\upsilon}(\Delta^1)^{op} \to \mathrm{N}_{\upsilon_0}(\Delta^1)^{op} \simeq \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^1$$

(see Lemma T.3.2.5.11).

For every object $A = (j, \langle n \rangle) \in \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, the forgetful functor $\mathrm{Sp}(\mathcal{C})_A^{\otimes} \to \mathcal{C}_A^{\otimes}$ can be identified with a product of n copies of the forgetful functor $\Omega_{\mathcal{C}_j}^{\infty} : \mathrm{Sp}(\mathcal{C}_j) \to \mathcal{C}_j$, and therefore admits a left adjoint $\Sigma_A^{\infty} : \mathcal{C}_A^{\otimes} \to \mathrm{Sp}(\mathcal{C})_A^{\otimes}$. It follows that u restricts to a coCartesian fibration $u_A : \mathcal{M}_A \to \{A\} \times \Delta^1$. In particular, for every object $X \in \mathcal{C}_A^{\otimes}$, we can choose a locally u-coCartesian morphism $\alpha : X \to \Sigma_A^{\infty}(X)$ in $\mathcal{M}$ which covers the canonical map $(A, 0) \to (A, 1)$ in $\Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^1$.

We first prove:

(i) Suppose we are given a locally q-coCartesian morphism $\beta : X' \to X$ in $\mathcal{C}^{\otimes}$, where $X \in \mathcal{C}_A^{\otimes}$ for some $A \in \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Let $\alpha_X : X \to \Sigma_A^{\infty}(X)$ be defined as above. Then $\alpha_X \circ \beta$ is a locally u-coCartesian morphism in $\mathcal{M}$.

To prove (i), let $\beta_0 : A' \to A$ denote the image of β in $\Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Fix an object $Y \in \mathrm{Sp}(\mathcal{C})_A^{\otimes}$. Let $\mathrm{Map}_{\mathcal{M}}^{\alpha_X \circ \beta}(X', Y)$ denote the inverse image of $u(\alpha_X \circ \beta)$ in the mapping space $\mathrm{Map}_{\mathcal{M}}(X', Y)$. We wish to show that composition with $\alpha_X \circ \beta$ induces a homotopy equivalence

$$\xi : \mathrm{Map}_{\mathrm{Sp}(\mathcal{C})_A^{\otimes}}(\Sigma_A^{\infty}(X), Y) \to \mathrm{Map}_{\mathcal{M}}^{\alpha_X \circ \beta}(X', Y).$$

Using the definition of Σ_A^{∞} and of the ∞-category $\mathcal{M}$, we can identify ξ with the map

$$\mathrm{Map}_{\mathcal{C}_A^{\otimes}}(X, U(Y)) \to \mathrm{Map}_{\mathcal{C}^{\otimes}}^{\beta}(X', U(Y))$$

given by composition with β. The desired result now follows from the fact that β is locally q-coCartesian.

Using (i), we next prove the following:

(ii) The map $u : \mathcal{M} \to \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^1$ is a locally coCartesian fibration.

To prove this, suppose we are given a morphism $\gamma_0 : (j', \langle n' \rangle, i') \to (j, \langle n \rangle, i)$ in $\Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^1$ and an object $X' \in \mathcal{M}$ lying over $(j', \langle n' \rangle, i')$. We wish to show that γ can be lifted to a locally u-coCartesian morphism $\gamma : X' \to Y$ in $\mathcal{M}$. If $i = i'$, then the desired result follows from the fact that $q : \mathcal{C}^{\otimes} \to \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and $(q \circ U) : \mathrm{Sp}(\mathcal{C})^{\otimes} \to \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ are locally coCartesian fibrations. Let us therefore assume that $i' = 0$ and $i = 1$. In this case, γ_0 factors as a composition

$$(j', \langle n' \rangle, 0) \xrightarrow{\beta_0} (j, \langle n \rangle, 0) \xrightarrow{\alpha_0} (j, \langle n \rangle, 1).$$

Since q is a locally coCartesian fibration, we can choose a locally q-coCartesian morphism $\beta : X' \to X$ lifting β_0. Let $\alpha_X : X \to \Sigma^\infty_{(j,\langle n \rangle)} X$ be as above. It follows from (i) that the composition $\alpha_X \circ \beta$ is a locally u-coCartesian morphism lifting γ.

Fix an integer $n \geq 1$ and let $S = \{1, \ldots, n\}$. Let $T = \Delta^2 \, \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Then T contains objects $t = (0, \langle n \rangle)$ and $t' = (2, \langle 1 \rangle)$. Let $\chi : \mathrm{Part}(S) \to \boldsymbol{\Delta}_T^{t,t'}$ be as in Notation 7.3.2.11, and let $\theta : \mathrm{N}(\boldsymbol{\Delta}_T^{t,t'}) \to \mathrm{Fun}(\mathrm{Sp}(\mathcal{C})^\otimes_{(0,\langle n \rangle)}, \mathrm{Sp}(\mathcal{C})_2)$ be the spray associated to the locally coCartesian fibration $(q \circ U) : \mathrm{Sp}(\mathcal{C})^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^2$. To prove that $\mathrm{Sp}(\mathcal{C})^\otimes$ is thin, we must show that the composite map

$$\phi : \mathrm{N}(\mathrm{Part}^0(S))^\triangleleft \simeq \mathrm{N}(\mathrm{Part}(S)) \overset{\chi}{\to} \mathrm{N}(\boldsymbol{\Delta}_T^{t,t'}) \overset{\theta}{\to} \mathrm{Fun}(\mathrm{Sp}(\mathcal{C})^\otimes_{(0,\langle n \rangle)}, \mathrm{Sp}(\mathcal{C})_2)$$

is a limit diagram in the ∞-category $\mathrm{Fun}(\mathrm{Sp}(\mathcal{C})^\otimes_{(0,\langle n \rangle)}, \mathrm{Sp}(\mathcal{C})_2)$. Note that ϕ takes values in the full subcategory of $\mathrm{Fun}(\mathrm{Sp}(\mathcal{C})^\otimes_{(0,\langle n \rangle)}, \mathrm{Sp}(\mathcal{C})_2)$ spanned by those maps which correspond to functors $\mathrm{Sp}(\mathcal{C}_0)^n \to \mathrm{Sp}(\mathcal{C}_2)$ which are excisive in each variable and commute with filtered colimits. Using Theorem 7.2.3.21, we are reduced to showing that the composite map

$$\phi' : \mathrm{N}(\mathrm{Part}(S)) \overset{\phi}{\to} \mathrm{Fun}(\mathrm{Sp}(\mathcal{C})^\otimes_{(0,\langle n \rangle)}, \mathrm{Sp}(\mathcal{C})_2) \to \mathrm{Fun}(\mathcal{C}_0^n, \mathrm{Sp}(\mathcal{C})_2)$$

is a limit diagram, where the second map is given by composition with the functor

$$\mathcal{C}_0^n \overset{\Sigma^\infty_{\mathbf{e}_0}}{\to} \mathrm{Sp}(\mathcal{C}_0)^n \simeq \mathrm{Sp}(\mathcal{C})^\otimes_{(0,\langle n \rangle)}.$$

Set $T^+ = \Delta^2 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^1$, so that T^+ contains objects $t^+ = (0, \langle n \rangle, 0)$ and $t'^+ = (2, \langle 1 \rangle, 1)$. Let $\mathcal{U} \subseteq \mathrm{Equiv}(S)$ be a downward-closed subset (that is, if an equivalence relation E on S belongs to $\mathcal{U}$, then any finer equivalent relation also belongs to $\mathcal{U}$). We define a functor $\chi_\mathcal{U} : \mathrm{Part}(S) \to \boldsymbol{\Delta}_{T^+}^{t^+, t'^+}$ as follows. To every chain of equivalence relations $(E_1 \subseteq \cdots \subseteq E_k)$ on S, $\chi_\mathcal{U}$ assigns the $(k+1)$-simplex of T^+ given by the chain of morphisms

$$(0, \langle n \rangle, 0) \to (1, (S/E_1)_*, i_1) \to (1, (S/E_2)_*, i_2) \to \cdots \to (1, (S/E_k)_*, i_k) \to (2, \langle 1 \rangle, 1),$$

where

$$i_j = \begin{cases} 0 & \text{if } E_j \in \mathcal{U} \\ 1 & \text{if } E_j \notin \mathcal{U}. \end{cases}$$

Let $\chi' : \mathrm{Part}(S) \to \boldsymbol{\Delta}_{T^+}^{t^+, t'^+}$ be the functor which carries a chain of equivalence relations $(E_1 \subseteq \cdots \subseteq E_k)$ to the $(k+2)$-simplex of T^+ given by the chain of morphisms

$$(0, \langle n \rangle, 0) \to (0, \langle n \rangle, 1) \to (1, (S/E_1)_*, 1) \to (1, (S/E_2)_*, 1) \to \cdots \to (1, (S/E_k)_*, 1) \to (2, \langle 1 \rangle, 1),$$

Let $\theta^+ : \mathrm{N}(\boldsymbol{\Delta}_{T^+}^{t^+, t'^+}) \to \mathrm{Fun}(\mathcal{C}_0^n, \mathrm{Sp}(\mathcal{C})_2)$ be the functor obtained by combining the equivalence $\mathcal{C}_0^n \simeq \mathcal{C}^\otimes_{(0,\langle n \rangle)}$ with the spray associated to the locally coCartesian fibration u. Unwinding the definition, we can identify ϕ' with the composition $\theta^+ \circ \chi'$. For every downward closed subset $\mathcal{U} \subseteq \mathrm{Equiv}(S)$, we let $\phi_\mathcal{U}$ denote the composite map

$$\mathrm{N}(\mathrm{Part}(S)) \overset{\chi_\mathcal{U}}{\to} \mathrm{N}(\boldsymbol{\Delta}_{T^+}^{t^+, t'^+}) \overset{\theta^+}{\to} \mathrm{Fun}(\mathcal{C}_0^n, \mathrm{Sp}(\mathcal{C})_2).$$

Let $D : \mathrm{Fun}(\mathcal{C}_0^n, \mathrm{Sp}(\mathcal{C})_2) \to \mathrm{Exc}_*(\mathcal{C}_0^n, \mathrm{Sp}(\mathcal{C})_2)$ be a left adjoint to the inclusion, given by the differentiation construction described in §7.2.3. We have an evident natural transformation $\chi_\emptyset \to \chi'$ of functors from $\mathrm{Part}(S)$ into $\boldsymbol{\Delta}_{T^+}^{t^+, t'^+}$, which induces a natural transformation $\iota : \phi_\emptyset \to \phi'$. Since ϕ' takes values in the full subcategory $\mathrm{Exc}_*(\mathcal{C}_0^n, \mathrm{Sp}(\mathcal{C})_2)$, ι induces a map $D \circ \phi_\emptyset \to \phi'$. Using Lemma 7.3.6.2, we deduce that this map is an equivalence: that is, for every chain of equivalence relations $\vec{E} = (E_1 \subseteq \cdots \subseteq E_k) \in \mathrm{Part}(S)$, the map $\iota_{\vec{E}} : \phi_\emptyset(\vec{E}) \to \phi'(\vec{E})$ exhibits $\phi'(\vec{E})$ as a differential of $\phi_\emptyset(\vec{E})$. We will deduce that ϕ' is a limit diagram from the following more general assertion:

(iii) For every downward-closed $\mathcal{U} \subseteq \mathrm{Equiv}(S)$, the composition $D \circ \phi_{\mathcal{U}}$ determines a limit diagram

$$\mathrm{N}(\mathrm{Part}(S)) \to \mathrm{Exc}_*(\mathcal{C}_0^n, \mathrm{Sp}(\mathcal{C})_2).$$

The proof of (iii) proceeds by descending induction on the cardinality of $\mathcal{U}$. Our base case is the following:

(iv) If $\mathcal{U} = \mathrm{Equiv}(S)$, then the functor $D \circ \phi_{\mathcal{U}}$ determines a limit diagram $\mathrm{N}(\mathrm{Part}(S)) \to \mathrm{Exc}_*(\mathcal{C}_0^n, \mathrm{Sp}(\mathcal{C})_2)$.

To carry out the inductive step, we will prove the following:

(v) Let $\mathcal{U} \subsetneq \mathrm{Equiv}(S)$ be a proper downward closed subset. Let E be an equivalence relation on S which is minimal among those elements which does not belong to $\mathcal{U}$, let $\mathcal{U}' = \mathcal{U} \cup \{E\}$, and assume that $D \circ \phi_{\mathcal{U}'}$ is a limit diagram in $\mathrm{Exc}_*(\mathcal{C}_0^n, \mathrm{Sp}(\mathcal{C})_2)$. Then $D \circ \phi_{\mathcal{U}}$ is a limit diagram in $\mathrm{Exc}_*(\mathcal{C}_0^n, \mathrm{Sp}(\mathcal{C})_2)$.

We now prove (iv). Let $\theta^- : \mathrm{N}(\boldsymbol{\Delta}_T^{t,t'}) \to \mathrm{Fun}(\mathcal{C}_{\langle n \rangle, 0}^{\otimes}, \mathcal{C}_2)$ be the spray associated to the locally coCartesian fibration $q : \mathcal{O}^{\otimes} \to T$, and let $\Sigma_{\mathcal{C}_2}^{\infty} : \mathcal{C}_2 \to \mathrm{Sp}(\mathcal{C})_2$ be a left adjoint to the functor $\mathrm{Sp}(\mathcal{C})_2 \to \mathcal{C}_2$ determined by p. Using (i), we deduce that $\phi_{\mathrm{Equiv}(S)}$ is equivalent to the composition

$$\mathrm{N}(\mathrm{Part}(S)) \xrightarrow{\psi} \mathrm{Fun}(\mathcal{C}_0^n, \mathcal{C}_2) \xrightarrow{\Sigma_{\mathcal{C}_2}^{\infty} \circ} \mathrm{Fun}(\mathcal{C}_0^n, \mathrm{Sp}(\mathcal{C})_2),$$

where ψ denotes the composition

$$\mathrm{N}(\mathrm{Part}(S)) \xrightarrow{\chi} \mathrm{N}(\boldsymbol{\Delta}_T^{t,t'}) \xrightarrow{\theta^-} \mathrm{Fun}(\mathcal{C}_{\langle n \rangle, 0}^{\otimes}, \mathcal{C}_2) \simeq \mathrm{Fun}(\mathcal{C}_0^n, \mathcal{C}_2).$$

Let $D' : \mathrm{Fun}(\mathcal{C}_0^n, \mathcal{C}_2) \to \mathrm{Exc}_*(\mathcal{C}_0^n, \mathcal{C}_2)$ be a left adjoint to the inclusion. The commutative diagram of ∞-categories

$$
\begin{array}{ccc}
\mathrm{Exc}_*(\mathcal{C}_0^n, \mathrm{Sp}(\mathcal{C})_2) & \xrightarrow{\Omega_{\mathcal{C}_2}^{\infty} \circ} & \mathrm{Exc}_*(\mathcal{C}_0^n, \mathcal{C}_2) \\
\downarrow & & \downarrow \\
\mathrm{Fun}(\mathcal{C}_0^n, \mathrm{Sp}(\mathcal{C})_2) & \xrightarrow{\Omega_{\mathcal{C}_2}^{\infty} \circ} & \mathrm{Fun}(\mathcal{C}_0^n, \mathcal{C}_2)
\end{array}
$$

determines a commutative diagram of left adjoint functors

$$
\begin{array}{ccc}
\mathrm{Exc}_*(\mathcal{C}_0^n, \mathrm{Sp}(\mathcal{C})_2) & \longleftarrow & \mathrm{Exc}_*(\mathcal{C}_0^n, \mathcal{C}_2) \\
{\scriptstyle D} \uparrow & & \uparrow {\scriptstyle D'} \\
\mathrm{Fun}(\mathcal{C}_0^n, \mathrm{Sp}(\mathcal{C})_2) & \xleftarrow[\Sigma_{\mathcal{C}_2}^{\infty} \circ] & \mathrm{Fun}(\mathcal{C}_0^n, \mathcal{C}_2),
\end{array}
$$

where the upper horizontal map is an equivalence by virtue of Proposition 1.4.2.22. Consequently, to show that $D \circ \phi_{\mathrm{Equiv}(S)} \simeq D \circ \Sigma_{\mathcal{C}_2}^{\infty}(\psi)$ is a limit diagram in $\mathrm{Exc}_*(\mathcal{C}_0^n, \mathrm{Sp}(\mathcal{C})_2)$, it will suffice to show that $D' \circ \psi$ is limit diagram in $\mathrm{Exc}_*(\mathcal{C}_0^n, \mathcal{C}_2)$. Using Corollary 7.2.3.24, we are reduced to proving that the diagram $\vec{\partial} \circ \psi$ is a limit diagram in $\mathrm{Exc}_*(\mathrm{Sp}(\mathcal{C}_0)^n, \mathrm{Sp}(\mathcal{C})_2)$. We will prove this by verifying the hypotheses of Theorem 7.3.5.5. Fix an equivalence relation E on S. We let $\mathrm{Equiv}_E(S)$ denote the subset of $\mathrm{Equiv}(S)$ spanned by those equivalence relations E' on S which refine E. We have a canonical isomorphism of partially ordered sets

$$\mathrm{Equiv}_E(S) = \prod_{S' \in S/E} \mathrm{Equiv}(S'),$$

where the product is taken over all equivalence classes in S (which we regard as subsets of S). The inclusion $\mathrm{Equiv}_E(S) \hookrightarrow \mathrm{Equiv}(S)$ admits a right adjoint, which we can identify with the map

$$\eta : \mathrm{Equiv}(S) \to \prod_{S' \in S/E} \mathrm{Equiv}(S')$$

given by restricting an equivalence relation on S to each of the subsets $S' \in S/E$. Composing with η gives a map $\gamma : \mathrm{Part}(S) \to \prod_{S' \in S/E} \mathrm{Part}(S')$. We first verify:

(a) The map u restricts to a right cofinal map $\mathrm{Part}^0(S) \to \prod_{S' \in S/E} \mathrm{Part}^0(S')$. According to Theorem T.4.1.3.1, this is equivalent to the assertion that for every element $\{X_{S'}\}_{S' \in S/E} \in \prod_{S' \in S/E} \mathrm{Part}^0(S')$, the partially ordered set $P = \{X \in \mathrm{Part}^0(S) : \gamma(X) \leq \{X_{S'}\}\}$ has weakly contractible nerve.

We can identify each $X_{S'}$ with a nondegenerate simplex in $\mathrm{N}(\mathrm{Equiv}(S'))$. Unwinding the definitions, we see that the simplicial set $\mathrm{N}(P)$ is isomorphic to the barycentric subdivision of the simplicial set $Q = \mathrm{N}(\mathrm{Equiv}(S)) \times_{\prod_{S' \in S/E} \mathrm{N}(\mathrm{Equiv}(S'))} \times \prod_{S' \in S/E} X_{S'}$, which is a full subcategory of $\mathrm{N}(\mathrm{Equiv}(S))$. Let Q_0 denote the full subcategory of Q spanned by those equivalence relations on S which refine E. The inclusion $Q_0 \hookrightarrow Q$ admits a right adjoint and is therefore a weak homotopy equivalence. We are therefore reduced to proving that Q_0 is weakly contractible. It now suffices to observe that the projection map $Q_0 \to \prod_{S' \in S/E} X_{S'}$ is an isomorphism, so that Q_0 is a product of simplices.

For each $S' \in S/E$, let $t_{S'} = (0, S'_*) \in T$, let $\theta_{S'} : \mathrm{N}(\boldsymbol{\Delta}_T^{t_{S'},t'}) \to \mathrm{Fun}(\mathcal{C}_{(0,S'_*)}^\otimes, \mathcal{C}_2)$ be the spray associated to the locally coCartesian fibration q. Composing with the functor $\mathrm{Part}(S') \to \mathrm{N}(\boldsymbol{\Delta}_T^{t_{S'},t'})$ and using the identification $\mathcal{C}_{(0,S'_*)}^\otimes \simeq \mathcal{C}_0^{S'}$, we obtain a map $\psi^{S'} : \mathrm{N}(\mathrm{Part}(S')) \to \mathrm{Fun}(\mathcal{C}_0^{S'}, \mathcal{C}_2)$. Using (3), we see that the composite map

$$\mathrm{N}(\mathrm{Part}(S)) \to \mathrm{N}(\mathrm{Part}(S')) \xrightarrow{\psi^{S'}} \mathrm{Fun}(\mathcal{C}_0^{S'}, \mathcal{C}_2)$$

is given by composing the functor $\psi : \mathrm{N}(\mathrm{Part}(S)) \to \mathrm{Fun}(\mathcal{C}_0^S, \mathcal{C}_2)$ with the map $\mathrm{Fun}(\mathcal{C}_0^S, \mathcal{C}_2) \to \mathrm{Fun}(\mathcal{C}_0^E, \mathcal{C}_2)$ determined by the inclusion

$$\mathcal{C}_0^{S'} \simeq \prod_{s \in S'} \mathcal{C}_0 \times \prod_{s \in S - S'} \{*\} \subseteq \mathcal{C}_0^n,$$

where $*$ denotes a zero object of $\mathcal{C}_0$ (see Remark 7.3.2.4).

To complete the verification of the hypotheses of Theorem 7.3.5.5, it suffices to observe the following:

(b) Each of the functors $\psi^{S'} \to \mathrm{N}(\mathrm{Part}(S')) \to \mathrm{Fun}(\mathcal{C}_0^{S'}, \mathcal{C}_2)$ is a limit diagram.

This follows immediately from our assumption that the corepresentable Δ^2-family of ∞-operads $\mathcal{O}^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^2$ is thin.

We now prove (v). Let $\mathcal{U} \subsetneq \mathrm{Equiv}(S)$ be a downward closed subset, let E be a minimal element of $\mathrm{Equiv}(S) - \mathcal{U}$, let $\mathcal{U}' = \mathcal{U} \cup \{E\}$, and assume that $D \circ \phi_{\mathcal{U}'}$ is a limit diagram. We wish to prove that $D \circ \phi_{\mathcal{U}}$ is a limit diagram. We define a functor

$$\chi_{\mathcal{U},\mathcal{U}'} : \mathrm{Part}(S) \to \boldsymbol{\Delta}_{T^+}^{t^+,t'^+}$$

as follows. If $\vec{E} = (E_1 \subseteq \cdots \subseteq E_k)$ is a chain of equivalence relations on S which does not contain the equivalence relation E, then we set $\chi_{\mathcal{U},\mathcal{U}'}(\vec{E}) = \chi_{\mathcal{U}}(\vec{E}) = \chi_{\mathcal{U}'}(\vec{E})$. Otherwise, there is a unique integer $p \leq k$ such that $E_p = E$. We define $\chi_{\mathcal{U},\mathcal{U}'}(J)$ to be the simplex of T^+ given by the chain of morphisms

$$
\begin{aligned}
(0, \langle n \rangle, 0) &\to (1, (S/E_1)_*, 0) \\
&\to \cdots \\
&\to (1, (S/E_p)_*, 0) \\
&\to (1, (S/E_p)_*, 1) \\
&\to \cdots \\
&\to (1, (S/E_k)_*, 1) \\
&\to (2, \langle 1 \rangle, 1).
\end{aligned}
$$

Composing with θ^+, we obtain a functor $\phi_{\mathcal{U},\mathcal{U}'} : \mathrm{N}(\mathrm{Part}(S)) \to \mathrm{Fun}(\mathcal{C}_0^n, \mathrm{Sp}(\mathcal{C})_2)$. Moreover, we have evident natural transformations $\chi_{\mathcal{U}} \to \chi_{\mathcal{U},\mathcal{U}'} \leftarrow \chi_{\mathcal{U}'}$, which induce natural transformations

$$\phi_{\mathcal{U}} \xrightarrow{\epsilon} \phi_{\mathcal{U},\mathcal{U}'} \xleftarrow{\epsilon'} \phi_{\mathcal{U}'}.$$

Using (i), we deduce that the map ϵ is an equivalence. Consequently, to complete the proof of (v), it will suffice to show that $D \circ \phi_{\mathcal{U},\mathcal{U}'}$ is a limit diagram.

Let $P \subseteq [1] \times \mathrm{Part}(S)$ be the subset consisting of those pairs $(i, \vec{E})$ where either $i = 0$ or $\vec{E}$ contains E. We define a map $\chi_P : P \to \mathbf{\Delta}_{T^+}^{t^+, t'^+}$ by the formula

$$\chi_P(i, \vec{E}) = \begin{cases} \chi_{\mathcal{U},\mathcal{U}'}(\vec{E}) & \text{if } i = 1 \\ \chi_{\mathcal{U}'}(\vec{E}) & \text{otherwise.} \end{cases}$$

Note that composition with θ^+ determines a map $\phi_P : \mathrm{N}(P) \to \mathrm{Fun}(\mathcal{C}_0^n, \mathrm{Sp}(\mathcal{C})_2)$. Let P^0 denote the partially ordered subset of P obtained by removing the least element $(0, \emptyset)$. By assumption, the restriction of $D \circ \phi_P$ to $\mathrm{N}(\{0\} \times \mathrm{Part}(S))$ is a limit diagram. The inclusion $\mathrm{N}(\{0\} \times \mathrm{Part}^0(S)) \hookrightarrow \mathrm{N}(P^0)$ admits a right adjoint (given by the projection $P^0 \to \mathrm{Part}^0(S)$) and is therefore right cofinal. It follows that $D \circ \phi_P$ is a limit diagram.

Define partially ordered subsets $P'' \subseteq P' \subseteq P$ as follows:

- A pair $(i, \vec{E}) \in P$ belongs to P' if either $i = 1$ or $\vec{E}$ does not contain E as a least element.

- A pair $(i, \vec{E}) \in P$ belongs to P'' if either $i = 1$ or $\vec{E}$ does not contain E.

Set $P'^0 = P' \cap P^0$ and $P''^0 = P'' \cap P^0$. Note that the projection map $P'' \to \mathrm{Part}(S)$ is an isomorphism of partially ordered sets, and that the restriction of ϕ_P to $\mathrm{N}(P'')$ coincides with $\phi_{\mathcal{U},\mathcal{U}'}$. To complete the proof, it will suffice to show that the restriction of $D \circ \phi_P$ to $\mathrm{N}(P'')$ is a limit diagram. The inclusion $\mathrm{N}(P''^0) \to \mathrm{N}(P'^0)$ admits a right adjoint (given by $(i, \vec{E}) \mapsto \begin{cases} (0, \vec{E} - \{E\}) & \text{if } i = 0 \\ (1, \vec{E}) & \text{if } i = 1. \end{cases}$), and is therefore right cofinal. We are therefore reduced to proving that the restriction of $D \circ \phi_P$ to $\mathrm{N}(P')$ is a limit diagram. Since $D \circ \phi_P$ is a limit diagram, this is a consequence of the following assertion:

(vi) The diagram $D \circ \phi_P$ is a right Kan extension of its restriction to $\mathrm{N}(P')$.

To prove (vi), fix an object $(0, \vec{E}) \in P - P'$; we will show that $D \circ \phi_P$ is a right Kan extension of $(D \circ \phi_P)| \mathrm{N}(P')$ at $(0, \vec{E})$. We can identify $\vec{E}$ with a chain of equivalence relations

$$E = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_k$$

on S. Let $P'_{\geq \vec{E}} = \{(i, \vec{E}') \in P' : \vec{E} \subseteq \vec{E}'\}$. We wish to prove that the canonical map

$$D(\phi_P(0, \vec{E})) \to \varprojlim_{(i, \vec{E}') \in P'_{\geq \vec{E}'}} D(\phi_P(i, \vec{E}'))$$

is an equivalence. Let $Q \subseteq P'_{\geq \vec{E}}$ be the partially ordered subset spanned by those pairs $(i, \vec{E}')$, where $\vec{E}' \cap \{E' \in \mathrm{Equiv}(S) : E \subseteq E'\} = \vec{E}$. The construction $(i, \vec{E}') \mapsto (i, \vec{E}' - \{E' \in \mathrm{Equiv}(S) : E \subseteq E' \notin \vec{E}\})$ determines a right adjoint to the inclusion $Q \hookrightarrow P'_{\geq \vec{E}}$, so that the inclusion $\mathrm{N}(Q) \to \mathrm{N}(P'_{\geq \vec{E}})$ is right cofinal. It will therefore suffice to prove that the canonical map

$$D(\phi_P(0, \vec{E})) \to \varprojlim_{(i, \vec{E}') \in Q} D(\phi_P(i, \vec{E}'))$$

is an equivalence.

Let $\mathrm{Part}'(S)$ denote the subset of $\mathrm{Part}(S)$ consisting of those linearly ordered subsets of $\mathrm{Equiv}(S)$ which contain E as a largest element. The construction $\vec{E}' \mapsto \vec{E}' \cup \vec{E}$ induces a bijection from $\mathrm{Part}'(S)$ to the subset of $\mathrm{Part}(S)$ consisting of those chains which contain $\vec{E}$ as a final segment. Let $\mathrm{Part}'^0(S)$ denote the

subset of $\mathrm{Part}'(S)$ consisting of linearly ordered subsets which have at least two elements. Unwinding the definitions, we are reduced to proving that the diagram σ :

$$
\begin{array}{ccc}
D\phi_P(0,\vec{E}) & \longrightarrow & \varprojlim_{\vec{E}'\in\mathrm{Part}'^0(S)} D\phi_P(0,\vec{E}'\cup\vec{E}) \\
\downarrow & & \downarrow \\
D\phi_P(1,\vec{E}) & \longrightarrow & \varprojlim_{\vec{E}'\in\mathrm{Part}'^0(S)} D\phi_P(1,\vec{E}'\cup\vec{E})
\end{array}
$$

is a pullback square in the ∞-category $\mathrm{Fun}(\mathcal{C}_0^n, \mathrm{Sp}(\mathcal{C})_2)$.

Let t'' denote the vertex $(1,(S/E)_*)$ of $\Delta^2\times\mathrm{N}(\mathfrak{Fin}_*)$. The locally coCartesian fibration $q : \mathcal{C}^\otimes \to \Delta^2\times\mathrm{N}(\mathfrak{Fin}_*)$ determines a spray $\hat{\theta} : \mathrm{N}(\mathbf{\Delta}_T^{t,t''}) \to \mathrm{Fun}(\mathcal{C}_{(0,\langle n\rangle)}^\otimes, \mathcal{C}_{(1,(S/\sim)_*)}^\otimes)$. Let $\chi'' : \mathrm{Part}'(S) \to \mathbf{\Delta}_T^{t,t''}$ be the map which carries a chain of equivalence relations

$$ E_0' \subseteq E_1' \subseteq \cdots \subseteq E_{k'}' = E $$

to the simplex corresponding to the chain of maps

$$ t = (0,\langle n\rangle) \to (1,(S/E_0')_*,1) \to \cdots \to ((S/E_{k'}')_*,1). $$

Composing χ'' and $\hat{\theta}$ and using the identifications $\mathcal{C}_{(0,\langle n\rangle)}^\otimes \simeq \mathcal{C}_0^n$ and $\mathcal{C}_{(1,(S/E)_*)}^\otimes \simeq \mathcal{C}_1^{S/E}$, we we obtain a diagram $\psi_E : \mathrm{N}(\mathrm{Part}'(S)) \to \mathrm{Fun}(\mathcal{O}_0^n, \mathcal{O}_1^{S/E})$.

For every nonempty subset $S' \subseteq S$, let $E_{S'}$ denote the restriction of the equivalence relation E to S', and let $\mathrm{Part}'(S')$ denote the subset of $\mathrm{Part}(S')$ consisting of those linearly ordered subsets of $\mathrm{Equiv}(E)$ which contain $E_{S'}$ as a maximal element. Consider the vertices $t_{S'} = (0,S_*')$ and $t_{S'}'' = (1,\langle 1\rangle)$ of T, and let $\hat{\theta}_{S'} : \mathrm{N}(\mathbf{\Delta}_T^{t_E,t_E''}) \to \mathrm{Fun}(\mathcal{C}_{(0,S_*')}^\otimes, \mathcal{C}_1)$ be the spray associated to the locally coCartesian fibration $q : \mathcal{C}^\otimes \to T$. Let $\chi_{S'} : \mathrm{Part}'(S') \to \mathbf{\Delta}_T^{t_E,t_E''}$ be the functor which carries a chain of equivalence relations

$$ E_0' \subseteq E_1' \subseteq \cdots \subseteq E_{k'}' $$

to the simplex given by the chain of maps

$$ t_E = (0,S_*') \to ((1,S'/E_0')_*) \to \cdots \to (1,(S'/E_{k'}')_*). $$

Composing $\chi_{S'}$ and $\theta_{S'}$, we obtain a functor $\psi_{S'} : \mathrm{N}(\mathrm{Part}'(S')) \to \mathrm{Fun}(\mathcal{C}_0^{S'}, \mathcal{C}_1)$. The inclusion $S' \hookrightarrow S$ determines a map $\mathrm{Equiv}(S) \to \mathrm{Equiv}(S')$, which induces a restriction map $r_{S'} : \mathrm{Part}'(S) \to \mathrm{Part}'(S')$. Unwinding the definitions, we see that ψ_E is equivalent to the composition

$$ \mathrm{N}(\mathrm{Part}'(S)) \overset{\prod_{S'\in S/E} r_{S'}}{\longrightarrow} \prod_{S'\in S/E} \mathrm{N}(\mathrm{Part}'(S')) \overset{\prod_{S'\in S/E} \psi_{S'}}{\longrightarrow} \prod_{S'\in S/E} \mathrm{Fun}(\mathcal{C}_0^{S'}, \mathcal{C}_1) \to \mathrm{Fun}(\mathcal{C}_0^n, \mathcal{C}_1^{S/E}). $$

We now divide the proof into two cases. Suppose first that $k = 0$, so that $\vec{E} = \{E\}$. The edge $(1,(S/E)_*) \to (2,\langle 1\rangle)$ of T determines a functor $G : \mathcal{C}_1^{S/E} \to \mathcal{C}_2$. Unwinding the definitions, we obtain canonical equivalences

$$ \phi_P(0,\vec{E}'\cup\vec{E}) \simeq \Sigma_{\mathcal{C}_2}^\infty \circ G \circ \prod_{S'\in S/E} \psi_{S'}(r_{S'}\vec{E}') $$

$$ \phi_P(1,\vec{E}'\cup\vec{E}) \simeq \vec{\partial}(G) \circ \prod_{S'\in S/E} \Sigma_{\mathcal{C}_1}^\infty \circ \psi_{S'}(r_{S'}\vec{E}'). $$

To prove that σ is a pullback diagram, it will suffice to show that the square

$$\vec{\partial}(G \circ \prod_{S' \in S/E}(\psi_{S'} \circ r'_S)(\{E\})) \longrightarrow \varprojlim_{\vec{E}' \in \mathrm{Part}'^0(S)} \vec{\partial}(G \circ \prod_{S' \in S/E}(\psi_{S'} \circ r_{S'})(\vec{E}))$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\vec{\partial}(G) \circ \prod_{S' \in S/E} \vec{\partial}((\psi_{S'} \circ r_{S'})(\{E\})) \longrightarrow \varprojlim_{\vec{E}' \in \mathrm{Part}'^0(S)} \vec{\partial}(G) \circ \prod_{S' \in S/E} \vec{\partial}(\psi_{S'} \circ r_{S'})(\vec{E}))$$

is a pullback square in $\mathrm{Exc}_*(\mathrm{Sp}(\mathcal{C}_0)^n, \mathrm{Sp}(\mathcal{C}_2))$. We will deduce this from Theorem 7.3.5.6.

Choose an equivalence relation $E' \subsetneq E$ on S. If $S' \subseteq S$, we let $E'_{S'}$ denote the restriction of the equivalence relation E' to S'. Unwinding the definitions, we see that for each $S' \in S/E$ and each $\vec{E} \in \mathrm{Part}'(S)$, the functor $(\psi_{S'} \circ r_{S'})(\vec{E}')^{E'_{S'}}$ canonically equivalent to the composition

$$\mathcal{C}_0^{S'} \simeq \prod_{S'' \in S'/E'_{S'}} \mathcal{C}_0^{E'} \overset{\prod(\psi_{S''} \circ r_{S''})(\vec{E}')}{\to} \mathcal{C}_1^{S'/E'_{S'}} \overset{\times}{\to} \mathcal{C}_1,$$

where the final functor is given by the iterated product. To satisfy the hypotheses of Theorem 7.3.5.6, it will suffice to verify the following:

(vii) The restriction maps $r_{S''}$ determine a right cofinal map $\mathrm{N}(\mathrm{Part}'^0(S)) \to \prod_{S'' \in S/E'} \mathrm{N}(\mathrm{Part}'(S''))$.

Fix elements $\{\vec{E}'_{S''} \in \mathrm{Part}'(S'')\}$ for $S'' \in S/E'$. Let $\mathrm{Part}''^0(S)$ denote the subset of $\mathrm{Part}'^0(S)$ consisting of those elements $\vec{E}'$ such that $r_{S''}(\vec{E}') \subseteq \vec{E}'_{S'}$ for $S'' \in S/E'$. According to Theorem T.4.1.3.1, it will suffice to show that the partially ordered set $\mathrm{Part}''^0(S)$ has weakly contractible nerve.

Let $\mathrm{Equiv}_E(S)$ denote the subset of $\mathrm{Equiv}(S)$ consisting of equivalence relations which refine E, and define $\mathrm{Equiv}_{E'}(S) \subseteq \mathrm{Equiv}_E(S)$ similarly. For every downward-closed subset $\mathcal{V} \subseteq \mathrm{Equiv}_E(S) - (\{E\} \cup \mathrm{Equiv}_{E'}(S))$, we let $\mathrm{Part}''^0_{\mathcal{V}}(S)$ denote the subset of $\mathrm{Part}''^0(S)$ consisting of those linearly ordered subsets $\vec{E}' \subseteq \mathrm{Equiv}(S)$ which do not intersect $\mathcal{V}$. We will prove that $\mathrm{Part}''^0_{\mathcal{V}}(S)$ has weakly contractible nerve by descending induction on the cardinality of $\mathcal{V}$. Taking $\mathcal{V} = \emptyset$, this will complete the proof of (vi). We first treat the base case $\mathcal{V} = \mathrm{Equiv}_E(S) - (\{E\} \cup \mathrm{Equiv}_{E'}(S))$ Let R denote the subset of $\mathrm{Part}''^0_{\mathcal{V}}(S)$ consisting of those subsets $\vec{E}' \subseteq \mathrm{Equiv}(S)$ which contain E'. The inclusion $R \subseteq \mathrm{Part}''^0_{\mathcal{V}}(S)$ admits a left adjoint, given by $\vec{E}' \mapsto \vec{E}' \cup \{E'\}$. It will therefore suffice to show that $\mathrm{N}(R)$ is weakly contractible. This is clear, since R contains a least element (given by the linearly ordered subset $\{E' \subsetneq E\} \subseteq \mathrm{Equiv}(S)$).

We now carry out the inductive step. Assume that $\mathrm{Part}''^0_{\mathcal{V}}(S)$ has weakly contractible nerve for some nonempty subset $\mathcal{V} \subseteq \mathrm{Equiv}_E(S) - (\{E\} \cup \mathrm{Equiv}_{E'}(S))$. Let E'' be a maximal element of $\mathcal{V}$ and let $\mathcal{V}' = \mathcal{V} - \{E''\}$. We will prove that $\mathrm{Part}''^0_{\mathcal{V}'}(S)$ has weakly contractible nerve by showing that the inclusion

$$i : \mathrm{N}(\mathrm{Part}''^0_{\mathcal{V}}(S)) \hookrightarrow \mathrm{N}(\mathrm{Part}''^0_{\mathcal{V}'}(S))$$

is a weak homotopy equivalence. Let R denote the subset of $\mathrm{Part}''^0_{\mathcal{V}'}(S)$ consisting of those linearly ordered subsets $\vec{E}' \subseteq \mathrm{Equiv}(S)$ with the following property: if E'' belongs to $\vec{E}'$, then the common refinement $E' \cap E''$ belongs to $\vec{E}'$. Then the inclusion map i factors as a composition

$$\mathrm{N}(\mathrm{Part}''^0_{\mathcal{V}}(S)) \overset{i'}{\hookrightarrow} \mathrm{N}(R) \overset{i''}{\hookrightarrow} \mathrm{N}(\mathrm{Part}''^0_{\mathcal{V}'}(S)).$$

Here i' admits a right adjoint and i'' admits a left adjoint. It follows that i' and i'' are weak homotopy equivalences, so that i is a weak homotopy equivalence. This completes the proof of (vii).

We now return to the proof of (vi) in the case where $k > 0$. Let $W = S/E_1$. Given an element $w \in W$, let S_w denote its inverse image in S and let E_w be the restriction of E to S_w. The map of pointed finite sets $(S/E)_* \to W_*$ determines functors

$$G_w : \prod_{E \in S_w/E} \mathcal{C}_1 \to \mathcal{C}_1$$

. Let $H : \mathrm{Sp}(\mathcal{C})_1^W \to \mathrm{Sp}(\mathcal{C})_2$ be the functor given by the composition

$$\mathrm{Sp}(\mathcal{C})_1^{S/E_1} \to \mathrm{Sp}(\mathcal{C})_1^{S/E_2} \to \cdots \to \mathrm{Sp}(\mathcal{C})_1^{S/\sim_k} \to \mathrm{Sp}(\mathcal{C})_2$$

determined by $\vec{E}$. Unwinding the definitions, we obtain canonical equivalences

$$\phi_P(0, \vec{E}' \cup \vec{E}) \simeq H \circ \prod_{w \in W} (\Sigma_{\mathcal{C}_1}^\infty \circ G_w \circ \prod_{S' \in S_w/E_w} \psi_{S'}(r_{S'}\vec{E}')$$

$$\phi_P(1, \vec{E}' \cup \vec{E}) \simeq H \circ \prod_{w \in W} (\vec{\partial}(G_w) \circ \prod_{S' \in S_w/E_w} \Sigma_{\mathcal{C}_1}^\infty \circ \circ \psi_{S'}(r_{S'}(\vec{E}))$$

for $\vec{E}' \in \mathrm{Part}'(S)$. Since $\mathrm{Sp}(\mathcal{C})^\otimes$ is a stable Δ^2-family of ∞-operads, the functor H is exact in each variable. Using Lemma 7.3.6.2 and Corollary 7.2.3.24, we see that σ is a pullback diagram if and only if the diagram

$$H \circ \textstyle\prod_{w \in W} \vec{\partial}(G_w \circ \prod_{S' \in S_w/E_w} (\psi_{S'} \circ r_{S'})(\{E\}))$$

$$\varprojlim_{\vec{E}' \in \mathrm{Part}'^0(S)} H \circ \textstyle\prod_{w \in W} \vec{\partial}(G_w \circ \prod_{S' \in S_w/E_w} (\psi_{S'} \circ r_{S'})(\vec{E}))$$

$$H \circ \textstyle\prod_{w \in W} \vec{\partial}(G_w) \circ \prod_{S' \in S_w/E_w} \vec{\partial}(\psi_{S'} \circ r_{S'})(\{E\}))$$

$$\varprojlim_{\vec{E}' \in \mathrm{Part}'^0(S)} H \circ \textstyle\prod_{w \in W} \vec{\partial}(G_w) \circ \prod_{S' \in S_w/E_w} \vec{\partial}(\psi_E \circ r_E)(I))$$

is a pullback square in $\mathrm{Exc}_*(\mathrm{Sp}(\mathcal{C})_0^n, \mathrm{Sp}(\mathcal{C})_2)$.

Define $F, F' : \mathrm{N}(\mathrm{Part}'(S)) \to \mathrm{Exc}_*(\mathrm{Sp}(\mathcal{C})_0^n, \mathrm{Sp}(\mathcal{C})_2)$ by the formulas

$$F(\vec{E}) = H \circ \prod_{w \in W} \vec{\partial}(G_w \circ \prod_{S' \in S_w/E_w} (\psi_{S'} \circ r_{S'})(\vec{E})) \qquad F'(\vec{E}) = H \circ \prod_{w \in W} \vec{\partial}(G_w) \circ \prod_{S' \in S_w/E_w} \mathrm{Diff}(\psi_{S'} \circ r_{S'})(\vec{E})).$$

We wish to prove that the evident natural transformation $F \to F'$ induces an equivalence $\mathrm{tfib}(F) \to \mathrm{tfib}(F')$ in the stable ∞-category $\mathrm{Exc}_*(\mathrm{Sp}(\mathcal{C})_0^n, \mathrm{Sp}(\mathcal{C})_2)$. To study this map, we need the following observation:

(viii) Let $(\prod_{w \in W} \mathrm{Part}'(S_w))^0$ denote the subset of $(\prod_{w \in W} \mathrm{Part}'(S_w))$ obtained by removing the least element. Then the restriction maps The restriction maps $\{r_{S_w} : \mathrm{Part}'(S) \to \mathrm{Part}'(S_w)\}$ induce a right cofinal functor

$$\mathrm{N}(\mathrm{Part}'^0(S)) \to \mathrm{N}(\prod_{w \in W} \mathrm{Part}'(S_w))^0.$$

To prove (viii), let us fix an element $\{\vec{E}'_w\}_{w \in W}$ of $\prod_{w \in W} \mathrm{Part}'(S_w)$ which is not the least element. Using the criterion of Theorem T.4.1.3.1, we are reduced to proving that the partially ordered set $Z = \{\vec{E}' \in \mathrm{Part}'^0(S) : (\forall w \in W)[r_{S_w}(\vec{E}') \subseteq \vec{E}'_w]\}$ has weakly contractible nerve. Let $Z'' \subseteq \mathrm{Part}(S)$ be the subset consisting of those nonempty chains $\vec{E}' = (E'_0 \subseteq \ldots \subseteq E'_{k'}) \in \mathrm{Part}(S)$ such that $E'_{k'} \subseteq E$ and $r_{S_w}(\vec{E}') \subseteq \vec{E}'_w$ for each $w \in W$, and let $Z' \subseteq Z''$ be the subset obtained by removing the one-element chain $\{E\}$. The inclusion $Z \hookrightarrow Z'$ has a left adjoint (given by the construction $\vec{E}' \mapsto \vec{E}' \cup \{E\}$) and therefore induces a weak homotopy equivalence $\mathrm{N}(Z) \to \mathrm{N}(Z')$. We are therefore reduced to proving that $\mathrm{N}(Z')$ is weakly contractible. Note that each $\vec{E}'_w$ determines an embedding $\Delta^{a_w} \hookrightarrow \mathrm{Equiv}(S_w)$. Since $\{\vec{E}'_w\}_{w \in W}$ is not the

least element of $\prod_{w\in W} \operatorname{Part}'(S_w)$, at least one of the integers a_w is nonzero. Unwinding the definitions, we can identify $\operatorname{N}(Z'')$ with the barycentric subdivision of the product $\prod_{w\in W} \Delta^{a_w}$. The weak contractibility of $\operatorname{N}(Z')$ now follows from Lemma 7.3.6.1.

For every subset $S' \subseteq S_w$, we let $r^w_{S'} : \operatorname{Part}'(S_w) \to \operatorname{Part}'(S')$ denote the evident restriction map. Define functors $F_w, F'_w : \operatorname{N}(\operatorname{Part}'(S_w)) \to \operatorname{Fun}(\operatorname{Sp}(\mathcal{C})^n_0, \operatorname{Sp}(\mathcal{C})_1)$ by the formulas

$$F_w(\vec{E}'_w) = \vec{\partial}(G_w \circ \prod_{S'\in S_w/E_w} (\psi_{S'} \circ r^w_{S'})(\vec{E}'_w)) \qquad F'_w(\{\vec{E}'_w\}) = \vec{\partial}(G_w) \circ \prod_{S'\in S_w/E_w} \vec{\partial}(\psi_{S'} \circ r^w_{S'})(\vec{E}'_w)).$$

Let

$$F_0 : \prod_{w\in W} \operatorname{N}(\operatorname{Part}'(S_w)) \to \operatorname{Fun}(\operatorname{Sp}(\mathcal{C})^n_0, \operatorname{Sp}(\mathcal{C})_2)$$

be given by the formula $F_0(\{\vec{E}_w\}) = H \circ \prod_{w\in W} F_w(\vec{E}_w)$, and define F'_0 similarly. Then we can recover F and F' as the composite functors

$$\operatorname{Part}'(S) \to \prod_{w\in W} \operatorname{Part}'(S_w) \xrightarrow{F_0} \operatorname{Fun}(\operatorname{Sp}(\mathcal{C})^n_0, \operatorname{Sp}(\mathcal{C})_2)$$

$$\operatorname{Part}'(S) \to \prod_{w\in W} \operatorname{Part}'(S_w) \xrightarrow{F'_0} \operatorname{Fun}(\operatorname{Sp}(\mathcal{C})^n_0, \operatorname{Sp}(\mathcal{C})_2).$$

We have an evident commutative diagram

$$
\begin{array}{ccc}
\operatorname{tfib} F_0 & \longrightarrow & \operatorname{tfib} F'_0 \\
\downarrow & & \downarrow \\
\operatorname{tfib} F & \longrightarrow & \operatorname{tfib} F',
\end{array}
$$

and assertion $(viii)$ guarantees that the vertical maps are equivalences. We are therefore reduced to proving that the map $\rho : \operatorname{tfib} F_0 \to \operatorname{tfib} F'_0$ is an equivalence. Since the functor H is exact in each variable, we can identify ρ with the canonical map $H \circ \prod_{w\in W}\{\operatorname{tfib} F_w\}_{w\in W} \to H \circ \prod_{w\in W}\{\operatorname{tfib} F'_w\}_{w\in W}$. It will therefore suffice to show that for each $w \in W$, the map $\operatorname{tfib} F_w \to \operatorname{tfib} F'_w$ is an equivalence. Equivalently, we wish to show that the diagram

$$
\begin{array}{ccc}
\vec{\partial}(G \circ \prod_{S'\in S_w/E_w} (\psi_{S'} \circ r^w_{S'})(\{E_w\})) & \longrightarrow & \varprojlim_{\vec{E}'_w \in \operatorname{Part}'^0(S_w)} \vec{\partial}(G \circ \prod_{S'\in S_w/E_w} (\psi_{S'} \circ r^w_E)(\vec{E}'_w)) \\
\downarrow & & \downarrow \\
\vec{\partial}(G) \circ \prod_{S'\in S_w/E_w} \vec{\partial}((\psi_{S'} \circ r^w_{S'})(\{E_w\})) & \longrightarrow & \varprojlim_{\vec{E}'_w \in \operatorname{Part}'^0(S_w)} \vec{\partial}(G) \circ \prod_{S'\in S_w/E_w} \vec{\partial}(\psi_{S'} \circ r^w_{S'})(\vec{E}'_w))
\end{array}
$$

is a pullback square in $\operatorname{Fun}(\operatorname{Sp}(\mathcal{C})^n_0, \operatorname{Sp}(\mathcal{C})_1)$. The proof now proceeds exactly as in the case $k = 0$ (with some minor changes in notation). $\qquad\square$

Chapter 8

Algebra in the Stable Homotopy Category

Let Sp denote the ∞-category of spectra. In §6.3.2, we saw that Sp admits a symmetric monoidal structure (the *smash product* symmetric monoidal structure) which is characterized up to equivalence by the requirement that the sphere spectrum $S \in \mathrm{Sp}$ is the unit object and the tensor product functor $\otimes : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ preserves colimits separately in each variable. This operation can be regarded as a homotopy theoretic analogue of the usual tensor product of abelian groups. In this chapter, we will undertake a systematic study of commutative and noncommutative algebra in the ∞-category Sp.

We begin in §8.1 by introducing the notion of an $\mathbb{E}_k$-*ring*, for $0 \le k \le \infty$. To guide the reader's intuition, we offer the following table of analogies:

Ordinary Algebra	∞-Categorical Algebra
Set	Space
Abelian group	Spectrum
Tensor product of abelian groups	Smash product of spectra
Associative Ring	$\mathbb{E}_1$-Ring
Commutative Ring	$\mathbb{E}_\infty$-Ring

Our goal in §8.2 is to show that these analogies are fairly robust, in the sense that many of the basic tools used in commutative and noncommutative algebra can be generalized to the setting of structured ring spectra.

The remainder of this chapter is devoted to studying the deformation theory of $\mathbb{E}_k$-rings. In §8.3, we will introduce a general formalism describing the *relative cotangent complex* $L_{B/A}$ of a morphism $\phi : A \to B$ in a presentable ∞-category $\mathcal{C}$. When $\mathcal{C}$ is the ∞-category of $\mathbb{E}_\infty$-rings, then we can identify $L_{B/A}$ with an object of the ∞-category $\mathrm{Mod}_B(\mathrm{Sp})$ of B-*module spectra*. In §8.4, we will see that many questions about a map $\phi : A \to B$ of $\mathbb{E}_\infty$-rings can be reduced to questions about $L_{B/A}$: that is, to problems in *stable* homotopy theory. In §8.5, we will apply this technology to obtain a classification of étale morphisms between structured ring spectra.

Warning 8.0.0.1. Let R be an associative ring, and let M and N be right and left modules over R, respectively. Then we can regard R as a discrete $\mathbb{E}_1$-algebra in the ∞-category of spectra, and M and N as discrete module spectra over R. Consequently, we can compute the tensor product $M \otimes_R N$ either in the ∞-category of spectra or in the ordinary category of abelian groups. Unless otherwise specified, the notation

$M \otimes_R N$ will indicate the relative tensor product in the ∞-category of spectra. We will denote the ordinary algebraic tensor product by $\mathrm{Tor}_0^R(M, N)$. These are generally different from one another: for example, the spectrum $M \otimes_R N$ need not be discrete. In fact, in §8.2 we will see that there are canonical isomorphisms $\pi_i(M \otimes_R N) \simeq \mathrm{Tor}_i^R(M, N)$ (Corollary 8.2.1.22). In particular, the algebraic tensor product $\mathrm{Tor}_0^R(M, N)$ can be realized as the 0th homotopy group of the spectrum $M \otimes_R N$.

In cases where the groups $\mathrm{Tor}_i^R(M, N)$ vanish for $i > 0$ (for example, if either M or N is a projective module over R), we will generally not distinguish in notation between $M \otimes_R N$ and $\mathrm{Tor}_0^R(M, N)$.

Remark 8.0.0.2. The theory of structured ring spectra plays an important role in modern stable homotopy theory. There is a vast literature on the subject, which we will not attempt to review here. We refer the readers to [51] for a foundational approach using the language of model categories, rather than the language of ∞-categories which we employ in this book.

8.1   Structured Ring Spectra

In this section, we introduce homotopy-theoretic analogues of some elementary notions from commutative and noncommutative algebra. Our starting point is the following:

Definition 8.1.0.1. Let $0 \le k \le \infty$, and let Sp denote the ∞-category of spectra (which we regard as endowed with the smash product monoidal structure of §6.3.2). Let $\mathbb{E}_k^\otimes$ denote the ∞-operad of little k-cubes (Definition 5.1.0.2). An $\mathbb{E}_k$-*ring* is an $\mathbb{E}_k$-algebra object of Sp. We let $\mathrm{Alg}^{(k)}$ denote the ∞-category $\mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Sp})$ of $\mathbb{E}_k$-rings.

In the special case $k = \infty$, we will agree that $\mathbb{E}_k^\otimes$ denotes the commutative ∞-operad $\mathrm{Comm}^\otimes = \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ (see Corollary 5.1.1.5) and we let CAlg denote the ∞-category $\mathrm{CAlg}(\mathrm{Sp}) = \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp})$ of $\mathbb{E}_\infty$-rings.

Remark 8.1.0.2. The terminology of Definition 8.1.0.1 is somewhat nonstandard. What we call $\mathbb{E}_k$-rings are often called $\mathbb{E}_k$-*ring spectra*. In the special case $k = \infty$, the term *commutative ring spectrum* is sometimes used, though some authors reserve this term for commutative algebra objects in the homotopy category hSp (that is, spectra which are equipped with a multiplication which is commutative and associative up to homotopy, rather than up to coherent homotopy). In the special case $k = 1$, many authors refer to $\mathbb{E}_1$-rings as *associative ring spectra* or A_∞-*ring spectra*.

Remark 8.1.0.3. Let R be an $\mathbb{E}_k$-ring for $0 \le k \le \infty$. We will say that R is *discrete* if its underlying spectrum is discrete: that is, if $\pi_i R \simeq 0$ for $i \neq 0$. The ∞-category of discrete $\mathbb{E}_k$-rings can be identified with the ∞-category of $\mathbb{E}_k$-algebra objects of $\mathrm{Sp}^\heartsuit$, which is equivalent to the nerve of the ordinary category of abelian groups. It follows that when $k = 1$, the ∞-category of discrete $\mathbb{E}_1$-rings is equivalent to the nerve of the category of associative rings. Using Corollary 5.1.1.7, we see that the ∞-category of discrete $\mathbb{E}_k$-rings is equivalent to the nerve of the category of commutative rings whenever $k \ge 2$.

Remark 8.1.0.4. If R is an $\mathbb{E}_k$-ring, then we can view R as an $\mathbb{E}_{k'}$-ring for any $k' \le k$. More precisely, the sequence of maps
$$\mathbb{E}_0^\otimes \hookrightarrow \mathbb{E}_1^\otimes \to \cdots \to \mathbb{E}_\infty^\otimes = \mathrm{Comm}^\otimes$$
induces forgetful functors
$$\mathrm{CAlg} \to \cdots \to \mathrm{Alg}^{(2)} \to \mathrm{Alg}^{(1)} \to \mathrm{Alg}^{(0)}.$$
If $R \in \mathrm{Alg}^{(k)}$, we will generally abuse notation by identifying R with its image in $\mathrm{Alg}^{(k')}$ for $k' \le k$.

Remark 8.1.0.5. Using Remark 6.3.2.22, we can identify the ∞-category $\mathrm{Alg}^{(1)}$ of $\mathbb{E}_1$-rings with the full subcategory of $\mathrm{Alg}(\mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp}))$ spanned by those monads T on Sp which preserve small colimits.

We will begin our study of $\mathbb{E}_k$-rings by considering the case $k = 1$. According to Example 5.1.0.7, we can identify the ∞-category $\mathrm{Alg}^{(1)}$ of $\mathbb{E}_1$-rings with the ∞-category $\mathrm{Alg} = \mathrm{Alg}(\mathrm{Sp})$. In other words, we can think of an $\mathbb{E}_1$-ring as a spectrum A equipped with a multiplication $A \otimes A \to A$, which is associative up

to coherent homotopy. The technology of Chapter 4 provides us with a robust theory of (left and right) modules over $\mathbb{E}_1$-rings, which we study in §8.1.1.

For any $\mathbb{E}_1$-ring A, the ∞-category $\mathrm{LMod}_A = \mathrm{LMod}_A(\mathrm{Sp})$ is stable and compactly generated. In §8.1.2 we will prove a converse to this statement (due to Schwede and Shipley): if $\mathcal{C}$ is a presentable stable ∞-category which is generated by a single compact object $\mathbf{1} \in \mathcal{C}$, then we can describe $\mathcal{C}$ as the ∞-category of left modules over the endomorphism object $R = \mathrm{End}_{\mathcal{C}}(\mathbf{1})$ (see Theorem 8.1.2.1). Moreover, we show that promoting R to an $\mathbb{E}_{k+1}$-ring is equivalent to promoting $\mathcal{C}$ to an $\mathbb{E}_k$-monoidal ∞-category (having $\mathbf{1}$ as unit object). In this case, it makes sense to consider $\mathbb{E}_k$-algebra objects of $\mathcal{C}$, which we refer to as $\mathbb{E}_k$-*algebras over R*. In §8.1.3, we will study the theory of $\mathbb{E}_k$-algebras over an $\mathbb{E}_{k+1}$-ring R, and prove a technical result (Theorem 8.1.3.1) which implies that many natural constructions are compatible with change of R.

Using Theorem 4.1.4.4, we conclude that $\mathrm{Alg}^{(1)}$ is equivalent to the underlying ∞-category of strictly associative monoids in any sufficiently nice monoidal model category of spectra (see Example 4.1.4.6). If we are interested in studying algebras over a *discrete* $\mathbb{E}_\infty$-ring R, then much more concrete models are available. In §8.1.4, we will show that the theory of $\mathbb{E}_1$-algebras over R is equivalent to the theory of differential graded algebras (Proposition 8.1.4.6). When R contains the field $\mathbf{Q}$ of rational numbers, there is a similar description of the ∞-category of $\mathbb{E}_\infty$-algebras over R (Proposition 8.1.4.11).

8.1.1 $\mathbb{E}_1$-Rings and Their Modules

Let $R \in \mathrm{Alg}^{(1)}$ be an $\mathbb{E}_1$-ring. In this section, we will introduce the theory of *R-module spectra*. This can be regarded as a generalization of homological algebra: if R is an ordinary ring (regarded as a discrete $\mathbb{E}_1$-ring via Proposition 8.1.3.18), then the homotopy category of R-module spectra coincides with the classical *derived category* of R (Proposition 8.1.1.15); in particular, the theory of R-module spectra is a generalization of the usual theory of R-modules.

Notation 8.1.1.1. According to Example 5.1.0.7, there is a trivial Kan fibration of ∞-operads $q : \mathbb{E}_1^\otimes \to \mathcal{A}\mathrm{ss}^\otimes$. We will fix a section of q. For any symmetric monoidal ∞-category $\mathcal{C}$, composition with this section induces an equivalence of ∞-categories $\theta : \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$. We will generally abuse notation by identifying an $\mathbb{E}_1$-algebra object $A \in \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$ with its image in $\mathrm{Alg}(\mathcal{C})$. In particular, we will denote the ∞-categories $\mathrm{LMod}_{\theta(A)}(\mathcal{C})$ and $\mathrm{RMod}_{\theta(A)}(\mathcal{C})$ defined in §4.2.1 by $\mathrm{LMod}_A(\mathcal{C})$ and $\mathrm{RMod}_A(\mathcal{C})$, respectively.

If A is an $\mathbb{E}_\infty$-ring, then we let Mod_A denote the ∞-category $\mathrm{Mod}_A^{\mathbb{E}_\infty}(\mathrm{Sp})$, so that we have canonical equivalences

$$\mathrm{LMod}_A \leftarrow \mathrm{Mod}_A \to \mathrm{RMod}_A$$

(see Proposition 4.4.1.4).

Definition 8.1.1.2. Let R be an $\mathbb{E}_1$-ring. We let LMod_R denote the ∞-category $\mathrm{LMod}_R(\mathrm{Sp})$. We will refer to LMod_R as the ∞-*category of left R-module spectra*. Similarly, we let RMod_R denote the ∞-category $\mathrm{RMod}_R(\mathrm{Sp})$ of *right R-module spectra*.

Remark 8.1.1.3. If R is an $\mathbb{E}_1$-ring, we will often refer to (left or right) R-module spectra simply as (left or right) *R-modules*.

For the sake of definiteness, we will confine our attention to the study of left module spectra throughout this section; the theory of right module spectra can be treated in an entirely parallel way.

Our first goal is to prove that the ∞-category of modules over an $\mathbb{E}_1$-ring is stable. This is a consequence of the following more general assertion:

Proposition 8.1.1.4. *Let $\mathcal{C}$ be an ∞-category equipped with a monoidal structure and a left action on an ∞-category $\mathcal{M}$. Assume that $\mathcal{M}$ is a stable ∞-category, and let $R \in \mathrm{Alg}(\mathcal{C})$ be such that the functor $M \mapsto R \otimes M$ is exact. Then $\mathrm{LMod}_R(\mathcal{M})$ is a stable ∞-category. Moreover, if $\mathcal{N}$ is an arbitrary stable ∞-category, then a functor $\mathcal{N} \to \mathrm{LMod}_R(\mathcal{M})$ is exact if and only if the composite functor $\mathcal{N} \to \mathrm{LMod}_R(\mathcal{M}) \to \mathcal{M}$ is exact. In particular, the forgetful functor $\mathrm{LMod}_R(\mathcal{M}) \to \mathcal{M}$ is exact.*

Proof. This is a special case of Proposition 4.2.3.4.   □

Corollary 8.1.1.5. *Let R be an $\mathbb{E}_1$-ring. Then the ∞-categories LMod_R and RMod_R are stable.*

We next consider connectivity properties of $\mathbb{E}_1$-rings and their modules. Let R be an $\mathbb{E}_1$-ring. We will generally not distinguish notationally between R and its underlying spectrum. In particular, for each $n \in \mathbf{Z}$, we let $\pi_n R$ denote the nth homotopy group of the underlying spectrum. We observe that $\pi_n R$ can be identified with the set $\pi_0 \mathrm{Map}_{\mathrm{Sp}}(S[n], R)$, where S denotes the sphere spectrum. Since S is the identity for the smash product, there is a canonical equivalence $S \otimes S \simeq S$; using the fact that $\otimes$ is exact in each variable, we deduce the existence of equivalences $S[n] \otimes S[m] \simeq S[n+m]$ for all $n, m \in \mathbf{Z}$. The multiplication map

$$\mathrm{Map}_{\mathrm{Sp}}(S[n], R) \times \mathrm{Map}_{\mathrm{Sp}}(S[m], R) \to \mathrm{Map}_{\mathrm{Sp}}(S[n] \otimes S[m], R \otimes R) \to \mathrm{Map}_{\mathrm{Sp}}(S[n+m], R)$$

determines a bilinear map $\pi_n R \times \pi_m R \to \pi_{n+m} R$. It is not difficult to see that these maps endow $\pi_* R = \bigoplus_n \pi_n R$ with the structure of a graded associative ring, which depends functorially on R. In particular, $\pi_0 R$ is an ordinary associative ring, and each $\pi_n R$ has the structure of a $\pi_0 R$-bimodule.

Remark 8.1.1.6. If R admits the structure of an $\mathbb{E}_k$-ring for $k \geq 2$, we can say a bit more. In this case, the multiplication on R is commutative (up to homotopy). It follows that the multiplication on $\pi_* R$ is *graded commutative*. That is, for $x \in \pi_n R$ and $y \in \pi_m R$, we have $xy = (-1)^{nm} yx$. Here the sign results from the fact that the composition

$$S[n+m] \simeq S[n] \otimes S[m] \overset{\sigma}{\simeq} S[m] \otimes S[n] \simeq S[n+m]$$

is given by the sign $(-1)^{nm}$. In particular, the homotopy group $\pi_0 R$ is equipped with the structure of a commutative ring, and every other homotopy group $\pi_n R$ has the structure of a module over $\pi_0 R$.

We will need the following basic result:

Lemma 8.1.1.7. *The t-structure on the ∞-category Sp determined by the class of connective objects is compatible with the smash product symmetric monoidal structure (in the sense of Example 2.2.1.3). In other words, the full subcategory $\mathrm{Sp}^{\mathrm{cn}} \subseteq \mathrm{Sp}$ spanned by the connective objects is closed under smash products and contains the unit object. Consequently, the monoidal structure on Sp determines a monoidal structure on $\mathrm{Sp}^{\mathrm{cn}}$.*

Proof. The ∞-category $\mathrm{Sp}^{\mathrm{cn}}$ is the smallest full subcategory of Sp which contains the sphere spectrum $S \in \mathrm{Sp}$ and is stable under colimits and extensions. Let $\mathcal{C}$ be the full subcategory of Sp spanned by those spectra X such that, for all $Y \in \mathrm{Sp}^{\mathrm{cn}}$, $X \otimes Y$ is connective. We wish to prove that $\mathrm{Sp}^{\mathrm{cn}} \subseteq \mathcal{C}$. Since the smash product preserves colimits separately in each variable, we conclude that $\mathcal{C}$ is closed under colimits and extensions in Sp. It will therefore suffice to prove that $S \in \mathcal{C}$. This is clear, since S is the unit object of Sp.   □

Recall that a spectrum X is said to be *connective* if $\pi_n X \simeq 0$ for $n < 0$. We will say that an $\mathbb{E}_k$-ring R is *connective* if its underlying spectrum is connective. We let $\mathrm{Alg}^{(k),\mathrm{cn}}$ denote the full subcategory of $\mathrm{Alg}^{(k)}$ spanned by the connective $\mathbb{E}_k$-rings. We can equivalently define $\mathrm{Alg}^{(k),\mathrm{cn}}$ to be the ∞-category $\mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Sp}^{\mathrm{cn}})$, where $\mathrm{Sp}^{\mathrm{cn}}$ denotes the full subcategory of Sp spanned by the connective spectra (the full subcategory $\mathrm{Sp}^{\mathrm{cn}} \subset \mathrm{Sp}$ inherits a symmetric monoidal structure in view of Proposition 2.2.1.1 and Lemma 8.1.1.7). In the special case $k = \infty$, we will denote this ∞-category $\mathrm{CAlg}_{\mathbb{E}_k}(\mathrm{Sp}^{\mathrm{cn}})$ by $\mathrm{CAlg}^{\mathrm{cn}}$.

Let M be a left R-module spectrum for some $\mathbb{E}_1$-ring R. We will generally abuse notation by identifying M with its image in Sp. In particular, we define the homotopy groups $\{\pi_n M\}_{n \in \mathbf{Z}}$ of M to be the homotopy groups of the underlying spectrum. The action map $R \otimes M \to M$ induces bilinear maps $\pi_n R \times \pi_m M \to \pi_{n+m} M$, which endow the sum $\pi_* M = \bigoplus_{n \in \mathbf{Z}} \pi_n M$ with the structure of a graded left module over $\pi_* R$. We will say that M is *connective* if its underlying spectrum is connective; that is, if $\pi_n M \simeq 0$ for $n < 0$.

Remark 8.1.1.8. When restricted to connective $\mathbb{E}_k$-rings, the functor Ω^∞ detects equivalences: if $f : A \to B$ is a morphism in $\mathrm{Alg}^{(k),\mathrm{cn}}$ such that $\Omega^\infty(f)$ is an equivalence, then f is an equivalence. We observe that the functor $\Omega^\infty : \mathrm{Alg}^{(k),\mathrm{cn}} \to \mathcal{S}$ is a composition of a pair of functors $\mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Sp}^{\mathrm{cn}}) \to \mathrm{Sp}^{\mathrm{cn}} \to \mathcal{S}$, both of which preserve sifted colimits (Corollary 3.2.3.2 and 1.4.3.9) and admit left adjoints. It follows from Theorem 6.2.2.5 that $\mathrm{Alg}^{(k),\mathrm{cn}}$ can be identified with the ∞-category of modules over a suitable monad on $\mathcal{S}$. In other words, we can view connective $\mathbb{E}_k$-rings as spaces equipped with some additional structures. Roughly speaking, these additional structures consist of an addition and multiplication which satisfy the axioms for a ring (commutative if $k \geq 2$), up to coherent homotopy.

The functor $\Omega^\infty : \mathrm{Alg}^{(k)} \to \mathcal{S}$ is not conservative: a map of $\mathbb{E}_k$-rings $f : A \to B$ which induces a homotopy equivalence of underlying spaces need not be an equivalence in $\mathrm{Alg}^{(k)}$. We observe that f is an equivalence of $\mathbb{E}_k$-rings if and only if it is an equivalence of spectra; that is, if and only if $\pi_n(f) : \pi_n A \to \pi_n B$ is an isomorphism of abelian groups for all $n \in \mathbf{Z}$. However, $\Omega^\infty(f)$ is a homotopy equivalence of spaces provided only that $\pi_n(f)$ is an isomorphism for $n \geq 0$; this is generally a weaker condition.

Remark 8.1.1.9. Roughly speaking, if we think of an $\mathbb{E}_1$-ring R as a space equipped with the structure of an associative ring up to coherent homotopy, then a left R-module can be thought of as another space which has an addition and a left action of R, up to coherent homotopy in the same sense. This intuition is really only appropriate in the case where R and M are connective, since the homotopy groups in negative degree have no simple interpretation in terms of underlying spaces.

If R is a connective $\mathbb{E}_1$-ring, the formation homotopy groups of a left R-module M can be interpreted in terms of an appropriate t-structure on LMod_R.

Notation 8.1.1.10. If R is an $\mathbb{E}_1$-ring, we let $\mathrm{LMod}_R^{\geq 0}$ be the full subcategory of LMod_R spanned by those left R-modules M for which $\pi_n M \simeq 0$ for $n < 0$, and $\mathrm{LMod}_R^{\leq 0}$ the full subcategory of LMod_R spanned by those R-modules M for which $\pi_n M \simeq 0$ for $n > 0$.

Notation 8.1.1.11. Let R be an $\mathbb{E}_1$-ring, and let M and N be left R-modules. We let $\mathrm{Ext}_R^i(M, N)$ denote the abelian group $\pi_0 \mathrm{Map}_{\mathrm{LMod}_R}(M, N[i])$.

Remark 8.1.1.12. Suppose that R is an associative ring, regarded as a discrete $\mathbb{E}_1$-ring, and let M and N be discrete left R-modules. Then the abelian groups $\mathrm{Ext}_R^i(M, N)$ of Notation 8.1.1.11 can be identified with the usual Yoneda Ext-groups, computed in the abelian category of (discrete) left R-modules. This is a consequence of Proposition 8.1.1.15, proven below.

Proposition 8.1.1.13. *Let R be a connective $\mathbb{E}_1$-ring. Then:*

(1) *The full subcategory $\mathrm{LMod}_R^{\geq 0} \subseteq \mathrm{LMod}_R$ is the smallest full subcategory which contains R (regarded as an R-module in the natural way; see Example 4.2.1.17) and is stable under small colimits.*

(2) *The subcategories $\mathrm{LMod}_R^{\geq 0}, \mathrm{LMod}_R^{\leq 0}$ determine an accessible t-structure on LMod_R (see §1.2.1).*

(3) *The t-structure described in (2) is both left and right complete, and the functor π_0 determines an equivalence of the heart $\mathrm{LMod}_R^{\heartsuit}$ with the (nerve of the) ordinary category of (discrete) $\pi_0 R$-modules.*

(4) *The subcategories $\mathrm{LMod}_R^{\geq 0}, \mathrm{LMod}_R^{\leq 0} \subseteq \mathrm{LMod}_R$ are stable under small products and small filtered colimits.*

Proof. According to Proposition 1.4.4.11, there exists an accessible t-structure $(\mathrm{LMod}_R', \mathrm{LMod}_R'')$ with the following properties:

(a) An object $M \in \mathrm{LMod}_R$ belongs to LMod_R'' if and only if $\mathrm{Ext}_R^i(R, M) \simeq 0$ for $i < 0$.

(b) The ∞-category LMod_R' is the smallest full subcategory of LMod_R which contains the object R and is stable under extensions and small colimits.

Corollary 4.2.4.8 implies that R (regarded as an object of LMod_R) corepresents the composition $\mathrm{LMod}_R \to$ $\mathrm{Sp} \xrightarrow{\Omega^\infty} \mathcal{S}$. It follows that $\mathrm{LMod}''_R = \mathrm{LMod}_R^{\leq 0}$. Because the forgetful functor $\mathrm{LMod}_R \to \mathrm{Sp}$ preserves small colimits (Corollary 4.2.3.5), we conclude that $\mathrm{LMod}_R^{\geq 0}$ is stable under extensions and small colimits. Since R is connective, $R \in \mathrm{LMod}_R^{\geq 0}$, so that $\mathrm{LMod}'_R \subseteq \mathrm{LMod}_R^{\geq 0}$. Let $\mathcal{C}$ be the smallest full subcategory of LMod_R which contains R and is stable under small colimits, so that $\mathcal{C} \subseteq \mathrm{LMod}'_R$. We will complete the proof of (1) and (2) by showing that $\mathcal{C} = \mathrm{LMod}_R^{\geq 0}$.

Let $M \in \mathrm{LMod}_R^{\geq 0}$. We will construct a diagram

$$M(0) \to M(1) \to M(2) \to \dots$$

in $(\mathrm{LMod}_R)_{/M}$ with the following properties:

(i) Let $i \geq 0$, and let $K(i)$ be a fiber of the map $M(i) \to M$. Then $\pi_j K(i) \simeq 0$ for $j < i$.

(ii) The R-module $M(0)$ is a coproduct of copies of R.

(iii) For $i \geq 0$, there is a pushout diagram

$$
\begin{array}{ccc}
F[i] & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
M(i) & \longrightarrow & M(i+1),
\end{array}
$$

where F is a coproduct of copies of R.

We begin by choosing $M(0)$ to be any coproduct of copies of R equipped with a map $M(0) \to M$ which induces a surjection $\pi_0 M(0) \to \pi_0 M$; for example, we can take $M(0)$ to be a coproduct of copies of R indexed by $\pi_0 M$. Let us now suppose that the map $f : M(i) \to M$ has been constructed, with $K(i) = \mathrm{fib}(f)$ such that $\pi_j K(i) \simeq 0$ for $j < i$. We now choose F to be a coproduct of copies of R and a map $g : F[i] \to K(i)$ which induces a surjection $\pi_0 F \to \pi_i K(i)$. Let h denote the composite map $F[i] \to K(i) \to M(i)$, and let $M(i+1) = \mathrm{cofib}(h)$. The canonical nullhomotopy of $K(i) \to M(i) \to M$ induces a factorization

$$M(i) \to M(i+1) \xrightarrow{f'} M$$

of f. We observe that there is a canonical equivalence $\mathrm{fib}(f') \simeq \mathrm{cofib}(g)$, so that $\pi_j \mathrm{fib}(f') \simeq 0$ for $j \leq i$.

Let $M(\infty)$ be the colimit of the sequence $\{M(i)\}$, and let K be the fiber of the canonical map $M(\infty) \to M$. Then K can be identified with a colimit of the sequence $\{K(i)\}_{i \geq 0}$. Since the formation of homotopy groups is preserves filtered colimits, we conclude that $\pi_j K \simeq \mathrm{colim}\, \pi_j K(i) \simeq 0$. Thus $M(\infty) \simeq M$, so that $M \in \mathcal{C}$ as desired.

Assertion (4) follows from the corresponding result for Sp, since the forgetful functor $\mathrm{LMod}_R \to \mathrm{Sp}$ preserves all limits and colimits (Corollaries 4.2.3.3 and 4.2.3.5). Since $\mathrm{LMod}_R \to \mathrm{Sp}$ is a conservative functor, an R-module M is zero if and only if $\pi_n M$ is zero for all $n \in \mathbf{Z}$. It follows from Proposition 1.2.1.19 that LMod_R is both right and left complete.

Let F be the functor from $\mathrm{LMod}_R^{\geq 0}$ to the (nerve of the) ordinary category of left $\pi_0 R$-modules, given by $M \mapsto \pi_0 M$. It is easy to see that F preserves colimits, and that the restriction of F to $\mathrm{LMod}_R^\heartsuit$ is an exact functor. We wish to prove that $F_0 = F|\mathrm{LMod}_R^\heartsuit$ is an equivalence. We first show that the restriction of F_0 is fully faithful. Fix $N \in \mathrm{LMod}_R^\heartsuit$, and let $\mathcal{D}$ be the full subcategory of $\mathrm{LMod}_R^{\geq 0}$ spanned by those objects M for which the map $\pi_0 \mathrm{Map}_{\mathrm{LMod}_R}(M, N) \to \mathrm{Hom}(F(\tau_{\leq 0} M), F(N))$ is bijective, where the right hand side indicates the group of $\pi_0 R$-module homomorphims. It is easy to see that $\mathcal{D}$ is stable under colimits and contains R. The first part of the proof shows that $\mathcal{D} = \mathrm{LMod}_R^{\geq 0}$. In particular, F_0 is fully faithful.

It remains to show that F_0 is essentially surjective. Since F_0 is fully faithful and exact, the essential image of F_0 is closed under the formation of cofibers. It will therefore suffice to show that every free left $\pi_0 R$-module belongs to the essential image of F_0. Since F_0 preserves coproducts, it will suffice to show that $\pi_0 R$ itself belongs to the essential image of F_0. We now conclude by observing that $F_0(\tau_{\leq 0} R) \simeq \pi_0 R$. $\square$

Warning 8.1.1.14. Let R be an associative ring, which we can identify with a discrete $\mathbb{E}_1$-ring (Proposition 8.1.3.18). The theory of R-module spectra (Definition 8.1.1.2) does *not* agree with the usual theory of R-modules. Instead, Proposition 8.1.1.13 allows us to identify the usual category of R-modules with the ∞-category of *discrete R-module spectra*.

Let R be a connective $\mathbb{E}_1$-ring, let $\mathcal{A}$ be the abelian category of left modules over the (ordinary) ring $\pi_0 R$. Then $\mathcal{A}$ has enough projective objects, so we can consider the derived ∞-category $\mathcal{D}^-(\mathcal{A})$ described in §1.3.2. Part (3) of Proposition 8.1.1.13 determines an equivalence $\mathrm{N}(\mathcal{A}) \simeq \mathrm{LMod}_R^{\heartsuit}$. Using Proposition 1.3.3.12, we deduce the existence of an (essentially unique) right t-exact functor $\theta : \mathcal{D}^-(\mathcal{A}) \to \mathrm{LMod}_R$.

Proposition 8.1.1.15. *Let R be a connective $\mathbb{E}_1$-ring, and let $\theta : \mathcal{D}^-(\mathcal{A}) \to \mathrm{LMod}_R$ be the functor constructed above. The following conditions are equivalent:*

(1) *The $\mathbb{E}_1$-ring R is discrete. That is, $\pi_i R \simeq 0$ for $i > 0$.*

(2) *The functor θ is fully faithful, and induces an equivalence of $\mathcal{D}^-(\mathcal{A})$ with the ∞-category of right bounded objects of LMod_R.*

Proof. Let $P \in \mathcal{A}$ be the projective object corresponding to the free left $\pi_0 R$-module on one generator. Then, for $M \in \mathcal{D}^-(\mathcal{A})$, we have a canonical isomorphism $\mathrm{Ext}^0_{\mathcal{D}^-(\mathcal{A})}(P, M) \simeq \pi_0 M$. If (2) is satisfied, then we deduce the existence of a canonical isomorphisms

$$\mathrm{Ext}^0_R(\theta(P), M) \simeq \pi_0 M \simeq \mathrm{Ext}^0_R(R, M)$$

for $M \in \mathrm{LMod}_R^{\geq 0}$. Thus $\theta(P)$ and R are isomorphic in the homotopy category hLMod_R. Since $\theta(P)$ is discrete, we conclude that R is discrete, which proves (1).

For the converse, let us suppose that R is discrete. Let us regard (the nerve of) $\mathcal{A}$ as a full subcategory of both $\mathcal{D}^-(\mathcal{A})$ and LMod_R. For $M, N \in \mathcal{A}$, let $\mathrm{Ext}^i_{\mathcal{A}}(M, N)$ denote the abelian group $\pi_0 \mathrm{Map}_{\mathcal{D}^-(\mathcal{A})}(M, N[i])$ (in other words, $\mathrm{Ext}^i_{\mathcal{A}}(M, N)$ is the classical Yoneda Ext-group computed in the abelian category $\mathcal{A}$). We claim that the canonical map $\mathrm{Ext}^i_{\mathcal{A}}(M, N) \to \mathrm{Ext}^i_R(M, N)$ is an isomorphism. For $i < 0$, both sides vanish. The proof in general goes by induction on i, the case $i = 0$ being trivial. For $i > 0$, we choose an exact sequence

$$0 \to K \to P \to M \to 0$$

in $\mathcal{A}$, where P is a free $\pi_0 R$-module. We have a commutative diagram of abelian groups with exact rows

$$
\begin{array}{ccccccc}
\mathrm{Ext}^{i-1}_{\mathcal{A}}(P, N) & \longrightarrow & \mathrm{Ext}^{i-1}_{\mathcal{A}}(K, N) & \longrightarrow & \mathrm{Ext}^i_{\mathcal{A}}(M, N) & \longrightarrow & \mathrm{Ext}^i_{\mathcal{A}}(P, N) \\
\downarrow{\scriptstyle\psi_1} & & \downarrow{\scriptstyle\psi_2} & & \downarrow{\scriptstyle\psi_3} & & \downarrow \\
\mathrm{Ext}^{i-1}_R(P, N) & \longrightarrow & \mathrm{Ext}^{i-1}_R(K, N) & \longrightarrow & \mathrm{Ext}^i_R(M, N) & \longrightarrow & \mathrm{Ext}^i_R(P, N).
\end{array}
$$

We wish to show that ψ_3 is an isomorphism. Since ψ_1 and ψ_2 are bijective by the inductive hypothesis, it will suffice to show that $\mathrm{Ext}^i_{\mathcal{A}}(P, N) \simeq 0 \simeq \mathrm{Ext}^i_R(P, N)$. The first equivalence follows from the fact that P is a projective object of $\mathcal{A}$. For the second, we observe that as an object of LMod_R, P coincides with a coproduct of copies of R (in virtue of assumption (1)). Consequently, $\mathrm{Ext}^i_R(P, N)$ can be identified with a product of copies of $\pi_{-i} N$, which vanishes since $i > 0$ and $N \in \mathrm{LMod}_R^{\geq 0}$.

Now suppose that $M \in \mathcal{A}$, and consider the full subcategory $\mathcal{C} \subseteq \mathcal{D}^-(\mathcal{A})$ spanned by those objects N for which the canonical map $\mathrm{Ext}^i_{\mathcal{D}^-(\mathcal{A})}(M, N) \to \mathrm{Ext}^i_R(\theta(M), \theta(N))$ is an isomorphism for all $i \in \mathbf{Z}$. Applying the five lemma to the relevant long exact sequences, we conclude that $\mathcal{C}$ is stable under extensions in $\mathcal{D}^-(\mathcal{A})$. The above argument shows that $\mathcal{C}$ contains the heart of $\mathcal{D}^-(\mathcal{A})$; it therefore contains the full subcategory $\mathcal{D}^b(\mathcal{A})$ of bounded object of $\mathcal{D}^-(\mathcal{A})$.

Now let $\mathcal{C}' \subseteq \mathcal{D}^-(\mathcal{A})$ spanned by those objects M having the property that for *every* $N \in \mathcal{D}^b(\mathcal{A})$, the canonical map $\mathrm{Ext}^i_{\mathcal{D}^-(\mathcal{A})}(M, N) \to \mathrm{Ext}^i_R(\theta(M), \theta(N))$ is an isomorphism for $i \in \mathbf{Z}$. Repeating the above argument, we conclude that $\mathcal{D}^b(\mathcal{A}) \subseteq \mathcal{C}'$. In particular, the restriction $\theta| \mathcal{D}^b(\mathcal{A})$ is fully faithful.

We claim that the essential image of $\theta| \mathcal{D}^b(\mathcal{A})$ consists of precisely the t-bounded objects of LMod_R. Let $M \in \mathrm{LMod}_R$ be a t-bounded object. We wish to prove that M belongs to the essential image of θ. Without loss of generality, we may suppose that $M \in \mathrm{LMod}_R^{\geq 0}$. Since M is t-bounded, we have also $M \in \mathrm{LMod}_R^{\leq n}$ for some $n \geq 0$. We now work by induction on n. If $n = 0$, then M belongs to the heart of LMod_R and the result is obvious. If $n > 0$, then we have a fiber sequence

$$\tau_{\geq n}M \to M \to \tau_{\leq n-1}M.$$

Since θ is exact and fully faithful, it will suffice to show that $\tau_{\geq n}M[-n]$ and $\tau_{\leq n-1}M$ belong to the essential image of θ, which follows from the inductive hypothesis.

The preceding argument shows that θ induces an equivalence $\mathcal{D}^b(\mathcal{A}) \to \mathrm{LMod}_R^b$ between the full subcategories of bounded objects. We now conclude by observing that both $\mathcal{D}^-(\mathcal{A})$ and LMod_R are left complete. $\quad\square$

Remark 8.1.1.16. Let R and $\mathcal{A}$ be as in Proposition 8.1.1.15, and assume that R is discrete. Let $\mathcal{D}(\mathcal{A})$ be the unbounded derived ∞-category of $\mathcal{A}$ (Definition 1.3.5.8), so that Proposition 1.3.5.24 allows us to identify $\mathcal{D}^-(\mathcal{A})$ with the full subcategory of $\mathcal{D}(\mathcal{A})$ spanned by the right bounded objects. Since $\mathcal{D}(\mathcal{A})$ and LMod_R are both right complete (Propositions 1.3.5.21 and 8.1.1.13), the fully faithful embedding $\mathcal{D}(\mathcal{A})^- \to \mathrm{LMod}_R$ induces an equivalence of ∞-categories $\mathcal{D}(\mathcal{A}) \simeq \mathrm{LMod}_R$. In other words, the ∞-category of left R-module spectra can be identified with the derived ∞-category of the abelian category of (discrete) R-modules.

8.1.2 Recognition Principles

Let R be a commutative ring and let $\mathcal{A}$ denote the abelian category of (discrete) R-modules. We will regard R as a discrete E_∞-ring, and let Mod_R denote the ∞-category of R-module spectra as in Notation 8.1.1.1. According to Remark 8.1.1.16, we can identify Mod_R with the derived ∞-category $\mathcal{D}(\mathcal{A})$ of chain complexes of R-modules. Since R is commutative, we can regard Mod_R as a symmetric monoidal ∞-category. In this section, we will show that the symmetric monoidal structure on Mod_R is determined by the symmetric monoidal structure on the ordinary category $\mathrm{Ch}(\mathcal{A})$ of chain complexes with values in $\mathcal{A}$.

We begin with a much more general question. Given a stable ∞-category $\mathcal{C}$ (such as the derived ∞-category $\mathcal{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$), under what circumstances can $\mathcal{C}$ be realized as the ∞-category RMod_R of right modules over an $\mathbb{E}_1$-ring R? This question is addressed by the following result of Schwede and Shipley:

Theorem 8.1.2.1. *[Schwede-Shipley [128]] Let $\mathcal{C}$ be a stable ∞-category. Then $\mathcal{C}$ is equivalent to RMod_R, for some $\mathbb{E}_1$-ring R, if and only if $\mathcal{C}$ is presentable and there exists a compact object $C \in \mathcal{C}$ which generates $\mathcal{C}$ in the following sense: if $D \in \mathcal{C}$ is an object having the property that $\mathrm{Ext}_{\mathcal{C}}^n(C, D) \simeq 0$ for all $n \in \mathbf{Z}$, then $D \simeq 0$.*

Proof. Suppose first that $\mathcal{C} \simeq \mathrm{RMod}_R$, and let $C = R$ (regarded as a left module over itself). Then $\mathcal{C}$ is presentable, C is a compact object of $\mathcal{C}$, and $\mathrm{Ext}_{\mathcal{C}}^n(C, D) \simeq \pi_{-n}D$ for every object $D \in \mathcal{C}$. It follows that $D \simeq 0$ if and only if $\mathrm{Ext}_{\mathcal{C}}^n(C, D) \simeq 0$ for all integers n, so that C generates $\mathcal{C}$.

Conversely, suppose that $\mathcal{C}$ is presentable and let $C \in \mathcal{C}$ be a compact generator. Let Pr^{L} denote the symmetric monoidal ∞-category of presentable ∞-categories, so that we can regard $\mathcal{C}$ as a right module over the ∞-category Sp (see Proposition 6.3.2.18). We will complete the proof by showing that the pair $(\mathcal{C}, C)$ lies in the image of the fully faithful embedding $\mathrm{Alg}(\mathrm{Sp}) \to \mathrm{Mod}_{\mathrm{Sp}}(\mathrm{Pr}^{\mathrm{L}})_{\mathrm{Sp}/}$ of Proposition 6.3.5.7. Corollary T.5.5.2.9 guarantees that the functor $F : \mathrm{Sp} \to \mathcal{C}$ given by $X \mapsto X \otimes C$ admits a right adjoint G. According to Proposition 6.3.5.7, it will suffice to show the following:

(a) The functor G preserves geometric realizations of simplicial objects.

(b) The functor G is conservative.

(c) For every object $D \in \mathcal{C}$ and every spectrum $X \in \mathrm{Sp}$, the canonical map $\theta_X : X \otimes G(D) \to G(X \otimes D)$ is an equivalence of spectra.

To prove (a), it suffices to show that G preserves all small colimits. Since G is exact, this is equivalent to the requirement that G preserves small filtered colimits, which follows from our assumption that C is compact. To prove (b), suppose we are given a map $\alpha : D \to D'$ such that $G(\alpha)$ is an equivalence. Let D'' be the cofiber of α. Since G is exact, we deduce that $G(D'') \simeq 0$, so that $\pi_n G(D'') \simeq \mathrm{Ext}_{\mathcal{C}}^{-n}(C, D'')$ vanishes for every integer n. Our assumption that C generates $\mathcal{C}$ implies that $D'' \simeq 0$, so that α is an equivalence.

It remains to prove (c). Fix an object $D \in \mathcal{C}$, and let $\mathcal{X} \subseteq \mathrm{Sp}$ be the full subcategory spanned by those spectra X for which the map θ_X is an equivalence. Since G preserves small colimits, the ∞-category $\mathcal{X} \subseteq \mathrm{Sp}$ is stable under small colimits. To prove that $\mathcal{X} = \mathrm{Sp}$, it suffices to show that $S[n] \in \mathcal{X}$ for every integer n, where S denotes the sphere spectrum. Since the functor G is exact, we can reduce to the case $n = 0$, where the result is obvious. $\square$

Remark 8.1.2.2. Let $\mathcal{C}$ be a stable ∞-category, and let $X \in \mathcal{C}$ be an object. Then it is possible to extract from $\mathcal{C}$ an $\mathbb{E}_1$-ring spectrum $\mathrm{End}_{\mathcal{C}}(X)$ with the property that $\pi_n \mathrm{End}_{\mathcal{C}}(X) \simeq \mathrm{Ext}_{\mathcal{C}}^{-n}(X, X)$ for all $n \in \mathbf{Z}$, and the ring structure on $\pi_* \mathrm{End}_{\mathcal{C}}(X)$ is given by composition in the triangulated category $h\mathcal{C}$. We will describe the argument in the case where $\mathcal{C}$ is presentable (the general case can be reduced to this case by first replacing $\mathcal{C}$ by a small subcategory which contains X, and then enlarging $\mathcal{C}$ by formally adjoining filtered colimits). According to Remark 6.3.2.20, the ∞-category $\mathcal{C}$ is naturally left-tensored over Sp. Proposition 4.2.1.33 implies that $\mathcal{C}$ is also enriched over Sp, so that there exists a morphism object $\mathrm{Mor}_{\mathcal{C}}(X, X)$. The object $\mathrm{End}_{\mathcal{C}}(X) = \mathrm{Mor}_{\mathcal{C}}(X, X)$ can be lifted to a final object of the monoidal ∞-category $\mathcal{C}^+[X]$, and can therefore be lifted to $\mathrm{Alg}(\mathrm{Sp}) \simeq \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$. The identification of the homotopy groups of $\mathrm{End}_{\mathcal{C}}(X)$ follows from the homotopy equivalence $\mathrm{Map}_{\mathrm{Sp}}(S[n], \mathrm{End}_{\mathcal{C}}(X)) \simeq \mathrm{Map}_{\mathcal{C}}(S[n] \otimes X, X)$.

Remark 8.1.2.3. Let $\mathcal{C}$ be a presentable stable ∞-category containing an object C. The $\mathbb{E}_1$-ring R appearing in the proof of Theorem 8.1.2.1 can be identified with the endomorphism algebra $\mathrm{End}_{\mathcal{C}}(C)$ described in Remark 8.1.2.2.

In the situation of Theorem 8.1.2.1, the $\mathbb{E}_1$-ring R is determined up to equivalence by the pair $(\mathcal{C}, C)$, but not by the ∞-category $\mathcal{C}$ alone. As in classical Morita theory, an equivalence between module categories RMod_R and $\mathrm{RMod}_{R'}$ need not result from an equivalence between R and R'. However, every equivalence between RMod_R and $\mathrm{RMod}_{R'}$ is obtained by tensor product with a suitable R-R'-bimodule spectrum. In fact, we have the following more general result:

Proposition 8.1.2.4. *Let R and R' be $\mathbb{E}_1$-rings, and let $\mathrm{Fun}^{\mathrm{L}}(\mathrm{RMod}_R, \mathrm{RMod}_{R'})$ be the ∞-category of functors from RMod_R to $\mathrm{RMod}_{R'}$ which preserve small colimits. Then the relative tensor product functor $\otimes_R : \mathrm{RMod}_R \times_R \mathrm{BMod}_{R'}(\mathrm{Sp}) \to \mathrm{RMod}_{R'}$ induces an equivalence of ∞-categories*

$$_R\mathrm{BMod}_{R'}(\mathrm{Sp}) \to \mathrm{Fun}^{\mathrm{L}}(\mathrm{RMod}_R, \mathrm{RMod}_{R'}).$$

Proof. Combine Proposition 6.3.2.18, Theorem 6.3.4.1, and Theorem 4.3.2.7. $\square$

If we wish to recover the $\mathbb{E}_1$-ring from the ∞-category RMod_R of right R-modules, we should consider not only RMod_R but also the distinguished object R (regarded as a right module over itself), whose endomorphism algebra can be identified with R. More generally, for any $k \geq 1$, we can recover an $\mathbb{E}_k$-ring R from the ∞-category of right R-modules, regarded as an $\mathbb{E}_{k-1}$-monoidal ∞-category. Before stating the precise result, we need to introduce a bit of terminology.

Notation 8.1.2.5. Fix a section s of the trivial Kan fibration $\mathbb{E}_1^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$ of Example 5.1.0.7, so that composition with s determines a map of ∞-operads $\mathcal{RM}^{\otimes} \to \mathbb{E}_1^{\otimes}$ and therefore a bifunctor of ∞-operads $\mathcal{RM}^{\otimes} \times \mathbb{E}_{k-1}^{\otimes} \to \mathbb{E}_k^{\otimes}$ for each $k > 0$. If $\mathcal{C}$ is an $\mathbb{E}_k$-monoidal ∞-category and $R \in \mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})$, then we let $\mathrm{RMod}_R(\mathcal{C})$ denote the fiber product $\mathrm{Alg}_{\mathcal{RM}/\mathbb{E}_k}(\mathcal{C}) \times_{\mathrm{Alg}_{\mathcal{A}\mathrm{ss}/\mathbb{E}_k}} \{R\}$. If we assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and that the tensor product on $\mathcal{C}$ preserves geometric realizations of simplicial objects, then the constructions of §6.3.3 show that $\mathrm{RMod}_R(\mathcal{C})$ inherits the structure of an $\mathbb{E}_{k-1}$-monoidal ∞-category.

Proposition 8.1.2.6. *Let $k \geq 1$. The construction $R \mapsto \mathrm{RMod}_R^{\otimes}$ determines a fully faithful embedding from the ∞-category $\mathrm{Alg}^{(k)}$ of $\mathbb{E}_k$-rings to the ∞-category $\mathrm{Alg}_{\mathbb{E}_{k-1}}(\mathrm{Pr}^{\mathrm{L}})$ of $\mathbb{E}_{k-1}$-monoidal presentable ∞-categories. An $\mathbb{E}_{k-1}$-monoidal ∞-category $\mathcal{C}^{\otimes} \to \mathbb{E}_{k-1}^{\otimes}$ belongs to the essential image of this embedding if and only if the following conditions are satisfied:*

(1) *The ∞-category $\mathcal{C}$ is stable and presentable, and if $k > 1$ then the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable.*

(2) *The unit object $\mathbf{1} \in \mathcal{C}$ is compact.*

(3) *The object $\mathbf{1}$ generates $\mathcal{C}$ in the following sense: if $C \in \mathcal{C}$ is an object such that $\mathrm{Ext}_{\mathcal{C}}^i(\mathbf{1}, C) \simeq 0$ for all integers i, then $C \simeq 0$.*

Proof. The full faithfulness follows from Corollary 6.3.5.17 and Proposition 6.3.2.18. The description of the essential image follows as in the proof of Theorem 8.1.2.1.   □

Proposition 8.1.2.6 is also valid (with the same proof) in the the limiting case $k = \infty$:

Proposition 8.1.2.7. *The construction $R \mapsto \mathrm{Mod}_R^{\otimes}$ determines a fully faithful embedding from the ∞-category CAlg of $\mathbb{E}_{\infty}$-rings to the ∞-category $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ of presentable symmetric monoidal ∞-categories. A symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ belongs to the essential image of this embedding if and only if the following conditions are satisfied:*

(1) *The ∞-category $\mathcal{C}$ is stable and presentable and the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable.*

(2) *The unit object $\mathbf{1} \in \mathcal{C}$ is compact.*

(3) *The object $\mathbf{1}$ generates $\mathcal{C}$ in the following sense: if $C \in \mathcal{C}$ is an object such that $\mathrm{Ext}_{\mathcal{C}}^i(\mathbf{1}, C) \simeq 0$ for all integers i, then $C \simeq 0$.*

Our next goal is to use Proposition 8.1.2.7 to address the question raised at the beginning of this section. Suppose that R is a commutative ring, and let $\mathcal{A}$ be the category of chain complexes of R-modules. We will show that the equivalence of ∞-categories $\mathcal{D}(\mathcal{A}) \simeq \mathrm{Mod}_R$ provided by Remark 8.1.1.16 can be promoted to an equivalence of symmetric monoidal ∞-categories. To formulate this result more precisely, we need to define a suitable symmetric monoidal structure on $\mathcal{D}(\mathcal{A})$. Roughly speaking, it is given by the tensor product of chain complexes over R. To analyze this tensor product more explicitly, it is convenient to introduce an appropriate model structure on $\mathrm{Ch}(\mathcal{A})$.

Proposition 8.1.2.8. *Let R be an associative ring and let $\mathcal{A}$ be the abelian category of (discrete) right R-modules. Then the category $\mathrm{Ch}(\mathcal{A})$ admits a left proper combinatorial model structure, which can be described as follows:*

(W) *A map of chain complexes $f : M_* \to N_*$ is a weak equivalence if it is a quasi-isomorphism: that is, if it induces an isomorphism on homology.*

(F) *A map of chain complexes $f : M_* \to N_*$ is a fibration if each of the maps $M_i \to N_i$ is surjective.*

(C) *A map of chain complexes $f : M_* \to N_*$ is a cofibration if and only if it has the left lifting property with respect to every map g which is simultaneously a fibration and a weak equivalence.*

Proof. For every integer n, we let $E(n)_*$ denote the chain complex

$$\cdots \to 0 \to R \xrightarrow{\mathrm{id}} R \to 0 \to \cdots$$

which is nontrivial only in degrees n and $n - 1$, and we let $\partial E(n)_*$ denote the subcomplex consisting of the module R concentrated in degree $n - 1$. Let C_0 be the collection of all monomorphisms of chain complexes

$\{\partial\, E(n)_* \hookrightarrow E(n)_*\}_{n\in\mathbf{Z}}$. Let C be the smallest weakly saturated collection of morphisms containing C_0. We first show that that there is a model structure on $\mathrm{Ch}(\mathcal{A})$ whose class of cofibrations is given by C and whose weak equivalences are quasi-isomorphisms. For this, it suffices to verify the hypotheses of Proposition T.A.2.6.13.

Note that every morphism in C is a cofibration with respect to the model structure of Proposition 1.3.5.3. It follows from Proposition 1.3.5.3 that the class of weak equivalences in $\mathrm{Ch}(\mathcal{A})$ is perfect and is stable under pushouts by morphisms in C. It therefore suffices to show that if $f : M_* \to N_*$ is a morphism which has the right lifting property with respect to every morphism in C_0, then f is a quasi-isomorphism. Unwinding the definitions, our assumption guarantees that if $x \in N_i$ and $y \in M_{i-1}$ satisfy $f(y) = dx$, then there exists an element $\overline{x} \in M_i$ with $d\overline{x} = y$ and $f(\overline{x}) = x$. Taking x to be an arbitrary cycle and $y = 0$, we deduce that f induces a surjection $\mathrm{H}_i(M) \to \mathrm{H}_i(N)$. To prove injectivity, choose a homology class $\eta \in \mathrm{H}_i(M)$ represented by $y \in M_i$, and suppose that the image of η in $\mathrm{H}_i(N)$ vanishes. Then $f(y) = dx$ for some $x \in M_{i+1}$, and our hypothesis guarantees the existence of an element $\overline{x} \in M_{i+1}$ with $d\overline{x} = 0$, so that $\eta = 0$ as desired.

To complete the proof, it suffices to show that the model structure we have constructed satisfies the requirements of Proposition 8.1.2.8: that is, we must show that a map $f : M_* \to N_*$ is a fibration if and only if it is degreewise surjective. Assume first that f is a fibration. We wish to show that for every integer n, f has the right lifting property with respect to the inclusion $g : 0 \hookrightarrow E(n)_*$. The map g is clearly a quasi-isomorphism. Moreover, g can be obtained as a composition of maps

$$0 \xrightarrow{g'} \partial\, E(n)_* \xrightarrow{g''} E(n)_*$$

where g'' belongs to C_0 and g' is a pushout of the morphism $\partial\, E(n-1)_* \to E(n-1)_*$ belonging to C_0. It follows that g is a trivial cofibration, so that f has the right lifting property with respect to g by virtue of our assumption that f is a fibration.

Now suppose that $f : M_* \to N_*$ is degreewise surjective; we wish to show that f is a fibration. Let $g : P_* \to Q_*$ be a trivial cofibration in $\mathrm{Ch}(\mathcal{A})$. For every pair of chain complexes of right R-modules X_* and Y_*, let $\mathrm{Map}(X_*, Y_*)$ be the chain complex of abelian groups given by Definition 1.3.2.1. We wish to show that the map

$$\phi : \mathrm{Map}(Q_*, M_*)_* \to \mathrm{Map}(P_*, M_*)_* \times_{\mathrm{Map}(P_*, N_*)_*} \mathrm{Map}(Q_*, N_*)_*$$

is surjective on 0-cycles. Since g is a trivial cofibration, each of the maps $P_n \to Q_n$ is a split monomorphism, and each quotient $F_n = Q_n/P_n$ is a projective right R-module. We therefore obtain a diagram of exact sequences (of chain complexes of abelian groups)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Map}(F_*, M_*)_* & \longrightarrow & \mathrm{Map}(Q_*, M_*)_* & \longrightarrow & \mathrm{Map}(P_*, M_*)_* & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle\theta} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Map}(F_*, N_*)_* & \longrightarrow & \mathrm{Map}(Q_*, N_*)_* & \longrightarrow & \mathrm{Map}(P_*, N_*)_*. & \longrightarrow & 0
\end{array}
$$

Since F_* is degreewise projective and f is degreewise surjective, the map θ is an epimorphism. It follows from a diagram chase that the map ϕ degreewise surjective, and that $\ker(\phi) \simeq \ker(\theta)$. Let $K_* = \ker(\phi)$, so that $K_* \simeq \mathrm{Map}(F_*, \ker(f))$. Let x be a 0-cycle in $\mathrm{Map}(P_*, M_*)_* \times_{\mathrm{Map}(P_*, N_*)_*} \mathrm{Map}(Q_*, N_*)_*$, and write $x = \phi(\widetilde{x})$ for $\widetilde{x} \in \mathrm{Map}(Q_*, M_*)_0$. Then $d\widetilde{x}$ is a (-1)-cycle of K. If we can write $d\widetilde{x} = dy$ for some $y \in K_0$, then $\widetilde{x} - y$ is a 0-cycle of $\mathrm{Map}(Q_*, M_*)$ lifting x. It will therefore suffice to show that the chain complex K_* is acyclic. For this, it suffices to show that the chain complex F_* has a contracting homotopy.

We will prove the following more general assertion: for every cofibrant object $Z_* \in \mathrm{Ch}(\mathcal{A})$ and every map of chain complexes $u : Z_* \to F_*$, where exists a nullhomotopy for u: that is, a collection of maps $h : Z_m \to F_{m+1}$ satisfying $dh + hd = u$. Taking u to be the identity map id_{F_*}, we will obtain the desired result. Without loss of generality we may assume that Z_* is the colimit of a transfinite sequence of chain complexes

$$0 = Z(0)_* \to Z(1)_* \to Z(2)_* \to \cdots$$

where each of the maps $Z(\alpha)_* \to Z(\alpha+1)_*$ is the pushout of an inclusion $\partial E(n_\alpha)_* \hookrightarrow E(n_\alpha)_*$. We construct a compatible family of maps $h_\alpha : Z(\alpha)_m \to F_{m+1}$ satisfying $dh_\alpha + h_\alpha d = u|Z(\alpha)_m$ using induction on α. When α is a limit ordinal (including the case $\alpha = 0$), there is nothing to prove. Let us therefore assume that h_α has been constructed, and explain how to define $h_{\alpha+1}$. By assumption, the chain complex $Z(\alpha+1)_*$ is freely generated by $Z(\alpha)_*$ together with an additional element x in degree n_α, satisfying $dx = y \in Z(\alpha)_{n_\alpha-1}$. The element $u(x) - h_\alpha(y) \in F_{n_\alpha}$ satisfies

$$d(u(x) - h_\alpha(y)) = u(dx) - dh_\alpha(y) - h_\alpha(dy) = u(y) - u(y) = 0,$$

so that $u(x) - h_\alpha(y)$ is a cycle. Since $g : P_* \to Q_*$ is a trivial cofibration, the chain complex F_* is acyclic. We may therefore find an element $z \in F_{n_\alpha+1}$ with $dz = u(x) - h_\alpha(y)$. We now complete the construction by defining $h_{\alpha+1}$ so that $h_{\alpha+1}(x) = z$ and $h_{\alpha+1}|Z(\alpha)_* = h_\alpha$. □

Remark 8.1.2.9. Let R be an associative ring and $\mathcal{A}$ the abelian category of (discrete) R-modules. We will refer to the model structure on $\mathrm{Ch}(\mathcal{A})$ described in Proposition 8.1.2.8 as the *projective model structure* on the category $\mathrm{Ch}(\mathcal{A})$. It generally does *not* agree with the model structure of Proposition 1.3.5.3. However, these model structures have the same weak equivalences. It follows that the underlying ∞-category of $\mathrm{Ch}(\mathcal{A})$ does not depend on which model structure we consider (it is the ∞-category obtained from $\mathrm{Ch}(\mathcal{A})$ by formally inverting all weak equivalences). We will denote this ∞-category by $\mathcal{D}(\mathcal{A})$ in what follows.

Remark 8.1.2.10. Let R be a field (not necessarily commutative). Then for every monomorphism $M \to N$ of right R-modules, the inclusion of the chain complex

$$\cdots \to 0 \to M \to N \to 0 \to \cdots$$

into

$$\cdots \to 0 \to N \xrightarrow{\mathrm{id}} N \to 0 \to \cdots$$

is a pushout of coproducts of generating cofibrations appearing in Proposition 8.1.2.8. It follows that the model structures of Proposition 1.3.5.3 and 8.1.2.8 coincide. In particular, every object of $\mathrm{Ch}(\mathcal{A})$ is cofibrant with respect to the projective model structure.

Proposition 8.1.2.11. *Let R be a commutative ring, let $\mathcal{A}$ denote the abelian category of (discrete) R-modules, and regard $\mathrm{Ch}(\mathcal{A})$ as a symmetric monoidal category via the tensor product of chain complexes (see Remark 1.2.3.21). Then $\mathrm{Ch}(\mathcal{A})$ is a symmetric monoidal model category, with respect to the projective model structure of Proposition 8.1.2.8.*

Proof. It is easy to see that the unit object of $\mathrm{Ch}(\mathcal{A})$ (given by the module R, considered as a chain complex concentrated in degree zero) is cofibrant. Suppose we are given cofibrations $f : M_* \to M'_*$ and $g : N_* \to N'_*$; we must show that the induced map

$$f \wedge g : (M_* \otimes N'_*) \coprod_{M_* \otimes N_*} (M'_* \otimes N_*) \to M'_* \otimes N'_*$$

is a cofibration, which is trivial if either f or g is trivial. We first show that $f \wedge g$ is a cofibration. Without loss of generality, we may assume that both f and g are generating cofibrations, having the form

$$\partial E(m)_* \to E(m)_* \qquad \partial E(n)_* \to E(n)_*$$

for some integers m and n (for an explanation of this notation, see the proof of Proposition 8.1.2.8). Unwinding the definitions, we see that $f \wedge g$ is a pushout of the generating cofibration $\partial E(m+n)_* \to E(m+n)_*$, and therefore a cofibration.

Now suppose that f is a trivial cofibration; we wish to show that $f \wedge g$ is a trivial cofibration. If we regard f as fixed, then the collection of morphisms g for which $f \wedge g$ is a trivial cofibration is weakly saturated. We may therefore assume that g is a generating trivial cofibration of the form $\partial E(n)_* \to E(n)_*$. In this case, the map $f \wedge g$ is an injection whose cokernel is isomorphic (after a shift) to the cokernel of f. Since f is a quasi-isomorphism which is degreewise injective, the chain complex $\mathrm{coker}(f)$ is acyclic, so that $f \wedge g$ is also a quasi-isomorphism. □

Remark 8.1.2.12. Combining Proposition 8.1.2.11 with Example 4.1.3.6, we conclude that if $\mathcal{A}$ is the abelian category of modules over a commutative ring R, then the derived ∞-category $\mathcal{D}(\mathcal{A})$ inherits a symmetric monoidal structure. This symmetric monoidal structure is determined uniquely (up to equivalence) by the requirement that the functor $\mathrm{N}(\mathrm{Ch}(\mathcal{A})^{\mathrm{o}}) \to \mathcal{D}(\mathcal{A})$ can be promoted to a symmetric monoidal functor; here $\mathrm{Ch}(\mathcal{A})^{\mathrm{o}}$ denotes the full subcategory of $\mathrm{Ch}(\mathcal{A})$ spanned by those objects which are cofibrant with respect to the model structure of Proposition 8.1.2.8.

We are now ready to address the question raised at the beginning of this section.

Theorem 8.1.2.13. *Let R be a commutative ring, let $\mathcal{A}$ denote the abelian category of R-modules, and regard R as a discrete $\mathbb{E}_\infty$-ring. Then there is a canonical equivalence of symmetric monoidal ∞-categories $\mathrm{Mod}_R \to \mathcal{D}(\mathcal{A})$; here we regard $\mathcal{D}(\mathcal{A})$ as a symmetric monoidal ∞-category as in Remark 8.1.2.12.*

Proof. The ∞-category $\mathcal{D}(\mathcal{A})$ is presentable by Proposition 1.3.4.22, and the tensor product on $\mathcal{D}(\mathcal{A})$ preserves colimits separately in each variable since it is induced by a left Quillen bifunctor. The ∞-category $\mathcal{D}(\mathcal{A})$ is stable by Remark 8.1.2.9 and the results of §1.3.2. Let R be the unit object of $\mathcal{D}(\mathcal{A})$. For any $M_* \in \mathcal{D}(\mathcal{A})$, we have canonical equivalences $\mathrm{H}_n(M) \simeq \mathrm{Ext}_{\mathcal{D}(\mathcal{A})}^{-n}(R, M_*)$. It follows that R is a compact generator for $\mathcal{D}(\mathcal{A})$, so that Proposition 8.1.2.7 yields a symmetric monoidal equivalence $\mathcal{D}(\mathcal{A}) \simeq \mathrm{Mod}_A$ for some $\mathbb{E}_\infty$-ring A. Here we can regard A as the endomorphism algebra of $R \in \mathcal{D}(\mathcal{A})$, so that

$$\pi_n A \simeq \mathrm{Ext}_{\mathcal{D}(\mathcal{A})}^{-n}(R, R) \simeq \begin{cases} R & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we deduce that A is a discrete $\mathbb{E}_\infty$-ring which can be identified with R, so that we have a symmetric monoidal equivalence $\mathcal{D}(\mathcal{A}) \simeq \mathrm{Mod}_R$. $\square$

8.1.3 Change of Ring

Let $\mathcal{A}b$ denote the category of abelian groups and let $R \in \mathrm{CAlg}(\mathcal{A}b)$ be a commutative ring. Suppose that A is an R-algebra: that is, an associative ring equipped with a map $\phi : R \to A$ whose image is contained in the center of A. Suppose that M is a left A-module. We may then regard M as an R-module (via the homomorphism ϕ). The action of A on M is determined by a map of abelian groups $\phi : A \otimes M \to M$. This map is R-bilinear: for every triple of elements $a \in A$, $x \in M$, and $\lambda \in R$, we have

$$\lambda(ax) = (\lambda a)x = (a\lambda)x = a(\lambda x).$$

It follows that the map ϕ factors through the relative tensor product $A \otimes_R M$. We can rephrase this statement more categorically as follows: the associative ring A can be regarded as an associative algebra object of the category $\mathrm{Mod}_R(\mathcal{A}b)$ of R-modules, and M can be regarded as a left A-module object of the category $\mathrm{Mod}_R(\mathcal{A}b)$.

We begin this section by generalizing this observation to the ∞-categorical setting. Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $R \in \mathrm{CAlg}(\mathcal{C})$ be a commutative algebra object of $\mathcal{C}$. Under some mild assumptions, the ∞-category $\mathrm{Mod}_R(\mathcal{C})$ of R-module objects of $\mathcal{C}$ inherits the structure of a symmetric monoidal ∞-category. Moreover, the forgetful functor $\mathrm{Mod}_R(\mathcal{C}) \to \mathcal{C}$ is lax symmetric monoidal. It follows that every algebra object $A \in \mathrm{Alg}(\mathrm{Mod}_R(\mathcal{C}))$ determines an algebra $A' \in \mathrm{Alg}(\mathcal{C})$, and we have a forgetful functor $\mathrm{LMod}_A(\mathrm{Mod}_R(\mathcal{C})) \to \mathrm{LMod}_{A'}(\mathcal{C})$. We will show that this forgetful functor is an equivalence of ∞-categories. In fact, we do not even need to assume that the tensor product on $\mathcal{C}$ is fully commutative.

Theorem 8.1.3.1. *Let $\mathcal{C}$ be an $\mathbb{E}_2$-monoidal ∞-category. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and that the tensor product on $\mathcal{C}$ preserves geometric realizations separately in each variable. Let $R \in \mathrm{Alg}_{/\mathbb{E}_2}(\mathcal{C}) \simeq \mathrm{Alg}_{/\mathbb{E}_1}(\mathrm{Alg}_{/\mathbb{E}_1}(\mathcal{C}))$, so that the ∞-category $\mathrm{LMod}_R(\mathcal{C})$ inherits an $\mathbb{E}_1$-monoidal structure. For every algebra object $A \in \mathrm{Alg}_{/\mathbb{E}_1}(\mathrm{LMod}_R(\mathcal{C}))$ having image $A' \in \mathrm{Alg}_{\mathbb{E}_1/\mathbb{E}_2}(\mathcal{C})$, the forgetful functor $\theta : \mathrm{LMod}_A(\mathrm{LMod}_R(\mathcal{C})) \to \mathrm{LMod}_{A'}(\mathcal{C})$ is an equivalence of ∞-categories.*

Remark 8.1.3.2. With a bit more care, the conclusion of Theorem 8.1.3.1 remains valid for an arbitrary fibration of ∞-operads $p : \mathcal{C}^{\otimes} \to \mathbb{E}_2^{\otimes}$: we do not need to assume that p is a coCartesian fibration or that p is compatible with $N(\mathbf{\Delta})^{op}$-indexed colimits. This more general statement can be reduced to the statement of Theorem 8.1.3.1 by first replacing p by its $\mathbb{E}_2$-monoidal envelope (see §2.2.4) and then using Proposition 6.3.1.10.

Proof. In what follows, it will be convenient to think of the $\mathbb{E}_2$-monoidal structure on $\mathcal{C}$ as giving determining two tensor product operations $\otimes, \boxtimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, which are coherently associative and commute with one another. The operations $\otimes$ and $\boxtimes$ can be identified with one another by the Eckmann-Hilton argument (see Example 5.1.2.4), but the argument will be clearer if we do not exploit this.

We have a diagram of forgetful functors

$$
\begin{array}{ccc}
\mathrm{LMod}_A(\mathrm{LMod}_R(\mathcal{C})) & \xrightarrow{\quad\theta\quad} & \mathrm{LMod}_{A'}(\mathcal{C}) \\
& \searrow{\scriptstyle G} \qquad \swarrow{\scriptstyle G'} & \\
& \mathcal{C} &
\end{array}
$$

To show that θ is an equivalence of ∞-categories, it will suffice to show that this diagram satisfies the hypotheses of Corollary 6.2.2.14. Using Corollaries 4.2.3.2 and 4.2.3.5, we deduce that G and G' are conservative and preserve geometric realizations of simplicial objects. Corollary 4.2.4.8 implies that G' admits a left adjoint F', given informally by $M \mapsto A' \otimes M$. The functor G factors as a composition

$$
\mathrm{LMod}_A(\mathrm{LMod}_R(\mathcal{C})) \xrightarrow{G_1} \mathrm{LMod}_R(\mathcal{C}) \xrightarrow{G_2} \mathcal{C} \,.
$$

Using Corollary 4.2.4.8 again, we deduce that these functors admit left adjoints F_1 and F_2. The functor F_2 is given by $M \mapsto R \boxtimes M$, and Lemma 6.3.3.15 shows that the functor F_1 is given by $N \mapsto R \boxtimes_{R \otimes R} (A \otimes N)$. It follows that G admits a left adjoint $F = F_1 \circ F_2$. To complete the proof, it will suffice to show that for each $M \in \mathcal{C}$, the canonical map $F'(M) \to \theta F(M)$ is an equivalence in $\mathrm{LMod}_{A'}(\mathcal{C})$. Unwinding the definitions, we must show that the canonical map

$$
\alpha : A \otimes M \to R \boxtimes_{R \otimes R} (A \otimes (R \boxtimes N))
$$

is an equivalence in $\mathcal{C}$. We note that α factors as a composition of equivalences

$$
A \otimes M \simeq R \boxtimes_R (A \otimes M) \simeq R \boxtimes_{R \otimes \mathbf{1}} (A \otimes (\mathbf{1} \boxtimes M)) \simeq R \boxtimes_{R \otimes R} (A \otimes (R \boxtimes M)).
$$

$\square$

The equivalences in Theorem 8.1.3.1 are compatible with the formation of relative tensor products:

Proposition 8.1.3.3. *Let $\mathcal{C}$ be an $\mathbb{E}_2$-monoidal ∞-category. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and that the tensor product on $\mathcal{C}$ preserves geometric realizations separately in each variable. Let $R \in \mathrm{Alg}_{/\mathbb{E}_2}(\mathcal{C}) \simeq \mathrm{Alg}_{/\mathbb{E}_1}(\mathrm{Alg}_{/\mathbb{E}_1}(\mathcal{C}))$, so that the ∞-category $\mathrm{LMod}_R(\mathcal{C})$ inherits an $\mathbb{E}_1$-monoidal structure. Let $\theta : \mathrm{LMod}_R(\mathcal{C}) \to \mathcal{C}$ denote the forgetful functor. Suppose we are given an algebra object $A \in \mathrm{Alg}_{/\mathbb{E}_1}(\mathrm{LMod}_R(\mathcal{C}))$, and let A' be the image of A in $\mathrm{Alg}_{\mathbb{E}_1/\mathbb{E}_2}(\mathcal{C})$, so that θ induces forgetful functors $\theta_L : \mathrm{LMod}_A(\mathrm{LMod}_R(\mathcal{C})) \to \mathcal{C}$ and $\theta_R : \mathrm{RMod}_A(\mathrm{LMod}_R(\mathcal{C})) \to \mathrm{RMod}_A(\mathcal{C})$ (which are equivalences of ∞-categories by Proposition 8.1.3.1). For every pair of objects $M \in \mathrm{RMod}_A(\mathrm{LMod}_R(\mathcal{C}))$, $N \in \mathrm{LMod}_A(\mathrm{LMod}_R(\mathcal{C}))$, the canonical map*

$$
\phi_{M,N} : \theta_R(M) \otimes_{A'} \theta_L(N) \to \theta(M \otimes_A N)
$$

is an equivalence in $\mathcal{C}$.

Proof. Let us regard $N \in \mathrm{LMod}_A(\mathrm{LMod}_R(\mathcal{C}))$ as fixed, and let $\mathfrak{X} \subseteq \mathrm{RMod}_A(\mathrm{LMod}_R(\mathcal{C}))$ be the full subcategory spanned by those objects for which the map ϕ is an equivalence. Since the forgetful functor θ and the relative tensor product functors commute with geometric realization, we conclude that $\mathfrak{X}$ is stable under geometric realizations in $\mathrm{RMod}_A(\mathrm{LMod}_R(\mathcal{C}))$. Using Proposition 6.2.2.12, we are reduced to proving that $\mathfrak{X}$ contains the essential image of the functor F appearing in the proof of Theorem 8.1.3.1. Unwinding the definitions, we must show that if $M_0 \in \mathcal{C}$, then the canonical map

$$(M_0 \otimes A') \otimes_{A'} \theta_L(N) \to \theta((M_0 \otimes A) \otimes_A N)$$

is an equivalence. This is clear, since both sides can be identified with the absolute tensor product $M_0 \otimes N$. $\square$

Corollary 8.1.3.4. *Let $k \geq 1$ be an integer and let $\mathcal{C}$ be an $\mathbb{E}_{k+1}$-monoidal ∞-category. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and that the tensor product on $\mathcal{C}$ preserves geometric realizations of simplicial objects separately in each variable. Let R be an $\mathbb{E}_{k+1}$-algebra in $\mathcal{C}$, let $A \in \mathrm{Alg}_{/\mathbb{E}_k}(\mathrm{LMod}_R(\mathcal{C}))$, and let A' denote the image of A in $\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})$. Then the forgetful functor*

$$\mathrm{LMod}_A(\mathrm{LMod}_R(\mathcal{C})) \to \mathrm{LMod}_{A'}(\mathcal{C})$$

is an equivalence of $\mathbb{E}_{k-1}$-monoidal ∞-categories.

Proof. Combine Theorem 8.1.3.1, Proposition 8.1.3.3, and Theorem 4.3.4.28. $\square$

We now specialize to the setting of structured ring spectra.

Definition 8.1.3.5. Let $k \geq 0$, and let $R \in \mathrm{Alg}^{(k+1)} = \mathrm{Alg}_{/\mathbb{E}_{k+1}}(\mathrm{Sp})$ be an $\mathbb{E}_{k+1}$-ring. We let LMod_R denote the ∞-category $\mathrm{LMod}_R(\mathrm{Sp})$ of left R-module spectra, which we regard as an $\mathbb{E}_{k+1}$-monoidal ∞-category. We let $\mathrm{Alg}_R^{(k)}$ denote the ∞-category $\mathrm{Alg}_{\mathbb{E}_k}(\mathrm{LMod}_R)$. We will refer to $\mathrm{Alg}_R^{(k)}$ as the ∞-*category of $\mathbb{E}_k$-algebras over R*.

Example 8.1.3.6. Let $R = S$ be the sphere spectrum, regarded as a trivial $\mathbb{E}_{k+1}$-algebra object of Sp. Then the forgetful functor $\mathrm{LMod}_R \to \mathrm{Sp}$ is an equivalence of $\mathbb{E}_k$-monoidal ∞-categories. It follows that the forgetful functor $\mathrm{Alg}_R^{(k)} \to \mathrm{Alg}^{(k)}$ is an equivalence: that is, an $\mathbb{E}_k$-algebra over the sphere spectrum is just an $\mathbb{E}_k$-ring, in the sense of Definition 8.1.0.1.

Remark 8.1.3.7. Let $0 \leq k$, let R be an $\mathbb{E}_{k+1}$-ring, and let A be an $\mathbb{E}_k$-algebra over R. We will generally abuse notation by identifying A with its image under the forgetful functor $\mathrm{Alg}_R^{(k)} \to \mathrm{Alg}^{(k)}$. Theorem 8.1.3.1 gives an equivalence of ∞-categories

$$\mathrm{LMod}_A(\mathrm{LMod}_R(\mathrm{Sp})) \simeq \mathrm{LMod}_A(\mathrm{Sp}),$$

which is $\mathbb{E}_{k-1}$-monoidal if $k > 0$ (by Corollary 8.1.3.4). Consequently, we may speak unambiguously about (left) A-module spectra and their relative tensor product over A, without making reference to the underlying $\mathbb{E}_{k+1}$-ring R.

Variant 8.1.3.8. Definition 8.1.3.5 continues to make sense in the special case $k = \infty$: if R is an $\mathbb{E}_\infty$-ring, then the ∞-category $\mathrm{Mod}_R(\mathrm{Sp}) \simeq \mathrm{LMod}_R(\mathrm{Sp})$ inherits a symmetric monoidal structure, so that the ∞-category $\mathrm{CAlg}(\mathrm{LMod}_R(\mathrm{Sp}))$ is well-defined. We will denote this ∞-category by CAlg_R and refer to its objects as $\mathbb{E}_\infty$-*algebras over R*. Using Theorem 6.3.6.10 and Proposition 3.4.1.4, we obtain a canonical equivalence of ∞-categories $\mathrm{CAlg}_R \simeq \mathrm{CAlg}_{R/}$: that is, we can identify an $\mathbb{E}_\infty$-algebra over R with an $\mathbb{E}_\infty$-ring A together with a map $R \to A$.

Warning 8.1.3.9. Let $k < \infty$ and let R be an $\mathbb{E}_{k+1}$-ring. We can identify the unit object of LMod_R with R itself, so there is an evident forgetful functor

$$\mathrm{Alg}_R^{(k)} \simeq (\mathrm{Alg}_R^{(k)})_{R/} \to \mathrm{Alg}_{R/}^{(k)}$$

generalizing the equivalence described in Variant 8.1.3.8. However, this functor is generally *not* an equivalence: a map of $\mathbb{E}_k$-rings from R to another $\mathbb{E}_k$-ring A does not exhibit A as an $\mathbb{E}_k$-algebra object of LMod_R unless it factors through the center $\mathfrak{Z}_{\mathbb{E}_k}(A)$ (see Definition 6.1.4.10).

Let $0 \leq k \leq \infty$ and let R be a connective $\mathbb{E}_{k+1}$-ring. We will say that an $\mathbb{E}_k$-algebra $A \in \mathrm{Alg}_R^{(k)}$ is *connective* if its underlying spectrum is connective. We let $\mathrm{Alg}_R^{(k),\mathrm{cn}}$ denote the full subcategory of $\mathrm{Alg}_R^{(k)}$ spanned by the connective $\mathbb{E}_k$-algebras over R. Note that $\mathrm{Alg}_R^{(k),\mathrm{cn}}$ can be identified with the ∞-category of $\mathbb{E}_k$-algebra objects of the subcategory $\mathrm{LMod}_R^{\mathrm{cn}} \subseteq \mathrm{LMod}_R$. To study this notion, we need the following observation:

Lemma 8.1.3.10. *Let $0 \leq k \leq \infty$ and let R be a connective $\mathbb{E}_{k+1}$-ring. Then the t-structure on LMod_R (see Proposition 8.1.1.13) is compatible with the $\mathbb{E}_k$-monoidal structure.*

Proof. If $k = 0$, it suffices to show that the unit object of LMod_R is connective; this unit object is given by R, so the result is immediate. If $k \geq 1$, we must also show that for $M, N \in \mathrm{LMod}_R^{\mathrm{cn}}$, the tensor product $M \otimes_R N$ also belongs to $\mathrm{LMod}_R^{\mathrm{cn}}$. Since $\mathrm{LMod}_R^{\mathrm{cn}}$ is generated by R under small colimits, we can assume that $M = N = R$, in which case the result is obvious. $\square$

For any object $A \in \mathrm{Alg}_R^{(k)}$, we can find a closest approximation to A which belongs to $\mathrm{Alg}_R^{(k),\mathrm{cn}}$.

Definition 8.1.3.11. Let R be a connective $\mathbb{E}_{k+1}$-ring for $0 \leq k \leq \infty$ and let $A \in \mathrm{Alg}_R^{(k)}$. A *connective cover* of A is a morphism $\phi : A' \to A$ of $\mathbb{E}_k$-algebras over R with the following properties:

(1) The $\mathbb{E}_k$-algebra A' is connective.

(2) For every connective object $A'' \in \mathrm{Alg}_R^{(k)}$, composition with ϕ induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Alg}_R^{(k)}}(A'', A') \to \mathrm{Map}_{\mathrm{Alg}_R^{(k)}}(A'', A).$$

Remark 8.1.3.12. In the situation of Definition 8.1.3.11, we will generally abuse terminology and simply refer to A' as a connective cover of A, in the case where the map ϕ is implicitly understood.

Proposition 8.1.3.13. *Let $0 \leq k \leq \infty$, and let R be a connective $\mathbb{E}_{k+1}$-ring. Then:*

(1) *Every $\mathbb{E}_k$-algebra $A \in \mathrm{Alg}_R^{(k)}$ admits a connective cover.*

(2) *An arbitrary map $\phi : A' \to A$ of $\mathbb{E}_k$-algebras over R is a connective cover of A if and only if A' is connective and the induced map $\pi_n A' \to \pi_n A$ is an isomorphism for $n \geq 0$.*

(3) *The inclusion $\mathrm{Alg}_R^{(k),\mathrm{cn}} \subseteq \mathrm{Alg}_R^{(k)}$ admits a right adjoint G, which carries each $\mathbb{E}_k$-algebra $A \in \mathrm{Alg}_R^{(k)}$ to a connective cover A' of A.*

Proof of Proposition 8.1.3.13. Combine Proposition 2.2.1.1 with Lemma 8.1.3.10. $\square$

Recall that an object X of an ∞-category $\mathcal{C}$ is said to be *n-truncated* if the mapping spaces $\mathrm{Map}_{\mathcal{C}}(Y, X)$ are n-truncated, for every $Y \in \mathcal{C}$ (see §T.5.5.6). Let R be a connective $\mathbb{E}_{k+1}$-ring for $0 \leq k \leq \infty$. Corollary 3.2.3.5 implies that the ∞-categories $\mathrm{Alg}_R^{(k)}$ and $\mathrm{Alg}_R^{(k),\mathrm{cn}}$ are presentable for $0 \leq k \leq \infty$, so we have a good theory of truncation functors.

Proposition 8.1.3.14. *Let $0 \leq k \leq \infty$, let R be a connective $\mathbb{E}_{k+1}$-ring, and let $A \in \mathrm{Alg}_R^{(k)}$. The following conditions are equivalent:*

(1) *As an object of $\mathrm{Alg}_R^{(k),\mathrm{cn}}$, A is n-truncated.*

(2) *As an object of $\mathrm{LMod}_R^{\mathrm{cn}}$, A is n-truncated.*

(3) *As an object of* $\mathrm{Sp}^{\mathrm{cn}}$, A *is* n-*truncated.*

(4) *The space* $\Omega^{\infty}(R)$ *is* n-*truncated.*

(5) *For every* $m > n$, *the homotopy group* $\pi_m R$ *is trivial.*

Proof. The equivalence (4) $\Leftrightarrow$ (5) is easy (Remark T.5.5.6.4), and the equivalences (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) are explained in Warning 1.2.1.9. The implication (1) $\Rightarrow$ (2) follows from Proposition T.5.5.6.16, since the forgetful functor $\mathrm{Alg}_R^{(k),\mathrm{cn}} \to \mathrm{LMod}_R^{\mathrm{cn}}$ preserves small limits (Corollary 3.2.2.5).

We now prove that (2) $\Rightarrow$ (1). Assume that A is n-truncated as a left R-module spectrum. Let $T : (\mathrm{Alg}_R^{(k),\mathrm{cn}})^{op} \to \mathcal{S}$ be the functor represented by A. Let $\mathcal{C} \subseteq \mathrm{Alg}_R^{(k),\mathrm{cn}}$ be the full subcategory of $\mathrm{Alg}_R^{(k),\mathrm{cn}}$ spanned by those objects B such that $T(B)$ is n-truncated. We wish to prove that $\mathcal{C} = \mathrm{Alg}_R^{(k),\mathrm{cn}}$. Since T preserves limits (Proposition T.5.1.3.2) and the class of n-truncated spaces is stable under limits (Proposition T.5.5.6.5), we conclude that $\mathcal{C}$ is stable under small colimits in $\mathrm{Alg}_R^{(k),\mathrm{cn}}$. Let F be a left adjoint to the forgetful functor $\mathrm{Alg}_R^{(k),\mathrm{cn}} \to \mathrm{LMod}_R$. Proposition 6.2.2.12 implies that $\mathrm{Alg}_R^{(k),\mathrm{cn}}$ is generated under colimits by the essential image of F. Consequently, it will suffice to show that $F(M) \in \mathcal{C}$ for every $M \in \mathrm{LMod}_R$. Equivalently, we must show that the space $\mathrm{Map}_{\mathrm{Alg}_R^{(k),\mathrm{cn}}}(F(M), A) \simeq \mathrm{Map}_{\mathrm{LMod}_R}(M, A)$ is n-truncated, which follows from (2). $\square$

Let R be a connective $\mathbb{E}_{k+1}$-ring, let $\tau_{\leq n} : \mathrm{LMod}_R^{\mathrm{cn}} \to \mathrm{LMod}_R^{\mathrm{cn}}$ be the truncation functor on connective left R-module spectra, and let $\tau_{\leq n}^{(k)} : \mathrm{Alg}_R^{(k),\mathrm{cn}} \to \mathrm{Alg}_R^{(k),\mathrm{cn}}$ be the truncation functor on connective $\mathbb{E}_k$-algebras over R. Since the forgetful functor $\theta : \mathrm{Alg}_R^{(k),\mathrm{cn}} \to \mathrm{LMod}_R^{\mathrm{cn}}$ preserves n-truncated objects, there is a canonical natural transformation $\alpha : \tau_{\leq n} \circ \theta \to \theta \circ \tau_{\leq n}^{(k)}$. Our next goal is to show that α is an equivalence.

Proposition 8.1.3.15. *Let* $0 \leq k \leq \infty$, *let* R *be a connective* $\mathbb{E}_{k+1}$-*ring, and let* $n \geq 0$ *be an integer. Then:*

(1) *The localization functor* $\tau_{\leq n} : \mathrm{LMod}_R^{\mathrm{cn}} \to \mathrm{LMod}_R^{\mathrm{cn}}$ *is compatible with the* $\mathbb{E}_k$-*monoidal structure on* $\mathrm{LMod}_R^{\mathrm{cn}}$, *in the sense of Definition 2.2.1.6.*

(2) *The* $\mathbb{E}_k$-*monoidal structure on* $\mathrm{LMod}_R^{\mathrm{cn}}$ *induces an* $\mathbb{E}_k$-*monoidal structure on the* ∞-*category*

$$\mathrm{LMod}_R^{\mathrm{cn}} \cap (\mathrm{LMod}_R)_{\leq n}$$

and an identification

$$\mathrm{Alg}_{/\mathbb{E}_k}(\mathrm{LMod}_R^{\mathrm{cn}} \cap (\mathrm{LMod}_R)_{\leq n}) \simeq \tau_{\leq n}^{(k)} \mathrm{Alg}_R^{(k),\mathrm{cn}}.$$

(3) *For every connective* $\mathbb{E}_k$-*algebra* A *over* R, *the map of left* R-*module spectra* $\tau_{\leq n} A \to \tau_{\leq n}^{(k)} A$ *described above is an equivalence.*

Proof. Assertion (1) follows from Proposition 2.2.1.8 and Lemma 8.1.3.10. Assertions (2) and (3) follow from (1) together with Proposition 2.2.1.9. $\square$

More informally, Proposition 8.1.3.15 asserts that if A is connective $\mathbb{E}_k$-algebra over a connective $\mathbb{E}_{k+1}$-ring R, then for each $n \geq 0$ the truncation $\tau_{\leq n} A$ inherits the structure of an $\mathbb{E}_k$-algebra over R.

Remark 8.1.3.16. Let $k \geq 1$ and let R be a connective $\mathbb{E}_{k+1}$-ring. The $\mathbb{E}_k$-monoidal structure on LMod_R induces an $\mathbb{E}_k$-monoidal structure on the subcategory $\mathrm{LMod}_R^{\heartsuit}$ of discrete objects of LMod_R, which is equivalent the nerve of the ordinary category of discrete modules over $\pi_0 R$. This $\mathbb{E}_k$-monoidal structure is given by the usual tensor product of modules over the commutative ring $\pi_0 R$. This follows, for example, from Theorem 8.1.2.13. We will discuss this point at greater length in §8.2.1.

Definition 8.1.3.17. Let $0 \leq k \leq \infty$ and let R be a connective $\mathbb{E}_{k+1}$-ring. We say that an $\mathbb{E}_k$-algebra R is *discrete* if it is connective and 0-truncated. We let $\mathrm{Alg}_R^{(k),\mathrm{disc}}$ denote the full subcategory of $\mathrm{Alg}_R^{(k)}$ spanned by the discrete objects.

Since the mapping spaces in $\mathrm{Alg}_R^{(k),\mathrm{disc}}$ are 0-truncated, it follows that $\mathrm{Alg}_R^{(k),\mathrm{disc}}$ is equivalent to the nerve of an ordinary category. We next identify the relevant category.

Proposition 8.1.3.18. *Let $1 \leq k \leq \infty$, and let R be a connective $\mathbb{E}_{k+1}$-ring. If $k = 1$, then the construction $A \mapsto \pi_0 A$ induces an equivalence from $\mathrm{Alg}_R^{(k)}$ to the (nerve of the) ordinary category of discrete associative algebras over $\pi_0 R$. If $k \geq 2$, then the construction $A \mapsto \pi_0 A$ determines an equivalence from $\mathrm{Alg}_R^{(k)}$ to the (nerve of the) ordinary category of discrete commutative algebras over $\pi_0 R$.*

Proof. Using Proposition 8.1.3.15, we can identify $\mathrm{Alg}_R^{(k),\mathrm{disc}}$ with the ∞-category of $\mathbb{E}_k$-algebra objects of the heart $\mathrm{LMod}_R^\heartsuit$. Combining this with Remark 8.1.3.16, we see that $\mathrm{Alg}_R^{(1),\mathrm{disc}}$ can be identified with the nerve of the category of associative algebras over $\pi_0 R$. When $k \geq 2$, Remark 8.1.3.16 and Corollary 5.1.1.7 imply that $\mathrm{Alg}_{/\mathbb{E}_k}(\mathrm{LMod}_R^\heartsuit)$ can be identified with the nerve of the ordinary category of *commutative* $\pi_0 R$-algebras. $\qquad\qquad\square$

Let $0 \leq k \leq \infty$ and let R be a connective $\mathbb{E}_{k+1}$-ring. Since the t-structure on LMod_R is left complete (Proposition 8.1.1.13), the map

$$\mathrm{LMod}_R^{\mathrm{cn}} \to \varprojlim_n \mathrm{LMod}_R^{\mathrm{cn}} \cap (\mathrm{LMod}_R)_{\leq n}$$

is an equivalence of ∞-categories. The forgetful functor $\mathrm{Alg}_{/\mathbb{E}_k}(\widehat{\mathcal{C}at}_\infty) \to \widehat{\mathcal{C}at}_\infty$ preserves small limits (Corollary 3.2.2.5), so that $\mathrm{LMod}_R^{\mathrm{cn}}$ is also a limit of the sequence $\{\mathrm{LMod}_R^{\mathrm{cn}} \cap (\mathrm{LMod}_R)_{\leq n}\}$ in the ∞-category of $\mathbb{E}_k$-monoidal ∞-categories and therefore also in the ∞-category Op_∞ of ∞-operads. This immediately implies the following:

Proposition 8.1.3.19. *Let $0 \leq k \leq \infty$, and let R be a connective $\mathbb{E}_{k+1}$-ring. Then the canonical map*

$$\mathrm{Alg}_R^{(k),\mathrm{cn}} \to \varprojlim_n (\tau_{\leq n} \mathrm{Alg}_R^{(k),\mathrm{cn}})$$

is an equivalence of ∞-categories. In other words, Postnikov towers are convergent in the ∞-category $\mathrm{Alg}_R^{(k),\mathrm{cn}}$ of connective $\mathbb{E}_k$-algebras over R (see Definition T.5.5.6.23).

8.1.4   Algebras over Commutative Rings

Let R be a commutative ring, which we regard as a discrete $\mathbb{E}_\infty$-ring. Our goal in this section is to describe some explicit models for the ∞-category of $\mathbb{E}_1$-algebras over R and (when R contains the field $\mathbf{Q}$ of rational numbers) the ∞-category of $\mathbb{E}_\infty$-algebras over R. We begin by reviewing a bit of terminology.

Definition 8.1.4.1. Let R be a commutative ring. A *differential graded algebra over R* is a graded associative algebra A_* over R equipped with a differential $d : A_* \to A_{*-1}$ satisfying the following conditions:

- The square of the differential d is equal to zero.

- The map d is a (graded) derivation. That is, we have the Leibniz rule $d(xy) = (dx)y + (-1)^m x dy$ for $x \in A_m$, $y \in A_n$.

If A_* and B_* are differential graded algebras over R, then a *morphism of differential graded algebras* from A_* to B_* is a homomorphism of graded R-algebras $\phi : A_* \to B_*$ such that $\phi(dx) = d\phi(x)$. With this notion of morphism, the collection of differential graded algebras over R forms a category, which we will denote by $\mathbf{DGA}(R)$.

Remark 8.1.4.2. Let R be a commutative ring and let $\mathcal{A}$ be the abelian category of (discrete) R-modules. Then we can identify differential graded algebras over R with associative algebra objects in the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes of R-modules. This identification gives an equivalence of categories $\mathrm{Alg}(\mathrm{Ch}(\mathcal{A})) \simeq \mathbf{DGA}(R)$.

If we want to understand the structure of differential graded algebras over R, we should begin by studying the tensor product of chain complexes over R.

Proposition 8.1.4.3. *Let R be a commutative ring, let $\mathcal{A}$ be the category of R-modules, and regard $\mathrm{Ch}(\mathcal{A})$ as a symmetric monoidal model category with respect to the projective model structure of Proposition 8.1.2.8. Then $\mathrm{Ch}(\mathcal{A})$ satisfies the monoid axiom (see Definition 4.1.4.1).*

Proof. Let U be the collection of all morphisms in $\mathrm{Ch}(\mathcal{A})$ of the form $M_* \otimes N'_* \to M_* \otimes N_*$, where $N'_* \to N_*$ is a trivial cofibration. Let $\overline{U}$ be the weakly saturated class of morphisms generated by U. We wish to show that every morphism in $\overline{U}$ is a quasi-isomorphism. We will prove a stronger assertion: namely, every morphism in $\overline{U}$ is a trivial cofibration with respect to the model structure described in Proposition 1.3.5.3. For this, it suffices to show that every morphism in U is a trivial cofibration with respect to the model structure of Proposition 1.3.5.3.

Let M_* be an arbitrary object of $\mathrm{Ch}(\mathcal{A})$, and let $f : N'_{|ast} \to N_*$ be a trivial cofibration with respect to the projective model structure; we wish to show that the induced map $F : M_* \otimes N'_* \to M_* \otimes N_*$ is a trivial cofibration with respect to the model structure of Proposition 1.3.5.3. We have an exact sequence of chain complexes

$$0 \to N'_* \to N_* \to N''_* \to 0$$

where each of the maps

$$0 \to N'_i \to N_i \to N''_i \to 0$$

is split exact. It follows that the sequence of chain complexes

$$0 \to M_* \otimes N'_* \overset{F}{\to} M_* \otimes N_* \to M_* \otimes N''_* \to 0$$

is exact, so that F is a monomorphism. We must show that F is a quasi-isomorphism. Equivalently, we must show that the chain complex $M_* \otimes N''_*$ is acyclic. As in the proof of Proposition 8.1.2.8, we observe that the chain complex N''_* admits a contracting homotopy, so that $M_* \otimes N''_*$ also admits a contracting homotopy and is therefore acyclic. $\square$

Remark 8.1.4.4. Let A_* be a differential graded algebra over a commutative ring R. Then we can regard A_* as a chain complex of R-modules. We will denote the homology of this chain complex by $\mathrm{H}_*(A)$. The multiplication on A_* induces a multiplication on $\mathrm{H}_*(A)$, so that $\mathrm{H}_*(R)$ has the structure of a graded R-algebra.

We say that a map $\phi : A_* \to B_*$ of differential graded algebras is a *quasi-isomorphism* if it induces a quasi-isomorphism of chain complexes over R: that is, if and only if it induces an isomorphism of graded rings $\mathrm{H}_*(A) \to \mathrm{H}_*(B)$.

Proposition 8.1.4.5. *Let R be a commutative ring. Then there exists a combinatorial model structure on the category $\mathbf{DGA}(R)$ of differential graded algebras over R with the following properties:*

(W) *A morphism of differential graded algebras $\phi : A_* \to B_*$ is a weak equivalence if and only if it is a quasi-isomorphism.*

(F) *A morphism of differential graded algebras $\phi : A_* \to B_*$ is a fibration if and only if each of the maps $A_n \to B_n$ is surjective.*

Moreover, if R is a field, then the model category $\mathbf{DGA}(R)$ is left proper.

Proof. Combine Proposition 4.1.4.3, Proposition 8.1.4.3, and Proposition 8.1.2.8. The last assertion follows from Proposition 4.1.4.3 and Remark 8.1.2.10. $\square$

Proposition 8.1.4.6. *Let R be a commutative ring, let $\mathbf{DGA}(R)$ denote the category of differential graded algebras over R, let $\mathbf{DGA}(R)^c$ be the full subcategory of $\mathbf{DGA}(R)$ spanned by the cofibrant objects, and let W denote the collection of weak equivalences in $\mathbf{DGA}(R)^c$. Then there is a canonical equivalence of ∞-categories*

$$\mathrm{N}(\mathbf{DGA}(R)^c)[W^{-1}] \simeq \mathrm{Alg}_R^{(1)}.$$

In other words, we can identify the ∞-category of $\mathbb{E}_1$-algebras over R with the underlying ∞-category of the model category $\mathbf{DGA}(R)$ of differential graded R-algebras.

Proof. Combine Theorems 8.1.2.13 and 4.1.4.4. $\square$

We next prove an analogue of Proposition 8.1.4.6 for $\mathbb{E}_\infty$-algebras, assuming that the commutative ring R has characteristic zero.

Proposition 8.1.4.7. *Let R be a commutative ring containing the field $\mathbf{Q}$ of rational numbers and let $\mathcal{A}$ be the abelian category of discrete R-modules. Regard $\mathrm{Ch}(\mathcal{A})$ as endowed with the projective model structure of Proposition 8.1.2.8. Then $\mathrm{Ch}(\mathcal{A})$ is freely powered (see Definition 4.4.4.2).*

Proof. We must show that if $f : M_* \to N_*$ is a cofibration in $\mathrm{Ch}(\mathcal{A})$, then f is a power cofibration. Without loss of generality, we may assume that f is a generating cofibration of the form $\partial E(m)_* \to E(m)_*$ for some $m \in \mathbf{Z}$ (see the proof of Proposition 8.1.2.8 for an explanation of this notation). We wish to show that for each $n \geq 0$, the induced map $\phi : \square^n(f) \to E(m)_*^{\otimes n}$ is a projective cofibration in $\mathrm{Ch}(\mathcal{A})^{\Sigma_n}$, where Σ_n denotes the symmetric group on n letters. Note that ϕ is a pushout of the inclusion $\phi_0 : \partial E(nm)_* \to E(nm)_*$, where the symmetric group acts trivially on $E(nm)_*$ if m is even and by the sign representation if m is odd. In either case, the assumption that R contains the field $\mathbf{Q}$ guarantees that ϕ_0 is a retract of the projective cofibration

$$\partial E(nm)_* \otimes_R R[\Sigma_n] \to E(nm)_* \otimes_R R[\Sigma_n],$$

where $R[\Sigma_n]$ denotes the regular representation of Σ_n over R. $\square$

Definition 8.1.4.8. Let R be a commutative ring and let A_* be a differential graded algebra over R. We will say that A_* is a *commutative differential graded algebra* if for every pair of elements $x \in A_m$, $y \in A_n$, we have $xy = (-1)^{mn} yx$. We let $\mathbf{CDGA}(R)$ denote the full subcategory of $\mathbf{DGA}(R)$ spanned by the commutative differential graded algebras over R.

Remark 8.1.4.9. Let R be a commutative ring and let $\mathcal{A}$ be the abelian category of (discrete) R-modules. Then we can identify commutative differential graded algebras over R with commutative algebra objects in the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes of R-modules. This identification gives an equivalence of categories $\mathrm{CAlg}(\mathrm{Ch}(\mathcal{A})) \simeq \mathbf{CDGA}(R)$.

Combining Proposition 8.1.4.3, Proposition 8.1.4.7, and Proposition 4.4.4.6, we obtain the following:

Proposition 8.1.4.10. *Let R be a commutative ring which contains the field $\mathbf{Q}$ of rational numbers. Then there exists a combinatorial model structure on the category $\mathbf{CDGA}(R)$ of differential graded algebras over R with the following properties:*

(W) *A morphism of commutative differential graded algebras $\phi : A_* \to B_*$ is a weak equivalence if and only if it is a quasi-isomorphism.*

(F) *A morphism of commutative differential graded algebras $\phi : A_* \to B_*$ is a fibration if and only if each of the maps $A_n \to B_n$ is surjective.*

Using Theorems 8.1.2.13 and 4.4.4.7, we obtain:

Proposition 8.1.4.11. *Let R be a commutative ring and let $\mathbf{CDGA}(R)$ denote the category of commutative differential graded algebras over R. Assume that R contains the field $\mathbf{Q}$ of rational numbers, let $\mathbf{CDGA}(R)^c$ be the full subcategory of $\mathbf{CDGA}(R)$ spanned by the cofibrant objects, and let W denote the collection of weak equivalences in $\mathbf{CDGA}(R)^c$. Then there is a canonical equivalence of ∞-categories*

$$\mathrm{N}(\mathbf{CDGA}(R)^c)[W^{-1}] \simeq \mathrm{CAlg}_R.$$

In other words, we can identify the ∞-category of $\mathbb{E}_\infty$-algebras over R with the underlying ∞-category of the model category $\mathbf{CDGA}(R)$ of commutative differential graded R-algebras.

Propositions 8.1.4.6 and 8.1.4.11 provided concrete models for the ∞-categories of $\mathbb{E}_1$ and $\mathbb{E}_\infty$-algebras over discrete commutative rings. If we are willing to restrict our attention to *connective* algebras, then there is another concrete model available, provided by the theory of simplicial rings. We begin with a few preliminary remarks.

Let $\mathcal{C}$ be a presentable ∞-category. We recall that an object $C \in \mathcal{C}$ is said to be *compact and projective* if the corepresentable functor $\mathrm{Map}_\mathcal{C}(C, \bullet)$ preserves sifted colimits. We say that $\mathcal{C}$ is *projectively generated* if there exists a small collection of compact projective objects $\{C_\alpha\}$ of $\mathcal{C}$ which generates $\mathcal{C}$ under small colimits; see Definition T.5.5.8.23. In this case, we will say that $\{C_\alpha\}$ is a *set of compact projective generators* for $\mathcal{C}$.

Proposition 8.1.4.12. *Let $G : \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories. Assume that G preserves small limits, small sifted colimits, and is conservative. Then:*

(1) *The functor G admits a left adjoint F.*

(2) *The functor F carries compact projective objects of $\mathcal{D}$ to compact projective objects of $\mathcal{C}$.*

(3) *Let $\{D_\alpha\}$ be a set of compact projective generators for $\mathcal{D}$. Then $\{F(D_\alpha)\}$ is a set of compact projective generators for $\mathcal{C}$.*

(4) *If $\mathcal{D}$ is projectively generated, so is $\mathcal{C}$.*

Proof. Assertion (1) follows from Corollary T.5.5.2.9 and assertion (2) from the assumption that G preserves sifted colimits. To prove (3), let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ generated under small colimits by the objects $\{F(D_\alpha)\}$. Then the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ admits a right adjoint U (by Corollary T.5.5.2.9). To prove that U is an equivalence, it suffices to show that for each $C \in \mathcal{C}$, the map $U(C) \to C$ is an equivalence. Since G is conservative, it suffices to show that $G(U(C)) \to G(C)$ is an equivalence in $\mathcal{D}$. Because the objects $\{D_\alpha\}$ generate $\mathcal{D}$ under small colimits, we are reduced to proving that the map

$$\mathrm{Map}_\mathcal{D}(D_\alpha, G(U(C))) \simeq \mathrm{Map}_\mathcal{C}(F(D_\alpha), U(C)) \to \mathrm{Map}_\mathcal{C}(F(D_\alpha), C) \simeq \mathrm{Map}_\mathcal{D}(D_\alpha, G(C))$$

is a homotopy equivalence, which is clear. Assertion (4) is an immediate consequence of (3). $\square$

Corollary 8.1.4.13. *The ∞-category $\mathrm{Sp}^{\mathrm{cn}}$ of connective spectra is projectively generated: in fact, the sphere spectrum S is a compact projective generator for $\mathrm{Sp}^{\mathrm{cn}}$.*

Proof. Apply Proposition 8.1.4.12 to the 0th space functor $\Omega^\infty : \mathrm{Sp}_{\geq 0} \to \mathcal{S}$ and invoke Proposition 1.4.3.9. $\square$

Corollary 8.1.4.14. *Let $\mathcal{M}$ be a presentable ∞-category which is left-tensored over a monoidal ∞-category $\mathcal{C}$, and let $R \in \mathrm{Alg}(\mathcal{C})$ be an algebra object such that tensor product with R induces a functor $\mathcal{M} \to \mathcal{M}$ which commutes with small colimits. If $\mathcal{M}$ is projectively generated, then $\mathrm{LMod}_R(\mathcal{M})$ is projectively generated. Moreover, if $\{M_\alpha\}$ is a collection of compact projective generators for $\mathcal{M}$, then the free modules $\{R \otimes M_\alpha\}$ are compact projective generators for the ∞-category $\mathrm{LMod}_R(\mathcal{M})$.*

Proof. Apply Proposition 8.1.4.12 to the forgetful functor $\mathrm{LMod}_R(\mathcal{M}) \to \mathcal{M}$ and use Corollary 4.2.3.7. $\square$

Corollary 8.1.4.15. *Let R be a connective $\mathbb{E}_1$-ring. Then the ∞-category $\mathrm{LMod}_R^{\mathrm{cn}}$ is projectively generated; in fact, the object R (regarded as a left module over itself) is a compact projective generator for $\mathrm{LMod}_R^{\mathrm{cn}}$.*

Proof. Combine Corollaries 8.1.4.13 and 8.1.4.14. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 8.1.4.16. *Let $\mathcal{O}^{\otimes}$ be an ∞-operad and let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads. Assume that p is compatible with small colimits and that for each object $X \in \mathcal{O}$, the fiber $\mathcal{C}_X$ is a projectively generated ∞-category. Then the ∞-category $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ is projectively generated.*

For each $X \in \mathcal{O}$, let $\mathrm{Free}_X : \mathcal{C}_X \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ be a left adjoint to the evaluation functor, and let $\{C_{X,\alpha} \in \mathcal{C}_X\}$ is a collection of compact generators for the ∞-category $\mathcal{C}_X$. Then the collection of objects $\{\mathrm{Free}_X(C_{X,\alpha})\}$ (where X ranges over the objects of $\mathcal{O}$) is a collection of compact projective generators for the ∞-category $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$.

Proof. Apply Proposition 8.1.4.12 to the forgetful functor $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \prod_{X \in \mathcal{O}} \mathcal{C}_X$. $\qquad\qquad\square$

Corollary 8.1.4.17. *Let R be a connective $\mathbb{E}_{k+1}$-ring for $0 \leq k \leq \infty$. Then the ∞-category $\mathrm{Alg}_R^{(k),\mathrm{cn}}$ of connective $\mathbb{E}_k$-algebras over R is projectively generated. Moreover, if $\mathrm{Free} : \mathrm{LMod}_R \to \mathrm{Alg}_R^{(k),\mathrm{cn}}$ denotes a left adjoint to the forgetful functor, then $\mathrm{Free}(R)$ is a compact projective generator for $\mathrm{Alg}_R^{(k),\mathrm{cn}}$.*

Proof. Combine Corollaries 8.1.4.16 and 8.1.4.17. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let R be a commutative ring and let $M \simeq R^m$ be a free R-module of finite rank m. For every $n \geq 0$, the tensor power $M^{\otimes n}$ can be identified with a free module of rank m^n: here the tensor power can be computed either in the ordinary category of R-modules, or in the ∞-category $\mathrm{Mod}_R(\mathrm{Sp})$ of R-module spectra. It follows that the tensor algebra

$$T(M) = \bigoplus_{n \geq 0} M^{\otimes n}$$

can be identified with the free $\mathbb{E}_1$-algebra over R generated by M (see Proposition 4.1.1.14).

Let $\mathbf{Alg}_R$ denote the category of associative R-algebras (that is, associative algebras in the abelian category of R-modules) and let $\mathbf{Alg}_R^0$ denote the full subcategory of $\mathbf{Alg}_R$ spanned by objects of the form $T(M)$, where M is a free R-module of finite rank. According to Proposition 8.1.3.18, we can identify $\mathrm{N}(\mathbf{Alg}_R)$ with the full subcategory $\mathrm{Alg}_R^{(1),\mathrm{disc}} \subseteq \mathrm{Alg}_R^{(1)}$. Under this equivalence, $\mathrm{N}(\mathbf{Alg}_R^0)$ can be identified with the full subcategory of $\mathrm{Alg}_R^{(1)}$ given by finite coproducts of $T(R)$, which is a compact projective generator for $\mathrm{Alg}_R^{(1),\mathrm{cn}}$ by Corollary 8.1.4.17. It follows from Proposition T.5.5.8.25 that the fully faithful embedding $\mathrm{N}(\mathbf{Alg}_R^0) \to \mathrm{Alg}_R^{(1),\mathrm{cn}}$ extends to an equivalence of ∞-categories $\mathcal{P}_{\Sigma}(\mathrm{N}(\mathbf{Alg}_R^0)) \simeq \mathrm{Alg}_R^{(1),\mathrm{cn}}$. In particular, the ∞-category $\mathrm{Alg}_R^{(1),\mathrm{disc}}$ of discrete objects of $\mathrm{Alg}_R^{(1),\mathrm{cn}}$ can be identified with the full subcategory of $\mathcal{P}_{\Sigma}(\mathrm{N}(\mathbf{Alg}_R^0))$ spanned by those functors $\mathrm{N}(\mathbf{Alg}_R^0)^{op} \to \mathcal{S}$ which preserve finite products and take 0-truncated values. Passing to homotopy categories, we obtain an equivalence

$$\mathbf{Alg}_R \simeq \mathrm{hAlg}_R^{(1),\mathrm{disc}} \simeq \mathrm{Fun}'((\mathbf{Alg}_R^0)^{op}, \mathcal{S}\mathrm{et}),$$

where $\mathrm{Fun}'((\mathbf{Alg}_R^0)^{op}, \mathcal{S}\mathrm{et})$ denotes the full subcategory of $\mathrm{Fun}((\mathbf{Alg}_R^0)^{op}, \mathcal{S}\mathrm{et})$ spanned by those functors which preserve finite products. Applying Propositions T.5.5.9.1 and T.5.5.9.2, we obtain the following analogue of Proposition 8.1.4.6:

Proposition 8.1.4.18. *Let R be a commutative ring, let $\mathbf{Alg}_R$ be the category of (discrete) associative R-algebras, and let $\mathbf{A}$ denote the category of simplicial objects of $\mathbf{Alg}_R$. Then $\mathbf{A}$ admits a simplicial model structure which may be described as follows:*

(W) *A map of simplicial associative R-algebras $A_{\bullet} \to B_{\bullet}$ is a weak equivalence if and only if the underlying map of simplicial sets is a weak homotopy equivalence.*

(F) *A map of simplicial associative R-algebras $A_\bullet \to B_\bullet$ is a fibration if and only if the underlying map of simplicial sets is a Kan fibration.*

Moreover, the underlying ∞-category $\mathrm{N}(\mathbf{A}^o)$ is canonically equivalent to the ∞-category $\mathrm{Alg}_R^{(1),\mathrm{cn}}$ of connective $\mathbb{E}_1$-algebras over R.

Remark 8.1.4.19. Let A be a connective $\mathbb{E}_1$-algebra over a discrete commutative ring R. Proposition 8.1.4.18 allows us to identify A with a simplicial object $A_\bullet$ in the category of associative R-algebras. The geometric realization $|A_\bullet|$ is a topological associative R-algebra, which determines $A_\bullet$ (and therefore the original $\mathbb{E}_1$-algebra A) up to equivalence. In particular, we can think of connective $\mathbb{E}_1$-algebras over $\mathbf{Z}$ as topological associative rings.

Suppose now that R is a commutative ring which contains the field $\mathbf{Q}$ of rational numbers. Let $\mathcal{A}$ be the abelian category of R-modules and regard the category $\mathbf{CDGA}(R)$ as endowed with the model structure described in Proposition 8.1.4.10. The forgetful functor $\mathbf{CDGA}(R) \to \mathrm{Ch}(\mathcal{A})$ is a right Quillen functor, so its left adjoint $M_* \mapsto \mathrm{Sym}^*(M_*)$ is a left Quillen functor. If M is a free R-module, then M is a cofibrant object of $\mathrm{Ch}(\mathcal{A})$ (when regarded as a chain complex concentrated in degree zero). It follows that if Free : $\mathrm{Mod}_R \to \mathrm{CAlg}_R$ denotes a left adjoint to the forgetful functor, then Free carries the object $M \in \mathrm{N}(\mathcal{A}) \simeq \mathrm{Mod}_R^\heartsuit$ to the discrete commutative algebra $\mathrm{Sym}^*(M)$, which is a polynomial algebra over R.

We can now repeat the reasoning which precedes the statement of Proposition 8.1.4.18. Let $\mathbf{CAlg}_R$ denote the category of (discrete) commutative R-algebras and let Poly_R denote the full subcategory of $\mathbf{CAlg}_R$ spanned by objects of the form $R[x_1, \ldots, x_n] \simeq \mathrm{Sym}^*(R^n)$. According to Proposition 8.1.3.18, we can identify $\mathrm{N}(\mathbf{CAlg}_R)$ with the full subcategory $\mathrm{CAlg}_R^{\mathrm{disc}} \subseteq \mathrm{CAlg}_R^{(1)}$. Under this equivalence, $\mathrm{N}(\mathrm{Poly}_R)$ can be identified with the full subcategory of CAlg_R given by finite coproducts of $R[x] \simeq \mathrm{Free}(R)$, which is a compact projective generator for $\mathrm{CAlg}_R^{\mathrm{cn}}$ by Corollary 8.1.4.17. It follows from Proposition T.5.5.8.25 that the fully faithful embedding $\mathrm{N}(\mathrm{Poly}_R) \to \mathrm{CAlg}_R^{\mathrm{cn}}$ extends to an equivalence of ∞-categories $\mathcal{P}_\Sigma(\mathrm{N}(\mathrm{Poly}_R)) \simeq \mathrm{CAlg}_R^{\mathrm{cn}}$. In particular, the ∞-category $\mathrm{CAlg}_R^{\mathrm{disc}}$ of discrete objects of $\mathrm{CAlg}_R^{\mathrm{cn}}$ can be identified with the full subcategory of $\mathcal{P}_\Sigma(\mathrm{N}(\mathrm{Poly}_R))$ spanned by those functors $\mathrm{N}(\mathrm{Poly}_R)^{op} \to \mathcal{S}$ which preserve finite products and take 0-truncated values. Passing to homotopy categories, we obtain an equivalence

$$\mathbf{CAlg}_R \simeq \mathrm{hCAlg}_R^{\mathrm{disc}} \simeq \mathrm{Fun}'((\mathbf{CAlg}_R^0)^{op}, \mathcal{S}\mathrm{et}),$$

where $\mathrm{Fun}'((\mathrm{Poly}_R)^{op}, \mathcal{S}\mathrm{et})$ denotes the full subcategory of $\mathrm{Fun}((\mathrm{Poly}_R)^{op}, \mathcal{S}\mathrm{et})$ spanned by those functors which preserve finite products. Using Propositions T.5.5.9.1 and T.5.5.9.2, we obtain a commutative analogue of Proposition 8.1.4.18:

Proposition 8.1.4.20. *Let R be a commutative ring which contains the field $\mathbf{Q}$ of rational numbers, let $\mathbf{CAlg}_R$ be the category of (discrete) commutative R-algebras, and let $\mathbf{A}$ denote the category of simplicial objects of $\mathbf{CAlg}_R$. Then $\mathbf{A}$ admits a simplicial model structure which may be described as follows:*

(W) *A map of simplicial commutative R-algebras $A_\bullet \to B_\bullet$ is a weak equivalence if and only if the underlying map of simplicial sets is a weak homotopy equivalence.*

(F) *A map of simplicial commutative R-algebras $A_\bullet \to B_\bullet$ is a fibration if and only if the underlying map of simplicial sets is a Kan fibration.*

Moreover, the underlying ∞-category $\mathrm{N}(\mathbf{A}^o)$ is canonically equivalent to the ∞-category $\mathrm{CAlg}_R^{\mathrm{cn}}$ of connective $\mathbb{E}_\infty$-algebras over R.

Warning 8.1.4.21. For any commutative ring R, the category $\mathbf{A}$ of simplicial commutative R-algebras can be endowed with a model structure, with weak equivalences and fibrations as described in Proposition 8.1.4.20. Moreover, Proposition T.5.5.9.2 gives an equivalence $\mathrm{N}(\mathbf{A}^o) \simeq \mathcal{P}_\Sigma(\mathrm{N}(\mathrm{Poly}_R))$, so that the fully faithful embedding $\mathrm{N}(\mathrm{Poly}_R) \to \mathrm{CAlg}_R^{\mathrm{disc}} \subseteq \mathrm{CAlg}_R$ extends in an essentially unique way to a functor $\mathrm{N}(\mathbf{A}^o) \to \mathrm{CAlg}_R$ which preserves sifted colimits. However, this functor is generally not an equivalence unless R contains the field $\mathbf{Q}$ of rational numbers.

8.2   Properties of Rings and Modules

In §8.1, we introduced the theory of structured ring spectra and their modules. Our approach was rather abstract: the basic definitions were obtained by specializing the theory of algebras and modules over ∞-operads (developed in the earlier chapters of this book) to the case where the ambient symmetric monoidal ∞-category is the ∞-category of spectra. In this section, we consider some less formal aspects of the theory. In particular, we will show that several basic tools of noncommutative and homological algebra can be can be generalized to the setting of structured ring spectra.

We will begin by considering the relative tensor product construction introduced in §4.3.5. If R is an $\mathbb{E}_1$-ring, M a right R-module, and N a left R-module, then we can consider the tensor product spectrum $M \otimes_R N$. In §8.2.1, we address the question of computing the homotopy groups of $\pi_*(M \otimes_R N)$. Our main result (Proposition 8.2.1.19) asserts that $\pi_*(M \otimes_R N)$ can be computed by means of a spectral sequence, whose second page can be described in terms of the graded Tor-groups $\operatorname{Tor}_*^{\pi_* R}(\pi_* M, \pi_* N)$. Many questions about the theory of structured ring spectra can be reduced to an analysis of appropriate tensor products, for which this spectral sequence is an invaluable tool.

In §8.2.2, we define *flat* and *projective* modules over a connective $\mathbb{E}_1$-ring R. These definitions specialize to give the usual theory of flat and projective modules in the special case where R is discrete. Most of the familiar properties of flat and projective modules can be generalized to the nondiscrete case. For example, we prove a generalization of Lazard's theorem, which asserts that every flat R-module can be obtained as a filtered colimit of (finitely generated) projective R-modules (Theorem 8.2.2.15).

If R is a connective $\mathbb{E}_1$-ring, then a left R-module P is projective if every map of connective left R-modules $M \to N$ which induces a surjection $\pi_0 M \to \pi_0 N$ also induces a surjection $\pi_0 \operatorname{Map}_{\operatorname{LMod}_R}(P, M) \to \pi_0 \operatorname{Map}_{\operatorname{LMod}_R}(P, N)$. In §8.2.3, we will study the formally dual notion of an *injective* object of LMod_R (or, more generally, of any stable ∞-category $\mathcal{C}$ equipped with a t-structure). Our main result is that the homotopy category of injective left R-module spectra is equivalent to the ordinary category of injective modules over the ring $\pi_0 R$ (Theorem 8.2.3.4).

One of the most important constructions in commutative algebra is the formation of *localizations*: if R is a commutative ring, then we can associate to every multiplicatively closed subset $S \subseteq R$ a ring of fractions $R[S^{-1}]$ obtained by formally inverting the elements of S. In §8.2.4 we will review the theory of *Ore localization*, which extends this construction to noncommutative rings. We will then generalize the theory of Ore localization to the setting of $\mathbb{E}_1$-rings (and their modules).

For every $\mathbb{E}_1$-ring R, the ∞-category LMod_R is compactly generated. We will refer to the compact objects of LMod_R as *perfect R-modules*. There is a closely related notion of *almost perfect R-module*, which we will introduce in §8.2.5. We also introduce the definition of a *(left) Noetherian $\mathbb{E}_1$-ring R*, generalizing the classical theory of Noetherian rings. As in the classical case, the assumption that R is (left) Noetherian ensures that finiteness conditions on (left) R-modules behave well. Moreover, if we restrict our attention to $\mathbb{E}_\infty$-rings, then the condition of being Noetherian is robust: for example, we have an analogue of the Hilbert basis theorem (Proposition 8.2.5.31).

8.2.1   Free Resolutions and Spectral Sequences

Let R be an $\mathbb{E}_1$-ring. In §8.1.1, we introduced the notion of left and right modules over R. If M is a right R-module and N is a left R-module, then we let $M \otimes_R N$ denote the relative tensor product of M and N (in the ∞-category of spectra). Our goal in this section is to develop some techniques which, in favorable cases, allow us to compute the homotopy groups of the tensor product $M \otimes_R N$ in terms of classical homological algebra.

We begin by recalling a few definitions. Let R be an associative ring and let N be a (discrete) left module over R. A *free resolution* of N is an exact sequence of left R-modules

$$\cdots \to P_2 \to P_1 \to P_0 \to N \to 0,$$

where each P_i is a free left module over R. If M is a right R-module, we obtain a chain complex of abelian

groups

$$\cdots \to M \otimes_R P_2 \to M \otimes_R P_1 \to M \otimes_R P_0.$$

We denote the homology groups of this chain complex by $\mathrm{Tor}_i^R(M, N)$. In particular, the usual tensor product $M \otimes_R N$ can be identified with $\mathrm{Tor}_0^R(M, N)$.

The Tor groups $\mathrm{Tor}_i^R(M, N)$ are independent of the choice of free resolution P_* for the module N, up to canonical isomorphism, and depend functorially on the pair (M, N). To prove this, one can argue as follows: given any two projective resolutions P_* and Q_* of N, there exists a map of chain complexes of left R-modules

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & N \\
& & \downarrow{f_2} & & \downarrow{f_1} & & \downarrow{f_0} & & \downarrow{\mathrm{id}} \\
\cdots & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & N,
\end{array}
$$

which is unique up to chain homotopy. Consequently, one obtains a map of chain complexes of abelian groups

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & M \otimes_R P_2 & \longrightarrow & M \otimes_R P_1 & \longrightarrow & M \otimes_R P_0 \\
& & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & M \otimes_R Q_2 & \longrightarrow & M \otimes_R Q_1 & \longrightarrow & M \otimes_R Q_0.
\end{array}
$$

whose induced map on homology groups does not depend on the choice of f_*.

We would like to generalize some of the above ideas to the setting of ∞-categories. First, we need to recall a bit of terminology.

Notation 8.2.1.1. Let $\mathcal{C}$ be a presentable ∞-category, and let $X_\bullet$ be a simplicial object of $\mathcal{C}$. For each $n \geq 0$, we let $L_n(X)$ and $M_n(X)$ denote the nth latching and matching object of X, respectively (see §T.A.2.9).

Definition 8.2.1.2. Let $\mathcal{C}$ be a presentable ∞-category and let S be a collection of objects of $\mathcal{C}$. We will say that a simplicial object $X_\bullet$ of $\mathcal{C}$ is *S-free* if, for every integer n, there exists a map $F \to X_n$ in $\mathcal{C}$ which induces an equivalence $L_n(X) \coprod F \to X_n$, such that F is a coproduct of objects of S.

Let $C \in \mathcal{C}$ and let $X_\bullet$ be a simplicial object of $\mathcal{C}_{/C}$. We will say that $X_\bullet$ is an *S-hypercovering of C* if, for every object $Y \in S$ corepresenting a functor $\chi : \mathcal{C} \to \mathcal{S}$, the simplicial object $\chi(X_\bullet)$ is a hypercovering in the ∞-topos $\mathcal{S}_{/\chi(C)}$ (see Definition T.6.5.3.2).

Example 8.2.1.3. Let $\mathcal{A}$ be the category of left R-modules, for some associative ring R, and let $S = \{R\}$. Using Theorem 1.2.3.7, we can identify simplicial objects of $\mathrm{N}(\mathcal{A})$ with nonnegatively graded chain complexes of R-modules. Let $M_\bullet$ be a simplicial object of $\mathrm{N}(\mathcal{A})$ and let $P_* = N_*(M)$ be the corresponding chain complex. Then $M_\bullet$ is S-free if and only if each P_n is a free left R-module. A map $|M_\bullet| \to M$ exhibits $M_\bullet$ as an S-hypercovering of a left R-module M if and only if the associated chain complex

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

is exact.

Our first goal in this section is to establish some basic existence and uniqueness theorems for free resolutions. For existence, we have the following:

Proposition 8.2.1.4. *Let $\mathcal{C}$ be a presentable ∞-category and let S be a set of objects of $\mathcal{C}$. Then, for every object $C \in \mathcal{C}$, there exists an S-hypercovering $X_\bullet : \mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathcal{C}_{/C}$ whose image in $\mathcal{C}$ is S-free.*

Proof. We will construct a compatible sequence of functors $F^{\leq n} : \mathrm{N}(\boldsymbol{\Delta}_{+,\leq n})^{op} \to \mathcal{C}$ satisfying the following conditions:

(*a*) For each $n \geq 0$, there exists an object $Z \in \mathcal{C}$ which is a coproduct of objects belonging to S and a map $Z \to F^{\leq n}([n])$ which induces an equivalence $Z \coprod L_n(F^{\leq n-1}) \to F^{\leq n}([n])$; here $L_n(F^{\leq n-1})$ denotes the nth latching object defined in §T.A.2.9.

(*b*) For $n \geq 0$ and each $Y \in \mathcal{C}$, the map $\mathrm{Map}_{\mathcal{C}}(Y, F^{\leq n}([n])) \to \mathrm{Map}_{\mathcal{C}}(Y, M_n(F^{\leq n-1}))$ is surjective on connected components; here $M_n(F^{\leq n-1})$ denotes the nth matching object as defined in §T.A.2.9.

(*c*) We have $F^{\leq -1}([-1]) = C$.

Assuming that such a sequence can be constructed, the union $\bigcup_n F^{\leq n}$ defines a simplicial object of $\mathcal{C}_{/C}$ having the desired properties. The construction of the diagrams $F^{\leq n}$ proceeds by induction on n. If $n = -1$, then $F^{\leq n}$ is uniquely determined by condition (*c*). Otherwise, extending $F^{\leq n-1}$ to a diagram $F^{\leq n}$ is equivalent to factoring the canonical map $\alpha : L_n(F^{\leq n-1}) \to M_n(F^{\leq n-1})$ as a composition

$$L_n(F^{\leq n-1}) \xrightarrow{\alpha'} F^{\leq n}([n]) \xrightarrow{\alpha''} M_n(F^{\leq n-1})$$

(see Proposition T.A.2.9.14). To satisfy condition (*a*) we must have $F^{\leq n}([n]) \simeq L_n(X_{\bullet}^{\leq n-1}) \coprod Z$ for some $Z \in \mathcal{C}$ which is a coproduct of objects of S. Let $M = M_n(F^{\leq n-1})$. To supply a morphism α'' which satisfies (*b*), it suffices to give a map $\eta : Z \to M$ such that the induced map $\mathrm{Map}_{\mathcal{C}}(Y, Z) \to \mathrm{Map}_{\mathcal{C}}(Y, M)$ is surjective on connected components for each $Y \in S$. For this, we take Z to be the coproduct $\coprod_{\alpha} Y_{\alpha}$, where α ranges over all equivalence classes of pairs (Y_{α}, u_{α}) such that $Y_{\alpha} \in S$ and $u_{\alpha} : Y_{\alpha} \to M$ is a morphism in $\mathcal{C}$. $\qquad\square$

We next address the uniqueness properties of S-free resolutions.

Proposition 8.2.1.5. *Let $\mathcal{C}$ be a presentable ∞-category and let S be a set of objects of $\mathcal{C}$. Let C be an object $\mathcal{C}$, let $Y_{\bullet}$ be an S-hypercovering of C, and let $X_{\bullet}$ be a simplicial object of $\mathcal{C}_{/C}$ whose image in $\mathcal{C}$ is S-free. Then there exists a map $f : X_{\bullet} \to Y_{\bullet}$ of simplicial objects in $\mathcal{C}_{/C}$.*

Proof. We construct f as the amalgam of a compatible sequence of maps $f^{\leq n} : X_{\bullet} | \mathrm{N}(\mathbf{\Delta}_{\leq n}) \to Y_{\bullet} | \mathrm{N}(\mathbf{\Delta}_{\leq n})$. Assume that $f^{\leq n-1}$ has already been constructed. Using Proposition T.A.2.9.14, we are reduced to the problem of solving the lifting problem

$$
\begin{array}{ccc}
L_n(X) & \longrightarrow & Y_n \\
\downarrow & & \downarrow \\
X_n & \longrightarrow & M_n(Y)
\end{array}
$$

in $\mathcal{C}_{/C}$. Since $X_{\bullet}$ is S-free, we can write X_n as a coproduct $L_n(X) \coprod F$, where F is a coproduct of objects of S. It then suffices to show that every map $F \to M_n(Y)$ can be lifted to a map $F \to Y_n$, which follows immediately from our assumption that $Y_{\bullet}$ is an S-hypercovering of C. $\qquad\square$

In the situation of Proposition 8.2.1.5, the map f is generally not unique. However, one can show that f is unique up to homotopy, in the following precise sense:

Definition 8.2.1.6. Let $\mathcal{C}$ be an ∞-category and let $X_{\bullet}, Y_{\bullet} : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{C}$ be simplicial objects. Let $\delta^* : \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_{/[1]})^{op}, \mathcal{C})$ be the functor given by composition with the forgetful functor $\mathbf{\Delta}_{/[1]} \to \mathbf{\Delta}$. A *simplicial homotopy* from $X_{\bullet}$ to $Y_{\bullet}$ is a morphism $h : \delta^*(X_{\bullet}) \to \delta^*(Y_{\bullet})$.

The inclusion maps $\{0\} \hookrightarrow [1] \hookleftarrow \{1\}$ induce forgetful functors

$$i_0^*, i_1^* : \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_{/[1]})^{op}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{C}).$$

Consequently, a simplicial homotopy h from $X_{\bullet}$ to $Y_{\bullet}$ determines maps of simplicial objects

$$f = i_0^*(h) : X_{\bullet} \to Y_{\bullet} \qquad g = i_1^*(h) : X_{\bullet} \to Y_{\bullet}.$$

In this case, we will say that h is a *simplicial homotopy from f to g*. We will say that f and g are *simplicially homotopic* if there is a simplicial homotopy from f to g.

Warning 8.2.1.7. The relation of simplicial homotopy introduced in Definition 8.2.1.6 is neither symmetric nor transitive in general.

Proposition 8.2.1.8. *Let $\mathcal{C}$ be a presentable ∞-category containing an object C, and let S be a set of objects of $\mathcal{C}$. Let $X_\bullet$ be a simplicial object of $\mathcal{C}_{/C}$ which is S-free and $Y_\bullet$ an S-hypercovering of C, and suppose we are given a pair of maps $f, g : X_\bullet \to Y_\bullet$ between simplicial objects of $\mathcal{C}_{/C}$. Then there is a simplicial homotopy from f to g.*

Proof. We employ the notation of Definition 8.2.1.6. Let K^0 be the full subcategory of $\mathrm{N}(\mathbf{\Delta}_{/[1]})^{op}$ spanned by the constant maps $\sigma : [n] \to [1]$ and let $p_0 : K^0 \to \mathrm{N}(\mathbf{\Delta})^{op}$ be the projection map, so that f and g determine a natural transformation $h^0 : p_0 \circ X_\bullet) \to p_0 \circ Y_\bullet$. We wish to show that h^0 can be extended to a natural transformation $\delta^*(X_\bullet) \to \delta^*(Y_\bullet)$. Let $\{\sigma_i : [n_i] \to [1]\}_{i \geq 0}$ be an enumeration of the objects of $\mathbf{\Delta}_{/[1]}$ which do not belong to K^0, having the property that $n_i < n_j$ implies $i < j$. For $j \geq 1$, let K^j denote the full subcategory of $\mathrm{N}(\mathbf{\Delta}_{[1]})^{op}$ spanned by the objects of K^0 together with the objects $\{\sigma_i\}_{i<j}$, and let $p_j : K^j \to \mathrm{N}(\mathbf{\Delta})^{op}$ be the projection map. We will show that h^0 can be extended to a compatible family of natural transformations $h^j : p_j \circ X_\bullet) \to p_j \circ Y_\bullet)$. The construction proceeds by induction on j. Assume that h^j has already been constructed; we wish to show that h^j can be extended to a natural transformation $h^{j+1} : p_{j+1} \circ X_\bullet \to p_{j+1} \circ Y_\bullet)$. The collection of injective and surjective maps endow $(\mathbf{\Delta}_{/[1]})^{op}$ with the structure of a Reedy category. Using Proposition T.A.2.9.14, we are reduced to solving a lifting problem of the form

$$
\begin{array}{ccc}
L_{n_j}(X) & \longrightarrow & Y_{n_j} \\
\downarrow & & \downarrow \\
X_{n_j} & \longrightarrow & M_{n_j}(Y)
\end{array}
$$

in the ∞-category $\mathcal{C}_{/C}$. Since $X_\bullet$ is S-free, we can write $X_{n_j} \simeq L_{n_j}(X) \coprod F$ where $F \in \mathcal{C}_{/C}$ is a coproduct of objects belonging to S. It therefore suffices to show that every map $F \to M_{n_j}(Y)$ can be lifted to a map $F \to Y_{n_j}$, which follows from our assumption that Y is an S-hypercovering. $\square$

Remark 8.2.1.9. In the situation of Proposition 8.2.1.8, the simplicial homotopy h between f and g is not unique. It is possible to address this issue by introducing a notion of higher (simplicial) homotopy. The proof of Proposition 8.2.1.8 then adapts to show that h is unique up to higher homotopy, which is itself unique up to a higher homotopy, and so forth. Since these results will not be needed in what follows, we leave them to the reader's imagination.

Remark 8.2.1.10. Let $\mathcal{A}$ be an abelian category and let $A_\bullet$ and $B_\bullet$ be simplicial objects of $\mathcal{A}$. Suppose we are given maps of simplicial objects $f, g : A_\bullet \to B_\bullet$ which are simplicially homotopic. Then the induced maps of unnormalized chain complexes $C_*(A) \to C_*(B)$ are chain homotopic: that is, there exists maps $h_n : N_n(A) \to N_{n+1}(B)$ such that $d \circ h_n + h_{n-1} \circ d = f - g$, where we adopt the convention that $h_{-1} = 0$. It follows that f and g induce the same map on homology objects $\mathrm{H}_*(A) \to \mathrm{H}_*(B)$.

To prove this, suppose that H is a simplicial homotopy from f to g. We then define h_n to be the sum of the maps

$$
A_n \xrightarrow{s_i} A_{n+1} \xrightarrow{H(\sigma_i)} B_{n+1},
$$

where s_i denotes the ith degeneracy map (induced by the unique surjection $a : [n+1] \to [n]$ satisfying $a(i) = a(i+1)$) and $\sigma_i \in \mathbf{\Delta}_{/[1]}$ denotes the map given by $\sigma_i(j) = \begin{cases} 0 & \text{if } j \leq i \\ 1 & \text{if } j > i. \end{cases}$

We now have the following general observation:

Proposition 8.2.1.11. *Let $\mathcal{C}$ be a presentable ∞-category containing a set of objects S and let $\mathcal{A}$ be an ordinary category. Let $\mathcal{C}'$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{C})$ spanned by the S-free simplicial objects and let $U : \mathcal{C}' \to \mathcal{C}$ be the geometric realization functor. Suppose we are given a functor $F_0 : \mathrm{h}\mathcal{C}' \to \mathcal{A}$ with the following property:*

($*$) Let $f, g : X_\bullet \to Y_\bullet$ be simplicially homotopic morphisms in $\mathcal{C}'$. Then $F_0(f) = F_0(g)$.

Then there exists a functor $F : h\mathcal{C} \to \mathcal{A}$ together with a natural transformation $\alpha : F_0 \to F \circ U$ having the following property: whenever $X_\bullet$ is an S-free S-hypercovering of an object $C \in \mathcal{C}$, the induced map $F_0(X_\bullet) \stackrel{\alpha}{\to} F(|X_\bullet|) \to F(C)$ is an isomorphism in $\mathcal{A}$. Moreover, the functor F (and natural transformation α) are unique up to (unique) isomorphism.

Proof. We can characterize the functor F abstractly as a left Kan extension of F_0 along the functor U. However, it is easy enough to give a concrete construction of F:

(a) For every object $C \in \mathcal{C}$, we can choose an S-free S-hypercovering $X_\bullet$ of C (Proposition 8.2.1.4). We then define $F(C)$ to be $F_0(X_\bullet) \in \mathcal{A}$.

(b) Let $u : C \to C'$ be a morphism in $\mathcal{C}$, and let $X_\bullet$ and $Y_\bullet$ be the S-free S-hypercoverings of C and C' chosen in (a). It follows from Proposition 8.2.1.5 that u can be lifted to a map of simplicial objects $\overline{u} : X_\bullet \to Y_\bullet$. We define $F(u) : F(C) \to F(C')$ to be the map $F_0(\overline{u}) : F_0(X_\bullet) \to F_0(Y_\bullet)$. It follows from Proposition 8.2.1.8 and assumption ($*$) that $F(u)$ does not depend on the choice of $\overline{u}$.

This construction determines a functor $F : h\mathcal{C} \to \mathcal{A}$. For each object $Y_\bullet \in \mathcal{C}'$, let $C = |Y_\bullet|$ and let $X_\bullet$ be the S-free S-hypercovering of C chosen in (a). Using Proposition 8.2.1.5, we can choose a map of simplicial objects $v : Y_\bullet \to X_\bullet$ in $\mathcal{C}_{/C}$. Let $\alpha(Y_\bullet) = F_0(v) : F_0(Y_\bullet) \to F_0(X_\bullet) = F(C)$. It follows from Proposition 8.2.1.8 and assumption ($*$) that the map $\alpha(C)$ does not depend on v, so that α determines a natural transformation of functors $F_0 \to F \circ U$.

We next claim that if $X_\bullet$ is an S-hypercovering of an object C, then the composite map $F_0(X_\bullet) \stackrel{\alpha(X_\bullet)}{\to} F(|X_\bullet|) \to F(C)$ is an isomorphism in $\mathcal{A}$. To prove this, let $Y_\bullet$ and $Z_\bullet$ denote the S-free S-hypercoverings of $|X_\bullet|$ and C chosen in (a). Choose a map of simplicial objects $u : X_\bullet \to Y_\bullet$ in $\mathcal{C}_{/|X_\bullet|}$ and a map of simplicial objects $v : Y_\bullet \to Z_\bullet$ in $\mathcal{C}_{/C}$; we wish to show that $F_0(v \circ u)$ is an isomorphism. To prove this, we use Proposition 8.2.1.5 to choose a map of simplicial objects $w : Z_\bullet \to X_\bullet$ in $\mathcal{C}_{/C}$. Then $w \circ (v \circ u)$ and $(v \circ u) \circ w$ are simplicially homotopic to the identity maps on $Z_\bullet$ and $X_\bullet$, respectively (Proposition 8.2.1.8). Using ($*$), we conclude that $F_0(w)$ is both right and left inverse to $F_0(v \circ u)$, so that $F_0(v \circ u)$ is an isomorphism. This completes the construction of F and α.

Now suppose that $F' : h\mathcal{C} \to \mathcal{A}$ is any other functor equipped with a natural transformation $\alpha' : F_0 \to F' \circ U$. There is a unique natural transformation $\beta : F \to F'$ such that the diagram

$$
\begin{array}{ccc}
 & F_0 & \\
{\scriptstyle \alpha}\swarrow & & \searrow{\scriptstyle \alpha'} \\
F \circ U & \xrightarrow{\ \ \beta'\ \ } & F' \circ U
\end{array}
$$

is commutative. The natural transformation β can be described explicitly as follows: if $C \in \mathcal{C}$ is an object and $X_\bullet$ be the S-free S-hypercovering of C chosen in (a), then β is the composite map

$$F(C) = F_0(X_\bullet) \to F'(|X_\bullet|) \to F'(C).$$

To complete the proof, it suffices to observe that if α' has the property that the composite map $F_0(X_\bullet) \to F'(C)$ is an isomorphism whenever $X_\bullet$ is an S-hypercovering of C, then the natural transformation β is an isomorphism. $\square$

Example 8.2.1.12. Let R be an associative ring, let $\mathcal{C}$ be the category of left R-modules, and let $\mathcal{A}$ be any abelian category.

Let $\mathcal{C}'$ denote the subcategory of $\mathrm{Ch}(\mathcal{C})$ consisting of nonnegatively graded chain complexes of free left R-modules, and suppose we are given an additive functor F from the category of free R-modules into $\mathcal{A}$.

Applying F termwise, we obtain a chain-complex valued functor $\mathcal{C}' \to \mathrm{Ch}(\mathcal{A})$ which we will also denote by F. For every integer n, the composite functor

$$\mathcal{C}' \xrightarrow{F} \mathrm{Ch}(\mathcal{A}) \xrightarrow{\mathrm{H}_n} \mathcal{A}$$

can be regarded as an $\mathcal{A}$-valued functor on simplicial objects in the category of free left R-modules. Remark 8.2.1.10 implies that this functor satisfies hypothesis $(*)$ of Proposition 8.2.1.11, and therefore induces a functor $L_n F : \mathcal{C} \to \mathcal{A}$. We refer to $L_n F$ as the *nth left derived functor of F*. Concretely, if N is a left R-module having a free resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to N,$$

then $L_n F(N)$ can be identified with the nth homology object of the chain complex

$$\cdots \to F(P_2) \to F(P_1) \to F(P_0).$$

Remark 8.2.1.13. In the special case where F is right exact and $\mathcal{A}$ has enough projective objects, the functors $L_n F$ are given by the composition

$$\mathrm{N}(\mathcal{C}) \longrightarrow \mathcal{D}^-(\mathcal{C}) \xrightarrow{LF} \mathcal{D}^-(\mathcal{A}) \xrightarrow{\pi_n} \mathrm{N}(\mathcal{A}),$$

where $LF : \mathcal{D}^-(\mathcal{C}) \to \mathcal{D}^-(\mathcal{A})$ is the left derived functor of F described in Example 1.3.3.4.

Example 8.2.1.14. Let R be an associative ring and let M be a right R-module. Then the usual tensor product construction $N \mapsto M \otimes_R N$ determines a functor from the category $\mathcal{C}$ of left R-modules to the category $\mathcal{A}b$ of abelian groups. The left derived functors of this construction are denoted by $\mathrm{Tor}_n^R(M, \bullet) : \mathcal{C} \to \mathcal{A}b$.

Variant 8.2.1.15. Let R be a graded associative ring and let $\mathcal{C}$ be the abelian category of graded left R-modules. For every integer n, let $R[n] \in \mathcal{C}$ denote the ring R itself, equipped with the shifted grading given by $R[n]_m \simeq R_{m-n}$. We say that a graded left R-module N is *graded-free* if it is isomorphic to a direct sum $\bigoplus_{\alpha \in A} R[n_\alpha]$ for some collection of integers $\{n_\alpha\}_{\alpha \in A}$. Let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ spanned by the graded-free R-modules.

Let **GrAb** denote the category of graded abelian groups. If M is a graded right R-module and N is a graded left R-module, then the algebraic tensor product $M \otimes_R N$ inherits a grading; we may therefore view the construction $N \mapsto M \otimes_R N$ as a functor from the category $\mathcal{C}$ to the category of graded abelian groups. Arguing as in Example 8.2.1.12, we can use Proposition 8.2.1.11 to define left derived functors

$$\mathrm{Tor}_n^R(M, \bullet) : \mathcal{C} \to \mathbf{GrAb}.$$

Concretely, the graded abelian group $\mathrm{Tor}_n^R(M, N)$ is given by nth homology of the chain complex

$$\cdots \to M \otimes_R P_2 \to M \otimes_R P_1 \to M \otimes_R P_0,$$

where P_* denotes a resolution of N by graded-free objects of $\mathcal{C}$.

Our next goal is to apply the formalism developed above to study the tensor product of modules over a ring spectrum. Let R be an $\mathbb{E}_1$-ring. Recall that the homotopy groups $\pi_* R$ have the structure of a graded ring and that if M is a (left or right) module spectrum over R, then $\pi_* M$ is a graded (left or right) module over $\pi_* R$. We can regard $\pi_* R$ as a kind of "first approximation" to R, and the ordinary tensor product of graded $\pi_* R$-modules as a "first approximation" to the relative tensor product over R. More precisely, suppose we are given objects $M \in \mathrm{RMod}_R$ and $N \in \mathrm{LMod}_R$, together with homotopy classes $x \in \pi_m M$ and $y \in \pi_n N$. Then x and y are represented by maps $R[m] \to R$ and $R[n] \to R$ in RMod_R and LMod_R, respectively. Combining these, we obtain a map of spectra

$$R[m+n] \simeq R[m] \otimes_R R[n] \xrightarrow{(x,y)} M \otimes_R N.$$

The image of the identity $1 \in \pi_0 R \simeq \pi_{m+n} R[m+n]$ is an element of $\pi_{m+n}(M \otimes_R N)$, which we will denote by $x \otimes y$. It is not difficult to see that the construction $(x, y) \mapsto x \otimes y$ is $\pi_* R$-bilinear, and therefore induces a map of graded abelian groups

$$\operatorname{Tor}_0^{\pi_* R}(\pi_* M, \pi_* N) \to \pi_*(M \otimes_R N).$$

This map is generally not an equivalence without some strong additional assumptions.

Definition 8.2.1.16. Let R be an $\mathbb{E}_1$-ring and let N be a left R-module spectrum. We say that M is *quasi-free* if $N \simeq \bigoplus_{\alpha \in A} R[n_\alpha]$ for some integers n_α. We let $\operatorname{LMod}_R^{\mathrm{qf}}$ denote the full subcategory of LMod_R spanned by the quasi-free left R-modules.

Proposition 8.2.1.17. *Let R be an $\mathbb{E}_1$-ring and let N be a quasi-free left R-module. For any R-module M, the map*

$$\operatorname{Tor}_0^{\pi_* R}(\pi_* M, \pi_* N) \to \pi_*(M \otimes_R N)$$

is an isomorphism of graded abelian groups.

Proof. Both sides are compatible with the formation of direct sums and suspensions. We may therefore assume that $N = R$, in which case the result is obvious. □

We can obtain information about the homotopy groups of an arbitrary tensor product $M \otimes_R N$ by resolving N by quasi-free R-modules.

Construction 8.2.1.18. Let R be an $\mathbb{E}_1$-ring and let $M \in \operatorname{RMod}_R$. If $P_\bullet$ is any simplicial object in LMod_R, then we can regard $M \otimes_R P_\bullet$ as a simplicial object in the ∞-category of Sp. Applying Remark 1.2.4.4, we obtain a spectral sequence of abelian groups $\{E_r^{p,q}, d_r\}_{r \geq 1}$ converging to $\pi_* M \otimes_R |P_\bullet|$, where $E_1^{*,q}$ is the normalized chain complex associated to the simplicial abelian group $\pi_q(M \otimes_R P_\bullet)$. If $P_\bullet'$ is another simplicial left R-module with associated spectral $\{E_r'^{p,q}, d_r'\}_{r \geq 1}$, then a map of simplicial objects $P_\bullet \to P_\bullet'$ induces a map of spectral sequences $\{E_r^{p,q}, d_r\}_{r \geq 1} \to \{E_r'^{p,q}, d_r'\}_{r \geq 1}$. If $f, g : P_\bullet \to P_\bullet'$ are simplicially homotopic maps of simplicial objects, then Remark 8.2.1.10 implies that the induced maps of chain complexes $E_1^{*,q} \to E_1'^{*,q}$ are chain homotopic, so that f and g induce the same map $E_r^{p,q} \to E_r'^{p,q}$ for $r = 2$ and therefore for $r \geq 2$.

Let S be the collection of all left R-modules of the form $R[n]$, where n is an integer, and let $\mathcal{C}'$ be the full subcategory of $\operatorname{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \operatorname{LMod}_R)$ spanned by the S-free simplicial objects. Let $\mathcal{A}$ denote the abelian category whose objects are spectral sequences $\{E_r^{p,q}, d_r\}_{r \geq 2}$ which begin at the second page. The above construction determines a functor $F_0 : \mathrm{h}\mathcal{C}' \to \mathcal{A}$ which carries simplicially homotopic morphisms to the same morphism in $\mathcal{A}$. Applying Proposition 8.2.1.11, we obtain a functor $F : \mathrm{hLMod}_R \to \mathcal{A}$ together with a natural transformation $F_0(P_\bullet) \to F(|P_\bullet|)$ which is an equivalence whenever $P_\bullet$ is an S-hypercovering of $|P_\bullet|$.

Let R be an $\mathbb{E}_1$-ring and suppose we are given modules $M \in \operatorname{RMod}_R$, $N \in \operatorname{LMod}_R$. Let $\{E_r^{p,q}, d_r\}_{r \geq 2}$ be the spectral sequence determined by Construction 8.2.1.18. Unwinding the definitions, we see that $\{E_r^{p,q}, d_r\}_{r \geq 2}$ is the spectral sequence associated to the simplicial spectrum $M \otimes_R P_\bullet$, where $P_\bullet$ is an S-free S-hypercovering of N. A choice of $P_\bullet$ gives a well-defined E_1-page for the spectral sequence, where for each $q \in \mathbf{Z}$, the chain complex $(E_1^{*,q}, d_1)$ is the normalized chain complex associated to the simplicial abelian group $\pi_q(M \otimes_R P_\bullet)$. Since $P_\bullet$ is S-free, each P_n is a quasi-free left R-module, so Proposition 8.2.1.17 gives an isomorphism of graded abelian groups

$$\operatorname{Tor}_0^{\pi_* R}(\pi_* M, \pi_* P_\bullet) \to \pi_*(M \otimes_R P_\bullet).$$

Since $P_\bullet$ is an S-hypercovering of N, the normalized chain complex associated to $\pi_* P_\bullet$ is a resolution of $\pi_* N$ by graded-free left $\pi_* R$-modules. Passing to homology, we obtain a canonical isomorphism

$$E_2^{p,*} \simeq \operatorname{Tor}_p^{\pi_* R}(\pi_* M, \pi_* N).$$

We have proven the following:

Proposition 8.2.1.19. *Let R be an $\mathbb{E}_1$-ring, let M be a right A-module, and let N be a left A-module. We regard $\pi_* M$ and $\pi_* N$ as graded modules over the graded ring $\pi_* R$. Then there exists a spectral sequence $\{E_r^{p,q}, d_r\}_{r \geq 2}$ with E_2-page $E_2^{p,q} = \mathrm{Tor}_p^{\pi_* A}(\pi_* M, \pi_* N)_q$ which converges (as in the proof of Proposition 1.2.2.14) to $\pi_{p+q}(M \otimes_R N)$.*

Remark 8.2.1.20. The spectral sequence $\{E_r^{p,q}, d_r\}$ of Proposition 8.2.1.19 depends functorially on the triple (R, M, N).

Remark 8.2.1.21. Let R, M, and N be as in Proposition 8.2.1.19. We have constructed a spectral sequence $\{E_r^{p,q}, d_r\}_{r \geq 2}$ by choosing a resolution $P_\bullet$ of N by quasi-free left R-modules. Using exactly the same reasoning, we can construct another spectral sequence $\{E_r'^{p,q}, d_r\}_{r \geq 2}$ by choosing a resolution $Q_\bullet$ of M by quasi-free right R-modules. In fact, these two spectral sequences are canonically isomorphic. Both can be identified with the spectral sequence associated to the simplicial spectrum $[n] \mapsto Q_n \otimes_R P_n$.

We now give a few simple applications of Proposition 8.2.1.19.

Corollary 8.2.1.22. *Let R be an $\mathbb{E}_1$-ring, M a right R-module, and N a left R-module. Suppose that R, M, and N are discrete. Then there exists a canonical isomorphism*

$$\pi_n(M \otimes_R N) \simeq \mathrm{Tor}_n^{\pi_0 R}(\pi_0 M, \pi_0 N).$$

Corollary 8.2.1.23. *Let A be a connective $\mathbb{E}_1$-ring, M a connective right R-module, and R a connective left A-module. Then:*

(1) *The relative tensor product $M \otimes_R N$ is connective.*

(2) *There is a canonical isomorphism $\pi_0(M \otimes_R N) \simeq \pi_0 M \otimes_{\pi_0 R} \pi_0 N$ in the category of abelian groups.*

Proof. This follows from the spectral sequence of Proposition 8.2.1.19, since $E_2^{p,q}$ vanishes for $p < 0$ or $q < 0$, while $E_2^{0,0} \simeq \pi_0 M \otimes_{\pi_0 R} \pi_0 N$. $\square$

Variant 8.2.1.24. Let R be an $\mathbb{E}_1$-ring. Using Propositions 6.3.2.18 and 4.2.1.33, we see that the ∞-category LMod_R is canonically enriched over the ∞-category of spectra. The formation of morphism objects determines a bifunctor

$$\mathrm{Mor}_R : \mathrm{LMod}_R^{op} \times \mathrm{LMod}_R \to \mathrm{Sp}.$$

Arguing as in Construction 8.2.1.18, we can associate to every pair of objects $M, N \in \mathrm{LMod}_R$ a spectral sequence $\{E_r^{p,q}, d_r\}_{r \geq 2}$ in the *opposite* of the category of abelian groups, whose second page is given by $E_2^{p,q} \simeq \mathrm{Ext}_{\pi_* R}^p(\pi_* M, \pi_* N)_{-q}$. Since the t-structure on the ∞-category Sp is not compatible with sequential limits, some care must be taken with the convergence of this spectral sequence. In good cases, one can show that it converges to $\pi_{-p-q} \mathrm{Mor}_R(M, N)$. For example, if M and R are connective and $\pi_n N \simeq 0$ for $n \gg 0$, a strong convergence statement can be deduced from Proposition 1.2.4.5.

8.2.2 Flat and Projective Modules

Let R be a connective $\mathbb{E}_1$-ring. In this section, we will see that there is a good theory of flat and projective R-modules, which reduces to the classical theory in the case where R is discrete.

Definition 8.2.2.1. Let R be an $\mathbb{E}_1$-ring. We will say that a left R-module M is *free* if is equivalent to a coproduct of copies of R (where we view R as a left module over itself). We will say that a free left module M is *finitely generated* if it can be written as a finite coproduct of copies of R.

Suppose that R is connective. We will say that a map $f : M \to N$ of connective left R-modules is *surjective* if it induces a surjection of $\pi_0 R$-modules $\pi_0 M \to \pi_0 N$.

Warning 8.2.2.2. The notion of free module introduced in Definition 8.2.2.1 is different from the general notion of freeness considered in §4.2.4. If M_0 is a spectrum, then the tensor product $R \otimes M_0$ is generally not free as a left R-module (in the sense of Definition 8.2.2.1), unless we assume that M_0 is a coproduct of copies of the sphere spectrum.

Remark 8.2.2.3. Using the long exact sequence of homotopy groups associated to an exact triangle, we conclude that a map $f : M \to N$ of connective modules over a connective $\mathbb{E}_1$-ring is surjective if and only if $\mathrm{fib}(f)$ is also connective.

Definition 8.2.2.4. Let R be a connective $\mathbb{E}_1$-ring. We will say that a left R-module P is *projective* if it is a projective object of the ∞-category $\mathrm{LMod}_R^{\mathrm{cn}}$ of connective left R-modules, in the sense of Definition T.5.5.8.18.

Remark 8.2.2.5. The terminology of Definition 8.2.2.4 is potentially ambiguous: a projective left R-module is typically not projective as an object of LMod_R. However, there is little risk of confusion, since the ∞-category LMod_R has no nonzero projective objects.

The following criterion for projectivity is often convenient:

Proposition 8.2.2.6. *Let $\mathcal{C}$ be a stable ∞-category equipped with a left-complete t-structure. Let $P \in \mathcal{C}_{\geq 0}$. The following conditions are equivalent:*

(1) *The object P is projective in $\mathcal{C}_{\geq 0}$.*

(2) *For every $Q \in \mathcal{C}_{\geq 0}$, the abelian group $\mathrm{Ext}_{\mathcal{C}}^1(P, Q)$ vanishes.*

(3) *For every $Q \in \mathcal{C}_{\geq 0}$ and every integer $i > 0$, the abelian group $\mathrm{Ext}_{\mathcal{C}}^i(P, Q)$ vanishes.*

(4) *For every $Q \in \mathcal{C}^{\heartsuit}$ and every integer $i > 0$, the abelian group $\mathrm{Ext}_{\mathcal{C}}^i(P, Q)$ vanishes.*

(5) *Given a fiber sequence*
$$N' \to N \to N'',$$
where $N', N, N'' \in \mathcal{C}_{\geq 0}$, the induced map $\mathrm{Ext}_{\mathcal{C}}^0(P, N) \to \mathrm{Ext}_{\mathcal{C}}^0(P, N'')$ is surjective.

Proof. It follows from Lemma 1.3.3.11 that $\mathcal{C}_{\geq 0}$ admits geometric realizations for simplicial objects, so that condition (1) makes sense. We first show that $(1) \Rightarrow (2)$. Let $f : \mathcal{C} \to \mathcal{S}$ be the functor corepresented by P. Let $M_\bullet$ be a Čech nerve for the morphism $0 \to Q[1]$, so that $M_n \simeq Q^n \in \mathcal{C}_{\geq 0}$. Then $Q[1]$ can be identified with the geometric realization $|M_\bullet|$. Since P is projective, $f(Q[1])$ is equivalent to the geometric realization $|f(M_\bullet)|$. We have a surjective map $* \simeq \pi_0 f(M_0) \to \pi_0 |f(M_\bullet)|$, so that $\pi_0 f(Q[1]) = \mathrm{Ext}_{\mathcal{C}}^1(P, Q) = 0$.

We now show that $(2) \Rightarrow (1)$. Proposition 1.4.2.22 implies that f is homotopic to a composition
$$\mathcal{C} \xrightarrow{F} \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathcal{S},$$

where F is an exact functor. Applying (2), we deduce that F is right t-exact (Definition 1.3.3.1). Lemma 1.3.3.11 implies that the induced map $\mathcal{C}_{\geq 0} \to \mathrm{Sp}^{\mathrm{conn}}$ preserves geometric realizations of simplicial objects. Applying Proposition 1.4.3.9, we conclude that $f | \mathcal{C}_{\geq 0}$ preserves geometric realizations as well.

The implications $(3) \Rightarrow (2)$ and $(3) \Rightarrow (4)$ are obvious. The implication $(2) \Rightarrow (3)$ follows by replacing Q by $Q[i-1]$, and $(2) \Rightarrow (5)$ follows immediately from the exactness of the sequence
$$\mathrm{Ext}_{\mathcal{C}}^0(P, N) \to \mathrm{Ext}_{\mathcal{C}}^0(P, N'') \to \mathrm{Ext}_{\mathcal{C}}^1(P, N').$$

We next show that $(5) \Rightarrow (2)$. Let $Q \in \mathcal{C}_{\geq 0}$ and let $\eta \in \mathrm{Ext}_{\mathcal{C}}^1(P, Q)$. Then η classifies a fiber sequence
$$Q \to Q' \xrightarrow{g} P.$$

Since $Q, P \in \mathcal{C}_{\geq 0}$, we have $Q' \in \mathcal{C}_{\geq 0}$ as well. Invoking (5), we deduce that g admits a section, so that $\eta = 0$.

We now complete the proof by showing that $(4) \Rightarrow (2)$. Let $Q \in \mathcal{C}_{\geq 0}$. For every integer $n \geq 0$, we have a fiber sequence

$$(\pi_n Q)[n] \to \tau_{\leq n} Q \to \tau_{\leq n-1} Q,$$

which gives rise to an exact sequence of abelian groups

$$\mathrm{Ext}_{\mathcal{C}}^{i+n}(P, \pi_n Q) \to \mathrm{Ext}_{\mathcal{C}}^i(P, \tau_{\leq n} Q) \to \mathrm{Ext}_{\mathcal{C}}^i(P, \tau_{\leq n-1} Q) \to \mathrm{Ext}_{\mathcal{C}}^{i+n+1}(P, \pi_n Q).$$

It follows from condition (4) that the tower of abelian groups $\{\mathrm{Ext}_{\mathcal{C}}^i(P, \tau_{\leq n} Q)\}_{n \geq 0}$ is constant for $n > -i$. Since $\mathcal{C}$ is left complete, we have $Q \simeq \varprojlim \tau_{\leq n} Q$ so that $\mathrm{Ext}_{\mathcal{C}}^i(P, Q) \simeq \mathrm{Ext}_{\mathcal{C}}^i(P, \tau_{\leq n} Q)$ for any $n > -i$. In particular, for $i > 0$ we have $\mathrm{Ext}_{\mathcal{C}}^i(P, Q) \simeq \mathrm{Ext}_{\mathcal{C}}^i(P, \pi_0 Q) \simeq 0$. $\qquad\square$

Proposition 8.2.2.7. *Let R be a connective $\mathbb{E}_1$-ring, and let P be a connective left R-module. The following conditions are equivalent:*

(1) *The left R-module P is projective.*

(2) *There exists a free R-module M such that P is a retract of M.*

Proof. Suppose first that P is projective. Choose a map of left R-modules $p : M \to P$, where M is free and the induced map $\pi_0 M \to \pi_0 P$ is surjective (for example, we can take M to be a direct sum of copies of R indexed by the set $\pi_0 P$). Invoking Proposition 8.2.2.6, we deduce that p admits a section (up to homotopy), so that P is a retract of M. This proves (2). To prove the converse, we observe that the collection of projective left R-modules is stable under retracts. It will therefore suffice to show that every free left R-module is projective. This follows immediately from the characterization given in Proposition 8.2.2.6. $\qquad\square$

Remark 8.2.2.8. It follows from the proof of Proposition 8.2.2.7 that, if $\pi_0 P$ is a finitely generated left module over $\pi_0 R$, then we can choose M to be a finitely generated free R-module.

Corollary 8.2.2.9. *Let R be a connective $\mathbb{E}_1$-ring. The following conditions on a connective left R-module P are equivalent:*

(1) *The R-module P is projective, and $\pi_0 P$ is finitely generated as a $\pi_0 R$-module.*

(2) *The R-module P is a compact projective object of $(\mathrm{LMod}_R)_{\geq 0}$.*

(3) *There exists a finitely generated free R-module M such that P is a retract of M.*

Proof. The equivalence $(2) \Leftrightarrow (3)$ follows by applying Corollary 6.2.2.16 to the composition

$$(\mathrm{LMod}_R)_{\geq 0} \to \mathrm{Sp}_{\geq 0} \overset{\Omega^\infty}{\to} \mathcal{S},$$

and invoking Example T.5.5.8.24. The equivalence $(1) \Leftrightarrow (3)$ follows from Remark 8.2.2.8. $\qquad\square$

Recall that, if N is a (discrete) left module over an associative ring R, we say that N is *flat* if the functor $M \mapsto M \otimes_R N$ is exact.

Definition 8.2.2.10. Let M be a left module over an $\mathbb{E}_1$-ring R. We will say that M is *flat* if the following conditions are satisfied:

(1) The homotopy group $\pi_0 M$ is flat as a left module over $\pi_0 R$, in the usual sense.

(2) For each $n \in \mathbf{Z}$, the natural map $\pi_n R \otimes_{\pi_0 R} \pi_0 M \to \pi_n M$ is an isomorphism of abelian groups.

Remark 8.2.2.11. Let R be a connective $\mathbb{E}_1$-ring. Then every flat left R-module is also connective.

Remark 8.2.2.12. Let R be a discrete $\mathbb{E}_1$-ring. A left R-module M is flat if and only if M is discrete, and $\pi_0 M$ is flat over $\pi_0 R$ (in the sense of classical algebra). In other words, Definition 8.2.2.10 is compatible with the usual definition of flatness.

Proposition 8.2.2.13. *Let R be an $\mathbb{E}_1$-ring, M a right R-module, and N a left R-module. Suppose that N is flat. For each $n \in \mathbf{Z}$, the canonical map*

$$\operatorname{Tor}_0^{\pi_0 R}(\pi_n M, \pi_0 N) \to \pi_n(M \otimes_R N)$$

is an isomorphism of abelian groups.

Proof. If N is flat, then $\operatorname{Tor}_p^{\pi_* R}(\pi_* M, \pi_* N)$ vanishes for $p > 0$, and is isomorphic to $\operatorname{Tor}_0^{\pi_0 R}(\pi_* M, \pi_0 N)$ for $p = 0$. It follows that the spectral sequence of Proposition 8.2.1.19 degenerates at the second page and yields the desired result. $\square$

   Our next goal is to prove an analogue of Lazard's theorem, which characterizes the class of flat modules over a connective $\mathbb{E}_1$-ring.

Lemma 8.2.2.14. *Let R be an $\mathbb{E}_1$-ring.*

 (1) *The collection of flat left R-modules is stable under coproducts, retracts, and filtered colimits.*

 (2) *Every free left R-module is flat. If R is connective, then every projective left R-module is flat.*

Proof. Assertion (1) is obvious, and (2) follows from Proposition 8.2.2.6. $\square$

Theorem 8.2.2.15 (Lazard's Theorem). *Let R be a connective $\mathbb{E}_1$-ring, and let N be a connective left A-module. The following conditions are equivalent:*

 (1) *The left R-module N can be obtained as a filtered colimit of finitely generated free modules.*

 (2) *The left R-module N can be obtained as a filtered colimit of projective left A-modules.*

 (3) *The left R-module N is flat.*

 (4) *The functor $M \mapsto M \otimes_A N$ is left t-exact; in other words, it carries $(\operatorname{RMod}_R)_{\leq 0}$ into $\operatorname{Sp}_{\leq 0}$.*

 (5) *If M is a discrete right R-module, then $M \otimes_R N$ is discrete.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious, (2) $\Rightarrow$ (3) follows from Lemma 8.2.2.14, (3) $\Rightarrow$ (4) from Proposition 8.2.2.13, and (4) $\Rightarrow$ (5) from Corollary 8.2.1.23.
   We next show that (5) $\Rightarrow$ (3). Suppose that (5) is satisfied. The functor $M \mapsto M \otimes_R N$ is exact, and carries the heart of RMod_R to the category of abelian groups. It therefore induces an exact functor from the abelian category of right $\pi_0 R$-modules to the category of abelian groups. According to Corollary 8.2.1.23, this functor is given by (classical) tensor product with the left $\pi_0 R$-module $\pi_0 N$. From the exactness we conclude that $\pi_0 N$ is a flat $\pi_0 R$-module.
   We now prove that the natural map $\phi : \pi_n R \otimes_{\pi_0 R} \pi_0 N \to \pi_n N$ is an isomorphism for all $n \in \mathbf{Z}$. For $n < 0$, this follows from the assumption that both N and R are connective. If $n = 0$ there is nothing to prove. We may therefore assume that $n > 0$, and we work by induction on n. Let M be the discrete right R-module corresponding to $\pi_0 A$. The inductive hypothesis implies that $\operatorname{Tor}_p^{\pi_* A}(\pi_* M, \pi_* N)_q$ vanishes unless $p = q = 0$ or $q \geq n$. These Tor-groups can be identified with the E_2-terms of the spectral sequence of Proposition 8.2.1.19, which computes the homotopy groups of the discrete spectrum $M \otimes_R N$. Consequently, we have $E_\infty^{0,0} = E_2^{0,0} \simeq \pi_0 N$. A simple calculation shows that $E_2^{0,n} \simeq \operatorname{coker}(\phi)$, and that if ϕ is surjective then $E_2^{1,n} \simeq \ker(\phi)$. To complete the proof, it suffices to prove that $E_2^{i,n} \simeq *$ for $0 \leq i \leq 1$. To see this, we observe that the vanishing of the groups $E_2^{i-r,n+r-1}$ and $E_2^{i+r,n-r+1}$ for $r \geq 2$ implies that $E_2^{i,n} \simeq E_\infty^{i,n}$, and the latter is a subquotient of $\pi_{i+n}(M \otimes_R N)$, which vanishes in view of assumption (5).

To complete the proof, it will suffice to show that (3) implies (1). Let $\mathcal{C}$ be the full subcategory of LMod_R spanned by a set of representatives for all free, finitely generated R-modules. Then $\mathcal{C}$ is essentially small, and consists of compact projective objects of $(\mathrm{Mod}_R)_{\geq 0}$ which generate $(\mathrm{Mod}_R)_{\geq 0}$ under small colimits. It follows (see §T.5.5.8) that the inclusion $\mathcal{C} \subseteq \mathrm{LMod}_R$ induces an equivalence $\mathcal{P}_\Sigma(\mathcal{C}) \to \mathrm{LMod}_R^{\mathrm{cn}}$. Applying Lemma T.5.1.5.5, we conclude that the identity functor from $\mathrm{LMod}_R^{\mathrm{cn}}$ is a left Kan extension of its restriction to $\mathcal{C}$. It follows that for every connective left R-module N, the canonical diagram $\mathcal{C}_{/N} = \mathcal{C} \times_{\mathrm{LMod}_R} (\mathrm{LMod}_R)_{/N} \to \mathrm{LMod}_R$ has N as a colimit. To complete the proof, it will suffice to show that $\mathcal{C}_{/N}$ is filtered provided that N is flat.

According to Proposition T.5.3.1.13, it will suffice to verify the following conditions:

(i) For every finite collection of objects $\{X_i\}$ of $\mathcal{C}_{/N}$, there exists an object $X \in \mathcal{C}_{/N}$ together with morphisms $X_i \to X$.

(ii) For every pair $X, Y \in \mathcal{C}_{/N}$, every nonnegative integer $n \geq 0$, and every map $S^n \to \mathrm{Map}_{\mathcal{C}_{/N}}(X, Y)$ in the homotopy category $\mathcal{H}$, there exists a morphism $Y \to Z$ in $\mathcal{C}_{/N}$ such that the induced map $S^n \to \mathrm{Map}_{\mathcal{C}}(X, Z)$ is nullhomotopic.

Assertion (i) follows immediately from the stability of $\mathcal{C}$ under finite coproducts. We now prove (ii). Suppose given a pair of maps $f : X \to N$ and $g : Y \to N$, respectively. We have a homotopy fiber sequence

$$\mathrm{Map}_{\mathcal{C}_{/N}}(f, g) \to \mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathrm{LMod}_R}(X, N).$$

Since LMod_R is stable, $\mathrm{Map}_{\mathcal{C}_{/N}}(f, g)$ is a torsor for $\mathrm{Map}_{\mathrm{LMod}_R}(X, \mathrm{fib}(g))$. It follows that any map $S^n \to \mathrm{Map}_{\mathrm{LMod}_R}(X, \mathrm{fib}(g))$ determines a homotopy class $\eta \in \mathrm{Ext}_R^{-n}(X, \mathrm{fib}(g))$. We wish to prove that there exists a commutative diagram

$$
\begin{array}{ccc}
Y & \longrightarrow & Z \\
& g \searrow \quad \swarrow h & \\
& N &
\end{array}
$$

such that the image of η in $\mathrm{Ext}_R^{-n}(X, \mathrm{fib}(h))$ vanishes. Arguing iteratively, we can reduce to the case where $X \simeq A$, so that η can be identified with an element of $\pi_n \mathrm{fib}(g)$.

Let $\eta' \in \pi_n Y$ denote the image of η. Our first step is to choose a diagram as above with the property that the image of η' in $\pi_n Z$ vanishes. We observe that η' lies in the kernel of the natural map $\pi_n(g) : \pi_n Y \to \pi_n N \simeq \mathrm{Tor}_0^{\pi_0 R}(\pi_n A, \pi_0 N)$. The classical version of Lazard's theorem (see [93]) implies that $\pi_0 N$ is isomorphic to a filtered colimit of free left $\pi_0 R$-modules. It follows that there exists a commutative diagram

$$
\begin{array}{ccc}
& P & \\
\bar{k} \nearrow & & \searrow \bar{h} \\
\pi_0 Y & \xrightarrow{\ \pi_0 g\ } & \pi_0 N
\end{array}
$$

of left $\pi_0 R$-modules, where P is a finitely generated free module, and the image of η' in $\pi_n R \otimes_{\pi_0 R} P$ vanishes. Using the freeness of P, we can realize $\bar{h}$ as $\pi_0 h$, where $h : Z \to N$ is a morphism of left R-modules, and Z is a finitely generated free module with $\pi_0 Z \simeq P$. Similarly, we can realize $\bar{k}$ as $\pi_0 k$, where $k : Y \to Z$ is a morphism of left R-modules. Using the freeness of Y again, we conclude that the diagram

$$
\begin{array}{ccc}
& Z & \\
k \nearrow & & \searrow h \\
Y & \xrightarrow{\ g\ } & N
\end{array}
$$

commutes in the homotopy category hLMod_R, and can therefore be lifted to a commutative triangle in LMod_A. By construction, the image of η' in $\pi_n Z$ vanishes. Replacing Y by Z, we may reduce to the case where $\eta' = 0$.

We now invoke the exactness of the sequence

$$\pi_{n+1}N \to \pi_n \operatorname{fib}(g) \to \pi_n Y$$

to conclude that η is the image of a class $\eta'' \in \pi_{n+1}N \simeq \pi_{n+1}R \otimes_{\pi_0 R} \pi_0 N$. Invoking Lazard's theorem once more, we deduce the existence of a commutative diagram of left $\pi_0 A$-modules

where Q is a finitely generated free module, and η'' is the image of some element of $\pi_{n+1}R \otimes_{\pi_0 R} N$. Arguing as before, we may assume that the preceding diagram is induced by a commutative triangle of left A-modules

Replacing Y by Z, we may assume that η'' lies in the image of the map $\pi_{n+1}Y \to \pi_{n+1}N$. The exactness of the sequence

$$\pi_{n+1}Y \to \pi_{n+1}N \to \pi_n \operatorname{fib}(g)$$

now implies that $\eta = 0$, as desired. This completes the proof of the implication $(3) \Rightarrow (1)$. $\square$

We now study the behavior of flatness under base change.

Proposition 8.2.2.16. *Let $f : A \to B$ be a map of $\mathbb{E}_1$-rings, let $G : \mathrm{LMod}_B \to \mathrm{LMod}_A$ be the forgetful functor, and let $F : \mathrm{LMod}_A \to \mathrm{LMod}_B$ be a left adjoint to G (given by $M \mapsto B \otimes_A M$, in view of Proposition 4.3.7.14). Then:*

(1) *The functor F carries free (projective, flat) A-modules to free (projective, flat) B-modules.*

(2) *Suppose that B is free (projective, flat) as a left A-module (that is, $G(B)$ is free, projective, or flat). Then G carries free (projective, flat) B-modules to free (projective, flat) A-modules.*

(3) *Suppose that, for every $n \geq 0$, the map f induces an isomorphism $\pi_n A \to \pi_n B$. Then F induces an equivalence of categories $\mathrm{LMod}_A^{\flat} \to \mathrm{LMod}_B^{\flat}$; here $\mathrm{LMod}_A^{\flat}$ denotes the full subcategory of LMod_A spanned by the flat left A-modules, and $\mathrm{LMod}_B^{\flat}$ is defined likewise.*

Proof. Assertions (1) and (2) are obvious. To prove (3), we first choose A' to be a connective cover of A (see Proposition 8.1.3.13). We have a homotopy commutative triangle of ∞-categories

It therefore suffices to prove the analogous assertion for the morphisms $A' \to A$ and $A' \to B$. In other words, we may reduce to the case where A is connective.

Since A is connective, the ∞-category Mod_A admits a t-structure. Let F' denote the composite functor

$$(\mathrm{LMod}_A)_{\geq 0} \subseteq \mathrm{LMod}_A \xrightarrow{F} \mathrm{Mod}_B .$$

Then F' has a right adjoint, given by the composition $G' = \tau_{\geq 0} \circ G$. Assertion (1) implies that F' preserves flatness, and a simple calculation of homotopy groups shows that G' preserves flatness as well. Consequently, F' and G' induce adjoint functors

$$\mathrm{LMod}_A^\flat \underset{G''}{\overset{F''}{\rightleftarrows}} \mathrm{LMod}_B^\flat \, .$$

It now suffices to show that the unit and counit of the adjunction are equivalences. In other words, we must show:

(*i*) For every flat left A-module M, the unit map $M \to \tau_{\geq 0} B \otimes_A M$ is an equivalence. For this, it suffices to show that $\pi_i M \simeq \pi_i \tau_{\geq 0} B \otimes_A M$ is an isomorphism for $i \in \mathbf{Z}$. If $i < 0$, then both groups vanish, so there is nothing to prove. If $i \geq 0$, then we must show that $\pi_i M \simeq \pi_i(B \otimes_A M)$, which follows immediately from Proposition 8.2.2.13 and the assumption that $\pi_i A \simeq \pi_i B$.

(*ii*) For every flat left B-module N, the counit map $B \otimes_A \tau_{\geq 0} G(N) \to N$ is an equivalence. In other words, we must show that for each $j \in \mathbf{Z}$, the map $\pi_j(B \otimes_A \tau_{\geq 0} G(N)) \to \pi_j N$ is an isomorphism of abelian groups. Since $G(N)$ is flat over A, Proposition 8.2.2.13 implies that the left side is given by $\pi_j B \otimes_{\pi_0 A} \pi_0 N$. The desired result now follows immediately from our assumption that N is flat.

$\square$

In general, if M is a flat left module over an $\mathbb{E}_1$-ring R, then then "global" properties of M as an R-module are often controlled by "local" properties of $\pi_0 R$, viewed as a left module over the discrete ring $\pi_0 R$. Our next pair of results illustrates this principle.

Lemma 8.2.2.17. *Let R be an $\mathbb{E}_1$-ring and let $f : M \to N$ be a map of flat left R-modules. Then f is an equivalence if and only if it induces an isomorphism $\pi_0 M \to \pi_0 N$.*

Proof. This follows immediately from the definition of flatness. $\square$

Proposition 8.2.2.18. *Let R be a connective $\mathbb{E}_1$-ring. A flat left R-module M is projective if and only if $\pi_0 M$ is a projective module over $\pi_0 R$.*

Proof. Suppose first that $\pi_0 M$ is freely generated by elements $\{\eta_i\}_{i \in I}$. Let $P = \bigoplus_{i \in I} R$, and let $f : P \to M$ be a map represented by $\{\eta_i\}_{i \in I}$. By construction, f induces an isomorphism $\pi_0 P \to \pi_0 M$. Lemma 8.2.2.17 implies that f is an equivalence, so that M is free.

In the general case, there exists a free $\pi_0 R$-module F_0 and a direct sum decomposition $F_0 \simeq N_0 \oplus \pi_0 M$. Replacing F_0 by $\oplus_{n \geq 0} F_0$ if necessary, we may assume that N_0 is itself free. The projection map $F_0 \to \pi_0 M$ is induced by a map $g : F \to M$ of left R-modules, where F is free. Then g induces a surjection $\pi_0 F \simeq F_0 \to \pi_0 M$. Using the flatness of M, we conclude that the maps

$$\pi_i F \simeq \pi_i R \otimes_{\pi_0 R} \pi_0 F \to \pi_i R \otimes_{\pi_0 R} \pi_0 M \simeq \pi_i M$$

are also surjective. Let N be a fiber of g, so that we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_i R \otimes_{\pi_0 R} \pi_0 N & \longrightarrow & \pi_i R \otimes_{\pi_0 R} \pi_0 F & \longrightarrow & \pi_i R \otimes_{\pi_0 R} \pi_0 M & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi''} & & \\
0 & \longrightarrow & \pi_i N & \longrightarrow & \pi_i F & \longrightarrow & \pi_i M & \longrightarrow & 0.
\end{array}
$$

Using the flatness of F and M, we deduce that the upper row is short exact, and that the maps ϕ and ϕ'' are isomorphisms. The snake lemma implies that ϕ' is an isomorphism; moreover, $\pi_0 N$ is isomorphic to the kernel of a surjection between flat $\pi_0 R$-modules, and is therefore itself flat. It follows that N is a flat R-module. Since $\pi_0 N$ is free, the first part of the proof shows that N is itself free.

Let $p : N \to F$ denote the natural map. Since $\pi_0 M$ is projective, the inclusion $\pi_0 N \subseteq \pi_0 F$ is split. Since F is free, we can lift this splitting to a morphism $q : F \to N$. Then $q \circ p : N \to N$ induces the identity map from $\pi_0 N$ to itself. Since N is free, we conclude that there is a homotopy $q \circ p \simeq \mathrm{id}_N$. It follows that N is a direct summand of F in the homotopy category $h\mathcal{C}$. Consequently, $M \simeq \mathrm{cofib}(p)$ can be identified with the complementary summand, and is therefore projective. $\qquad\square$

Corollary 8.2.2.19. *Let $f : R \to R'$ be a map of connective $\mathbb{E}_1$-rings. Let $\mathrm{Proj}(R)$ denote the full subcategory of LMod_R spanned by the projective left R-modules, and define $\mathrm{Proj}(R')$ similarly. Suppose that f induces an isomorphism $\pi_0 R \to \pi_0 R'$. Then the base change functor $M \mapsto R' \otimes_R M$ induces an equivalence of homotopy categories*

$$\phi : \mathrm{hProj}(R) \to \mathrm{hProj}(R').$$

Proof. We first show that the functor ϕ is fully faithful. For this, we must show that if P and Q are projective left R-modules, then the canonical map

$$\mathrm{Ext}_R^0(P, Q) \to \mathrm{Ext}_{R'}^0(R' \otimes_R P, R' \otimes_R Q)$$

is bijective. Without loss of generality, we may suppose that P is free. In this case, the left hand side can be identified with a product of copies of $\pi_0 Q$, while the right hand side can be identified with a product of copies of $\pi_0(R' \otimes_R Q)$. Since Q is connective, the latter module can be identified with $\mathrm{Tor}_0^{\pi_0 R}(\pi_0 R', \pi_0 Q)$ (Corollary 8.2.1.23), which is isomorphic to $\pi_0 Q$ in view of our assumption that f induces an isomorphism $\pi_0 R \to \pi_0 R'$.

We now prove that ϕ is essentially surjective. Let $\overline{P}$ be a projective R'-module. Then there exists a free R'-module $\overline{F}$ and an idempotent map $\overline{e} : \overline{F} \to \overline{F}$, so that $\overline{P}$ can be identified with the colimit of the sequence

$$\overline{F} \xrightarrow{\overline{e}} \overline{F} \xrightarrow{\overline{e}} \dots.$$

Choose a free left R-module F and an equivalence $\phi(F) \simeq \overline{F}$. Using the first part of the proof, we deduce the existence of a map $e : F \to F$ (not necessarily idempotent) such that the diagram

$$
\begin{array}{ccc}
\phi(F) & \xrightarrow{\ \phi(e)\ } & \phi(F) \\
\downarrow & & \downarrow \\
\overline{F} & \xrightarrow{\ \overline{e}\ } & \overline{F}
\end{array}
$$

commutes up to homotopy. Since the functor $M \mapsto R' \otimes_R M$ preserves colimits, we deduce that $\overline{P}$ is equivalent to $\phi(P)$, where P denotes the colimit of the sequence

$$F \xrightarrow{e} F \xrightarrow{e} \dots.$$

To complete the proof, it will suffice to show that P is projective. In view of Proposition 8.2.2.18, it will suffice to show that $\pi_0 P$ is a projective module over the ordinary associative ring $\pi_0 R$, and that P is a flat R-module. The first assertion follows from the isomorphism

$$\pi_0 P \simeq \pi_0(R' \otimes_R P) \simeq \pi_0 \overline{P},$$

and the second from the observation that the collection of flat left R-modules is stable under filtered colimits (Lemma 8.2.2.14). $\qquad\square$

Remark 8.2.2.20. Let A be an $\mathbb{E}_1$-ring, and let P be a projective left A-module. Then P is a finitely generated projective left A-module if and only if $\pi_0 P$ is finitely generated as a (discrete) left module over $\pi_0 A$. The "only if" direction is obvious. For the converse, suppose that $\pi_0 P$ is generated by a finite set of elements $\{x_i\}_{i \in I}$. Let M be the (finitely generated) free module on a set of generators $\{X_i\}_{i \in I}$, so that we have a canonical map $\phi : M \to P$. Since P is projective and ϕ induces a surjection $\pi_0 M \to \pi_0 P$, the map ϕ splits (Proposition 8.2.2.6), so that P is a direct summand of M.

Remark 8.2.2.21. Let R be an $\mathbb{E}_2$-ring, and regard the ∞-category LMod_R as a monoidal ∞-category (whose monoidal structure is given by relative tensor product over R). Then $\pi_0 R$ is a commutative ring. The proof of Proposition 8.2.1.19 gives a spectral sequence $\{E_r^{p,q}, d_r\}_{r \geq 2}$ in the category of $\pi_0 R$-modules with
$$E_2^{p,q} = \mathrm{Tor}_p^{\pi_* R}(\pi_* M, \pi_* N)_q,$$
which converges to $\pi_{p+q}(M \otimes_R N)$. If M or N is flat over R, then $E_2^{p,q}$ vanishes for $p \neq 0$. It follows that this spectral sequence degenerates at the E_2-page, and we get a canonical isomorphism
$$\pi_*(M \otimes_R N) \simeq E_2^{0,*} \simeq \mathrm{Tor}_0^{\pi_* R}(\pi_* M, \pi_* N).$$
It follows that if M and N are both flat over R, then the relative tensor product $M \otimes_R N$ is again flat over R (this can also be deduced from Theorem 8.2.2.15). Since the unit object $R \in \mathrm{LMod}_R$ is flat, Proposition 2.2.1.1 implies that the full subcategory $\mathrm{LMod}_R^\flat \subseteq \mathrm{LMod}_R$ inherits the structure of a monoidal ∞-category.

If R is an $\mathbb{E}_{k+1}$-ring, then $\mathrm{LMod}_R^\flat$ inherits the structure of an $\mathbb{E}_k$-monoidal ∞-category. If A is an $\mathbb{E}_k$-algebra over R, then we will say that A is *flat over* R if A is flat when regarded as a left R-module. We let $\mathrm{Alg}_R^{(k),\flat}$ denote the full subcategory of $\mathrm{Alg}_R^{(k)}$ spanned by the flat $\mathbb{E}_k$-algebras. If $k = \infty$, we will denote $\mathrm{Alg}_R^{(k),\flat}$ by $\mathrm{CAlg}_R^\flat$.

Remark 8.2.2.22. Let R be an $\mathbb{E}_\infty$-ring. Combining Corollary 3.4.1.7 with Remark 8.2.2.21, we can identify the ∞-category $\mathrm{CAlg}_R^\flat$ with the full subcategory of $\mathrm{CAlg}_{R/}$ spanned by those morphisms $\phi : R \to A$ which exhibit A as a flat module (either left or right) over R.

Remark 8.2.2.23. Let R be an $\mathbb{E}_{k+1}$-ring and let A be a flat $\mathbb{E}_k$-algebra over R. If R is connective, then A is also connective; if R is discrete, then A is also discrete.

From Proposition 8.2.2.16, we immediately deduce the following:

Proposition 8.2.2.24. *Let* $f : R \to R'$ *be a map of* $\mathbb{E}_{k+1}$*-rings. Suppose that* f *induces an isomorphism* $\pi_i R \to \pi_i R'$ *for all* $i \geq 0$*. Then the tensor product functor* $A \mapsto R' \otimes_R A$ *induces an equivalence from the* ∞*-category* $\mathrm{Alg}_R^{(k),\flat}$ *of flat* $\mathbb{E}_k$*-algebras over* R *to the* ∞*-category* $\mathrm{Alg}_R'^{(k),\flat}$ *of flat* $\mathbb{E}_k$*-algebras over* R'*.*

8.2.3 Injective Objects of Stable ∞-Categories

In §8.2.2, we studied projective objects in the ∞-category of (connective) left modules over an $\mathbb{E}_1$-ring R. In this section, we discuss the formally dual notion of an *injective* object of a stable ∞-category equipped with a t-structure. We begin with a restatement of Proposition 8.2.2.6:

Proposition 8.2.3.1. *Let* $\mathcal{C}$ *be a stable* ∞*-category equipped with a right-complete t-structure. Let* $I \in \mathcal{C}_{\leq 0}$*. The following conditions are equivalent:*

(1) *When viewed as an object of* $(\mathcal{C}_{\leq 0})^{op}$*,* I *is projective. In other words, for every cosimplicial object* $X^\bullet$ *of* $\mathcal{C}_{\leq 0}$*, the map*
$$|\mathrm{Map}_{\mathcal{C}}(X^\bullet, I)| \to \mathrm{Map}_{\mathcal{C}}(\varprojlim X^\bullet, I)$$
is a homotopy equivalence.

(2) *For every* $X \in \mathcal{C}_{\leq 0}$*, the abelian group* $\mathrm{Ext}_{\mathcal{C}}^1(X, I)$ *vanishes.*

(3) *For every* $X \in \mathcal{C}_{\leq 0}$ *and every integer* $i > 0$*, the abelian group* $\mathrm{Ext}_{\mathcal{C}}^i(X, I)$ *vanishes.*

(4) *For every* $X \in \mathcal{C}^\heartsuit$ *and every integer* $i > 0$*, the abelian group* $\mathrm{Ext}_{\mathcal{C}}^i(X, I)$ *vanishes.*

(5) *Given a fiber sequence*
$$N' \to N \to N'',$$
where $N', N, N'' \in \mathcal{C}_{\leq 0}$*, the induced map* $\mathrm{Ext}_{\mathcal{C}}^0(N', I) \to \mathrm{Ext}_{\mathcal{C}}^0(N, I)$ *is surjective.*

Proof. Apply Proposition 8.2.3.1 to the ∞-category $\mathcal{C}^{op}$.   □

Definition 8.2.3.2. Let $\mathcal{C}$ be a stable ∞-category equipped with a right-complete t-structure. We will say that an object $I \in \mathcal{C}$ is *injective* if I belongs to $\mathcal{C}_{\leq 0}$ and satisfies the equivalent conditions of Proposition 8.2.3.1.

Let $\mathcal{C}$ be a stable ∞-category equipped with a complete t-structure. At a formal level, the notion of an injective object of $\mathcal{C}_{\leq 0}$ is entirely dual to the notion of a projective object of $\mathcal{C}_{\geq 0}$. However, these two notions behave very differently in practice. For example, if $\mathcal{C}$ is the ∞-category of left modules over an $\mathbb{E}_1$-ring R, then the class of projective objects of $\mathcal{C}_{\geq 0}$ admits a concrete description: according to Proposition 8.2.2.7, they are precisely the left R-modules which arise as direct summands of free left R-modules. The injective objects of $\mathcal{C}_{\leq 0}$ are much more difficult to describe. However, we will see in a moment that in some sense, the problem is really one of classical homological algebra: the classification of injective objects of the stable ∞-category $\mathcal{C}$ is equivalent to classifying the injective objects in the abelian category of discrete modules over $\pi_0 R$:

Proposition 8.2.3.3. *Let $\mathcal{C}$ be a presentable stable ∞-category equipped with a right-complete t-structure, and assume that $\mathcal{C}_{\leq 0}$ is closed under small filtered colimits. Then:*

(a) *Let I be an injective object of $\mathcal{C}_{\leq 0}$ and let $X \in \mathcal{C}^{\heartsuit}$. Then for $n \geq 0$, the abelian groups $\operatorname{Ext}^i_{\mathcal{C}}(X, \tau_{\geq -n}I)$ vanish for $0 < i \leq n+1$.*

(b) *If I is an injective object of $\mathcal{C}_{\leq 0}$, then $\pi_0 I$ is an injective object in the abelian category $h\mathcal{C}^{\heartsuit}$.*

(c) *Let X and I be objects of $\mathcal{C}_{\leq 0}$, and suppose we are given a map $f_0 : \pi_0 X \to \pi_0 I$. If I is injective, then f_0 can be extended to a map $f : X \to I$. Moreover, f is unique up to homotopy.*

(d) *Let $f_0 : \pi_0 X \to \pi_0 I$ be as in (2). Assume that I is injective and that f_0 is an isomorphism in the abelian category $h\mathcal{C}^{\heartsuit}$, and let $f : X \to I$ be a lifting of f_0. Then f is an equivalence if and only if X is injective.*

Proof. We first prove (a). We have a fiber sequence

$$\tau_{\geq -n}I \to I \to \tau_{\leq -n-1}I$$

and therefore an exact sequence of abelian groups

$$\operatorname{Ext}^{i-1}_{\mathcal{C}}(X, \tau_{\leq -n-1}I) \to \operatorname{Ext}^i_{\mathcal{C}}(X, \tau_{\geq -n}I) \to \operatorname{Ext}^i_{\mathcal{C}}(X, I).$$

Since $X \in \mathcal{C}_{\geq 0}$ and $\tau_{\leq -n-1}I \in \mathcal{C}_{\leq -n-1}$, the groups $\operatorname{Ext}^{i-1}_{\mathcal{C}}(X, \tau_{\leq -n-1}I)$ vanish for $i \leq n+1$, and the injectivity I implies that $\operatorname{Ext}^i_{\mathcal{C}}(X, I) \simeq 0$ for $i > 0$. It follows that $\operatorname{Ext}^i_{\mathcal{C}}(X, \tau_{\geq -n}I) \simeq 0$ for $0 < i \leq n+1$.

To prove (b), suppose we are given a fiber sequence

$$M' \to M \to M'',$$

in $\mathcal{C}^{\heartsuit}$. We wish to show that every map $M' \to \pi_0 I$ can be extended to a map $M \to \pi_0 I$, which follows from the vanishing of $\operatorname{Ext}^1_{\mathcal{C}}(M'', \pi_0 I)$ (which is a special case of (a)).

We now prove (C). Let $I \in \mathcal{C}_{\leq 0}$ be injective and let $X \in \mathcal{C}_{\leq 0}$ be arbitrary. We have a fiber sequence

$$\pi_0 X \to X \to \tau_{\leq -1}X.$$

It follows from characterization (5) of Proposition 8.2.3.1 that every map

$$f_0 \in \operatorname{Map}_{\mathcal{C}}(\pi_0 X, \pi_0 I) \simeq \operatorname{Map}_{\mathcal{C}}(\pi_0 X, I)$$

factors (up to homotopy) through a map $f \in \operatorname{Map}_{\mathcal{C}}(X, I)$. To prove that f is unique up to homotopy, it suffices to show that the homotopy fibers of the map $\operatorname{Map}_{\mathcal{C}}(X, I) \to \operatorname{Map}_{\mathcal{C}}(\pi_0 X, I)$ are connected. These

homotopy fibers are torsors for the space $\mathrm{Map}_{\mathcal{C}}(\tau_{\leq -1}X, I)$, which is connected since $\mathrm{Ext}^0_{\mathcal{C}}(\tau_{\leq -1}X, I) \simeq \mathrm{Ext}^1_{\mathcal{C}}(\tau_{\leq 0}(X[1]), I) \simeq 0$.

Now consider assertion (*d*). The "only if" direction is obvious: if $f : X \to I$ is an equivalence and I is injective, then X is also injective. To prove the converse, let us assume that $f : X \to I$ is a map between injective objects of $\mathcal{C}_{\leq 0}$ which induces an equivalence $f_0 : \pi_0 X \to \pi_0 I$. Let $g_0 : \pi_0 I \to \pi_0 X$ be a homotopy inverse to f_0. Since X is injective, assertion (*b*) implies that g_0 can be lifted to a map $g : I \to X$. Then the composition $g \circ f : X \to X$ induces the identity map $\pi_0 X \to \pi_0 X$. It follows from (*b*) that $g \circ f$ is homotopic to id_X. The same argument shows that $f \circ g$ is homotopic to id_I, so that g is a homotopy inverse to f and therefore f is an equivalence. $\square$

If R is a connective $\mathbb{E}_1$-ring, then Corollary 8.2.2.19 asserts that the homotopy category of the ∞-category of projective left R-modules is equivalent to the ordinary category of projective modules over $\pi_0 R$. Our next goal is to prove an analogous result for injective modules:

Theorem 8.2.3.4. *Let $\mathcal{C}$ be a presentable stable ∞-category equipped with a right-complete t-structure, and assume that $\mathcal{C}_{\leq 0}$ is closed under filtered colimits. Let $\mathcal{C}^{\mathrm{inj}}$ denote the full subcategory of $\mathcal{C}_{\leq 0}$ spanned by the injective objects. Let $\mathcal{A} = \mathrm{h}\mathcal{C}^{\heartsuit}$ be the heart of $\mathcal{C}$, and let $\mathcal{A}^{\mathrm{inj}}$ be the full subcategory of $\mathcal{A}$ spanned by the injective objects. Then the construction $Q \mapsto \pi_0 Q$ determines an equivalence of categories $\theta : \mathrm{h}\mathcal{C}^{\mathrm{inj}} \to \mathcal{A}^{\mathrm{inj}}$.*

The proof of Theorem 8.2.3.4 will require some preliminaries.

Lemma 8.2.3.5. *Let α be an ordinal and let $(\alpha) = \{\beta : \beta < \alpha\}$ be the collection of ordinals smaller than α. Let $F : \mathrm{N}(\alpha) \to \mathcal{S}$ be a functor with the following property: for every ordinal $\beta < \alpha$, the map $F(\beta) \to \varprojlim_{\gamma < \beta} F(\gamma)$ has connected homotopy fibers. Then $\varprojlim_{\beta < \alpha} F(\beta)$ is connected.*

Proof. Using Proposition T.4.2.4.4, we may assume without loss of generality that F arises from a diagram $X : (\alpha) \to \mathrm{Set}_{\Delta}$ which is fibrant with respect to the injective model structure. Then $\varprojlim_{\beta < \alpha} F(\beta)$ is represented by the Kan complex $\varprojlim_{\beta < \alpha} X(\beta)$ (Theorem T.4.2.4.1). The assumption that X is fibrant is equivalent to the requirement that each of the maps $\theta_{\beta} : X(\beta) \to \varprojlim_{\gamma < \beta} X(\gamma)$ is a Kan fibration, and we are given that each of the maps θ_{β} has connected homotopy fibers. Let x and y be vertices of $\varprojlim_{\beta < \alpha} X(\beta)$, and let x_{β} and y_{β} denote the images of x and y in $X(\beta)$ for $\beta < \alpha$. To show that x and y belong to the same path component of $\varprojlim_{\beta < \alpha} X(\beta)$, we must construct a compatible system of edges $\{e_{\beta} : x_{\alpha} \to y_{\beta}\}$ in $X(\beta)$. The construction proceeds by induction on β. Assume that the edges $\{e_{\gamma}\}_{\gamma < \beta}$ have been constructed, thereby determining an edge $e' : x' \to y'$ in $\varprojlim_{\gamma < \beta} X(\gamma)$. Since θ_{β} is a Kan fibration, we can choose an edge $\overline{e}' : x_{\beta} \to \overline{y}'$ in $X(\beta)$ lying over e'. Since the fiber $X(\beta)_{y'}$ is path connected, we can choose a path $\overline{e}'' : \overline{y}' \to y_{\beta}$ in $X(\beta)_{y'}$. Using the fact that θ_{β} is a Kan fibration again, we conclude that there exists a 2-simplex σ:

$$
\begin{array}{ccc}
 & \overline{y}' & \\
\overline{e}' \nearrow & & \searrow \overline{e}'' \\
x_{\beta} & \xrightarrow[\;e_{\beta}\;]{} & y_{\beta}
\end{array}
$$
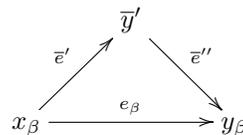
lying over the degenerate 2-simplex associated to e', which produces the desired edge e_{β} lying over e'. $\square$

Lemma 8.2.3.6. *Let $\mathcal{C}$ be a presentable ∞-category equipped with a right complete t-structure, and assume that $\mathcal{C}_{\leq 0}$ is closed under filtered colimits. Then there exists a small collection of objects $\{X_i \in \mathcal{C}^{\heartsuit}\}_{i \in I}$ such that an object $Q \in \mathcal{C}_{\leq 0}$ is injective if and only if the abelian groups $\mathrm{Ext}^n_{\mathcal{C}}(X_i, Q)$ vanish for all $i \in I$ and $n > 0$.*

Proof. Choose a set of objects $\{Y_j\}_{j \in J}$ which generates the abelian category $\mathrm{h}\mathcal{C}^{\heartsuit}$ under small colimits, and let $\{X_i\}_{i \in I}$ be a collection of representatives for all isomorphism classes of quotients of the objects $\{Y_j\}_{j \in J}$.

Since $h\mathcal{C}^\heartsuit$ is a Grothendieck abelian category, the collection I is small (see the proof of Proposition 1.3.5.3). We claim that this collection of objects has the desired property.

Let Q be an object of $\mathcal{C}_{\leq 0}$ such that $\operatorname{Ext}^n_\mathcal{C}(X_i, Q) \simeq 0$ for $i \in I$ and $n > 0$; we wish to prove that Q is injective. To prove this, consider an arbitrary object $Z \in \mathcal{C}^\heartsuit$ and integer $n > 0$; we will show that $\operatorname{Ext}^n_\mathcal{C}(Z, Q) \simeq 0$. To this end, we first construct a transfinite sequence of subobjects $Z_\alpha \subseteq Z$ as follows. Assume that α is an ordinal that the subobjects $\{Z_\beta \subseteq Z\}_{\beta < \alpha}$ have been constructed. If the induced map $\phi : \varinjlim Z_\beta \to Z$ is not an isomorphism, then there exists an index $j \in J$ and a map $Y_j \to Z$ which does not factor through ϕ. We define Z_α to be the image of the map $Y_j \oplus (\varinjlim_{\beta < \alpha} Z_\beta) \to Z$.

The proof of Proposition 1.3.5.3 shows that the collection of isomorphism classes of subobjects of Z is small, so this process must eventually stop: that is, we have $\varinjlim_{\beta < \alpha} Z_\beta \simeq Z$ for some ordinal α. Then $\operatorname{Ext}^n_\mathcal{C}(Z, Q) \simeq \pi_0 \varprojlim_{\beta < \alpha} \operatorname{Map}_\mathcal{C}(Z_\beta[-n], Q)$. To prove that this group vanishes, it suffices (by Lemma 8.2.3.5) to show that for each $\beta < \alpha$, the map

$$\theta : \operatorname{Map}_\mathcal{C}(Z_\beta[-n], Q) \to \operatorname{Map}_\mathcal{C}(\varinjlim_{\gamma < \beta} Z_\gamma[-n], Q)$$

has connected homotopy fibers. By construction, the map $\varinjlim_{\gamma < \beta} Z_\gamma \to Z_\beta$ is a monomorphism in $h\mathcal{C}^\heartsuit$ whose cokernel is given by X_i for some $i \in I$. We then have a fiber sequence

$$X_i[-n-1] \to \varinjlim_{\gamma < \beta} Z_\gamma[-n] \to Z_\beta[-n]$$

so that θ is a pullback of the map $\theta' : * \to \operatorname{Map}_\mathcal{C}(X_i[-n-1], Q)$. It therefore suffices to show that θ' has connected homotopy fibers: that is, that the homotopy $\pi_1 \operatorname{Map}_\mathcal{C}(X_i[-n-1], Q) \simeq \operatorname{Ext}^n(X_i, Q)$ vanishes, which follows from our assumption on I. $\qquad\square$

Proposition 8.2.3.7. *Let $\mathcal{C}$ be a presentable stable ∞-category equipped with a right-complete t-structure, and assume that $\mathcal{C}_{\leq 0}$ is closed under filtered colimits. Let Q_0 be an injective object of the abelian category $h\mathcal{C}^\heartsuit$. Then there exists a map $\phi : Q_0 \to Q$ in $\mathcal{C}_{\leq 0}$, where I is an injective object of $\mathcal{C}$ and ϕ induces an equivalence $Q_0 \to \pi_0 Q$.*

Proof. Let $\{X_i\}_{i \in I}$ be as in Lemma 8.2.3.6. Since $\mathcal{C}$ is presentable, we can choose a regular cardinal κ such that each of the objects X_i is κ-compact. We will extend the object Q_0 to a transfinite sequence of objects $\{Q_\alpha \in \mathcal{C}_{\leq 0}\}_{\alpha < \kappa}$ with the following properties:

(a) If $\lambda < \kappa$ is a nonzero limit ordinal, then $Q_\lambda \simeq \varinjlim_{\alpha < \lambda} Q_\lambda$.

(b) The map $Q_0 \to Q_\alpha$ induces an equivalence $Q_0 \to \pi_0 Q_\alpha$ for each $\alpha < \kappa$.

(c) Let $\alpha < \kappa$ and let $\eta \in \operatorname{Ext}^n_\mathcal{C}(X_i, Q_\alpha)$ for some $i \in I$ and some $n > 0$. Then the image of η vanishes in $\operatorname{Ext}^n_\mathcal{C}(X_i, Q_{\alpha+1})$.

Assuming that such a construction is possible, let $Q = \varinjlim_{\alpha < \kappa} Q_\alpha$. Then the natural map $Q_0 \to Q$ induces an equivalence $Q_0 \simeq \pi_0 Q$ by virtue of (b) (together with the fact that the t-structure on $\mathcal{C}$ is compatible with filtered colimits). We claim that Q is injective. To prove this, consider an arbitrary class $\eta \in \operatorname{Ext}^n_\mathcal{C}(X_i, Q)$ for $i \in I$ and $n > 0$. Since X_i is κ-compact, η is the image of a class $\eta_0 \in \operatorname{Ext}^n_\mathcal{C}(X_i, Q_\alpha)$ for some $\alpha < \kappa$. The image of η_0 in $\operatorname{Ext}^n_\mathcal{C}(X_i, Q_{\alpha+1})$ vanishes by (c), so that $\eta = 0$ as desired.

It remains to construct the sequence Q_α. We proceed by induction on α, the case where α is a limit ordinal being prescribed by condition (a). Let us therefore suppose that Q_α has been defined. Let S be the set of all triples (n, i, η), where $\eta \in \operatorname{Ext}^n_\mathcal{C}(X_i, Q_\alpha)$. Choose a well-ordering of the set S having order type β. For each $\gamma < \beta$, let $\eta_\gamma \in \operatorname{Ext}^{n_\gamma}_\mathcal{C}(X_{i_\gamma}, Q_\alpha)$ denote the corresponding class. We will construct a transfinite sequence of objects $\{P_\gamma\}_{\gamma \leq \beta+1}$ of $\mathcal{C}_{\leq 0}$ with the following properties:

(a') We have $P_0 = Q_\alpha$. If $\lambda \leq \beta + 1$ is a limit ordinal, then $P_\lambda \simeq \varinjlim_{\gamma < \lambda} P_\gamma$.

(b') For each $\gamma \leq \beta + 1$, the map $Q_0 \to P_\gamma$ induces an equivalence $Q_0 \simeq \pi_0 P_\gamma$.

(c') For each $\gamma \leq \beta$, the image of η_γ in $\mathrm{Ext}^{n_\gamma}(X_{i_\gamma}, P_{\gamma+1})$ vanishes.

Assuming that this construction is possible, we can complete the proof by setting $Q_{\alpha+1} = P_{\beta+1}$.

The construction of the objects P_γ proceeds by induction on γ. When γ is a limit ordinal, the definition of P_γ is determined by (a'). Let us therefore assume that P_γ has been constructed. Let $n = n_\gamma$ and $i = i_\gamma$, and let η denote the image of η_γ in $\mathrm{Ext}^n_{\mathcal{C}}(X_i, P_\gamma)$. Then η determines a map $\phi : X_i[-n] \to P_\gamma$. If $n > 1$, we define $Q_{\gamma+1}$ to be the cofiber of ϕ; it is then clear that $P_{\gamma+1}$ has the desired properties. Let us therefore assume that $n = 1$, so that ϕ induces a map $\psi : X_i \to \pi_{-1} P_\gamma$ in the abelian category $\mathrm{h}\mathcal{C}^\heartsuit$. Let K denote the fiber of ψ, so that the composite map

$$K[-1] \to X_i[-1] \to \pi_{-1} P_\gamma \to \tau_{\leq -1} P_\gamma$$

is nullhomotopic. It follows that the map $K[-1] \to X_i[-1] \xrightarrow{\phi} P_\gamma$ factors through some map $\xi : K[-1] \to \pi_0 P_\gamma \simeq Q_0$. The map ξ determines an extension

$$0 \to Q_0 \to E \to K \to 0$$

in the abelian category $\mathrm{h}\mathcal{C}^\heartsuit$. Since Q_0 is an injective object of $\mathrm{h}\mathcal{C}^\heartsuit$, this extension splits so that $\xi = 0$. Let X_i/K denote the cofiber of the map $K \to X_i$, so that ϕ factors as a composition

$$X_i[-1] \to (X_i/K)[-1] \xrightarrow{\phi'} P_\gamma.$$

We now define $P_{\gamma+1}$ to be the cofiber of ϕ'. Then $P_{\gamma+1}$ satisfies condition (c') by construction. To verify (b'), we observe that there is an exact sequence

$$0 \to \pi_0 Q_\gamma \to \pi_0 Q_{\gamma+1} \to X_i/K \xrightarrow{\psi'} \pi_{-1} Q_\gamma,$$

where ψ' is the map induced by ψ. Since K is the fiber of ψ, the map ψ' is injective so that $\pi_0 Q_{\gamma+1} \simeq \pi_0 Q_\gamma \simeq Q_0$ as desired. $\qquad\square$

Proof of Theorem 8.2.3.4. Proposition 8.2.3.3 implies that θ is fully faithful. The essential surjectivity follows from Proposition 8.2.3.7. $\qquad\square$

Remark 8.2.3.8. In the situation of Theorem 8.2.3.4, Remark 1.3.5.23 implies that the inclusion $\mathrm{N}(\mathcal{A}) \to \mathcal{C}$ extends in an essentially unique way to a t-exact functor $F : \mathcal{D}^+(\mathcal{A}) \to \mathcal{C}$. Using the dual of Proposition 1.3.3.7, we see that F is fully faithful if and only if it carries injective objects of $\mathcal{A}$ to injective objects of $\mathcal{C}$. In other words, F is fully faithful if and only if the injective objects of $\mathcal{C}$ belong to the heart $\mathcal{C}^\heartsuit$.

Example 8.2.3.9. Let Sp denote the ∞-category of spectra, endowed with the t-structure described in Proposition 1.4.3.6. Then the heart $\mathrm{Sp}^\heartsuit$ is equivalent to the category of abelian groups. An abelian group A is injective if and only if it *divisible*: that is, if and only if the multiplication map $A \xrightarrow{n} A$ is surjective for every integer n. Theorem 8.2.3.4 implies that for every injective abelian group A, there is an injective spectrum I_A with $\pi_0 I_A \simeq A$. Moreover, the spectrum I_A is determined uniquely (up to a connected space of choices). In particular, there exists an injective spectrum I with $\pi_0 I \simeq \mathbf{Q}/\mathbf{Z}$, which is unique up to equivalence. The spectrum I is called the *Brown-Comenetz dual of the sphere spectrum*. It is characterized up to homotopy equivalence by the following universal property in the stable homotopy category hSp: for every spectrum X, there is a canonical isomorphism $\mathrm{Ext}^n_{\mathrm{Sp}}(X, I) \simeq \mathrm{Hom}(\pi_n X, \mathbf{Q}/\mathbf{Z})$, where the latter Hom is computed in the category of abelian groups. In particular, the homotopy groups of I are dual to the stable homotopy groups of spheres.

8.2.4   Localizations and Ore Conditions

Let R be a commutative ring and let S be a subset of R which contains the unit and is closed under multiplication. In this case, one can define a commutative ring $R[S^{-1}]$ together with a ring homomorphism $\phi : R \to R[S^{-1}]$ with the following universal property: if R' is any other commutative ring, the composition with ϕ induces a bijection from the set of ring homomorphisms $R[S^{-1}] \to R'$ to the set of ring homomorphisms from R into R' which carry each element of S to an invertible element in R'. The ring $R[S^{-1}]$ can be described concretely as follows: elements of $R[S^{-1}]$ are equivalence classes of symbols $\frac{x}{s}$, where $x \in R$ and $s \in S$; here we let $\frac{x}{s} = \frac{x'}{s'}$ if and only if there exists an element $t \in S$ such that $xs't = x'st$.

Now suppose that R is an associative ring and that S is a multiplicatively closed subset of R. By general nonsense, one can construct a new associative ring $R[S^{-1}]$ by freely adjoining to R multiplicative inverses for the elements of S. There is generally no easy description of the ring $R[S^{-1}]$: for example, it can be very difficult to decide whether the map $R \to R[S^{-1}]$ is injective. However, we can do much better if we are willing to introduce a suitable assumption on S.

Definition 8.2.4.1. Let R be an associative ring and let S be a subset of R. We say that S satisfies the *left Ore condition* if the following hold:

 (a) The set S contains the unit element of R and is closed under multiplication.

 (b) For every pair of elements $x \in R$, $s \in S$, there exist elements $y \in R$ and $t \in S$ such that $tx = ys$.

 (c) Let $x \in R$ be an element for which there exists $s \in S$ such that $xs = 0$. Then there exists an element $t \in S$ such that $tx = 0$.

Remark 8.2.4.2. Conditions (b) and (c) of Definition 8.2.4.1 are automatic if the ring R is commutative, or more generally if the set S is contained in the center of R.

Remark 8.2.4.3. We say that a set $S \subseteq R$ satisfies the *right Ore condition* if it satisfies condition (a) of Definition 8.2.4.1, together with the following versions of (b) and (c):

 (b') For every pair of elements $x \in R$, $s \in S$, there exist elements $y \in R$ and $t \in S$ such that $xt = sy$.

 (c') Let $x \in R$ be an element for which there exists $s \in S$ such that $sx = 0$. Then there exists an element $t \in S$ such that $xt = 0$.

Equivalently, $S \subseteq R$ satisfies the right Ore condition if it satisfies the left Ore condition when viewed as a subset of R^{rev}, the same ring with the opposite multiplication.

Construction 8.2.4.4. Let R be an associative ring, let $S \subseteq R$ be a subset which satisfies the left Ore condition, and let M be a left R-module. We define a relation $\sim$ on the product $S \times M$ as follows: we have $(s, x) \sim (s', x')$ if there exists elements $a, a' \in R$ such that the products as and $a's'$ belong to S, $as = a's'$, and $ax = a'x'$. It is obvious that this relation is symmetric and reflexive. It is also reflexive: if $(s, x) \sim (s', x') \sim (s'', x'')$, then we can choose elements $a, a', b', b'' \in R$ such that

$$as = a's' \in S \quad b's' = b''s'' \in S \quad ax = a'x' \quad b'x' = b''x''.$$

Since S satisfies condition (b) of Definition 8.2.4.1, we can find elements $u \in S$, $c \in R$ such that $ua's' = cb's'$. Thus $ua' - cb'$ is annihilated by right multiplication by $s' \in S$; using condition (c), we deduce that $ua' - cb'$ is annihilated by left multiplication by some element $t \in S$. Then

$$tuas = tua's' = tcb's' = tcb''s'',$$

and this product belongs to S since $t, u, as \in S$ and S is closed under multiplication. To prove that $(s, x) \sim (s'', x'')$, it suffices to observe that

$$tuax = tua'x' = tcb'x' = tcb''x''.$$

Let $S^{-1}M$ denote the quotient of $S \times M$ by the equivalence relation $\sim$. We denote the image of a pair (s, x) in $S^{-1}M$ by $s^{-1}x$. We refer to $S^{-1}M$ as the *left module of fractions* associated to the pair (S, M). There is a canonical map $\phi : M \to S^{-1}M$, given by $\phi(x) = 1^{-1}x$.

We now summarize the some of the main properties of Construction 8.2.4.4.

Proposition 8.2.4.5. *Let R be an associative ring, let $S \subseteq R$ be a subset which satisfies the left Ore condition, and let M be a left module over R. Then:*

(1) *The set $S^{-1}M$ admits a unique R-module structure satisfying the following pair of conditions:*

    (a) *The canonical map $\phi : M \to S^{-1}M$ is an R-module homomorphism.*

    (b) *For every pair of elements $s \in S$, $x \in M$, we have $\phi(x) = s(s^{-1}x)$.*

(2) *For each $s \in S$, multiplication by s induces a bijection from $S^{-1}M$ to itself.*

(3) *Let N be another left R-module, and assume that for each $s \in S$, left multiplication by s induces a bijection from N to itself. For every R-module homomorphism $\psi : M \to N$, there exists a unique R-module homomorphism $\psi' : S^{-1}M \to N$ such that the diagram*

$$
\begin{array}{ccc}
 & S^{-1}M & \\
\phi \nearrow & & \searrow \psi' \\
M & \xrightarrow{\quad \psi \quad} & N
\end{array}
$$

    is commutative.

(4) *Suppose that $M = R$. Then there is a unique associative ring structure on $S^{-1}M = S^{-1}R$ such that $\phi : R \to S^{-1}R$ is a ring homomorphism satisfying condition (b) above.*

(5) *The ring homomorphism ϕ carries each element of S to an invertible element of $S^{-1}R$.*

(6) *Let A be an associative ring, and let $\psi : R \to A$ be a ring homomorphism with the property that for each $s \in S$, the image $\psi(s) \in A$ is invertible. Then there exists a unique ring homomorphism $\psi' : S^{-1}R \to A$ such that the diagram*

$$
\begin{array}{ccc}
 & S^{-1}R & \\
\phi \nearrow & & \searrow \psi' \\
R & \xrightarrow{\quad \psi \quad} & A
\end{array}
$$

    is commutative.

Remark 8.2.4.6. Let R be an associative ring and let $S \subseteq R$ be a subset. If S satisfies the left Ore condition, then we can define a left ring of fractions $S^{-1}R$ as in Proposition 8.2.4.5. If S also satisfies the right Ore condition, then the dual version of Proposition 8.2.4.5 allows us to construct a right ring of fractions RS^{-1}. It follows from assertion (5) of Proposition 8.2.4.5 that there is a canonical isomorphism $S^{-1}R \simeq RS^{-1}$.

It is not difficult (though somewhat tedious) to prove Proposition 8.2.4.5 by direct calculation. In this section, we will give a different proof of a more general result, where the associative ring R is replaced by an $\mathbb{E}_1$-ring and M by a left R-module spectrum.

Remark 8.2.4.7. Let R be an $\mathbb{E}_1$-ring, and regarded $\pi_* R$ as a graded associative ring. Let S be a set of homogeneous elements of $\pi_* R$ which satisfies the left Ore condition. In particular, we have the following:

(b) For every pair of elements $x \in \pi_* R$, $s \in S$, there exist elements $y \in \pi_* R$ and $t \in S$ such that $tx = ys$.

(c) Let $x \in \pi_* R$ be an element for which there exists $s \in S$ such that $xs = 0$. Then there exists an element $t \in S$ such that $tx = 0$.

Note that it suffices to verify conditions (b) and (c) in the case where x is a homogeneous element of $\pi_* R$. Moreover, in case (b), we may assume that the element y is also homogeneous.

Definition 8.2.4.8. Let R be an $\mathbb{E}_1$-ring and let $S \subseteq \pi_* R$ be a set of homogeneous elements which contains the unit and is closed under multiplication. Suppose that M is a left R-module spectrum. We will say that M is *S-nilpotent* if, for every $x \in \pi_n M$, there exists an element $s \in S$ such that $sx = 0$.

Lemma 8.2.4.9. *Let R be an $\mathbb{E}_1$-ring and let $S \subseteq \pi_* R$ be a set of homogeneous elements which contains the unit and is closed under multiplication. Suppose we are given a fiber sequence*

$$M' \xrightarrow{\psi} M \xrightarrow{\phi} M''$$

in LMod_R. *If any two of the left A-modules M, M', and M'' are S-nilpotent, then so it the third.*

Remark 8.2.4.10. Let R be an $\mathbb{E}_1$-ring and let $S \subseteq \pi_* R$ be a set of homogeneous elements which contains the unit and is closed under products. It is easy to see that the collection of S-nilpotent left R-modules is closed under shifts and (possibly infinite) coproducts. Combining this observation with Lemma 8.2.4.9, we deduce that the collection of S-nilpotent left R-modules is a stable subcategory of LMod_R, which is closed under small colimits.

Proof of Lemma 8.2.4.9. For the sake of definiteness, suppose that M' and M'' are S-nilpotent; we will show that M is S-nilpotent. Let $x \in \pi_n M$ for some integer n. Since M'' is S-nilpotent, the image $\phi(x) \in \pi_n M''$ is annihilated by multiplication by some element $s \in S$ having degree d. Then $\phi(sx) = s\phi(x) = 0$, so that sx belongs to to the kernel of the map $\pi_{n+d} M \to \pi_{n+d} M''$. The exactness of the sequence

$$\pi_{n+d} M' \to \pi_{n+d} M \to \pi_{n+d} M''$$

implies that $sx = \psi(y)$ for some element $y \in \pi_{n+d} M'$. Since M' is S-nilpotent, there exists an element $t \in S$ such that $ty = 0$. Then $(ts)x = t(sx) = t\psi(y) = \psi(ty) = 0$. $\qquad\qquad\square$

The left Ore condition of Definition 8.2.4.1 has a convenient reformulation in terms of S-nilpotent modules.

Lemma 8.2.4.11. *Let R be an $\mathbb{E}_1$-ring and let $S \subseteq \pi_* R$ be a set of homogeneous elements which contains the unit and is closed under multiplication. The following conditions are equivalent:*

(1) *The set S satisfies the left Ore condition.*

(2) *For every element $s \in S$ of degree d, if R/Rs denotes the cofiber of the map of left R-modules $R[d] \to R$ given by right multiplication by s, then R/Rs is S-nilpotent.*

Proof. Assume first that (1) is satisfied. Let $x \in \pi_n R/Rs$; we wish to show that x is annihilated by some element of S. Using the exact sequence

$$\pi_n R/Rs \xrightarrow{\phi} \pi_{n-d-1} R \xrightarrow{s} \pi_{n-1} R,$$

we deduce that $\phi(x) \in \pi_{n-d-1} R$ is annihilated by right multiplication by s. Using condition (c) of Definition 8.2.4.1, we deduce that there exists an element $t \in S$ of degree d' such that $0 = t\phi(x) \in \pi_{n+d'-d-1}$. Using the exactness of the sequence

$$\pi_{n+d'} R \xrightarrow{\psi} \pi_{n+d'} R/Rs \to \pi_{n+d'-d-1} R$$

we conclude that $tx = \psi(y)$ for some $y \in \pi_{n+d'} R$. Using condition (b) of Definition 8.2.4.1 (and Remark 8.2.4.7), we conclude that there exists an element $u \in S$ of degree d'' and an element $z \in \pi_{n+d'+d''-d} R$ such

that $uy = zs$. It follows that the image of uy in $\pi_{n+d'+d''}R/Rs$ vanishes, so that $(ut)x = 0$. This completes the proof of (2).

Now suppose that (2) is satisfied. We will show that S satisfies conditions (b) and (c) of Definition 8.2.4.1. Suppose first that $x \in \pi_* R$ and $s \in S$; we wish to show that there exists $y \in \pi_* R$ and $t \in S$ such that $tx = ys$. As noted in Remark 8.2.4.7, we may assume without loss of generality that x is homogeneous; say $x \in \pi_n R$. Let $s \in S$ have degree d. The image of x in $\pi_n R/Rs$ is annihilated by multiplication by some homogeneous element $t \in S$ of degree d'. It follows that tx belongs to the kernel of the map $\pi_{n+d'} R \to \pi_{n+d'} R/Rs$, and therefore to the image of the map $\pi_{n+d'-d} R \xrightarrow{s} \pi_{n+d'} R$ given by right multiplication by s.

We now verify (c). As noted in Remark 8.2.4.7, it suffices to show that if $x \in \pi_n R$ is annihilated right multiplication by some element $s \in S$ of degree d, then $tx = 0$ for some $t \in S$. The exactness of the sequence

$$\pi_{n+d+1}R/Rs \xrightarrow{\phi} \pi_n R \xrightarrow{s} \pi_{n+d}R$$

shows that $x = \phi(y)$ for some $y \in \pi_{n+d+1}R/Rs$. Since R/Rs is S-nilpotent, there exists an element $t \in S$ such that $ty = 0$, from which it follows immediately that $tx = t\phi(y) = 0$. $\qquad\square$

Lemma 8.2.4.12. *Let R be an $\mathbb{E}_1$-ring and let $S \subseteq \pi_* R$ be a set of homogeneous elements which contains the unit and is closed under products. Let M be a left R-module which is S-nilpotent. Then there exists a map of left R-modules $f : N \to M$ with the following properties:*

(1) *The left R-module N is isomorphic to a coproduct of left R-modules of the form $(R/Rs_\alpha)[n_\alpha]$, where the s_α are elements of S and R/Rs_α is defined as in Lemma 8.2.4.11.*

(2) *The map f induces a surjection of graded abelian groups $\pi_* N \to \pi_* M$.*

Proof. Let $y \in \pi_n M$. Then y is classified by a map of left R-modules $f_y : R[n] \to M$. Since M is S-nilpotent, we have $sy = 0$ for some element $s \in S$ of degree d. It follows that the composition

$$R[n+d] \xrightarrow{s} R[n] \xrightarrow{f_y} M$$

is nullhomotopic, so that f_y factors as a composition

$$R[n] \to (R/Rx)[n] \xrightarrow{f'_y} M.$$

Taking the coproduct of the maps f'_y over all homogeneous elements $y \in \pi_* M$, we obtain the desired map $f : N \to M$. $\qquad\square$

Lemma 8.2.4.13. *Let R be an $\mathbb{E}_1$-ring and let $S \subseteq \pi_* R$ be a set of homogeneous elements which satisfies the left Ore condition. Let M be an S-nilpotent left R-module. Then there exists a sequence*

$$0 = M_0 \to M_1 \to M_2 \to \cdots$$

of left R-modules with the following properties:

(1) *For each $i \geq 0$, the fiber of the map $M_i \to M_{i+1}$ is a coproduct of left A-modules of the form $(R/Rs_\alpha)[n_\alpha]$, for some elements $s_\alpha \in S$ and integers n_α.*

(2) *The colimit $\varinjlim M_i$ is equivalent to M (as an object of LMod_A).*

Proof. We construct S-nilpotent left R-modules M_i and maps $f_i : M_i \to M$ using induction on i. When $i = 0$, set $M_i = 0$, so that $f_i : 0 \to M$ is determined up to a contractible space of choices. For the inductive step, assume that $f_i : M_i \to M$ has been constructed. Since M_i and M are S-nilpotent, the fiber K_i of f_i is also S-nilpotent. We may therefore choose a map $g_i : N_i \to K_i$ which induces a surjection on homotopy groups, where N_i is a coproduct of left A-modules of the form $(A/As)[n]$. Since S satisfies the left Ore

condition, Remark 8.2.4.10 and Lemma 8.2.4.11 imply that N_i is S-nilpotent. We define M_{i+1} to be a cofiber of the composite map $N_i \xrightarrow{g_i} K_i \to M_i$. By construction, f_i admits a canonical factorization

$$M_i \longrightarrow M_{i+1} \xrightarrow{f_{i+1}} M.$$

Lemma 8.2.4.9 guarantees that M_{i+1} is also S-nilpotent.

The above construction obviously satisfies condition (1). To prove (2), note that for each i we have an exact triangle

$$N_i \to K_i \to K_{i+1}.$$

Since the first map is surjective on homotopy, we conclude that the induced map $\pi_* K_i \to \pi_* K_{i+1}$ vanishes. It follows that $\pi_* \varinjlim K_i \simeq \varinjlim \pi_* K_i \simeq 0$, so that the map $\varinjlim M_i \simeq M$ is an equivalence. $\square$

Proposition 8.2.4.14. *Let R be an $\mathbb{E}_1$-ring and let $S \subseteq \pi_* R$ be a set of homogeneous elements satisfying the left Ore condition. Let N be a left R-module. The following conditions are equivalent:*

(1) *For every element $s \in S$, left multiplication by s induces an isomorphism of graded abelian groups from $\pi_* N$ to itself.*

(2) *For every element $s \in S$ and every integer n, the mapping space $\mathrm{Map}_{\mathrm{LMod}_A}(R/Rs[n], N)$ is contractible.*

(3) *For every S-nilpotent object $M \in \mathrm{LMod}_A$, the mapping space $\mathrm{Map}_{\mathrm{LMod}_A}(M, N)$ is contractible.*

Proof. Let $s \in S$ be an element of degree d. The fiber sequence

$$R[d] \xrightarrow{s} R \to R/Rs$$

determines a long exact sequence

$$\mathrm{Ext}_R^n(R/Rs, N) \to \pi_{-n} N \xrightarrow{s} \pi_{d-n} N \to \mathrm{Ext}_R^{n+1}(R/Rs, N),$$

from which the equivalence (1) $\Leftrightarrow$ (2) follows immediately. The implication (3) $\Rightarrow$ (2) follows from Lemma 8.2.4.11. We now prove (2) $\Rightarrow$ (3). Let $M \in \mathrm{LMod}_R$ be S-nilpotent, and choose a sequence

$$0 = M_0 \to M_1 \to M_2 \to \cdots$$

satisfying the hypotheses of Lemma 8.2.4.13. Since $M \simeq \varinjlim M_i$, we have

$$\mathrm{Map}_{\mathrm{LMod}_R}(M, N) \simeq \varprojlim \mathrm{Map}_{\mathrm{LMod}_R}(M_i, N).$$

It will therefore suffice to show that each mapping space $\mathrm{Map}_{\mathrm{LMod}_R}(M_i, N)$ is contractible. We proceed by induction on i, the case $i = 0$ being obvious. Assume that $\mathrm{Map}_{\mathrm{LMod}_R}(M_i, N)$ is contractible. We have a fiber sequence

$$\mathrm{Map}_{\mathrm{LMod}_R}(M_{i+1}, N) \to \mathrm{Map}_{\mathrm{LMod}_R}(M_i, N) \to \mathrm{Map}_{\mathrm{LMod}_R}(N_i, N),$$

where N_i is a coproduct of left R-modules of the form $(R/Rs_\alpha)[n_\alpha]$. It therefore suffices to show that $\mathrm{Map}_{\mathrm{LMod}_R}(N_i, N) \simeq \prod_\alpha \mathrm{Map}_{\mathrm{LMod}_R}((R/Rs_\alpha)[n], N)$ is contractible, which follows from (2). $\square$

Definition 8.2.4.15. Let R be an $\mathbb{E}_1$-ring and let $S \subseteq \pi_* R$ be a set of homogeneous elements satisfying the left Ore condition. We will say that a left R-module N is S-*local* if it satisfies the equivalent conditions of Proposition 8.2.4.14. We let $\mathrm{LMod}_R^{Nil(S)}$ denote the full subcategory of LMod_R spanned by the S-nilpotent left R-modules, and $\mathrm{LMod}_R^{Loc(S)}$ the full subcategory of LMod_R spanned by the S-local R-modules.

Remark 8.2.4.16. It follows immediately from characterization (1) of Proposition 8.2.4.14 that $\mathrm{LMod}_R^{Loc(S)}$ is a stable subcategory of LMod_R, which is closed under small coproducts and therefore under arbitrary small colimits.

In the situation of Definition 8.2.4.15, Remark 8.2.4.10 implies that $\mathrm{LMod}_R^{Nil(S)}$ is closed under small colimits in LMod_R. Moreover, there exists a set of objects which generate $\mathrm{LMod}_R^{Nil(S)}$ under small colimits: for example, the objects of the form $R/Rs[n]$, where $s \in S$ (Lemma 8.2.4.13). It follows that the ∞-category $\mathrm{LMod}_R^{Nil(S)}$ is presentable. Using Corollary T.5.5.2.9, we deduce that the inclusion $\mathrm{LMod}_R^{Nil(S)} \to \mathrm{LMod}_R$ admits a right adjoint G. If $M \in \mathrm{LMod}_R^{Nil(S)}$ and $N \in \mathrm{LMod}_R$, then the counit map $u : G(N) \to N$ induces an equivalence $\mathrm{Ext}_R^i(M, G(N)) \to \mathrm{Ext}_R^i(M, N)$ for all integers i. It follows that the cofiber of u is S-local. We can summarize the situation as follows:

Proposition 8.2.4.17. *Let R be an $\mathbb{E}_1$-ring and let $S \subseteq \pi_* R$ be a set of homogeneous elements satisfying the left Ore condition. Then $\mathrm{LMod}_R^{Nil(S)}$ and $\mathrm{LMod}_R^{Loc(S)}$ are stable subcategories of LMod_R, which are closed under small colimits. Moreover, the pair $(\mathrm{LMod}_R^{Nil(S)}, \mathrm{LMod}_R^{Loc(S)})$ determines an accessible t-structure on LMod_R (with trivial heart).*

Remark 8.2.4.18. In the situation of Proposition 8.2.4.17, every object $M \in \mathrm{LMod}_R$ determines a fiber sequence
$$M' \to M \to M'',$$
where M' is S-nilpotent and M'' is S-local. This fiber sequence is determined by M up to contractible ambiguity, so we can regard M' and M'' as functors of M. We will denote M'' by $S^{-1}M$, and refer to it as the *S-localization* of M.

Our next goal is to describe the S-localization of a left R-module M more explicitly. In particular, we will see that the homotopy groups of $S^{-1}M$ can be computed using a graded analogue of Construction 8.2.4.4.

Construction 8.2.4.19. Let R be an $\mathbb{E}_1$-ring, let $S \subseteq \pi_* R$ be a set of homogeneous elements of $\pi_* R$ which satisfy the left Ore condition, and let M be a left R-module. For every integer d, let S_d denote the subset of S consisting of homogeneous elements of degree d. Let $\phi : M \to S^{-1}M$ be the canonical map. Let $s \in S_d$ and let $x \in \pi_{n+d}M$. Since left multiplication by s on $\pi_* M$ is bijective, there exists a unique element $\psi_{d,n}(s, x) \in \pi_n S^{-1}M$ such that $s\psi_{d,n}(s, x) = \phi(x)$.

Proposition 8.2.4.20. *Let R be an $\mathbb{E}_1$-ring, let $S \subseteq \pi_* R$ be a set of homogeneous elements of $\pi_* R$ which satisfy the left Ore condition, and let M be a left R-module. Then:*

(1) *Every element $x \in \pi_n S^{-1}M$ has the form $\psi_{d,n}(s, y)$ for some $s \in S_d$ and $y \in \pi_{n+d}M$. Here we employ the notation of Construction 8.2.4.19.*

(2) *Suppose we are given elements $s \in S_d$, $s' \in S_{d'}$, $y \in \pi_{n+d}M$ and $y' \in \pi_{n+d'}M$. Then $\psi_{d,n}(s, y) = \psi_{d',n}(s', y')$ if and only if there exist an integer m and homogeneous elements $z \in \pi_{m-d}R$, $z' \in \pi_{m-d'}R$ such that $zs = z's' \in S_m$ and $zy = z'y' \in \pi_{n+m}M$.*

Proof. We have a fiber sequence
$$M \xrightarrow{\beta} S^{-1}M \xrightarrow{\alpha} M',$$
where M' is S-nilpotent. Let $x \in \pi_n S^{-1}M$. Since M' is S-nilpotent, there exists an element $s \in S_d$ such that $\alpha(sx) = s\alpha(x) = 0 \in \pi_{n+d}M'$. It follows that $sx = \beta(y)$ for some $y \in \pi_{n+d}M$: that is, $x = \psi_{d,n}(s, y)$. This proves (1).

We now prove (2). The "if" direction is obvious. For the converse, suppose that $\psi_{d,n}(s, y) = x = \psi_{d',n}(s', y')$. Since S satisfies the left Ore condition, we can choose homogeneous elements $u \in S_{k-d}$, $v \in \pi_{k-d'}R$ such that $us = vs'$. Then
$$\beta(uy) = u\beta(y) = u(sx) = (us)x = (vs')x = v(s'x) = v\beta(y') = \beta(vy'),$$
so that $uy - vy' \in \pi_{n+k}M$ is annihilated by β. It follows that $uy - vy'$ is the image of an element $\xi \in \pi_{n+k+1}M'$. Since M' is S-nilpotent, we can choose an element $t \in S_{k'}$ such that $t\xi = 0$, so that $tuy = tvy'$. Set $m = k + k'$, $z = tu$, and $z' = tu'$. Then $z \in S_{m-d}$ so that $zs \in S_m$, and $zs = z's'$ since $us = vs'$. We have $zy = tuy = tvy' = z'y'$, thereby completing the proof of (2). $\square$

Proof of Parts (1), (2), *and* (3) *of Proposition 8.2.4.5.* Let R be an associative ring, let M be a (discrete) left R-module, and let $S \subseteq R$ be a subset which satisfies the left Ore condition. Let X denote the quotient of $S \times M$ by the equivalence relation of Definition 8.2.4.4; for $(s, x) \in S \times M$, we will denote the image of (s, x) in X by $s^{-1}x$. Let $S^{-1}M$ denote the S-localization of M as an R-module spectrum. Proposition 8.2.4.20 implies that $S^{-1}M$ is discrete; we will abuse notation by identifying $S^{-1}M$ with the ordinary R-module $\pi_0(S^{-1}M)$. Proposition 8.2.4.20 also determines a bijection of sets $\psi : X \to S^{-1}M$, given by the formula $s^{-1}x \mapsto \psi_{0,0}(s, x)$. This proves the existence assertion of (1). Assertions (2) follows from the fact that $S^{-1}M$ is S-local, and assertion (3) from the fact $S^{-1}M$ is the S-localization of M.

To complete the proof, it suffices to prove the uniqueness of the R-module structure on X constructed above. Suppose we are given some other left R-module structure on X satisfying the following conditions:

(a) The canonical map $\phi : M \to S^{-1}M$ is an R-module homomorphism.

(b) For every pair of elements $s \in S$, $x \in M$, we have $\phi(x) = s(s^{-1}x)$.

We claim that X is S-local (when regarded as a discrete R-module spectrum): in other words, we claim that for $s \in S$, the left multiplication map $X \xrightarrow{s} X$ is invertible. We first show that this map is injective. Suppose that $s(t^{-1}x) = 0$ for some element $t^{-1}x \in S$. Since S satisfies the left Ore condition, we can choose elements $u \in S$, $a \in R$ such that $as = ut$. Then

$$0 = a(s(t^{-1}x)) = (as)(t^{-1}x) = (ut)(t^{-1}x) = u(t(t^{-1}x)) = u\phi(x) = \phi(ux),$$

so there exists $v \in S$ such that $v(ux) = (vu)x$ vanishes. This immediately implies that $t^{-1}x = 0$ in X.

We now claim that left multiplication by s is surjective. To prove this, let $t^{-1}x$ be an element of X. We claim that $t^{-1}x = s((ts)^{-1}x)$. Since multiplication by t is injective, it suffices to show that $t(t^{-1}x) = (ts)((ts)^{-1}x)$. It follows from (b) that both sides are given by $\phi(x)$. This completes the proof that X is S-local. It follows that ϕ admits a unique factorization

$$M \xrightarrow{u} S^{-1}M \xrightarrow{\theta} X,$$

where u denotes the canonical map from M to its S-localization. To complete the proof, it will suffice to show that the map θ is inverse to the bijection of sets $\psi : X \to S^{-1}M$ constructed above. We will show that $\theta \circ \psi = \mathrm{id}_X$ (since ψ is bijective, it follows that $\psi \circ \theta$ is the identity map on $S^{-1}M$). By construction, $\psi(s^{-1}x)$ is the unique element of $S^{-1}M$ satisfying $s\psi(s^{-1}x) = \phi(x)$. It follows that $s(\theta \circ \psi)(s^{-1}x) = \phi(x) = s(s^{-1}x)$. Since multiplication by s is injective on X, we conclude that $(\theta \circ \psi)(s^{-1}x) = s^{-1}x$. $\square$

Remark 8.2.4.21. The proof given above shows that the notation $S^{-1}M$ introduced in Construction 8.2.4.4 (where M is a discrete module over an associative ring) is compatible with our notation for S-localizations introduced in Remark 8.2.4.18.

Definition 8.2.4.22. Let R be an $\mathbb{E}_1$-ring and let $S \subseteq \pi_*R$ be a set of homogeneous elements which satisfies the left Ore condition. We will say that a left R-module M is *S-complete* if $\mathrm{Ext}^i_R(N, M) \simeq 0$ for every S-local left R-module N. We let $\mathrm{LMod}_R^{Comp(S)}$ denote the full subcategory of LMod_R spanned by the S-complete R-modules.

Proposition 8.2.4.23. *Let R be an $\mathbb{E}_1$-ring and let $S \subseteq \pi_*R$ be a set of homogeneous elements which satisfies the left Ore condition. Then:*

(1) *The ∞-category $\mathrm{LMod}_R^{Comp(S)}$ is a stable subcategory of LMod_R.*

(2) *The inclusion $\mathrm{LMod}_R^{Comp(S)} \to \mathrm{LMod}_R$ admits a left adjoint F.*

(3) *For every $M \in \mathrm{LMod}_R$, there exists a fiber sequence*

$$M' \to M \to M''$$

where M' is S-local and M'' is S-complete.

(4) *The pair of subcategories* $(\mathrm{LMod}_R^{Loc(S)}, \mathrm{LMod}_R^{Comp(S)})$ *is a t-structure on* LMod_R *(with trivial heart).*

(5) *The functor F induces an equivalence of ∞-categories* $\mathrm{LMod}_R^{Nil(S)} \to \mathrm{LMod}_R^{Comp(S)}$.

Proof. Assertion (1) is obvious. Note that Proposition 8.2.4.17 implies that $\mathrm{LMod}_R^{Loc(S)}$ is a presentable ∞-category which is closed under small colimits in LMod_R. It follows from Corollary T.5.5.2.9 that the full subcategory inclusion $\phi : \mathrm{LMod}_R^{Loc(S)} \to \mathrm{LMod}_R$ admits a right adjoint ψ. For every object $M \in \mathrm{LMod}_R$, the counit map $v : \psi M \to M$ induces an isomorphism $\mathrm{Ext}_R^*(N, \psi M) \to \mathrm{Ext}_R^*(N, M)$ for every S-local object N. It follows that the cofiber of v is S-complete, which proves (3). Assertions (2) and (4) follow immediately from (3).

We now prove (5). Let F denote a left adjoint to the inclusion $\mathrm{LMod}_R^{Comp(S)} \to \mathrm{LMod}_R$ and let G denote a right adjoint to the inclusion $\mathrm{LMod}_R^{Nil(S)} \to \mathrm{LMod}_R$. We will abuse notation by identifying F and G with their restrictions to $\mathrm{LMod}_R^{Nil(S)}$ and $\mathrm{LMod}_R^{Comp(S)}$, respectively. It follows that we have an adjunction

$$\mathrm{LMod}_R^{Nil(S)} \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{LMod}_R^{Comp(S)} \ .$$

We claim that this adjunction exhibits the functors F and G as homotopy inverse. We first show that the unit map $u : \mathrm{id} \to G \circ F$ is an equivalence. Let $M \in \mathrm{LMod}_R^{Nil(S)}$, so that we have a fiber sequence

$$\psi(M) \to M \to F(M).$$

We wish to show that the map $M \to (G \circ F)(M)$ is an equivalence. Since $M \simeq G(M)$, it will suffice to show that $G(\psi(M)) \simeq 0$. This is clear, since $\psi(M)$ is S-local. To prove that the counit map $F \circ G \to \mathrm{id}$ is an equivalence, consider an arbitrary S-complete object $N \in \mathrm{LMod}_R$. We have a fiber sequence

$$G(N) \to N \to S^{-1}N.$$

Since N is S-complete, we have $N \simeq F(N)$. Consequently, to prove that the unit map $(F \circ G)(N) \to N$ is an equivalence, it will suffice to show that $F(S^{-1}N) \simeq 0$, which follows immediately from the observation that $S^{-1}N$ is S-local. $\qquad\square$

In the case where R is connective, the subcategories of LMod_R defined above interact nicely with the t-structure on LMod_R.

Proposition 8.2.4.24. *Let R be a connective $\mathbb{E}_\infty$-ring and let $S \subseteq \pi_* R$ be a set of homogeneous elements which satisfies the left Ore condition. Then:*

(1) *Let M be an S-nilpotent R-module. Then the truncations $\tau_{\geq n}M$ and $\tau_{\leq n}M$ are S-nilpotent for every integer n. It follows that the t-structure on LMod_R determines an accessible t-structure*

$$(\mathrm{LMod}_R^{Nil(S)} \cap (\mathrm{LMod}_R)_{\geq 0}, \mathrm{LMod}_R^{Nil(S)} \cap (\mathrm{LMod}_R)_{\leq 0})$$

on $\mathrm{LMod}_R^{Nil(S)}$, which is both right and left complete.

(2) *Let $G : \mathrm{LMod}_R \to \mathrm{LMod}_R^{Nil(S)}$ be a right adjoint to the inclusion. Then G is left t-exact.*

(3) *The localization functor $M \mapsto S^{-1}M$ is left t-exact (when regarded as a functor from LMod_R to itself).*

Proof. Assertion (1) is immediate from the definition. Since the inclusion functor $\mathrm{LMod}_R^{Nil(S)} \to \mathrm{LMod}_R$ is left t-exact, its right adjoint is left t-exact; this proves (2). Assertion (3) follows easily from Proposition 8.2.4.20. $\qquad\square$

If we are localizing with respect to a collection of elements S of degree zero in $\pi_* R$, we can be more precise.

Proposition 8.2.4.25. *Let R be a connective $\mathbb{E}_1$-ring and let $S \subseteq \pi_0 R$ be a subset which satisfies the left Ore condition as a subset of $\pi_* R$. Then:*

(1) *Let M be an R-module. Then M is S-complete if and only if each homotopy group $\pi_n M$ is S-complete, when regarded as a discrete R-module. It follows that the t-structure on LMod_R determines an accessible t-structure $(\mathrm{LMod}_R^{Loc(S)} \cap (\mathrm{LMod}_R)_{\geq 0}, \mathrm{LMod}_R^{Loc(S)} \cap (\mathrm{LMod}_R)_{\leq 0})$ on $\mathrm{LMod}_R^{Loc(S)}$, which is both right and left complete.*

(2) *The localization functor $M \mapsto S^{-1}M$ is t-exact (when regarded as a functor from LMod_R to itself).*

Proof. Assertion (1) follows immediately from the definitions, and assertion (2) is a consequence of Proposition 8.2.4.20. $\qquad\square$

Let R be an $\mathbb{E}_1$-ring and let $S \subseteq \pi_* R$ be a set of homogeneous elements which satisfies the left Ore condition. Since the inclusion $\mathrm{LMod}_R^{Loc(S)} \to \mathrm{LMod}_R$ preserves filtered colimits, the left adjoint functor $M \mapsto S^{-1}M$ carries compact objects of LMod_R to compact objects of $\mathrm{LMod}_R^{Loc(S)}$ (Proposition T.5.5.7.2). In particular, $S^{-1}R$ is a compact object of $\mathrm{LMod}_R^{Loc(S)}$. For any S-local left R-module M, we have isomorphisms

$$\mathrm{Ext}_R^n(S^{-1}R, M) \simeq \mathrm{Ext}_R^n(R, M) \simeq \pi_{-n}M.$$

It follows that these groups vanish if and only if $M \simeq 0$, so that $S^{-1}R$ is a compact generator for the stable ∞-category $\mathrm{LMod}_R^{Loc(S)}$. Let $R[S^{-1}]$ denote the $\mathbb{E}_1$-ring classifying endomorphisms of $S^{-1}R$ in $\mathrm{LMod}_R^{Loc(S)}$. Using Theorem 8.1.2.1 (and Remark 8.1.2.3), we obtain an equivalence of stable ∞-categories $\mathrm{LMod}_R^{Loc(S)} \simeq \mathrm{LMod}_{R[S^{-1}]}$ carrying $S^{-1}R$ to $R[S^{-1}]$. Composing this with the localization functor $M \mapsto S^{-1}M$, we obtain a colimit-preserving functor

$$\Phi : \mathrm{LMod}_R \to \mathrm{LMod}_{R[S^{-1}]}$$

carrying R to $R[S^{-1}]$. Using Theorem 6.3.5.5, we see that Φ is induced by a map of $\mathbb{E}_1$-rings $\phi : R \to R[S^{-1}]$, which is well-defined up to a contractible space of choices.

Remark 8.2.4.26. We can summarize the above discussion more informally as follows: if R is an $\mathbb{E}_1$-ring and $S \subseteq \pi_* R$ is a set of homogeneous elements satisfying the left Ore condition, then the spectrum $S^{-1}R$ admits the structure of an $\mathbb{E}_1$-ring, and the left R-module structure on $S^{-1}R$ arises from a map of $\mathbb{E}_1$-rings.

Proposition 8.2.4.27. *Let R be an $\mathbb{E}_1$-ring, let $S \subseteq \pi_* R$ be a set of homogeneous elements satisfying the left Ore condition, and let $\phi : R \to R[S^{-1}]$ be defined as above. If A is any $\mathbb{E}_1$-ring, then composition with ϕ induces a fully faithful map of Kan complexes $\theta : \mathrm{Map}_{\mathrm{Alg}^{(1)}}(R[S^{-1}], A) \to \mathrm{Map}_{\mathrm{Alg}^{(1)}}(R, A)$, whose essential image is the collection of $\mathbb{E}_1$-rings $\psi : R \to A$ such that $\psi(s)$ is invertible in $\pi_* A$, for each $s \in S$.*

Proof. Let $F : \mathrm{LMod}_R \to \mathrm{LMod}_R^{Loc(S)}$ be a left adjoint to the inclusion, given by $M \mapsto S^{-1}M$. Since F is a localization functor, composition with F induces a fully faithful embedding $\mathrm{Fun}^{\mathrm{L}}(\mathrm{LMod}_R^{Loc(S)}, \mathrm{LMod}_A) \to \mathrm{Fun}^{\mathrm{L}}(\mathrm{LMod}_R, \mathrm{LMod}_A)$, where $\mathrm{Fun}^{\mathrm{L}}(\mathrm{LMod}_R, \mathrm{LMod}_A)$ denotes the full subcategory of $\mathrm{Fun}(\mathrm{LMod}_R, \mathrm{LMod}_A)$ spanned by those functors which preserve small colimits, and $\mathrm{Fun}^{\mathrm{L}}(\mathrm{LMod}_R^{Loc(S)}, \mathrm{LMod}_A)$ is defined similarly.

Using Theorem 6.3.5.5 and the equivalence $\mathrm{LMod}_{R[S^{-1}]} \simeq \mathrm{LMod}_R^{Loc(S)}$, we deduce immediately that θ is fully faithful. Moreover, a map of $\mathbb{E}_1$-rings $\psi : R \to A$ belongs to the essential image of θ if and only if it satisfies the following condition:

(∗) For every map of left R-modules $M \to M'$ which induces an equivalence $S^{-1}M \to S^{-1}M'$, the induced map $A \otimes_R M \to A \otimes_R M'$ is an equivalence.

Passing to fibers, we can reformulate condition (∗) as follows:

(∗′) For every S-nilpotent R-module M, we have $A \otimes_R M \simeq 0$.

Since the collection of S-nilpotent R-modules M is generated under colimits by R-modules of the form $(R/Rs)[n]$ where $s \in S$ is a homogeneous element of degree d (Lemma 8.2.4.13), it suffices to verify condition $(*')$ in the case $M = R/Rs$. In this case, $A \otimes_R M$ is the cofiber of the map $A[d] \xrightarrow{s} A$ given by multiplication by s, so that $A \otimes_R M \simeq 0$ if and only if $\psi(s) \in \pi_d A$ is invertible in $\pi_* A$. $\square$

Using Proposition 8.2.4.27, we can easily recover the classical theory of left rings of fractions:

Proof of Assertions (4), (5), *and* (6) *of Proposition 8.2.4.5.* Let R be an associative ring and let $S \subseteq R$ satisfy the left Ore condition. Let X denote the quotient of $S \times M$ by the equivalence relation of Definition 8.2.4.4; for $(s, x) \in S \times M$, we will denote the image of (s, x) in X by $s^{-1}x$. Let $\phi_0 : R \to X$ be the map given by $\phi_0(x) = 1^{-1}x$, and let $\phi : R \to R[S^{-1}]$ be the $\mathbb{E}_1$-rings appearing in Proposition 8.2.4.27. Proposition 8.2.4.20 implies that $R[S^{-1}]$ is discrete; we will abuse notation and identify $R[S^{-1}]$ with the ordinary associative ring $\pi_0 R[S^{-1}]$. Proposition 8.2.4.20 also determines a bijection of sets $\psi : X \to R[S^{-1}]$, which is characterized by the requirement that $\phi(s)\psi(s^{-1}x) = \phi(x)$ for all $s \in S$, $x \in R$. Taking $s = 1$, we deduce that $\phi = \psi \circ \phi_0$. There is a unique associative ring structure on X so that ψ is an isomorphism of rings, so that $\phi_0 : R \to X$ is a ring homomorphism satisfying $\phi_0(s)(s^{-1}x) = \phi_0(x)$. This proves the existence clause of (4).

Let us regard X as a left R-module via the map ϕ_0. This left R-module structure must coincide with the left R-module structure of part (1), so that (2) implies that multiplication by $\phi_0(s)$ induces a bijection $X \to X$ for each $s \in S$. It follows that ϕ_0 carries each element of S to an invertible element in X, which proves (5). Assertion (6) follows immediately from Proposition 8.2.4.27.

It remains only to prove the uniqueness clause of (4). Suppose that X is endowed with an arbitrary associative ring structure such that ϕ_0 is a ring homomorphism satisfying $\phi_0(s)(s^{-1}x) = \phi_0(x)$ for $s \in S$ and $x \in R$. Then the induced left R-module structure on X must coincide with the left R-module structure appearing in assertion (1), so that assertion (2) implies that left multiplication by $\phi_0(s)$ induces a bijection from X to itself for each $s \in S$. In other words, ϕ_0 carries each element of S to an invertible element of X, so that Proposition 8.2.4.27 gives a unique ring homomorphism $\theta : R[S^{-1}] \to X$ satisfying $\theta \circ \phi = \phi_0$. We will complete the proof by showing that θ is an inverse of the bijection $\psi : X \to R[S^{-1}]$, so that $\psi = \theta^{-1}$ is also a ring isomorphism. For this, we compute

$$\phi_0(s)(\theta \circ \psi)(s^{-1}x) = (\theta \circ \phi)(s)(\theta \circ \psi)(s^{-1}x) = \theta(\phi(s)\psi(s^{-1}x)) = (\theta \circ \phi)(x) = \phi_0(x) = \phi_0(s)(s^{-1}x).$$

Since $\phi_0(s)$ is an invertible element of X, we conclude that $(\theta \circ \psi)(s^{-1}x) = s^{-1}x$ for every element $s^{-1}x \in X$, as desired. $\square$

8.2.5   Finiteness Properties of Rings and Modules

In this section, we will discuss some finiteness conditions on the ∞-categories of modules over an $\mathbb{E}_1$-ring R. We begin by introducing the definition of a *perfect* R-module. Roughly speaking, an R-module M is perfect if it can be obtained as a successive extension of finitely many (possibly shifted) copies of R, or is a retract of such an R-module. Alternatively, we can describe the class of perfect R-modules as the compact objects of the ∞-category LMod_R (Proposition 8.2.5.2).

In general, the condition that an R-module be perfect is very strong. For example, if R is a discrete commutative ring and M is a finitely generated (discrete) module over R, then M need not be perfect when viewed as an object of the stable ∞-category LMod_R (though this is true if R is a regular Noetherian ring of finite Krull dimension). To remedy the situation, we will introduce the weaker notion of an *almost perfect* R-module. This notion has a closer relationship with finiteness conditions in the classical theory of rings and modules. For example, we will show that under some mild assumptions on R, a left R-module M is almost perfect if and only if each homotopy group $\pi_i M$ is a finitely presented left module over $\pi_0 R$, and $\pi_i M \simeq 0$ for $i \ll 0$ (Proposition 8.2.5.17), at least when R itself satisfies a suitable finiteness condition (that is, when R is *left coherent* in the sense of Definition 8.2.5.13).

Definition 8.2.5.1. Let R be an $\mathbb{E}_1$-ring. We let $\mathrm{LMod}_R^{\mathrm{perf}}$ denote the smallest stable subcategory of LMod_R which contains R (regarded as a left module over itself) and is closed under retracts. Similarly, we let $\mathrm{RMod}_R^{\mathrm{perf}}$ denote the smallest stable subcategory of RMod_R which contains R and is closed under retracts. We will say that a left (right) R-module M is *perfect* if it belongs to $\mathrm{LMod}_R^{\mathrm{perf}}$ ($\mathrm{RMod}_R^{\mathrm{perf}}$).

Proposition 8.2.5.2. *Let R be an $\mathbb{E}_1$-ring. Then:*

(1) *The ∞-category LMod_R is compactly generated.*

(2) *An object of LMod_R is compact if and only if it is perfect.*

Proof. The compact objects of LMod_R form a stable subcategory which is closed under the formation of retracts. Moreover, Corollary 4.2.4.8 implies $R \in \mathrm{LMod}_R$ corepresents the functor given by the composition

$$\mathrm{LMod}_R = \mathrm{LMod}_R(\mathrm{Sp}) \longrightarrow \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathcal{S}$$

of two functors which each preserve filtered colimits. It follows that R is compact as a left R-module, from which it immediately follows that all perfect objects of LMod_R are compact.

According to Proposition T.5.3.5.11, the inclusion $f : \mathrm{LMod}_R^{\mathrm{perf}} \subseteq \mathrm{LMod}_R$ induces a fully faithful functor $F : \mathrm{Ind}(\mathrm{LMod}_R^{\mathrm{perf}}) \to \mathrm{LMod}_R$. To complete the proof, it will suffice to show that F is essentially surjective. Since f is right exact, F preserves all colimits (Proposition T.5.5.1.9), so the essential image of F is stable under colimits. If F is not essentially surjective, then Proposition 1.4.4.11 implies that there exists a nonzero $N \in \mathrm{LMod}_R$ such that $\mathrm{Map}_{\mathrm{LMod}_R}(N', N) \simeq *$ for all N' belonging to the essential image of F. In particular, taking $N' = R[n]$, we conclude that $\pi_n N \simeq *$. It follows that N is a zero object of LMod_R, contrary to our assumption. $\qquad\square$

Let R be an $\mathbb{E}_1$-ring. It follows from Remark 6.3.4.8 that the the ∞-categories LMod_R and RMod_R are duals of one another (in the symmetric monoidal ∞-category of presentable stable ∞-categories). In this special case, it is easy to verify the duality directly:

Proposition 8.2.5.3. *Let R be an $\mathbb{E}_1$-ring. The relative tensor product functor*

$$\otimes_R : \mathrm{RMod}_R \times \mathrm{LMod}_R \to \mathrm{Sp}$$

induces fully faithful embeddings

$$\theta : \mathrm{RMod}_R \to \mathrm{Fun}(\mathrm{LMod}_R, \mathrm{Sp}) \quad \theta' : \mathrm{LMod}_R \to \mathrm{Fun}(\mathrm{RMod}_R, \mathrm{Sp}).$$

A functor $f : \mathrm{LMod}_R \to \mathrm{Sp}$ ($g : \mathrm{RMod}_R \to \mathrm{Sp}$) belongs to the essential image of θ (θ') if and only if f (g) preserves small colimits.

Proof. Let $\mathcal{C}$ be the full subcategory of $\mathrm{Fun}(\mathrm{LMod}_R, \mathrm{Sp})$ spanned by those functors which preserve small colimits. Proposition T.5.5.3.8 implies that $\mathcal{C}$ is presentable, and Proposition 4.3.5.14 implies that θ factors through $\mathcal{C}$. We will show that θ induces an equivalence $G : \mathrm{RMod}_R \to \mathcal{C}$; the analogous assertion for left modules follows by the same argument.

Proposition 8.2.5.2 implies that LMod_R is equivalent to the ∞-category $\mathrm{Ind}(\mathrm{LMod}_R^{\mathrm{perf}})$. It follows from Propositions T.4.2.3.11, T.5.5.1.9 and 1.1.4.1 that $\mathcal{C}$ is equivalent to the ∞-category $\mathrm{Fun}^{\mathrm{Ex}}(\mathrm{LMod}_R^{\mathrm{perf}}, \mathrm{Sp})$ of exact functors from $\mathrm{LMod}_R^{\mathrm{perf}}$ to spectra. In particular, for every perfect left R-module N, evaluation on N induces a functor $\mathcal{C} \to \mathrm{Sp}$ that preserves all small limits and colimits.

Let $G' : \mathcal{C} \to \mathrm{Sp}$ be given by evaluation on R, regarded as a (perfect) left module over itself. If $\alpha : f \to f'$ is a natural transformation of functors from $\mathrm{LMod}_R^{\mathrm{perf}}$ to Sp, then the full subcategory of $\mathrm{hLMod}_R^{\mathrm{perf}}$ spanned by objects C such that $\alpha(C) : f(C) \to f'(C)$ is an equivalence is a triangulated subcategory of $\mathrm{hLMod}_R^{\mathrm{perf}}$ which is stable under retracts. If follows that G' is *conservative*: if $G'(\alpha)$ is an equivalence, then α is an

equivalence. Since G' preserves small limits, we deduce also that G' detects small limits: if $\bar{p} : K^{\triangleleft} \to \mathcal{C}$ is such that $G' \circ \bar{p}$ is a limit diagram in Sp, then $\bar{p}$ is a limit diagram in $\mathcal{C}$. Similarly, G' detects small colimits.

The composite functor $G' \circ G : \mathrm{RMod}_R \to \mathrm{Sp}$ can be identified with the forgetful functor, in view of Corollary 4.2.4.8. It follows that $G' \circ G$ preserves all limits and colimits (Corollaries 4.2.3.3 and 4.2.3.5). Since G' detects small limits and colimits, we deduce that G preserves small limits and colimits. Corollary T.5.5.2.9 implies that G and G' admit left adjoints, which we will denote by F and F'.

Choose unit and counit transformations

$$ u : \mathrm{id} \to G \circ F, \quad v : F \circ G \to \mathrm{id}\,. $$

We wish to prove that u and v are equivalences. Since $G' \circ G$ detects equivalences, the functor G detects equivalences, so that v is an equivalence if and only if $G(v) : G \circ F \circ G \to G$ is an equivalence. Since $G(v)$ has a section determined by u, it will suffice to prove that u is an equivalence.

Let $\mathcal{C}'$ be the full subcategory of $\mathcal{C}$ spanned by those objects $f \in \mathcal{C}$ for which the map $u(f) : f \to (G \circ F)(f)$ is an equivalence. Since G and F preserve small colimits, we deduce that $\mathcal{C}'$ is stable under shifts and colimits in $\mathcal{C}$. Proposition 6.2.2.12 implies that $\mathcal{C}$ is generated, under geometric realizations, by the essential image of F'. Let $\mathrm{Sp}' \subseteq \mathrm{Sp}$ be the inverse image of $\mathcal{C}'$ under F'. Since F' is a colimit-preserving functor, we deduce that $\mathrm{Sp}' \subseteq \mathrm{Sp}$ is closed under shifts and colimits. Since Sp is generated under colimits by the objects $S[n]$, where $n \in \mathbf{Z}$ and $S \in \mathrm{Sp}$ denotes the sphere spectrum, it will suffice to show that $S \in \mathrm{Sp}'$.

Corollary 4.2.4.8 allows us to identify $(F \circ F')(S)$ with $R \otimes S \simeq R$, regarded as a left module over itself. It follows that $(G \circ F)(F'(S))$ can be identified with the forgetful functor $f_0 : \mathrm{LMod}_R \to \mathrm{Sp}$. We are reduced to proving that the unit map $S \to R \simeq f_0(R)$ induces an equivalence $F'(S) \simeq f_0$ in $\mathcal{C}$.

Applying Proposition 1.4.2.22, we can identify $\mathcal{C}$ with the full subcategory $\mathcal{D} \subseteq \mathrm{Fun}(\mathrm{LMod}_R^{\mathrm{perf}}, \mathcal{S}) = \mathcal{P}(\mathrm{LMod}_R^{\mathrm{perf,op}})$ spanned by those functors which preserve finite limits. Under this equivalence, f_0 corresponds to the composition $\mathrm{LMod}_R^{\mathrm{perf}} \subseteq \mathrm{LMod}_R \to \mathrm{Sp} \xrightarrow{\Omega^{\infty}} \mathcal{S}$, while $F'(S)$ corresponds to the image of $* \in \mathcal{S}$ under the composition $\mathcal{S} \xrightarrow{\Sigma^{\infty}} \mathrm{Sp} \xrightarrow{F'} \mathcal{C} \simeq \mathcal{D}$ which is the left adjoint to the the functor $\mathcal{D} \to \mathcal{S}$ given by evaluation at $R \in \mathrm{LMod}_R^{\mathrm{perf}}$. To complete the proof, it will suffice to show that the unit $1 \in \pi_0 R$ exhibits the composite functor $\mathrm{LMod}_R \to \mathrm{Sp} \xrightarrow{\Omega^{\infty}} \mathcal{S}$ as corepresented by $R \in \mathrm{LMod}_R$. In other words, we must show that for every $M \in \mathrm{LMod}_R$, the canonical map $\mathrm{Map}_{\mathrm{LMod}_R}(R, M) \to \Omega^{\infty} M$ is a homotopy equivalence. Using Corollary 4.2.4.8, we can reduce to the case where R is the unit object of $S \in \mathrm{Alg}(\mathrm{Sp})$. We are therefore reduced to proving that if $M \in \mathrm{Sp}$, then the canonical map $\mathrm{Map}_{\mathrm{Sp}}(S, M) \to \Omega^{\infty} M$ is an equivalence, which is clear. $\square$

Proposition 8.2.5.4. *Let R be an $\mathbb{E}_1$-ring, and let M be a left R-module. The following conditions are equivalent:*

(1) *The left R-module M is perfect.*

(2) *The left R-module M is a compact object of LMod_R.*

(3) *There exists a right R-module $M^{\vee}$ such that the composition $\mathrm{LMod}_R \xrightarrow{M^{\vee} \otimes_R \bullet} \mathrm{Sp} \xrightarrow{\Omega^{\infty}} \mathcal{S}$ is equivalent to the functor corepresented by M.*

Moreover, if these conditions are satisfied, then the object $M^{\vee} \in \mathrm{RMod}_R$ is also perfect.

Proof. The equivalence (1) $\Leftrightarrow$ (2) is Proposition 8.2.5.2. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{LMod}_R, \mathrm{Sp})$ spanned by those functors which are continuous and exact. Proposition 8.2.5.3 yields an equivalence of ∞-categories $\mathrm{RMod}_R \to \mathcal{C}$. According to Propositions 1.4.2.22 and 1.4.4.4, composition with Ω^{∞} induces a fully faithful embedding $\mathcal{C} \to \mathrm{Fun}(\mathrm{LMod}_R, \mathcal{S})$, whose essential image $\mathcal{C}'$ consists precisely of those functors which are continuous and left exact. The functor co-represented by M is automatically left-exact, and is continuous if and only if M is compact. This proves that (2) $\Leftrightarrow$ (3).

Let $j : (\mathrm{LMod}_R)^{op} \to \mathrm{Fun}(\mathrm{LMod}_R, \mathcal{S})$ denote the dual Yoneda embedding, so that j restricts to a map $j' : (\mathrm{LMod}_R^{\mathrm{perf}})^{op} \to \mathcal{C}'$. Composing j' with a homotopy inverse to the equivalence $\mathrm{RMod}_R \to \mathcal{C} \to \mathcal{C}'$, we

obtain a "dualization" map $(\mathrm{LMod}_R^{\mathrm{perf}})^{op} \to \mathrm{RMod}_R$, which we will indicate by $M \mapsto M^\vee$. Let $\mathcal{D} \subseteq \mathrm{LMod}_R^{\mathrm{perf}}$ be the full subcategory spanned by those objects M such that $M^\vee$ is perfect. We wish to show that $\mathcal{D} = \mathrm{LMod}_R^{\mathrm{perf}}$. The functor $M \mapsto M^\vee$ is exact, and $\mathrm{RMod}_R^{\mathrm{perf}}$ is a stable subcategory of RMod_R which is closed under retracts. It follows that $\mathcal{D}$ is a stable subcategory of LMod_R which is closed under retracts. Since $A^\vee \simeq A$ belongs to $\mathcal{D}$, we conclude that $\mathrm{LMod}_R^{\mathrm{perf}} \subseteq \mathcal{D}$. $\qquad\square$

Corollary 8.2.5.5. *Let R be a connective $\mathbb{E}_1$-ring, and let M be a perfect left R-module. Then:*

(1) *The homotopy groups $\pi_m M$ vanish for $m \ll 0$.*

(2) *Suppose that $\pi_m M \simeq 0$ for all $m < k$. Then $\pi_k M$ is a finitely presented module over $\pi_0 A$.*

Proof. We have an equivalence $M \simeq \varinjlim(\tau_{\geq -n} M)$. Since M is compact (Proposition 8.2.5.2), we conclude that M is a retract of some $\tau_{\geq -n} M$, so that $\pi_m M \simeq 0$ for $m < -n$. This proves (1). To prove (2), we observe that the inclusion of the heart of LMod_R into $\mathrm{LMod}_R^{\mathrm{cn}}$ preserves filtered colimits, so the right adjoint $\tau_{\leq 0} : \mathrm{LMod}_R^{\mathrm{cn}} \to \mathrm{LMod}_R^\heartsuit$ preserves compact objects. It now suffices to observe that the compact objects in the ordinary category of $\pi_0 R$-modules are precisely the finitely presented modules. $\qquad\square$

In the situation of Proposition 8.2.5.4, we will refer to $M^\vee$ as a *dual* of M. We next prove that this notion of duality determines an antiequivalence between the ∞-categories $\mathrm{LMod}_R^{\mathrm{perf}}$ and $\mathrm{RMod}_R^{\mathrm{perf}}$.

Lemma 8.2.5.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and let $F : \mathcal{C} \times \mathcal{D} \to \mathcal{S}$ be a bifunctor. The following conditions are equivalent:*

(1) *The induced map $f : \mathcal{C} \to \mathrm{Fun}(\mathcal{D}, \mathcal{S}) = \mathcal{P}(\mathcal{D}^{op})$ is fully faithful, and the essential image of f coincides with the essential image of the Yoneda embedding $\mathcal{D}^{op} \to \mathcal{P}(\mathcal{D}^{op})$.*

(2) *The induced map $f' : \mathcal{D} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S}) = \mathcal{P}(\mathcal{C}^{op})$ is fully faithful, and the essential image of f coincides with the essential image of the Yoneda embedding $\mathcal{C}^{op} \to \mathcal{P}(\mathcal{C}^{op})$.*

Proof. Let

$$G_\mathcal{C} : \mathcal{C} \times \mathcal{C}^{op} \to \mathcal{S}, \quad G_\mathcal{D} : \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{S}$$

be the maps used in the definition of the Yoneda embeddings (see §T.5.1.3). Then (1) is equivalent to the existence of an equivalence $\alpha : \mathcal{D}^{op} \to \mathcal{C}$ such that the composition $\mathcal{D}^{op} \times \mathcal{D} \longrightarrow \mathcal{C} \times \mathcal{D} \xrightarrow{F} \mathcal{S}$ is homotopic to $G_\mathcal{D}$. If β is a homotopy inverse to α, then the composition $\mathcal{C} \times \mathcal{C}^{op} \xrightarrow{\beta^{op}} \mathcal{C} \times \mathcal{D} \xrightarrow{F} \mathcal{S}$ is homotopic to $G_\mathcal{C}$, which proves (2). The converse follows by the same argument. $\qquad\square$

We will say that a functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{S}$ is a *perfect pairing* if it satisfies the hypotheses of Lemma 8.2.5.6. In this case, F determines an equivalence between $\mathcal{C}$ and $\mathcal{D}^{op}$, well-defined up to homotopy. The proof of Proposition 8.2.5.4 yields the following:

Proposition 8.2.5.7. *Let R be an $\mathbb{E}_1$-ring. Then the bifunctor*

$$\mathrm{RMod}_R^{\mathrm{perf}} \times \mathrm{LMod}_R^{\mathrm{perf}} \xrightarrow{\otimes_R} \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathcal{S}$$

is a perfect pairing.

Let R be an $\mathbb{E}_1$-ring, and let M be a left R-module. Roughly speaking, M is perfect if it can built from finitely many copies of R (by forming shifts, extensions, and retracts). This is a very strong condition which is often violated in practice. For example, suppose that R is a commutative Noetherian ring and that M is a discrete R-module, such that $\pi_0 M$ is finitely generated over R in the sense of classical commutative algebra. In this case, we can choose a resolution

$$\ldots \to P_2 \to P_1 \to P_0 \to \pi_0 M$$

where each P_i is a free R-module of finite rank. However, we cannot usually guarantee that $P_n \simeq 0$ for $n \gg 0$; in general this is possible only when R is regular ([50]). Consequently, the module M is not generally perfect as an R-module spectrum (see Example 8.2.5.25 below). Nevertheless, the fact that we can choose each P_i to be of finite rank can be regarded as a weaker finiteness condition on M, which we now formulate.

Definition 8.2.5.8. Let $\mathcal{C}$ be a compactly generated ∞-category. We will say that an object $C \in \mathcal{C}$ is *almost compact* if $\tau_{\leq n} C$ is a compact object of $\tau_{\leq n} \mathcal{C}$ for all $n \geq 0$.

Remark 8.2.5.9. Let $\mathcal{C}$ be a compactly generated ∞-category. Then every compact object of $\mathcal{C}$ is almost compact.

Definition 8.2.5.10. Let R be a connective $\mathbb{E}_1$-ring. We will say that a left R-module M is *almost perfect* if there exists an integer k such that $M \in (\mathrm{LMod}_R)_{\geq k}$ and is almost compact as an object of $(\mathrm{LMod}_R)_{\geq k}$. We let $\mathrm{LMod}_R^{\mathrm{aperf}}$ denote the full subcategory of LMod_R spanned by the almost perfect left R-modules.

Proposition 8.2.5.11. *Let R be a connective $\mathbb{E}_1$-ring. Then:*

(1) *The full subcategory $\mathrm{LMod}_R^{\mathrm{aperf}} \subseteq \mathrm{LMod}_R$ is closed under translations and finite colimits, and is therefore a stable subcategory of LMod_R.*

(2) *The full subcategory $\mathrm{LMod}_R^{\mathrm{aperf}} \subseteq \mathrm{LMod}_R$ is closed under the formation of retracts.*

(3) *Every perfect left R-module is almost perfect.*

(4) *The full subcategory $(\mathrm{LMod}_R^{\mathrm{aperf}})_{\geq 0} \subseteq \mathrm{LMod}_R$ is closed under the formation of geometric realizations of simplicial objects.*

(5) *Let M be a left R-module which is connective and almost perfect. Then M can be obtained as the geometric realization of a simplicial left R-module $P_\bullet$ such that each P_n is a free R-module of finite rank.*

Proof. Assertions (1) and (2) are obvious, and (3) follows from Proposition 8.2.5.2 and Corollary 8.2.5.5. To prove (4), it suffices to show that the collection of compact objects of $\tau_{\leq n} \mathrm{LMod}_R^{\mathrm{cn}}$ is closed under geometric realizations, which follows from Lemma 1.3.3.10.

We now prove (5). In view of Theorem 1.2.4.1 and Remark 1.2.4.3, it will suffice to show that M can be obtained as the colimit of a sequence

$$D(0) \xrightarrow{f_1} D(1) \xrightarrow{f_2} D(2) \to \ldots$$

where each $\mathrm{cofib}(f_n)[-n]$ is a free R-module of finite rank; here we agree by convention that f_0 denotes the zero map $0 \to D(0)$. The construction goes by induction. Suppose that the diagram

$$D(0) \to \ldots \to D(n) \xrightarrow{g} M$$

has already been constructed, and that $N = \mathrm{fib}(g)$ is n-connective. Part (1) implies that N is almost perfect, so that the bottom homotopy group $\pi_n N$ is a compact object in the category of left $\pi_0 A$-modules. It follows that there exists a map $\beta : Q[n] \to N$, where Q is a free left R-module of finite rank, and β induces a surjection $\pi_0 Q \to \pi_n N$. We now define $D(n+1)$ to be the cofiber of the composite map $Q(n) \xrightarrow{\beta} N \to D(n)$, and construct a diagram

$$D(0) \to \ldots \to D(n) \to D(n+1) \xrightarrow{g'} M.$$

Using the octahedral axiom, we obtain a fiber sequence

$$Q[n] \to \mathrm{fib}(g) \to \mathrm{fib}(g'),$$

and the associated long exact sequence of homotopy groups proves that $\mathrm{fib}(g')$ is $(n+1)$-connective.

In particular, we conclude that for fixed m, the maps $\pi_m D(n) \to \pi_m M$ are isomorphisms for $n \gg 0$, so that the natural map $\varinjlim D(n) \to M$ is an equivalence of left R-modules, as desired. $\qquad\square$

Using Proposition 8.2.5.11, we can give the following characterization of the ∞-category of (connective) almost perfect modules over a connective $\mathbb{E}_1$-ring.

Corollary 8.2.5.12. *Let R be a connective $\mathbb{E}_1$-ring, let $\mathcal{C}$ denote the full subcategory of LMod_R spanned by those left A-modules which are connective and almost perfect, and let $\mathcal{C}^0 \subseteq \mathcal{C}$ denote the full subcategory of $\mathcal{C}$ spanned by the objects $\{R^n\}_{n \geq 0}$. Let $\mathcal{D}$ be an arbitrary ∞-category which admits geometric realizations for simplicial objects, and let $\mathrm{Fun}_\sigma(\mathcal{C}, \mathcal{D})$ be the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which preserve geometric realizations of simplicial objects. Then the restriction functor $\mathrm{Fun}_\sigma(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}^0, \mathcal{D})$ is an equivalence of ∞-categories.*

Proof. Let $\mathcal{C}'$ be the smallest full subcategory of $\mathcal{P}(\mathcal{C}^0)$ which contains the essential image of the Yoneda embedding and is stable under geometric realizations of simplicial objects, and let $j : \mathcal{C}^0 \to \mathcal{C}'$ be the Yoneda embedding. Using Remark T.5.3.5.9, we conclude that composition with j induces an equivalence $\mathrm{Fun}_\sigma(\mathcal{C}', \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}^0, \mathcal{D})$ for any ∞-category $\mathcal{D}$ which admits geometric realizations of simplicial objects. In particular, the inclusion $\mathcal{C}^0 \subseteq \mathcal{C}$ extends (up to homotopy) to a functor $F : \mathcal{C}' \to \mathcal{C}$ which commutes with geometric realizations. To complete the proof, it will suffice to show that F is an equivalence of ∞-categories. Using the fact that each R^n is a projective object of $\mathrm{LMod}_R^{\mathrm{cn}}$, we deduce that F is fully faithful. Part (5) of Proposition 8.2.5.11 implies that F is essentially surjective. $\qquad\square$

For a general connective $\mathbb{E}_1$-ring R, the t-structure on LMod_R does not restrict to a t-structure on the full subcategory $\mathrm{LMod}_R^{\mathrm{aperf}}$. One might naively expect the heart of $\mathrm{LMod}_R^{\mathrm{aperf}}$ to be equivalent to the ordinary category of finitely presented $\pi_0 R$-modules. In general, this is not an abelian category. We can correct this defect by introducing an appropriate hypothesis on R. We begin by recalling a definition from classical algebra.

Definition 8.2.5.13. An associative ring R is *left coherent* if every finitely generated left ideal of R is finitely presented (as a left R-module).

Example 8.2.5.14. An associative ring R is *left Noetherian* if every left ideal in R is finitely generated. Every left Noetherian ring is left coherent. An infinitely generated polynomial ring $\mathbf{Z}[x_1, x_2, x_3, \ldots]$ is an example of a (left) coherent ring which is not (left) Noetherian.

For completeness, we include a proof of the following classical result:

Lemma 8.2.5.15. *Let R be a left coherent associative ring. Then:*

(1) *Every finitely generated submodule of R^n is finitely presented.*

(2) *Every finitely generated submodule of a finitely presented left R-module is finitely presented.*

(3) *If $f : M \to N$ is a map of finitely presented left R-modules, then $\ker(f)$ and $\mathrm{coker}(f)$ are finitely presented.*

Proof. We first make the following elementary observations, which do not require the assumption that R is left coherent:

(a) Suppose $f : M \to N$ is an epimorphism of left R-modules. If M is finitely generated and N is finitely presented, then $\ker(f)$ is finitely generated.

(b) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of left R-modules. If M' and M'' are finitely presented, then M is finitely presented.

We now prove (1) using induction on n. When $n = 0$ there is nothing to prove. Suppose that $n > 0$ and that $M \subseteq R^n$ is finitely generated. Form a diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & R^{n-1} & \longrightarrow & R^n & \longrightarrow & R & \longrightarrow & 0
\end{array}
$$

where the vertical maps are monomorphisms. Then M'' can be identified with a finitely generated left ideal of R. Since R is left coherent, we conclude that M'' is finitely presented. Using (a), we deduce that M' is itself finitely generated. The inductive hypothesis now implies that M' is finitely presented, so that we can use (b) to conclude that M is finitely presented.

We next prove (2). Suppose that $f : M \to N$ is a monomorphism, where N is finitely presented and M is finitely generated. Choose an epimorphism $g : R^n \to N$, and form a pullback diagram

$$
\begin{array}{ccc}
K & \longrightarrow & R^n \\
\downarrow & & \downarrow \\
M & \longrightarrow & N.
\end{array}
$$

Then K can be identified with the kernel of the induced map $R^n \to N/M$, and is therefore finitely generated. Part (1) implies that K is finitely presented. The induced map $K \to M$ is an epimorphism, whose kernel is isomorphic to $\ker(g)$ and is therefore finitely generated. It follows that M is finitely presented, as desired.

We now prove (3). It is clear that $\operatorname{coker}(f)$ is finitely presented (this does not require the left coherence of R). We next show that $\ker(f)$ is finitely presented. The image of f is a finitely generated submodule of N, and therefore finitely presented by (2). Consequently, we may replace N by $\operatorname{im}(f)$, and thereby reduce to the case where f is an epimorphism. We now apply (a) to deduce that $\ker(f)$ is finitely generated. Invoking (2) again, we conclude that $\ker(f)$ is finitely presented as desired. $\qquad\square$

Definition 8.2.5.16. Let R be an $\mathbb{E}_1$-ring. We will say that R is *left coherent* if the following conditions are satisfied:

(1) The $\mathbb{E}_1$-ring R is connective.

(2) The associative ring $\pi_0 R$ is left coherent (in the sense of Definition 8.2.5.13).

(3) For each $n \geq 0$, the homotopy group $\pi_n R$ is finitely presented as a left module over $\pi_0 R$.

Proposition 8.2.5.17. *Let R be an $\mathbb{E}_1$-ring and M a left R-module. Suppose that R is left coherent. Then M is almost perfect if and only if the following conditions are satisfied:*

(i) *For $m \ll 0$, $\pi_m M = 0$.*

(ii) *For every integer m, $\pi_m M$ is finitely presented as a $\pi_0 R$-module.*

Proof. Without loss of generality, we may assume that M is connective. Suppose first that M is almost perfect. We will prove by induction on n that $\pi_n M$ is finitely presented as a $\pi_0 R$-module. If $n = 0$, this simply reduces to the observation that the compact objects of the ordinary category of left $\pi_0 R$-modules are precisely the finitely presented $\pi_0 R$-modules. In particular, we can choose a finitely generated free R-module P and a map $\alpha : P \to M$ which induces a surjection $\pi_0 P \to \pi_0 M$. Since R is left coherent, the homotopy groups $\pi_m P$ are finitely presented $\pi_0 R$-modules. Let $K = \operatorname{fib}(\alpha)$. Then K is connective by construction, and almost perfect by Proposition 8.2.5.11. The inductive hypothesis implies that $\pi_i K$ is finitely presented for $0 \leq i < n$.

We have a short exact sequence

$$0 \to \operatorname{coker}(\pi_n K \to \pi_n P) \to \pi_n M \to \ker(\pi_{n-1} K \to \pi_{n-1} P) \to 0.$$

Using Lemma 8.2.5.15, we deduce that the outer terms are finitely generated, so that $\pi_n M$ is finitely generated. Applying the same reasoning, we conclude that $\pi_n K$ is finitely generated, so that $\operatorname{coker}(\pi_n K \to \pi_n P)$ is finitely presented. Using the exact sequence again, we conclude that $\pi_n M$ is finitely presented.

Now suppose that the connective left R-module M satisfies condition (ii). We will prove that M can be obtained as the geometric realization of a simplicial left R-module $P_\bullet$ such that each P_n is a free A-module of finite rank. As in the proof of Proposition 8.2.5.11, it will suffice to show that M is the colimit of a sequence

$$D(0) \xrightarrow{f_1} D(1) \xrightarrow{f_2} D(2) \to \dots$$

where each $\mathrm{cofib}(f_n)[-n]$ is a free R-module of finite rank. Supposing that the partial sequence

$$D(0) \to \ldots \to D(n) \xrightarrow{g} M$$

has been constructed, with the property that $\mathrm{fib}(g)$ is n-connective. If $\pi_n \mathrm{fib}(g)$ is finitely generated as a $\pi_0 R$-module, then we can proceed as in the proof of Proposition 8.2.5.11. To verify this, we observe that $D(n)$ is almost perfect and therefore satisfies (ii) (by the first part of the proof). We now use the exact sequence

$$0 \to \mathrm{coker}(\pi_{n+1}D(n) \to \pi_{n+1}M) \to \pi_n \mathrm{fib}(g) \to \ker(\pi_n D(n) \to \pi_n M) \to 0$$

to conclude that $\pi_n \mathrm{fib}(g)$ is finitely presented. $\qquad\square$

Proposition 8.2.5.18. *Let R be a connective $\mathbb{E}_1$-ring. The following conditions are equivalent:*

(1) *The $\mathbb{E}_1$-ring R is left coherent.*

(2) *For every left R-module M, if M is almost perfect, then $\tau_{\geq 0} M$ is almost perfect.*

(3) *The pair of ∞-categories*

$$(\mathrm{LMod}_R^{\mathrm{aperf}} \cap \mathrm{LMod}_R^{\geq 0}, \mathrm{LMod}_R^{\mathrm{aperf}} \cap \mathrm{LMod}_R^{\leq 0})$$

determines a t-structure on $\mathrm{LMod}_R^{\mathrm{aperf}}$.

Proof. The implication $(1) \Rightarrow (2)$ follows from the description of almost perfect modules given in Proposition 8.2.5.17. The equivalence $(2) \Leftrightarrow (3)$ is obvious. We will show that $(3) \Rightarrow (1)$.

Suppose that (3) is satisfied. We note that the first non-vanishing homotopy group of any almost perfect R-module is a finitely presented module over $\pi_0 R$. Applying (2) to the module $R[-n]$, we deduce that $\pi_n R$ is a finitely presented $\pi_0 R$-module. To complete the proof, it suffices to show that $\pi_0 R$ is left coherent.

Let $A = \pi_0 R$, and regard A as a discrete left R-module. Using condition (3), we deduce that A is almost perfect. Let $I \subseteq A$ be a finitely generated left ideal. Then I is the image of a map $f : A^n \to A$. Let K denote the fiber of f (in the ∞-category LMod_R). Then K is almost perfect. We have a short exact sequence

$$0 \to \pi_0 K \to A^n \to I \to 0.$$

Since K is almost perfect, condition (3) implies that $\pi_0 K$ is finitely generated as an A-module, so that I is finitely presented as a A-module. This completes the proof of (1). $\qquad\square$

Remark 8.2.5.19. Let R be a left coherent $\mathbb{E}_1$-ring, and regard $\mathrm{LMod}_R^{\mathrm{aperf}}$ as endowed with the t-structure described in Proposition 8.2.5.18. Then $\mathrm{LMod}_R^{\mathrm{aperf}}$ is right bounded and left complete, and the functor $M \mapsto \pi_0 M$ determines an equivalence from the heart of $\mathrm{LMod}_R^{\mathrm{aperf}}$ to the (nerve of the) category of finitely presented left modules over $A = \pi_0 R$.

We now study the interaction between finiteness and flatness properties of modules.

Proposition 8.2.5.20. *Let R be a connective $\mathbb{E}_1$-ring, and let M be a connective left R-module. The following conditions are equivalent:*

(1) *The left R-module M is a retract of a finitely generated free R-module.*

(2) *The left R-module M is flat and almost perfect.*

Proof. The implication $(1) \Rightarrow (2)$ is obvious. Conversely, suppose that M is flat and almost perfect. Then $\pi_0 M$ is a left module over $\pi_0 R$ which is finitely presented and flat, and therefore projective. Using Proposition 8.2.2.18, we deduce that M is projective. Choose a map $f : P \to M$, where P is a free left R-module of finite rank and the induced map $\pi_0 P \to \pi_0 M$ is surjective. Since M is projective, the map f splits, so that M is a summand of P. This proves (1). $\qquad\square$

It follows from Proposition 8.2.5.20 that if an almost perfect left R-module M is flat, then M is perfect. We conclude with a mild generalization of this statement, where the flatness hypothesis is relaxed.

Definition 8.2.5.21. Let R be a connective $\mathbb{E}_1$-ring. We will say that a left R-module M has Tor-*amplitude* $\leq n$ if, for every discrete right R-module N, the homotopy groups $\pi_i(N \otimes_R M)$ vanish for $i > n$. We will say that M is *of finite* Tor-*amplitude* if it has Tor-amplitude $\leq n$ for some integer n.

Remark 8.2.5.22. In view of Theorem 8.2.2.15, a connective left R-module M has Tor-amplitude ≤ 0 if and only if M is flat.

Proposition 8.2.5.23. *Let R be a connective $\mathbb{E}_1$-ring.*

(1) *If M is a left R-module of* Tor-*amplitude* $\leq n$, *then $M[k]$ has* Tor-*amplitude* $\leq n + k$.

(2) *Let*
$$M' \to M \to M''$$
be a fiber sequence of left R-modules. If M' and M'' have Tor-*amplitude* $\leq n$, *then so does M.*

(3) *Let M be a left R-module of* Tor-*amplitude* $\leq n$. *Then any retract of M has* Tor-*amplitude* $\leq n$.

(4) *Let M be an almost perfect left module over R. Then M is perfect if and only if M has finite* Tor-*amplitude.*

(5) *Let M be a left module over R having* Tor-*amplitude* $\leq n$. *For every $N \in (\mathrm{RMod}_R)_{\leq 0}$, the homotopy groups $\pi_i(N \otimes_R M)$ vanish for $i > n$.*

Proof. The first three assertions follow immediately from the exactness of the functor $N \mapsto N \otimes_R M$. It follows that the collection left R-modules of finite Tor-amplitude is stable under retracts and finite colimits, and contains the module $R[n]$ for every integer n. This proves the "only if" direction of (4). For the converse, let us suppose that M is almost perfect and of finite Tor-amplitude. We wish to show that M is perfect. We first apply (1) to reduce to the case where M is connective. The proof now goes by induction on the Tor-amplitude n of M. If $n = 0$, then M is flat and we may conclude by applying Proposition 8.2.5.20. We may therefore assume $n > 0$.

Since M is almost perfect, there exists a free left R-module P of finite rank and a fiber sequence
$$M' \to P \xrightarrow{f} M$$
where f is surjective. To prove that M is perfect, it will suffice to show that P and M' are perfect. It is clear that P is perfect, and it follows from Proposition 8.2.5.11 that M' is almost perfect. Moreover, since f is surjective, M' is connective. We will show that M' is of Tor-amplitude $\leq n - 1$; the inductive hypothesis will then imply that M is perfect, and the proof will be complete.

Let N be a discrete right R-module. We wish to prove that $\pi_k(N \otimes_R M') \simeq 0$ for $k \geq n$. Since the functor $N \otimes_R \bullet$ is exact, we obtain for each k an exact sequence of homotopy groups
$$\pi_{k+1}(N \otimes_R M) \to \pi_k(N \otimes_R M') \to \pi_k(N \otimes_R P).$$

The left entry vanishes in virtue of our assumption that M has Tor-amplitude $\leq n$. We now complete the proof of (4) by observing that $\pi_k(N \otimes_R P)$ is a finite direct sum of copies of $\pi_k N$, and therefore vanishes because $k \geq n > 0$ and N is discrete.

We now prove (5). Assume that M has Tor-amplitude $\leq m$. Let $N \in (\mathrm{RMod}_R)_{\leq 0}$; we wish to prove that $\pi_i(N \otimes_R M) \simeq 0$ for $i > n$. Since $N \simeq \varinjlim \tau_{\geq -m} N$, it will suffice to prove the vanishing after replacing N by $\tau_{\geq -m} N$ for every integer m. We may therefore assume that $N \in (\mathrm{RMod}_R)_{\geq -m}$ for some $m \geq 0$. We proceed by induction on m. When $m = 0$, the desired result follows immediately from our assumption on M. If $m > 0$, we have a fiber sequence
$$\tau_{\geq 1-m} N \to N \to (\pi_{-m} N)[-m],$$

hence an exact sequence

$$\pi_i((\tau_{\geq 1-m}N) \otimes_R M) \to \pi_i(N \otimes_R M) \to \pi_{i+m}(\pi_{-m}N \otimes_R M).$$

If $i > n$, then the first group vanishes by the inductive hypothesis, and the second by virtue of our assumption that M has Tor-amplitude $\leq n$. $\qquad\square$

Remark 8.2.5.24. Let R be a connective $\mathbb{E}_1$-ring, and let $\mathcal{C}$ be the smallest stable subcategory of LMod_R which contains all finitely generated projective modules. Then $\mathcal{C} = \mathrm{LMod}_R^{\mathrm{perf}}$. The inclusion $\mathcal{C} \subseteq \mathrm{LMod}_R^{\mathrm{perf}}$ is obvious. To prove the converse, we must show that every object $M \in \mathrm{LMod}_R^{\mathrm{perf}}$ belongs to $\mathcal{C}$. Invoking Corollary 8.2.5.5, we may reduce to the case where M is connective. We then work by induction on the (necessarily finite) Tor-amplitude of M. If M is of Tor-amplitude ≤ 0, then M is flat and the desired result follows from Proposition 8.2.5.20. In the general case, we choose a finitely generated free R-module P and a map $f : P \to M$ which induces a surjection $\pi_0 P \to \pi_0 M$ (which is possible in view of Corollary 8.2.5.5). As in the proof of Proposition 8.2.5.23, we may conclude that that fiber K of f is a connective perfect module of smaller Tor-amplitude than that of M, so that $K \in \mathcal{C}$ by the inductive hypothesis. Since $P \in \mathcal{C}$ and $\mathcal{C}$ is stable under the formation of cofibers, we conclude that $M \in \mathcal{C}$ as desired.

Example 8.2.5.25. Let R be an associative ring which we regard as a discrete $\mathbb{E}_1$-ring. Let $\mathrm{LMod}_R^- \subseteq \mathrm{LMod}_R$ be the full subcategory consisting of bounded-above objects, and let $M \in \mathrm{LMod}_R^-$. Using an inverse to the functor θ of Proposition 8.1.1.15, we can identify any $M \in \mathrm{LMod}_R^-$ with a (bounded above) complex $P_\bullet$ in the abelian category of (discrete) R-modules. It follows from Remark 8.2.5.24 that M is perfect if and only if $P_\bullet$ can be chosen to have only finitely many terms, each of which is finitely generated over R.

We conclude this section by discussing some finiteness conditions for algebras over structured ring spectra.

Definition 8.2.5.26. Let $1 \leq k \leq \infty$ and let R be a connective $\mathbb{E}_{k+1}$-ring. We let $\mathrm{Free} : \mathrm{LMod}_R \to \mathrm{Alg}_R^{(k)}$ denote a left adjoint to the forgetful functor. Note that Free carries connective R-modules to connective $\mathbb{E}_k$-algebras over R. Let A be a connective $\mathbb{E}_k$-algebra over R. We say that A is:

- *finitely generated and free* if there exists a finitely generated free left R-module M and an equivalence $A \simeq \mathrm{Free}(M)$ in $\mathrm{Alg}_R^{(k)}$. We let $\mathrm{Alg}_R^{(k),\mathrm{free}}$ denote the full subcategory spanned by the finitely generated free algebras.

- *of finite presentation* if A belongs to the smallest full subcategory of $\mathrm{Alg}_R^{(k)}$ which contains $\mathrm{Alg}_R^{(k),\mathrm{free}}$ and is stable under finite colimits.

- *locally of finite presentation* if A is a compact object of $\mathrm{Alg}_R^{(k)}$.

- *almost of finite presentation* if A is an almost compact object of $\mathrm{Alg}_R^{(k)}$ (see Definition 8.2.5.8): that is, if $\tau_{\leq n}A$ is a compact object of $\tau_{\leq n}\mathrm{Alg}_R^{(k)}$ for all $n \geq 0$.

The basic properties of Definition 8.2.5.26 can be summarized as follows:

Proposition 8.2.5.27. *Let $1 \leq k \leq \infty$ and let R be a connective $\mathbb{E}_{k+1}$-ring. Then:*

(1) *The ∞-category $\mathrm{Alg}_R^{(k),\mathrm{cn}}$ is projectively generated (Definition T.5.5.8.23). Moreover, a connective R-algebra A is a compact projective object of $\mathrm{Alg}_R^{(k),\mathrm{cn}}$ if and only if A is a retract of a finitely generated free commutative R-algebra.*

(2) *Let $\mathrm{Alg}_R^{(k),\mathrm{fp}} \subseteq \mathrm{Alg}_R^{(k)}$ denote the full subcategory spanned by those $\mathbb{E}_k$-algebras over R which are connective and of finite presentation. Then the inclusion induces an equivalence $\mathrm{Ind}(\mathrm{Alg}_R^{(k),\mathrm{fp}}) \simeq \mathrm{Alg}_R^{(k)}$.*

(3) *The ∞-category $\mathrm{Alg}_R^{(k),\mathrm{cn}}$ of connective $\mathbb{E}_k$-algebras over R is compactly generated. The compact objects of $\mathrm{Alg}_R^{(k),\mathrm{cn}}$ are those connective $\mathbb{E}_k$-algebras which are locally of finite presentation over R.*

(4) *A connective $\mathbb{E}_k$-algebra A over R is almost of finite presentation if and only if, for each $n \geq 0$, there exists an R-algebra A' of finite presentation such that $\tau_{\leq n} A$ is a retract of $\tau_{\leq n} A'$.*

Proof. Assertion (1) from Corollary 8.1.4.17 and its proof. Assertion (2) then follows from Proposition T.5.3.5.11, (3) from Lemma T.5.4.2.4, and (4) from Corollary T.5.5.7.4. $\qquad\square$

Remark 8.2.5.28. Let $R \to R'$ be a map of connective $\mathbb{E}_{k+1}$-rings, let $A \in \mathrm{Alg}_R^{(k),\mathrm{cn}}$ be a connective $\mathbb{E}_k$-algebra over R, and let $A' \simeq R' \otimes_R A$ be the image of A in $\mathrm{Alg}_{R'}^{(k),\mathrm{cn}}$. If A is free and finitely generated (finitely presented, almost finitely presented) over R, then A' is free and finitely generated (finitely presented, almost finitely presented) over R'. In the first three cases, this is obvious. The last two follow from assertions (3) and (4) of Proposition 8.2.5.27, together with the observation that $\tau_{\leq n} A' \simeq \tau_{\leq n}(R' \otimes_R \tau_{\leq n} A)$.

Remark 8.2.5.29. Suppose given a commutative diagram

$$
\begin{array}{ccc}
 & B & \\
\nearrow & & \searrow \\
A & \longrightarrow & C
\end{array}
$$
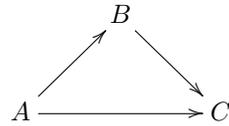
of $\mathbb{E}_\infty$-rings, where B is of locally of finite presentation over A. Then C is locally of finite presentation over B if and only if C is locally of finite presentation over A. This follows immediately from Propositions 8.2.5.27 and T.5.4.5.15. In §8.4.3, we will prove the analogue of this statement for morphisms which are almost of finite presentation.

Definition 8.2.5.30. Let R be a connective $\mathbb{E}_k$-ring for $2 \leq k \leq \infty$. We will say that R is *coherent* if it is left coherent, when regarded as an $\mathbb{E}_1$-ring: that is, if $\pi_0 R$ is a coherent ring (in the sense of ordinary commutative algebra) and each homotopy group $\pi_i R$ is a finitely presented module over $\pi_0 R$. We will say that R is *Noetherian* if R is coherent and the ordinary commutative ring $\pi_0 R$ is Noetherian.

The following result relates some of the absolute and relative finiteness conditions introduced above:

Proposition 8.2.5.31 (Hilbert Basis Theorem)**.** *Let $f : R \to R'$ be a map of connective $\mathbb{E}_\infty$-rings. Suppose that R is Noetherian. Then R' is almost of finite presentation as an R-algebra if and only if the following conditions are satisfied:*

(1) *The ring $\pi_0 R'$ is a finitely generated $\pi_0 R$-algebra.*

(2) *The $\mathbb{E}_\infty$-ring R' is Noetherian.*

Proof. We first prove the "only if" direction. We first note that $\pi_0 R' = \tau_{\leq 0} R'$ is a compact object in the ordinary category of commutative $\pi_0 R$-algebras. This proves (1). The classical Hilbert basis theorem implies that $\pi_0 R'$ is Noetherian. It remains only to show that each $\pi_n R'$ is a finitely generated module over $\pi_0 R'$. This condition depends only on the truncation $\tau_{\leq n} R'$. In view of part (4) of Proposition 8.2.5.27, we may assume that R' is a finitely presented commutative R-algebra.

Let $\mathcal{C}$ be the full subcategory of CAlg_R spanned by those connective $\mathbb{E}_\infty$-algebras over R which satisfy (1) and (2). To show that $\mathcal{C}$ contains all finitely presented R-algebras, it will suffice to show that $\mathcal{C}$ is stable under finite colimits, and contains the free commutative R-algebra on a single generator. We first prove the stability under finite colimits. In view of Corollary T.4.4.2.4, it will suffice to show that $R \in \mathcal{C}$ and that $\mathcal{C}$ is stable under pushouts. The inclusion $R \in \mathcal{C}$ is obvious (since R is Noetherian by assumption). Suppose given a pushout diagram

$$
\begin{array}{ccc}
A & \longrightarrow & A' \\
\downarrow & & \downarrow \\
A'' & \longrightarrow & A' \otimes_A A''
\end{array}
$$
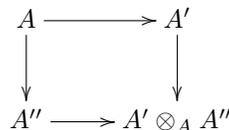
in CAlg_R, where $A, A', A'' \in \mathcal{C}$. We wish to prove that $A' \otimes_A A'' \in \mathcal{C}$. According to Corollary 8.2.1.23, the commutative ring $B = \pi_0(A' \otimes_A A'')$ is canonically isomorphic to the classical tensor product $\pi_0 A' \otimes_{\pi_0 A} \pi_0 A''$, and is therefore a finitely generated commutative $\pi_0 R$-algebra. Thus $A' \otimes_A A''$ satisfies (1). Moreover, Proposition 8.2.1.19 yields a convergent spectral sequence

$$E_2^{p,q} = \mathrm{Tor}_p^{\pi_* A}(\pi_* A', \pi_* A'')_q \Rightarrow \pi_{p+q}(A' \otimes_A A'').$$

It is not difficult to see that this is a spectral sequence of modules over the commutative ring B. Each of the B-modules $E_2^{p,q}$ is a finitely generated B-module. It follows that the subquotients $E_\infty^{p,q}$ are likewise finitely generated over B. Consequently, each homotopy group $\pi_n(A' \otimes_A A'')$ has a finite filtration by finitely generated B-modules, and is therefore itself finitely generated over B. This proves that $A' \otimes_A A''$ satisfies (2), and therefore belongs to the ∞-category $\mathcal{C}$.

Let $R\{X\} = \mathrm{Sym}^*(R)$ denote the free commutative R-algebra on a single generator. We wish to show that $R\{X\} \in \mathcal{C}$. We first treat the case where R is discrete. Let k denote the commutative ring $\pi_0 R$. Proposition 3.1.3.11 supplies canonical isomorphisms

$$\pi_m R\{X\} \simeq \pi_m(\coprod_n \mathrm{Sym}^n(R)) \simeq \oplus_{n \geq 0} \mathrm{H}_m(B\Sigma_n; k).$$

In particular, $\pi_0 R\{X\}$ is canonically isomorphic to the polynomial ring $k[x]$, which proves (1). To prove (2), we must show that each of the groups $\oplus_{n \geq 0} \mathrm{H}_m(\Sigma_n; k)$ is a finitely generated $k\{x\}$-module. Since the homology of a finite group can be computed using a finite complex, each of the groups $\mathrm{H}_m(B\Sigma_n; k)$ is individually a finite k-module. Moreover, multiplication by x is given by the maps

$$\eta_{m,n} : \mathrm{H}_m(B\Sigma_n; k) \to \mathrm{H}_m(B\Sigma_{n+1}; k)$$

induced by the inclusions $\Sigma_n \subseteq \Sigma_{n+1}$. To complete the proof, it will suffice to show that for fixed m, $\eta_{m,n}$ is surjective for $n \gg 0$. This follows from Nakaoka's homological stability theorem for the symmetric groups (for a simple proof, we refer to the reader to [85]).

Let us now treat the general case. Let B denote the polynomial ring $\pi_0 R\{X\} = (\pi_0 R)[X]$. Then B is finitely generated over $\pi_0 R$, so $R\{X\}$ satisfies (1). We wish to prove that $R\{X\}$ satisfies (2). Assume otherwise, and choose m minimal such that $\pi_m R\{X\}$ is not a finitely generated B-module; note that m is necessarily positive. Set $R_0 = \tau_{\leq 0}$. Then the argument above proves that the tensor product $R_0 \otimes_R R\{X\} \simeq R_0\{X\}$ belongs to $\mathcal{C}$. Invoking Proposition 8.2.1.19 again, we obtain a convergent spectral sequence of B-modules

$$E_2^{p,q} = \mathrm{Tor}_p^{\pi_* R}(\pi_0 R, \pi_*(R\{X\}))_q \Rightarrow \pi_{p+q}(R_0\{X\}).$$

Using the minimality of m, we deduce that $E_2^{p,q}$ is finitely generated over B for all $q < m$. It follows that $E_\infty^{0,m}$ is the quotient of $E_2^{0,m}$ by a finitely generated submodule. Since $E_\infty^{0,m}$ is a submodule of the finitely generated B-module $\pi_m(R_0\{X\})$, we conclude that $E_2^{0,m}$ is itself finitely generated over B. But $E_2^{0,m}$ contains $\pi_m(R\{X\})$ as a summand, contradicting our choice of m. This completes the proof of the "only if" direction.

Now suppose that R' satisfies (1) and (2). We will construct a sequence of maps $\psi_n : A_n \to R'$ in CAlg_R with the property that the maps ψ_n induce isomorphisms $\pi_m A_n \to \pi_m R'$ for $m < n$ and surjections $\pi_n A_n \to \pi_n R'$. To construct A_0, we invoke assumption (1): choose a finite set of elements $x_1, \ldots, x_i \in \pi_0 R'$ which generate $\pi_0 R'$ as an $\pi_0 R$-algebra. These elements determine a map of R-algebras $\psi_0 : \mathrm{Sym}^*(R^i) \to R'$ with the desired property.

Let us now suppose that $\psi_n : A_n \to R'$ has already been constructed. Let K denote the fiber of the map ψ_n, formed in the ∞-category of A_n-modules. Our assumption on ψ_n implies that $\pi_i K \simeq 0$ for $i < n$. Let $B = \pi_0 A_n$. Since A_n is almost of finite presentation over R, the first part of the proof shows that B is a Noetherian ring and that each of the homotopy groups $\pi_i A_n$ is a finitely generated B-module. Moreover, since the map $B \to \pi_0 R'$ is surjective and R' satisfies (2), we conclude that each of the homotopy groups $\pi_i R'$ is a finitely generated B-module. Using the long exact sequence

$$\ldots \to \pi_{n+1} R' \to \pi_n K \to \pi_n A_n \to \ldots$$

we deduce that $\pi_n K$ is a finitely generated B-module. Consequently, there exists a finitely generated free A_n-module M and a map $M[n] \to K$ which is surjective on π_n. Let $f : \operatorname{Sym}^*(M[n]) \to A_n$ be the induced map, and form a pushout diagram

$$
\begin{array}{ccc}
\operatorname{Sym}^*(M[n]) & \xrightarrow{\ f\ } & A_n \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_0'} \\
A_n & \longrightarrow & A_{n+1},
\end{array}
$$

where f_0 classifies the zero from $M[n]$ to A_n. By construction, we have a canonical homotopy from $\psi_n \circ f$ to $\psi_n \circ f_0$, which determines a map $\psi_{n+1} : A_{n+1} \to R'$. We observe that there is a fiber sequence of A_n-modules

$$\operatorname{fib}(f_0') \to \operatorname{fib}(\psi_n) \to \operatorname{fib}(\psi_{n+1}).$$

To show that ψ_{n+1} has the desired properties, it suffices to show that $\pi_i \operatorname{fib}(\psi_{n+1}) \simeq 0$ for $i \leq n$. Using the long exact sequence associated to the fiber sequence above, we may reduce to proving the following pair of assertions:

(a) The homotopy groups $\pi_i \operatorname{fib}(f_0')$ vanish for $i < n$.

(b) The canonical map $\pi_n \operatorname{fib}(f_0') \to \pi_n K$ is surjective.

We now observe that there is an equivalence $\operatorname{fib}(f_0') \simeq \operatorname{fib}(f_0) \otimes_{\operatorname{Sym}^*(M[n])} A_n$. In view of Corollary 8.2.1.23, it will suffice to prove the same assertions after replacing f_0' by f_0. Using Proposition 3.1.3.11, we obtain a canonical equivalence

$$\operatorname{fib}(f_0) \simeq \bigoplus_{m>0} \operatorname{Sym}^m(M[n]).$$

Because the ∞-category $\operatorname{Mod}_{A_n}^{\operatorname{cn}}$ is closed under colimits in Mod_{A_n}, it follows that the homotopy groups $\pi_i \operatorname{Sym}^m(M[n])$ vanish for $i < mn$. This proves (a). To prove (b), it will suffice to show that the composite map

$$M[n] \simeq \operatorname{Sym}^1(M[n]) \to \bigoplus_{m>0} \operatorname{Sym}^m(M[n]) \to K$$

induces a surjection on π_n, which follows immediately from our construction. $\qquad \square$

8.3 The Cotangent Complex Formalism

Let R be a connective $\mathbb{E}_\infty$-ring. Then $\pi_0 R$ is an ordinary commutative ring. We can view $\pi_0 R$ as the underlying commutative ring of R, so that the $\mathbb{E}_\infty$-ring R is determined by $\pi_0 R$ together with some additional information, which is somehow encoded in the higher homotopy groups of R. It often useful to think of this information as "infinitesimal" in nature, so that the truncation map $R \to \pi_0 R$ (which kills the higher homotopy groups of R) is somehow very close to being an equivalence. Our goal in this section is to develop some technology which will allow us to exploit this idea: namely, the theory of the *cotangent complex* of an $\mathbb{E}_\infty$-ring.

We begin by reviewing a bit of classical commutative algebra. Suppose we are given a pair of commutative rings A and B, and that we wish to study ring homomorphisms from A to B. Writing A as a quotient of a polynomial ring (perhaps on infinitely many generators), this amounts to finding the solutions to some set of polynomial equations in B. This problem is generally very difficult. However, if we fix a ring a homomorphism $\phi : A \to B$, it is often not very difficult to describe all ring homomorphisms $\phi' : A \to B$ which are sufficiently "close" to ϕ. For example, suppose we are given an ideal $I \subseteq B$ such that $I^2 = 0$, and that we wish to classify ring homomorphisms $\phi' : A \to B$ such that $\phi(x) \cong \phi'(x)$ modulo I for all $x \in A$. In this case, we can write $\phi'(x) = \phi(x) + dx$ for some map $d : A \to I$. Since $I^2 = 0$, the condition that ϕ' is a ring homomorphism translates into a relatively simple condition on d: namely, it must satisfy the Leibniz

rule $d(xy) = \phi(x)dy + \phi(y)dx$. In summary, the classification of ring homomorphisms $\phi' : A \to B$ which are congruent to ϕ modulo I requires only that we solve a set of linear equations, rather than a set of polynomial equations. This is quite a bit more tractable.

To study the situation more systematically, it is convenient to introduce a bit of terminology. Let A be a commutative ring and let M be an A-module. A *derivation* from A into M is a map $d : A \to M$ satisfying the conditions

$$d(x + y) = dx + dy \qquad d(xy) = xdy + ydx.$$

The collection of derivations of A into M forms an abelian group, which we will denote by $\mathrm{Der}(A, M)$. If A is fixed, then the functor $M \mapsto \mathrm{Der}(A, M)$ is corepresented by an A-module Ω_A, called the A-module of *absolute Kähler differentials*. One can construct Ω_A explicitly as a quotient of the free module generated by symbols $\{dx\}_{x \in A}$ by the submodule generated by the elements $\{d(x + y) - dx - dy, d(xy) - x(dy) - y(dx)\}_{x,y \in A}$.

Our goal in this section is to introduce an analogue of the construction $A \mapsto \Omega_A$, where we replace the commutative ring A with an arbitrary $\mathbb{E}_\infty$-ring. More precisely, if A is an $\mathbb{E}_\infty$-ring, we will define an A-module spectrum L_A, which we call the *absolute cotangent complex* of A. By construction, L_A will enjoy the following universal property: for any A-module M, there is a bijection between homotopy classes of maps $L_A \to M$ with homotopy classes of *derivations* of A into M.

To make this idea precise, we need a new definition of derivation: the definition for ordinary commutative rings given above is given in terms of equations, and does not generalize easily to the ∞-categorial setting. Instead, let us take our cue from the preceding discussion. If $\phi, \phi' : A \to B$ are two ring homomorphisms between commutative rings A and B which are congruent modulo an ideal $I \subseteq B$ with $I^2 = 0$, then $\phi' - \phi$ is a derivation from A into I. Every derivation from a commutative ring A into an A-module M arises in this way. To see this, we can take B to be the direct sum $A \oplus M$, equipped with the ring structure given by $(a, m)(a', m') = (aa', am' + a'm)$. There is a natural inclusion $\phi : A \to B$, and we can identify M with an ideal of B satisfying $M^2 = 0$, so that $\mathrm{Der}(A, M)$ can be identified with the set of ring homomorphsims from A into B which are congruent to ϕ modulo M: that is, with the collection of sections of the projection map $A \oplus M \to A$. This description of $\mathrm{Der}(A, M)$ does not directly require writing any equations: instead, it depends on our ability to endow the direct sum $A \oplus M$ with the structure of a commutative ring.

To describe the situation a little bit more systematically, let Ring denote the category of commutative rings, and Ring^+ the category of pairs (A, M), where A is a commutative ring and M is an A-module. A morphism in the category Ring^+ is a pair of maps $(f, f') : (A, M) \to (B, N)$, where $f : A \to B$ is a ring homomorphism and $f' : M \to N$ is a map of A-modules, (here we regard N as an A-module via transport of structure along f). Let $G : \mathrm{Ring}^+ \to \mathrm{Ring}$ be the square-zero extension functor given by the formula $(A, M) \mapsto A \oplus M$. Then the functor G admits a left adjoint F, which is described by the formula $F(A) = (A, \Omega_A)$. Here Ω_A is the A-module of Kähler differentials defined above.

We now make two observations:

(1) In addition to the functor G, there is a forgetful functor $G' : \mathrm{Ring}^+ \to \mathrm{Ring}$, given by $(A, M) \mapsto A$. Moreover, there is a natural transformation of functors from G to G', which can itself be viewed as a functor from Ring^+ into the category $\mathrm{Fun}([1], \mathrm{Ring})$ of arrows in Ring.

(2) For every commutative ring A, the fiber $G'^{-1}\{A\}$ is an abelian category (namely, the category of A-modules).

We wish to produce an analogous theory of derivations in the case where the category Ring is replaced by an arbitrary presentable ∞-category $\mathcal{C}$. What is the proper analogue of Ring^+ in this general situation? Observation (1) suggests that we should choose another ∞-category $\mathcal{C}^+$ equipped with a functor $\mathcal{C}^+ \to \mathrm{Fun}(\Delta^1, \mathcal{C})$. Observation (2) suggests that the fibers of composite map

$$\phi : \mathcal{C}^+ \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{1\}, \mathcal{C}) \simeq \mathcal{C}$$

should be "abelian" in some sense. There is a good ∞-categorical analogue of the theory of abelian categories: the theory of *stable* ∞-categories introduced in Chapter 1. It is therefore natural to require that the the

fibers of ϕ be stable. In §8.3.1, we will see that there is a canonical choice for the ∞-category $\mathcal{C}^+$ with these properties. We will refer to this canonical choice as the *tangent bundle* to $\mathcal{C}$ and denote by $T_{\mathcal{C}}$. Roughly speaking, an object of $T_{\mathcal{C}}$ consists of a pair (A, M), where $A \in \mathcal{C}$ and $M \in \mathrm{Sp}(\mathcal{C}^{/A})$.

Once we have established the theory of tangent bundles, we can proceed to define the analogue of the Kähler differentials functor. Namely, for any presentable ∞-category $\mathcal{C}$, we will define the *cotangent complex functor* $L : \mathcal{C} \to T_{\mathcal{C}}$ to be a left adjoint to the forgetful functor

$$T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{0\}, \mathcal{C}) \simeq \mathcal{C}.$$

However, it is important to exercise some care here: in the algebraic situation, we want to make sure that the cotangent complex L_A of an $\mathbb{E}_\infty$-ring produces an A-module. In other words, we want to ensure that the composition

$$\mathcal{C} \xrightarrow{L} T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{1\}, \mathcal{C}) \simeq \mathcal{C}$$

is the identity functor. We will construct a functor L with this property using a notion of *relative adjunction*, which we explain in §8.3.2.

Given an object $A \in \mathcal{C}$ and $M \in T_{\mathcal{C}} \times_{\mathcal{C}} \{A\}$, we can define the notion of a *derivation* of A into M. This can be described either as map from L_A into M in the ∞-category $T_{\mathcal{C}} \times_{\mathcal{C}} \{A\}$, or as a section of the canonical map $G(M) \to A$ in $\mathcal{C}$. For many purposes, it is convenient to work in an ∞-category containing both $\mathcal{C}$ and $T_{\mathcal{C}}$, in which the morphisms are given by derivations. Such an ∞-category is readily available: namely, the correspondence associated to the pair of adjoint functors $\mathcal{C} \underset{G}{\overset{L}{\rightleftarrows}} T_{\mathcal{C}}$, where G and L are defined as above. We will call this ∞-category the *tangent correspondence* to $\mathcal{C}$; an explicit construction will be given in §8.3.6.

In the classical theory of Kähler differentials, it is convenient to consider the absolute Kähler differentials Ω_A of a commutative ring A, but also the module of relative Kähler differentials $\Omega_{B/A}$ associated to a ring homomorphism $A \to B$. In §8.3.3 we will introduce an analogous *relative* version of the cotangent complex L. We will then establish some of the basic formal properties of the relative cotangent complex. For example, given a sequence of commutative ring homomorphisms $A \to B \to C$, there is an associated short exact sequence

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

Corollary 8.3.3.6 provides an ∞-categorical analogue of this statement: for every commutative diagram
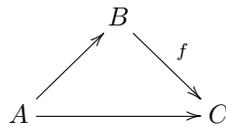
in a presentable ∞-category $\mathcal{C}$, there is an associated fiber sequence

$$f^* L_{B/A} \to L_{C/A} \to L_{C/B}$$

in the stable ∞-category $\mathrm{Sp}(\mathcal{C}^{/C})$.

In §8.3.4, we will specialize our abstract theory of the cotangent complex to the setting where $\mathcal{C}$ is the ∞-category CAlg of $\mathbb{E}_\infty$-rings. In this case, we will show that the tangent bundle $T_{\mathcal{C}}$ can be identified with the ∞-category $\mathrm{Mod}(\mathrm{Sp})$ consisting of pairs (A, M), where $A \in \mathrm{CAlg}$ is an $\mathbb{E}_\infty$-ring and $M \in \mathrm{Mod}_A$ is an A-module spectrum. In particular, we can view the cotangent complex L_A (or the relative cotangent complex $L_{A/B}$, if $\phi : B \to A$ is a map of $\mathbb{E}_\infty$-rings) as an A-module spectrum.

There are many variations on the theme described above. For example, if A is an $\mathbb{E}_k$-ring for some integer k, then $\mathrm{Sp}(\mathrm{Alg}_{/A}^{(k)})$ can be identified with the ∞-category $\mathrm{Mod}_A^{\mathbb{E}_k}(\mathrm{Sp})$ of $\mathbb{E}_k$-modules over A. When $k < \infty$, the results of §6.3.7 can be used to obtain useful explicit formula for the cotangent complex $L_A \in \mathrm{Mod}_A^{\mathbb{E}_k}(\mathrm{Sp})$, which we explain in §8.3.5.

Roughly speaking, we can regard the cotangent complex L_A of an $\mathbb{E}_\infty$-ring A as an object which controls the "infinitesimal" behavior of A. Since the difference between a connective $\mathbb{E}_\infty$-ring A and the ordinary commutative ring $\pi_0 A$ is purely "infinitesimal," one can often reduce questions about A to questions about the ordinary commutative ring $\pi_0 A$ and questions about the cotangent complex L_A. We will exploit this principle systematically in §8.4.

Remark 8.3.0.1. If A is an $\mathbb{E}_\infty$-ring, the homotopy groups of the cotangent complex L_A are often called the *topological André-Quillen homology groups* of A, and have been studied by many authors. We refer the reader to [10], [11], [8], and [104].

Warning 8.3.0.2. The classical theory of André-Quillen homology is obtained by forming the nonabelian left derived functor of the Kähler differentials functor $A \mapsto \Omega_A$. If A is a commutative ring, then this construction does not generally recover the cotangent complex L_A studied in this book, unless we assume that A contains the field $\mathbf{Q}$ of rational numbers.

8.3.1   Stable Envelopes and Tangent Bundles

Let $p : \mathcal{C} \to \mathcal{D}$ be a categorical fibration of ∞-categories. In §7.2.2, we introduced a new ∞-category $\mathrm{Stab}(p)$ whose objects are pairs (D, X), where D is an object of $\mathcal{D}$ and X is a spectrum object of the fiber $\mathcal{C}_D$ (see Construction 7.2.2.2). This definition is sensible only in cases where the fibers of p are pointed ∞-categories. In this section, we will discuss a slightly different construction, given by applying fiberwise application of the stabilization construction $\mathcal{E} \mapsto \mathrm{Sp}(\mathcal{E})$ procedure of Definition 1.4.2.8. We will then apply this construction to define the *tangent bundle* $T_\mathcal{C}$ of a presentable ∞-category $\mathcal{C}$.

Definition 8.3.1.1. Let $\mathcal{C}$ be a presentable ∞-category. A *stable envelope* of $\mathcal{C}$ is a categorical fibration $u : \mathcal{C}' \to \mathcal{C}$ with the following properties:

 (*i*) The ∞-category $\mathcal{C}'$ is stable and presentable.

 (*ii*) The functor u admits a left adjoint.

 (*iii*) For every presentable stable (pointed) ∞-category $\mathcal{E}$, composition with u induces an equivalence of ∞-categories $\mathrm{Fun}^{\mathrm{R}}(\mathcal{E}, \mathcal{C}') \to \mathrm{Fun}^{\mathrm{R}}(\mathcal{E}, \mathcal{C})$. Here $\mathrm{Fun}^{\mathrm{R}}(\mathcal{E}, \mathcal{C}')$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{E}, \mathcal{C}')$ spanned by those functors which admit left adjoints, and $\mathrm{Fun}^{\mathrm{R}}(\mathcal{E}, \mathcal{C})$ is defined similarly.

More generally, suppose that $p : \mathcal{C} \to \mathcal{D}$ is a presentable fibration. A *stable envelope* of p is a categorical fibration $u : \mathcal{C}' \to \mathcal{C}$ with the following properties:

 (1) The composition $p \circ u$ is a presentable fibration.

 (2) The functor u carries $(p \circ u)$-Cartesian morphisms of $\mathcal{C}'$ to p-Cartesian morphisms of $\mathcal{C}$.

 (3) For every object $D \in \mathcal{D}$, the induced map $\mathcal{C}'_D \to \mathcal{C}_D$ is a stable envelope of $\mathcal{C}'_D$.

Remark 8.3.1.2. Let $\mathcal{C}$ be a presentable ∞-category, so that the projection $p : \mathcal{C} \to \Delta^0$ is a presentable fibration. It follows immediately from the definitions that a map $u : \mathcal{C}' \to \mathcal{C}$ is a stable envelope of $\mathcal{C}$ if and only if u is a stable envelope of p.

Let $p : \mathcal{C} \to \mathcal{D}$ be a presentable fibration, and let $u : \mathcal{C}' \to \mathcal{C}$ be a stable envelope of p. We will often abuse terminology by saying that $\mathcal{C}'$ is a a *stable envelope* of p, or that u *exhibits* $\mathcal{C}'$ *as a stable envelope of* p. In the case where $\mathcal{D} \simeq \Delta^0$, we will say instead that $\mathcal{C}'$ *is a stable envelope of* $\mathcal{C}$, or that u *exhibits* $\mathcal{C}'$ *as a stable envelope of* $\mathcal{C}$.

Remark 8.3.1.3. Suppose given a pullback diagram of simplicial sets

$$
\begin{array}{ccc}
\mathcal{C}_0 & \longrightarrow & \mathcal{C} \\
{\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p} \\
\mathcal{D}_0 & \longrightarrow & \mathcal{D}
\end{array}
$$

where p (and therefore also p_0) is a presentable fibration. If $u : \mathcal{C}' \to \mathcal{C}$ is a stable envelope of the presentable fibration p, then the induced map $\mathcal{C}' \times_{\mathcal{C}} \mathcal{C}_0 \to \mathcal{C}_0$ is a stable envelope of the presentable fibration p_0.

Example 8.3.1.4. Let $\mathcal{C}$ be a presentable ∞-category. Then the map $\Omega^{\infty}_{\mathcal{C}} : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ exhibits $\mathrm{Sp}(\mathcal{C})$ as a stable envelope of $\mathcal{C}$. This follows immediately from Corollary 1.4.4.5.

Example 8.3.1.5. Let $p : \mathcal{C} \to \mathcal{D}$ be a presentable fibration, and let $\mathrm{Stab}(p)$ be defines as in Construction 7.2.2.2. Using Proposition 7.2.2.13, we see that the map $\Omega^{\infty}_p : \mathrm{Stab}(p) \to \mathcal{C}$ exhibits $\mathrm{Stab}(p)$ as a stable envelope of p.

Remark 8.3.1.6. Let $\mathcal{C}$ be a presentable ∞-category. A stable envelope of $\mathcal{C}$ is determined uniquely up to equivalence by the universal property given in Definition 8.3.1.1, and is therefore equivalent to $\mathrm{Sp}(\mathcal{C})$. More precisely, suppose we are given a commutative diagram

$$
\begin{array}{ccc}
\mathcal{C}' & \overset{w}{\longrightarrow} & \mathcal{C}'' \\
& {\scriptstyle u}\searrow \quad \swarrow{\scriptstyle v} & \\
& \mathcal{C} &
\end{array}
$$

in which u and v are stable envelopes of $\mathcal{C}$. Then the functor w is an equivalence of ∞-categories (observe that in this situation, the functor w automatically admits a left adjoint by virtue of Proposition 1.4.4.4).

Our next goal is to establish a relative version of Remark 8.3.1.6. First, we need to introduce a bit of notation. Suppose we are given a diagram

$$
\begin{array}{ccc}
\mathcal{C} & & \mathcal{D} \\
{\scriptstyle p}\searrow & & \swarrow{\scriptstyle q} \\
& \mathcal{E} &
\end{array}
$$

of ∞-categories, where p and q are presentable fibrations. We let $\mathrm{Fun}^R_{\mathcal{E}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{E}}(\mathcal{C}, \mathcal{D})$ spanned by those functors $G : \mathcal{C} \to \mathcal{D}$ with the following properties:

(i) The functor G carries p-Cartesian edges of $\mathcal{C}$ to q-Cartesian edges of $\mathcal{D}$.

(ii) For each object $E \in \mathcal{E}$, the induced functor $G_E : \mathcal{C}_E \to \mathcal{D}_E$ admits a left adjoint.

We let $\mathrm{Fun}^R_{\mathcal{E}}(\mathcal{C}, \mathcal{D})^{\simeq}$ denote the largest Kan complex contained in $\mathrm{Fun}^R_{\mathcal{E}}(\mathcal{C}, \mathcal{D})$.

Proposition 8.3.1.7. *Let $p : \mathcal{C} \to \mathcal{D}$ be a presentable fibration of ∞-categories. Then there exists a functor $u : \mathcal{C}' \to \mathcal{C}$ with the following properties:*

(1) *The functor u is a stable envelope of the presentable fibration p.*

(2) *Let $q : \mathcal{E} \to \mathcal{D}$ be a presentable fibration, and assume that each fiber of q is a stable ∞-category. Then composition with u induces a trivial Kan fibration*

$$
\mathrm{Fun}^R_{\mathcal{D}}(\mathcal{E}, \mathcal{C}')^{\simeq} \to \mathrm{Fun}^R_{\mathcal{D}}(\mathcal{E}, \mathcal{C})^{\simeq}.
$$

(3) *Let $v : \mathcal{E} \to \mathcal{C}$ be any stable envelope of p. Then v factors as a composition*

$$\mathcal{E} \xrightarrow{\overline{v}} \mathcal{C}' \xrightarrow{u} \mathcal{C},$$

where $\overline{v}$ is an equivalence of ∞-categories.

Remark 8.3.1.8. Assertion (2) of Proposition 8.3.1.7 implies the stronger property that the map

$$\mathrm{Fun}^R_{\mathcal{D}}(\mathcal{E}, \mathcal{C}') \to \mathrm{Fun}^R_{\mathcal{D}}(\mathcal{E}, \mathcal{C})$$

is a trivial Kan fibration, but we will not need this fact.

Proof. Let $\mathrm{R}\mathcal{P}\mathrm{r}$ denote the ∞-category whose objects are presentable ∞-categories and whose morphisms are functors which admit left adjoints (see §T.5.5.3), and let $\mathrm{R}\mathcal{P}\mathrm{r}^{\mathrm{stab}}$ be the full subcategory of $\mathrm{R}\mathcal{P}\mathrm{r}$ spanned by those presentable ∞-categories which are stable. It follows from Corollary 1.4.4.5 that the inclusion $\mathrm{R}\mathcal{P}\mathrm{r}^{\mathrm{stab}} \subseteq \mathrm{R}\mathcal{P}\mathrm{r}$ admits a right adjoint, given by the construction $\mathcal{X} \mapsto \mathrm{Sp}(\mathcal{X})$. Let us denote this right adjoint by G.

The presentable fibration p is classified by a functor $\chi : \mathcal{D}^{op} \to \mathrm{R}\mathcal{P}\mathrm{r}$. Let α denote the counit transformation $G \circ \chi \to \chi$. Then α is classified by a map $u : \mathcal{C}' \to \mathcal{C}$ of presentable fibrations over $\mathcal{D}$. Making a fibrant replacement if necessary, we may suppose that u is a categorical fibration. Assertion (1) now follows immediately from the construction.

To prove (2), let us suppose that the presentable fibration q is classified by a functor $\chi' : \mathcal{D}^{op} \to \mathrm{R}\mathcal{P}\mathrm{r}$. Using Theorem T.3.2.0.1 and Proposition T.4.2.4.4, we deduce the existence of a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Map}_{\mathrm{Fun}(\mathcal{D}^{op},\mathrm{R}\mathcal{P}\mathrm{r})}(\chi', G \circ \chi) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}(\mathcal{D}^{op},\mathrm{R}\mathcal{P}\mathrm{r})}(\chi',\chi) \\
\downarrow & & \downarrow \\
\mathrm{Fun}^{R,\sim}_{\mathcal{D}}(\mathcal{E}, \mathcal{C}') & \longrightarrow & \mathrm{Fun}^{R,\sim}_{\mathcal{D}}(\mathcal{E}, \mathcal{C})
\end{array}
$$

in the homotopy category of spaces, where the vertical arrows are homotopy equivalences. Since the fibers of q are stable, χ' factors through $\mathrm{R}\mathcal{P}\mathrm{r}^{\mathrm{stab}} \subseteq \mathrm{R}\mathcal{P}\mathrm{r}$, so the upper horizontal arrow is a homotopy equivalence. It follows that the lower horizontal arrow is a homotopy equivalence as well. Since u is a categorical fibration, the lower horizontal arrow is also a Kan fibration, and therefore a trivial Kan fibration.

We now prove assertion (3). The existence of $\overline{v}$ (and its uniqueness up to homotopy) follows immediately from (2). To prove that $\overline{v}$ is an equivalence, we first invoke Corollary T.2.4.4.4 to reduce to the case where $\mathcal{D}$ consists of a single vertex. In this case, the result follows from Remark 8.3.1.6. □

Definition 8.3.1.9. Let $\mathcal{C}$ be a presentable ∞-category. A *tangent bundle to* $\mathcal{C}$ is a functor $T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ which exhibits $T_{\mathcal{C}}$ as the stable envelope of the presentable fibration $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{1\}, \mathcal{C}) \simeq \mathcal{C}$.

In the situation of Definition 8.3.1.9, we will often abuse terminology by referring to $T_{\mathcal{C}}$ as the *tangent bundle* to $\mathcal{C}$. We note that $T_{\mathcal{C}}$ is determined up equivalence by $\mathcal{C}$. Roughly speaking, we may think of an object of $T_{\mathcal{C}}$ as a pair (A, M), where A is an object of $\mathcal{C}$ and M is a spectrum object of $\mathcal{C}_{/A}$. In the case where $\mathcal{C}$ is the ∞-category of $\mathbb{E}_{\infty}$-rings, we can identify M with an A-module (Corollary 8.3.4.14). In this case, the functor $T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ associates to (A, M) the projection morphism $A \oplus M \to A$. Our terminology is justified as follows: we think of this morphism as a "tangent vector" in the ∞-category $\mathcal{C}$, relating the object A to the "infinitesimally near" object $A \oplus M$ (this analogy with differential geometry is based on suggestions of Tom Goodwillie).

For some purposes, it is convenient to have an explicit construction for the tangent bundle $T_{\mathcal{C}}$ of a presentable ∞-category $\mathcal{C}$. The following is a variation on Example 8.3.1.5:

Proposition 8.3.1.10. *Let $\mathcal{C}$ be a presentable ∞-category, and let $e : \mathrm{Exc}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ be the functor which carries an excisive functor $X : \mathcal{S}^{\mathrm{fin}}_* \to \mathcal{C}$ to the map $X(S^0) \to X(*)$. Then e exhibits $T_{\mathcal{C}}$ as a tangent bundle to $\mathcal{C}$.*

Proof. Let $\mathcal{E}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1 \times \mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$ spanned by those functors $F : \Delta^1 \times \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}$ with the following properties:

(i) The restriction $F|(\{0\} \times \mathcal{S}_*^{\mathrm{fin}})$ belongs to $T_{\mathcal{C}}$.

(ii) The canonical map $F(0, *) \to F(1, *)$ is an equivalence in $\mathcal{C}$.

(iii) For every pointed finite space X, the canonical map $F(1, X) \to F(1, *)$ is an equivalence in $\mathcal{C}$.

If we let $\mathcal{S}_*^{\mathrm{fin}}{}'$ denote the full subcategory of $\Delta^1 \times \mathcal{S}_*^{\mathrm{fin}}$ spanned by $\{0\} \times \mathcal{S}_*^{\mathrm{fin}}$ together with the object $(1, *)$, then (iii) is equivalent to the assertion that F is a right Kan extension of $F|\mathcal{S}_*^{\mathrm{fin}}{}'$, and ($ii$) is equivalent to the assertion that $F|\mathcal{S}_*^{\mathrm{fin}}{}'$ is a left Kan extension of $F|(\{0\} \times \mathcal{S}_*^{\mathrm{fin}})$. It follows from Proposition T.4.3.2.15 that restriction along the inclusion $\{0\} \times \mathcal{S}_*^{\mathrm{fin}} \hookrightarrow \Delta^1 \times \mathcal{S}_*^{\mathrm{fin}}$ induces a trivial Kan fibration $\phi : \mathcal{E} \to T_{\mathcal{C}}$.

Let $c : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}_*^{\mathrm{fin}}$ denote the constant functor taking the value $* \in \mathcal{S}_*^{\mathrm{fin}}$. There is a unique natural transformation $\mathrm{id} \to c$, classified by a map $h : \Delta^1 \times \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}_*^{\mathrm{fin}}$. Composition with h determines a map $T_{\mathcal{C}} \to \mathcal{E}$ which is a section of the trivial Kan fibration ϕ. The map p factors as a composition

$$T_{\mathcal{C}} \to \mathcal{E} \xrightarrow{p'} \mathrm{Fun}(\Delta^1, \mathcal{C}),$$

where p' is given by evaluation at the object $S^0 \in \mathcal{S}_*^{\mathrm{fin}}$. It will therefore suffice to show that p' exhibits $\mathcal{E}$ as a tangent bundle to $\mathcal{C}$.

Let $\mathcal{E}' = \mathcal{E} \times_{\mathrm{Fun}(\{1\} \times \mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})} \mathcal{C}$. Using Proposition T.4.3.2.15, we deduce that the inclusion $\mathcal{E}' \hookrightarrow \mathcal{E}$ is an equivalence of ∞-categories. Example 8.3.1.5 implies that the composite map $\mathcal{E}' \to \mathcal{E} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ exhibits $\mathcal{E}'$ as a stable envelope of $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$, from which it follows immediately that p' exhibits $\mathcal{E}$ as a tangent bundle to $\mathcal{C}$. $\square$

Corollary 8.3.1.11. *Let $\mathcal{C}$ be a presentable ∞-category. Then the tangent bundle $T_{\mathcal{C}}$ is also a presentable ∞-category.*

Proof. It follows from Proposition 8.3.1.10 (and Lemmas T.5.5.4.19, T.5.5.4.17, and T.5.5.4.18) that $T_{\mathcal{C}}$ can be realized as an accessible localization of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$. $\square$

It follows from Corollary 8.3.1.11 that if $\mathcal{C}$ is a presentable ∞-category, then the tangent bundle $T_{\mathcal{C}}$ admits small limits and colimits. The following result describes these limits and colimits in more detail:

Proposition 8.3.1.12. *Let $\mathcal{C}$ be a presentable ∞-category, let $T_{\mathcal{C}}$ be a tangent bundle to $\mathcal{C}$, and let p denote the composition*

$$T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{1\}, \mathcal{C}) \simeq \mathcal{C}.$$

Then:

(1) *A small diagram $\overline{q} : K^{\triangleright} \to T_{\mathcal{C}}$ is a colimit diagram if and only if $\overline{q}$ is a p-colimit diagram and $p \circ \overline{q}$ is a colimit diagram in $\mathcal{C}$.*

(2) *A small diagram $\overline{q} : K^{\triangleleft} \to T_{\mathcal{C}}$ is a limit diagram if and only if $\overline{q}$ is a p-limit diagram and $p \circ \overline{q}$ is a limit diagram in $\mathcal{C}$.*

Proof. We will prove (1); assertion (2) will follow from the same argument. The "if" direction follows from Proposition T.4.3.1.5. The converse then follows from the uniqueness of colimit diagrams and the following assertion:

($*$) Let K be a small simplicial set, and let $q : K \to T_{\mathcal{C}}$ be a diagram. Then q admits an extension $\overline{q} : K^{\triangleright} \to T_{\mathcal{C}}$ such that $\overline{q}$ is a p-colimit diagram, and $p \circ \overline{q}$ is a colimit diagram in $\mathcal{C}$.

To prove ($*$), we first invoke the assumption that $\mathcal{C}$ is presentable to deduce the existence of a colimit diagram $\overline{q}_0 : K^{\triangleright} \to \mathcal{C}$ extending $p \circ q$. It then suffices to show that we can lift $\overline{q}_0$ to a p-colimit diagram in $T_{\mathcal{C}}$; this follows from that fact that p is a presentable fibration. $\square$

We now study the functorial properties of the construction of tangent bundles.

Lemma 8.3.1.13. *Let $\mathcal{C}$ be a presentable ∞-category, let $T_{\mathcal{C}} = \mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$ the tangent bundle of $\mathcal{C}$ given in Proposition 8.3.1.10, and let $L : \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \to T_{\mathcal{C}}$ be a left adjoint to the inclusion. A morphism $\alpha : X \to Y$ in $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$ is an L-equivalence if and only if the following conditions are satisfied:*

(1) *The map $X(*) \to Y(*)$ is an equivalence in $\mathcal{C}$.*

(2) *Let $X', Y' : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}_{X(*)//Y(*)}$ be the functors determined by X and Y, and choose left Kan extensions $\overline{X}, \overline{Y} : \mathcal{S}_* \to \mathcal{C}_{X(*)//Y(*)}$ of X' and Y', respectively. Then the canonical map $\overline{X} \to \overline{Y}$ induces an equivalence $\partial \overline{X} \to \partial \overline{Y}$ in $\mathrm{Sp}(\mathcal{C}_{X(*)//Y(*)})$ (see Definition 7.2.1.1).*

Proof. Assume first that α is an L-equivalence. For each $K \in \mathcal{C}$, let $Z_K \in \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$ be the constant functor taking the value K. Note that Z_K is a right Kan extension of its restriction to $\{*\} \subseteq \mathcal{S}_*^{\mathrm{fin}}$. Since $Z_K \in T_{\mathcal{C}}$, composition with α induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}}(Y(*), K) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})}(Y, Z_K) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})}(X, Z_K) \simeq \mathrm{Map}_{\mathcal{C}}(X(*), K).$$

Since K is arbitrary, condition (1) follows. To prove (2), let $\mathcal{C}'$ denote the pointed ∞-category $\mathcal{C}_{X(*)//Y(*)}$ and let $\overline{Z} : \mathcal{S}_* \to \mathcal{C}'$ be an arbitrary strongly excisive functor which commutes with sequential colimits. We wish to prove that the canonical map

$$\theta : \mathrm{Map}_{\mathrm{Fun}(\mathcal{S}_*, \mathcal{C}')}(\overline{Y}, \overline{Z}) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{S}_*, \mathcal{C}')}(\overline{X}, \overline{Z})$$

is a homotopy equivalence. Let $Z' = \overline{Z}|\,\mathcal{S}_*^{\mathrm{fin}}$. Since $\overline{X}$ and $\overline{Y}$ are left Kan extensions of X' and Y', respectively, we can identify θ with the map

$$\theta_0 : \mathrm{Map}_{\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}')}(Y', Z') \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}')}(X', Z').$$

The map Z' induces a functor $Z : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}$. To prove that θ_0 is an equivalence, it suffices to show that the underlying map $\mathrm{Map}_{\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})}(Y, Z) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})}(X, Z)$ is a homotopy equivalence. This follows from our assumption that α is an L-equivalence and the observation that $Z \in T_{\mathcal{C}}$.

We now prove the converse. Assume that conditions (1) and (2) are satisfied; we wish to show that α is an L-equivalence. Fix an object $Z \in T_{\mathcal{C}}$. We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Map}_{\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})}(Y, Z) & \xrightarrow{\ \phi\ } & \mathrm{Map}_{\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})}(X, Z) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathcal{C}}(Y(*), Z(*)) & \xrightarrow{\ \phi_0\ } & \mathrm{Map}_{\mathcal{C}}(X(*), Z(*))
\end{array}
$$

and we wish to show that ϕ is a homotopy equivalence. Condition (1) implies that ϕ_0 is a homotopy equivalence. It will therefore suffice to show that ϕ induces a homotopy equivalence after passing to the homotopy fibers over any point $\eta \in \mathrm{Map}_{\mathcal{C}}(Y(*), Z(*))$. Let $K = Z(*)$ and let $\mathcal{C}'$ be defined as above, so that Z determines a functor $Z' : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}'$ given on objects by $Z'(U) = Z(U) \times_{Z(*)} Y(*)$. We observe that the induced map of homotopy fibers is given by $\phi' : \mathrm{Map}_{\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}')}(Y', Z') \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}')}(X', Z')$. Let $\overline{Z} : \mathcal{S}_* \to \mathcal{C}'$ be a left Kan extension of Z', so that ϕ' is equivalent to the map

$$\overline{\phi} : \mathrm{Map}_{\mathrm{Fun}(\mathcal{S}_*, \mathcal{C}')}(\overline{Y}, \overline{Z}) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{S}_*, \mathcal{C}')}(\overline{X}, \overline{Z}).$$

Since $Z \in T_{\mathcal{C}}$, the functor Z' is strongly excisive, so that $\overline{Z}$ is also strongly excisive. It follows from assumption (2) that $\overline{\phi}$ is a homotopy equivalence, as desired. $\qquad\square$

Proposition 8.3.1.14. *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories which preserves filtered colimits. Let $T_{\mathcal{C}} \subseteq \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$ and $T_{\mathcal{D}} = \mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{D})$ be the tangent bundle of Proposition 8.3.1.10, and let $L : \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \to T_{\mathcal{C}}$ and $L' : \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{D}) \to T_{\mathcal{D}}$ denote left adjoints to the inclusion functor. Let $F : \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{D})$ be given by composition with f. Then F carries L-equivalents to L'-equivalences (and therefore induces a functor $T_{\mathcal{C}} \to T_{\mathcal{D}}$).*

Proof. Combine Corollary 7.2.1.24 with the criterion of Lemma 8.3.1.13. $\qquad\square$

Example 8.3.1.15. Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads. Assume that:

(*i*) For every object $X \in \mathcal{O}$, the ∞-category $\mathcal{C}_X$ is presentable.

(*ii*) For every operation $\phi \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y)$, the associated functor $\phi_! : \prod_{i \in I} \mathcal{C}_{X_i} \to \mathcal{C}_Y$ preserves sequential colimits.

The projection map $p' : \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}^{\otimes}) \times_{\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{O}^{\otimes})} \mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes}$ is again a coCartesian fibration of ∞-operads. Let $T_{\mathcal{C}}$ denote the full subcategory of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \times_{\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{O})} \mathcal{O}$ spanned by those objects which correspond to functors $F : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}_X$ which carry pushout squares to pullback squares (for some $X \in \mathcal{O}$), and let $T_{\mathcal{C}}^{\otimes}$ denote the corresponding full subcategory of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}^{\otimes}) \times_{\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{O}^{\otimes})} \mathcal{O}^{\otimes}$ (see Definition 2.2.1). Using Propositions 8.3.1.14 and 2.2.1.9, we conclude that the forgetful functor $T_{\mathcal{C}}^{\otimes} \to \mathcal{O}^{\otimes}$ is again a coCartesian fibration of ∞-operads. Taking $\mathcal{O}^{\otimes} = \mathbb{E}_k^{\otimes}$ for $1 \le k \le \infty$, we obtain the following result:

(∗) Let $\mathcal{C}$ be a presentable ∞-category equipped with an $\mathbb{E}_k$-monoidal structure, such that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves sequential colimits. Then the tangent bundle $T_{\mathcal{C}}$ inherits an $\mathbb{E}_k$-monoidal structure.

8.3.2 Relative Adjunctions

Let $\mathcal{C}$ be a presentable ∞-category and let $T_{\mathcal{C}}$ denote the tangent bundle of $\mathcal{C}$ (see Definition 8.3.1.9). Our goal in this section is to produce a left adjoint to the composite functor

$$T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{0\}, \mathcal{C}) \simeq \mathcal{C}.$$

The existence of the desired left adjoint can be deduced easily from the adjoint functor theorem (Corollary T.5.5.2.9). However, we will later need more detailed information about L. To obtain this information, it is convenient to formulate a *relative* version of the theory of adjoint functors.

Proposition 8.3.2.1. *Suppose we are given a commutative diagram*

$$\begin{array}{ccc} \mathcal{C} & \xleftarrow{\quad G \quad} & \mathcal{D} \\ & {}_{q}\searrow \quad \swarrow_{p} & \\ & \mathcal{E} & \end{array}$$
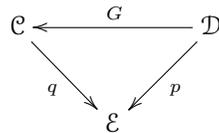
of ∞-categories, where the maps p and q are categorical fibrations. The following conditions are equivalent:

(1) *The functor G admits a left adjoint F. Moreover, for every object $C \in \mathcal{C}$, the functor q carries the unit map $u_C : C \to GFC$ to an equivalence in $\mathcal{E}$.*

(2) *There exists a functor $F : \mathcal{C} \to \mathcal{D}$ and a natural transformation $u : \mathrm{id}_{\mathcal{C}} \to G \circ F$ which exhibits F as a left adjoint to G, and has the property that $q(u)$ is the identity transformation from q to itself (in particular, $q \circ F = p$).*

Proof. We first construct a correspondence associated to the functor G. Let $X = \mathcal{C} \coprod_{\mathcal{D} \times \{0\}} (\mathcal{D} \times \Delta^1)$. Using the small object argument, we can construct a factorization

$$X \xrightarrow{i} \mathcal{M} \xrightarrow{r} \mathcal{E} \times \Delta^1$$

where i is inner anodyne and r is an inner fibration. Moreover, we may assume that the maps

$$\mathcal{C} \to \mathcal{M} \times_{\Delta^1} \{0\}$$

$$\mathcal{D} \to \mathcal{M} \times_{\Delta^1} \{1\}$$

are isomorphisms of simplicial sets. We will henceforth identify $\mathcal{C}$ and $\mathcal{D}$ with full subcategories of $\mathcal{M}$ via these isomorphisms.

The functor G admits a left adjoint F if and only if the projection $r' : \mathcal{M} \to \Delta^1$ is a coCartesian fibration. We will show that conditions (1) and (2) Proposition 8.3.2.1 are equivalent to the existence of F, together with the following additional requirement:

(3) The projection $r'' : \mathcal{M} \to \mathcal{E}$ carries r'-coCartesian morphisms to equivalences in $\mathcal{E}$.

We will assume for the remainder of the proof that r' is a coCartesian fibration. The implications $(2) \Rightarrow (1)$ is obvious. We next prove that $(1) \Rightarrow (3)$. Let $\alpha : C \to D$ be an r'-coCartesian morphism in $\mathcal{M}$; we wish to prove that $r''(\alpha)$ is an equivalence in $\mathcal{E}$. We may assume that $C \in \mathcal{C}$ and $D \in \mathcal{D}$ (otherwise, α is itself an equivalence and the result is obvious). The map α fits into a commutative diagram

$$
\begin{array}{ccc}
 & G(D) & \\
 \beta \nearrow & & \searrow \gamma \\
C & \xrightarrow{\quad \alpha \quad} & D
\end{array}
$$

where $r''(\gamma)$ is degenerate. Consequently, to prove that $r''(\alpha)$ is an equivalence, it suffices to show that $r''(\beta) = q(\beta)$ is an equivalence. This follows from (1), since β can be identified with the unit map u_C.

We now complete the proof by showing that $(3) \Rightarrow (2)$. Assume that (3) is satisfied. We will construct a commutative diagram

$$
\begin{array}{ccc}
\mathcal{C} \times \{0\} & \hookrightarrow & \mathcal{M} \\
\downarrow & {\overset{h}{\nearrow}} & \downarrow r \\
\mathcal{C} \times \Delta^1 & \longrightarrow & \mathcal{E} \times \Delta^1.
\end{array}
$$

with the following property: for every object $C \in \mathcal{C}$, the functor h carries $\{C\} \times \Delta^1$ to an r'-coCartesian morphism of $\mathcal{M}$. To construct β, we work simplex-by-simplex on $\mathcal{C}$. Let us first consider the case of zero-dimensional simplices. Fix an object $C \in \mathcal{C}$ and choose an r'-coCartesian morphism $\alpha : C \to D$ in $\mathcal{M}$. Assumption (3) guarantees that $r''(\alpha)$ is an equivalence in $\mathcal{E}$, so there exists a commutative diagram σ :

$$
\begin{array}{ccc}
 & r''(D) & \\
 r''(\alpha) \nearrow & & \searrow \beta \\
r''(C) & \xrightarrow{\quad \text{id} \quad} & r''(C)
\end{array}
$$

in the ∞-category $\mathcal{E}$. Since p is a categorical fibration, we can lift β to an equivalence $\overline{\beta} : D \to D'$ in $\mathcal{D}$. Since r is an inner fibration, we can further lift σ to a 2-simplex

$$
\begin{array}{ccc}
 & D & \\
 \alpha \nearrow & & \searrow \overline{\beta} \\
C & \xrightarrow{\quad \gamma \quad} & D'.
\end{array}
$$

in $\mathcal{M}$. Since $\overline{\beta}$ is an equivalence, the map γ is equivalent to α and therefore r'-coCartesian; we can therefore define the restriction $h|(\{C\} \times \Delta^1)$ to coincide with γ.

To handle simplices of larger dimension, we need to solve mapping problems of the form

$$
\begin{array}{ccc}
(\Delta^n \times \{0\}) \coprod_{\partial \Delta^n \times \{0\}} (\partial \Delta^n \times \Delta^1) & \xrightarrow{\ j\ } & \mathcal{M} \\
\downarrow & \nearrow & \downarrow r \\
\Delta^n \times \Delta^1 & \longrightarrow & \mathcal{E} \times \Delta^1,
\end{array}
$$

where $n > 0$ and the map j carries $\{0\} \times \Delta^1$ to an r'-coCartesian morphism in $\mathcal{M}$. Note that condition (3) guarantees that any r'-coCartesian morphism in $\mathcal{M}$ is also r-coCartesian (Proposition T.2.4.1.3). The existence of the required extension now follows from Proposition T.2.4.1.8.

We now define $F : \mathcal{C} \to \mathcal{D}$ to be the restriction of h to $\mathcal{C} \times \{1\}$. Together with the evident inclusion $\mathcal{D} \times \Delta^1 \hookrightarrow \mathcal{M}$, the map h determines a commutative diagram

$$
\begin{array}{ccc}
& G \circ F & \\
{\scriptstyle u}\nearrow & & \searrow{\scriptstyle h'} \\
\mathrm{id}_{\mathcal{C}} & \xrightarrow{\ \ h\ \ } & F
\end{array}
$$

in the ∞-category $\mathrm{Fun}_{\mathcal{E}}(\mathcal{C}, \mathcal{M})$. The natural transformation h' is evidently r'-coCartesian; using Proposition T.2.4.1.3 we deduce that h' is r-coCartesian so that there exists a dotted arrow u as indicated in the diagram, thereby proving that condition (2) is satisfied. $\square$

Definition 8.3.2.2. Suppose we are given a commutative diagram of ∞-categories

$$
\begin{array}{ccc}
\mathcal{C} & \xleftarrow{\ \ G\ \ } & \mathcal{D} \\
{\scriptstyle q}\searrow & & \swarrow{\scriptstyle p} \\
& \mathcal{E}, &
\end{array}
$$

where p and q are categorical fibrations. We will say that G *admits a left adjoint relative to* $\mathcal{E}$ if the equivalent conditions of Proposition 8.3.2.1 are satisfied.

Remark 8.3.2.3. In the situation of Proposition 8.3.2.1, if F and $u : \mathrm{id}_{\mathcal{C}} \to G \circ F$ are as in condition (2), then we will say that u is the *unit for an adjunction between F and G relative to* $\mathcal{E}$, or that u *exhibits F as a left adjoint of G relative to* $\mathcal{E}$.

Remark 8.3.2.4. Given a commutative diagram

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ \ F\ \ } & \mathcal{D} \\
{\scriptstyle q}\searrow & & \swarrow{\scriptstyle p} \\
& \mathcal{E}, &
\end{array}
$$

we can define an evident dual condition that F *admit a right adjoint G relative to* $\mathcal{E}$. In this case, the functor G a left adjoint relative to $\mathcal{E}$. Indeed, we claim that for every object $C \in \mathcal{C}$, the functor q carries the unit map $u : C \to (G \circ F)(C)$ to an equivalence in $\mathcal{E}$. To prove this, it suffices to show that p carries $F(u)$ to an equivalence in $\mathcal{E}$. But the map $F(u)$ fits into a commutative diagram

$$
\begin{array}{ccc}
& (F \circ G \circ F)(C) & \\
{\scriptstyle F(u)}\nearrow & & \searrow{\scriptstyle v} \\
F(C) & \xrightarrow{\ \mathrm{id}_{F(C)}\ } & F(C).
\end{array}
$$

By a two-out-of-three argument, it suffices to show that $p(\mathrm{id}_{F(C)})$ is an equivalence in $\mathcal{E}$ (which is obvious) and that $p(v)$ is an equivalence in $\mathcal{E}$ (which follows from our assumption that F admits a right adjoint relative to G).

Proposition 8.3.2.5. *Suppose we are given a commutative diagram*

$$\begin{array}{ccc} \mathcal{C} & \xleftarrow{\quad G \quad} & \mathcal{D} \\ & {}_{q}\searrow \quad \swarrow_{p} & \\ & \mathcal{E} & \end{array}$$
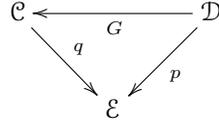
of ∞-categories, where the maps p and q are categorical fibrations. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor with $pF = q$ and $u : \mathrm{id}_{\mathcal{C}} \to G \circ F$ a natural transformation which exhibits F as a left adjoint of G relative to $\mathcal{E}$. Then, for every functor $\mathcal{E}' \to \mathcal{E}$, if we let $F' : \mathcal{C} \times_{\mathcal{E}} \mathcal{E}' \to \mathcal{D} \times_{\mathcal{E}} \mathcal{E}'$ and $G' : \mathcal{D} \times_{\mathcal{E}} \mathcal{E}' \to \mathcal{C} \times_{\mathcal{E}} \mathcal{E}'$ denote the induced functors and $u' : \mathrm{id} \to G' \circ F'$ the induced natural transformation, then u' exhibits F' as a left adjoint to G' relative to $\mathcal{E}'$. In particular, for every object $E \in \mathcal{E}$, the induced natural transformation $u_E : \mathrm{id}_{\mathcal{C}_E} \to G_E \circ F_E$ is the unit of an adjunction between the ∞-categories $\mathcal{C}_E$ and $\mathcal{D}_E$.

Proof. Fix objects $C' \in \mathcal{C} \times_{\mathcal{E}} \mathcal{E}'$ and $D' \in \mathcal{D} \times_{\mathcal{E}} \mathcal{E}'$ having images $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Let E'_0 and E'_1 denote the images of C' and D' in $\mathcal{E}'$, and let E_0 and E_1 denote their images in $\mathcal{E}$. We wish to prove that the composite map

$$\mathrm{Map}_{\mathcal{D} \times_{\mathcal{E}} \mathcal{E}'}(F'(C'), D') \to \mathrm{Map}_{\mathcal{C} \times_{\mathcal{E}} \mathcal{E}'}((G' \circ F')(C'), G'(D')) \xrightarrow{u'} \mathrm{Map}_{\mathcal{C} \times_{\mathcal{E}} \mathcal{E}'}(C', G'(D'))$$

is a homotopy equivalence. This map fits into a homotopy coherent diagram

$$\begin{array}{ccccc} \mathrm{Map}_{\mathcal{D} \times_{\mathcal{E}} \mathcal{E}'}(F'(C'), D') & \longrightarrow & \mathrm{Map}_{\mathcal{C} \times_{\mathcal{E}} \mathcal{E}'}(C', G'(D')) & \longrightarrow & \mathrm{Map}_{\mathcal{E}'}(E'_0, E'_1) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{D}}(F(C), D) & \xrightarrow{\quad \phi \quad} & \mathrm{Map}_{\mathcal{C}}(C, G(D)) & \longrightarrow & \mathrm{Map}_{\mathcal{E}}(E_0, E_1). \end{array}$$

The right square and the outer rectangle are homotopy pullback diagrams, so that the left square is also a homotopy pullback diagram. It therefore suffices to show that the map ϕ is a homotopy equivalence, which follows from our assumption that u is the unit of an adjunction between F and G. $\qquad\square$

We now establish some useful criteria for establishing the existence of relative adjoints.

Proposition 8.3.2.6. *Suppose given a commutative diagram*

$$\begin{array}{ccc} \mathcal{C} & \xleftarrow{\quad G \quad} & \mathcal{D} \\ & {}_{q}\searrow \quad \swarrow_{p} & \\ & \mathcal{E} & \end{array}$$
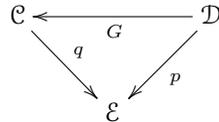
of ∞-categories, where p and q are locally Cartesian categorical fibrations. Then G admits a left adjoint relative to $\mathcal{E}$ if and only if the following conditions are satisfied:

(1) *For every object $E \in \mathcal{E}$, the induced map of fibers $G_E : \mathcal{D}_E \to \mathcal{C}_E$ admits a left adjoint.*

(2) *The functor G carries locally p-Cartesian morphisms in $\mathcal{D}$ to locally q-Cartesian morphisms in $\mathcal{C}$.*

Proof. Suppose first that $u : \mathrm{id}_{\mathcal{C}} \to G \circ F$ exhibits $F : \mathcal{C} \to \mathcal{D}$ as a left adjoint to G relative to $\mathcal{E}$. Proposition 8.3.2.5 implies that condition (1) is satisfied. To prove (2), let $\alpha : D \to D'$ be a locally p-Cartesian morphism in $\mathcal{D}$; we wish to prove that $G(\alpha)$ is locally q-Cartesian. The map $p(\alpha)$ determines a 1-simplex $\Delta^1 \to \mathcal{E}$. Replacing $\mathcal{C}$ and $\mathcal{D}$ by their pullbacks $\mathcal{C} \times_{\mathcal{E}} \Delta^1$ and $\mathcal{D} \times_{\mathcal{E}} \Delta^1$ (and invoking Proposition 8.3.2.5 once more),

we can reduce to the case where $\mathcal{E} = \Delta^1$. Let $C \in \mathcal{C} \times_{\Delta^1} \{0\}$; we wish to prove that composition with $G(\alpha)$ induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}}(C, G(D)) \to \mathrm{Map}_{\mathcal{C}}(C, G(D')).$$

This is equivalent to the requirement that composition with α induce a homotopy equivalence

$$\mathrm{Map}_{\mathcal{D}}(F(C), D) \to \mathrm{Map}_{\mathcal{D}}(F(C), D'),$$

which follows from the observation that $F(C) \in \mathcal{D} \times_{\Delta^1} \{0\}$ (since α is assumed to be locally p-coCartesian).

Now suppose that (1) and (2) are satisfied. We will prove that G satisfies the first criterion of Proposition 8.3.2.1. In other words, we must show that for each $C \in \mathcal{C}$, there exists an object $D \in \mathcal{D}$ and a map $u : C \to G(D)$ satisfying the following pair of conditions:

(i) For every object $D' \in \mathcal{D}$, composition with u induces an equivalence

$$\mathrm{Map}_{\mathcal{D}}(D, D') \to \mathrm{Map}_{\mathcal{C}}(C, G(D')).$$

(ii) The morphism $q(u)$ is an equivalence in $\mathcal{E}$.

To construct u, we let E denote the image of the object C in the ∞-category $\mathcal{E}$. Assumption (1) implies that $G_E : \mathcal{D}_E \to \mathcal{C}_E$ admits a left adjoint F_E. In particular, there exists an object $D = F_E(C) \in \mathcal{D}_E$ and a morphism $u : C \to G(D)$ in $\mathcal{C}_E$ which satisfies the following modified version of condition (i):

(i') For every object $D' \in \mathcal{D}_E$, composition with u induces an equivalence

$$\mathrm{Map}_{\mathcal{D}_E}(D, D') \to \mathrm{Map}_{\mathcal{C}_E}(C, G(D')).$$

It is obvious that u satisfies condition (ii). We will prove that condition (i) is satisfied. Let $D' \in \mathcal{D}$ be arbitrary: we wish to prove that the map $\phi : \mathrm{Map}_{\mathcal{D}}(D, D') \to \mathrm{Map}_{\mathcal{C}}(C, G(D'))$ is a homotopy equivalence. Let E' denote the image of D' in $\mathcal{E}$. It will suffice to show that ϕ induces a homotopy equivalence after passing to the homotopy fiber over any point $\alpha \in \mathrm{Map}_{\mathcal{E}}(E, E')$. Choose a locally p-coCartesian morphism $D'' \to D'$ in $\mathcal{D}$ lying over α. Condition (2) guarantees that the induced map $G(D'') \to G(D')$ is locally q-Cartesian. Using Proposition T.2.4.4.2, we can identify the map of homotopy fibers ϕ_α with the map $\mathrm{Map}_{\mathcal{D}_E}(D, D'') \to \mathrm{Map}_{\mathcal{C}_E}(C, G(D''))$, which is a homotopy equivalence by virtue of (i'). $\square$

Corollary 8.3.2.7. *Suppose we are given a commutative diagram*

of ∞-operads, where p and q are coCartesian fibrations. Assume that, for every object $X \in \mathcal{O}$, the induced map of fibers $F_X : \mathcal{C}_X \to \mathcal{D}_X$ admits a right adjoint G_X. Then F admits a right adjoint G relative to $\mathcal{O}^{\otimes}$. Moreover, G is a map of ∞-operads.

Proof. Let $X \in \mathcal{O}_{\langle n \rangle}^{\otimes}$, and choose inert morphisms $X \to X_i$ covering the maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \leq i \leq n$. Then the induced map F_X is homotopic to the composition

$$\mathcal{C}_X \simeq \prod_{1 \leq i \leq n} \mathcal{C}_{X_i} \overset{\prod F_{X_i}}{\to} \prod_{1 \leq i \leq n} \mathcal{D}_{X_i} \simeq \mathcal{D}_X.$$

It follows that F_X admits a right adjoint G_X, given by the product of the right adjoints G_{X_i} to the functors F_{X_i}. Applying the dual version of Proposition 8.3.2.6, we deduce that F admits a right adjoint G relative to $\mathcal{O}^{\otimes}$. The description of G_X given above shows that G carries inert morphisms of $\mathcal{D}^{\otimes}$ to inert morphisms of $\mathcal{C}^{\otimes}$. $\square$

Example 8.3.2.8. Let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category, let $\mathcal{M}$ and $\mathcal{N}$ be ∞-categories left tensored over $\mathcal{C}$, let $F \in \mathrm{LinFun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$, and let $f : \mathcal{M} \to \mathcal{N}$ be the functor underlying F. The action of $\mathcal{C}$ on $\mathcal{M}$ and $\mathcal{N}$ can be encoded by coCartesian fibrations of ∞-operads $\mathcal{M}^{\otimes} \to \mathcal{LM}^{\otimes} \leftarrow \mathcal{N}^{\otimes}$, and F determines an $\mathcal{LM}$-monoidal functor $F^{\otimes} : \mathcal{M}^{\otimes} \to \mathcal{N}^{\otimes}$. Suppose that f admits a right adjoint g. Corollary 8.3.2.7 implies that $F^{\otimes}$ admits a right adjoint $G^{\otimes}$ relative to $\mathcal{LM}^{\otimes}$, and that $G^{\otimes}$ is a map of ∞-operads. It follows that $F^{\otimes}$ and $G^{\otimes}$ induce adjoint functors

$$\mathrm{LMod}(\mathcal{M}) \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathrm{LMod}(\mathcal{N}).$$

such that the diagram

$$
\begin{array}{ccc}
\mathrm{LMod}(\mathcal{M}) & \xleftarrow{\ \psi\ } & \mathrm{LMod}(\mathcal{N}) \\
\downarrow & & \downarrow \\
\mathcal{M} & \xleftarrow{\ g\ } & \mathcal{N}
\end{array}
$$

commutes up to (canonical) homotopy. The adjunction between ϕ and ψ is relative to the ∞-category $\mathrm{Alg}(\mathcal{C})$. In particular, for every algebra object $A \in \mathrm{Alg}(\mathcal{C})$ we obtain adjoint functors

$$\mathrm{LMod}_A(\mathcal{M}) \underset{\psi_A}{\overset{\phi_A}{\rightleftarrows}} \mathrm{LMod}_A(\mathcal{N}).$$

Remark 8.3.2.9. In the situation of Example 8.3.2.8, suppose that $C \in \mathcal{C}$ and $N \in \mathcal{N}$. The counit map $F(G(N)) \to N$ induces a map

$$F(C \otimes G(N)) \simeq C \otimes F(G(N)) \to C \otimes N$$

which is adjoint to a map $C \otimes G(N) \to G(C \otimes N)$. If this map is an equivalence for every pair $(C, N) \in \mathcal{C} \times \mathcal{N}$, then the functor $G^{\otimes}$ is an $\mathcal{LM}$-monoidal functor: that is, we can regard G as a $\mathcal{C}$-linear functor from $\mathcal{N}$ to $\mathcal{M}$.

Example 8.3.2.10. Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}_0 \subseteq \mathcal{C}$. Let T be a monad on $\mathcal{C}$, and suppose that the action of T carries $\mathcal{C}_0$ to itself. Let $\mathrm{Fun}_0(\mathcal{C}, \mathcal{C})$ be the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ spanned by those functors U such that $U(\mathcal{C}_0) \subseteq \mathcal{C}_0$. Then $\mathrm{Fun}_0(\mathcal{C}, \mathcal{C})$ is stable under composition, and therefore inherits a monoidal structure from the monoidal structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ (see §2.2.1). Then T is an algebra object of $\mathrm{Fun}_0(\mathcal{C}, \mathcal{C})$, so that T determines a monad on $\mathcal{C}_0$ (which we will also denote by T) via the evident monoidal functor $\mathrm{Fun}_0(\mathcal{C}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{C}_0, \mathcal{C}_0)$. The inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ is $\mathrm{Fun}_0(\mathcal{C}, \mathcal{C})$-linear, and therefore induces a fully faithful embedding $\mathrm{Mod}_T(\mathcal{C}_0) \to \mathrm{Mod}_T(\mathcal{C})$ (whose essential image is the full subcategory $\mathcal{C}_0 \times_{\mathcal{C}} \mathrm{Mod}_T(\mathcal{C}) \subseteq \mathrm{Mod}_T(\mathcal{C}))$. Suppose that the inclusion $\mathcal{C}_0 \subseteq \mathcal{C}$ admits a right adjoint g. It then follows from Example 8.3.2.8 that the inclusion $\mathrm{Mod}_T(\mathcal{C}_0) \to \mathrm{Mod}_T(\mathcal{C})$ admits a right adjoint G, and that the diagram

$$
\begin{array}{ccc}
\mathrm{Mod}_T(\mathcal{C}_0) & \xleftarrow{\ G\ } & \mathrm{Mod}_T(\mathcal{C}) \\
\downarrow & & \downarrow \\
\mathcal{C}_0 & \xleftarrow{\ g\ } & \mathcal{C}
\end{array}
$$

commutes up to canonical homotopy.

There is a similar criterion for detecting the existence of relative left adjoints in the setting of locally coCartesian fibrations:

Proposition 8.3.2.11. *Suppose we are given a commutative diagram of ∞-categories*

$$
\begin{array}{ccc}
\mathcal{C} & \xleftarrow{\quad G \quad} & \mathcal{D} \\
& \searrow{\scriptstyle q} \quad \swarrow{\scriptstyle p} & \\
& \mathcal{E} &
\end{array}
$$
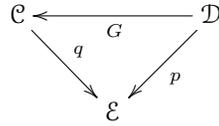
where p and q are locally coCartesian categorical fibrations. Then G admits a left adjoint relative to $\mathcal{E}$ if and only if the following conditions are satisfied:

(1) *For each object $E \in \mathcal{E}$, the induced map $G_E : \mathcal{D}_E \to \mathcal{C}_E$ admits a left adjoint F_E.*

(2) *Let $C \in \mathcal{C}$ be an object and let $\alpha : q(C) \to E'$ be a morphism in $\mathcal{E}$. Let $\overline{\alpha} : F_E(C) \to D$ be a locally p-coCartesian morphism in $\mathcal{D}$ lying over α, and let $\beta : C \to G(D)$ be the composition of the unit map $C \to (G \circ F_E)(C)$ with $G(\beta)$. Choose a factorization of β as a composition*

$$C \xrightarrow{\beta'} C' \xrightarrow{\beta''} G(D)$$

where β' is a locally q-coCartesian morphism lifting α and β'' is a morphism in $\mathcal{C}_{E'}$. Then β'' induces an equivalence $F_{E'}(C') \to D$ in the ∞-category $\mathcal{D}_{E'}$.

Proof. Suppose first that $u : \mathrm{id}_{\mathcal{C}} \to G \circ F$ exhibits $F : \mathcal{C} \to \mathcal{D}$ as a left adjoint to G relative to $\mathcal{E}$. Proposition 8.3.2.5 implies that condition (1) is satisfied. In the situation of condition (2), we can identify F_E and $F_{E'}$ with the restrictions of F. Under these identifications, the map $\phi : F_{E'}(C') \to D$ adjoint to β'' fits into a commutative diagram

$$
\begin{array}{ccc}
& F(C) & \\
{\scriptstyle F(\beta')} \swarrow & & \searrow {\scriptstyle \overline{\alpha}} \\
F(C') & \xrightarrow{\quad \phi \quad} & D.
\end{array}
$$
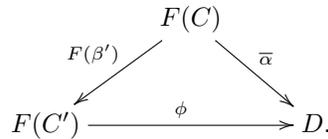
Consequently, ϕ is an equivalence if and only if $F(\beta')$ is locally p-coCartesian. We now complete the proof by observing that F admits a right adjoint relative to $\mathcal{E}$ (Remark 8.3.2.4) and therefore carries locally q-coCartesian morphisms in $\mathcal{C}$ to locally p-coCartesian morphisms in $\mathcal{D}$ (apply Proposition 8.3.2.6 after passing to opposite ∞-categories).

Conversely, suppose that conditions (1) and (2) are satisfied. The argument proceeds as in the proof of Proposition 8.3.2.6. We must show that for each $C \in \mathcal{C}$, there exists an object $D \in \mathcal{D}$ and a map $u : C \to G(D)$ satisfying the following pair of conditions:

(*i*) For every object $D' \in \mathcal{D}$, composition with u induces an equivalence

$$\mathrm{Map}_{\mathcal{D}}(D, D') \to \mathrm{Map}_{\mathcal{C}}(C, G(D')).$$

(*ii*) The morphism $q(u)$ is an equivalence in $\mathcal{E}$.

To construct u, we let E denote the image of the object C in the ∞-category $\mathcal{E}$. Assumption (1) implies that $G_E : \mathcal{D}_E \to \mathcal{C}_E$ admits a left adjoint F_E. In particular, there exists an object $D = F_E(C) \in \mathcal{D}_E$ and a morphism $u : C \to G(D)$ in $\mathcal{C}_E$ which satisfies the following modified version of condition (*i*):

(*i'*) For every object $D' \in \mathcal{D}_E$, composition with u induces an equivalence

$$\mathrm{Map}_{\mathcal{D}_E}(D, D') \to \mathrm{Map}_{\mathcal{C}_E}(C, G(D')).$$

It is obvious that u satisfies condition (ii). We will complete the proof by showing that u satisfies (i). Let $D' \in \mathcal{D}$ be arbitrary: we wish to prove that the map $\phi : \mathrm{Map}_{\mathcal{D}}(D, D') \to \mathrm{Map}_{\mathcal{C}}(C, G(D'))$ is a homotopy equivalence. Let E' denote the image of D' in $\mathcal{E}$. It will suffice to show that ϕ induces a homotopy equivalence after passing to the homotopy fiber over any point $\alpha \in \mathrm{Map}_{\mathcal{E}}(E, E')$. Choose a locally p-coCartesian morphism $\overline{\alpha} : D \to D''$ lying over α, and factor the composition $\beta : C \to G(D) \to G(D'')$ as a composition

$$ C \xrightarrow{\beta'} C' \xrightarrow{\beta''} G(D'') $$

where β' is a locally q-coCartesian lift of α and β'' is a morphism in $\mathcal{C}_{E'}$. Using Proposition T.2.4.4.2, we can identify the homotopy fiber ϕ_α with the induced map

$$ \mathrm{Map}_{\mathcal{D}_{E'}}(D'', D') \longrightarrow \mathrm{Map}_{\mathcal{C}_{E'}}(G(D''), G(D')) \xrightarrow{\circ \beta''} \mathrm{Map}_{\mathcal{C}_{E'}}(C', G(D')). $$

Identifying the latter space with $\mathrm{Map}_{\mathcal{D}_{E'}}(F_{E'}(C'), D')$, we see that the map ϕ_α is induced by composition with the map $F_{E'}(C') \to D''$ adjoint to β'', which is an equivalence by virtue of (2). $\qquad\square$

Corollary 8.3.2.12. *Suppose we are given a commutative diagram*

of ∞-operads, where p and q are coCartesian fibrations. Assume that:

(1) *For every object $X \in \mathcal{O}$, the induced map of fibers $G_X : \mathcal{D}_X \to \mathcal{C}_X$ admits a left adjoint F_X.*

(2) *For every operation $\phi \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y)$ in $\mathcal{O}^{\otimes}$, if we let $\phi_!^{\mathcal{C}}$ and $\phi_!^{\mathcal{D}}$ denote the associated functors*

$$ \prod_{i \in I} \mathcal{C}_{X_i} \to \mathcal{C}_Y \qquad \prod_{i \in I} \mathcal{D}_{X_i} \to \mathcal{D}_Y, $$

then the evident natural transformation

$$ F_Y \circ \phi_!^{\mathcal{C}} \to \phi_!^{\mathcal{D}} \circ \left(\prod_{i \in I} F_{X_i}\right) $$

is an equivalence of functors from $\prod_i \mathcal{C}_{X_i}$ to $\mathcal{D}_Y$.

Then G admits a left adjoint F relative to $\mathcal{O}^{\otimes}$. Moreover, F is a $\mathcal{O}^{\otimes}$-monoidal functor.

Proof. The existence of F follows from Proposition 8.3.2.11. Using Proposition 8.3.2.6, we deduce that F carries p-coCartesian morphisms in $\mathcal{C}^{\otimes}$ to q-coCartesian morphisms in $\mathcal{D}^{\otimes}$ and is therefore a $\mathcal{O}$-monoidal functor. $\qquad\square$

Remark 8.3.2.13. In the situation of Corollary 8.3.2.12, Proposition 8.3.2.5 implies that F and G induce adjoint functors $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \rightleftarrows \mathrm{Alg}_{/\mathcal{O}} \mathcal{D}$.

We now apply the theory of relative adjunctions to the study of tangent bundles.

Definition 8.3.2.14. Let $\mathcal{C}$ be a presentable ∞-category, and consider the associated diagram

where q is given by evaluation at $\{1\} \subseteq \Delta^1$. The functor G carries p-Cartesian morphisms to q-Cartesian morphisms, and for each object $A \in \mathcal{C}$ the induced map $G_A : \mathrm{Sp}(\mathcal{C}^{/A}) \to \mathcal{C}^{/A}$ admits a left adjoint Σ^∞. Applying Proposition 8.3.2.6, we conclude that G admits a left adjoint relative to $\mathcal{C}$, which we will denote by F. The *absolute cotangent complex functor* $L : \mathcal{C} \to T_{\mathcal{C}}$ is defined to be the composition

$$\mathcal{C} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \xrightarrow{F} T_{\mathcal{C}},$$

where the first map is given by the diagonal embedding. We will denote the value of L on an object $A \in \mathcal{C}$ by $L_A \in \mathrm{Sp}(\mathcal{C}^{/A})$, and will refer to L_A as the *cotangent complex of A*.

Remark 8.3.2.15. Let $\mathcal{C}$ be a presentable ∞-category. Since the diagonal embedding $\mathcal{C} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ is a left adjoint to the evaluation map $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{0\}, \mathcal{C}) \simeq \mathcal{C}$, we deduce that the absolute cotangent complex functor $L : \mathcal{C} \to T_{\mathcal{C}}$ is left adjoint to the composition

$$T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{0\}, \mathcal{C}) \simeq \mathcal{C}.$$

Remark 8.3.2.16. The terminology of Definition 8.3.2.14 is slightly abusive, since the tangent bundle $T_{\mathcal{C}}$ and the functor L are only well-defined up to equivalence. It would perhaps be more accurate to refer to $L : \mathcal{C} \to T_{\mathcal{C}}$ as *an* absolute cotangent functor. However, L and $T_{\mathcal{C}}$ are well-defined up to a contractible space of choices, so we will tolerate the ambiguity.

Remark 8.3.2.17. Let $\mathcal{C}$ be a presentable ∞-category containing an object A. We observe that the fiber of the tangent bundle $T_{\mathcal{C}}$ over $A \in \mathcal{C}$ can be identified with the ∞-category $\mathrm{Sp}(\mathcal{C}_{/A})$. Under this identification, the object $L_A \in \mathrm{Sp}(\mathcal{C}_{/A})$ corresponds to the image of $\mathrm{id}_A \in \mathcal{C}_{/A}$ under the suspension spectrum functor $\Sigma^\infty_+ : \mathcal{C}_{/A} \to \mathrm{Sp}(\mathcal{C}_{/A})$.

Remark 8.3.2.18. Let $\mathcal{C}$ be a presentable ∞-category. Since the cotangent complex functor L is a left adjoint, it carries colimit diagrams in $\mathcal{C}$ to colimit diagrams in $T_{\mathcal{C}}$. In view of Proposition 8.3.1.12, we see that L also carries small colimit diagrams in $\mathcal{C}$ to p-colimit diagrams in $T_{\mathcal{C}}$, where p denotes the composition

$$T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{1\}, \mathcal{C}) \simeq \mathcal{C}.$$

Remark 8.3.2.19. Let $\mathcal{C}$ be a presentable ∞-category, and let A be an initial object of $\mathcal{C}$. Using Remark 8.3.2.18, we deduce that L_A is an initial object of the tangent bundle $T_{\mathcal{C}}$. Equivalently, L_A is a zero object of the stable ∞-category $\mathrm{Sp}(\mathcal{C}^{/A})$.

8.3.3   The Relative Cotangent Complex

Let $\mathcal{C}$ be a presentable ∞-category. In §8.3.2, we defined the absolute cotangent complex functor $L : \mathcal{C} \to T_{\mathcal{C}}$, which associates to each $A \in \mathcal{C}$ an object $L_A \in \mathrm{Sp}(\mathcal{C}^{/A})$. For many applications, it is convenient to consider also a *relative cotangent complex* associated to a morphism $f : A \to B$ in $\mathcal{C}$. In this section, we will define the relative cotangent complex $L_{B/A}$ and establish some of its basic properties.

Definition 8.3.3.1. Let $\mathcal{C}$ be a presentable ∞-category and let $p : T_{\mathcal{C}} \to \mathcal{C}$ be a tangent bundle to $\mathcal{C}$. A *relative cofiber sequence* in $T_{\mathcal{C}}$ is a diagram σ:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

in $T_{\mathcal{C}}$ with the following properties:

(1) The map $p \circ \sigma$ factors through the projection $\Delta^1 \times \Delta^1 \to \Delta^1$, so that the vertical arrows above become degenerate in $\mathcal{C}$.

(2) The diagram σ is a pushout square. (Since condition (1) implies that $p \circ \sigma$ is a pushout square, this is equivalent to the requirement that σ be a p-colimit diagram; see Proposition 8.3.1.12).

Let $\mathcal{E}$ denote the full subcategory of

$$\mathrm{Fun}(\Delta^1 \times \Delta^1, T_{\mathcal{C}}) \times_{\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})} \mathrm{Fun}(\Delta^1, \mathcal{C})$$

spanned by the relative cofiber sequences. There is an evident forgetful functor $\psi : \mathcal{E} \to \mathrm{Fun}(\Delta^1, T_{\mathcal{C}})$, given by restriction to the upper half of the diagram. Invoking Proposition T.4.3.2.15 twice, we deduce that ψ is a trivial Kan fibration.

The *relative cotangent complex functor* is defined to be the composition

$$\mathrm{Fun}(\Delta^1, \mathcal{C}) \overset{L}{\to} \mathrm{Fun}(\Delta^1, T_{\mathcal{C}}) \overset{s}{\to} \mathcal{E} \overset{s'}{\to} T_{\mathcal{C}},$$

where s is a section of ψ and s' is given by evaluation at the vertex $\{1\} \times \{1\} \subseteq \Delta^1 \times \Delta^1$.

We will denote the image of a morphism $f : A \to B$ under the relative cotangent complex functor by $L_{B/A} \in T_{\mathcal{C}} \times_{\mathcal{C}} \{B\} \simeq \mathrm{Sp}(\mathcal{C}^{B/})$.

Remark 8.3.3.2. Let $\mathcal{C}$ and $p : T_{\mathcal{C}} \to \mathcal{C}$ be as in Definition 8.3.3.1. By definition, the relative cotangent complex of a morphism $f : A \to B$ fits into a relative cofiber sequence

$$\begin{array}{ccc}
L_A & \longrightarrow & L_B \\
\downarrow & & \downarrow \\
0 & \longrightarrow & L_{B/A}
\end{array}$$

in the ∞-category $T_{\mathcal{C}}$. Using Proposition T.4.3.1.9, we deduce the existence of a cofiber sequence

$$f_! L_A \to L_B \to L_{B/A}$$

in the stable ∞-category $\mathrm{Sp}(\mathcal{C}^{/B}) \simeq T_{\mathcal{C}} \times_{\mathcal{C}} \{B\}$; here $f_! : \mathrm{Sp}(\mathcal{C}^{/A}) \to \mathrm{Sp}(\mathcal{C}^{/B})$ denotes the functor induced by the coCartesian fibration p.

Remark 8.3.3.3. Let $\mathcal{C}$ be a presentable ∞-category containing a morphism $f : A \to B$. If A is an initial object of $\mathcal{C}$, then the canonical map $L_B \to L_{B/A}$ is an equivalence. This follows immediately from Remark 8.3.3.2, since the absolute cotangent complex L_A vanishes (Remark 8.3.2.19). We will sometimes invoke this equivalence implicitly, and ignore the distinction between the relative cotangent complex $L_{B/A}$ and the absolute cotangent complex L_B.

Remark 8.3.3.4. Let $\mathcal{C}$ be a presentable ∞-category containing a morphism $f : A \to B$. If f is an equivalence, then the relative cotangent complex $L_{B/A}$ is a zero object of $\mathrm{Sp}(\mathcal{C}^{/B})$. This follows immediately from Remark 8.3.3.2.

We next study the fiber sequence of cotangent complexes associated to a triple of morphisms $A \to B \to C$.

Proposition 8.3.3.5. *Let $\mathcal{C}$ be a presentable ∞-category, let $T_{\mathcal{C}}$ be a tangent bundle to $\mathcal{C}$. Suppose given a commutative diagram*

$$\begin{array}{ccc}
 & B & \\
\nearrow & & \searrow \\
A & \longrightarrow & C
\end{array}$$

in $\mathcal{C}$*. The resulting square*

$$
\begin{array}{ccc}
L_{B/A} & \xrightarrow{\;f\;} & L_{C/A} \\
\downarrow & & \downarrow \\
L_{B/B} & \longrightarrow & L_{C/B}
\end{array}
$$

is a pushout diagram in $T_{\mathcal{C}}$ *(and therefore a relative cofiber sequence, in view of Remark 8.3.3.4).*

Proof. We have a commutative diagram

$$
\begin{array}{ccc}
L_A & \longrightarrow L_B & \longrightarrow L_C \\
\downarrow & \downarrow & \downarrow \\
L_{A/A} & \longrightarrow L_{B/A} & \longrightarrow L_{C/A} \\
& \downarrow & \downarrow \\
& L_{B/B} & \longrightarrow L_{C/B}
\end{array}
$$

in the ∞-category $T_{\mathcal{C}}$. Here $L_{A/A}$ and $L_{B/B}$ are zero objects in the fibers $\mathrm{Sp}(\mathcal{C}^{/A})$ and $\mathrm{Sp}(\mathcal{C}^{/B})$, respectively (Remark 8.3.3.4). By construction, the upper left square and both large rectangles in this diagram are coCartesian. It follows first that the upper right square is coCartesian, and then that the lower right square is coCartesian as desired. $\qquad\square$

Corollary 8.3.3.6. *Let* $\mathcal{C}$ *be a presentable* ∞*-category containing a commutative triangle*

$$
\begin{array}{ccc}
 & B & \\
 \nearrow & & \searrow {\scriptstyle f} \\
A & \longrightarrow & C,
\end{array}
$$

and let $f_{!} : \mathrm{Sp}(\mathcal{C}^{/B}) \to \mathrm{Sp}(\mathcal{C}^{/C})$ *denote the induced map. Then we have a canonical cofiber sequence*

$$
f_{!}L_{B/A} \to L_{C/A} \to L_{C/B}
$$

in the ∞*-category* $\mathrm{Sp}(\mathcal{C}^{/C})$*.*

Our next result records the behavior of the relative cotangent complex under base change.

Proposition 8.3.3.7. *Let* $\mathcal{C}$ *be a presentable* ∞*-category,* $T_{\mathcal{C}}$ *a tangent bundle to* $\mathcal{C}$*, and* p *the composite map*

$$
T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{1\}, \mathcal{C}) \simeq \mathcal{C}.
$$

Suppose given a pushout diagram

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow {\scriptstyle f} \\
A' & \longrightarrow & B'
\end{array}
$$

in $\mathcal{C}$*. Then the induced map* $\beta : L_{B/A} \to L_{B'/A'}$ *is a* p*-coCartesian morphism in* $T_{\mathcal{C}}$*.*

Proof. Using Definition 8.3.3.1, we deduce the existence of a map between relative cofiber sequences in $T_{\mathcal{C}}$, which we can depict as a cubical diagram τ:

$$
\begin{array}{ccc}
L_A & \longrightarrow & L_B \\
& \searrow & & \searrow \\
& 0_A & \longrightarrow & L_{B/A} \\
\downarrow & & \downarrow \\
L_{A'} & \longrightarrow & L_{B'} \\
& \searrow & & \searrow \\
& 0_{A'} & \longrightarrow & L_{B'/A'}.
\end{array}
$$

Let $K \subseteq \Delta^1 \times \Delta^1 \times \Delta^1$ denote the full simplicial subset obtained by omitting the final vertex. Let $K_0 \subseteq K$ be obtained by omitting the vertex $v = \{1\} \times \{1\} \times \{0\}$ such that $\tau(v) = L_{B'}$, and let $K_1 \subseteq K$ be obtained by omitting the vertex $w = \{1\} \times \{0\} \times \{1\}$ such that $\tau(w) = L_{B/A}$. By construction, τ is a p-left Kan extension of $\tau|K_1$. Using Proposition T.4.3.2.8, we conclude that τ is a p-colimit diagram.

Remark 8.3.2.18 implies that the square

$$
\begin{array}{ccc}
L_A & \longrightarrow & L_B \\
\downarrow & & \downarrow \\
L_{A'} & \longrightarrow & L_{B'}
\end{array}
$$

is a p-colimit diagram, so that $\tau|K$ is a p-left Kan extension of $\tau|K_0$. Invoking Proposition T.4.3.2.8 again, we deduce that τ is a p-left Kan extension of $\tau|K_0$. It follows that τ restricts to a p-colimit square:

$$
\begin{array}{ccc}
0_A & \longrightarrow & L_{B/A} \\
\downarrow & & \downarrow \\
0_{A'} & \longrightarrow & L_{B'/A'}.
\end{array}
$$

Proposition T.4.3.1.9 implies that the induced square

$$
\begin{array}{ccc}
0 & \longrightarrow & f_! L_{B/A} \\
\downarrow & & \downarrow{\scriptstyle \alpha} \\
0 & \longrightarrow & L_{B'/A'}
\end{array}
$$

is a pushout square in $\mathrm{Sp}(\mathcal{C}^{/B'})$; in other words, the map α is an equivalence. This is simply a reformulation of the assertion that β is p-coCartesian. $\square$

There is another way to view the relative cotangent complex: if we fix an object $A \in \mathcal{C}$, then the functor $B \mapsto L_{B/A}$ can be identified with the *absolute* cotangent complex for the ∞-category $\mathcal{C}_{A/}$. The rest of this section will be devoted to justifying this assertion. These results will not be needed elsewhere in this paper, and may be safely omitted by the reader. We begin by describing the tangent bundle to an ∞-category of the form $\mathcal{C}_{A/}$.

Proposition 8.3.3.8. *Let $\mathcal{C}$ be a presentable ∞-category containing an object A, and let $\mathcal{D} = \mathcal{C}_{A/}$. Let $T_{\mathcal{C}}$ and $T_{\mathcal{D}}$ denote tangent bundles to $\mathcal{C}$ and $\mathcal{D}$, respectively. Then there is a canonical equivalence*

$$T_{\mathcal{D}} \simeq T_{\mathcal{C}} \times_{\mathcal{C}} \mathcal{D}$$

of presentable fibrations over $\mathcal{D}$.

Proposition 8.3.3.8 a relative version of the following more elementary observation:

Lemma 8.3.3.9. *Let $\mathcal{C}$ be an ∞-category which admits finite limits and let A be an object of $\mathcal{C}$. The forgetful functor $\mathcal{C}_{A/} \to \mathcal{C}$ induces equivalences of ∞-categories*

$$f : (\mathcal{C}_{A/})_* \to \mathcal{C}_* \qquad g : \mathrm{Sp}(\mathcal{C}_{A/}) \to \mathrm{Sp}(\mathcal{C}).$$

Proof. We will prove that f is an equivalence; the assertion that g is an equivalence is an obvious consequence. Let 1 denote a final object of $\mathcal{C}$. Using Proposition T.1.2.13.8, we deduce that $\mathcal{C}_{A/}$ admits a final object, given by a morphism $u : A \to 1$. Using Lemma T.7.2.2.8, we deduce the existence of a commutative diagram

$$
\begin{array}{ccc}
\mathcal{C}_{u/} & \xrightarrow{f'} & \mathcal{C}_{1/} \\
\downarrow & & \downarrow \\
(\mathcal{C}_{A/})_* & \xrightarrow{f} & \mathcal{C}_*,
\end{array}
$$

where the vertical arrows are equivalences. It follows that f is an equivalence if and only if f' is an equivalence. But f is a trivial Kan fibration, since the inclusion $\{1\} \subseteq \Delta^1$ is right anodyne. $\square$

Proof of Proposition 8.3.3.8. Let $\mathcal{E} = \mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathrm{Fun}(\{1\}, \mathcal{C})} \mathcal{D}$, so that we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Fun}(\Delta^1, \mathcal{D}) & \xrightarrow{\quad f \quad} & \mathcal{E} \\
& \searrow^{q} \quad \swarrow^{q'} & \\
& \mathcal{D}, &
\end{array}
$$

where q and q' are presentable fibrations. We first claim that f carries q-limit diagrams to q'-limit diagrams. In view of Propositions T.4.3.1.9 and T.4.3.1.10, it will suffice to verify the following pair of assertions:

(*i*) For each object $\overline{B} \in \mathcal{D}$, corresponding to a morphism $A \to B$ in $\mathcal{C}$, the induced map of fibers

$$f_{\overline{B}} : \mathcal{D}^{/\overline{B}} \to \mathcal{C}^{/B}$$

preserves limits.

(*ii*) The map f carries q-Cartesian morphisms to q'-Cartesian morphisms.

To prove (*i*), we observe that $f_{\overline{B}}$ is equivalent to the forgetful functor $(\mathcal{C}_{/B})_{A/} \to \mathcal{C}_{/B}$, which preserves limits by Proposition T.1.2.13.8. Assertion (*ii*) is equivalent to the requirement that the forgetful functor $\mathcal{D} \to \mathcal{C}$ preserves pullback diagrams, which follows again from Proposition T.1.2.13.8.

Using Remark 8.3.1.3, we can identify $T_{\mathcal{C}} \times_{\mathcal{C}} \mathcal{D}$ with the stable envelope of the presentable fibration q'. It follows from the universal property of Proposition 8.3.1.7 that the map f fits into a commutative diagram

$$
\begin{array}{ccc}
T_{\mathcal{D}} & \xrightarrow{\overline{f}} & T_{\mathcal{C}} \times_{\mathcal{C}} \mathcal{D} \\
\downarrow & & \downarrow \\
\mathrm{Fun}(\Delta^1, \mathcal{D}) & \xrightarrow{\quad f \quad} & \mathcal{E} \, .
\end{array}
$$

To complete the proof, we will show that $\overline{f}$ is an equivalence. In view of Corollary T.2.4.4.4, it will suffice to show that for each $\overline{B} \in \mathcal{D}$ classifying a map $A \to B$ in $\mathcal{C}$, the induced map $\mathrm{Sp}(\mathcal{D}^{/\overline{B}}) \to \mathrm{Sp}(\mathcal{C}^{/B})$ is an equivalence of ∞-categories. This follows immediately from Lemma 8.3.3.9. $\square$

We now wish to study the relationship between the cotangent complex functors of $\mathcal{C}$ and $\mathcal{C}_{A/}$, where A is an object of $\mathcal{C}$. For this, it is convenient to introduce a bit of terminology.

Definition 8.3.3.10. Let $F, F' : \mathcal{C} \to \mathcal{D}$ be a functors from an ∞-category $\mathcal{C}$ to an ∞-category $\mathcal{D}$, and let $\alpha : F \to F'$ be a natural transformation. We will say that α is *coCartesian* if, for every morphism $C \to C'$ in $\mathcal{C}$, the induced diagram

$$
\begin{array}{ccc}
F(C) & \longrightarrow & F(C') \\
\downarrow{\scriptstyle \alpha_C} & & \downarrow{\scriptstyle \alpha_{C'}} \\
F'(C) & \longrightarrow & F'(C')
\end{array}
$$

is a pushout square in $\mathcal{D}$.

The basic properties of the class of coCartesian natural transformations are summarized in the following lemma:

Lemma 8.3.3.11. (1) *Let $F, F', F'' : \mathcal{C} \to \mathcal{D}$ be functors between ∞-categories, and let $\alpha : F \to F'$ and $\beta : F' \to F''$ be natural transformations. If α is coCartesian, then β is coCartesian if and only if $\beta \circ \alpha$ is coCartesian.*

(2) *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories, let $G, G' : \mathcal{D} \to \mathcal{E}$ be a pair of functors, and let $\alpha : G \to G'$ be a natural transformation. If α is coCartesian, then so is the induced transformation $GF \to G'F$.*

(3) *Let $F, F' : \mathcal{C} \to \mathcal{D}$ be a pair of functors between ∞-categories, let $G : \mathcal{D} \to \mathcal{E}$ another functor, and let $\alpha : F \to F'$ be a natural transformation. If α is coCartesian and G preserves all pushout squares which exist in $\mathcal{D}$, then the induced transformation $GF \to GF'$ is coCartesian.*

Definition 8.3.3.12. We will say that a commutative diagram of ∞-categories

$$
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\ H\ } & \mathcal{C} \\
\downarrow{\scriptstyle G} & & \downarrow{\scriptstyle G'} \\
\mathcal{D}' & \xrightarrow{\ H'\ } & \mathcal{C}'
\end{array}
$$

is *rectilinear* if the following conditions are satisfied:

(1) The functors G and G' admit left adjoints, which we will denote by F and F' respectively.

(2) The identity map $H'G \simeq G'H$ induces a coCartesian natural transformation $F'H' \to HF$.

Remark 8.3.3.13. The condition of being rectilinear is closely related to the condition of being *left adjointable*, as defined in §T.7.3.1.

Proposition 8.3.3.14. *Let $\mathcal{C}$ be a presentable ∞-category containing an object A and let $\mathcal{D} = \mathcal{C}_{A/}$. Let $G : T_{\mathcal{C}} \to \mathcal{C}$ denote the composite map*

$$
T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{0\}, \mathcal{C}) \simeq \mathcal{C},
$$

and let $G' : T_{\mathcal{D}} \to \mathcal{D}$ be defined similarly, so that we have a commutative diagram

$$
\begin{array}{ccc}
T_{\mathcal{D}} & \longrightarrow & T_{\mathcal{C}} \\
\downarrow & & \downarrow \\
\mathcal{D} & \longrightarrow & \mathcal{C}
\end{array}
$$

(*see the proof of Proposition 8.3.3.8*). *Then the above diagram is rectilinear.*

Corollary 8.3.3.15. *Let $\mathcal{C}$ and $\mathcal{D} = \mathcal{C}_{A/}$ be as in Proposition 8.3.3.14, and let $L^{\mathcal{C}} : \mathcal{C} \to T_{\mathcal{C}}$ and $L^{\mathcal{D}} : \mathcal{D} \to T_{\mathcal{D}}$ be cotangent complex functors for $\mathcal{C}$ and $\mathcal{D}$, respectively. Then:*

(1) *Let $p : \mathcal{D} \to \mathcal{C}$ be the projection, and let $q : T_{\mathcal{D}} \to T_{\mathcal{C}}$ be the induced map. Then there is a coCartesian natural transformation $L^{\mathcal{C}} \circ p \to q \circ L^{\mathcal{D}}$.*

(2) *There is a pushout diagram of functors*

$$
\begin{array}{ccc}
L_A^{\mathcal{C}} & \longrightarrow & L^{\mathcal{C}} \circ p \\
\downarrow & & \downarrow \\
0 & \longrightarrow & q \circ L^{\mathcal{D}}.
\end{array}
$$

Here the terms in the left hand column indicate the constant functors taking the values $L_A^{\mathcal{C}}, 0 \in \mathrm{Sp}(\mathcal{C}^{/A}) \subseteq T_{\mathcal{C}}$.

(3) *The functor $q \circ L^{\mathcal{D}} : \mathcal{D} \to T_{\mathcal{C}}$ can be identified with the functor $B \mapsto L_{B/A}$.*

Proof. Assertion (1) is merely a reformulation of Proposition 8.3.3.14. To prove (2), we let $e : \mathcal{D} \to \mathcal{D}$ denote the constant functor taking the value $\mathrm{id}_A \in \mathcal{D}$, so that we have a natural transformation $\alpha : e \to \mathrm{id}_{\mathcal{D}}$. Applying the coCartesian transformation of (1) to α yields the desired diagram, since $L^{\mathcal{D}} \circ e$ vanishes by Remark 8.3.2.19. Assertion (3) follows immediately from (2) and the definition of the relative cotangent complex. $\qquad\square$

To prove Proposition 8.3.3.14, we observe that the square in question fits into a commutative diagram

$$
\begin{array}{ccc}
T_{\mathcal{D}} & \longrightarrow & T_{\mathcal{C}} \\
\downarrow & & \downarrow \\
\mathrm{Fun}(\Delta^1, \mathcal{D}) & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{C}) \\
\downarrow & & \downarrow \\
\mathrm{Fun}(\{0\}, \mathcal{D}) & \longrightarrow & \mathrm{Fun}(\{0\}, \mathcal{C}).
\end{array}
$$

It will therefore suffice to prove the following three results:

Lemma 8.3.3.16. *Suppose given a commutative diagram of ∞-categories*

$$
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\ H\ } & \mathcal{C} \\
{\scriptstyle G_0}\downarrow & & \downarrow{\scriptstyle G_0'} \\
\mathcal{D}' & \xrightarrow{\ H'\ } & \mathcal{C}' \\
{\scriptstyle G_0'}\downarrow & & \downarrow{\scriptstyle G_1'} \\
\mathcal{D}'' & \xrightarrow{\ H''\ } & \mathcal{C}''
\end{array}
$$

If the upper and lower squares are rectilinear, then the outer square is rectilinear.

Lemma 8.3.3.17. *Let $p : \mathcal{D} \to \mathcal{C}$ be a functor between ∞-categories. Then the commutative diagram*

$$
\begin{array}{ccc}
\mathrm{Fun}(\Delta^1, \mathcal{D}) & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{C}) \\
\downarrow{\scriptstyle G} & & \downarrow{\scriptstyle G'} \\
\mathrm{Fun}(\{0\}, \mathcal{D}) & \longrightarrow & \mathrm{Fun}(\{0\}, \mathcal{C})
\end{array}
$$

is rectilinear.

Lemma 8.3.3.18. *Let $\mathcal{C}$ be a presentable ∞-category containing an object A, and let $\mathcal{D} = \mathcal{C}_{A/}$. Then the diagram*

$$
\begin{array}{ccc}
T_{\mathcal{D}} & \longrightarrow & T_{\mathcal{C}} \\
\downarrow & & \downarrow \\
\mathrm{Fun}(\Delta^1, \mathcal{D}) & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{C})
\end{array}
$$

(see the proof of Proposition 8.3.3.8) is rectilinear.

Proof of Lemma 8.3.3.16. We observe that $G_1 G_0$ admits a left adjoint $L_0 L_1$, where L_0 and L_1 are left adjoints to G_0 and G_1, respectively. Similarly, $G_1' G_0'$ admits a left adjoint $L_0' L_1'$. It remains only to show that the composite transformation

$$
L_0 L_1 H'' \to L_0 H' L_1' \to H L_0' L_1'
$$

is coCartesian, which follows from Lemma 8.3.3.11. $\qquad\square$

Proof of Lemma 8.3.3.17. For any ∞-category $\mathcal{C}$, the evaluation functor $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{0\}, \mathcal{C}) \simeq \mathcal{C}$ has a left adjoint given by the diagonal embedding $\delta_{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$. In the situation of Lemma 8.3.3.17, we obtain a *strictly commutative* diagram of adjoint functors

$$
\begin{array}{ccc}
\mathrm{Fun}(\Delta^1, \mathcal{D}) & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{C}) \\
\uparrow{\scriptstyle \delta_{\mathcal{D}}} & & \uparrow{\scriptstyle \delta_{\mathcal{C}}} \\
\mathcal{D} & \longrightarrow & \mathcal{C}\,.
\end{array}
$$

It now suffices to observe that that any invertible natural transformation is automatically coCartesian. $\square$

To prove Lemma 8.3.3.18, we once again break the work down into two steps. First, we need a bit of terminology:

Notation 8.3.3.19. For every ∞-category $\mathcal{C}$, we let $P_*(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\Delta^2, \mathcal{C})$ spanned by those diagrams

$$
\begin{array}{ccc}
& B & \\
\nearrow & & \searrow \\
A & \xrightarrow{\quad f \quad} & C
\end{array}
$$

such that f is an equivalence. If $\mathcal{C}$ is presentable, then the evaluation map

$$
P_*(\mathcal{C}) \to \mathrm{Fun}(\Delta^{\{1,2\}}, \mathcal{C}) \simeq \mathrm{Fun}(\Delta^1, \mathcal{C})
$$

exhibits $P_*(\mathcal{C})$ as a pointed envelope of the presentable fibration $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{1\}, \mathcal{C}) \simeq \mathcal{C}$.

Now let $p : \mathcal{D} \to \mathcal{C}$ be as in Lemma 8.3.3.18. The proof of Proposition 8.3.3.8 gives a commutative diagram

$$
\begin{array}{ccc}
T_{\mathcal{D}} & \longrightarrow & T_{\mathcal{C}} \\
\downarrow & & \downarrow \\
P_*(\mathcal{D}) & \longrightarrow & P_*(\mathcal{C}) \\
\downarrow & & \downarrow \\
\mathrm{Fun}(\Delta^1, \mathcal{D}) & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{C}).
\end{array}
$$

We wish to prove that the outer square is rectilinear. In view of Lemma 8.3.3.16, it will suffice to prove the upper and bottom squares are rectilinear. For the upper square, we observe that Proposition 8.3.3.8 gives a homotopy pullback diagram

$$
\begin{array}{ccc}
T_{\mathcal{D}} & \longrightarrow & T_{\mathcal{C}} \\
\downarrow & & \downarrow \\
P_*(\mathcal{D}) & \longrightarrow & P_*(\mathcal{C}) \\
\downarrow & & \downarrow \\
\mathcal{C} & \longrightarrow & \mathcal{D} .
\end{array}
$$

Lemma 8.3.3.18 is therefore a consequence of the following pair of results:

Lemma 8.3.3.20. *Suppose given a commutative diagram*

$$
\begin{array}{ccc}
\mathcal{D} & \longrightarrow & \mathcal{C} \\
\downarrow & & \downarrow {\scriptstyle G} \\
\mathcal{D}' & \longrightarrow & \mathcal{C}' \\
\downarrow & & \downarrow \\
\mathcal{D}'' & \longrightarrow & \mathcal{C}''
\end{array}
$$

of ∞-categories, where each square is homotopy Cartesian. If G admits a left adjoint relative to $\mathcal{C}''$, then the upper square is rectilinear.

Lemma 8.3.3.21. *Let $\mathcal{C}$ be a presentable ∞-category containing an object A, let $\mathcal{D} = \mathcal{C}_{A/}$. Then the diagram*

$$
\begin{array}{ccc}
P_*(\mathcal{D}) & \longrightarrow & P_*(\mathcal{C}) \\
\downarrow {\scriptstyle G'} & & \downarrow {\scriptstyle G} \\
\mathrm{Fun}(\Delta^1, \mathcal{D}) & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{C})
\end{array}
$$

is rectilinear.

Proof of Lemma 8.3.3.20. Without loss of generality, we may assume that every map in the diagram

$$
\begin{array}{ccc}
\mathcal{D} & \longrightarrow & \mathcal{C} \\
\downarrow {\scriptstyle G'} & & \downarrow {\scriptstyle G} \\
\mathcal{D}' & \longrightarrow & \mathcal{C}' \\
\downarrow & & \downarrow \\
\mathcal{D}'' & \longrightarrow & \mathcal{C}''
\end{array}
$$

is a categorical fibration, and that each square is a pullback in the category of simplicial sets. Let F be a left adjoint to G relative to $\mathcal{C}''$, and choose a counit map $v : F \circ G \to \mathrm{id}_{\mathcal{C}}$ which is compatible with the projection to $\mathcal{C}''$ (so that v can be identified with a morphism in the ∞-category $\mathrm{Map}_{\mathcal{C}''}(\mathcal{C}, \mathcal{C})$). Let $F' : \mathcal{D}' \to \mathcal{D}$ be the map induced by F, so that v induces a natural transformation $F' \circ G' \to \mathrm{id}_{\mathcal{D}}$, which is easily verified to be the counit of an adjunction. It follows that we have a strictly commutative diagram

$$
\begin{array}{ccc}
\mathcal{D} & \longrightarrow & \mathcal{C} \\
{\scriptstyle F'}\big\uparrow & & \big\uparrow{\scriptstyle F} \\
\mathcal{D}' & \longrightarrow & \mathcal{C}' .
\end{array}
$$

To complete the proof it suffices to observe that any invertible natural transformation is automatically coCartesian. $\square$

Proof of Lemma 8.3.3.21. The forgetful functor $G : P_*(\mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ has a left adjoint F. We can identify F with the functor which carries a diagram $B \to C$ in $\mathcal{C}$ to the induced diagram

$$
\begin{array}{ccc}
 & B \coprod C & \\
\nearrow & & \searrow \\
C \xrightarrow{\quad \mathrm{id} \quad} & & C,
\end{array}
$$

regarded as an object of $P_*(\mathcal{C})$. Similarly, G' has a left adjoint F', which carries a diagram $A \to B \to C$ to the induced diagram

$$
\begin{array}{ccc}
 & B \coprod_A C & \\
\nearrow & & \searrow \\
A \longrightarrow C \xrightarrow{\quad \mathrm{id} \quad} & & C.
\end{array}
$$

We observe that a diagram in $P_*(\mathcal{C})$ is a pushout square if and only if it determines a pushout square in $\mathcal{C}$ after evaluating at each vertex in Δ^2. Unwinding the definition, we see that the Lemma 8.3.3.21 is equivalent to the following elementary assertion: for every commutative diagram

$$
\begin{array}{ccc}
A \longrightarrow B \longrightarrow C \\
\searrow \quad \downarrow \qquad \downarrow \\
B' \longrightarrow C'
\end{array}
$$

in $\mathcal{C}$, the induced diagram

$$
\begin{array}{ccc}
B \coprod C & \longrightarrow & B \coprod_A C \\
\downarrow & & \downarrow \\
B' \coprod C' & \longrightarrow & B' \coprod_A C'
\end{array}
$$

is a pushout square. $\square$

8.3.4   Tangent Bundles to ∞-Categories of Algebras

Let A be a commutative ring, and let M be an A-module. Then the direct sum $A \oplus M$ inherits the structure of a commutative ring, with multiplication described by the formula

$$
(a, m)(a', m') = (aa', am' + a'm).
$$

We wish to describe an analogous construction in the case where A is an $\mathbb{E}_\infty$-ring and M is a module *spectrum* over A. In this context we cannot define a ring structure on $A \oplus M$ simply by writing formulas: we must obtain $A \oplus M$ in some other way. We begin by listing some features which we expect of this construction:

(a) The square-zero extension $A \oplus M$ admits a projection map $A \oplus M \to A$.

(b) The square-zero extension $A \oplus M$ depends functorially on M. In other words, it is given by a functor

$$G : \mathrm{Mod}_A \to \mathrm{CAlg}_{/A}.$$

(c) The underlying spectrum of $A \oplus M$ can be identified (functorially) with the usual coproduct of A and M in the ∞-category of Sp.

Condition (c) automatically implies that the functor G preserves limits. Since the ∞-category Mod_A is stable, the functor G would then be equivalent to a composition

$$\mathrm{Mod}_A \xrightarrow{G'} \mathrm{Sp}(\mathrm{CAlg}_{/A}) \xrightarrow{\Omega^\infty} \mathrm{CAlg}_{/A}.$$

In fact, we will prove something stronger: the functor G' is an equivalence of ∞-categories. Let us describe a functor F' which is homotopy inverse to G'. Let X be an object of $\mathrm{Sp}(\mathrm{CAlg}_{/A})$. Then the 0th space of X is a *pointed* object of $\mathrm{CAlg}_{/A}$, which we can identify with an *augmented A-algebra*: that is, an $\mathbb{E}_\infty$-ring B which fits into a commutative diagram

$$
\begin{array}{ccc}
 & B & \\
 \nearrow & & \searrow{\scriptstyle f} \\
A & \xrightarrow{\;\;\mathrm{id}\;\;} & A.
\end{array}
$$
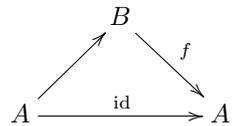
We now observe that in this situation, the fiber of f inherits the structure of an A-module. We can therefore define a functor $F' : \mathrm{Sp}(\mathrm{CAlg}_{/A}) \to \mathrm{Mod}_A$ by taking $F'(X)$ to be the fiber of f.

We now have an approach to defining the desired functor G. Namely, we first construct the functor $F' : \mathrm{Sp}(\mathrm{CAlg}_{/A}) \to \mathrm{Mod}_A$ described above. If we can prove that F' is an equivalence of ∞-categories, then we can define G' to be a homotopy inverse to F', and G to be the composition of G' with the 0th space functor $\Omega^\infty : \mathrm{Sp}(\mathrm{CAlg}_{/A}) \to \mathrm{CAlg}_{/A}$.

Our goal in this section is to flesh out the ideas sketched above. It will be convenient to work in a bit more generality: rather than only considering commutative algebras, we consider algebras over an arbitrary coherent ∞-operad. We begin with some generalities.

Definition 8.3.4.1. Let $\mathcal{O}^\otimes$ be an ∞-operad. We will say that a map $q : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ is a *stable $\mathcal{O}$-monoidal ∞-category* if the following conditions are satisfied:

(1) The map q is a coCartesian fibration of ∞-operads.

(2) For each object $X \in \mathcal{O}$, the fiber $\mathcal{C}_X$ is a stable ∞-category.

(3) For every morphism $\alpha \in \mathrm{Mul}_\mathcal{O}(\{X_i\}, Y)$, the associated functor $\alpha_! : \prod_i \mathcal{C}_{X_i} \to \mathcal{C}_Y$ is exact separately in each variable.

Remark 8.3.4.2. Let $\mathcal{O}^\otimes$ be an ∞-operad and let $q : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a stable $\mathcal{O}$-monoidal ∞-category. Then the ∞-category $\mathrm{Fun}_\mathcal{O}(\mathcal{O}, \mathcal{C})$ of sections of the restricted map $q_0 : \mathcal{C} \to \mathcal{O}$ is stable: this follows immediately from Proposition T.5.4.7.11.

Definition 8.3.4.3. Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad, and let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads. An *augmented $\mathcal{O}$-algebra object of* $\mathcal{C}$ is a morphism $f : A \to A_0$ in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ such that A_0 is an initial object of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. (In view of Proposition 3.2.1.8, this is equivalent to the requirement that $A_0(0) \to A_0(X)$ is q-coCartesian whenever $0 \to X$ is a morphism in $\mathcal{O}^{\otimes}$ with $0 \in \mathcal{O}^{\otimes}_{\langle 0 \rangle}$.) We let $\mathrm{Alg}^{\mathrm{aug}}_{\mathcal{O}}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}))$ spanned by the augmented $\mathcal{O}$-algebra objects in $\mathcal{C}$.

Suppose further that $\mathcal{C}^{\otimes}$ is a stable $\mathcal{O}$-monoidal ∞-category, so that $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ is stable (Remark 8.3.4.2). Let $\theta : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ denote the restriction functor. Given an augmented $\mathcal{O}$-algebra object $A \to A_0$ of $\mathcal{C}$, we define the *augmentation ideal* to the the fiber of the induced morphism $\theta(A) \to \theta(A_0)$. The formation of augmentation ideals determines a functor

$$\mathrm{Alg}^{\mathrm{aug}}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C}).$$

Remark 8.3.4.4. Let $\mathcal{O}^{\otimes}$ be a small ∞-operad and let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a presentable $\mathcal{O}$-monoidal ∞-category. It follows from Proposition T.5.4.7.11 that the ∞-category $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ is presentable, and that for each object $X \in \mathcal{O}$ the evaluation functor $e_X : \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ preserves small limits and small colimits. It follows from Corollary T.5.5.2.9 that e_X admits both a left and a right adjoint, which we will denote by $(e_X)_!$ and $(e_X)_*$.

The following result characterizes the augmentation ideal functor by a universal property:

Proposition 8.3.4.5. *Let $\mathcal{O}^{\otimes}$ be a small unital ∞-operad and let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a presentable stable $\mathcal{O}$-monoidal ∞-category. Let $0_{\mathcal{C}}$ denote a zero object of the stable ∞-category $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$, and let $1_{\mathcal{C}}$ denote an initial object of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. Then there exists a pair of adjoint functors $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C}) \underset{g}{\overset{f}{\rightleftarrows}} \mathrm{Alg}^{\mathrm{aug}}_{\mathcal{O}}(\mathcal{C})$ with the following properties:*

(1) *The functor f is given by composition*

$$\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C}) \simeq \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})^{/0_{\mathcal{C}}} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{/1_{\mathcal{C}}} \simeq \mathrm{Alg}^{\mathrm{aug}}_{\mathcal{O}}(\mathcal{C}),$$

where the middle map is induced by a left adjoint F to the forgetful functor $G : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$. Here we implicitly invoke the identification $1_{\mathcal{C}} \simeq F(0_{\mathcal{C}})$; note that the existence of F follows from Corollary 3.1.3.5.

(2) *The functor $g : \mathrm{Alg}^{\mathrm{aug}}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ is the augmentation ideal functor.*

(3) *Let X and Y be objects of $\mathcal{O}^{\otimes}$, and let $(e_X)_! : \mathcal{C}_X \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ and $e_Y : \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ be as in Remark 8.3.4.4. Then the composition*

$$\mathcal{C}_X \overset{(e_X)_!}{\longrightarrow} \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C}) \overset{f}{\longrightarrow} \mathrm{Alg}^{\mathrm{aug}}_{\mathcal{O}}(\mathcal{C}) \overset{g}{\longrightarrow} \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C}) \overset{e_Y}{\longrightarrow} \mathcal{C}_Y$$

is equivalent to the functor $C \mapsto \coprod_{n>0} \mathrm{Sym}^n_{\mathcal{O},Y}(C)$ (see Construction 3.1.3.7).

Proof. The existence of the functor g and assertion (2) follow from Proposition T.5.2.5.1, together with the definition of the augmentation ideal functor. Invoking (2), we deduce that there is a fiber sequence

$$g \circ f \to G \circ F \overset{h}{\to} \underline{G}(1_{\mathcal{C}})$$

in the stable ∞-category of functors from $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ to itself, where $\underline{G}(1_{\mathcal{C}})$ denotes the constant functor taking the value $G(1_{\mathcal{C}})$. Theorem The results of §3.1.3 guarantee that $e_Y \circ G \circ F \circ (e_X)_!$ can be idenitifed with the functor $\coprod_{n \geq 0} \mathrm{Sym}^n_{\mathcal{O},Y}$. We observe that the map h is split by the inclusion $\mathrm{Sym}^0_{\mathcal{O},Y} \to \coprod_{n \geq 0} \mathrm{Sym}^n_{\mathcal{O},Y}$, so that we obtain an identification of $g \circ f$ with the complementary summand $\coprod_{n>0} \mathrm{Sym}^n_{\mathcal{O},Y}$. $\square$

Remark 8.3.4.6. Let $\operatorname{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C}) \underset{g}{\overset{f}{\rightleftarrows}} \operatorname{Alg}_{\mathcal{O}}^{\operatorname{aug}}(\mathcal{C})$ be as in Proposition 8.3.4.5, and let $X, Y \in \mathcal{O}$. Unwinding the definitions, we see that the unit map $\operatorname{id} \to g \circ f$ induces a functor $e_Y \circ (e_X)_! \to e_Y \circ g \circ f \circ (e_X)_!$. This can be identified with the inclusion of the summand $\operatorname{Sym}_{\mathcal{O},Y}^1 \to \coprod_{n>0} \operatorname{Sym}_{\mathcal{O},Y}^n$.

The main result of this section is the following:

Theorem 8.3.4.7. *Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad and let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a stable $\mathcal{O}$-monoidal ∞-category. Then the augmentation ideal functor $G : \operatorname{Alg}_{\mathcal{O}}^{\operatorname{aug}}(\mathcal{C}) \to \operatorname{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ induces an equivalence of ∞-categories $\operatorname{Sp}(\operatorname{Alg}_{\mathcal{O}}^{\operatorname{aug}}(\mathcal{C})) \to \operatorname{Sp}(\operatorname{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})) \simeq \operatorname{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$.*

The proof of Theorem 8.3.4.7 will use some ideas from the calculus of functors.

Lemma 8.3.4.8. *Let K be a simplicial set. Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits, and let $\mathcal{D}$ be stable ∞-category which admits sequential colimits and K-indexed colimits. Then the derivative functor $\partial : \operatorname{Fun}_*(\mathcal{C}, \mathcal{D}) \to \operatorname{Exc}(\mathcal{C}, \mathcal{D})$ preserves K-indexed colimits.*

Proof. Since $\mathcal{D}$ is stable, the loop functor $\Omega_{\mathcal{D}}$ is an equivalence of ∞-categories. It follows that $\Omega_{\mathcal{D}}$ preserves K-indexed colimits. We observe that $\operatorname{Exc}(\mathcal{C}, \mathcal{D})$ is the full subcategory of $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which are right exact; it follows that $\operatorname{Exc}(\mathcal{C}, \mathcal{D})$ is stable under K-indexed colimits in $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$. Similarly, $\operatorname{Fun}_*(\mathcal{C}, \mathcal{D})$ is stable under K-indexed colimits in $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$; we therefore conclude that K-indexed colimits in $\operatorname{Fun}_*(\mathcal{C}, \mathcal{D})$ and $\operatorname{Exc}(\mathcal{C}, \mathcal{D})$ are computed pointwise. The desired result now follows from the formula for computing the derivative given in Example 7.1.1.28 $\qquad\square$

Remark 8.3.4.9. Let $\mathcal{C} \to \mathcal{O}$ be a presentable fibration of ∞-categories, where $\mathcal{O}$ is small. For each $X \in \mathcal{O}$, let $(e_X)_!$ denote a left adjoint to the evaluation functor $e_X : \operatorname{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C}) \to \mathcal{C}_X$. Then the essential images of the functors $(e_X)_!$ generate the ∞-category $\mathcal{D} = \operatorname{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ under small colimits. To prove this, let $\mathcal{D}_0$ denote the smallest full subcategory of $\mathcal{D}$ containing the essential image of each $(e_X)_!$ and closed under small colimits in $\mathcal{D}$. Since the essential image of each $(e_X)_!$ is generated under small colimits by a small collection of objects, we deduce that $\mathcal{D}_0 \subseteq \mathcal{D}$ is presentable. Let D be an object of $\mathcal{D}$; we wish to prove that $D \in \mathcal{D}_0$. Let $\chi : \mathcal{D}^{op} \to \mathcal{S}$ be the functor represented by D. The composite functor

$$\chi|\,\mathcal{D}_0 : \mathcal{D}_0^{op} \to \mathcal{D}^{op} \to \mathcal{S}$$

preserves small limits, and is therefore representable by an object $D_0 \in \mathcal{D}_0$ (Proposition T.5.5.2.2). We therefore obtain a map $f : D_0 \to D$ which exhibits D_0 as a $\mathcal{D}_0$-colocalization of D. In particular, for each $X \in \mathcal{O}$ and each $C \in \mathcal{C}_X$, composition with f induces a homotopy equivalence

$$\operatorname{Map}_{\mathcal{C}_X}(C, D_0(X)) \simeq \operatorname{Map}_{\mathcal{D}}((e_X)_!(C), D_0) \to \operatorname{Map}_{\mathcal{D}}((e_X)_!(C), D) \simeq \operatorname{Map}_{\mathcal{C}_X}(C, D(X)).$$

This proves that $e_X(f)$ is an equivalence for each $X \in \mathcal{O}$, so that f is an equivalence and $D \in \mathcal{D}_0$ as required.

Proposition 8.3.4.10. *Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad, and let $\mathcal{C}^{\otimes}$ be a presentable stable $\mathcal{O}$-monoidal ∞-category. Let $G : \operatorname{Alg}_{\mathcal{O}}^{\operatorname{aug}}(\mathcal{C}) \to \operatorname{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ be the augmentation ideal functor, and let F be a left adjoint to G. Then the unit map $\operatorname{id} \to GF$ induces an equivalence of derivatives $\alpha : \partial \operatorname{id} \to \partial(GF)$.*

Proof. We wish to show that for every object $M \in \operatorname{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$, the natural transformation α induces an equivalence

$$\alpha_M : M \simeq \partial(\operatorname{id})(M) \to \partial(GF)(M).$$

Since both sides are compatible with the formation of colimits in M, it will suffice to prove this in the case where $M = (e_X)_!(C)$ for some $X \in \mathcal{O}$ and some $C \in \mathcal{C}_X$ (Remark 8.3.4.9). Moreover, to prove that α_M is an equivalence, it suffices to show that $e_Y(\alpha_M)$ is an equivalence in $\mathcal{C}_Y$, for each $Y \in \mathcal{O}$. In other words, it suffices to show that α induces an equivalence

$$\beta : e_Y \circ (e_X)_! \to e_Y \circ \partial(GF) \circ (e_X)_!.$$

Since the functors $(e_X)_!$ and e_Y are exact, we can identify the latter composition with $\partial(e_Y \circ G \circ F \circ (e_X))_!$ (Corollary 7.2.1.24).

According to Proposition 8.3.4.5, the functor $e_Y \circ G \circ F \circ (e_X)_!$ can be identified with the total symmetric power functor $C \mapsto \coprod_{n>0} \mathrm{Sym}^n_{\mathcal{O},Y}(C)$. According to Remark 8.3.4.6, we can express this as the coproduct of $e_Y \circ (e_X)_!$ with the functor T given by the formula $T(C) \simeq \coprod_{n \geq 2} \mathrm{Sym}^n_{\mathcal{O},Y}(C)$. In view of Lemma 8.3.4.8, it will suffice to show that each of the derivatives $\partial \mathrm{Sym}^n_{\mathcal{O},Y}$ is nullhomotopic for $n \geq 2$. We observe that $\mathrm{Sym}^n_{\mathcal{O},Y}$ can be expressed as a colimit of functors of the form

$$\mathcal{C}_X \xrightarrow{\delta} \mathcal{C}^n_X \xrightarrow{\gamma_!} \mathcal{C}_Y$$

where $\gamma_!$ denotes the functor associated to an operation $\gamma \in \mathrm{Mul}_{\mathcal{O}}(\{X\}_{1 \leq i \leq n})$. In view of Lemma 8.3.4.8, it suffices to show that each constituent $\partial(\delta \circ \gamma_!)$ is nullhomotopic, which follows from Proposition 7.1.3.10. □

Lemma 8.3.4.11. *Let $\mathcal{C}$ be a stable ∞-category, let $f : C \to D$ be a morphism in $\mathcal{C}$, and let $f^* : \mathcal{C}^{/D} \to \mathcal{C}^{/C}$ be the functor given by pullback along f. Then:*

(1) *The functor f^* is conservative.*

(2) *Let K be a weakly contractible simplicial set, and assume that $\mathcal{C}$ admits K-indexed colimits. Then the functor f^* preserves K-indexed colimits.*

Proof. Let $\mathcal{E}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}) \times_{\mathrm{Fun}(\{1\} \times \Delta^1, \mathcal{C})} \{f\}$ spanned by the pullback diagrams

$$
\begin{array}{ccc}
C' & \longrightarrow & D' \\
\downarrow & & \downarrow \\
C & \xrightarrow{\ f\ } & D.
\end{array}
$$

Since $\mathcal{C}$ admits pullbacks, Proposition T.4.3.2.15 implies that evaluation along $\Delta^1 \times \{1\}$ induces a trivial Kan fibration $\mathcal{E} \to \mathcal{C}^{/D}$. Let g denote a section of this trivial fibration. Then the functor f^* can be identified with the composition

$$\mathcal{C}^{/D} \xrightarrow{g} \mathcal{E} \xrightarrow{g'} \mathcal{C}^{/C},$$

where g' is given by evaluation along $\Delta^1 \times \{0\}$.

Let u be a morphism in $\mathcal{C}^{/D}$. Let σ denote the fiber of the morphism $g(u)$, formed in the stable ∞-category $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$. Then σ is a pullback diagram

$$
\begin{array}{ccc}
W & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Z
\end{array}
$$

in the ∞-category $\mathcal{C}$. The objects Y and Z are both zero, so the bottom horizontal map is an equivalence. It follows that the upper horizontal map is an equivalence. If $f^*(u)$ is an equivalence, then $W \simeq 0$. It follows that $X \simeq 0$, so that u is an equivalence in $\mathcal{C}^{/D}$. This completes the proof of (1).

To prove (2), let us choose a colimit diagram $\overline{p} : K^{\triangleright} \to \mathcal{C}^{/D}$. Let $\overline{q} = g \circ \overline{p}$. We wish to prove that $g' \circ \overline{q}$ is a colimit diagram in $\mathcal{C}^{/C}$. In view of Proposition T.1.2.13.8, it will suffice to show that $\overline{q}$ defines a colimit diagram in $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$. Let $q = \overline{q}|K$, and let $\sigma \in \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ be a colimit of q in $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$. Since the class of pushout diagrams in $\mathcal{C}$ is stable under colimits, we conclude that σ is a pushout diagram. Let σ' be the image under $\overline{q}$ of the cone point of $K^{\triangleright}$, let $\alpha : \sigma \to \sigma'$ be the map determined by $\overline{q}$, and let

$\tau \in \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ be the cofiber of α. We wish to prove that α is an equivalence, which is equivalent to the assertion that $\tau \simeq 0$. We may view τ as a pushout diagram

$$
\begin{array}{ccc}
W & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Z
\end{array}
$$

in $\mathcal{C}$. Since $\mathcal{C}$ is stable, this diagram is also a pullback. Consequently, it will suffice to show that the objects $X, Y, Z \in \mathcal{C}$ are equivalent to zero. For the object X, this follows from our assumption that $\overline{p}$ is a colimit diagram (and Proposition T.1.2.13.8). To show that Y and Z are zero, it suffices to observe that every constant map $K^{\triangleright} \to \mathcal{C}$ is a colimit diagram, because K is weakly contractible (Corollary T.4.4.4.10). $\qquad\square$

Lemma 8.3.4.12. *Suppose given an adjunction of ∞-categories*

$$
\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}
$$

where $\mathcal{C}$ is stable. Let C be an object of $\mathcal{C}$, and consider the induced adjunction

$$
\mathcal{C}_{/C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}_{/FC}
$$

(see Proposition T.5.2.5.1). Then:

(1) *If the functor G is conservative, then g is conservative.*

(2) *Let K be a weakly contractible simplicial set. Assume that $\mathcal{C}$ and $\mathcal{D}$ admit K-indexed colimits, that the functor G preserves K-indexed colimits, and that $\mathcal{C}$ is stable. Then the ∞-categories $\mathcal{D}^{/FC}$ and $\mathcal{C}^{/C}$ admit K-indexed colimits, and the functor g preserves K-indexed colimits.*

Proof. We first prove (1). Proposition T.5.2.5.1 shows that g can be written as a composition

$$
\mathcal{D}_{/FC} \overset{g'}{\to} \mathcal{C}_{/GFC} \overset{g''}{\to} \mathcal{C}_{/C},
$$

where g' is induced by G and g'' is given by pullback along the unit map $C \to GFC$. It will therefore suffice to show that g' and g'' are conservative. We have a commutative diagram of ∞-categories

$$
\begin{array}{ccc}
\mathcal{D}^{/FC} & \overset{g'}{\longrightarrow} & \mathcal{C}^{/GFC} \\
\downarrow & & \downarrow \\
\mathcal{D} & \overset{G}{\longrightarrow} & \mathcal{C}.
\end{array}
$$

Since the vertical functors detect equivalences and G is conservative, we deduce that g' is conservative. It follows from Lemma T.5.2.8.22 that g'' is conservative as well.

We now prove (2). Proposition T.1.2.13.8 implies that the ∞-categories $\mathcal{C}^{/C}$, $\mathcal{C}^{/GFC}$, and $\mathcal{D}^{/FC}$ admit K-indexed colimits. Consequently, it will suffice to show that g' and g'' preserve K-indexed colimits. For the functor g', this follows from Proposition T.1.2.13.8 and our assumption that G preserves K-indexed colimits. For the functor g'', we invoke Lemma 8.3.4.11. $\qquad\square$

Proof of Theorem 8.3.4.7. Enlarging the universe if necessary, we may suppose that $\mathcal{O}^{\otimes}$ and $\mathcal{C}^{\otimes}$ are small. The coCartesian fibration $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is classified by a map of ∞-operads $\chi : \mathcal{O}^{\otimes} \to \mathrm{Cat}_{\infty}^{\times}$. Let χ' denote the composition of χ with the ∞-operad map $\mathrm{Ind} : \mathrm{Cat}_{\infty}^{\times} \to \widehat{\mathrm{Cat}}_{\infty}^{\times}$ determined by Remark 6.3.1.8, and let

$\mathcal{C}'^{\otimes} \to \mathcal{O}^{\otimes}$ be the $\mathcal{O}$-monoidal ∞-category classified by χ'. Then we have a fully faithful functor $\mathcal{C}^{\otimes} \to \mathcal{C}'^{\otimes}$ which induces a homotopy pullback diagram

$$
\begin{array}{ccc}
\mathrm{Alg}_{\mathcal{O}}^{\mathrm{aug}}(\mathcal{C}) & \longrightarrow & \mathrm{Alg}_{\mathcal{O}}^{\mathrm{aug}}(\mathcal{C}') \\
\downarrow & & \downarrow \\
\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C}) & \longrightarrow & \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C}')
\end{array}
$$

where the horizontal maps are fully faithful inclusions. Passing to stable envelopes, we get a homotopy pullback diagram

$$
\begin{array}{ccc}
\mathrm{Sp}(\mathrm{Alg}_{\mathcal{O}}^{\mathrm{aug}}(\mathcal{C})) & \longrightarrow & \mathrm{Sp}(\mathrm{Alg}_{\mathcal{O}}^{\mathrm{aug}}(\mathcal{C}') \\
\downarrow & & \downarrow \\
\mathrm{Sp}(\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})) & \longrightarrow & \mathrm{Sp}(\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C}')).
\end{array}
$$

It will therefore suffice to show that the right vertical map is an equivalence. In other words, we may replace $\mathcal{C}^{\otimes}$ by $\mathcal{C}'^{\otimes}$ and thereby reduce to the case where $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a presentable stable $\mathcal{O}$-monoidal ∞-category.

The forgetful functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{C})$ is conservative (Lemma 3.2.2.6) and preserves geometric realizations of simplicial objects (Proposition3.2.3.1). It follows from Lemma 8.3.4.12 that G has the same properties. Using Theorem 6.2.2.5, we deduce that G exhibits $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{aug}}(\mathcal{C})$ as monadic over $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ (see Definition 6.2.2.4). The desired result now follows by combining Proposition 8.3.4.10 with Corollary 7.2.2.17. $\qquad\square$

In the special case where the ∞-operad $\mathcal{O}^{\otimes}$ is coherent, we can use Theorem 8.3.4.7 to describe other fibers of the tangent bundle of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$:

Theorem 8.3.4.13. *Let $\mathcal{O}^{\otimes}$ be a coherent ∞-operad, let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a stable $\mathcal{O}$-monoidal ∞-category, and let $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ be a $\mathcal{O}$-algebra object of $\mathcal{C}$. Then the ∞-category $\mathrm{Sp}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/A})$ is canonically equivalent to $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}))$.*

Corollary 8.3.4.14. *Let A be an $\mathbb{E}_{\infty}$-ring. There is a canonical equivalence of ∞-categories*

$$
\mathrm{Sp}(\mathrm{CAlg}_{/A}) \simeq \mathrm{Mod}_A.
$$

Remark 8.3.4.15. In the situation of Theorem 8.3.4.13, we have an evident functor

$$
\Omega^{\infty} : \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})) \simeq \mathrm{Sp}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/A}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/A}.
$$

This functor associates to each $M \in \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}))$ a commutative algebra object which we will denote by $A \oplus M$. The proof of Theorem 8.3.4.13 will justify this notation; that is, we will see that when regarded as an object of $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$, $A \oplus M$ can be canonically identified with the coproduct of A and M.

Proof. The desired equivalence is given by the composition

$$
\begin{array}{rcl}
\mathrm{Sp}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/A}) & \simeq & \mathrm{Sp}((\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/A})_{A/}) \\
& \simeq & \mathrm{Sp}((\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{A/})_{/A}) \\
& \overset{\phi}{\simeq} & \mathrm{Sp}(\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}))_{/A} \\
& \simeq & \mathrm{Sp}(\mathrm{Alg}_{\mathcal{O}}^{\mathrm{aug}}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}))) \\
& \overset{\phi'}{\simeq} & \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})).
\end{array}
$$

Here ϕ is the equivalence of Corollary 3.4.1.7, ϕ' is given by Proposition 8.3.4.7. $\qquad\square$

Remark 8.3.4.16. Let $\mathcal{C}^{\otimes}$ be a stable symmetric monoidal ∞-category (such that the tensor product on $\mathcal{C}$ is exact in each variable) let A be a commutative algebra object of $\mathcal{C}$, let M be an A-module, and let $A \oplus M$ denote the image of M under the composition

$$\mathrm{Mod}_A(\mathcal{C}) \simeq \mathrm{Sp}(\mathrm{CAlg}(\mathcal{C})_{/A}) \xrightarrow{\Omega^{\infty}} \mathrm{CAlg}(\mathcal{C})_{/A}.$$

We claim that the algebra structure on $A \oplus M$ is "square-zero" in the homotopy category $h\mathcal{C}$. In other words:

(1) The unit map $1_{\mathcal{C}} \to A \oplus M$ is homotopic to the composition of $1_{\mathcal{C}} \to A$ with the inclusion $A \to A \oplus M$.

(2) The multiplication

$$m : (A \otimes A) \oplus (A \otimes M) \oplus (M \otimes A) \oplus (M \otimes M) \simeq (A \oplus M) \otimes (A \oplus M) \to A \oplus M$$

is given as follows:

   (i) On the summand $A \otimes A$, the map m is homotopic to the composition of the multiplication map $A \otimes A \to A$ with the inclusion $A \to A \oplus M$.

   (ii) On the summands $A \otimes M$ and $M \otimes A$, the map m is given by composing the action of A on M with the inclusion $M \to A \oplus M$.

   (iii) On the summand $M \otimes M$, the map m is nullhomotopic.

Only assertion (iii) requires proof. For this, we will invoke the fact that the commutative algebra structure on $A \oplus M$ depends functorially on M. Consequently, for every A-module N we obtain a map $\psi_N : N \otimes N \to N$, which we must show to be nullhomotopic. Let M' and M'' be copies of the A-module M, which we will distinguish notationally for clarity, and let $f : M' \oplus M'' \to M$ denote the "fold" map which is the identity on each factor. Invoking the functoriality of ψ, we deduce that the map $\psi_M : M \otimes M \to M$ factors as a composition

$$M \otimes M = M' \otimes M'' \to (M' \oplus M'') \otimes (M' \oplus M'') \xrightarrow{\psi_{M' \oplus M''}} M' \oplus M'' \xrightarrow{f} M.$$

Consequently, to prove that ψ_M is nullhomotopic, it will suffice to show that $\phi = \psi_{M' \oplus M''}|(M' \otimes M'')$ is nullhomotopic. Let $\pi_{M'} : M' \oplus M'' \to M'$ and $\pi_{M''} : M \oplus M'' \to M''$ denote the projections onto the first and second factor, respectively. To prove that $\psi_{M' \oplus M''}$ is nullhomotopic, it suffices to show that $\pi_{M'} \circ \phi$ and $\pi_{M''} \circ \phi$ are nullhomotopic. We now invoke functoriality once more to deduce that $\pi_{M'} \circ \phi$ is homotopic to the composition

$$M' \otimes M'' \xrightarrow{(\mathrm{id}, 0)} M' \otimes M' \xrightarrow{\psi_{M'}} M'.$$

This composition is nullhomotopic, since the first map factors through $M' \otimes 0 \simeq 0$. The same argument shows that $\pi_{M''} \circ \phi$ is nullhomotopic, as desired.

Remark 8.3.4.17. Let A be an $\mathbb{E}_{\infty}$-ring, let M be an A-module, and let $A \oplus M$ denote the corresponding square-zero extension. As a graded abelian group, we may identify $\pi_*(A \oplus M)$ with the direct sum $(\pi_* A) \oplus (\pi_* M)$. It follows from Remark 8.3.4.16 that the multiplication on $\pi_*(A \oplus M)$ is given on homogeneous elements by the formula

$$(a, m)(a', m') = (aa', am' + (-1)^{\deg(a') \deg(m)} a' m).$$

In particular, if A is an ordinary commutative ring (viewed as a discrete $\mathbb{E}_{\infty}$-ring) and M is an ordinary A-module, then we can identify the discrete $\mathbb{E}_{\infty}$-ring $A \oplus M$ with the classical square-zero extension discussed in the introduction to this section.

We now prove a "global" version of Theorem 8.3.4.13:

Theorem 8.3.4.18. *Let $\mathcal{O}^{\otimes}$ be a coherent ∞-operad, and let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a presentable stable $\mathcal{O}$-monoidal ∞-category. Then there is a canonical equivalence*

$$\phi : T_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \times_{\mathrm{Fun}(\mathcal{O}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}))} \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathrm{Mod}^{\mathcal{O}}(\mathcal{C}))$$

of presentable fibrations over $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$.

In other words, we may view $T_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}$ as the ∞-category whose objects are pairs (A, M), where A is a $\mathcal{O}$-algebra object of $\mathcal{C}$ and M is an A-module. The idea of the proof is simple: we will define ϕ using a relative version of the augmentation ideal functor defined above. We will then show that ϕ is a map of Cartesian fibrations, so that the condition that ϕ be an equivalence can be checked fibrewise. We are then reduced to the situation of Theorem 8.3.4.13.

Proof. We will denote objects of $\mathcal{M} = \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \times_{\mathrm{Fun}(\mathcal{O}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}))} \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathrm{Mod}^{\mathcal{O}}(\mathcal{C}))$ by pairs (A, M), where $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ and $M \in \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathrm{Mod}^{\mathcal{O}}(\mathcal{C})$ is a module over A.

Let $\mathcal{E} = \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})) \times_{\mathrm{Fun}(\Delta^2, \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}))} \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ denote the ∞-category of diagrams of the form

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
{\scriptstyle \mathrm{id}}\downarrow & {\scriptstyle \mathrm{id}}\searrow \quad \downarrow & \\
A & \underset{\mathrm{id}}{\longrightarrow} & A,
\end{array}
$$

of $\mathcal{O}$-algebra objects of $\mathcal{C}$. The canonical map $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}^{\mathcal{O}}(\mathcal{C}))$ determines a section s of the projection

$$p : \mathcal{X} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}),$$

which we can think of informally as assigning to an algebra A the pair (A, A) where we regard A as a module over itself.

Let $\mathcal{D}$ denote the fiber product

$$\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{M}) \times_{\mathrm{Fun}(\Delta^1 \times \{1\}, \mathcal{M})} \mathrm{Fun}(\Delta^1 \times \{1\}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})),$$

so that we can identify objects of $\mathcal{D}$ with commutative squares

$$
\begin{array}{ccc}
(A, M) & \longrightarrow & (B, B) \\
\downarrow & & \downarrow \\
(A', M') & \longrightarrow & (B', B')
\end{array}
$$

in the ∞-category $\mathcal{M}$. Let $\overline{\mathcal{E}}$ denote the full subcategory of $\mathcal{E} \times_{\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}))} \mathcal{D}$ spanned by those squares

$$
\begin{array}{ccc}
(A, M) & \longrightarrow & (B, B) \\
\downarrow & & \downarrow \\
(A, M') & \longrightarrow & (A, A)
\end{array}
$$

which are p-limit diagrams, and such that M' is a zero object of $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C}))$. Invoking Proposition T.4.3.2.15 twice (and Theorem 3.4.3.1), we deduce that the projection map $\overline{\mathcal{E}} \to \mathcal{E}$ is a trivial Kan fibration. Let $r : \mathcal{E} \to \overline{\mathcal{E}}$ be a section of this projection, and let $r' : \overline{\mathcal{E}} \to \mathcal{X}$ be given by evaluation in the upper left hand corner. Let ψ denote the composition

$$\psi : \mathcal{E} \xrightarrow{r} \overline{\mathcal{E}} \xrightarrow{r'} \mathcal{M},$$

so that ψ carries a diagram

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\text{id} \downarrow & {\scriptstyle \text{id}} \searrow & \downarrow f \\
A & \underset{\text{id}}{\longrightarrow} & A,
\end{array}
$$

to the augmentation ideal $\mathrm{fib}(f)$, regarded as an A-module.

We observe that the restriction map $\mathcal{E} \to \mathrm{Fun}(\Delta^1 \times \{1\}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}))$ can be regarded as a pointed envelope of the presentable fibration

$$\mathrm{Fun}(\Delta^1 \times \{1\}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})) \to \mathrm{Fun}(\{1\} \times \{1\}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}).$$

Let $\Omega^{\infty}_* : T_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})} \to \mathcal{E}$ exhibit $T_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}$ as a tangent bundle to $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. Let ϕ denote the composition

$$T_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})} \xrightarrow{\Omega^{\infty}} \mathcal{E} \xrightarrow{\psi} \mathcal{M}.$$

To complete the proof, it will suffice to show that ϕ is an equivalence of ∞-categories.

By construction, we have a commutative diagram

$$
\begin{array}{ccccc}
T_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})} & \xrightarrow{\Omega^{\infty}_*} & \mathcal{E} & \xrightarrow{\phi_0} & \mathcal{M} \\
& {\scriptstyle q} \searrow & \downarrow {\scriptstyle q'} & {\scriptstyle q''} \swarrow & \\
& & \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}), & &
\end{array}
$$

with $\phi = \phi_0 \circ \Omega^{\infty}_*$, where q, q', and q'' are presentable fibrations. Since Ω^{∞}_* is a right adjoint relative to $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$, it carries q-Cartesian morphisms to q'-Cartesian morphisms. We observe that ϕ_0 carries q'-Cartesian morphisms to q''-Cartesian morphisms; in concrete terms, this merely translates into the observation that every pullback diagram

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
f \downarrow & & \downarrow f' \\
A' & \longrightarrow & B'
\end{array}
$$

in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is also a pullback diagram in $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ (Corollary 3.2.2.5), and therefore induces an equivalence $\mathrm{fib}(f) \simeq \mathrm{fib}(f')$ in $\mathcal{M}$. It follows that ϕ carries q-Cartesian morphisms to q''-Cartesian morphisms.

We now invoke Corollary T.2.4.4.4: the map ϕ is an equivalence of ∞-categories if and only if, for every commutative algebra object $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$, the induced map

$$\phi_A : \mathrm{Sp}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/A}) \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C}))$$

is an equivalence of ∞-categories. We now observe that ϕ_A can be identified with the augmentation ideal functor which appears in the proof of Theorem 8.3.4.13, and therefore an equivalence as required. $\qquad \square$

8.3.5 The Cotangent Complex of an $\mathbb{E}_k$-Algebra

Let $k \to A$ be a map of commutative rings. The multiplication map $A \otimes_k A \to A$ is a surjection whose kernel is an ideal $I \subseteq A \otimes_k A$. The quotient I/I^2 is an A-module, and there is a canonical k-linear derivation $d : A \to I/I^2$, which carries an element $a \in A$ to the image of $(a \otimes 1 - 1 \otimes a) \in I$. In fact, this derivation is *universal*: for any A-module M, composition with d induces a bijection $\mathrm{Hom}_A(I/I^2, M) \to \mathrm{Der}_k(A, M)$. In other words, the quotient I/I^2 can be identified with the module of Kähler differentials $\Omega_{A/k}$.

The above analysis generalizes in a straightforward way to the setting of associative algebras. Assume that k is a commutative ring and that A is an associative k-algebra. Let M be an A-bimodule (in the

category of k-modules: that is, we require $\lambda m = m\lambda$ for $m \in M$ and $\lambda \in k$). A *k-linear derivation* from A into M is a k-linear map $d : A \to M$ satisfying the Leibniz formula $d(ab) = d(a)b + ad(b)$. If we let I denote the kernel of the multiplication map $A \otimes_k A \to A$, then I has the structure of an A-bimodule, and the formula $d(a) = a \otimes 1 - 1 \otimes a$ defines a derivation from A into M. This derivation is again universal in the following sense:

($*$) For any bimodule M, composition with d induces a bijection of $\mathrm{Hom}(I, M)$ with the set of k-linear derivations from I into M.

If A is commutative, then I/I^2 is the universal A-module map which receives an A-bimodule homomorphism from I. Consequently, ($*$) can be regarded as a generalization of the formula $\Omega_{A/k} \simeq I/I^2$.

Our goal in this section is to obtain an ∞-categorical analogue of assertion ($*$). Rather than than working in the ordinary abelian category of k-modules, we will work with a symmetric monoidal stable ∞-category $\mathcal{C}$. In this case, we can consider algebra objects $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ for any coherent ∞-operad $\mathcal{O}^{\otimes}$. According to Theorem 8.3.4.18, we can identify $\mathrm{Sp}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/A})$ with the stable ∞-category $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$ of $\mathcal{O}$-algebra objects of $\mathcal{C}$. In particular, the absolute cotangent complex L_A can be identified an object of $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$. Our goal is to obtain a concrete description of L_A in the special case where $\mathcal{O}^{\otimes} = \mathbb{E}_k^{\otimes}$ is the ∞-operad of little k-cubes.

To motivate the description, let us consider first the case where $k = 1$. In this case, we can identify $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$ with the ∞-category of A-bimodule objects of $\mathcal{C}$ (see Theorem 4.3.4.28). Motivated by assertion ($*$), we might suppose that L_A can be identified with the fiber of the multiplication map $A \otimes A \to A$ (which we regard as a map of A-bimodules). The domain of this map is the *free* A-bimodule, characterized up to equivalence by the existence of a morphism $e : \mathbf{1} \to A \otimes A$ with the property that it induces homotopy equivalences

$$\mathrm{Map}_{\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})}(A \otimes A, M) \to \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, M)$$

(here and in what follows, we will identify A-module objects of $\mathcal{C}$ with their images in $\mathcal{C}$).

Assume now that $k \geq 0$ is arbitrary, that $\mathcal{C}$ is presentable, and that the tensor product on $\mathcal{C}$ preserves colimits separately in each variable. The forgetful functor $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ preserves small limits and colimits (Corollaries 3.4.3.2 and 3.4.4.6), and therefore admits a left adjoint $\mathrm{Free} : \mathcal{C} \to \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$ (Corollary T.5.5.2.9). We can formulate our main result as follows:

Theorem 8.3.5.1. *Let $\mathcal{C}^{\otimes}$ be a stable symmetric monoidal ∞-category and let $k \geq 0$. Assume that $\mathcal{C}$ is presentable and that the tensor product operation on $\mathcal{C}$ preserves colimits separately in each variable. For every $\mathbb{E}_k$-algebra object $A \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$, there is a canonical fiber sequence*

$$\mathrm{Free}(\mathbf{1}) \to A \to L_A[k]$$

in the stable ∞-category $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$. Here $\mathrm{Free} : \mathcal{C} \to \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$ denotes the free functor described above, and the map of A-modules $\mathrm{Free}(\mathbf{1}) \to A$ is determines by the unit map $\mathbf{1} \to A$ in the ∞-category $\mathcal{C}$.

Remark 8.3.5.2. A version of Theorem 8.3.5.1 is proven in [53].

Remark 8.3.5.3. If A is an $\mathbb{E}_k$-algebra object of $\mathcal{C}$, then we can think of an A-module $M \in \mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})$ as an object of $\mathcal{C}$ equipped with a commuting family of (left) actions of A parametrized by the $(k-1)$-sphere of rays in the Euclidean space $\mathbf{R}^k$ which emanate from the origin. This is equivalent to the action of a single associative algebra object of $\mathcal{C}$: namely, the topological chiral homology $\int_{S^{k-1}} A$ (see the discussion at the end of §5.3.3). The free module $\mathrm{Free}(\mathbf{1})$ can be identified with $\int_{S^{k-1}} A$ itself.

An equivalent formulation of Theorem 8.3.5.1 asserts the existence of a fiber sequence of A-modules

$$L_A \xrightarrow{\theta} \Omega^{k-1} \mathrm{Free}(\mathbf{1}) \xrightarrow{\theta'} \Omega^{k-1} A.$$

In particular, the map θ classifies a derivation d of A into $\Omega^{k-1} \mathrm{Free}(\mathbf{1})$. Informally, this derivation is determined by pairing the canonical S^{k-1}-parameter family of maps $A \to \int_{S^{k-1}} A$ with the fundamental class of S^{k-1}. Because the induced family of composite maps $A \to \int_{S^{k-1}} A \to A$ is constant, this derivation lands in the fiber of the map θ'. When $k = 1$, we can identify $F(\mathbf{1})$ with the tensor product $A \otimes A$, and our heuristic recovers the classical formula $d(a) = a \otimes 1 - 1 \otimes a$.

Remark 8.3.5.4. In the statement of Theorem 8.3.5.1, the shift $L_A[k]$ can be identified with the tensor product of L_A with the pointed space S^k, regarded as the one-point compactification of the Euclidean space $\mathbf{R}^k$. With respect to this identification, the fiber sequence of Theorem 8.3.5.1 can be constructed so as to be equivariant with respect to the group of self-homeomorphisms of $\mathbf{R}^k$ (which acts on the ∞-operad $\mathbb{E}_k$ up to coherent homotopy, as explained in §5.2.2). However, this equivariance is not apparent from the construction we present below.

We now explain how to deduce Theorem 8.3.5.1 from Theorem 6.3.7.5. Fix an $\mathbb{E}_k$-algebra $A \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$, and let $\mathcal{E} = \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})_{A/}$. Consider the functors $X, Y, Z : \mathcal{E} \to \mathcal{S}_*$ given informally by the formulas

$$X(f : A \to B) = \Omega^n \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_k}}(A, B) \quad Y(f : A \to B) = \mathfrak{Z}_{\mathbb{E}_k}(f)^\times \quad Z(f : A \to B) = B^\times.$$

Theorem 6.3.7.5 implies that these functors fit into a pullback diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & Z, \end{array}$$

where $* : \mathcal{E} \to \mathcal{S}_*$ is the constant diagram taking the value $*$. (In fact, we have a pullback diagram in the ∞-category of functors from $\mathcal{E}$ to the ∞-category $\mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$ of $\mathbb{E}_k$-spaces, but we will not need this).

Let $\mathcal{E}' = \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})_{A//A}$. Let $X' : \mathcal{E}' \to \mathcal{S}_*$ be the functor which assigns to a diagram

$$\begin{array}{ccc} & B & \\ {\scriptstyle f}\nearrow & & \searrow \\ A & \xrightarrow{\mathrm{id}_A} & A \end{array}$$

the fiber of the induced map $X(f) \to X(\mathrm{id}_A)$, and let Y' and Z' be defined similarly. Using Lemma T.5.5.2.3, we deduce the existence of a pullback diagram of functors

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ * & \longrightarrow & Z'. \end{array}$$

Let $\phi : \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})_{A//A}$ be the functor given informally by the formula $M \mapsto A \oplus M$ (that is, ϕ is the composition of the identification $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Sp}(\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})_{/A})$ with the functor $\Omega^\infty : \mathrm{Sp}(\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})_{/A}) \to \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})_{A//A}$). Let $X'' = X' \circ \phi$, and define Y'' and Z'' similarly. We have a pullback diagram of functors

$$\begin{array}{ccc} X'' & \longrightarrow & Y'' \\ \downarrow & & \downarrow \\ * & \longrightarrow & Z'' \end{array}$$

from $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$ to $\mathcal{S}_*$.

The functor Z'' carries an A-module M to the fiber of the map $(A \oplus M)^\times \to A^\times$, which can be identified with $\mathrm{Map}_{\mathcal{C}}(\mathbf{1}, M) \simeq \mathrm{Map}_{\mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})}(\mathrm{Free}(\mathbf{1}), M)$. In other words, the functor Z'' is corepresentable by the object $\mathrm{Free}(\mathbf{1}) \in \mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})$. Similarly, Theorem 6.1.4.27 implies that the functor Y'' is corepresentable by the object $A \in \mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})$. By definition, the functor X'' is corepresentable by the shifted cotangent complex

$L_A[k]$. Since the Yoneda embedding for $\mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})$ is fully faithful, we deduce the existence of a commutative diagram of representing objects

$$
\begin{array}{ccc}
L_A[k] & \longleftarrow & A \\
\uparrow & & \uparrow \\
0 & \longleftarrow & \mathrm{Free}(\mathbf{1})
\end{array}
$$

which is evidently a pushout square. This yields the desired fiber sequence

$$\mathrm{Free}(\mathbf{1}) \to A \to L_A[k].$$

in $\mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})$.

Remark 8.3.5.5. The fiber sequence of Theorem 8.3.5.1 can be chosen to depend functorially on A (this follows from a more careful version of the construction above). We leave the details to the reader.

Remark 8.3.5.6. Let A be a commutative algebra object of $\mathcal{C}$. Then A can be regarded as an $\mathbb{E}_k$-algebra object of $\mathcal{C}$ for every nonnegative integer k. When regarded as an $\mathbb{E}_k$-algebra object, A has a cotangent complex which we will denote by $L_A^{(k)}$ (to emphasize the dependence on k). The topological chiral homology (see Definition 5.3.2.6) $\int_{S^{k-1}} A$ can be identified with the tensor product $A \otimes S^{k-1}$ (Theorem 5.3.3.8), which is the $(k-1)$-fold (unreduced) suspension $\Sigma^{k-1} A$ of A, regarded as an object of $\mathrm{CAlg}(\mathcal{C})_{/A}$. According to Theorem 8.3.5.1, we have a canonical identification $L_A^{(k)} \simeq \mathrm{fib}(\Omega^{k-1}\Sigma^{k-1}(A) \to A)$ in the ∞-category $\mathcal{C}$. Since the ∞-operad CRing is equivalent to the colimit of the ∞-operads $\mathbb{E}_k$ (see Corollary 5.1.1.5), we conclude that the *commutative* algebra cotangent complex L_A can be computed as the colimit $\varinjlim_k L_A^{(k)}$. Combining this observation with the above identification, we obtain an alternative "derivation" of the formula $\Omega^\infty \Sigma^\infty \simeq \varinjlim_k \Omega^k \Sigma^k$.

Example 8.3.5.7. Let $\mathcal{C}$ be as in Theorem 8.3.5.1 and let $\epsilon : A \to \mathbf{1}$ be an augmented $\mathbb{E}_k$-algebra object of $\mathcal{C}$ (see Example 6.1.4.14). Theorem 6.1.4.27 guarantees the existence of a Koszul dual $\mathbb{D}(A) = \mathfrak{Z}_{\mathbb{E}_k}(\epsilon)$. Moreover, as an object of the underlying ∞-category $\mathcal{C}$, $\mathbb{D}(A)$ can be identified with a morphism object $\mathrm{Mor}_{\mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})}(A, \mathbf{1})$. Combining this observation with the fiber sequence of Theorem 8.3.5.1 (and observing that the morphism object $\mathrm{Mor}_{\mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})}(\mathrm{Free}(\mathbf{1}), \mathbf{1})$ is equivalent to $\mathbf{1}$), we obtain a fiber sequence

$$\mathrm{Mor}_{\mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})}(L_A[k], \mathbf{1}) \to \mathbb{D}(A) \overset{\theta}{\to} \mathbf{1}$$

in $\mathcal{C}$. The map θ underlies the augmentation on $\mathbb{D}(A)$ described in Example 6.1.4.14; we may therefore view $\mathrm{Mor}_{\mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})}(L_A[k], \mathbf{1})$ as the "augmentation ideal" of the Koszul dual $\mathbb{D}(A)$.

In heuristic terms, we can view the $\mathbb{E}_k$-algebra A as determining a "noncommutative scheme" $\mathbf{Spec}\, A$, which is equipped with a point given by the augmentation ϵ. We can think of L_A as a version of the cotangent bundle of $\mathbf{Spec}\, A$, and $\mathrm{Mor}_{\mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})}(L_A, \mathbf{1})$ as a version of the tangent space to $\mathbf{Spec}\, A$ at the point determined by ϵ. The above analysis shows that, up to a shift by k, this "tangent space" itself is the augmentation ideal in a *different* augmented $\mathbb{E}_k$-algebra object of $\mathcal{C}$ (namely, the Koszul dual algebra $\mathbb{D}(A)$). We will return to this perspective in a future work.

8.3.6   The Tangent Correspondence

Let $\mathcal{C}$ be an ∞-category, $T_{\mathcal{C}}$ a tangent bundle to $\mathcal{C}$ (Definition 8.3.1.9), and $L : \mathcal{C} \to T_{\mathcal{C}}$ the associated cotangent complex functor (Definition 8.3.2.14). Then there exists a coCartesian fibration $p : \mathcal{M} \to \Delta^1$ with $\mathcal{M} \times_{\Delta^1} \{0\} \simeq \mathcal{C}$, $\mathcal{M} \times_{\Delta^1} \{1\} \simeq T_{\mathcal{C}}$, such that the associated functor $\mathcal{C} \to T_{\mathcal{C}}$ can be identified with L (see §T.5.2.1). We will refer to $\mathcal{M}$ as a *tangent correspondence* to $\mathcal{C}$. The tangent correspondence will play an essential role in §8.4. For this reason, we devote the present section to giving an explicit construction of a tangent correspondence to $\mathcal{C}$, which we will denote by $\mathcal{M}^T(\mathcal{C})$.

Remark 8.3.6.1. Since the cotangent complex functor L admits a right adjoint, the coCartesian fibration $p : \mathcal{M} \to \Delta^1$ considered above is also a Cartesian fibration, associated to the composite functor

$$T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{0\}, \mathcal{C}) \simeq \mathcal{C}.$$

Recall that a *correspondence* between a pair of ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is an ∞-category $\mathcal{M}$ equipped with a functor $p : \mathcal{M} \to \Delta^1$ and isomorphisms $\mathcal{C} \simeq \mathcal{M} \times_{\Delta^1} \{0\}$ and $\mathcal{D} \simeq \mathcal{M} \times_{\Delta^1} \{1\}$. If p is a Cartesian fibration, then a correspondence determines a functor $\mathcal{D} \to \mathcal{C}$, which is well-defined up to homotopy. It is therefore reasonable to think of a correspondence as a "generalized functor". Our first result describes how to compose these "generalized functors" with ordinary functors.

Lemma 8.3.6.2. *Suppose given sequence of maps $A \xrightarrow{f} B \to \Delta^1$ in the category of simplicial sets. Let A_1 denote the fiber product $A \times_{\Delta^1} \{1\}$, and define B_1 similarly. If f is a categorical equivalence, then the induced map $A_1 \to B_1$ is a categorical equivalence.*

Proof. This follows immediately from the definition, since $\mathfrak{C}(A_1)$ and $\mathfrak{C}(B_1)$ can be identified with the full simplicial subcategories of $\mathfrak{C}(A)$ and $\mathfrak{C}(B)$ lying over the object $\{1\} \in \mathfrak{C}(\Delta^1)$. $\square$

Proposition 8.3.6.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and let $p : \mathcal{M} \to \Delta^1$ be a correspondence from $\mathcal{C}$ to $\mathcal{D}$. Let $G : \mathcal{D}' \to \mathcal{D}$ be a categorical fibration of simplicial sets. We define a new simplicial set $\mathcal{M}'$ equipped with a map $p' : \mathcal{M}' \to \mathcal{M}$, so that the following universal property is satisfied: for every map of simplicial sets $A \to \Delta^1$, we have a pullback diagram of sets*

$$
\begin{array}{ccc}
\mathrm{Hom}_{\Delta^1}(A, \mathcal{M}') & \longrightarrow & \mathrm{Hom}(A \times_{\Delta^1} \{1\}, \mathcal{D}') \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\Delta^1}(A, \mathcal{M}) & \longrightarrow & \mathrm{Hom}(A \times_{\Delta^1} \{1\}, \mathcal{D}).
\end{array}
$$

Then:

(1) *The map $\mathcal{M}' \to \mathcal{M}$ is an inner fibration of simplicial sets.*

(2) *The simplicial set $\mathcal{M}'$ is an ∞-category.*

(3) *Let $f : C \to D'$ be a morphism in $\mathcal{M}'$ from an object of $\mathcal{C}$ to an object of $\mathcal{D}'$. Then f is a $(p \circ p')$-Cartesian morphism of $\mathcal{M}'$ if and only if $p'(f)$ is a p-Cartesian morphism of $\mathcal{M}$.*

(4) *Assume that the map $\mathcal{M} \to \Delta^1$ is a Cartesian fibration, associated to a functor $G' : \mathcal{D} \to \mathcal{C}$. Then the composite map $\mathcal{M}' \to \mathcal{M} \to \Delta^1$ is a Cartesian fibration, associated to the functor $G' \circ G$.*

Proof. We first prove (1). We wish to show that the projection $\mathcal{M}' \to \mathcal{M}$ has the right lifting property with respect to every inclusion $A \to B$ which is a categorical equivalence of simplicial sets. Fix a map $\alpha : B \to \Delta^1$; we must show that it is possible to solve any mapping problem of the form

$$
\begin{array}{ccc}
A \times_{\Delta^1} \{1\} & \longrightarrow & \mathcal{D}' \\
{\scriptstyle i} \downarrow & & \downarrow {\scriptstyle G} \\
B \times_{\Delta^1} \{1\} & \longrightarrow & \mathcal{D}.
\end{array}
$$

Since G is assumed to be a categorical fibration, it will suffice to show that i is a categorical equivalence, which follows from Lemma 8.3.6.2. This completes the proof of (1). Assertion (2) follows immediately.

We now prove (3). Let $\overline{f}$ denote the image of f in $\mathcal{M}$. We have a commutative diagram of simplicial sets

$$
\begin{array}{ccc}
 & \mathcal{M}_{/\overline{f}} & \\
{\scriptstyle\phi}\nearrow & & \searrow{\scriptstyle\psi} \\
\mathcal{M}'_{/f} & \xrightarrow{\hspace{3cm}} & \mathcal{C}_{/C} \, .
\end{array}
$$

We observe that f is $(p\circ p')$-Cartesian if and only if $(\psi\circ\phi)$ is a trivial Kan fibration, and that $\overline{f}$ is p-Cartesian if and only if ψ is a trivial Kan fibration. The desired equivalence now follows from the observation that ϕ is an isomorphism.

To prove (4), let us suppose that we are given a map $h : \mathcal{D} \times \Delta^1 \to \mathcal{M}$ which is a p-Cartesian natural transformation from G' to $\mathrm{id}_{\mathcal{D}}$. Using the definition of $\mathcal{M}'$, we see that the composition

$$
\mathcal{D}' \times \Delta^1 \to \mathcal{D} \times \Delta^1 \xrightarrow{h} \mathcal{M}
$$

can be lifted uniquely to a map $h' : \mathcal{D}' \times \Delta^1 \to \mathcal{M}'$ which is a natural transformation from $G' \circ G$ to $\mathrm{id}_{\mathcal{D}'}$. It follows from (3) that h' is a $(p \circ p')$-Cartesian transformation, so that $(p \circ p')$ is a Cartesian fibration associated to the functor $G' \circ G$. $\qquad\square$

We now describe an important example of a correspondence.

Notation 8.3.6.4. Let $K \subseteq \Delta^1 \times \Delta^1$ denote the full subcategory spanned by the vertices $\{i\} \times \{j\}$ where $i \leq j$ (so that K is isomorphic to a 2-simplex Δ^2). For every simplicial set A equipped with a map $f : A \to \Delta^1$, we let $\overline{A}$ denote the inverse image of K under the induced map

$$
\Delta^1 \times A \to \Delta^1 \times \Delta^1.
$$

Note that the map $A \xrightarrow{(f,\mathrm{id})} \Delta^1 \times A$ factors through $\overline{A}$; we will denote the resulting inclusion by $\psi_A : A \to \overline{A}$.

Let $\mathcal{C}$ be an ∞-category. The *fundamental correspondence of* $\mathcal{C}$ is a simplicial set $\mathcal{M}^0(\mathcal{C})$ equipped with a map $p : \mathcal{M}^0(\mathcal{C}) \to \Delta^1$, characterized by the following universal property: for every map of simplicial sets $A \to \Delta^1$, we have a canonical bijection of sets

$$
\mathrm{Hom}_{\Delta^1}(A, \mathcal{M}^0(\mathcal{C})) \simeq \mathrm{Hom}(\overline{A}, \mathcal{C}).
$$

The inclusions $\psi_A : A \to \overline{A}$ determine a map $q : \mathcal{M}^0(\mathcal{C}) \to \mathcal{C}$. Together p and q determine a map $\mathcal{M}^0(\mathcal{C}) \to \mathcal{C} \times \Delta^1$, which we will call the *fundamental projection*.

Remark 8.3.6.5. Let $\mathcal{C}$ be an ∞-category, and let $\mathcal{M}^0(\mathcal{C})$ be its fundamental correspondence. Then the fiber $\mathcal{M}^0(\mathcal{C}) \times_{\Delta^1} \{0\}$ is canonically isomorphic to $\mathcal{C}$, and the fiber $\mathcal{M}^0(\mathcal{C}) \times_{\Delta^1} \{1\}$ is canonically isomorphic to $\mathrm{Fun}(\Delta^1, \mathcal{C})$. We will generally abuse terminology, and use these isomorphisms identify $\mathcal{C}$ and $\mathrm{Fun}(\Delta^1, \mathcal{C})$ with subsets of $\mathcal{M}^0(\mathcal{C})$. The map $q : \mathcal{M}^0(\mathcal{C}) \to \mathcal{C}$ is given by the identity on $\mathcal{C}$, and by evaluation at $\{1\} \subseteq \Delta^1$ on $\mathrm{Fun}(\Delta^1, \mathcal{C})$.

Proposition 8.3.6.6. *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{M}^0(\mathcal{C})$ be the fundamental correspondence of $\mathcal{C}$, and let $\pi : \mathcal{M}^0(\mathcal{C}) \to \mathcal{C} \times \Delta^1$ denote the fundamental projection, and $p : \mathcal{M}^0(\mathcal{C}) \to \Delta^1$ the composition of π with projection onto the second factor. Then:*

(1) *The fundamental projection π is a categorical fibration. In particular, $\mathcal{M}^0(\mathcal{C})$ is an ∞-category.*

(2) *The map p is a Cartesian fibration.*

(3) *Let $A \in \mathcal{C} \subseteq \mathcal{M}^0(\mathcal{C})$, and let $(f : B \to C) \in \mathrm{Fun}(\Delta^1, \mathcal{C}) \subseteq \mathcal{M}^0(\mathcal{C})$. Let $\alpha : A \to f$ be a morphism in $\mathcal{M}^0(\mathcal{C})$, corresponding to a commutative diagram*

$$
\begin{array}{ccc}
A & \xrightarrow{\ \overline{\alpha}\ } & B \\
 & \searrow & \downarrow{\scriptstyle f} \\
 & & C
\end{array}
$$

in $\mathcal{C}$. Then α is p-Cartesian if and only if $\overline{\alpha}$ is an equivalence in $\mathcal{C}$.

(4) *The Cartesian fibration p is associated to the functor $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ given by evaluation at the vertex $\{0\} \in \Delta^1$.*

(5) *The map p is also a coCartesian fibration, associated to the diagonal inclusion $\mathcal{C} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$.*

The proof will require a few lemmas. In what follows, we will employ the conventions of Notation 8.3.6.4.

Lemma 8.3.6.7. *Let A be a simplicial set equipped with a map $A \to \Delta^1$, and let*

$$
\widetilde{A} = (A \times \{0\}) \coprod_{A_1 \times \{0\}} (A_1 \times \Delta^1) \subseteq \overline{A}.
$$

Then the inclusion $\widetilde{A} \subseteq \overline{A}$ is a categorical equivalence.

Proof. The functors $A \mapsto \widetilde{A}$ and $A \mapsto \overline{A}$ both commute with colimits. Since the class of categorical equivalences is stable under filtered colimits, we may reduce to the case where A has only finitely many simplices. We now work by induction on the dimension n of A, and the number of nondegenerate simplices of dimension n. If A is empty there is nothing to prove; otherwise there exists a pushout diagram

$$
\begin{array}{ccc}
\partial \Delta^n & \longrightarrow & \Delta^n \\
\downarrow & & \downarrow \\
A' & \longrightarrow & A.
\end{array}
$$

This induces homotopy pushout diagrams

$$
\begin{array}{ccc}
\overline{\partial \Delta^n} & \longrightarrow & \overline{\Delta^n} \\
\downarrow & & \downarrow \\
\overline{A'} & \longrightarrow & \overline{A}
\end{array}
\qquad
\begin{array}{ccc}
\widetilde{\partial \Delta^n} & \longrightarrow & \widetilde{\Delta^n} \\
\downarrow & & \downarrow \\
\widetilde{A'} & \longrightarrow & \widetilde{A}.
\end{array}
$$

It will therefore suffice to prove the lemma after replacing A by A', $\partial \Delta^n$, or Δ^n. In the first two cases this follows from the inductive hypothesis. We may therefore assume that $A = \Delta^n$. In particular, A is an ∞-category. The composite map

$$
\overline{A} \subseteq A \times \Delta^1 \to \Delta^1
$$

is a Cartesian fibration associated to the inclusion $i : A_1 \to A$, and $\widetilde{A}$ can be identified with the mapping cylinder of i. The desired result now follows from Proposition T.3.2.2.10. $\qquad\square$

Lemma 8.3.6.8. *Suppose given maps of simplicial sets $A \xrightarrow{f} B \to \Delta^1$. If f is a categorical equivalence, then the induced map $\overline{A} \to \overline{B}$ is a categorical equivalence.*

Proof. Let $\widetilde{A}$ and $\widetilde{B}$ be defined as in Lemma 8.3.6.7. We have a commutative diagram

$$
\begin{array}{ccc}
\widetilde{A} & \xrightarrow{\widetilde{f}} & \widetilde{B} \\
\downarrow & & \downarrow \\
\overline{A} & \xrightarrow{\overline{f}} & \overline{B},
\end{array}
$$

where the vertical maps are categorical equivalences by Lemma 8.3.6.7. It will therefore suffice to show that $\widetilde{f}$ is a categorical equivalence. The map $\widetilde{f}$ determines a map of homotopy pushout diagrams

$$
\begin{array}{ccc}
A_1 \times \{0\} \longrightarrow A \times \{0\} \\
\downarrow \qquad\qquad \downarrow \\
A_1 \times \Delta^1 \longrightarrow \widetilde{A}
\end{array}
\qquad
\begin{array}{ccc}
B_1 \times \{0\} \longrightarrow B \times \{0\} \\
\downarrow \qquad\qquad \downarrow \\
B_1 \times \Delta^1 \longrightarrow \widetilde{B}.
\end{array}
$$

It therefore suffices to show that the map $A_1 \to B_1$ is a categorical equivalence, which follows from Lemma 8.3.6.2. $\qquad\square$

Proof of Proposition 8.3.6.6. We first prove (1). Consider a lifting problem

$$
\begin{array}{ccc}
A & \longrightarrow & \mathcal{M}^0(\mathcal{C}) \\
\downarrow{\scriptstyle i} & \nearrow & \downarrow{\scriptstyle \pi} \\
B & \longrightarrow & \mathcal{C} \times \Delta^1,
\end{array}
$$

where i is a monomorphism of simplicial sets. We must show that this lifting problem has a solution if i is a categorical equivalence. Unwinding the definitions (and using the conventions of Notation 8.3.6.4, we are reduced to showing that $\mathcal{C}$ has the extension property with respect to the inclusion $j : \overline{A} \coprod_A B \to \overline{B}$. For this, it suffices to show that j is a categorical equivalence. Since the Joyal model structure is left proper, it will suffice to show that the inclusion $\overline{A} \to \overline{B}$ is a categorical equivalence, which follows from Lemma 8.3.6.8.

We next prove (3). Let us identify α with a 2-simplex in $\mathcal{C}$. Unwinding the definitions, we see that α is p-Cartesian if and only if the map $\phi : \mathcal{C}_{/\alpha} \to \mathcal{C}_{/f}$ is a trivial Kan fibration. In view of Proposition T.1.2.4.3, this is equivalent to the requirement that the map $A \to B$ be an equivalence in $\mathcal{C}_{/C}$, which is equivalent to the requirement that $\overline{\alpha}$ be an equivalence in $\mathcal{C}$ (Proposition T.1.2.13.8).

We now prove (2). Since p is the composition of π with the projection map $\mathcal{C} \times \Delta^1 \to \Delta^1$, we deduce immediately that p is an inner fibration. To show that p is a Cartesian fibration, it will suffice to show that for every object $X \in \mathcal{M}^0(\mathcal{C})$ and every morphism $\overline{\alpha} : y \to p(x)$ in Δ^1, there exists a p-Cartesian morphism $\alpha : Y \to X$ lifting $\overline{\alpha}$. If $\overline{\alpha}$ is degenerate, we can choose α to be degenerate. We may therefore assume that $X \in \mathrm{Fun}(\Delta^1, \mathcal{C})$ classifies a map $B \to C$ in $\mathcal{C}$. We can then choose α to classify the diagram

$$
\begin{array}{ccc}
B & \xrightarrow{\mathrm{id}} & B \\
& \searrow & \downarrow \\
& & C.
\end{array}
$$

It follows from (3) that α is p-Cartesian.

Let $G : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ denote the functor given by evaluation at the vertex $\{0\}$. To prove (4), we must exhibit a p-Cartesian natural transformation $h : \Delta^1 \times \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{M}^0(\mathcal{C})$ from G to $\mathrm{id}_{\mathrm{Fun}(\Delta^1, \mathcal{C})}$. We now choose h to classify the composite map

$$
K \times \mathrm{Fun}(\Delta^1, \mathcal{C}) \xrightarrow{(h_0, \mathrm{id})} \Delta^1 \times \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}
$$

where K is defined as in Notation 8.3.6.4, and $h_0 : K \simeq \Delta^2 \to \Delta^1$ is the map which collapses the edge $\Delta^{\{0,1\}} \subseteq \Delta^2$. It follows from (3) that h is a Cartesian transformation with the desired properties.

We now prove (5). Let $F : \mathcal{C} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ denote the diagonal embedding. The $G \circ F = \mathrm{id}_{\mathcal{C}}$. The identity map $\mathrm{id}_{\mathcal{C}} \to G \circ F$ is the unit for an adjunction between G and F. Thus p is also a coCartesian fibration, associated to the functor F, as desired. $\square$

Definition 8.3.6.9. Let $\mathcal{C}$ be a presentable ∞-category and let $G : T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ be a tangent bundle to $\mathcal{C}$. We define the *tangent correspondence* $\mathcal{M}^T(\mathcal{C})$ to be the result of applying the construction of Proposition 8.3.6.3 using the fundamental correspondence $\mathcal{M}^0(\mathcal{C})$ and the functor G. By construction, $\mathcal{M}^T(\mathcal{C})$ is equipped with a projection map $\pi : \mathcal{M}^T(\mathcal{C}) \to \Delta^1 \times \mathcal{C}$.

Remark 8.3.6.10. The terminology of Definition 8.3.6.9 is slightly abusive: the tangent correspondence $\mathcal{M}^T(\mathcal{C})$ depends on a choice of tangent bundle $T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$. However, it is easy to eliminate this ambiguity: for example, we can use an explicit construction of $T_{\mathcal{C}}$ (see Proposition 8.3.1.10).

The following result is an immediate consequence of Propositions 8.3.6.6, Proposition 8.3.6.3, and the definition of the cotangent complex functor L:

Proposition 8.3.6.11. *Let $\mathcal{C}$ be a presentable ∞-category. Then:*

(1) *The projection $\mathcal{M}^T(\mathcal{C}) \to \Delta^1 \times \mathcal{C}$ is a categorical fibration.*

(2) *The composite map $p : \mathcal{M}^T(\mathcal{C}) \to \Delta^1 \times \mathcal{C} \to \Delta^1$ is a Cartesian fibration, associated to the functor*

$$T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{0\}, \mathcal{C}) \simeq \mathcal{C}.$$

(3) *The map p is also a coCartesian fibration, associated to the cotangent complex functor $L : \mathcal{C} \to T_{\mathcal{C}}$.*

8.4 Deformation Theory

In §8.3, we studied the general formalism of cotangent complexes. For every presentable ∞-category $\mathcal{C}$, we defined the tangent bundle $T_{\mathcal{C}}$ and a relative cotangent complex functor $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to T_{\mathcal{C}}$

$$(f : A \to B) \mapsto L_{B/A} \in \mathrm{Sp}(\mathcal{C}^{/B}).$$

In this section, we will specialize to the situation where $\mathcal{C}$ is the ∞-category CAlg of $\mathbb{E}_\infty$-rings. In this case, Theorem 8.3.4.18 allows us to identify the tangent bundle $T_{\mathcal{C}}$ with the ∞-category of pairs (A, M), where A is an $\mathbb{E}_\infty$-ring and M is an A-module. We will henceforth use this identification to view the relative cotangent complex $L_{B/A}$ as taking its value in the ∞-category Mod_B of B-module spectra.

The basic idea we emphasize in this section is that the theory of the relative cotangent complex "controls" the deformation theory of $\mathbb{E}_\infty$-rings. In §8.4.1, we will make this precise by introducing the notion of a *square-zero extension* of $\mathbb{E}_\infty$-rings. To every map $\phi : A \to B$ of $\mathbb{E}_\infty$-rings and every map of B-modules $\eta : L_{B/A} \to M$, we will associate a new A-algebra B^η equipped with a map $B^\eta \to B$: roughly speaking, B^η is given by the fiber of the A-linear derivation $B \to M$ determined by η. We say that a map $\widetilde{B} \to B$ is a square-zero extension if it arises via this construction. Our main result asserts that a large class of morphisms can be obtained as square zero extensions: for example, the Postnikov tower of a connective $\mathbb{E}_\infty$-ring B is given by successive square-zero extensions

$$\cdots \to \tau_{\leq 2} B \to \tau_{\leq 1} B \to \tau_{\leq 0} B.$$

Suppose we are given a square-zero extension $\mathbb{E}_\infty$-rings $\widetilde{A} \to A$. In this case, there is a close relationship between $\mathbb{E}_\infty$-algebras over $\widetilde{A}$ and $\mathbb{E}_\infty$-algebras over A. Every $\mathbb{E}_\infty$-algebra $\widetilde{B}$ over $\widetilde{A}$ determines an $\mathbb{E}_\infty$-algebra $B = \widetilde{B} \otimes_{\widetilde{A}} A$. Under some mild connectivity assumptions, we will see that $\widetilde{B}$ can be recovered as a

square-zero extension of B. This leads to an algebraic version of Kodaira-Spencer theory, which reduces the classification of $\mathbb{E}_\infty$-algebras over $\widetilde{A}$ to the classification over A, together with a "linear" problem involving the relative cotangent complex (see Theorem 8.4.2.7).

In §8.4.3, will study connectivity and finiteness properties of the relative cotangent complex $L_{B/A}$ associated to a morphism $\phi : A \to B$ between connective $\mathbb{E}_\infty$-rings. It is not difficult to show that finiteness properties of f are inherited by the relative cotangent complex $L_{B/A}$. For example, if f is of finite presentation, then the relative cotangent complex $L_{B/A}$ is a perfect B-module. We will see that the converse holds under some mild additional assumptions (Theorem 8.4.3.18).

Remark 8.4.0.1. There is a voluminous literature on deformation theory in the setting of ordinary commutative algebra and in algebraic geometry. Some references include [96], [74], and [75].

8.4.1   Square-Zero Extensions

Let R be a commutative ring. A *square-zero extension* of R is a commutative ring $\widetilde{R}$ equipped with a surjection $\phi : \widetilde{R} \to R$, with the property that the product of any two elements in $\ker(\phi)$ is zero. In this case, the kernel $M = \ker(\phi)$ inherits the structure on R-module.

Let $\widetilde{R}$ be a square-zero extension of a commutative ring R by an R-module M. There exists a ring homomorphism
$$(R \oplus M) \times_R \widetilde{R} \to \widetilde{R},$$
given by the formula
$$(r, m, \widetilde{r}) \mapsto \widetilde{r} + m.$$
This map exhibits $\widetilde{R}$ as endowed with an *action* of $R \oplus M$ in the category of commutative rings with a map to R (we observe that $R \oplus M$ has the structure of an abelian group object in this category). Consequently, in some sense square-zero extensions of R by M can be viewed as *torsors* for the trivial square-zero extension $R \oplus M$.

In general, if $\phi : \widetilde{R} \to R$ is a square-zero extension of R by $M \simeq \ker(\phi)$, we say that $\widetilde{R}$ is *trivial* if ϕ admits a left inverse. In this case, a choice of left inverse to ϕ determines an isomorphism $\widetilde{R} \simeq R \oplus M$. Such an isomorphism need not exist (for example, we could take $R = \mathbf{Z}/p\mathbf{Z}$ and $\widetilde{R} = \mathbf{Z}/p^2\mathbf{Z}$), and need not be unique. However, any two sections of ϕ differ by a derivation from R into M, which is classified by an R-linear map from the module of Kähler differentials Ω_R into M. Conversely, any derivation of R into M determines an automorphism of $\widetilde{R}$ (whether $\widetilde{R}$ is trivial or not), which permutes the set of sections of ϕ. Consequently, we deduce that the *automorphism group* of the trivial square zero extension of R by M can be identified with the group of R-module homomorphisms $\mathrm{Ext}_R^0(\Omega_R, M)$.

It is tempting to try to pursue this analogy further, and to try identify the *isomorphism classes* of square-zero extensions of R by M with the higher Ext-group $\mathrm{Ext}_R^1(\Omega_R, M)$. Given an extension class $\eta \in \mathrm{Ext}_R^1(\Omega_R, M)$, we can indeed construct a square-zero extension $\widetilde{R}$ of R by M. Indeed, let us view η as defining an exact sequence
$$0 \to M \to \widetilde{M} \xrightarrow{f} \Omega_R \to 0$$
in the category of R-modules. We now form a pullback diagram
$$
\begin{array}{ccc}
\widetilde{R} & \longrightarrow & R \\
\downarrow & & \downarrow{\scriptstyle d} \\
\widetilde{M} & \longrightarrow & \Omega_R
\end{array}
$$
in the category of abelian groups. We can identify elements of $\widetilde{R}$ with pairs $(r, \widetilde{m})$, where $r \in R$ and $\widetilde{m} \in \widetilde{M}$ satisfy the equation $f(\widetilde{m}) = dr$. The abelian group $\widetilde{R}$ admits a ring structure, given by the formula
$$(r, \widetilde{m})(r', \widetilde{m}') = (rr', r'\widetilde{m} + r\widetilde{m}').$$

It is easy to check that $\widetilde{R}$ is a square-zero extension of R by M. However, not every square-zero extension of R by M can be obtained from this construction. In order to obtain *all* square-zero extensions of R, it is necessary to replace the module of Kähler differentials Ω_R by a more refined invariant, such as the $\mathbb{E}_\infty$-ring cotangent complex L_R.

Our goal in this section is to study analogues of all of the ideas sketched above in the setting of $\mathbb{E}_\infty$-rings. Roughly speaking, we will mimic the above construction to produce a functor $\Phi : \mathcal{D}\mathrm{er} \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg})$. Here $\mathcal{D}\mathrm{er}$ denotes an ∞-category of triples (A, M, η), where A is an $\mathbb{E}_\infty$-ring, M is an A-module, and $\eta : A \to M[1]$ is a derivation (which we can identify with a morphism of A-modules L_A into $M[1]$). The functor Φ carries (A, M, η) to a map $A^\eta \to A$; here we will refer to A^η as *the square-zero extension of A classified by η.*

Using this definition, it follows more or less tautologically that square-zero extensions of an $\mathbb{E}_\infty$-ring A are "controlled" by the absolute cotangent complex of L_A. For example, if L_A vanishes, then every square-zero extension of A by an A-module M is equivalent to the trivial extension $A \oplus M$ constructed in §8.3.4. The trouble with this approach is that it is not obvious how to give an *intrinsic* characterization of the class of square-zero extensions. For example, suppose that $f : \widetilde{A} \to A$ is a square-zero extension of A by an A-module M. We then have a canonical identification $M \simeq \mathrm{fib}(f)$ in the ∞-category of $\widetilde{A}$-modules. However, in general there is no way to recover the A-module structure on $\mathrm{fib}(f)$ from the morphism f alone. In other words, the functor Φ described above fails to be fully faithful. We can remedy the situation by studying a more restricted class of morphisms between $\mathbb{E}_\infty$-rings, which we call *n-small extensions*. This collection of morphisms has two important features:

(i) Given a map $f : \widetilde{A} \to A$, it is easy to decide whether or not f is an n-small extension. Namely, one must check that the fiber $\mathrm{fib}(f)$ has certain connectivity properties, and that a certain bilinear map $\pi_n \mathrm{fib}(f) \times \pi_n \mathrm{fib}(f) \to \pi_{2n} \mathrm{fib}(f)$ vanishes.

(ii) On the class of n-small extensions of $\mathbb{E}_\infty$-rings, one can construct an inverse to the functor Φ (Theorem 8.4.1.26). In particular, every n-small extension is a square-zero extension.

In conjunction, (i) and (ii) imply that square-zero extensions exist in abundance. For example, if A is a connective $\mathbb{E}_\infty$-ring, then the Postnikov tower

$$\ldots \to \tau_{\leq 2} A \to \tau_{\leq 1} A \to \tau_{\leq 0} A$$

is a sequence of square-zero extensions.

We begin by defining the notion of a square-zero extension in an arbitrary presentable ∞-category $\mathcal{C}$. Although we are primarily interested in the case where $\mathcal{C} = \mathrm{CAlg}$ is the ∞-category of $\mathbb{E}_\infty$-rings, the theory we develop here also has many applications in "nonlinear" settings. For example, when $\mathcal{C}$ is the ∞-category of spaces, it can be regarded as a generalization of classical obstruction theory.

Definition 8.4.1.1. Let $\mathcal{C}$ be a presentable ∞-category, and let $p : \mathcal{M}^T(\mathcal{C}) \to \Delta^1 \times \mathcal{C}$ denote a tangent correspondence to $\mathcal{C}$ (see Definition 8.3.6.9). A *derivation* in $\mathcal{C}$ is a map $f : \Delta^1 \to \mathcal{M}^T(\mathcal{C})$ such that $p \circ f$ coincides with the inclusion $\Delta^1 \times \{A\} \subseteq \Delta^1 \times \mathcal{C}$, for some $A \in \mathcal{C}$. In this case, we will identify f with a morphism $\eta : A \to M$ in $\mathcal{M}^T(\mathcal{C})$, where $M \in T_\mathcal{C} \times_\mathcal{C} \{A\} \simeq \mathrm{Sp}(\mathcal{C}^{/A})$. We will also say that $\eta : A \to M$ is a *derivation of A into M*.

We let $\mathcal{D}\mathrm{er}(\mathcal{C})$ denote the fiber product $\mathrm{Fun}(\Delta^1, \mathcal{M}^T(\mathcal{C})) \times_{\mathrm{Fun}(\Delta^1, \Delta^1 \times \mathcal{C})} \mathcal{C}$. We will refer to $\mathcal{D}\mathrm{er}(\mathcal{C})$ as *the ∞-category of derivations in $\mathcal{C}$.*

Remark 8.4.1.2. In the situation of Definition 8.4.1.1, let $L : \mathcal{C} \to T_\mathcal{C}$ be a cotangent complex functor. A derivation $\eta : A \to M$ can be identified with a map $d : L_A \to M$ in the fiber $T_\mathcal{C} \times_\mathcal{C} \{A\} \simeq \mathrm{Sp}(\mathcal{C}^{/A})$. We will often abuse terminology by identifying η with d, and referring to d as a *derivation of A into M*.

Definition 8.4.1.3. Let $\mathcal{C}$ be a presentable ∞-category, and let $p : \mathcal{M}^T(\mathcal{C}) \to \Delta^1 \times \mathcal{C}$ be a tangent correspondence for $\mathcal{C}$. An *extended derivation* is a diagram σ

$$
\begin{array}{ccc}
\widetilde{A} & \xrightarrow{\ f\ } & A \\
\downarrow & & \downarrow{\scriptstyle \eta} \\
0 & \longrightarrow & M
\end{array}
$$

in $\mathcal{M}^T(\mathcal{C})$ with the following properties:

(1) The diagram σ is a pullback square.

(2) The objects $\widetilde{A}$ and A belong to $\mathcal{C} \subseteq \mathcal{M}^T(\mathcal{C})$, while 0 and M belong to $T_{\mathcal{C}} \subseteq \mathcal{M}^T(\mathcal{C})$.

(3) Let $\overline{f} : \Delta^1 \to \mathcal{C}$ be the map which classifies the morphism f appearing in the diagram above, and let $e : \Delta^1 \times \Delta^1 \to \Delta^1$ be the unique map such that $e^{-1}\{0\} = \{0\} \times \{0\}$. Then the diagram

$$
\begin{array}{ccc}
\Delta^1 \times \Delta^1 & \xrightarrow{\ \sigma\ } \mathcal{M}^T(\mathcal{C}) \xrightarrow{\ p\ } & \Delta^1 \times \mathcal{C} \\
\downarrow{\scriptstyle e} & & \downarrow \\
\Delta^1 & \xrightarrow{\hspace{3.5cm}\overline{f}\hspace{3.5cm}} & \mathcal{C}
\end{array}
$$

is commutative.

(4) The object $0 \in T_{\mathcal{C}}$ is a zero object of $\mathrm{Sp}(\mathcal{C}^{/A})$. Equivalently, 0 is a p-initial vertex of $\mathcal{M}^T(\mathcal{C})$.

We let $\widetilde{\mathcal{D}\mathrm{er}}(\mathcal{C})$ denote the full subcategory of

$$
\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{M}^T(\mathcal{C})) \times_{\mathrm{Fun}(\Delta^1 \times \Delta^1, \Delta^1 \times \mathcal{C})} \mathrm{Fun}(\Delta^1, \mathcal{C})
$$

spanned by the extended derivations.

If σ is an extended derivation in $\mathcal{C}$, then η is a derivation in $\mathcal{C}$. We therefore obtain a restriction functor

$$
\widetilde{\mathcal{D}\mathrm{er}}(\mathcal{C}) \to \mathcal{D}\mathrm{er}(\mathcal{C}).
$$

Let $\mathcal{C}$ and $\mathcal{M}^T(\mathcal{C})$ be above, and let

$$
\sigma \in \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{M}^T(\mathcal{C})) \times_{\mathrm{Fun}(\Delta^1 \times \Delta^1, \Delta^1 \times \mathcal{C})} \mathrm{Fun}(\Delta^1, \mathcal{C}).
$$

Then σ automatically satisfies conditions (2) and (3) of Definition 8.4.1.3. Moreover, σ satisfies condition (4) if and only if σ is a p-left Kan extension of $\sigma|\{1\} \times \Delta^1$ at the object $\{0\} \times \{1\}$. Invoking Proposition T.4.3.2.15 twice, we deduce the following:

Lemma 8.4.1.4. *Let $\mathcal{C}$ be a presentable ∞-category. Then the forgetful $\psi : \widetilde{\mathcal{D}\mathrm{er}}(\mathcal{C}) \to \mathcal{D}\mathrm{er}(\mathcal{C})$ is a trivial Kan fibration.*

Notation 8.4.1.5. Let $\mathcal{C}$ be a presentable ∞-category. We let $\Phi : \mathcal{D}\mathrm{er}(\mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ denote the composition

$$
\mathcal{D}\mathrm{er}(\mathcal{C}) \to \widetilde{\mathcal{D}\mathrm{er}}(\mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C}),
$$

where the first map is a section of the trivial fibration $\widetilde{\mathcal{D}\mathrm{er}}(\mathcal{C}) \to \mathcal{D}\mathrm{er}(\mathcal{C})$, and the second map is induced by the inclusion $\Delta^1 \times \{0\} \subseteq \Delta^1 \times \Delta^1$. In other words, Φ associates to every derivation $\eta : A \to M$ a map $f : \widetilde{A} \to A$ which fits into a pullback diagram

$$
\begin{array}{ccc}
\widetilde{A} & \xrightarrow{\ f\ } & A \\
\downarrow & & \downarrow{\scriptstyle \eta} \\
0 & \longrightarrow & M
\end{array}
$$

in the ∞-category $\mathcal{M}^T(\mathcal{C})$.

Definition 8.4.1.6. Let $\mathcal{C}$ be a presentable ∞-category, and let $\Phi : \mathcal{D}er(\mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ be the functor described in Notation 8.4.1.5. We will denote the image of a derivation $(\eta : A \to M) \in \mathcal{D}er(\mathcal{C})$ under the functor Φ by $(A^\eta \to A)$.

Let $f : \widetilde{A} \to A$ be a morphism in $\mathcal{C}$. We will say that f is a *square-zero extension* if there exists a derivation $\eta : A \to M$ in $\mathcal{C}$ and an equivalence $\widetilde{A} \simeq A^\eta$ in the ∞-category $\mathcal{C}^{/A}$. In this case, we will also say that $\widetilde{A}$ is a *square-zero extension of A by $M[-1]$*.

Remark 8.4.1.7. Let $\eta : A \to M$ be a derivation in a presentable ∞-category $\mathcal{C}$, and let $A \oplus M$ denote the image of M under the functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}^{/A}) \to \mathcal{C}$. Using Proposition T.4.3.1.9, we conclude that there is a pullback diagram

$$\begin{array}{ccc} A^\eta & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle d_\eta} \\ A & \xrightarrow{d_0} & A \oplus M \end{array}$$

in the ∞-category $\mathcal{C}$. Here we identify d_0 with the map associated to the zero derivation $L_A \to M$.

Remark 8.4.1.8. In the situation of Remark 8.4.1.7, let B be another object of $\mathcal{C}$. We have a pullback diagram of mapping spaces

$$\begin{array}{ccc} \mathrm{Map}_\mathcal{C}(B, A^\eta) & \longrightarrow & \mathrm{Map}_\mathcal{C}(B, A) \\ \downarrow & & \downarrow \\ \mathrm{Map}_\mathcal{C}(B, A) & \longrightarrow & \mathrm{Map}_\mathcal{C}(B, A \oplus M). \end{array}$$

Fix a map $\phi : B \to A$ in $\mathcal{C}$, and let $\phi^* : \mathrm{Sp}(\mathcal{C}^{/B}) \to \mathrm{Sp}(\mathcal{C}^{/A})$ denote the functor induced by ϕ, so that ϕ determines a map $\eta' : \phi^* L_B \to L_A \xrightarrow{\eta} M$. It follows that the fiber product $\mathrm{Map}_\mathcal{C}(B, A^\eta) \times_{\mathrm{Map}_\mathcal{C}(B,A)} \{\phi\}$ can be identified with the space of paths from 0 to η' in $\mathrm{Map}_{\mathrm{Sp}(\mathcal{C}^{/A})}(\phi^* L_B, M)$.

Example 8.4.1.9. Let $\mathcal{C}$ be a presentable ∞-category containing an object A. Let $M \in \mathrm{Sp}(\mathcal{C}^{/A})$, and let $\eta : A \to M$ be the derivation classified by the zero map $L_A \to M$ in $\mathrm{Sp}(\mathcal{C}^{/A})$. Since the functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}^{/A}) \to \mathcal{C}^{/A}$ preserves small limits, we conclude from Remark 8.4.1.7 that the square-zero extension A^η can be identified with $\Omega^\infty M[-1]$. In particular, if $M = 0$, then the canonical map $A^\eta \to A$ is an equivalence, so we can identify A^η with A.

Warning 8.4.1.10. Let $\mathcal{C}$ be a presentable ∞-category, and let $f : \widetilde{A} \to A$ be a morphism in $\mathcal{C}$. Suppose f is a square-zero extension, so that there exists a map $\eta : L_A \to M$ in $\mathrm{Sp}(\mathcal{C}_{/A})$ and an equivalence $\widetilde{A} \simeq A^\eta$. In this situation, the object M and the map η need not be uniquely determined, even up to equivalence. However, this is true in some favorable situations; see Theorem 8.4.1.26.

Example 8.4.1.11. Suppose we are given a fibration of simply connected spaces

$$F \to E \xrightarrow{f} B,$$

such that $\pi_k F \simeq *$ for all $k \neq n$. In this case, the fibration f is *classified* by a map η from B into an Eilenberg-MacLane space $K(A, n+1)$, where $A = \pi_n F$. It follows that we have a homotopy pullback diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & B \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{id},\eta)} \\ B & \xrightarrow{(\mathrm{id},0)} & B \times K(A, n+1). \end{array}$$

The space $B \times K(A, n+1)$ is an infinite loop object of the ∞-category of spaces over B: it has deloopings given by $K(A, n+m)$ for $m \geq 1$. Consequently, the above diagram exhibits E as a square-zero extension of B in the ∞-category of spaces.

In fact, using a slightly more sophisticated version of the same construction, one can show that the same result holds without any assumptions of simple-connectedness; moreover it is sufficient that the homotopy groups of F be confined to a small range, rather than a single degree.

Our ultimate goal in this section is to show that, in the setting of $\mathbb{E}_\infty$-rings, square-zero extensions exist in abundance. For example, if A is a connective $\mathbb{E}_\infty$-ring, then the Postnikov tower

$$\ldots \to \tau_{\leq 2} A \to \tau_{\leq 1} A \to \tau_{\leq 0} A,$$

consists of square-zero extensions.

We begin by considering the case of associative algebra objects. Let $\mathcal{C}$ be a presentable stable ∞-category, and assume that $\mathcal{C}$ is equipped with monoidal structure such that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. According to Theorem 8.3.4.13, for any associative algebra object $A \in \mathcal{C}$, we have a canonical equivalence

$$\mathrm{Sp}(\mathrm{Alg}(\mathcal{C})_{/A}) \simeq \mathrm{Mod}_A^{Ass}(\mathcal{C}) \simeq {}_A\mathrm{BMod}_A(\mathcal{C}).$$

If A is an associative algebra object of $\mathcal{C}$, we let L_A denote its absolute cotangent complex (viewed as an object of ${}_A\mathrm{BMod}_A(\mathcal{C})$). Given a map of associative algebras $f : A \to B$, we let $L_{B/A} \in {}_B\mathrm{BMod}_B(\mathcal{C})$ denote the relative cotangent complex of f.

Remark 8.4.1.12. Applying Theorem 8.3.5.1 in the monoidal ∞-category ${}_A\mathrm{BMod}_A(\mathcal{C})$, we deduce the existence of a canonical fiber sequence

$$L_{B/A} \to B \otimes_A B \to B.$$

Let $f : A \to B$ be a map of associative algebra objects of $\mathcal{C}$. The fiber I of f can be identified with the limit of a diagram

$$A \xrightarrow{f} B \leftarrow 0,$$

which we can view as a diagram of nonunital algebra objects of the monoidal ∞-category ${}_A\mathrm{BMod}_A(\mathcal{C})$ of A-bimodule objects of $\mathcal{C}$. It follows that I inherits the structure of a nonunital algebra in ${}_A\mathrm{BMod}_A(\mathcal{C})$. In particular, I has the structure of an A-bimodule, and there is a natural multiplication map $m : I \otimes_A I \to I$.

Remark 8.4.1.13. In the above situation, the multiplication $I \otimes_A I$ is given by the composition

$$I \otimes_A I \to I \otimes_A A \simeq I.$$

Our next result can be regarded as a partial justification for the terminology "square-zero extension".

Proposition 8.4.1.14. *Let $\mathcal{C}$ be a presentable stable ∞-category equipped with a monoidal structure which preserves colimits separately in each variable, let $f : A^\eta \to A$ be a square-zero extension in $\mathrm{Alg}(\mathcal{C})$, and let I denote the fiber of f. Then the multiplication map $\theta : I \otimes_{A^\eta} I \to I$ is nullhomotopic (as a map of A^η-bimodules).*

Proof. Without loss of generality, we may assume that A^η is a square-zero extension classified by a derivation $\eta : L_A \to M[1]$, for some A-A-bimodule $M \in {}_A\mathrm{BMod}_A(\mathcal{C})$. We have a pullback diagram of associative algebras

$$
\begin{array}{ccc}
A^\eta & \longrightarrow & A \\
\downarrow & & \downarrow \\
A & \xrightarrow{\ f'\ } & A \oplus M[1].
\end{array}
$$

Then $I \simeq M$, and the multiplication map θ factors as a composition

$$I \otimes_{A^\eta} I \to M \otimes_A M \xrightarrow{\theta'} M$$

where θ' is the map of A-bimodules determined by the multiplication on the fiber of f'. According to Remark 8.4.1.13, θ' is obtained from the map $\theta'' : M \to A$ by tensoring over A with M. It will therefore suffice to show that θ'' is nullhomotopic (as a map of A-bimodules). This follows from the observation that f' admits a left inverse (as a morphism in $\mathrm{Alg}(\mathcal{C})_{A/}$, and therefore also as a map of A-bimodules). $\qquad\square$

Proposition 8.4.1.15. *Let $\mathcal{C}$ be a presentable stable ∞-category equipped with a monoidal structure for which the tensor product preserves small colimits separately in each variable. Let $f : A \to B$ be a map of associative algebra objects of $\mathcal{C}$, let $\eta : B \to B \oplus L_{B/A}$ be the universal derivation, and factor f as a composition $A \xrightarrow{f'} B^\eta \xrightarrow{f''} B$. Then there is a fiber sequence of A-A-bimodules*

$$\mathrm{fib}(f) \otimes_A \mathrm{fib}(f) \xrightarrow{\alpha} \mathrm{fib}(f) \xrightarrow{\beta} \mathrm{fib}(f''),$$

where α is given by the multiplication on $\mathrm{fib}(f)$.

Proof. Remark 8.4.1.12 supplies an identification of $L_{B/A}$ with the fiber of the multiplication map $m : B \otimes_A B \to B$. Note that m admits a left inverse s, given by

$$B \simeq B \otimes_A A \xrightarrow{\mathrm{id} \otimes f} B \otimes_A B.$$

It follows that $L_{B/A} \simeq \mathrm{fib}(m) \simeq \mathrm{cofib}(s) \simeq B \otimes_A \mathrm{cofib}(f)$. In particular, $\mathrm{fib}(f'') \simeq L_{B/A}[-1] \simeq B \otimes_A \mathrm{fib}(f)$. Under this identification, the map β is obtained from f by tensoring over A with $\mathrm{fib}(f)$. It follows that $\mathrm{fib}(\beta) \simeq \mathrm{fib}(f) \otimes_A \mathrm{fib}(f)$. Using Remark 8.4.1.13, we see that the induced map $\mathrm{fib}(f) \otimes_A \mathrm{fib}(f) \simeq \mathrm{fib}(\beta) \to \mathrm{fib}(f)$ coincides with α. $\qquad\square$

Remark 8.4.1.16. Let $f : A \to B$ be as in Proposition 8.4.1.14. It follows that the composite map

$$\mathrm{fib}(f) \otimes_A \mathrm{fib}(f) \xrightarrow{\alpha} \mathrm{fib}(f) \xrightarrow{\beta} \mathrm{fib}(f'')$$

is canonically nullhomotopic. We can describe the nullhomotopy explicitly as follows. The factorization $f = f'' \circ f'$ determines a commmutative diagram

$$
\begin{array}{ccc}
\mathrm{fib}(f) \otimes_A \mathrm{fib}(f) & \xrightarrow{\alpha} & \mathrm{fib}(f) \\
\downarrow & & \downarrow{\scriptstyle\beta} \\
\mathrm{fib}(f'') \otimes_{B^\eta} \mathrm{fib}(f'') & \xrightarrow{\alpha'} & \mathrm{fib}(f'')
\end{array}
$$

in $_A\mathrm{BMod}_A(\mathcal{C})$. Proposition 8.4.1.14 supplies a canonical nullhomotopy of α', whence a canonical nullhomotopy of $\beta \circ \alpha$.

Remark 8.4.1.17. Let $f : A \to B$ be as in Proposition 8.4.1.15. Using Proposition 8.4.1.15, we obtain a fiber sequence

$$\mathrm{fib}(f) \otimes_A \mathrm{fib}(f) \to A \to B^\eta.$$

We can summarize the situation informally as follows: the universal square-zero extension B^η of B through which f factors has the form $A/(I \otimes_A I)$, where $I = \mathrm{fib}(f)$.

We now introduce a special class of morphisms between $\mathbb{E}_1$-algebras, which we call *small extensions*.

Definition 8.4.1.18. Let $\mathcal{C}$ be a stable presentable ∞-category equipped with a monoidal structure and a t-structure. Assume that the unit object $\mathbf{1} \in \mathcal{C}$ belongs to $\mathcal{C}_{\geq 0}$, that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable, and that $\otimes$ carries $\mathcal{C}_{\geq 0} \times \mathcal{C}_{\geq 0}$ into $\mathcal{C}_{\geq 0}$.

Let $f : A \to B$ be a map of associative algebra objects of $\mathcal{C}$ and let $n \geq 0$. We will say that f is an *n-connective extension* if $A \in \mathcal{C}_{\geq 0}$ and $\mathrm{fib}(f) \in \mathcal{C}_{\geq n}$. We will say that f is an *n-small extension* if the following additional conditions are satisfied:

(1) The fiber $\mathrm{fib}(f)$ belongs to $\mathcal{C}_{\leq 2n}$.

(2) The multiplication map $\mathrm{fib}(f) \otimes_A \mathrm{fib}(f) \to \mathrm{fib}(f)$ is nullhomotopic.

We let $\mathrm{Fun}_{n-con}(\Delta^1, \mathrm{Alg}(\mathcal{C}))$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{Alg}(\mathcal{C}))$ spanned by the n-connective extensions, and $\mathrm{Fun}_{n-sm}(\Delta^1, \mathrm{Alg}(\mathcal{C}))$ the full subcategory of $\mathrm{Fun}_{n-con}(\Delta^1, \mathrm{Alg}(\mathcal{C}))$ spanned by the n-small extensions.

Remark 8.4.1.19. Let $\mathcal{C}$ be as in Definition 8.4.1.18 and let $f : A \to B$ be an n-connective extension for $n \geq 0$. Since A and $\mathrm{fib}(f)$ belong to $\mathcal{C}_{\geq 0}$, we deduce that $B \in \mathcal{C}_{\geq 0}$. Moreover, the map $\pi_0 A \to \pi_0 B$ is an epimorphism in the abelian category $\mathcal{C}^{\heartsuit}$.

Remark 8.4.1.20. Let $\mathcal{C}$ be as in Definition 8.4.1.18, and let $f : A \to B$ be an n-connective extension such that $\mathrm{fib}(f) \in \mathcal{C}_{\leq 2n}$. Since $\mathrm{fib}(f) \otimes_A \mathrm{fib}(f) \in \mathcal{C}_{\geq 2n}$, the multiplication map $\mathrm{fib}(f) \otimes_A \mathrm{fib}(f) \to \mathrm{fib}(f)$ is nullhomotopic if and only if it induces the zero map $\pi_{2n}(\mathrm{fib}(f) \otimes_A \mathrm{fib}(f)) \to \pi_{2n} \mathrm{fib}(f)$ in the abelian category $\mathcal{C}^{\heartsuit}$. In other words, condition (2) of Definition 8.4.1.18 is equivalent to the vanishing of a certain map

$$\pi_n \mathrm{fib}(f) \otimes \pi_n \mathrm{fib}(f) \to \pi_{2n} \mathrm{fib}(f)$$

in $\mathcal{C}^{\heartsuit}$.

Example 8.4.1.21. Let $\mathcal{C} = \mathrm{Sp}$ be the ∞-category of spectra, and let A be an associative ring, which we regard as a discrete algebra object of $\mathcal{C}$. A map $f : \widetilde{A} \to A$ in $\mathrm{Alg}(\mathcal{C})$ is a 0-small extension if and only if the following conditions are satisfied:

(a) The algebra object $\widetilde{A} \in \mathrm{Alg}(\mathcal{C})$ is discrete.

(b) The map f induces a surjection of associative rings $\pi_0 \widetilde{A} \to \pi_0 A$.

(c) If $I \subseteq \pi_0 \widetilde{A}$ is the kernel of the ring homomorphism of (b), then $I^2 = 0 \subseteq \pi_0 \widetilde{A}$.

In other words, the theory of 0-small extensions of discrete associative algebras in $\mathcal{C}$ is equivalent to the classical theory of square-zero extensions between ordinary associative rings.

Notation 8.4.1.22. Let $\mathcal{C}$ be as in Definition 8.4.1.18. We let Der denote the ∞-category $\mathrm{Der}(\mathrm{Alg}(\mathcal{C}))$ of derivations in $\mathrm{Alg}(\mathcal{C})$. Using Theorem 8.3.4.13, we can identify objects of Der with pairs $(A, \eta : L_A \to M[1])$ where A is an algebra object of $\mathcal{C}$ and η is a map of A-A-bimodules. We let Der_{n-con} denote the full subcategory of Der spanned by those pairs $(A, \eta : L_A \to M[1])$ such $A \in \mathcal{C}_{\geq 0}$ and $M \in \mathcal{C}_{\geq n}$. We let Der_{n-sm} denote the full subcategory of Der_{n-con} spanned by those pairs $(A, \eta : L_A \to M[1])$ such that $M \in \mathcal{C}_{\leq 2n}$.

We can now state a preliminary version of our main result:

Theorem 8.4.1.23. *Let $\mathcal{C}$ be as in Definition 8.4.1.18, and let $\Phi : \mathrm{Der} \to \mathrm{Fun}(\Delta^1, \mathrm{Alg}(\mathcal{C}))$ be the functor of Notation 8.4.1.5, given informally by the formula*

$$(A, \eta : L_A \to M[1]) \mapsto (A^{\eta} \to A).$$

For each $n \geq 0$, the functor Φ induces an equivalence of ∞-categories

$$\Phi_{n-sm} : \mathrm{Der}_{n-sm} \to \mathrm{Fun}_{n-sm}(\Delta^1, \mathrm{Alg}(\mathcal{C})).$$

Proof. Let $A \in \mathrm{Alg}(\mathcal{C})$ and let $M \in {_A}\mathrm{BMod}_A(\mathcal{C})$. For any derivation $\eta : L_A \to M[1]$, we can identify the fiber of square-zero extension $A^\eta \to A$ with M (as an object of $\mathcal{C}$). It follows immediately that the functor $\Phi : \mathcal{D}\mathrm{er} \to \mathrm{Fun}(\Delta^1, \mathrm{Alg}(\mathcal{C}))$ restricts to a functor $\Phi_{n-con} : \mathcal{D}\mathrm{er}_{n-con} \to \mathrm{Fun}_{n-con}(\Delta^1, \mathrm{Alg}(\mathcal{C}))$. For any square-zero extension $f : A^\eta \to A$, the induced multiplication $\mathrm{fib}(f) \otimes_{A^\eta} \mathrm{fib}(f) \to \mathrm{fib}(f)$ is nullhomotopic (Proposition 8.4.1.14), so that Φ also restricts to a functor $\Phi_{n-sm} : \mathcal{D}\mathrm{er}_{n-sm} \to \mathrm{Fun}_{n-sm}(\Delta^1, \mathrm{Alg}(\mathcal{C}))$.

The functor Φ admits a left adjoint $\Psi : \mathrm{Fun}(\Delta^1, \mathrm{Alg}(\mathcal{C})) \to \mathcal{D}\mathrm{er}$, given informally by the formula

$$(\widetilde{A} \to A) \mapsto (A, \eta : L_A \to L_{A/\widetilde{A}}).$$

Assume that $f : \widetilde{A} \to A$ is an n-connective extension. Remark 8.4.1.12 implies that $L_{A/\widetilde{A}}$ can be identified with the fiber of the multiplication map $m : A \otimes_{\widetilde{A}} A \to A$, and therefore with the cofiber of the section

$$s : A \simeq A \otimes_{\widetilde{A}} \widetilde{A} \to A \otimes_{\widetilde{A}} A$$

of m. Thus $L_{A/\widetilde{A}}[-1] \simeq \mathrm{fib}(f) \otimes_{\widetilde{A}} A$. Since $\mathrm{fib}(f)$ is n-connective and A and $\widetilde{A}$ are connective, we deduce that $L_{A/\widetilde{A}}$ is n-connective. It follows that Ψ restricts to a functor $\Psi_{n-con} : \mathrm{Fun}_{n-con}(\Delta^1, \mathrm{Alg}(\mathcal{C})) \to \mathcal{D}\mathrm{er}_{n-con}$.

Let $i : \mathcal{D}\mathrm{er}_{n-sm} \hookrightarrow \mathcal{D}\mathrm{er}_{n-con}$ be the inclusion functor. Then i admits a left adjoint τ, given informally by the formula

$$\tau(A, L_A \to M[1]) = (A, L_A \to (\tau_{\leq 2n} M)[1]).$$

It follows that Φ_{n-sm} also admits a left adjoint Ψ_{n-sm}, given by the composition $\tau \circ \Psi_{n-con}$. The functor Φ_{n-sm} is clearly conservative. To prove that Φ_{n-sm} is an equivalence of ∞-categories, it will suffice to show that the unit transformation $u : \mathrm{id} \to \Phi_{n-sm} \circ \Psi_{n-sm}$ is an equivalence. In other words, we must show that if $f : \widetilde{A} \to A$ is an n-small extension, then the transformation $u_f : f \to (\Phi \circ \tau \circ \Psi)(f)$ is an equivalence. Let $\eta_0 : L_{A/\widetilde{A}} \to LA/\widetilde{A}$ be the universal deformation, and let η be the truncation map $L_{A/\widetilde{A}} \to \tau_{\leq 2n+1} L_{A/\widetilde{A}}$. We wish to show that the composite map $\widetilde{A} \to A^{\eta_0} \to A^\eta$ is an equivalence. We have a commutative diagram

$$
\begin{array}{ccccc}
\widetilde{A} & \xrightarrow{g} & A^{\eta_0} & \longrightarrow & A^\eta \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f''} \\
A & \longrightarrow & A & \longrightarrow & A.
\end{array}
$$

By construction, we have $\mathrm{fib}(f'') \simeq \tau_{\leq 2n} \mathrm{fib}(f')$. It will therefore suffice to show that g induces an equivalence $\mathrm{fib}(f) \simeq \tau_{\leq 2n} \mathrm{fib}(f')$. We have a fiber sequence

$$\mathrm{fib}(g) \xrightarrow{\alpha} \mathrm{fib}(f) \xrightarrow{\beta} \mathrm{fib}(f').$$

Proposition 8.4.1.15 allows us to identify α with the multiplication map $m : \mathrm{fib}(f) \otimes_{\widetilde{A}} \mathrm{fib}(f) \to \mathrm{fib}(f)$. Since f is a small extension, m is nullhomotopic, so that β induces an equivalence $\mathrm{fib}(f') \simeq \mathrm{fib}(f) \oplus (\mathrm{fib}(f) \otimes_{\widetilde{A}} \mathrm{fib}(f))[1]$. Since $\mathrm{fib}(f) \in \mathcal{C}_{\leq 2n} \cap \mathcal{C}_{\geq n}$, $(\mathrm{fib}(f) \otimes_{\widetilde{A}} \mathrm{fib}(f))[1] \in \mathcal{C}_{\geq 2n+1}$, so that the composite map $\mathrm{fib}(f) \to \mathrm{fib}(f') \to \tau_{\leq 2n} \mathrm{fib}(f)$ is an equivalence as desired. $\square$

Remark 8.4.1.24. In the situation of Theorem 8.4.1.23, the full subcategory $\mathrm{Fun}_{n-sm}(\Delta^1, \mathrm{Alg}(\mathcal{C}))$ is a localization of $\mathrm{Fun}_{n-con}(\Delta^1, \mathrm{Alg}(\mathcal{C}))$. Indeed, we claim that the functor $\Phi_{n-sm} \circ \tau \circ \Psi^{n-con}$ is a left adjoint to the inclusion $\mathrm{Fun}_{n-sm}(\Delta^1, \mathrm{Alg}(\mathcal{C})) \hookrightarrow \mathrm{Fun}_{n-con}(\Delta^1, \mathrm{Alg}(\mathcal{C}))$. Using Theorem 8.4.1.23, we are reduced to proving that $\tau \circ \Psi_{n-con} : \mathrm{Fun}^{n-con}(\Delta^1, \mathrm{Alg}(\mathcal{C})) \to \mathcal{D}\mathrm{er}^{n-sm}$ is left adjoint to composition

$$\mathcal{D}\mathrm{er}_{n-sm} \hookrightarrow \mathcal{D}\mathrm{er}_{n-con} \xrightarrow{\Phi} \mathrm{Fun}_{n-con}(\Delta^1, \mathrm{Alg}(\mathcal{C})),$$

which is evident.

We now generalize the above discussion to the case of $\mathbb{E}_k$-algebras, for $k \geq 1$.

Notation 8.4.1.25. Let $1 \leq k \leq \infty$ and let $\mathcal{C}$ be presentable stable ∞-category equipped with a t-structure. Assume also that $\mathcal{C}$ is an $\mathbb{E}_k$-monoidal ∞-category, the unit object $\mathbf{1}$ belongs to $\mathcal{C}_{\geq 0}$, and that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable and carries $\mathcal{C}_{\geq 0} \times \mathcal{C}_{\geq 0}$ into $\mathcal{C}_{\geq 0}$.

We let $\mathrm{Alg}^{(k)}(\mathcal{C})$ denote the ∞-category $\mathrm{Alg}_{/\mathbb{E}_k}(\mathcal{C})$ and $\mathrm{Alg}(\mathcal{C})$ the ∞-category $\mathrm{Alg}_{\mathbb{E}_1/\mathbb{E}_k}(\mathcal{C})$, so that the inclusion $\mathbb{E}_1 \hookrightarrow \mathbb{E}_k$ determines a forgetful functor $\theta : \mathrm{Alg}^{(k)}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$. For $n \geq 0$, we say that a morphism $f : A \to B$ in $\mathrm{Alg}^{(k)}(\mathcal{C})$ is an *n-small extension* if $\theta(f)$ is an n-small extension. We let $\mathrm{Fun}_{n-sm}(\Delta^1, \mathrm{Alg}^{(k)}(\mathcal{C}))$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{Alg}^{(k)}(\mathcal{C}))$ spanned by the n-small extensions.

For $A \in \mathrm{Alg}^{(k)}(\mathcal{C})$, we let $L_A^{(k)} \in \mathrm{Sp}(\mathrm{Alg}^{(k)}(\mathcal{C})_{/A}) \simeq \mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})$ denote its cotangent complex as an object of $\mathrm{Alg}^{(k)}(\mathcal{C})$. Let $\mathrm{Der}^{(k)}$ denote the ∞-category $\mathrm{Der}(\mathrm{Alg}^{(k)}(\mathcal{C}))$ of derivations in $\mathrm{Alg}^{(k)}(\mathcal{C})$, so that the objects of $\mathrm{Der}^{(k)}$ can be identified with pairs $(A, \eta : L_A^{(k)} \to M[1])$ where A is an $\mathbb{E}_k$-algebra object of $\mathcal{C}$ and η is a morphism in $\mathrm{Mod}_A^{\mathbb{E}_k}(\mathcal{C})$. We let $\mathrm{Der}_{n-sm}^{(k)}$ denote the full subcategory of $\mathrm{Der}^{(k)}$ spanned by those pairs $(A, \eta : L_A^{(k)} \to M[1])$ such that A is connective and the image of M belongs to $\mathcal{C}_{\geq n} \cap \mathcal{C}_{\leq 2n}$.

We have the following generalization of Theorem 8.4.1.23:

Theorem 8.4.1.26. *Let $\mathcal{C}$ be as in Notation 8.4.1.25 and let $\Phi^{(k)} : \mathrm{Der}^{(k)} \to \mathrm{Fun}(\Delta^1, \mathrm{Alg}^{(k)}(\mathcal{C}))$ be the functor of Notation 8.4.1.5 for $1 \leq k \leq \infty$. For each $n \geq 0$, the functor $\Phi^{(k)}$ induces an equivalence of ∞-categories*

$$\Phi_{n-sm}^{(k)} : \mathrm{Der}_{n-sm}^{(k)} \to \mathrm{Fun}_{n-sm}(\Delta^1, \mathrm{Alg}^{(k)}(\mathcal{C})).$$

Corollary 8.4.1.27. *Let $\mathcal{C}$ be as in Notation 8.4.1.25. Then every n-small extension in $\mathrm{Alg}^{(k)}(\mathcal{C})$ is a square-zero extension.*

Corollary 8.4.1.28. *Let $\mathcal{C}$ be as in Notation 8.4.1.25, and let $A \in \mathrm{Alg}^{(k)}(\mathcal{C}_{\geq 0})$. Then every map in the Postnikov tower*

$$\ldots \to \tau_{\leq 3} A \to \tau_{\leq 2} A \to \tau_{\leq 1} A \to \tau_{\leq 0} A$$

is a square-zero extension.

Remark 8.4.1.29. Corollary 8.4.1.28 underscores the importance of the cotangent complex in the study of algebraic structures. For example, suppose we wish to understand the space of maps $\mathrm{Map}_{\mathrm{Alg}^{(k)}}(A, B)$ between two connective $\mathbb{E}_k$-rings A and B. This space can be realized as the homotopy inverse limit of the mapping spaces $\mathrm{Map}_{\mathrm{Alg}^{(k)}}(A, \tau_{\leq n} B)$. In the case $n = 0$, this is simply the discrete set of ring homomorphisms from $\pi_0 A$ to $\pi_0 B$. For $n > 0$, Corollary 8.4.1.28 implies the existence of a pullback diagram

$$
\begin{array}{ccc}
\tau_{\leq n} B & \longrightarrow & \tau_{\leq n-1} B \\
\downarrow & & \downarrow \\
\tau_{\leq n-1} B & \longrightarrow & \tau_{\leq n-1} B \oplus (\pi_n B)[n+1].
\end{array}
$$

This reduces us to the study of $\mathrm{Map}_{\mathrm{Alg}^{(k)}}(A, \tau_{\leq n-1} B)$ and the "linear" problem of understanding derivations from A into $(\pi_n B)[n+1]$. This linear problem is controlled by the cotangent complex $L_A^{(k)}$.

Lemma 8.4.1.30. *Let $\mathcal{C}$ be a stable monoidal ∞-category equipped with a t-structure. Assume that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is exact in each variable and carries $\mathcal{C}_{\geq 0} \times \mathcal{C}_{\geq 0}$ into $\mathcal{C}_{\geq 0}$. Let $n \geq 0$, and suppose we are given a finite collection of morphisms $\{p_i : B_i \to A_i\}_{1 \leq i \leq m}$ in $\mathcal{C}$ such that each $A_i \in \mathcal{C}_{\geq 0}$ and each $\mathrm{fib}(p_i)$ is $(n+1)$-connective. Let f denote the induced map*

$$\mathrm{fib}(\bigotimes_i p_i) \to \prod_{1 \leq j \leq m} \mathrm{fib}(\mathrm{id}_{A_1} \otimes \cdots \otimes \mathrm{id}_{A_{j-1}} \otimes p_j \otimes \mathrm{id}_{A_{j+1}} \otimes \cdots \otimes \mathrm{id}_{A_m}).$$

Then $\mathrm{fib}(f) \in \mathcal{C}_{\geq 2n+2}$.

Proof. We proceed by induction on n. When $n = 0$, $\mathrm{fib}(f) \simeq 0$ and the result is obvious. Assume therefore that $n > 0$. Let $p = \bigotimes_{1 \leq i \leq m} p_i$ and let $p' = \bigotimes_{1 < i \leq n} p_i$. We have a transformation of fiber sequences

$$
\begin{array}{ccc}
B_1 \otimes \mathrm{fib}(p') & \xrightarrow{\ g\ } & \prod_{1 < j \leq m} \mathrm{fib}(\mathrm{id}_{A_1} \otimes \cdots \otimes p_j \otimes \cdots \otimes \mathrm{id}_{A_m}) \\
\downarrow & & \downarrow \\
\mathrm{fib}(p) & \xrightarrow{\ f\ } & \prod_{1 \leq j \leq m} \mathrm{fib}(\mathrm{id}_{A_1} \otimes \cdots \otimes p_j \otimes \cdots \otimes \mathrm{id}_{A_m}) \\
\downarrow & & \downarrow \\
\mathrm{fib}(p_1) \otimes A_2 \otimes \cdots \otimes A_n & \longrightarrow & \mathrm{fib}(p_1 \otimes \mathrm{id}_{A_2} \otimes \cdots \otimes \mathrm{id}_{A_m}).
\end{array}
$$

Since the tensor product on $\mathcal{C}$ is exact in each variable, the bottom horizontal map is an equivalence, so that $\mathrm{fib}(f) \simeq \mathrm{fib}(g)$. The map g factors as a composition

$$
B_1 \otimes \mathrm{fib}(p') \xrightarrow{\ \mathrm{id}_{B_1} \otimes f'\ } \prod_{1 < j \leq m} B_1 \otimes A_2 \otimes \cdots \otimes \mathrm{fib}(p_j) \otimes \cdots \otimes A_m \xrightarrow{\ g'\ } \prod_{1 < j \leq m} A_1 \otimes A_2 \otimes \cdots \otimes \mathrm{fib}(p_j) \otimes \cdots \otimes A_m.
$$

The inductive hypothesis implies that $\mathrm{fib}(f') \in \mathcal{C}_{\geq 2n+2}$. Since $A_1 \in \mathcal{C}_{\geq 0}$ and $\mathrm{fib}(p_1) \in \mathcal{C}_{\geq n+1}$, the object B_1 belongs to $\mathcal{C}_{\geq 0}$, so that $B_1 \otimes \mathrm{fib}(f') \in \mathcal{C}_{\geq 2n+2}$. We are therefore reduced to proving that $\mathrm{fib}(g') \in \mathcal{C}_{\geq 2n+2}$. The map g' factors as a product of maps g'_j with $\mathrm{fib}(g'_j) \simeq \mathrm{fib}(p_1) \otimes A_2 \otimes \cdots \otimes \mathrm{fib}(p_j) \otimes \cdots \otimes A_m$. Since each A_i belongs to $\mathcal{C}_{\geq 0}$ and $\mathrm{fib}(p_1), \mathrm{fib}(p_j) \in \mathcal{C}_{\geq n+1}$, we conclude that $\mathrm{fib}(g'_j) \in \mathcal{C}_{\geq 2n+2}$. It follows that $\mathrm{fib}(g') \simeq \prod_{1 < j \leq m} \mathrm{fib}(g'_j) \in \mathcal{C}_{\geq 2n+2}$ as desired. $\square$

Proof of Theorem 8.4.1.26. When $k = 1$, the desired result follows immediately from Theorem 8.4.1.23 (since the ∞-operads $\mathbb{E}_1^{\otimes}$ and $\mathcal{A}ss^{\otimes}$ are equivalent; see Example 5.1.0.7). We will reduce the general case to the case $k = 1$ using Theorem 5.1.2.2. We will assume for notational simplicity that $k \geq 2$ is finite (the case $k = \infty$ can be treated by the same method, or by passage to the limit over finite k). Using the bifunctor of ∞-operads $\mathbb{E}_{k-1}^{\otimes} \times \mathbb{E}_1^{\otimes} \to \mathbb{E}_k^{\otimes}$ of Construction 5.1.2.1, we regard $\mathrm{Alg}(\mathcal{C})$ as an $\mathbb{E}_{k-1}$-monoidal ∞-category. Note that the tensor product on $\mathrm{Alg}(\mathcal{C})$ preserves small sifted colimits, and in particular filtered colimits. According to Example 8.3.1.15, the tangent bundle $T_{\mathrm{Alg}(\mathcal{C})}$ inherits an $\mathbb{E}_{k-1}$-monoidal structure, which induces in turn an $\mathbb{E}_{k-1}$-monoidal structure on $\mathcal{D}\mathrm{er}^{(1)}$. The tensor product on $\mathcal{D}\mathrm{er}^{(1)}$ is given informally by the formula

$$
(A, \eta : L_A \to M[1]) \otimes (A', \eta' : L_{A'} \to M'[1]) \simeq (A \otimes A', \eta : L_{A \otimes A'} \to (A \otimes M')[1] \oplus (M \otimes A')[1]);
$$

here we can think of η as the differential on $A \otimes A'$ determined by the Leibniz rule. We regard $\mathrm{Fun}(\Delta^1, \mathrm{Alg}(\mathcal{C}))$ as endowed with the $\mathbb{E}_{k-1}$-monoidal structure given by pointwise tensor product (see Remark 2.1.3.4), so that the functor $\Phi^{(1)} : \mathcal{D}\mathrm{er}^{(1)} \to \mathrm{Fun}(\Delta^1, \mathrm{Alg}(\mathcal{C}))$ is lax $\mathbb{E}_{k-1}$-monoidal. Let $(\mathcal{D}\mathrm{er}^{(1)}_{n-sm})^{\otimes} \subseteq (\mathcal{D}\mathrm{er}^{(1)})^{\otimes}$ be the ∞-suboperad of $\mathcal{D}\mathrm{er}^{\otimes}$ determined by the full subcategory $\mathcal{D}\mathrm{er}_{n-sm} \subseteq \mathcal{D}\mathrm{er}$ (as explained in §2.2.1). Define $\mathrm{Fun}_{n-sm}(\Delta^1, \mathrm{Alg}(\mathcal{C}))^{\otimes}$ similarly, so that $\Phi^{(1)}_{n-sm}$ induces a map of ∞-operads

$$
\Phi' : (\mathcal{D}\mathrm{er}^{(1)}_{n-sm})^{\otimes} \to \mathrm{Fun}_{n-sm}(\Delta^1, \mathrm{Alg}(\mathcal{C}))^{\otimes}.
$$

We have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{D}\mathrm{er}^{(k)}_{n-sm} & \xrightarrow{\ \Phi^{(k)}_{n-sm}\ } & \mathrm{Fun}_{n-sm}(\Delta^1, \mathrm{Alg}^{(k)}(\mathcal{C})) \\
\downarrow & & \downarrow \\
\mathrm{Alg}_{/\mathbb{E}_{k-1}}(\mathcal{D}\mathrm{er}^{(1)}_{n-sm}) & \longrightarrow & \mathrm{Alg}_{/\mathbb{E}_{k-1}}(\mathrm{Fun}_{n-sm}(\Delta^1, \mathrm{Alg}(\mathcal{C}))).
\end{array}
$$

Using Theorem 5.1.2.2, we deduce that the vertical maps are equivalences of ∞-categories. The lower horizontal map is given by composition with Φ'; it will therefore suffice to show that Φ' is an equivalence of ∞-operads.

Theorem 8.4.1.26 guarantees that Φ' induces an equivalence of underlying ∞-categories

$$\Phi_{n-sm} : \mathrm{Der}^{(1)}_{n-sm} \to \mathrm{Fun}_{n-sm}(\Delta^1, \mathrm{Alg}(\mathcal{C})).$$

In particular, Φ_{n-sm} is essentially surjective. It will therefore suffice to show that Φ' is fully faithful. Unwinding the definitions, we must show that for every sequence of objects $\{(A_i, \eta_i : L_{A_i} \to M_i) \in \mathrm{Der}^{(1)}_{n-sm}\}_{1 \le i \le m}$ and every object $(B, \eta' : L_B \to N) \in \mathrm{Der}^{(1)}_{n-sm}$, if $f_i : A_i^{\eta_i} \to A_i$ and $g : B^{\eta'} \to B$ denote their images under $\Phi^{(1)}$, then $\Phi^{(1)}$ induces a homotopy equivalence

$$\theta : \mathrm{Map}_{\mathrm{Der}^{(1)}}\Big(\bigotimes_{1 \le i \le m} (A_i, \eta_i), (B, \eta) \Big) \to \mathrm{Hom}_{\mathrm{Fun}(\Delta^1, \mathrm{Alg}(\mathcal{C}))}\Big(\bigotimes_{1 \le i \le n} f_i, g \Big).$$

Let $\Psi^{(1)} : \mathrm{Fun}(\Delta^1, \mathrm{Alg}(\mathcal{C})) \to \mathrm{Der}^{(1)}$ be the left adjoint to $\Phi^{(1)}$ and $\tau : \mathrm{Der}_{n-con} \to \mathrm{Der}_{n-sm}$ a left adjoint to the inclusion, as in Theorem 8.4.1.26. The codomain of θ is homotopy equivalent to the mapping space $\mathrm{Map}_{\mathrm{Der}^{(1)}}(\Psi(\otimes_i f_i), (B, \eta'))$. Since η' is n-small, it will suffice to show that if $v : \Psi(\otimes_i f_i) \to \bigotimes_i (A_i, \eta_i)$ is the evident counit map, then $\tau(v)$ is an equivalence in $\mathrm{Der}^{(1)}$. Let $A = \bigotimes_i A_i$ and $A' = \bigotimes_i A_i^{\eta_i}$, so we can identify v with a map of A-A-bimodules

$$v : L_{A/A'} \to \prod_i A_1 \otimes \cdots \otimes A_{i-1} \otimes M_i \otimes A_{i+1} \otimes \cdots \otimes A_m.$$

We wish to show that $\tau_{\le 2n+1}(v)$ is an equivalence in $\mathcal{C}$.

Using Remark 8.4.1.12, we can identify $L_{A/A'}$ with the cofiber of $\otimes_{1 \le i \le n} p_i$, where each p_i is the multiplication map $A_i \otimes_{A_i^{\eta_i}} A_i \to A_i$. Unwinding the definitions, we see that v is is a product of maps $\{v_j\}_{1 \le j \le m}$, each of which is given by the composition

$$
\begin{aligned}
\mathrm{fib}(\otimes_i p_i) \quad &\overset{v_j'}{\to} \quad \mathrm{fib}(\mathrm{id}_{A_1} \otimes \cdots \otimes \mathrm{id}_{A_{j-1}} \otimes p_j \otimes \mathrm{id}_{A_{j+1}} \otimes \cdots \otimes \mathrm{id}_{A_m} \\
&\simeq \quad A_1 \otimes \cdots \otimes A_{j-1} \otimes \mathrm{fib}(p_j) \otimes A_{j+1} \otimes \cdots \otimes A_m \\
&\overset{v_j''}{\to} \quad A_1 \otimes \cdots A_{j-1} \otimes M_j \otimes A_{j+1} \otimes \cdots \otimes A_m.
\end{aligned}
$$

Here each v_j'' is induced by the map $t_j : \mathrm{fib}(p_j) \simeq L_{A_j/A_j^{\eta_j}} \to M_j$ determined by the adjunction of $\Phi^{(1)}$ and $\Psi^{(1)}$. The proof of Theorem 8.4.1.26 shows that $\tau_{\le 2n+1}(t_j)$ is an equivalence. Since each A_i is connective, it follows that $\tau_{\le 2n+1}(v_j'')$ is an equivalence. To complete the proof, it will suffice to show that if

$$v' : \mathrm{fib}(\otimes_i p_i) \to \prod_{1 \le j \le m} A_1 \otimes \cdots \otimes A_{j-1} \otimes \mathrm{fib}(p_j) \otimes A_{j+1} \otimes \cdots \otimes A_m$$

is the product of the maps v_j', then $\tau_{\le 2n+1}(v')$ is an equivalence in $\mathcal{C}$. This follows from Lemma 8.4.1.30, since $\mathrm{fib}(m_i)$ is $(n+1)$-connective for $1 \le i \le m$. $\qquad\square$

8.4.2 Deformation Theory of $\mathbb{E}_\infty$-Algebras

Suppose we are given a map of commutative rings $\phi : \widetilde{A} \to A$. Let B be a flat commutative A-algebra. A *deformation of B over $\widetilde{A}$* is a pair $(\widetilde{B}, \alpha)$, where $\widetilde{B}$ is a flat commutative $\widetilde{A}$-algebra and α is an isomorphism of A-algebras $\widetilde{B} \otimes_{\widetilde{A}} A \simeq B$. One of the basic problems of *deformation theory* is to classify the deformations of B over $\widetilde{A}$. In the special case where $\widetilde{A}$ is a square-zero extension of A by an A-module $M \simeq \ker(\phi)$, this can be translated into a linear problem involving the cotangent complexes of B and A. In this section, we will explain this translation in a more general context. We begin by introducing some terminology.

Definition 8.4.2.1. Let A be an $\mathbb{E}_\infty$-ring and suppose we are given a square-zero extension $\widetilde{A}$ of A by an A-module M. If $B \in \mathrm{CAlg}_A$, then a *deformation of B to $\widetilde{A}$* is a pair $(\widetilde{B}, \alpha)$, where $\widetilde{B} \in \mathrm{CAlg}_{\widetilde{A}}$ and α is an equivalence $\widetilde{B} \otimes_{\widetilde{A}} A \simeq B$ in CAlg_A.

Remark 8.4.2.2. In Definition 8.4.2.1, it is not necessary to require that $\widetilde{B}$ be flat over $\widetilde{A}$. If A is connective and $\widetilde{A}$ is a square-zero extension of A by a connective A-module M, then a deformation $\widetilde{B}$ of B will be flat over $\widetilde{A}$ if and only if B is flat over A. The "only if" direction is obvious. For the converse, suppose that B is flat over A: we claim that for every discrete $\widetilde{A}$-module N, the relative tensor product $\widetilde{B} \otimes_{\widetilde{A}} N$ is discrete. To prove this, let $I \subseteq \pi_0 \widetilde{A}$ be the kernel of the surjective map $\pi_0 \widetilde{A} \to \pi_0 A$, so that we have a short exact sequence of modules over $\pi_0 \widetilde{A}$:

$$0 \to IN \to N \to N/IN \to 0$$

It will therefore suffice to show that the tensor products $\widetilde{B} \otimes_{\widetilde{A}} IN$ and $\widetilde{B} \otimes_{\widetilde{A}} N/IN$ are discrete. Replacing N by IN or N/IN, we can reduce to the case where $IN = 0$, so that N has the structure of an A-module. Then $\widetilde{B} \otimes_{\widetilde{A}} N \simeq B \otimes_A N$ is discrete by virtue of the assumption that B is flat over A.

In the situation of Definition 8.4.2.1, the assumption that $\widetilde{A}$ is a square-zero extension of A by M implies that $\widetilde{A} \simeq A^\eta$ for some A-linear map $\eta : L_A \to M[1]$. Given a map of $\mathbb{E}_\infty$-rings $A \to B$, we get a map of B-modules $\eta_B : B \otimes_A L_A \to B \otimes_A M[1]$. Our main result can be summarized informally as follows (see Proposition 8.4.2.5 below for a precise formulation):

($*$) Assume that A, B, and M are connective. Then giving a deformation of $\widetilde{B}$ of B over $\widetilde{A}$ is equivalent to providing a factorization of η_B as a composition

$$B \otimes_A L_A \to L_B \xrightarrow{\eta'} B \otimes_A M[1].$$

In this case, the corresponding extension is given by $\widetilde{B} = B^{\eta'}$.

In particular, B admits a deformation over $\widetilde{A}$ if and only if the composite map

$$L_{B/A}[-1] \to B \otimes_A L_A \xrightarrow{\eta_B} B \otimes_A M[1]$$

vanishes.

Remark 8.4.2.3. More concretely, we see that a square-zero extension of A by an A-module M and a map of $\mathbb{E}_\infty$-rings $A \to B$ determine an *obstruction class* in $\mathrm{Ext}^2_B(L_{B/A}, B \otimes_A M)$, given by the composition $L_{B/A}[-1] \to B \otimes_A L_A \to B \otimes_A M[1]$. If A and M are connective, then this obstruction class vanishes if and only if B admits a deformation over $\widetilde{A}$. In this case, the collection of equivalence classes of extensions is a torsor for the abelian group $\mathrm{Ext}^1_B(L_{B/A}, B \otimes_A M)$.

This is a precise analogue of the situation in the classical deformation theory of algebraic varieties. Suppose given a smooth morphism of smooth, projective varieties $X \to Y$ over a field k. Given a first-order deformation $\widetilde{Y}$ of Y, we encounter an obstruction in $\mathrm{H}^2(X; T_{X/Y})$ to extending $\widetilde{Y}$ to a first-order deformation of X. If this obstruction vanishes, then the set of isomorphism classes of extensions is naturally a torsor for the cohomology group $\mathrm{H}^1(X; T_{X/Y})$ (see [88]).

In order to formulate assertion ($*$) more precisely, we need to introduce a bit of terminology.

Notation 8.4.2.4. Throughout this section, we let $\mathcal{D}\mathrm{er} = \mathcal{D}\mathrm{er}(\mathrm{CAlg})$ denote the ∞-category of derivations in CAlg (see Definition 8.4.1.1). More informally, the objects of $\mathcal{D}\mathrm{er}$ are triples (A, M, η), where A is an $\mathbb{E}_\infty$-ring, M is an A-module spectrum, and $\eta : A \to M[1]$ is a derivation. In what follows, we will generally abuse notation by identifying the triple (A, M, η) with the underlying map $\eta : A \to M[1]$. We let $A^\eta = \mathrm{fib}(\eta)$ denote the corresponding square-zero extension of A.

We define a subcategory $\mathcal{D}\mathrm{er}^+ \subseteq \mathcal{D}\mathrm{er}$ as follows:

(i) An object $(\eta : A \to M[1]) \in \mathcal{D}\mathrm{er}$ belongs to $\mathcal{D}\mathrm{er}^+$ if and only if both A and M are connective. Equivalently, η belongs to $\mathcal{D}\mathrm{er}^+$ if both A and A^η are connective, and the map of commutative rings $\pi_0 A^\eta \to \pi_0 A$ is surjective.

(ii) Let $f : (\eta : A \to M[1]) \to (\eta' : B \to N[1])$ be a morphism in $\mathcal{D}\mathrm{er}$ between objects which belong to $\mathcal{D}\mathrm{er}^+$. Then f belongs to $\mathcal{D}\mathrm{er}^+$ if and only if the induced map $B \otimes_A M \to N$ is an equivalence of B-modules.

We can now formulate $(*)$ more precisely:

Proposition 8.4.2.5. *Let A be a connective $\mathbb{E}_\infty$-ring, M a connective A-module, and $\eta : A \to M[1]$ a derivation. Then the functor Φ of Notation 8.4.1.5 induces an equivalence of ∞-categories*

$$\mathcal{D}\mathrm{er}^+_{\eta/} \to \mathrm{CAlg}^{\mathrm{cn}}_{A^\eta},$$

given on objects by $(\eta' : B \to N[1]) \mapsto B^{\eta'}$.

We will deduce Proposition 8.4.2.5 from a slightly stronger result, whose statement will require a bit more notation.

Notation 8.4.2.6. We define a subcategory $\mathrm{Fun}^+(\Delta^1, \mathrm{CAlg})$ as follows:

(i) An object $f : \widetilde{A} \to A$ of $\mathrm{Fun}(\Delta^1, \mathrm{CAlg})$ belongs to $\mathrm{Fun}^+(\Delta^1, \mathrm{CAlg})$ if and only if both A and $\widetilde{A}$ are connective, and f induces a surjection $\pi_0 \widetilde{A} \to \pi_0 A$.

(ii) Let $f, g \in \mathrm{Fun}^+(\Delta^1, \mathrm{CAlg})$, and let $\alpha : f \to g$ be a morphism in $\mathrm{Fun}(\Delta^1, \mathrm{CAlg})$. Then α belongs to $\mathrm{Fun}^+(\Delta^1, \mathrm{CAlg})$ if and only if it classifies a pushout square in the ∞-category CAlg.

Theorem 8.4.2.7. *Let $\Phi : \mathcal{D}\mathrm{er} \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg})$ be the functor defined in Notation 8.4.1.5, given by $(\eta : A \to M[1]) \mapsto A^\eta$. Then Φ induces a functor $\Phi^+ : \mathcal{D}\mathrm{er}^+ \to \mathrm{Fun}^+(\Delta^1, \mathrm{CAlg})$. Moreover, the functor Φ^+ factors as a composition*

$$\mathcal{D}\mathrm{er}^+ \xrightarrow{\Phi^+_0} \overline{\mathcal{D}\mathrm{er}}^+ \xrightarrow{\Phi^+_1} \mathrm{Fun}^+(\Delta^1, \mathrm{CAlg}),$$

where Φ^+_0 is an equivalence of ∞-categories and Φ^+_1 is a left fibration.

Proof of Proposition 8.4.2.5. Let $\eta : A \to M[1]$ be an object of $\mathcal{D}\mathrm{er}^+$. Theorem 8.4.2.7 implies that Φ induces an equivalence $\mathcal{D}\mathrm{er}^+_{\eta/} \to \mathrm{Fun}^+(\Delta^1, \mathrm{CAlg})_{\Phi(\eta)/}$. It now suffices to observe that the evaluation map $\mathrm{Fun}^+(\Delta^1, \mathrm{CAlg})_{\Phi(\eta)/} \to \mathrm{CAlg}^{\mathrm{cn}}_{A^\eta}$ is a trivial Kan fibration. $\qquad\square$

The proof of Theorem 8.4.2.7 will require a few lemmas.

Lemma 8.4.2.8. *Let*

be a commutative diagram in the ∞-category $\mathcal{D}\mathrm{er}$. If f and f' belong to $\mathcal{D}\mathrm{er}^+$, then so does f''.

Proof. This follows immediately from Proposition T.2.4.1.7. $\qquad\square$

Lemma 8.4.2.9. *Let $f : (\eta : A \to M[1]) \to (\eta' : B \to N[1])$ be a morphism in $\mathcal{D}\mathrm{er}^+$. If the induced map $A^\eta \to B^\eta$ is an equivalence of $\mathbb{E}_\infty$-rings, then f is an equivalence.*

Proof. The morphism f determines a map of fiber sequences

$$
\begin{array}{ccccc}
A^{\eta} & \longrightarrow & A & \longrightarrow & M[1] \\
\downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} \\
B^{\eta'} & \longrightarrow & B & \longrightarrow & N[1]
\end{array}
$$

Since the left vertical map is an equivalence, we obtain an equivalence $\alpha : \operatorname{cofib}(f_0) \simeq \operatorname{cofib}(f_1)$. To complete the proof, it will suffice to show that $\operatorname{cofib}(f_0)$ vanishes. Suppose otherwise. Since $\operatorname{cofib}(f_0)$ is connective, there exists some smallest integer n such that $\pi_n \operatorname{cofib}(f_0) \neq 0$. In particular, $\operatorname{cofib}(f_0)$ is n-connective.

Since f induces an equivalence $B \otimes_A M \to N$, the cofiber $\operatorname{cofib}(f_1)$ can be identified with $\operatorname{cofib}(f_0) \otimes_A M[1]$. Since M is connective, we deduce that $\operatorname{cofib}(f_1)$ is $(n+1)$-connective. Using the equivalence α, we conclude that $\operatorname{cofib}(f_0)$ is $(n+1)$-connective, which contradicts our assumption that $\pi_n \operatorname{cofib}(f_0) \neq 0$. $\square$

Lemma 8.4.2.10. *Let $\mathcal{D}_0 \subseteq \mathcal{D}$ be small ∞-categories, and let $p : \mathcal{M} \to \mathcal{C}$ be a presentable fibration. Then:*

(1) *The induced map*
$$
q : \operatorname{Fun}(\mathcal{D}, \mathcal{M}) \to \operatorname{Fun}(\mathcal{D}, \mathcal{C}) \times_{\operatorname{Fun}(\mathcal{D}_0, \mathcal{C})} \operatorname{Fun}(\mathcal{D}_0, \mathcal{M})
$$
is a coCartesian fibration.

(2) *A morphism in $\operatorname{Fun}(\mathcal{D}, \mathcal{M})$ is q-coCartesian if and only if the induced functor $f : \mathcal{D} \times \Delta^1 \to \mathcal{M}$ is a p-left Kan extension of its restriction to $(\mathcal{D} \times \{0\}) \coprod_{\mathcal{D}_0 \times \{0\}} (\mathcal{D} \times \Delta^1)$.*

Proof. The "if" direction of (2) follows immediately from Lemma T.4.3.2.12. Since every diagram

$$
\begin{array}{ccc}
(\mathcal{D} \times \{0\}) \coprod_{\mathcal{D}_0 \times \{0\}} (\mathcal{D} \times \Delta^1) & \longrightarrow & \mathcal{M} \\
\cap \downarrow & \nearrow & \downarrow{\scriptstyle p} \\
\mathcal{D} \times \Delta^1 & \longrightarrow & \mathcal{C}
\end{array}
$$

admits an extension as indicated, which is a p-left Kan extension, assertion (1) follows immediately. The "only if" direction of (2) then follows from the uniqueness properties of q-coCartesian morphisms. $\square$

Proof of Theorem 8.4.2.7. We let $\widetilde{\mathcal{D}\mathrm{er}}$ denote the ∞-category of extended derivations in CAlg (so that the forgetful functor $\widetilde{\mathcal{D}\mathrm{er}} \to \mathcal{D}\mathrm{er}$ is a trivial Kan fibration), and $\mathcal{M}^T$ a tangent correspondence for the ∞-category CAlg. Form a pullback diagram

$$
\begin{array}{ccc}
\widetilde{\mathcal{D}\mathrm{er}}^{+} & \longrightarrow & \widetilde{\mathcal{D}\mathrm{er}} \\
\downarrow{\scriptstyle u^+} & & \downarrow{\scriptstyle u} \\
\mathcal{D}\mathrm{er}^{+} & \longrightarrow & \mathcal{D}\mathrm{er}\,.
\end{array}
$$

Since the map u is a trivial Kan fibration, u^+ is also a trivial Kan fibration.

Let $\mathcal{X}$ denote the full subcategory of $\operatorname{Fun}(\Delta^1 \times \Delta^1, \mathrm{CAlg})$ spanned by those diagrams

$$
\begin{array}{ccc}
\widetilde{A} & \longrightarrow & A \\
\downarrow & & \downarrow{\scriptstyle \alpha} \\
A' & \xrightarrow{\ \beta\ } & A''.
\end{array}
$$

such that α and β are equivalences. The diagonal inclusion $\Delta^1 \subseteq \Delta^1 \times \Delta^1$ induces a map $\epsilon : \mathcal{X} \to$ $\mathrm{Fun}(\Delta^1, \mathrm{CAlg})$. Using Proposition T.4.3.2.15, we deduce that this map is a trivial Kan fibration. The map ϵ has a section υ, which carries a morphism $\widetilde{A} \to A$ to the commutative diagram

$$
\begin{array}{ccc}
\widetilde{A} & \longrightarrow & A \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
A & \xrightarrow{\ \mathrm{id}\ } & A.
\end{array}
$$

It follows that υ is also an equivalence.

Let $\overline{\mathcal{D}\mathrm{er}}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{M}^T)$ spanned by those pullback diagrams

$$
\begin{array}{ccc}
\widetilde{A} & \longrightarrow & A \\
\downarrow & & \downarrow{\scriptstyle \eta} \\
0 & \xrightarrow{\ \gamma\ } & M
\end{array}
$$

such that the objects $\widetilde{A}$ and A belong to $\mathrm{CAlg} \subseteq \mathcal{M}^T$, the objects 0 and M belong to T_{CAlg}, the maps η and γ induce equivalences in CAlg, and 0 is a p-initial object of T_{CAlg}. We have a homotopy pullback diagram of ∞-categories

$$
\begin{array}{ccc}
\widetilde{\mathcal{D}\mathrm{er}} & \xrightarrow{\ \upsilon'\ } & \overline{\mathcal{D}\mathrm{er}} \\
\downarrow & & \downarrow \\
\mathrm{Fun}(\Delta^1, \mathrm{CAlg}) & \xrightarrow{\ \upsilon\ } & \mathcal{X}.
\end{array}
$$

Since υ is a categorical equivalence, we conclude that υ' is also a categorical equivalence.

The functor Φ is defined to be a composition

$$
\mathcal{D}\mathrm{er} \xrightarrow{s} \widetilde{\mathcal{D}\mathrm{er}} \xrightarrow{\upsilon'} \overline{\mathcal{D}\mathrm{er}} \xrightarrow{s'} \mathcal{X} \xrightarrow{s''} \mathrm{Fun}(\Delta^1, \mathrm{CAlg}),
$$

where s is a section to u and s'' is the map which carries a diagram

$$
\begin{array}{ccc}
\widetilde{A} & \longrightarrow & A \\
\downarrow{\scriptstyle \alpha} & & \downarrow \\
A' & \longrightarrow & A''
\end{array}
$$

to the map α. We define Φ_0^+ and Φ_1^+ to be the restrictions of $\upsilon' \circ s$ and $s'' \circ s'$, respectively. To complete the proof, it will suffice to show that s' induces a left fibration of simplicial sets $\overline{\mathcal{D}\mathrm{er}}^+ \to \mathrm{Fun}^+(\Delta^1, \mathrm{CAlg})$. To prove this, we will describe the ∞-category $\overline{\mathcal{D}\mathrm{er}}^+$ in another way.

Let $\overline{\mathcal{D}}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{M}^T)$ spanned by morphisms of the form $\eta_0 : \widetilde{A} \to 0$, satisfying the following conditions:

(i) The object $\widetilde{A}$ belongs to $\mathrm{CAlg} \subseteq \mathcal{M}^T$.

(ii) Let $f : \widetilde{A} \to A'$ be the image of η_0 under the map $\mathcal{M}^T \to \mathrm{CAlg}$. Then $\widetilde{A}$ and A' are connective, and f induces a surjection $\pi_0 \widetilde{A} \to \pi_0 A'$.

(iii) The object 0 belongs to $T_{\mathrm{CAlg}} \subseteq \mathcal{M}^T$. Moreover, 0 is a zero object of $T_{\mathrm{CAlg}} \times_{\mathrm{CAlg}} \{A'\} \simeq \mathrm{Mod}_{A'}$.

Using Proposition T.4.3.2.15, we deduce that projection map $\psi_0 : \overline{\mathcal{D}} \to \mathrm{Fun}'(\Delta^1, \mathrm{CAlg})$ is a trivial Kan fibration, where $\mathrm{Fun}'(\Delta^1, \mathrm{CAlg})$ denotes the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{CAlg})$ spanned by those morphisms $\widetilde{A} \to A$ which satisfy condition (ii).

Let $\mathcal{D}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{M})$ spanned by those diagrams

$$
\begin{array}{ccc}
\widetilde{A} & \xrightarrow{\ e\ } & A \\
\Big\downarrow{\eta_0} & & \Big\downarrow{\eta} \\
0 & \xrightarrow{\ \gamma\ } & M
\end{array}
$$

satisfying properties (i), (ii), and (iii) above, where $A \in \mathrm{CAlg} \subseteq \mathcal{M}^T$ and $M \in T_{\mathrm{CAlg}} \subseteq \mathcal{M}^T$. Restriction to the left half of the diagram yields a forgetful functor $\psi_1 : \mathcal{D} \to \overline{\overline{\mathcal{D}}}$, which fits into a pullback square

$$
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\hspace{3cm}} & \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{M}) \\
\Big\downarrow{\psi_1} & & \Big\downarrow{\psi_1'} \\
\overline{\mathcal{D}} & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{M}^T) \times_{\mathrm{Fun}(\Delta^1, \Delta^1)} \mathrm{Fun}(\Delta^1 \times \Delta^1, \Delta^1).
\end{array}
$$

Applying Lemma 8.4.2.10 to the presentable fibration $\mathcal{M}^T \to \Delta^1$, we conclude that ψ_1' is a coCartesian fibration. It follows that ψ_1 is also a coCartesian fibration, and that a morphism in $\mathcal{D}$ is ψ_1-coCartesian if and only if it satisfies criterion (2) in the statement of Lemma 8.4.2.10.

We define subcategories $\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \mathcal{D}$ as follows:

- Every object of $\mathcal{D}$ belongs to $\mathcal{D}_1$.

- A morphism f in $\mathcal{D}$ belongs to $\mathcal{D}_1$ if and only if $(\psi_0 \circ \psi_1)(f)$ belongs to $\mathrm{Fun}^+(\Delta^1, \mathrm{CAlg})$, and f is ψ_1-coCartesian. Since ψ_0 is a trivial Kan fibration, this is equivalent to the requirement that f is $\psi_1 \circ \psi_0$-coCartesian.

- We define $\mathcal{D}_0$ to be the full subcategory of is the full subcategory of $\mathcal{D}_1$ spanned by those diagrams

$$
\begin{array}{ccc}
\widetilde{A} & \longrightarrow & A \\
\Big\downarrow & & \Big\downarrow{\eta} \\
0 & \xrightarrow{\ \gamma\ } & M
\end{array}
$$

which are pullback diagrams in $\mathcal{M}^T$, such that η and γ induce equivalences in CAlg.

Using Corollary T.2.4.2.5, we deduce immediately that $\psi_0 \circ \psi_1$ induces a left fibration $\psi : \mathcal{D}_1 \to \mathrm{Fun}^+(\Delta^1, \mathcal{C})$. To complete the proof, it will suffice to verify the following:

(1) The subcategory $\mathcal{D}_0 \subseteq \mathcal{D}_1$ is a *cosieve* in $\mathcal{D}_1$. That is, if $f : X \to Y$ is a morphism in $\mathcal{D}_1$ and X belongs to $\mathcal{D}_0$, then Y also belongs to $\mathcal{D}_0$. It follows immediately that ψ restricts to a left fibration $\mathcal{D}_0 \to \mathrm{Fun}^+(\Delta^1, \mathrm{CAlg})$.

(2) We have an equality $\mathcal{D}_0 = \overline{\mathcal{D}\mathrm{er}}^+$ of subcategories of $\overline{\mathcal{D}\mathrm{er}}$.

In order to prove these results, we will need to analyze the structure of a morphism $f : X \to Y$ in the ∞-category $\mathcal{D}$ in more detail. Let us suppose that $X, Y \in \mathcal{D}$ classify diagrams

$$
\begin{array}{ccccccccc}
\widetilde{A} & \longrightarrow & A & & & & \widetilde{B} & \longrightarrow & B \\
\Big\downarrow & & \Big\downarrow & & & & \Big\downarrow & & \Big\downarrow \\
0 & \longrightarrow & M & & & & 0' & \longrightarrow & N
\end{array}
$$

in $\mathcal{M}^T$, lying over diagrams

$$
\begin{array}{ccc}
\widetilde{A} & \longrightarrow & A \\
\downarrow & & \downarrow \\
A' & \longrightarrow & A''
\end{array}
\qquad\qquad
\begin{array}{ccc}
\widetilde{B} & \longrightarrow & B \\
\downarrow & & \downarrow \\
B' & \longrightarrow & B''
\end{array}
$$

in E_∞. Unwinding the definitions, we see that the morphism f belongs to $\mathcal{D}_1$ if and only if the following conditions are satisfied:

(a) The morphism $\psi(f)$ belongs to $\mathrm{Fun}^+(\Delta^1, \mathrm{CAlg})$. In other words, the diagram

$$
\begin{array}{ccc}
\widetilde{A} & \longrightarrow & \widetilde{B} \\
\downarrow & & \downarrow \\
A' & \longrightarrow & B'
\end{array}
$$

is a pushout square of $\mathbb{E}_\infty$-rings.

(b) The diagram

$$
\begin{array}{ccc}
\widetilde{A} & \longrightarrow & \widetilde{B} \\
\downarrow & & \downarrow \\
A & \longrightarrow & B
\end{array}
$$

is a pushout square of $\mathbb{E}_\infty$-rings.

(c) The diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & M \\
\downarrow & & \downarrow{\scriptstyle j} \\
0' & \longrightarrow & N
\end{array}
$$

is a pushout square in T_{CAlg}. Unwinding the definitions, this is equivalent to the requirement that the diagram

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
A'' & \longrightarrow & B''
\end{array}
$$

is a pushout square of $\mathbb{E}_\infty$-rings, and that the induced map $M \otimes_{A''} B'' \to N$ is an equivalence of B''-modules.

We observe that (b) and (c) are simply a translation of the requirement that f satisfies criterion (2) of Lemma 8.4.2.10.

We now prove (1). Suppose that $X \in \mathcal{D}_0$; we wish to prove that $Y \in \mathcal{D}_0$. It follows from (c) that the map $B \to B''$ is an equivalence. To prove that the map $B' \to B''$ is an equivalence, we consider the commutative diagram

$$
\begin{array}{ccccc}
\widetilde{A} & \longrightarrow & A' & \longrightarrow & A'' \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{B} & \longrightarrow & B' & \longrightarrow & B''.
\end{array}
$$

From (a) we deduce that the left square is a pushout, and from (b) and (c) together we deduce that the large rectangle is a pushout. It follows that the right square is a pushout as well. Since the map $A' \to A''$ is

an equivalence (in virtue of our assumption that $X \in \mathcal{D}_0$), we conclude that $B' \to B''$ is an equivalence as desired.

To complete the proof that $Y \in \mathcal{D}_0$, it will suffice to show that Y is a pullback diagram. This is equivalent to the assertion that the induced diagram Y':

$$
\begin{array}{ccc}
\widetilde{B} & \longrightarrow & B' \\
\downarrow & & \downarrow \\
B & \longrightarrow & B'' \oplus N
\end{array}
$$

is a pullback diagram of $\mathbb{E}_\infty$ $\widetilde{B}$-algebras. Since the forgetful functor $\mathrm{CAlg}(\mathrm{Mod}_{\widetilde{B}}) \to \mathrm{Mod}_{\widetilde{B}}$ preserves limits, it will suffice to show that Y' is a pullback diagram in the ∞-category of $\widetilde{B}$-modules.

Let X' denote the diagram

$$
\begin{array}{ccc}
\widetilde{A} & \longrightarrow & A' \\
\downarrow & & \downarrow \\
A & \longrightarrow & A'' \oplus M
\end{array}
$$

determined by X. Since $X \in \mathcal{D}_0$, X' is a pullback diagram of E_∞-rings, and therefore a pullback diagram of $\widetilde{A}$-modules. Since the relative tensor product functor $\otimes_{\widetilde{A}} \widetilde{B}$ is exact, it will suffice to show that the map $f : X \to Y$ induces an equivalence $X' \otimes_{\widetilde{A}} \widetilde{B} \to Y'$. In other words, it suffices to show that each of the induced diagrams

$$
\begin{array}{ccc}
\widetilde{A} \longrightarrow A \\
\downarrow \quad\quad \downarrow \\
\widetilde{B} \longrightarrow B
\end{array}
\qquad
\begin{array}{ccc}
\widetilde{A} \longrightarrow A' \\
\downarrow \quad\quad \downarrow \\
\widetilde{B} \longrightarrow B'
\end{array}
\qquad
\begin{array}{ccc}
\widetilde{A} \longrightarrow A'' \oplus M \\
\downarrow \quad\quad\quad \downarrow \\
\widetilde{B} \longrightarrow B'' \oplus N
\end{array}
$$

is a pushout square of $\mathbb{E}_\infty$-rings. For the left and middle squares, this follows from (a) and (b). The rightmost square fits into a commutative diagram

$$
\begin{array}{ccccc}
\widetilde{A} & \longrightarrow & A & \longrightarrow & A'' \oplus M \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{B} & \longrightarrow & B & \longrightarrow & B'' \oplus N
\end{array}
$$

where the left part of the diagram is a pushout square by (b) and the right square is a pushout by (c). This completes the proof that $Y \in \mathcal{D}_0$, so that $\mathcal{D}_0 \subseteq \mathcal{D}_1$ is a cosieve as desired.

We now prove (2). We first show that the subcategories

$$
\mathcal{D}_0, \overline{\mathcal{D}\mathrm{er}}^+ \subseteq \overline{\mathcal{D}\mathrm{er}}
$$

consist of the same objects. Let $X \in \overline{\mathcal{D}\mathrm{er}}$ be given by a diagram

$$
\begin{array}{ccc}
\widetilde{A} & \longrightarrow & A' \\
\downarrow & & \downarrow \\
0 & \longrightarrow & M,
\end{array}
$$

projecting to a diagram

$$
\begin{array}{ccc}
\widetilde{A} & \longrightarrow & A' \\
\downarrow & & \downarrow \\
A & \longrightarrow & A''
\end{array}
$$

in the ∞-category CAlg. Then X belongs to $\mathcal{D}_0$ if and only if the both A and $\widetilde{A}$ are connective, and the map $\pi_0 \widetilde{A} \to \pi_0 A$ is surjective. On the other hand, X belongs to $\widetilde{\mathcal{D}\mathrm{er}}^+$ if and only if both A' and $M[-1]$ are connective. The equivalence of these conditions follows immediately from the observation that A and A' are equivalent, and the long exact sequence of homotopy groups associated to the fiber sequence

$$\widetilde{A} \to A \to M.$$

Now let us suppose that $f : X \to Y$ is a morphism in $\overline{\mathcal{D}\mathrm{er}}$, where both X and Y belong to $\mathcal{D}_0$. We wish to show that f belongs to $\mathcal{D}_0$ if and only if f belongs to $\overline{\mathcal{D}\mathrm{er}}^+$. We observe that f belongs to $\mathcal{D}_0$ if and only if f satisfies the conditions (a), (b), and (c) described above. On the other hand, f belongs to $\overline{\mathcal{D}\mathrm{er}}^+$ if and only if the induced map $M \otimes_{A''} B'' \to N$ is an equivalence. Since this follows immediately from condition (c), we conclude that we have inclusions

$$\mathcal{D}_0 \subseteq \widetilde{\mathcal{D}\mathrm{er}}^+ \subseteq \widetilde{\mathcal{D}\mathrm{er}}.$$

To prove the reverse inclusion, let $f : X \to Y$ be a morphism in $\overline{\mathcal{D}\mathrm{er}}^+$. We wish to show that f belongs to $\mathcal{D}_0$. In other words, we must show that f satisfies conditions (a), (b), and (c). Since the maps

$$A \to A'' \leftarrow A' \qquad B \to B'' \leftarrow B'$$

are equivalences, condition (c) is automatic and conditions (a) and (b) are equivalent to one another. We are therefore reduced to the problem of showing that the diagram

$$
\begin{array}{ccc}
\widetilde{A} & \longrightarrow & \widetilde{B} \\
\downarrow & & \downarrow \\
A & \longrightarrow & B
\end{array}
$$

is a pushout square.

The image $\Phi(f)$ can be factored as a composition $g' \circ g''$, corresponding to a diagram

$$
\begin{array}{ccccc}
\widetilde{A} & \longrightarrow & \widetilde{B} & \xrightarrow{\mathrm{id}} & \widetilde{B} \\
\downarrow & & \downarrow & & \downarrow \\
A & \longrightarrow & C & \longrightarrow & B,
\end{array}
$$

where the left square is a pushout. Let $f' : X \to Z$ be a ψ-coCartesian lift of g', so that f is homotopic to some composition $X \xrightarrow{f'} Z \xrightarrow{f''} Y$. We observe that f' belongs to $\mathcal{D}_0$. It will therefore suffice to show that f'' belongs to $\mathcal{D}_0$ as well. Lemma 8.4.2.8 implies that f'' belongs to $\overline{\mathcal{D}\mathrm{er}}^+$. We may therefore replace f by f'' and thereby reduce to the situation where f induces an equivalence $\widetilde{A} \to \widetilde{B}$. In this case, condition (a) is equivalent to the assertion that f induces an equivalence $A \to B$, which follows from Lemma 8.4.2.9. $\square$

8.4.3   Connectivity and Finiteness of the Cotangent Complex

Let $\phi : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings. According to Remark 8.3.3.4, the relative cotangent complex $L_{B/A}$ vanishes whenever ϕ is an equivalence. We may therefore regard $L_{B/A}$ as a measure of a failure of f to be an equivalence. Our goal in this section is to prove that this is a good measure in (at least) two respects:

(1) The relative cotangent complex $L_{B/A}$ is highly connected if ϕ is highly connected. Moreover, the converse holds if ϕ induces an isomorphism of commutative rings $\pi_0 A \to \pi_0 B$ (Corollary 8.4.3.2).

(2) If B is locally of finite presentation over A (Definition 8.2.5.26), then the relative cotangent complex $L_{B/A}$ is perfect. Conversely, if $L_{B/A}$ is perfect and $\pi_0 B$ is a finitely presented $\pi_0 A$-algebra, then B is locally of finite presentation over A. Similar results hold if we replace "locally of finite presentation" by "almost of finite presentation" and "perfect" by "almost perfect" (Theorem 8.4.3.18).

We begin with the following more precise formulation of (1):

Theorem 8.4.3.1. *Let $f : A \to B$ be a morphism between connective $\mathbb{E}_\infty$-rings. Assume that $\mathrm{cofib}(f)$ is n-connective for $n \geq 0$. Then there is a canonical $(2n)$-connective map of B-modules $\epsilon_f : B \otimes_A \mathrm{cofib}(f) \to L_{B/A}$.*

We will give the proof of Theorem 8.4.3.1 later in this section. First, let us deduce some consequences.

Corollary 8.4.3.2. *Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings. Assume that $\mathrm{cofib}(f)$ is n-connective, for $n \geq 0$. Then the relative cotangent complex $L_{B/A}$ is n-connective. The converse holds provided that f induces an isomorphism $\pi_0 A \to \pi_0 B$.*

Proof. Let $\epsilon_f : B \otimes_A \mathrm{cofib}(f) \to L_{B/A}$ be the map described in Theorem 8.4.3.1, so that we have a fiber sequence of B-modules:

$$\mathrm{fib}(\epsilon_f) \to B \otimes_A \mathrm{cofib}(f) \to L_{B/A}$$

To prove that $L_{B/A}$ is n-connective, it suffices to show that $\mathrm{cofib}(f) \otimes_A B$ is n-connective and that $\mathrm{fib}(\epsilon_f)$ is $(n-1)$-connective. The first assertion is obvious, and the second follows from Theorem 8.4.3.1 since $2n \geq n - 1$.

To prove the converse, let us suppose that $\mathrm{cofib}(f)$ is *not* n-connective. We wish to show that $L_{B/A}$ is not n-connective. Let us assume that n is chosen as small as possible, so that $\mathrm{cofib}(f)$ is $(n-1)$-connective. By assumption, f induces an isomorphism $\pi_0 A \to \pi_0 B$, so we must have $n \geq 2$. Applying Theorem 8.4.3.1, we conclude that ϵ_f is $(2n-2)$-connective. Since $n \geq 2$, we deduce in particular that ϵ_f is n-connective, so that the map $\pi_{n-1}(B \otimes_A \mathrm{cofib}(f)) \to \pi_{n-1} L_{B/A}$ is an isomorphism. Since $\mathrm{cofib}(f)$ is $(n-1)$-connective and $\pi_0 A \simeq \pi_0 B$, the map $\pi_{n-1} \mathrm{cofib}(f) \to \pi_{n-1}(B \otimes_A \mathrm{cofib}(f))$ is an isomorphism. It follows that $\pi_{n-1} \mathrm{cofib}(f) \to \pi_{n-1} L_{B/A}$ is also an isomorphism, so that $\pi_{n-1} L_{B/A}$ is nonzero. $\square$

Corollary 8.4.3.3. *Let A be a connective $\mathbb{E}_\infty$-ring. Then the absolute cotangent complex L_A is connective.*

Proof. Apply Corollary 8.4.3.2 to the unit map $S \to A$ in the case $n = 0$. $\square$

Corollary 8.4.3.4. *Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings. Then f is an equivalence if and only if the following conditions are satisfied:*

(1) *The map f induces an isomorphism $\pi_0 A \to \pi_0 B$.*

(2) *The relative cotangent complex $L_{B/A}$ vanishes.*

Corollary 8.4.3.5. *Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings. Assume that $\mathrm{cofib}(f)$ is n-connective for $n \geq 0$. Then the induced map $L_f : L_A \to L_B$ has n-connective cofiber. In particular, the canonical map $\pi_0 L_A \to \pi_0 L_{\pi_0 A}$ is an isomorphism.*

Proof. The map L_f factors as a composition

$$L_A \xrightarrow{g} B \otimes_A L_A \xrightarrow{g'} L_B.$$

We observe that $\mathrm{cofib}(g) \simeq \mathrm{cofib}(f) \otimes_A L_A$. Since the cotangent complex L_A is connective (Corollary 8.4.3.3) and $\mathrm{cofib}(f)$ is n-connective, we conclude that $\mathrm{cofib}(g)$ is n-connective. It will therefore suffice to show that $\mathrm{cofib}(g') \simeq L_{B/A}$ is n-connective. Let $\epsilon_f : B \otimes_A \mathrm{cofib}(f) \to L_{B/A}$ be as in Theorem 8.4.3.1, so we have a fiber sequence

$$B \otimes_A \mathrm{cofib}(f) \to L_{B/A} \to \mathrm{cofib}(\epsilon_f).$$

It therefore suffices to show that $B \otimes_A \mathrm{cofib}(f)$ and $\mathrm{cofib}(\epsilon_f)$ are n-connective. The first assertion follows immediately from the n-connectivity of $\mathrm{cofib}(f)$, and the second from Theorem 8.4.3.1 since $2n + 1 \geq n$. $\square$

Corollary 8.4.3.6. *Let $f : A \to B$ be a map of $\mathbb{E}_\infty$-rings, and suppose that $\mathrm{cofib}(f)$ is n-connective. Then there exists a canonical $(2n-1)$-connective map of A-modules $\mathrm{cofib}(f) \to L_{B/A}$.*

Proof. We have a fiber sequence

$$\mathrm{cofib}(f) \xrightarrow{\delta} B \otimes_A \mathrm{cofib}(f) \to \mathrm{cofib}(f) \otimes_A \mathrm{cofib}(f),$$

so that the map δ is $(2n-1)$-connective. If ϵ_f denotes the map appearing in the statement of Theorem 8.4.3.1, then $\epsilon_f \circ \delta$ is a $(2n-1)$-connective map $\mathrm{cofib}(f) \to L_{B/A}$. $\square$

The theory of the cotangent complex of $\mathbb{E}_\infty$-rings can be regarded as a homotopy-theoretic analogue of the classical theory of Kähler differentials. We now apply Theorem 8.4.3.1 to make this analogy more explicit. First, we need a bit of terminology. Recall that if R is a commutative ring, then the module of (absolute) Kähler differentials is the free R-module generated by the symbols $\{dr\}_{r \in R}$, subject to the relations

$$d(rr') = r\,dr' + r'\,dr$$

$$d(r + r') = dr + dr'.$$

We denote this R-module by Ω_R. Given a map of commutative rings $\eta : R' \to R$, we let $\Omega_{R/R'}$ denote the quotient of Ω_R by the submodule generated by the elements $\{d\eta(r')\}_{r' \in R'}$.

Remark 8.4.3.7. Let $\eta : R' \to R$ be a homomorphism of commutative rings. Then we have a canonical short exact sequence

$$\mathrm{Tor}_0^{R'}(\Omega_{R'}, R) \to \Omega_R \to \Omega_{R/R'} \to 0$$

in the category of R-modules.

Lemma 8.4.3.8. *Let A be a discrete $\mathbb{E}_\infty$-ring. Then there is a canonical isomorphism*

$$\pi_0 L_A \simeq \Omega_{\pi_0 A}$$

in the category of (discrete) $\pi_0 A$-modules.

Proof. It will suffice to show that $\pi_0 L_A$ and $\Omega_{\pi_0 A}$ corepresent the same functor on the ordinary category of modules over the commutative ring $\pi_0 A$. Let M be a $\pi_0 A$-module, which we will identify with the corresponding discrete A-module. We have homotopy equivalences

$$\mathrm{Map}_{\mathrm{Mod}_A}(\pi_0 L_A, M) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(L_A, M) \simeq \mathrm{Map}_{\mathrm{CAlg}_{/A}}(A, A \oplus M).$$

Since A and M are both discrete, the space on the right is homotopy equivalent to the discrete set of ring homomorphisms from $\pi_0 A$ to $\pi_0(A \oplus M)$ which reduce to the identity on $\pi_0 A$. These are simply derivations from $\pi_0 A$ into M in the usual sense, which are classified by $(\pi_0 A)$-module homomorphisms from $\Omega_{\pi_0 A}$ into M. $\square$

Proposition 8.4.3.9. *Let $f : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings. Then:*

(1) *The relative cotangent $L_{B/A}$ is connective.*

(2) *As a $\pi_0 B$-module, $\pi_0 L_{B/A}$ is canonically isomorphic to the module of relative Kähler differentials $\Omega_{\pi_0 B/\pi_0 A}$.*

Proof. Assertion (1) follows from Corollary 8.4.3.3 and the existence of a fiber sequence

$$L_A \otimes_A B \to L_B \to L_{B/A}.$$

Associated to this fiber sequence we have a long exact sequence

$$\pi_0(L_A \otimes_A B) \xrightarrow{g} \pi_0 L_B \to \pi_0 L_{B/A} \to \pi_{-1}(L_A \otimes_A B) \simeq 0$$

of discrete $\pi_0 B$-modules. Consequently, we may identify $\pi_0 L_{B/A}$ with the cokernel of the map g.

Using Corollary 8.4.3.5 and Lemma 8.4.3.8, we can identify $\pi_0 L_A$ and $\pi_0 L_B$ with the modules $\Omega_{\pi_0 A}$ and $\Omega_{\pi_0 B}$, respectively. Using Corollary 8.2.1.23, we can identify $\pi_0(L_A \otimes_A B)$ with the discrete $\pi_0 B$-module $\mathrm{Tor}_0^{\pi_0 A}(\Omega_{\pi_0 A}, \pi_0 B)$. The desired result now follows from the short exact sequence of Remark 8.4.3.7. $\square$

We now turn to the proof of Theorem 8.4.3.1. We begin with a construction of the map ϵ_f. For later applications, it will be useful to work in a slightly more general setting.

Construction 8.4.3.10. Let $\mathcal{C}$ denote a symmetric monoidal stable ∞-category equipped with a t-structure which satisfies the following conditions:

(i) The ∞-category $\mathcal{C}$ is presentable.

(ii) The tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable.

(iii) The full subcategory $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ contains the unit object and is closed under tensor products.

Let $f : A \to B$ be a morphism of commutative algebra objects of $\mathcal{C}$, so that f induces a map of B-module objects $\eta : L_B \to L_{B/A}$. We let B^η denote the associated square-zero extension of B by $L_{B/A}[-1]$. Since the restriction of η to L_A is nullhomotopic, the map f factors as a composition

$$A \xrightarrow{f'} B^\eta \xrightarrow{f''} B.$$

We therefore obtain a map of A-modules $\mathrm{cofib}(f) \to \mathrm{cofib}(f'')$, which is adjoint to a map of B-modules

$$\epsilon_f : B \otimes_A \mathrm{cofib}(f) \to \mathrm{cofib}(f'') \simeq L_{B/A}.$$

Remark 8.4.3.11. The ∞-category of spectra (endowed with the smash product monoidal structure) satisfies conditions (i) through (iii) of Construction 8.4.3.10. However, these conditions are also satisfied in other situations of interest: for example, $\mathcal{C}$ might be the ∞-category of sheaves of spectra on a topological space X.

We can now state the following more precise version of Theorem 8.4.3.1:

Theorem 8.4.3.12. *Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category equipped with a t-structure satisfying conditions (i) through (iii) of Construction 8.4.3.10. Let $f : A \to B$ be a morphism of commutative algebra objects of $\mathcal{C}_{\geq 0}$, and assume that $\mathrm{cofib}(f) \in \mathcal{C}_{\geq n}$. Let $\epsilon_f : B \otimes_A \mathrm{cofib}(f) \to L_{B/A}$ be as in Construction 8.4.3.10. Then $\mathrm{fib}(\epsilon_f) \in \mathcal{C}_{\geq 2n}$.*

The proof of Theorem 8.4.3.12 will require some preliminaries.

Remark 8.4.3.13. Let $\mathcal{C}$ be as in Construction 8.4.3.10, and let $M \in \mathcal{C}_{\geq m}$. Then each tensor power $M^{\otimes k}$ belongs to $\mathcal{C}_{\geq km}$. Using the closure of $\mathcal{C}_{\geq km}$ under colimits, we conclude that the symmetric power $\mathrm{Sym}^k(M)$ belongs to $\mathcal{C}_{\geq km}$.

Proposition 8.4.3.14. *Let $\mathcal{C}$ be as in Construction 8.4.3.10, let $M \in \mathcal{C}$, and let $A = \mathrm{Sym}^* M$ denote the commutative algebra object of $\mathcal{C}$ freely generated by M. Then there is a canonical equivalence $L_A \simeq A \otimes M$ in the ∞-category of A-modules.*

Proof. For every A-module N, we have a chain of homotopy equivalences

$$
\begin{aligned}
\mathrm{Map}_{\mathrm{Mod}_A(\mathcal{C})}(A \otimes M, N) &\simeq \mathrm{Map}_{\mathcal{C}}(M, N) \\
&\simeq \mathrm{Map}_{\mathcal{C}_{/A}}(M, A \oplus N) \\
&\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathcal{C})_{/A}}(A, A \oplus N) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_A(\mathcal{C})}(L_A, N).
\end{aligned}
$$

It follows that $M \otimes A$ and L_A corepresent the same functor in the homotopy category $\mathrm{hMod}_A(\mathcal{C})$, and are therefore equivalent. $\qquad\square$

Our connectivity estimates for the cotangent complex all hinge on the following basic observation:

Lemma 8.4.3.15. *Let $\mathcal{C}$ be as in Construction 8.4.3.10 and let $\mathrm{Sym}^* : \mathcal{C} \to \mathrm{CAlg}(\mathcal{C})$ denote a left adjoint to the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$. Let $f : A \to B$ be a morphism in $\mathrm{CAlg}(\mathcal{C}_{\geq 0})$ and assume that $\mathrm{cofib}(f)$ is n-connective for some $n \geq 0$. Then there exists an object $M \in \mathcal{C}_{\geq n-1}$ and a commutative diagram*

in $\mathrm{CAlg}(\mathcal{C})$, where the upper square is a pushout, $A' \in \mathrm{CAlg}(\mathcal{C}_{\geq 0})$, $\mathrm{cofib}(f') \in \mathcal{C}_{\geq n+1}$, and ϕ is adjoint to the zero map $M \to \mathbf{1}$ in $\mathcal{C}$. Here $\mathbf{1}$ denotes the unit object of $\mathcal{C}$, regarded as an initial object of $\mathrm{CAlg}(\mathcal{C})$.

Proof. We will abuse notation by not distinguishing between the objects $A, B \in \mathrm{CAlg}(\mathcal{C})$ and their images in $\mathcal{C}$. Let $M = \mathrm{fib}(f)$, so that we have a pushout diagram of spectra

Invoking the universal property of Sym^*, we obtain a commutative diagram

in the ∞-category $\mathrm{CAlg}(\mathcal{C})$, where the upper square is a pushout. We observe that $\mathrm{Sym}^* 0$ is equivalent to the unit object $\mathbf{1}$. Let A' denote the tensor product $A \otimes_{\mathrm{Sym}^* M} \mathbf{1}$ so that we obtain a commutative diagram

as above. Since A' can also be identified with the tensor product

$$A \otimes_{\mathrm{Sym}^* A} \mathrm{Sym}^* B,$$

we conclude that A' is connective. The only nontrivial point is to verify that $\mathrm{cofib}(f') \in \mathcal{C}_{\geq n+1}$. Suppose first that $n = 0$; in this case, we wish to show that f' induces an epimorphism $\pi_0 A' \to \pi_0 B$ in the heart of $\mathcal{C}$. To prove this, we observe that the counit map

$$f'' : \mathrm{Sym}^* B \to B$$

factors through f'. The map f'' induces an epimorphism on all homotopy groups, because the underlying morphism in $\mathcal{C}$ admits a section.

We now treat the more generic case $n \geq 1$. Let I denote the fiber of the projection map $\mathrm{Sym}^* M \to \mathbf{1}$, so that we have a map of fiber sequences

$$
\begin{array}{ccccc}
A \otimes_{\mathrm{Sym}^* M} I & \longrightarrow & A & \longrightarrow & A' \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle f'} \\
M & \longrightarrow & A & \longrightarrow & B
\end{array}
$$

in the ∞-category $\mathcal{C}$. Consequently, we obtain an equivalence of spectra $\mathrm{cofib}(f') \simeq \mathrm{cofib}(g)[1]$, so it will suffice to show that $\mathrm{cofib}(g) \in \mathcal{C}_{\geq n}$. Using Proposition 3.1.3.11, we can identify I with the coproduct $\oplus_{i>0} \mathrm{Sym}^i(M)$. The map g admits a section, given by the composition

$$M \simeq \mathrm{Sym}^1(M) \to I \to A \otimes_{\mathrm{Sym}^* M} I.$$

We may therefore identify the $\mathrm{fib}(g)$ with a summand of the tensor product $A \otimes_{\mathrm{Sym}^* M} I$. It will now suffice to show that this tensor product is $(n-1)$-connective. Since A and $\mathrm{Sym}^* M$ are connective, it will suffice to show that I is $(n-1)$-connective. This follows immediately from Remark 8.4.3.13. $\qquad \square$

Lemma 8.4.3.16. *Let $\mathcal{C}$ be as in Construction 8.4.3.10. Let $f : A \to B$ be a morphism in $\mathrm{CAlg}(\mathcal{C}_{\geq 0})$, and choose objects $M, N \in \mathrm{Mod}_B(\mathcal{C}_{\geq 0})$. If $\mathrm{cofib}(f)$ is n-connective, then the induced map $\theta : M \otimes_A N \to M \otimes_B N$ is n-connective.*

Proof. We note that the fiber $\mathrm{fib}(\theta)$ can be identified with the geometric realization of a simplicial object $C_\bullet$ of $\mathcal{C}$, given by $C_\bullet = \mathrm{fib}(\mathrm{Bar}_A(M, N)_\bullet) \to \mathrm{Bar}_B(M, N)_\bullet)$. Using the spectral sequence of Proposition 1.2.4.5, we are reduced to showing that each of the objects C_m belongs to $\mathcal{C}_{\geq n-m}$. When $m = 0$, C_m vanishes and the result is obvious. Assume therefore that $m > 0$; we claim that $C_m \in \mathcal{C}_{\geq n-1} \subseteq \mathcal{C}_{\geq n-m}$. That is, we claim that the canonical map

$$M \otimes A^{\otimes m} \otimes N \to M \otimes B^{\otimes m} \otimes N$$

is $(n-1)$-connective. This follows immediately from the $(n-1)$-connectivity of f. $\qquad \square$

Lemma 8.4.3.17. *Let $\mathcal{C}$ be as in Construction 8.4.3.10 and let $f : A \to B$ be a morphism in $\mathrm{CAlg}(\mathcal{C}_{\geq 0})$. Suppose that $n \geq 0$ and that f induces an equivalence $\tau_{\leq n} A \to \tau_{\leq n} B$. Then $\tau_{\leq n} L_{B/A} \simeq 0$.*

Proof. It will suffice to show that f induces an equivalence $\tau_{\leq n}(B \otimes_A L_A) \to \tau_{\leq n} L_B$. To this end, choose an arbitrary object $M \in \mathrm{Mod}_B(\mathcal{C})_{\leq n}$; we wish to show that the canonical map

$$\theta : \mathrm{Map}_{\mathrm{Mod}_B(\mathcal{C})}(L_B, M) \to \mathrm{Map}_{\mathrm{Mod}_A(\mathcal{C})}(L_A, M)$$

is a homotopy equivalence. Equivalently, we must show that the map

$$\theta' : \mathrm{Map}_{\mathrm{CAlg}(\mathcal{C})_{/B}}(B, B \oplus M) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{C})_{/B}}(A, B \oplus M)$$

is an equivalence. Since A and B belong to $\mathrm{CAlg}(\mathcal{C}_{\geq 0})$, we may replace M by $\tau_{\geq 0}M$ and thereby reduce to the case where $M \in \mathcal{C}_{\geq 0}$. We have a pullback diagram

$$
\begin{array}{ccc}
B \oplus M & \longrightarrow & \tau_{\leq n}B \oplus M \\
\downarrow & & \downarrow \\
B & \longrightarrow & \tau_{\leq n}B,
\end{array}
$$

so that θ' is equivalent to the map

$$
\theta'' : \mathrm{Map}_{\mathrm{CAlg}(\mathcal{C})_{/\tau_{\leq n}B}}(B, (\tau_{\leq n}B) \oplus M) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{C})_{/\tau_{\leq n}B}}(A, \tau_{\leq n}B \oplus M)
$$

Since $\tau_{\leq n}B$ and $\tau_{\leq n}B \oplus M$ belong to $\mathrm{CAlg}(\mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq n})$, the condition that f induces an equivalence $\tau_{\leq n}A \to \tau_{\leq n}B$ ensures that θ'' is a homotopy equivalence. $\qquad\square$

Proof of Theorem 8.4.3.12. Let us say that a morphism $f : A \to B$ in $\mathrm{CAlg}(\mathcal{C})$ is *n-good* if $\mathrm{fib}(\epsilon_f) \in \mathcal{C}_{\geq 2n}$. We make the following observations:

(a) Suppose given a commutative triangle

in $\mathrm{CAlg}(\mathcal{C}_{\geq 0})$. If f and g are n-good and $\mathrm{cofib}(f), \mathrm{cofib}(g) \in \mathcal{C}_{\geq n}$, then h is n-good. To prove this, we consider the diagram of C-modules

$$
\begin{array}{ccc}
C \otimes_A \mathrm{cofib}(f) & \longrightarrow C \otimes_A \mathrm{cofib}(h) \longrightarrow & C \otimes_A \mathrm{cofib}(g) \\
\downarrow{\scriptstyle \epsilon'} & \downarrow{\scriptstyle \epsilon_h} & \downarrow{\scriptstyle \epsilon''} \\
C \otimes_B L_{B/A} & \longrightarrow \quad L_{C/A} \quad \longrightarrow & L_{C/B}.
\end{array}
$$

Here $\epsilon' = \mathrm{id}_C \otimes \epsilon_f$, so that $\mathrm{fib}(\epsilon') \in \mathcal{C}_{\geq 2n}$. It will therefore suffice to show that $\mathrm{fib}(\epsilon'') \in \mathcal{C}_{\geq 2n}$. We can write ϵ'' as a composition

$$
C \otimes_A \mathrm{cofib}(g) \overset{\phi}{\to} C \otimes_B \mathrm{cofib}(g) \overset{\epsilon_g}{\to} L_{C/B},
$$

so that we have a fiber sequence

$$
\mathrm{fib}(\phi) \to \mathrm{fib}(\epsilon'') \to \mathrm{fib}(\epsilon_g).
$$

It will therefore suffice to show that $\mathrm{fib}(\phi) \in \mathcal{C}_{\geq 2n}$. This follows from Lemma 8.4.3.16, since $\mathrm{cofib}(f)$ and $\mathrm{cofib}(g)$ are n-connective.

(b) Suppose given a pushout diagram

$$
\begin{array}{ccc}
A & \overset{f}{\longrightarrow} & B \\
\downarrow & & \downarrow \\
A' & \overset{f'}{\longrightarrow} & B'
\end{array}
$$

in $\mathrm{CAlg}(\mathcal{C})$, where $B, B' \in \mathrm{CAlg}(\mathcal{C}_{\geq 0})$. If f is n-good, then so is f'. This follows immediately from the equivalence $\mathrm{fib}(\epsilon_{f'}) \simeq B' \otimes_B \mathrm{fib}(\epsilon_f)$.

(c) Let $f : A \to B$ be an arbitrary morphism of $\mathbb{E}_\infty$-rings. Then the domain $B \otimes_A \text{cofib}(f)$ of the morphism ϵ_f can be identified with the cofiber of the map $B \to B \otimes_A B$ given by the inclusion of the second factor. This map admits a left homotopy inverse (given by the multiplication on B).

(d) Let $M \in \mathcal{C}_{\geq n-1}$, let $\mathbf{1}$ denote the unit object of $\mathcal{C}$, and consider the map $f : \text{Sym}^* M \to \mathbf{1}$ in $\text{CAlg}(\mathcal{C})$ which is adjoint to the zero map $M \to \mathbf{1}$ in $\mathcal{C}$. Then f is n-good. To prove this, we will explicitly compute both the domain and codomain of ϵ_f.

Using Corollary 8.3.3.6 we obtain a fiber sequence

$$\mathbf{1} \otimes_{\text{Sym}^* M} L_{\text{Sym}^* M} \to L_{\mathbf{1}} \to L_{\mathbf{1}/\text{Sym}^* M}$$

in $\text{Mod}_{\mathbf{1}}(\mathcal{C}) = \mathcal{C}$. Using Proposition 8.4.3.14, we may rewrite this fiber sequence

$$M \to 0 \to L_{\mathbf{1}/\text{Sym}^* M}$$

so that the codomain of ϵ_f is given by $L_{\mathbf{1}/\text{Sym}^* \mathbf{M}} \simeq M[1]$.

We next observe that the pushout diagram

$$
\begin{array}{ccc}
M & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & M[1]
\end{array}
$$

induces an equivalence of $\mathbb{E}_\infty$-rings $\mathbf{1} \otimes_{\text{Sym}^* M} \mathbf{1} \simeq \text{Sym}^* M[1]$. Invoking (c), we deduce that $\mathbf{1} \otimes_{\text{Sym}^* M} \text{cofib}(f)$ can be identified with the cofiber of the unit map $\mathbf{1} \to \text{Sym}^* M[1]$. Using Proposition 3.1.3.11, we can identify this fiber with the direct sum $\oplus_{i>0} \text{Sym}^i(M[1])$.

We now observe that the composition

$$M[1] \simeq \text{Sym}^1(M[1])[-1] \to \oplus_{i>0} \text{Sym}^i(M[1])[-1] \xrightarrow{\epsilon_f} M[1]$$

is homotopic to the identity. Consequently, the fiber of ϵ_f can be identified with the direct sum $\oplus_{i \geq 2} \text{Sym}^i(M[1])$. To complete the proof that $\text{fib}(\epsilon_f) \in \mathcal{C}_{\geq 2n}$, it will suffice to show that each symmetric power $\text{Sym}^i(M[1])$ belongs to $\mathcal{C}_{\geq 2n}$ for $i \geq 2$. This follows immediately from Remark 8.4.3.13.

(e) If $f : A \to B$ is a morphism in $\text{CAlg}(\mathcal{C}_{\geq 0})$ which induces an equivalence $\tau_{\leq 2n-1} A \to \tau_{\leq 2n-1} B$, then f is n-good. To prove this, we note that $B \otimes_A \text{cofib}(f)$ and $L_{B/A}[1]$ both belong to $\mathcal{C}_{\geq 2n}$ (see Lemma 8.4.3.17).

We are now ready to proceed with the proof of Theorem 8.4.3.1. Let $f : A \to B$ be a morphism in $\text{CAlg}(\mathcal{C}_{\geq 0})$ and suppose that $\text{cofib}(f) \in \mathcal{C}_{\geq n}$; we wish to show that f is n-good. Applying Lemma 8.4.3.15 repeatedly, we deduce the existence of a sequence of objects

$$A_n \to A_{n+1} \to A_{n+2} \to \dots$$

in $\text{CAlg}(\mathcal{C})_{/B}$, with the following properties:

(i) The object A_n coincides with A (as an object of $\text{CAlg}(\mathcal{C})_{/B}$).

(ii) For $m \geq n$, the cofiber of the map $A_m \to B$ belongs to $\mathcal{C}_{\geq m}$, and $A_m \in \text{CAlg}(\mathcal{C}_{\geq 0})$.

(iii) For each $m \geq n$, there exists an object $M \in \mathcal{C}_{\geq m-1}$ and a pushout diagram

$$
\begin{array}{ccc}
\text{Sym}^* M & \xrightarrow{\phi_m} & \mathbf{1} \\
\downarrow & & \downarrow \\
A_m & \xrightarrow{g_{m,m+1}} & A_{m+1},
\end{array}
$$

where $g_{j,k}$ denotes the morphism in $\mathrm{CAlg}(\mathcal{C})$ underlying the map from A_j to A_k in our direct system, and ϕ_m is adjoint to the zero map $M \to \mathbf{1}$ in $\mathcal{C}$.

Using (e), we deduce that the map $A_{2n+1} \to B$ is good. Using (a), we are reduced to showing that the maps $g_{m,m+1}$ are good for $m \leq 2n$. Using (b) and (iii), we are reduced to showing that each of the morphisms ϕ_m is n-good, which follows immediately from (d). $\qquad\square$

We close this section by studying the finiteness properties of the relative cotangent complex $L_{B/A}$ for a map $A \to B$ of $\mathbb{E}_\infty$-rings. Our main result can be formulated as follows:

Theorem 8.4.3.18. *Let A be a connective $\mathbb{E}_\infty$-ring, and let B be a connective $\mathbb{E}_\infty$-algebra over A. Then:*

(1) *If B is locally of finite presentation over A, then $L_{B/A}$ is perfect as a B-module. The converse holds provided that $\pi_0 B$ is finitely presented as a $\pi_0 A$-algebra.*

(2) *If B is almost of finite presentation over A, then $L_{B/A}$ is almost perfect as a B-module. The converse holds provided that $\pi_0 B$ is finitely presented as a $\pi_0 A$-algebra.*

As an immediate consequence, we deduce the following analogue of Remark 8.2.5.29:

Corollary 8.4.3.19. *Suppose given a commutative diagram*
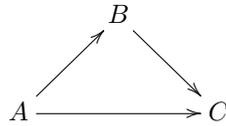
of connective $\mathbb{E}_\infty$-rings. Assume furthermore that B is of almost of finite presentation over A. Then C is almost of finite presentation over A if and only if C is almost of finite presentation over B.

Proof of Theorem 8.4.3.18. We first prove the forward implications. It will be convenient to phrase these results in a slightly more general form. Suppose given a commutative diagram σ:
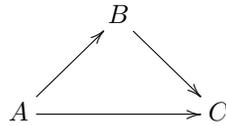
of connective $\mathbb{E}_\infty$-rings, and let $F(\sigma) = L_{B/A} \otimes_B C$. We will show:

(1′) If B is locally of finite presentation as an $\mathbb{E}_\infty$-algebra over A, then $F(\sigma)$ is perfect as a C-module.

(2′) if B is almost of finite presentation as an $\mathbb{E}_\infty$-algebra over A, then $F(\sigma)$ is almost perfect as a C-module.

We will obtain the forward implications of (1) and (2) by applying these results in the case $B = C$.

We first observe that the construction $\sigma \mapsto F(\sigma)$ defines a functor from $\mathrm{CAlg}_{A//C}$ into Mod_C. Using Remark 8.3.2.18 and Proposition T.4.3.1.10, we deduce that this functor preserves colimits. Since the collection of finitely presented C-modules is closed under finite colimits and retracts, it will suffice to prove (1′) in the case where B is finitely generated and free. In this case, $B = \mathrm{Sym}_A^* M$ for some finitely generated free A-module M. Using Proposition 8.4.3.14, we deduce that $F(\sigma) \simeq M \otimes_A C$ is a finitely generated free C-module, as desired.

We now prove (2′). It will suffice to show that for each $n \geq 0$, there exists a commutative diagram
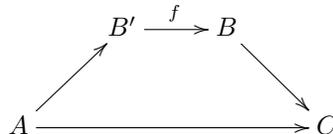
such that $L_{B'/A} \otimes_{B'} C$ is perfect, and the induced map

$$\tau_{\leq n}(L_{B'/A} \otimes_{B'} C) \to \tau_{\leq n}(L_{B/A} \otimes_B C)$$

is an equivalence. To guarantee the latter condition, it suffices to choose B' so that the relative cotangent complex $L_{B/B'}$ is n-connective. Using Corollary 8.4.3.2, it suffices to guarantee that f is $(n+1)$-connective. Moreover, assertion $(1')$ implies that $L_{B'/A} \otimes_{B'} C$ will be perfect so long as B' is locally of finitely presentation as an A-algebra. The existence of a commutative A-algebra with the desired properties now follows from Proposition 8.2.5.27.

We now prove the reverse implication of (2). Assume that $L_{B/A}$ is almost perfect and that $\pi_0 B$ is a finitely presented as a (discrete) $\pi_0 A$-algebra. To prove (2), it will suffice to construct a sequence of maps

$$A \to B(-1) \to B(0) \to B(1) \to \ldots \to B$$

such that each $B(n)$ is locally of finite presentation as an A-algebra, and each map $f_n : B(n) \to B$ is $(n+1)$-connective. We begin by constructing $B(-1)$ with an even stronger property: the map f_{-1} induces an isomorphism $\pi_0 B(-1) \to \pi_0 B$. Choose a finite presentation

$$\pi_0 B \simeq (\pi_0 A)[x_1, \ldots, x_k]/(g_1, \ldots, g_m)$$

for the ordinary commutative ring $\pi_0 B$. Let M denote the free A-module generated by symbols $\{X_i\}_{1 \leq i \leq k}$, so that the elements $\{x_i\} \subseteq \pi_0 B$ determine a map of A-modules $M \to B$. Let $h : \mathrm{Sym}_A^*(M) \to B$ be the adjoint map. We observe that there is a canonical isomorphism $\pi_0(\mathrm{Sym}_A^*(M)) \simeq (\pi_0 A)[x_1, \ldots, x_k]$. It follows that the image of the induced map

$$\pi_0 \mathrm{fib}(h) \to \pi_0 \mathrm{Sym}_A^*(M)$$

can be identified with the ideal in $(\pi_0 A)[x_1, \ldots, x_k]$ generated by the elements $\{g_j\}_{1 \leq j \leq m}$. Choose elements $\{\overline{g}_j\}_{1 \leq j \leq m}$ in $\pi_0 \mathrm{fib}(h)$ lifting $\{g_j\}_{1 \leq j \leq m}$. Let N be the free A-module generated by symbols $\{G_j\}_{1 \leq j \leq m}$, so that the elements $\{\overline{g}_j\}_{1 \leq j \leq m}$ determine a map of A-modules $N \to \mathrm{fib}(h)$. This map classifies a commutative diagram of A-modules

$$\begin{array}{ccc} N & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathrm{Sym}_A^*(M) & \xrightarrow{\ h\ } & B. \end{array}$$

Adjoint to this, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}_A^* N & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathrm{Sym}_A^*(M) & \longrightarrow & B \end{array}$$

in CAlg_A. Let $B(-1)$ denote the tensor product

$$A \otimes_{\mathrm{Sym}_A^* N} \mathrm{Sym}_A^* M.$$

Then the above diagram classifies a map of commutative A-algebras $f_{-1} : B(-1) \to B$. By construction, $B(-1)$ is of finite presentation over A, and f_{-1} induces an isomorphism

$$\pi_0 B(-1) \simeq (\pi_0 A)[x_1, \ldots, x_k]/(g_1, \ldots, g_m) \simeq \pi_0 B.$$

We now proceed in an inductive fashion. Assume that we have already constructed a connective A-algebra $B(n)$ which is of finite presentation over A, and an $(n+1)$-connective morphism $f_n : B(n) \to B$ of

commutative A-algebras. Moreover, we assume that the induced map $\pi_0 B(n) \to \pi_0 B$ is an isomorphism (if $n \geq 0$ this is automatic; for $n = -1$ it follows from the specific construction given above). We have a fiber sequence of B-modules

$$L_{B(n)/A} \otimes_{B(n)} B \to L_{B/A} \to L_{B/B(n)}.$$

By assumption, $L_{B/A}$ is almost perfect. Assertion $(2')$ implies that $L_{B(n)/A} \otimes_{B(n)} B$ is perfect. Using Proposition 8.2.5.11, we deduce that the relative cotangent complex $L_{B/B(n)}$ is almost perfect. Moreover, Corollary 8.4.3.2 ensures that $L_{B/B(n)}$ is $(n+2)$-connective. It follows that $\pi_{n+2} L_{B/B(n)}$ is a finitely generated as a (discrete) module over $\pi_0 B$. Using Theorem 8.4.3.1 and the bijectivity of the map $\pi_0 B(n) \to \pi_0 B$, we deduce that the canonical map

$$\pi_{n+1} \operatorname{fib}(f_n) \to \pi_{n+2} L_{B/B(n)}$$

is bijective. Choose a finitely generated projective $B(n)$-module M and a map $M[n+1] \to \operatorname{fib}(f_n)$ such that the composition

$$\pi_0 M \simeq \pi_{n+1} M[n+1] \to \pi_{n+1} \operatorname{fib}(f) \simeq \pi_{n+2} L_{B/B(n)}$$

is surjective (for example, we can take M to be a free $B(n)$-module indexed by a set of generators for the $\pi_0 B$-module $L_{B/B(n)}$). By construction, we have a commutative diagram of $B(n)$-modules

$$
\begin{array}{ccc}
M[n+1] & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
B(n) & \longrightarrow & B.
\end{array}
$$

Adjoint to this, we obtain a diagram

$$
\begin{array}{ccc}
\operatorname{Sym}^*_{B(n)}(M[n+1]) & \longrightarrow & B(n) \\
\downarrow & & \downarrow \\
B(n) & \longrightarrow & B.
\end{array}
$$

in the ∞-category of CAlg_A. We now define $B(n+1)$ to be the pushout

$$A \otimes_{\operatorname{Sym}^*_A M[n+1]} B(n),$$

and $f_{n+1} : B(n+1) \to B$ to be the induced map. It is clear that $B(n+1)$ is locally of finite presentation over $B(n)$, and therefore locally of finite presentation over A (Remark 8.2.5.29). To complete the proof of (2), it will suffice to show that the fiber of f_{n+1} is $(n+2)$-connective.

By construction, we have a commutative diagram

$$
\begin{array}{ccc}
 & \pi_0 B(n+1) & \\
 \overset{e'}{\nearrow} & & \overset{e''}{\searrow} \\
\pi_0 B(n) & \xrightarrow{\quad e \quad} & \pi_0 B
\end{array}
$$

where the map e' is surjective and e is bijective. It follows that e' and e'' are also bijective. In view of Corollary 8.4.3.2, it will now suffice to show $L_{B/B(n+1)}$ is $(n+3)$-connective. We have a fiber sequence of B-modules

$$L_{B(n+1)/B(n)} \otimes_{B(n+1)} B \to L_{B/B(n)} \to L_{B/B(n+1)}$$

Using Proposition 8.4.3.14 and Proposition 8.3.3.7, we conclude that $L_{B(n+1)/B(n)}$ is canonically equivalent to $M[n+2] \otimes_{B(n)} B(n+1)$. We may therefore rewrite our fiber sequence as

$$M[n+2] \otimes_{B(n)} B \to L_{B/B(n)} \to L_{B/B(n+1)}.$$

The inductive hypothesis and Corollary 8.4.3.2 guarantee that $L_{B/B(n)}$ is $(n+2)$-connective. The $(n+3)$-connectiveness of $L_{B/B(n+1)}$ is therefore equivalent to the surjectivity of the map

$$\pi_0 M \simeq \pi_{n+2}(M[n+2] \otimes_{B(n)} B) \to \pi_{n+2} L_{B/B(n)},$$

which is evident from our construction. This completes the proof of (2).

To complete the proof of (1), we use the same strategy but make a more careful choice of M. Let us assume that $L_{B/A}$ is perfect. It follows from the above construction that each cotangent complex $L_{B/B(n)}$ is likewise perfect. Using Proposition 8.2.5.23, we may assume $L_{B/B(-1)}$ is of Tor-amplitude $\leq k+2$ for some $k \geq 0$. Moreover, for each $n \geq 0$ we have a fiber sequence of B-modules

$$L_{B/B(n-1)} \to L_{B/B(n)} \to P[n+2] \otimes_{B(n)} B,$$

where P is finitely generated and projective, and therefore of Tor-amplitude ≤ 0. Using Proposition 8.2.5.23 and induction on n, we deduce that the Tor-amplitude of $L_{B/B(n)}$ is $\leq k+2$ for $n \leq k$. In particular, the B-module $\overline{M} = L_{B/B(k)}[-k-2]$ is connective and has Tor-amplitude ≤ 0. It follows from Remark 8.2.5.22 that $\overline{M}$ is a flat B-module. Invoking Proposition 8.2.5.20, we conclude that $\overline{M}$ is a finitely generated projective B-module. Using Corollary 8.2.2.19, we can choose a finitely generated projective $B(k)$-module M and an equivalance $M[n+2] \otimes_{B(k)} B \simeq L_{B/B(k)}$. Using this map in the construction outlined above, we guarantee that the relative cotangent complex $L_{B/B(k+1)}$ vanishes. It follows from Corollary 8.4.3.4 that the map $f_{k+1} : B(k+1) \to B$ is an equivalence, so that B is locally of finite presentation as an $\mathbb{E}_\infty$-algebra over A, as desired. $\qquad\square$

8.5 Étale Morphisms

In this section, we will develop an ∞-categorical generalization of the theory of étale morphisms between commutative rings. We begin by recalling a definition from commutative algebra.

Definition 8.5.0.1. Let $f : A \to B$ be a map of commutative rings. We say that f is *étale* if the following conditions are satisfied:

(1) The commutative ring B is finitely presented as an A-algebra.

(2) The map f exhibits B as a flat A-module.

(3) The multiplication map $p : B \otimes_A B \to B$ is the projection onto a summand: that is, there exists another map of commutative rings $q : B \otimes_A B \to R$ such that p and q induce an isomorphism $B \otimes_A B \to B \times R$.

Remark 8.5.0.2. Condition (3) of Definition 8.5.0.1 is equivalent to the following assertion:

($*$) There exists an idempotent element $e \in B \otimes_A B$ such that p induces an isomorphism $(B \otimes_A B)[\frac{1}{e}] \simeq B$.

Indeed, if $B \otimes_A B \simeq B \times R$, we can take e to be the preimage of the element $(1,0) \in B \times R$. Conversely, if ($*$) is satisfied, then we can take $R = (B \otimes_A B)[\frac{1}{1-e}]$.

Remark 8.5.0.3. For any étale map of commutative rings $f : A \to B$, the module of Kähler differentials $\Omega_{B/A}$ is trivial. Equivalently, for every B-module M, the projection map $p : B \oplus M \to B$ admits a unique section (as a map of A-modules). Indeed, if p has two sections s and s', then we get an induced map $f : B \otimes_A B \to B \oplus M$. If $e \in B \otimes_A B$ is the idempotent of Remark 8.5.0.2, then $f(e) = 1 + m$ for some $m \in M$ and is therefore invertible. It follows that f factors through the multiplication map $B \otimes_A B \to B$, so that $s = s'$.

Our goal in this section is to study the following generalization of Definition 8.5.0.1:

Definition 8.5.0.4. Let $2 \leq k \leq \infty$, and let $\phi : A \to B$ be a map of $\mathbb{E}_k$-rings. We will say that ϕ is *étale* if the following conditions are satisfied:

(1) The underlying map of commutative rings $\pi_0 A \to \pi_0 B$ is étale (in the sense of Definition 8.5.0.1).

(2) The map ϕ exhibits B as a flat (left or right) A-module.

Remark 8.5.0.5. Suppose given a pushout diagram of $\mathbb{E}_\infty$-rings

$$
\begin{array}{ccc}
A & \longrightarrow & A' \\
{\scriptstyle f}\downarrow & & \downarrow {\scriptstyle f'} \\
B & \longrightarrow & B'.
\end{array}
$$

If f is étale, then so is f'. The flatness of f follows from Proposition 8.2.2.16. Moreover, Proposition 8.2.2.13 ensures that the induced diagram

$$
\begin{array}{ccc}
\pi_0 A & \longrightarrow & \pi_0 A' \\
\downarrow & & \downarrow \\
\pi_0 B & \longrightarrow & \pi_0 B'
\end{array}
$$

is a pushout in the category of ordinary commutative rings. Since the left vertical map is étale, it follows that the right vertical map is étale, so that f' is likewise étale.

One of our main results can be stated as follows:

Theorem 8.5.0.6. *Let $2 \leq k \leq \infty$, let R be an $\mathbb{E}_{k+1}$-ring, and let A be an $\mathbb{E}_k$-algebra over R. Let $(\mathrm{Alg}_R^{(k)})_{A/}^{\acute{e}t}$ denote the full subcategory of $(\mathrm{Alg}_R^{(k)})_{A/}$ spanned by the étale morphisms. Then the construction $B \mapsto \pi_0 B$ induces an equivalence from $(\mathrm{Alg}_R^{(k)})_{A/}^{\acute{e}t}$ to the nerve of the ordinary category of étale $\pi_0 A$-algebras.*

More informally: given an $\mathbb{E}_k$-ring A and an étale morphism of commutative rings $\pi_0 A \to B_0$, there exists an étale map of $\mathbb{E}_k$-rings $f : A \to B$ and an isomorphism of $(\pi_0 A)$-algebras $\pi_0 B \simeq B_0$; moreover, B is determined uniquely up to equivalence.

Example 8.5.0.7. Let A be an $\mathbb{E}_k$-ring for $2 \leq k \leq \infty$ and let $x \in \pi_0 A$. Theorem 8.5.0.6 implies that there exists another $\mathbb{E}_k$-ring $A[x^{-1}]$, equipped with a map $A \to A[x^{-1}]$ which induces an isomorphism $(\pi_* A)[x^{-1}] \simeq \pi_*(A[x^{-1}])$. Using Proposition 8.2.4.20, we deduce that $A[x^{-1}]$ can be identified with the Ore localization $A[S^{-1}]$ where S is the multiplicatively closed subset $\{x^n\}_{n \geq 0} \subseteq \pi_0 A$.

In the case $k = \infty$, it is possible to give a very direct proof of Theorem 8.5.0.6 using the deformation theory developed in §8.4.2. We will adopt a more roundabout strategy which also works in the case $k < \infty$. We will begin in §8.5.1 by formulating and proving an analogue of Theorem 8.5.0.6 in the case $k = 1$, assuming that R is the sphere spectrum and that A is connective. In §8.5.2, we will generalize this result to the case where A is not assumed to be connective, and in §8.5.3 we will generalize to the case where R is arbitrary. Finally, in §8.5.4 we will prove Theorem 8.5.0.6 in general, reducing to the case $k = 1$ using a mechanism provided by Theorem 5.1.2.2.

8.5.1   Étale Morphisms of $\mathbb{E}_1$-Rings

Our goal in this section is to prove a version of Theorem 8.5.0.6 in the setting of $\mathbb{E}_1$-rings. The first step is to decide what we mean by an étale map $\phi : A \to B$ of $\mathbb{E}_1$-rings. The definition that we adopt requires some mild commutativity assumptions on A and B.

Definition 8.5.1.1. Let A be an $\mathbb{E}_1$-ring. We will say that A is *quasi-commutative* if the following condition is satisfied: for every $x \in \pi_0 A$ and every $y \in \pi_n A$, we have $xy = yx \in \pi_n A$.

Remark 8.5.1.2. Let A be an $\mathbb{E}_k$ ring for $2 \leq k \leq \infty$. Then the underlying $\mathbb{E}_1$-ring of A is quasi-commutative. To see this, it suffices to observe that for $x \in \pi_0 A$, the spectrum maps $l_x, r_x : A \to A$ given by left and right multiplication by x are homotopic to one another (since the multiplication $A \otimes A \to A$ is commutative up to homotopy).

Remark 8.5.1.3. Let A be quasi-commutative $\mathbb{E}_1$-ring. Then $\pi_0 A$ is a commutative ring.

We now adapt Definition 8.5.0.4 to the setting of $\mathbb{E}_1$-rings.

Definition 8.5.1.4. Let $\phi : A \to B$ be a morphism of $\mathbb{E}_1$-rings. We will say that ϕ is *étale* if the following conditions are satisfied:

(1) The $\mathbb{E}_1$-rings A and B are quasi-commutative.

(2) The morphism ϕ induces an étale homomorphism of commutative rings $\pi_0 A \to \pi_0 B$ (Definition 8.5.0.1).

(3) For every integer $n \in \mathbf{Z}$, the associated map $\mathrm{Tor}_0^{\pi_0 A}(\pi_n A, \pi_0 B) \to \pi_n B$ is an isomorphism of abelian groups.

Remark 8.5.1.5. In the situation of Definition 8.5.1.4, condition (1) guarantees that the left and right actions of $\pi_0 B$ on $\pi_n B$ agree, so that the map $\pi_n A \otimes_{\pi_0 A} \pi_0 B \to \pi_n B$ is unambiguously defined. In other words, if condition (1) is satisfied, then condition (3) is equivalent to either of the following assertions:

(3′) The map ϕ exhibits B as a flat left A-module.

(3″) The map ϕ exhibits B as a flat right A-module.

Remark 8.5.1.6. The collection of étale morphisms between $\mathbb{E}_1$-rings is closed under composition. In particular, every equivalence of quasi-commutative $\mathbb{E}_1$-rings is étale.

Remark 8.5.1.7. Suppose given a commutative diagram

$$
\begin{array}{ccc}
 & B & \\
{}^{f}\nearrow & & \searrow{}^{g} \\
A & \xrightarrow{\quad h \quad} & C
\end{array}
$$

of quasi-commutative $\mathbb{E}_1$-rings. If f is étale, then g is étale if and only if h is étale. The "only if" direction is Remark 8.5.1.6. For the converse, let us suppose that f and h are both étale. The induced maps $\pi_0 A \to \pi_0 B$ and $\pi_0 A \to \pi_0 C$ are both étale map of ordinary commutative rings, so that g also induces an étale map $\pi_0 B \to \pi_0 C$. We now observe that for $n \in \mathbf{Z}$, we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Tor}_0^{\pi_0 B}(\mathrm{Tor}_0^{\pi_0 A}(\pi_n A, \pi_0 B), \pi_0 C) & \longrightarrow & \mathrm{Tor}_0^{\pi_0 A}(\pi_n A, \pi_0 C) \\
\downarrow & & \downarrow \\
\mathrm{Tor}_0^{\pi_0 B}(\pi_n B, \pi_0 C) & \longrightarrow & \pi_n C.
\end{array}
$$

Since f and h are flat, the vertical maps are isomorphisms. The upper horizontal map is obviously an isomorphism, so the lower horizontal map is an isomorphism as well.

Remark 8.5.1.8. Let $\phi : A \to B$ be a morphism of $\mathbb{E}_k$-rings for $2 \leq k \leq \infty$. Then the underlying map of $\mathbb{E}_1$-rings satisfies condition (1) of Definition 8.5.1.4. It follows that ϕ is étale as a map of $\mathbb{E}_k$-rings (in the sense of Definition 8.5.0.4) if and only if it is étale as a map of $\mathbb{E}_1$-rings (in the sense of Definition 8.5.1.4).

Remark 8.5.1.9. Let A be an ordinary associative ring, regarded as a discrete $\mathbb{E}_1$-ring. Then A is quasi-commutative if and only if it is a commutative ring. A morphism $\phi : A \to B$ is étale (in the sense of Definition 8.5.1.4) if and only if B is discrete (as an $\mathbb{E}_1$-ring) and the underlying associative ring is a commutative ring which is étale over A, in the sense of Definition 8.5.0.1.

The primary objective in this section gives a classification of étale morphisms $\phi : A \to B$, in the case where A is a connective, quasi-commutative $\mathbb{E}_1$-ring. To state it, we need to introduce a bit of notation.

Notation 8.5.1.10. Let A be a quasi-commutative $\mathbb{E}_1$-ring. We let $\mathrm{Alg}_{A/}^{(1),\mathrm{\acute{e}t}}$ denote the full subcategory of $\mathrm{Alg}_{A/}^{(1)}$ spanned by the étale morphisms $\phi : A \to B$. If we are given a morphism of $\mathbb{E}_1$-rings $\psi : A \to C$, we let $\mathrm{Alg}_{A//C}^{(1),\mathrm{\acute{e}t}}$ denote the full subcategory of $\mathrm{Alg}_{A//C}^{(1)}$ spanned by those diagrams

$$
\begin{array}{ccc}
 & B & \\
{\scriptstyle\phi}\nearrow & & \searrow \\
A & \xrightarrow{\;\;\psi\;\;} & C
\end{array}
$$
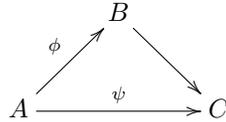
where ϕ is étale.

Let Ring denote the category of commutative rings. Given a commutative ring A, we let $\mathrm{Ring}_{A/}^{\mathrm{\acute{e}t}}$ denote the full subcategory of $\mathrm{Ring}_{A/}$ spanned by the étale ring homomorphisms $A \to B$. If we are given a map of commutative rings $\psi : A \to C$, we let $\mathrm{Ring}_{A//C}^{\mathrm{\acute{e}t}}$ denote the full subcategory of $\mathrm{Ring}_{A//C}$ spanned by those diagrams

$$
\begin{array}{ccc}
 & B & \\
{\scriptstyle\phi}\nearrow & & \searrow \\
A & \xrightarrow{\;\;\psi\;\;} & C
\end{array}
$$
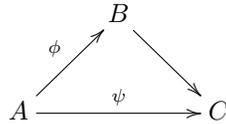
where ϕ is étale.

We can now state our main result:

Theorem 8.5.1.11. *Let A be an $\mathbb{E}_1$-ring which is connective and quasi-commutative. Then the forgetful functor $B \mapsto \pi_0 B$ determines an equivalence of ∞-categories*

$$
\mathrm{Alg}_{A/}^{(1),\acute{e}t} \to \mathrm{N}(\mathrm{Ring}_{\pi_0 A/}^{\acute{e}t}).
$$

The proof of Theorem 8.5.1.11 will occupy our attention for the remainder of this section. We begin by introducing a bit of terminology.

Definition 8.5.1.12. Let R be an $\mathbb{E}_1$-ring. We will say that an element $x \in \pi_0 R$ is *quasi-central* if the set $S = \{x^n\}_{n \geq 0}$ satisfies the left Ore condition (Definition 8.2.4.1).

Remark 8.5.1.13. Suppose that R is a quasi-commutative $\mathbb{E}_1$-ring. Then every element $x \in \pi_0 R$ is quasi-central. Let $R[x^{-1}]$ denote the localization $R[S^{-1}]$, where $S = \{x^n\}_{n \geq 0}$. Proposition 8.2.4.20 implies that the ring $\pi_* R[x^{-1}]$ is obtained from $\pi_* R$ by inverting the element x. It follows immediately that the localization map $R \to R[x^{-1}]$ is étale.

Lemma 8.5.1.14. *Let A and B be quasi-commutative $\mathbb{E}_1$-rings with connective covers $\widetilde{A}$ and $\widetilde{B}$. Then every element $x \in \pi_0(\widetilde{A} \otimes \widetilde{B})$ has quasi-central image in $\pi_0(A \otimes B)$. In particular, if A and B are connective, then every element of $\pi_0(A \otimes B)$ is quasi-central.*

Proof. We will say that a left $\widetilde{A} \otimes \widetilde{B}$-module P is x-*nilpotent* if it is $\{x^n\}_{n \geq 0}$-nilpotent, in the sense of Definition 8.2.4.8 (that is, if and only if every element of $\pi_* P$ is annihilated by left multiplication by power of x). If $M \in {}_{\widetilde{A}}\mathrm{BMod}_{\widetilde{A}}(\mathrm{Sp})$ and $N \in {}_{\widetilde{B}}\mathrm{BMod}_{\widetilde{B}}(\mathrm{Sp})$, then $M \otimes N$ has the structure of an $(\widetilde{A} \otimes \widetilde{B})$-bimodule spectrum. In this case, let $r_x^{M,N} : M \otimes N \to M \otimes N$ be the map given by right multiplication by x. Let us say that the pair (M, N) is *good* if the cofiber $\mathrm{cofib}(r_x^{M,N})$ is x-nilpotent (as a left $(\widetilde{A} \otimes \widetilde{B})$-module). According to Lemma 8.2.4.11, the image of x is quasi-central if and only if the pair (A, B) is good. The collection of all $N \in {}_B\mathrm{BMod}_B(\mathrm{Sp})$ such that (A, N) is good is closed under small colimits. It will therefore suffice to show

that each pair $(A, \tau_{\geq n} B)$ is good. Using the same reasoning, we are reduced to showing that every pair of the form $(\tau_{\geq m} A, \tau_{\geq n} B)$ is good. Note that for $p \leq q + m + n$, the natural map

$$\pi_p \operatorname{cofib}(r_x^{\tau_{\leq m+q} \tau_{\geq m} A, \tau_{\leq n+q} \tau_{\leq n} B}) \to \pi_p \operatorname{cofib}(r_x^{\tau_{\geq m} A, \tau_{\geq n} B})$$

is an isomorphism. It will therefore suffice to show that each of the pairs $(\tau_{\leq m+q} \tau_{\geq m} A, \tau_{\leq n+q} \tau_{\geq n} B)$ is good. Since the collection of x-nilpotent left modules is closed under shifts and extensions, we are reduced to proving that the pair (M, N) is good in the case $M = \pi_{m'} A$, $N = \pi_{n'} B$.

Note that the commutative ring $\pi_0(\widetilde{A} \otimes \widetilde{B})$ can be identified with $\operatorname{Tor}_0^{\mathbf{Z}}(\pi_0 A, \pi_0 B)$. In particular, $\pi_0(\widetilde{A} \otimes \widetilde{B})$ is a commutative ring, and we have commutative ring homomorphisms

$$\pi_0 A \xrightarrow{\phi} \pi_0(\widetilde{A} \otimes \widetilde{A}) \xleftarrow{\psi} \pi_0 B.$$

We claim that every element $x \in \pi_0(A \otimes B)$ satisfies the following condition:

(∗) Left and right multiplication by x induce homotopic maps from the spectrum $M \otimes N$ to itself.

The collection of those elements of $\pi_0(\widetilde{A} \otimes \widetilde{B})$ which satisfy (∗) is stable under sums. Consequently, it suffices to prove (∗) in the case $x = \phi(a)\psi(b)$, for some $a \in \pi_0 A$ and $b \in \pi_0 B$. The desired result then follows from the observation that left and right multiplication by a induce the same map from $M \simeq N$ to itself (since M and N are discrete, it suffices to check this at the level of homotopy groups, in which case it follows from our assumption that A and B are quasi-commutative).

Let $X = \operatorname{cofib}(r_x^{M,N})$. We have a long exact sequence

$$\pi_p(M \otimes N) \xrightarrow{f} \pi_p M \otimes N \to \pi_p X \to \pi_{p-1}(M \otimes N) \xrightarrow{f'} \pi_{p-1}(M \otimes N).$$

Condition (∗) guarantees that f and f' are given by left multiplication by x. It follows that the cokernel of f and the kernel of f' (in the ordinary category of abelian groups) are annihilated by left multiplication by x, so that $\pi_n X$ is annihilated by left multiplication by x^2.   □

We now show that if $\phi : A \to B$ is an étale map of connective $\mathbb{E}_1$-rings, then B is determined by A and $\pi_0 B$. More precisely, we have the following universal property:

Proposition 8.5.1.15. *Let $\phi : A \to B$ be an étale map of connective $\mathbb{E}_1$-rings and let C be an arbitrary quasi-commutative $\mathbb{E}_1$-ring. Then the diagram σ :*

$$
\begin{array}{ccc}
\operatorname{Map}_{\operatorname{Alg}^{(1)}}(B, C) & \longrightarrow & \operatorname{Map}_{\operatorname{Alg}^{(1)}}(A, C) \\
\downarrow & & \downarrow \\
\operatorname{Hom}_{\operatorname{Ring}}(\pi_0 B, \pi_0 C) & \longrightarrow & \operatorname{Hom}_{\operatorname{Ring}}(\pi_0 A, \pi_0 C)
\end{array}
$$

is a pullback square in $\mathcal{S}$.

Proof. Since A and B are connective, we may replace C by its connective cover $\tau_{\geq 0} C$ and thereby reduce to the case where C is connective. For every map $C' \to \tau_{\leq 0} C$ in $\operatorname{Alg}^{(1)}$, we let $\sigma_{C'}$ denote the commutative diagram

$$
\begin{array}{ccc}
\operatorname{Map}_{\operatorname{Alg}^{(1)}}(B, C') & \longrightarrow & \operatorname{Map}_{\operatorname{Alg}^{(1)}}(A, C') \\
\downarrow & & \downarrow \\
\operatorname{Map}_{\operatorname{Alg}^{(1)}}(B, \tau_{\leq 0} C) & \longrightarrow & \operatorname{Map}_{\operatorname{Alg}^{(1)}}(\pi_0 A, \tau_{\leq 0} C).
\end{array}
$$

Note that σ is equivalent to σ_C; it will therefore suffice to prove that σ_C is a pullback square in $\mathcal{S}$.

The collection of objects C' for which $\sigma_{C'}$ is a pullback square is closed under limits in $\mathrm{Alg}^{(1)}_{/\tau_{\leq 0}C}$. Since C can be realized as the limit of its Postnikov tower

$$\cdots \to \tau_{\leq 2}C \to \tau_{\leq 1}C \to \tau_{\leq 0}C$$

(Proposition 8.1.3.19), we may replace C by $\tau_{\leq n}C$ and thereby reduce to the case where C is n-truncated. We now proceed by induction on n, the case $n = 0$ being trivial. If $n > 0$, then the truncation map $C \to \tau_{\leq n-1}C$ is a square-zero extension (Corollary 8.4.1.28), so we have a pullback diagram

$$\begin{array}{ccc}
C & \longrightarrow & \tau_{\leq n-1}C \\
\downarrow & & \downarrow \\
\tau_{\leq n-1}C & \longrightarrow & (\tau_{\leq n-1}C) \oplus (\pi_n C)[n+1].
\end{array}$$

It will therefore suffice to prove that $\sigma_{C'}$ is a pullback diagram, where C' is the trivial square-zero extension of C by $(\pi_n C)[n+1]$. Unwinding the definitions, we are reduced to proving that for any ring homomorphism $\phi : \pi_0 B \to \pi_0 C$, the abelian groups $\mathrm{Ext}^k_{\mathcal{C}}(L_{B/A}, \pi_n C)$ are trivial, where $\mathcal{C}$ denotes the stable ∞-category ${}_B\mathrm{BMod}_B(\mathrm{Sp})$, $L_{B/A} \in \mathcal{C}$ is the relative cotangent complex of B over A, and we regard $\pi_n C$ as a discrete object of $\mathcal{C}$ via the ring homomorphism ϕ.

Corollary 6.3.6.12 furnishes an equivalence of ∞-categories $\mathcal{C} \simeq \mathrm{LMod}_T$, where T denotes the $\mathbb{E}_1$-ring $B^{\mathrm{rev}} \otimes B$. Since the map $\pi_0 A \to \pi_0 B$ is an étale homomorphism of commutative rings, there exists an idempotent element $e \in \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 B)$ such that $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 B)[\frac{1}{e}] \simeq B$ (Remark 8.5.0.2). The map $\pi_0 T \simeq \mathrm{Tor}_0^{\mathbf{Z}}(\pi_0 B, \pi_0 B) \to \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 B)$ is surjective, so we can lift e to an element $x \in \pi_0 T$. Since B is connective and quasi-commutative, Lemma 8.5.1.14 guarantees that x is quasi-central.

Using Remark 8.4.1.12, we can identify the relative cotangent complex $L_{B/A}$ with the cofiber of the multiplication map $B \otimes_A B \to B$. Using the flatness of B over A, we obtain an isomorphism

$$\pi_* L_{B/A}[-1] \simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_* B)[\frac{1}{1-e}].$$

In particular, $\pi_* L_{B/A}$ is annihilated by $x \in \pi_0 T$, so that $L_{B/A}$ is x-nilpotent. Since C is quasi-commutative, the left and right actions of $\pi_0 B$ on $\pi_n C$ (through the chosen homomorphism ϕ) coincide, so that the action of $\pi_0 T$ on $\pi_n C$ factors through the map

$$\pi_0 T \simeq \mathrm{Tor}_0^{\pi_0 R}(\pi_0 B, \pi_0 B) \to \pi_0 B.$$

It follows that $x \in \pi_0 T$ acts by the identity on $\pi_n C$, so that the groups $\mathrm{Ext}^k_{\mathcal{C}}(L_{B/A}, \pi_n C)$ vanish by Proposition 8.2.4.14. $\qquad\square$

Proof of Theorem 8.5.1.11. Let A be a connective quasi-commutative $\mathbb{E}_1$-ring; we wish to show that the forgetful functor $\theta : \mathrm{Alg}^{(1),\text{ét}}_{A/} \to \mathrm{N}(\mathrm{Ring}^{\text{ét}}_{\pi_0 A/})$ is an equivalence of ∞-categories. Using Proposition 8.1.3.19, we deduce that $\mathrm{Alg}^{(1),\text{ét}}_{A/}$ is equivalent to the homotopy limit of the tower of ∞-categories

$$\cdots \to \mathrm{Alg}^{(1),\text{ét}}_{\tau_{\leq 2}A/} \to \mathrm{Alg}^{(1),\text{ét}}_{\tau_{\leq 1}A/} \to \mathrm{Alg}^{(1),\text{ét}}_{\tau_{\leq 0}A/} \simeq \mathrm{N}(\mathrm{Ring}^{\text{ét}}_{\pi_0 A/}).$$

We may therefore assume without loss of generality that A is n-truncated. We proceed by induction on n, the case $n = 0$ being trivial. Assume therefore that $n > 0$. Proposition 8.5.1.15 guarantees that θ is fully faithful; it will therefore suffice to show that θ is essentially surjective. Fix an étale map of commutative rings $\phi_0 : \pi_0 A \to B_0$. Let $A' = \tau_{\leq n-1}A$. Using the inductive hypothesis, we can lift ϕ_0 to an étale morphism $\phi' : A' \to B'$ of $\mathbb{E}_1$-rings. According to Corollary 8.4.1.28, we conclude that the truncation map $A \to A'$ is a square-zero extension. Let $L_{A'}$ denote the cotangent complex of A (as an $\mathbb{E}_1$-ring), and write $A = A'^{\eta}$ for some derivation $\eta : L_{A'} \to M[n+1]$, where $M \in {}_{A'}\mathrm{BMod}_{A'}(\mathrm{Sp})$ corresponds to the discrete $(\pi_0 A)$-$(\pi_0 A)$-bimodule given by $\pi_n A$. Since A is quasi-commutative, the left and right actions of $\pi_0 A$ on M agree.

Let N denote the abelian group $B_0 \otimes_{\pi_0 A} \pi_n A$. We can regard N as a bimodule over B_0, where the left and right actions of B_0 on N agree. We may thereby identify N with a discrete object of $_{B'}\mathrm{BMod}_{B'}(\mathrm{LMod}_R)$. The map of A'-A'-bimodules $L_{A'} \xrightarrow{\eta} M[n+1] \to N[n+1]$ induced a map of B'-B'-bimodules $\eta' : B' \otimes_{A'} L_{A'} \otimes_{A'} B' \to N[n+1]$. Consider the cofiber diagram

$$B' \otimes_{A'} L_{A'} \otimes_{A'} B' \to L_{B'} \to L_{B'/A'}.$$

Write $_{B'}\mathrm{BMod}_{B'}(\mathrm{Sp}) \simeq \mathrm{LMod}_T$ and choose $x \in \pi_0 T$ as in the proof of Proposition 8.5.1.15, so that x is quasi-central and $L_{B'/A'}$ is x-nilpotent. Since the left and right actions of $\pi_0 B$ on N coincide, multiplication by x is homotopic to the identity on N. Using Proposition 8.2.4.14, we deduce that the groups $\mathrm{Ext}^i_T(L_{B'/A'}, N)$ are trivial. It follows that the map η' factors as a composition

$$B' \otimes_{A'} L_{A'} \otimes_{A'} B' \to L_{B'} \xrightarrow{\gamma} N.$$

The induced map $A \simeq A'^\eta \to B'^\gamma$ is an object $\mathrm{Alg}_{A/}^{(1),\text{ét}}$ whose image in $\mathrm{N}(\mathrm{Ring}_{\pi_0 A/}^{\text{ét}})$ is isomorphic to the original ring homomorphism $\phi : \pi_0 A \to B_0$. $\square$

For later use, we record the following hybrid of Theorem 8.5.1.11 and Proposition 8.5.1.15.

Theorem 8.5.1.16. *Let $\psi : A \to C$ be a map of connective, quasi-commutative $\mathbb{E}_1$-rings. Then the construction $B \mapsto \pi_0 B$ induces an equivalence of ∞-categories*

$$\mathrm{Alg}_{A//C}^{(1),\text{ét}} \to \mathrm{N}(\mathrm{Ring}_{\pi_0 A//\pi_0 C}^{\text{ét}}).$$

Proof. We have a commutative diagram of ∞-categories

$$
\begin{array}{ccc}
\mathrm{Alg}_{A//C}^{(1),\text{ét}} & \longrightarrow & \mathrm{N}(\mathrm{Ring}_{\pi_0 A//\pi_0 C}^{\text{ét}}) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\
\mathrm{Alg}_{A/}^{(1),\text{ét}} & \longrightarrow & \mathrm{N}(\mathrm{Ring}_{\pi_0 A/}^{\text{ét}}).
\end{array}
$$

The vertical maps are right fibrations, and the bottom horizontal map is an equivalence of ∞-categories (Proposition 8.5.1.11). It therefore suffices to show that for every étale map $A \to A'$, the functor θ induces a homotopy equivalence from the fiber of p over A' to the fiber of q over $\pi_0 A'$ (Corollary T.2.4.4.4). This follows immediately from Proposition 8.5.1.15. $\square$

8.5.2   The Nonconnective Case

Let A be a quasi-commutative $\mathbb{E}_1$-ring, and suppose we are given an étale homomorphism $\phi_0 : \pi_0 A \to B_0$ of commutative rings. In §8.5.1, we showed that when A is connective, there is an essentially unique way to lift ϕ_0 to an étale map $\phi : A \to B$ of quasi-commutative $\mathbb{E}_1$-rings (Theorem 8.5.1.11). Our goal in this section is to extend this result to the case where A is not assumed to be connective.

We begin by outlining our strategy. Let A be as above, and let $\tau_{\geq 0} A$ be its connective cover. Using Theorem 8.5.1.11, we deduce that ϕ_0 can be lifted to an étale map of connective $\mathbb{E}_1$-rings $\tau_{\geq 0} A \to B'$. In particular, B' is flat over $\tau_{\geq 0} A$. The tensor product $B = A \otimes_{\tau_{\geq 0} A} B'$ is a flat A-module, which is equipped with a canonical isomorphism $\pi_0 B \simeq \pi_0 B' \simeq B_0$. It is not obvious that B is an $\mathbb{E}_1$-ring: a relative tensor product of associative ring spectra does not generally inherit a ring structure. We will show that B admits an $\mathbb{E}_1$-structure by exploiting the quasi-commutativity of A. Before giving the details, we need to embark on a bit of a digression.

Definition 8.5.2.1. Let R be an associative ring, and let M be a (discrete) R-R-bimodule. For every element $x \in R$, let $l_x, r_x : M \to M$ be the endomorphisms given by left and right multiplication by x, respectively. We will say that M is *x-balanced* if the difference $l_x - r_x$ is locally nilpotent: that is, for each $y \in M$, we have $(l_x - r_x)^n(y) = 0$ for $n \gg 0$. We say that M is *balanced* if it is x-balanced for each $x \in R$.

Remark 8.5.2.2. Let R and M be as in Definition 8.5.2.1. Suppose that R is commutative, and let X denote the affine scheme given by the spectrum of R and identify M with a quasi-coherent sheaf on the product $X \times X$. Then M is balanced if and only if the restriction of M to the open set $X \times X - \Delta$ is zero, where $\Delta \subseteq X \times X$ denotes the image of the diagonal map $X \to X \times X$.

Remark 8.5.2.3. Let R be an associative ring, and let ${}_R\mathrm{BMod}_R(\mathcal{A}b)$ denote the abelian category of (discrete) R-R-bimodules. For each $x \in R$, let $\mathcal{C} \subseteq {}_R\mathrm{BMod}_R(\mathcal{A}b)$ be the full subcategory spanned by the x-balanced bimodules. Then $\mathcal{C}_x$ is an abelian subcategory of ${}_R\mathrm{BMod}_R(\mathcal{A}b)$, which is closed under colimits, extensions, and the formation of subobjects and quotient objects. It follows that the category $\bigcup_{x \in R} \mathcal{C}_x$ of balanced bimodules is also an abelian subcategory of ${}_R\mathrm{BMod}_R(\mathcal{A}b)$, closed under extensions, small colimits, passage to subobjects, and passage to quotient objects.

Remark 8.5.2.4. Let R be an associative ring and let x be an element of the center of R. For every right R-module M, multiplication by x defines an R-module endomorphism $M \xrightarrow{x} M$. Similarly, if N is a left R-module, then multiplication by x determines an A-linear map $N \xrightarrow{x} N$. For each $n \geq 0$, these endomorphisms induce the same map from $\mathrm{Tor}_n^R(M, N)$ to itself.

Suppose we are given associative rings A, B, and C. Let M be a discrete A-B-bimodule and let N a discrete B-C-bimodule. Then the relative tensor product $M \otimes_B N$ is an A-C-bimodule spectrum, with homotopy groups given by $\pi_n(M \otimes_B N) = \mathrm{Tor}_n^B(M, N)$. In particular, taking $A = B = C$, we see that if M and N are A-A-bimodules, then each of the abelian groups $\mathrm{Tor}_n^A(M, N)$ has the structure of an A-A-bimodule.

Lemma 8.5.2.5. *Let R be an associative ring, let x be an element of the center of A, and let M and N be discrete R-R-bimodules. If M and N are x-balanced, then $\mathrm{Tor}_n^R(M, N)$ is x-balanced.*

Proof. Let $l_x, r_x : M \to M$ be the maps given by left multiplication and right multiplication by x, respectively. For each integer $k \geq 0$, let M_k denote the kernel of $(l_x - r_x)^k$. Since x is central, M_k is an R-R submodule of M. Our assumption that M is x-balance implies that $M = \bigcup_k M_k$, so that $\mathrm{Tor}_n^R(M, N) \simeq \varinjlim \mathrm{Tor}_n^R(M_k, N)$. Since the collection of x-balanced submodules is closed under colimits, it will suffice to show that each $\mathrm{Tor}_n^R(M_k, N)$ is x-balanced. We proceed by induction on k. Using the exact sequences

$$\mathrm{Tor}_n^R(M_k, N) \to \mathrm{Tor}_n^R(M_{k+1}, N) \to \mathrm{Tor}_n^R(M_{k+1}/M_k, N),$$

we are reduced to proving that each of the bimodules $\mathrm{Tor}_n^R(M_{k+1}/M_k, N)$ is x-balanced. We may therefore replace M by M_{k+1}/M_k, and thereby reduce to the case where left and right multiplication by x induce the same map from M to itself. By the same argument, we can assume that left and right multiplication by x induce the same map from N to itself. In this case, Remark 8.5.2.4 immediately implies that left and right multiplication by x induce the same map from $\mathrm{Tor}_n^R(M, N)$ to itself, from which it follows immediately that $\mathrm{Tor}_n^R(M, N)$ is x-balanced. $\square$

Definition 8.5.2.6. Let A be a quasi-commutative $\mathbb{E}_1$-ring, and let $M \in {}_A\mathrm{BMod}_A(\mathrm{Sp})$. Let $x \in \pi_0 A$. We will say that M is x-*balanced* if every homotopy group $\pi_n M$ is x-balanced, in the sense of Definition 8.5.2.1. We will say that M is *balanced* if it is x-balanced, for each $x \in \pi_0 A$.

Remark 8.5.2.7. Let A be a quasi-commutative $\mathbb{E}_1$-ring, and let M be a discrete A-A-bimodule. Then M is balanced (x-balanced) in the sense of Definition 8.5.2.6 if and only if it is balanced (x-balanced) in the sense of Definition 8.5.2.1, when viewed as a bimodule over the commutative ring $\pi_0 A$.

Notation 8.5.2.8. Let A be a quasi-commutative $\mathbb{E}_1$-ring. We let ${}_A\mathrm{BMod}_A^{\mathrm{bal}}(\mathrm{Sp})$ denote the full subcategory of ${}_A\mathrm{BMod}_A(\mathrm{Sp})$ spanned by the balanced A-A-bimodules. It follows immediately from Remark 8.5.2.3 that ${}_A\mathrm{BMod}_A^{\mathrm{bal}}(\mathrm{Sp})$ is a stable subcategory of ${}_A\mathrm{BMod}_A(\mathrm{Sp})$, which is closed under small colimits.

Proposition 8.5.2.9. *Let A be a connective, quasi-commutative $\mathbb{E}_1$-ring. Then the full subcategory*

$$_A\mathrm{BMod}_A^{\mathrm{bal}}(\mathrm{Sp}) \subseteq {}_A\mathrm{BMod}_A(\mathrm{Sp})$$

contains A and is closed under the relative tensor product $\otimes_A$.

Proof. Since A is quasi-commutative, it is obvious that the unit object $A \in {}_A\mathrm{BMod}_A(\mathrm{Sp})$ is balanced. Suppose that $M, N \in {}_A\mathrm{BMod}_A(\mathrm{Sp})$ are balanced; we wish to show that $M \otimes_A N$ is also balanced. Fix $x \in \pi_0 A$; we will show that $M \otimes_A N$ is x-balanced. According to Proposition 8.2.1.19, there exists a spectral sequence $\{E_r^{p,q}, d_r\}_{r \geq 1}$ converging to $\pi_{p+q}(M \otimes_A N)$, with $E_2^{p,q} = \mathrm{Tor}_p^{\pi_* A}(\pi_* M, \pi_* N)_q$. Using the functoriality of the construction of this spectral sequence, we see that it is a spectral sequence of bimodules over the graded ring $\pi_* A$. It follows that $\pi_*(M \otimes_A N)$ admits an exhaustive filtration by bimodules over $\pi_* A$, and that the associated graded objects for this filtration are given by subquotients of bimodules of the form $\mathrm{Tor}_p^{\pi_* A}(\pi_* M, \pi_* N)$. Using Remark 8.5.2.3, we are reduced to showing that each of the $(\pi_* A)$-bimodules $\mathrm{Tor}_p^{\pi_* A}(\pi_* M, \pi_* N)$ is x-balanced, which follows from Lemma 8.5.2.5. $\qquad\square$

Lemma 8.5.2.10. *Let $\phi : A \to B$ be an étale homomorphism of commutative rings, and choose an element $\overline{e} \in \mathrm{Tor}_0^{\mathbf{Z}}(B, B)$ whose image $e \in R = \mathrm{Tor}_0^A(B, B)$ is an idempotent satisfying $\mathrm{Tor}_0^A(B, B)[e^{-1}] \simeq B$ (see Remark 8.5.0.2). Let M be a (discete) B-B-bimodule. Then M is balanced as a B-B-bimodule if and only if the following conditions are satisfied:*

(1) *As an A-A-bimodule, M is balanced.*

(2) *The canonical map $M \to M[\overline{e}^{-1}]$ is an isomorphism.*

Proof. Assume first that M is balanced as a B-B-bimodule. Condition (1) is obvious. To verify (2), write A as a union of finitely generated subrings A_α. The structure theory of étale morphisms of commutative rings implies that there exists an index α and an étale homomorphism $A_\alpha \to B_\alpha$ such that $B \simeq \mathrm{Tor}_0^{A_\alpha}(A, B_\alpha)$. Enlarging A_α if necessary, we may assume that $\overline{e}$ is the image of an element $\overline{e}' \in \mathrm{Tor}_0^{\mathbf{Z}}(B_\alpha, B_\alpha)$ whose image $e' \in \mathrm{Tor}_0^{A_\alpha}(B_\alpha, B_\alpha)$ satisfies $\mathrm{Tor}_0^{A_\alpha}(B_\alpha, B_\alpha)[e'^{-1}] \simeq B_\alpha$. Replacing A by A_α, B by B_α, and $\overline{e}$ by $\overline{e}'$, we may assume that the commutative ring A is finitely generated over $\mathbf{Z}$.

Let I be the kernel of the multiplication map $m : \mathrm{Tor}_0^{\mathbf{Z}}(B, B) \to B$. Since B is a finitely generated commutative ring, $\mathrm{Tor}_0^{\mathbf{Z}}(B, B)$ also a finitely generated commutative ring and therefore Noetherian, so the ideal I is finitely generated. Since M is balanced as a B-B-bimodule, every element of I has a locally nilpotent action on M. It follows that every element $x \in M$ is annihilated by some power of I. For each $k \geq 0$, let M_k denote the submodule of M which is annihilated by I^k. Then $M = \bigcup_k M_k$. Consequently, to prove (b), it suffices to show that each of the maps $M_k \to M_k[\overline{e}^{-1}]$ is an isomorphism. We proceed by induction on k, the case $k = 0$ being obvious. To handle the inductive step, we observe that there is a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M_k & \longrightarrow & M_{k+1} & \longrightarrow & M_{k+1}/M_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\psi} & & \\
0 & \longrightarrow & M_k[\overline{e}^{-1}] & \longrightarrow & M_{k+1}[\overline{e}^{-1}] & \longrightarrow & (M_{k+1}/M_k)[\overline{e}^{-1}] & \longrightarrow & 0.
\end{array}
$$

Using the inductive hypothesis and the snake lemma, we are reduced to proving that the map ψ is an isomorphism. We may therefore replace M by M_{k+1}/M_k, and thereby reduce to the case where the action of $\mathrm{Tor}_0^{\mathbf{Z}}(B, B)$ on M factors through $\mathrm{Tor}_0^{\mathbf{Z}}(B, B)/I \simeq B$. The desired result now follows from the observation that $m(\overline{e}) = 1$.

Now suppose that (1) and (2) are satisfied; we wish to prove that M is balanced as a B-B-bimodule. Choose $x \in B$; we will show that M is x-balanced. Arguing as above, we can reduce to the case where A is a finitely generated commutative ring. Let $m' : \mathrm{Tor}_0^{\mathbf{Z}}(A, A) \to A$ be the multiplication map and let J be the kernel of m'. Since $\mathrm{Tor}_0^{\mathbf{Z}}(A, A)$ is a finitely generated commutative ring, it is Noetherian. It follows that

J is finitely generated. Since M is balanced as an A-A-bimodule, the action of J on M is locally nilpotent. We may therefore write $M = \bigcup_{k \geq 0} M'_k$, where M'_k denotes the submodule of M which is annihilated by J^k. Since the collection of x-balanced bimodules is closed under filtered colimits, it will suffice to show that each M'_k is x-balanced. We proceed by induction on k. Since the collection of x-balanced bimodules is closed under extensions, we are reduced to proving that each quotient M'_{k+1}/M'_k is x-balanced. Replacing M by M'_{k+1}/M'_k, we may reduce to the case where $JM = 0$: that is, the left and right actions of A on M coincide. It follows that we may regard M as a module over the ring $\mathrm{Tor}_0^A(B,B)$. Using condition (2), we see that $e \in \mathrm{Tor}_0^A(B,B)$ acts invertibly on M: that is, the action of $\mathrm{Tor}_0^{\mathbf{Z}}(B,B)$ on M factors through the map $\mathrm{Tor}_0^{\mathbf{Z}}(B,B) \to \mathrm{Tor}_0^A(B,B)[e^{-1}] \simeq B$, so that the left and right actions of B on M coincide. In this case, it is obvious that M is x-balanced. $\square$

Remark 8.5.2.11. In the situation of Lemma 8.5.2.10, suppose that M satisfies condition (2) and that the left and right actions of A on M coincide. The proof of Lemma 8.5.2.10 shows that the left and right actions of B on M coincide.

Notation 8.5.2.12. Let B be a connective $\mathbb{E}_1$-ring and let $M \in {}_B\mathrm{BMod}_B(\mathcal{C})$, so that we can identify M with a left module over the $\mathbb{E}_1$-ring $B \otimes B^{\mathrm{rev}}$ (see Corollary 6.3.6.12). Suppose that $e \in \pi_0(B \otimes B^{\mathrm{rev}}) \simeq \mathrm{Tor}_0^{\mathbf{Z}}(\pi_0 B, \pi_0 B)$ is quasi-central: that is, the set $S = \{e^n\}_{n \geq 0} \subseteq \pi_0(B \otimes B^{\mathrm{rev}})$ satisfies the left Ore condition. We will denote the localization $M[S^{-1}]$ by $M[e^{-1}]$, and we will say that M is e-*local* if the unit map $M \to M[e^{-1}]$ is an equivalence.

In the situation of Notation 8.5.2.12, if B is quasi-commutative, then every element of $\pi_0(B \otimes B^{\mathrm{rev}})$ is quasi-central (Lemma 8.5.1.14). We immediately deduce the following generalization of Lemma 8.5.2.10.

Lemma 8.5.2.13. *Let $\phi : A \to B$ be an étale map of connective, quasi-commutative $\mathbb{E}_1$-rings, and let $M \in {}_B\mathrm{BMod}_B(\mathrm{Sp})$. Then M is balanced as a B-B-bimodule if and only if it is balanced as an A-A-bimodule and $\overline{e}$-local, where $\overline{e} \in \mathrm{Tor}_0^{\mathbf{Z}}(\pi_0 B, \pi_0 B)$ is chosen as in Lemma 8.5.2.10.*

Our next goal is to describe a "base-change" functor for balanced bimodules along an étale morphism.

Lemma 8.5.2.14. *Let $\phi : A \to B$ be an étale homomorphism of commutative rings and let $\overline{e}$ be as in Lemma 8.5.2.10. Let M be a balanced A-A-bimodule. Then the canonical map*

$$\theta_M : B \otimes_A M \to (B \otimes_A M \otimes_A B)[\overline{e}^{-1}]$$

is an isomorphism of left B-modules.

Proof. Arguing as in the proof of Lemma 8.5.2.10, we reduce to the case in which A is a finitely generated commutative ring. Let I denote the kernel of the multiplication map $\mathrm{Tor}_0^{\mathbf{Z}}(A,A) \to A$. Then I is a finitely generated ideal. Since every element of I has a locally nilpotent action on M, we conclude that $M = \bigcup_{k \geq 0} M_k$, where M_k denotes the submodule of M annihilated by I^k. It will therefore suffice to show that each of the maps θ_{M_k} is an isomorphism. We proceed by induction on k, the case $k = 0$ being trivial. To handle the inductive step, we use the short exact sequence

$$0 \to M_k \to M_{k+1} \to M_{k+1}/M_k \to 0$$

to reduce to the problem of showing that θ_{M_{k+1}/M_k} is an isomorphism. Replacing M by M_{k+1}/M_k, we reduce to the case where $IM = 0$: that is, the left and right actions of A on M are the same. In this case, we can identify θ_M with a map

$$B \otimes_A M \to \mathrm{Tor}_0^A(B,B)[e^{-1}] \otimes_A M.$$

This map is an isomorphism, since $\overline{e}$ was chosen so that $\mathrm{Tor}_0^A(B,B)[e^{-1}] \simeq B$. $\square$

Lemma 8.5.2.15. *Let $\phi : A \to B$ be an étale map between connective quasi-commutative $\mathbb{E}_1$-rings. Then the forgetful functor ${}_B\mathrm{BMod}_B^{\mathrm{bal}}(\mathrm{Sp}) \to {}_A\mathrm{BMod}_A^{\mathrm{bal}}(\mathrm{Sp})$ admits a left adjoint $F : {}_A\mathrm{BMod}_A^{\mathrm{bal}}(\mathrm{Sp}) \to {}_B\mathrm{BMod}_B^{\mathrm{bal}}(\mathrm{Sp})$. Moreover, for every object $M \in {}_A\mathrm{BMod}_A^{\mathrm{bal}}(\mathrm{Sp})$, the composite map*

$$B \otimes_A M \to B \otimes_A M \otimes_A B \to F(M)$$

is an equivalence of left B-modules.

Proof. Let $\overline{e} \in \pi_0(B \otimes B^{\mathrm{rev}}) \simeq \mathrm{Tor}_0^{\mathbf{Z}}(\pi_0 B, \pi_0 B)$ be as in Lemma 8.5.2.10. Using Lemma 8.5.2.13, we deduce that the functor F exists and is given on objects by the formula $F(M) = (B \otimes_A M \otimes_A B)[\overline{e}^{-1}]$. The last assertion follows from Lemma 8.5.2.14. $\qquad\square$

In the situation of Lemma 8.5.2.15, let us regard ${}_B\mathrm{BMod}_B^{\mathrm{bal}}(\mathrm{Sp})$ and ${}_A\mathrm{BMod}_A^{\mathrm{bal}}(\mathrm{Sp})$ as monoidal ∞-categories, so that the forgetful functor $G : {}_B\mathrm{BMod}_B^{\mathrm{bal}}(\mathrm{Sp}) \to {}_A\mathrm{BMod}_A^{\mathrm{bal}}(\mathrm{Sp})$ is lax monoidal. We therefore obtain a canonical map $F(M \otimes_A N) \to F(M) \otimes_B F(N)$ for every pair of objects $M, N \in {}_A\mathrm{BMod}_A^{\mathrm{bal}}(\mathrm{Sp})$. We claim that this map is an equivalence. To prove this, it will suffice (by Lemma 8.5.2.15) to show that the composite map

$$B \otimes_A (M \otimes_A N) \to F(M \otimes_A N) \to F(M) \otimes_B F(N)$$

is an equivalence. This map factors as a composition

$$B \otimes_A M \otimes_A N \xrightarrow{\psi} F(M) \otimes_A N \xrightarrow{\psi'} F(M) \otimes_B F(N),$$

where ψ and ψ' are equivalences by Lemma 8.5.2.15. A similar argument shows that the canonical map $F(A) \to B$ is an equivalence. Using Corollary 8.3.2.12, we deduce that F can be regarded as a monoidal functor from ${}_A\mathrm{BMod}_A^{\mathrm{bal}}(\mathrm{Sp})$ to ${}_B\mathrm{BMod}_B^{\mathrm{bal}}(\mathrm{Sp})$, and Remark 8.3.2.13 implies that composition with F induces a functor

$$\mathrm{Alg}({}_A\mathrm{BMod}_A^{\mathrm{bal}}(\mathrm{Sp})) \to \mathrm{Alg}({}_B\mathrm{BMod}_B^{\mathrm{bal}}(\mathrm{Sp}))$$

which is left adjoint to the forgetful functor. Combining this observation with Corollary 3.4.1.7, we obtain the following result:

Proposition 8.5.2.16. *Let $\phi : A \to B$ be an étale morphism between connective, quasi-commutative $\mathbb{E}_1$-rings. Let $\mathrm{Alg}_{A/}^{(1),\mathrm{bal}}$ denote the full subcategory of $\mathrm{Alg}_{A/}^{(1)}$ spanned by those morphisms of $\mathbb{E}_1$-rings $\psi : A \to A'$ which exhibit A' as a balanced A-A-bimodule, and let $\mathrm{Alg}_{B/}^{(1),\mathrm{bal}}$ be defined similarly. Then the forgetful functor $\mathrm{Alg}_{B/}^{(1),\mathrm{bal}} \to \mathrm{Alg}_{A/}^{(1),\mathrm{bal}}$ admits a left adjoint F. Moreover, for every $A' \in \mathrm{Alg}_{A/}^{(1),\mathrm{bal}}$, the canonical map $B \otimes_A A' \to F(A')$ is an equivalence of left B-modules.*

Remark 8.5.2.17. In the situation of Proposition 8.5.2.16, the map $\phi : A \to B$ exhibits B as a flat right module over A. It follows from Proposition 8.2.2.13 that for every $A' \in \mathrm{Alg}_{A/}^{(1),\mathrm{bal}}$, the canonical map $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_n A') \to \pi_n F(A')$ is an isomorphism for every integer n. It follows that $F(A')$ is flat as a right A'-module. If we assume in addition that A' is quasi-commutative and the natural map $\pi_0 A \to \pi_0 A'$ is an isomorphism, then we obtain an isomorphism $\pi_0 B \to \pi_0 F(A')$, and Remark 8.5.2.11 shows that the left and right actions of $\pi_0 F(A')$ on $\pi_n F(A')$ coincide for each n. It follows that $F(A')$ is quasi-commutative and that the map $A' \to F(A')$ is an étale morphism of $\mathbb{E}_1$-rings.

Proposition 8.5.2.18. *Let A be a quasi-commutative $\mathbb{E}_1$-ring. Suppose we are given an étale morphism of commutative rings $\phi_0 : \pi_0 A \to B_0$. Then there exists a map of $\mathbb{E}_1$-rings $\phi : A \to B$ with the following properties:*

(1) *The map ϕ is étale (in particular, B is quasi-commutative).*

(2) *There is an isomorphism of commutative rings $\pi_0 B \simeq B_0$ such that ϕ induces the map $\phi_0 : \pi_0 A \to B_0$.*

(3) *For every quasi-commutative $\mathbb{E}_1$-ring C, the canonical map*

$$\mathrm{Map}_{\mathrm{Alg}^{(1)}}(B,C) \to \mathrm{Map}_{\mathrm{Alg}^{(1)}}(A,C) \times_{\mathrm{Hom}_{\mathrm{Ring}}(\pi_0 A, \pi_0 C)} \mathrm{Hom}_{\mathrm{Ring}}(\pi_0 B, \pi_0 C).$$

Proof. Let A' be a connective cover of A. Using Theorem 8.5.1.11, we can choose an étale map of connective $\mathbb{E}_1$-rings $\phi' : A' \to B'$ and an isomorphism $\pi_0 B' \simeq B_0$. Let $F : \mathrm{Alg}^{(1),\mathrm{bal}}_{A'/} \to \mathrm{Alg}^{(1),\mathrm{bal}}_{B'/}$ be as in Proposition 8.5.2.16, and set $B = F(A)$. We have a commutative diagram of $\mathbb{E}_1$-rings

$$\begin{array}{ccc}
A' & \xrightarrow{\phi'} & B' \\
\downarrow & & \downarrow \\
A & \xrightarrow{\phi} & B.
\end{array}$$

It follows from Remark 8.5.2.17 that ϕ is étale and that $\pi_0 B \simeq \pi_0 B' \simeq B_0$. Using the definition of F, we note that for every map of $\mathbb{E}_1$-rings $A \to C$ which exhibits C as a balanced bimodule over A, we have $\mathrm{Map}_{\mathrm{Alg}^{(1)}_{A/}}(B,C) \simeq \mathrm{Map}_{\mathrm{Alg}^{(1)}_{A'/}}(B',C)$. If C is quasi-commutative, then it is automatically balanced as a bimodule, so that upper square in the diagram

$$\begin{array}{ccc}
\mathrm{Map}_{\mathrm{Alg}^{(1)}}(B,C) & \longrightarrow & \mathrm{Map}_{\mathrm{Alg}^{(1)}}(A,C) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathrm{Alg}^{(1)}}(B',C) & \longrightarrow & \mathrm{Map}_{\mathrm{Alg}^{(1)}}(A',C) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathrm{Ring}}(B_0, \pi_0 C) & \longrightarrow & \mathrm{Hom}_{\mathrm{Ring}}(\pi_0 A, \pi_0 C)
\end{array}$$

is a pullback square. The lower square is pullback by Proposition 8.5.1.15, so that the outer rectangle is also a pullback diagram. $\qquad\square$

Corollary 8.5.2.19. *Let $\phi : A \to B$ be an étale morphism between quasi-commutative $\mathbb{E}_1$-rings. Then, for every quasi-commutative $\mathbb{E}_1$-ring C, the canonical map*

$$\mathrm{Map}_{\mathrm{Alg}^{(1)}}(B,C) \to \mathrm{Map}_{\mathrm{Alg}^{(1)}}(A,C) \times_{\mathrm{Hom}_{\mathrm{Ring}}(\pi_0 A, \pi_0 C)} \mathrm{Hom}_{\mathrm{Ring}}(\pi_0 B, \pi_0 C)$$

is a homotopy equivalence.

Proof. Let $B_0 = \pi_0 B$ and let $\phi_0 : \pi_0 A \to B_0$ be the étale map of commutative rings induced by ϕ. Choose an étale morphism $\phi' : A \to B'$ satisfying the conclusions of Proposition 8.5.2.18. In particular, we see that the map $\phi : A \to B$ factors as a composition

$$A \xrightarrow{\phi'} B' \xrightarrow{\psi} B,$$

where ψ induces the identity map $\pi_0 B' \to \pi_0 B$. Since B' and B are both flat over A, we deduce that for every integer n the map

$$\pi_n B' \simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_n A, \pi_0 B') \to \mathrm{Tor}_0^{\pi_0 A}(\pi_n A, \pi_0 B) \simeq \pi_n B$$

is an isomorphism. It follows that ψ is an equivalence, so that $\phi : A \to B$ also satisfies the conclusions of Proposition 8.5.2.18. $\qquad\square$

We are now ready to prove a non-connective version of Theorem 8.5.1.16.

Theorem 8.5.2.20. *Let $\psi : A \to C$ be a map of quasi-commutative $\mathbb{E}_1$-rings. Then the construction $B \mapsto \pi_0 B$ induces an equivalence of ∞-categories*

$$\theta : \mathrm{Alg}^{(1),\acute{e}t}_{A//C} \to \mathrm{N}(\mathrm{Ring}^{\acute{e}t}_{\pi_0 A//\pi_0 C}).$$

Proof. We have a commutative diagram of ∞-categories

$$
\begin{array}{ccc}
\mathrm{Alg}^{(1),\acute{e}t}_{A//C} & \longrightarrow & \mathrm{N}(\mathrm{Ring}^{\acute{e}t}_{\pi_0 A//\pi_0 C}) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\
\mathrm{Alg}^{(1),\acute{e}t}_{A/} & \xrightarrow{\ \theta'\ } & \mathrm{N}(\mathrm{Ring}^{\acute{e}t}_{\pi_0 A/}).
\end{array}
$$

The vertical maps are right fibrations. The map θ' is essentially surjective by Proposition 8.5.2.18, and fully faithful by Corollary 8.5.2.19. It follows that θ' is an equivalence of ∞-categories. It therefore suffices to show that for every étale map $A \to B$, the functor θ induces a homotopy equivalence from the fiber of p over B to the fiber of q over $\pi_0 B$ (Corollary T.2.4.4.4). This follows immediately from Corollary 8.5.2.19. $\qquad\square$

8.5.3 Cocentric Morphisms

For every associative ring A, let $\mathfrak{Z}(A)$ denote the center of A. The construction $A \mapsto \mathfrak{Z}(A)$ is not functorial: if $\phi : A \to B$ is a morphism of commutative rings, then ϕ does not generally carry $\mathfrak{Z}(A)$ into $\mathfrak{Z}(B)$. However, we can guarantee that $\phi(\mathfrak{Z}(A)) \subseteq \mathfrak{Z}(B)$ if we are willing to assume that ϕ satisfies the following condition:

($*$) The inclusion $\mathfrak{Z}(B) \subseteq \mathfrak{Z}(\phi)$ is a bijection. In other words, an element $b \in B$ is central if and only if $b\phi(a) = \phi(a)b$ for all $a \in A$.

Condition ($*$) has the virtue of being phrased in terms of centralizers, and can therefore be generalized to an arbitrary monoidal ∞-category.

Definition 8.5.3.1. Let $\mathcal{C}$ be a monoidal ∞-category, let $\mathcal{M}$ be an ∞-category which is left-tensored over $\mathcal{M}$, and suppose we are given morphisms $A \xrightarrow{\phi} B \xrightarrow{\psi} C$. We will say that ϕ is *cocentric relative to* ψ if the following conditions are satisfied:

(i) There exists a centralizer $\mathfrak{Z}(\psi)$ of ψ.

(ii) The composite map

$$\mathfrak{Z}(\psi) \otimes A \xrightarrow{\mathrm{id}\,\otimes\,\phi} \mathfrak{Z}(\psi) \otimes B \longrightarrow C$$

exhibits $\mathfrak{Z}(\psi)$ as a centralizer of $\psi \circ \phi$.

We will say that a morphism $\phi : A \to B$ in $\mathcal{M}$ is *cocentric* if it is cocentric relative to id_B.

Example 8.5.3.2. Let $\mathcal{C}$ be the nerve of the category of associative rings. Regard $\mathcal{C}$ as a monoidal ∞-category via the usual tensor product of rings. A ring homomorphism $\phi : A \to B$ in $\mathcal{C}$ is cocentric if and only if it satisfies condition ($*$) above.

Example 8.5.3.3. In [45], the authors define a *centric* map of spaces to be a map $f : X \to Y$ which induces a homotopy equivalence $\mathrm{Map}_{\mathcal{S}}(X, X)_{\mathrm{id}_X} \to \mathrm{Map}_{\mathcal{S}}(X, Y)_f$. Here $\mathrm{Map}_{\mathcal{S}}(X, X)_{\mathrm{id}_X}$ denote the connected component of $\mathrm{Map}_{\mathcal{S}}(X, X)$ containing id_X, and $\mathrm{Map}_{\mathcal{S}}(X, Y)_f$ is defined similarly. Let $\mathcal{S}_{\geq 1}$ be the full subcategory of $\mathcal{S}$ spanned by the connected spaces, which we regard as endowed with the Cartesian monoidal structure, and regard $\mathcal{S}$ as left-tensored over $\mathcal{S}_{\geq 1}$ (via the Cartesian product). For every map of spaces $f : X \to Y$, the space $\mathrm{Map}_{\mathcal{S}}(X, Y)_f$ can be identified with the centralizer of f in $\mathcal{S}_{\geq 1}$. Consequently, f is centric if and only if it induces a homotopy equivalence $\mathfrak{Z}(\mathrm{id}_X) \to \mathfrak{Z}(f)$. This is precisely *dual* to the requirement of Definition 8.5.3.1.

In this section, we will develop the theory of cocentric morphisms and apply it to obtain a relative version of Theorem 8.5.2.20. Our first main result is that if $\phi : A \to B$ is a cocentric morphism, the ϕ induces a map from the center of A (provided that it exists) to the center of B.

Proposition 8.5.3.4. *Let $\mathcal{C}$ be a monoidal ∞-category containing an algebra object $R \in \mathrm{Alg}(\mathcal{C})$, let $\mathcal{M}$ be an ∞-category left-tensored over $\mathcal{C}$. Suppose we are given a diagram*

$$
\begin{array}{ccc}
\Lambda_0^n & \xrightarrow{\;U_0\;} & \mathrm{LMod}_R(\mathcal{M}) \\
\big\downarrow & {\scriptstyle U}\;\nearrow & \big\downarrow \\
\Delta^n & \xrightarrow{\;\;V\;\;} & \mathcal{M}
\end{array}
$$

for some integer $n > 0$. Let $A = V(0)$, $B = V(1)$, and $C = V(n)$, so that V induces morphisms $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ in $\mathcal{M}$. If ϕ is cocentric relative to ψ, then there exists a dotted arrow U as indicated, rendering the diagram commutative.

We will give the proof of Proposition 8.5.3.4 at the end of this section.

Remark 8.5.3.5. In the case $n = 1$, Proposition 8.5.3.4 asserts that if $\phi : A \to B$ is a cocentric morphism and A admits a left action of an algebra object $R \in \mathrm{Alg}(\mathcal{C})$, then B also admits a left action of R so that ϕ is a map of left R-modules. In particular, if there exists a center $\mathfrak{Z}(A)$ for A, then $\mathfrak{Z}(A)$ acts on B via some map $\mathfrak{Z}(A) \to \mathfrak{Z}(B)$. Applying Proposition 8.5.3.4 for $n > 0$, one can argue that this map is unique up to a contractible space of choices.

The basic example of interest to us is the following:

Proposition 8.5.3.6. *Let $\mathrm{Alg}^{(1)}$ denote the symmetric monoidal ∞-category of $\mathbb{E}_1$-rings, which we regard as left-tensored over itself. Suppose we are given morphisms $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ in $\mathrm{Alg}^{(1)}$ such that C is quasi-commutative and ϕ is étale (so that A and B are also quasi-commutative). Then ϕ is cocentric relative to ψ.*

Proof. Let us regard the ∞-categories $_B\mathrm{BMod}_B(\mathrm{Sp})$ and $_A\mathrm{BMod}_A(\mathrm{Sp})$ as left-tensored over the ∞-category of spectra. Using Theorem 6.1.4.27, we deduce that the centralizers $\mathfrak{Z}(\psi)$ and $\mathfrak{Z}(\psi \circ \phi)$ exist, and are given by

$$
\mathfrak{Z}(\psi) \simeq \mathrm{Mor}_{_B\mathrm{BMod}_B(\mathrm{Sp})}(B, C) \qquad \mathfrak{Z}(\psi \circ \phi) \simeq \mathrm{Mor}_{_A\mathrm{BMod}_A(\mathrm{Sp})}(A, C) \simeq \mathrm{Mor}_{_B\mathrm{BMod}_B(\mathrm{Sp})}(B \otimes_A B, C).
$$

To show that the canonical map $\mathfrak{Z}(\psi) \to \mathfrak{Z}(\psi \circ \phi)$ is an equivalence of spectra, it suffices to show that it induces an isomorphism on homotopy groups.

Let K denote the fiber of the map $B \otimes_A B \to B$; we wish to show that $\mathrm{Ext}^n_{_B\mathrm{BMod}_B(\mathrm{Sp})}(K, C) \simeq 0$ for every integer n. Since ϕ is étale, there exists an element $\overline{e} \in \mathrm{Tor}_0^{\mathbf{Z}}(\pi_0 B, \pi_0 B)$ whose image $e \in \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 B)$ is an idempotent satisfying $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 B)[e^{-1}] \simeq \pi_0 B$. According to Lemma 8.5.1.14, the image x of $\overline{e}$ in $\pi_0(B \otimes B^{\mathrm{rev}})$ is quasi-central. Since C is quasi-commutative, multiplication by x induces the identity map from $\pi_* C$ to itself. It will therefore suffice to show that K is x-nilpotent, which is clear (since the homotopy groups of K are annihilated by multiplication by $\overline{e}$). $\square$

Corollary 8.5.3.7. *Every étale morphism in $\mathrm{Alg}^{(1)}$ is cocentric.*

Corollary 8.5.3.8. *Let R be an $\mathbb{E}_2$-ring and let A be an $\mathbb{E}_1$-algebra over R. Suppose we are given an étale map $\phi : A \to B$ of $\mathbb{E}_1$-rings. Then there exists an R-algebra structure on B such that ϕ lifts to a morphism of $\mathbb{E}_1$-algebras over R.*

Proof. Combine Propositions 8.5.3.6 and 8.5.3.4 (in the case $n = 1$). $\square$

Corollary 8.5.3.9. *Let R be an $\mathbb{E}_2$-ring and let A be an $\mathbb{E}_1$-algebra over R. Suppose we are given a commutative diagram*

$$
\begin{array}{ccc}
\partial\Delta^m & \longrightarrow & (\mathrm{Alg}_R^{(1)})_{A/} \\
\downarrow & \nearrow & \downarrow{\scriptstyle q} \\
\Delta^m & \xrightarrow{\ U\ } & (\mathrm{Alg}^{(1)})_{A/}
\end{array}
$$

(here we have arranged that q is a categorical fibration). If the map $A \to U(0)$ is étale and the image of $U(n)$ in $\mathrm{Alg}^{(1)}$ is quasi-commutative, then there exists a dotted arrow as indicated, rendering the diagram commutative.

Proof. Combine Proposition 8.5.3.6 with Proposition 8.5.3.4 (in the case $n = m + 1$). $\qquad\square$

Corollary 8.5.3.10. *Let R be an $\mathbb{E}_2$-ring. Suppose we are given maps $\phi : A \to B$ and $\psi : A \to C$ of quasi-commutative $\mathbb{E}_1$-algebras over R, where ϕ is étale. Let $\theta : \mathrm{Alg}_R^{(1)} \to \mathrm{Alg}^{(1)}$ be the forgetful functor. Then θ induces a homotopy equivalence*

$$
\mathrm{Map}_{(\mathrm{Alg}_R^{(1)})_{A/}}(B, C) \to \mathrm{Map}_{\mathrm{Alg}_{\theta(A)/}^{(1)}}(\theta(B), \theta(C)).
$$

Notation 8.5.3.11. Let R be an $\mathbb{E}_2$-ring, and suppose we are given a map $\psi : A \to C$ of quasi-commutative $\mathbb{E}_1$-algebras over R. We let $(\mathrm{Alg}_R^{(1)})_{A//C}^{\text{ét}}$ denote the full subcategory of $(\mathrm{Alg}_R^{(1)})_{A//C}$ spanned by those diagrams

$$
\begin{array}{ccc}
 & B & \\
{\scriptstyle\phi}\nearrow & & \searrow \\
A & \xrightarrow[\psi]{\hspace{2cm}} & C
\end{array}
$$

where ϕ is étale.

Corollary 8.5.3.12. *Let R be an $\mathbb{E}_2$-ring and let A be a quasi-commutative $\mathbb{E}_1$-algebra over R. Then the forgetful functor*

$$
(\mathrm{Alg}_R^{(1)})_{A//C}^{\text{ét}} \to \mathrm{Alg}_{A//C}^{(1),\text{ét}}
$$

is an equivalence of ∞-categories.

Proof. Combine Corollaries 8.5.3.10 and 8.5.3.8. $\qquad\square$

Corollary 8.5.3.13. *Let R be an $\mathbb{E}_2$-ring and let $\psi : A \to C$ be a morphism of quasi-commutative $\mathbb{E}_1$-algebras over R. Then the construction $B \mapsto \pi_0 B$ induces an equivalence of ∞-categories*

$$
(\mathrm{Alg}_R^{(1)})_{A//C}^{\text{ét}} \to \mathrm{N}(\mathrm{Ring}_{\pi_0 A/ \ /\pi_0 C}^{\text{ét}}).
$$

Proof. Combine Corollary 8.5.3.12 with Theorem 8.5.2.20. $\qquad\square$

Proof of Proposition 8.5.3.4. Let $p : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes}$ be defined as in Notation 4.2.2.16. Then R determines a map $\mathrm{N}(\boldsymbol{\Delta})^{op} \to \mathcal{C}^{\otimes}$. We let $\mathcal{N}$ denote the fiber product $\mathcal{M}^{\otimes} \times_{\mathcal{C}^{\otimes}} \mathrm{N}(\boldsymbol{\Delta})^{op}$, so that the projection map $q : \mathcal{N} \to \mathrm{N}(\boldsymbol{\Delta})^{op}$ is a locally coCartesian fibration whose fibers are canonically equivalent to the ∞-category $\mathcal{M}$. Using Corollary 4.2.2.15 and Proposition T.A.2.3.1, we see that it suffices to solve the weakly equivalent lifting problem

$$
\begin{array}{ccc}
\Lambda_0^n & \xrightarrow{\ \overline{U}_0\ } & {}^{\Delta}\mathrm{LMod}_R(\mathcal{M}) \\
\downarrow & & \downarrow \\
\Delta^n & \xrightarrow{\ V\ } & \mathcal{M}.
\end{array}
$$

Here we can identify $^{\Delta}\mathrm{LMod}_R(\mathcal{M})$ with a full subcategory of the ∞-category $\mathrm{Fun}_{\mathrm{N}(\mathbf{\Delta})^{op}}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{N})$ of sections of q. Consequently, $\overline{U}_0$ determines a map $f : \mathrm{N}(\mathbf{\Delta})^{op} \times \Lambda_0^n \to \mathcal{N}$. To construct $\overline{U}$, we must find a suitable map $F : \mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^n \to \mathcal{N}$ extending f.

Let $X = \mathrm{N}(\mathbf{\Delta})^{op} \times \Lambda_0^n$, and let $X' = X \coprod_{\{[0]\} \times \Lambda_0^n}(\{[0]\} \times \Delta^n)$. We regard X' as a simplicial subset of $\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^n$. Amalgamating f and V, we obtain a map $f' : X' \to \mathcal{N}$. Let X'' denote the simplicial subset of $\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^n$ given by the union of X' with those simplices σ whose intersection with $\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^{\{1,\ldots,n\}}$ is contained in $\{[0]\} \times \Delta^{\{1,\ldots,n\}}$. We claim that the inclusion $X' \hookrightarrow X''$ is a categorical equivalence. Note that there is a pushout diagram

$$(\mathrm{N}(\mathbf{\Delta}_{[0]/})^{op} \star \partial\,\Delta^{\{1,\ldots,n\}}) \coprod_{\Lambda_0^n} \Delta^n \overset{i}{\longrightarrow} \mathrm{N}(\mathbf{\Delta}_{[0]/})^{op} \star \Delta^{\{1,\ldots,n\}}$$

$$X' \longrightarrow X''.$$

It will therefore suffice to show that the map i is a categorical equivalence: that is, that the diagram

$$\Lambda_0^n \longrightarrow \Delta^n$$

$$\mathrm{N}(\mathbf{\Delta}_{[0]/})^{op} \star \partial\,\Delta^{\{1,\ldots,n\}} \longrightarrow \mathrm{N}(\mathbf{\Delta}_{[0]/})^{op} \star \Delta^{\{1,\ldots,n\}}$$

is a homotopy pushout square (with respect to the Joyal model structure). This follows immediately from the observation that the inclusion $\{[0]\} \hookrightarrow \mathrm{N}(\mathbf{\Delta}_{[0]/})^{op}$ is right anodyne.

Let $X''' \subseteq \mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^n$ be the simplicial subset consisting of X together with all those nondegenerate simplices σ whose intersection with $\mathrm{N}(\mathbf{\Delta})^{op} \times \{1\}$ is contained in $\{[0]\} \times \{1\}$. To prove this, we let K denote the product $\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^{\{2,\ldots,n\}}$ and $K_0 \subseteq K$ the simplicial subset given by

$$(\mathrm{N}(\mathbf{\Delta})^{op} \times \partial\,\Delta^{\{2,\ldots,n\}}) \coprod_{\{[0]\} \times \partial\,\Delta^{\{2,\ldots,n\}}} (\{[0]\} \times \Delta^{\{2,\ldots,n\}}).$$

Since $[0]$ is an initial object of $\mathrm{N}(\mathbf{\Delta})^{op}$, the inclusion $K_0 \hookrightarrow K$ is left anodyne. It follows that the diagram

$$\mathrm{N}(\mathbf{\Delta}_{[0]/})^{op} \star K_0 \longrightarrow \mathrm{N}(\mathbf{\Delta}_{[0]/})^{op} \star K_0^{\triangleleft}$$

$$\mathrm{N}(\mathbf{\Delta}_{[0]/})^{op} \star K \longrightarrow \mathrm{N}(\mathbf{\Delta}_{[0]/})^{op} \star K^{\triangleleft}$$

is a homotopy pushout square (with respect to the Joyal model structure). We have a pushout diagram of simplicial sets

$$(\mathrm{N}(\mathbf{\Delta}_{[0]/})^{op} \star K_0^{\triangleleft}) \coprod_{\mathrm{N}(\mathbf{\Delta}_{[0]/})^{op} \star K_0}(\mathrm{N}(\mathbf{\Delta}_{[0]/})^{op} \star K) \longrightarrow \mathrm{N}(\mathbf{\Delta}_{[0]/})^{op} \star K^{\triangleleft}$$

$$X'' \longrightarrow X''',$$

so that the inclusion $X'' \hookrightarrow X'''$ is a categorical equivalence. It follows that the inclusion $X' \hookrightarrow X'''$ is also a categorical equivalence. Because q is a categorical fibration, the lifting problem depicted in the diagram

$$\begin{array}{ccc} X' & \overset{f'}{\longrightarrow} & \mathcal{N} \\ \downarrow & \overset{F_0}{\nearrow} & \downarrow q \\ X''' & \longrightarrow & \mathrm{N}(\mathbf{\Delta})^{op} \end{array}$$

admits a solution.

Let S be the collection of all nondegenerate simplices $\sigma : \Delta^m \to N(\mathbf{\Delta})^{op} \times \Delta^{\{1,\ldots,n\}}$ such that the induced map $\Delta^m \to \Delta^{\{1,\ldots,n\}}$ is surjective and $\sigma(0) = ([k], 1)$ for $k > 0$. If σ is a nondegenerate simplex $\Delta^m \to N(\mathbf{\Delta})^{op} \times \Delta^n$ which is not contained in X''', we let λ_σ denote the smallest integer such that $\sigma(\lambda_\sigma) = ([k], 1)$ for $k > 0$, and define the *tail* of σ to be the simplex $t(\sigma) = (\sigma|\Delta^{\{\lambda_\sigma, \lambda_{\sigma+1}, \ldots, m\}})$, so that $t(\sigma) \in S$. Choose a well-ordering of S such that $\sigma < \sigma'$ whenever σ has dimension smaller than that of σ'. Let α denote the order type of S, so that we have a bijection

$$\{\beta : \beta < \alpha\} \to S$$

$$\beta \mapsto \sigma_\beta.$$

For every ordinal $\beta \leq \alpha$, let $\mathcal{E}_\beta \subseteq N(\mathbf{\Delta})^{op} \times \Delta^n$ be the simplicial subset consisting of those nondegenerate simplices σ which either belong to X''' or satisfy $t(\sigma) = \sigma_\gamma$ for some $\gamma < \beta$, so that $\mathcal{E}_0 = X'''$. We will extend F_0 to a compatible family of maps $F_\beta \in \mathrm{Fun}_{N(\mathbf{\Delta})^{op}}(\mathcal{E}_\beta, \mathcal{N})$. The construction proceeds by induction on β. If β is a limit ordinal, we set $F_\beta = \bigcup_{\gamma < \beta} F_\gamma$.

Let us assume that F_β has been defined for some $\beta < \alpha$; our goal is to construct $F_{\beta+1}$. Let $\sigma = \sigma_\beta : \Delta^m \to N(\mathbf{\Delta})^{op} \times \Delta^n$. Let Y denote the full subcategory of $(N(\mathbf{\Delta})^{op} \times \Delta^n)_{/\sigma}$ spanned by those objects whose image in $N(\mathbf{\Delta})^{op} \times \Delta^n$ are either of the form $([k], 0)$ or $([0], 1)$, so that we have a pushout diagram of simplicial sets

$$
\begin{array}{ccc}
Y \star \partial\,\Delta^m & \longrightarrow & Y \star \Delta^m \\
\downarrow & & \downarrow \\
\mathcal{E}_\beta & \longrightarrow & \mathcal{E}_{\beta+1} .
\end{array}
$$

Consequently, to construct $F_{\beta+1}$ from F_β, it suffices to solve the lifting problem depicted in the diagram

$$
\begin{array}{ccc}
Y \star \partial\,\Delta^m & \longrightarrow & \mathcal{N} \\
\downarrow & \nearrow & \downarrow {\scriptstyle q} \\
Y \star \Delta^m & \longrightarrow & N(\mathbf{\Delta})^{op}.
\end{array}
$$

Let g denote the composite map $Y \to \mathcal{E}_\beta \overset{F_\beta}{\to} \mathcal{N}$, so that we can rewrite our lifting problem as follows:

$$
\begin{array}{ccc}
\partial\,\Delta^m & \longrightarrow & \mathcal{N}_{g/} \\
\downarrow & \nearrow & \downarrow {\scriptstyle q} \\
\Delta^m & \longrightarrow & (N(\mathbf{\Delta})^{op})_{qg/}.
\end{array}
$$

Let $\sigma(0) = ([k], 1)$ for some $k > 0$. Note that there is a left cofinal map $\Lambda_0^2 \to Y$, whose image in $N(\mathbf{\Delta})^{op} \times \Delta^n$ is the diagram

$$([k], 0) \leftarrow ([0], 0) \to ([0], 1).$$

Let $g_0 = g|\Lambda_0^2$. Since the map $\mathcal{N}_{g/} \to \mathcal{N}_{g_0/} \times_{(N(\mathbf{\Delta})^{op})_{qg_0/}} (N(\mathbf{\Delta})^{op})_{qg/}$ is a trivial Kan fibration, it suffices to solve the lifting problem depicted in the diagram τ :

$$
\begin{array}{ccc}
\partial\,\Delta^m & \overset{h_0}{\longrightarrow} & \mathcal{N}_{g_0/} \\
\downarrow & \nearrow & \downarrow {\scriptstyle q'} \\
\Delta^m & \longrightarrow & (N(\mathbf{\Delta})^{op})_{qg_0/}.
\end{array}
$$

We now treat the special case $m = 0$ (in which case we must also have $n = 1$). Let δ denote the unique map $[0] \to [k]$ in $N(\mathbf{\Delta})^{op}$. We are required to choose a commutative diagram

$$
\begin{array}{ccc}
F_\beta([0], 0) & \xrightarrow{F_\beta(\delta, \mathrm{id})} & F_\beta([k], 0) \\
\downarrow & & \downarrow \\
F_\beta([0], 1) & \xrightarrow{\quad u \quad} & X
\end{array}
$$

in the ∞-category $\mathcal{N}$ covering the diagram

$$
\begin{array}{ccc}
[0] & \xrightarrow{\delta} & [k] \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
[0] & \xrightarrow{\delta} & [k]
\end{array}
$$

in $N(\mathbf{\Delta})^{op}$. Note that q is a locally coCartesian fibration, and that $F_\beta(\delta, \mathrm{id})$ is a locally q-coCartesian morphism (since $\overline{U}_0(0)$ belongs to $^{\Delta}\mathrm{LMod}(\mathcal{M})$). We can therefore make our choice in such a way that u is also locally q-coCartesian. This strategy guarantees that our maps F_β satisfy the following additional condition:

    $(*)$ If $n = 1$ and $([k], 1) \in \mathcal{E}_\beta$, then F_β induces a locally q-coCartesian morphism $F_\beta([0], 1) \to F_\beta([k], 1)$.

Let us now treat the case where $m > 0$. Define objects $\overline{B}, \overline{C} \in \mathcal{N}_{g_0/}$ by $\overline{B} = h_0(0)$ and let $\overline{C} = h_0(m)$. To prove that the lifting problem depicted in the diagram τ admits a solution, it suffices to show that the mapping space $\mathrm{Map}_{\mathcal{N}_{g_0/}}(\overline{B}, \overline{C})$ is contractible. Let us abuse notation by identifying each fiber of q with the ∞-category $\mathcal{M}$, so that (by virtue of $(*)$) we may assume that $\overline{B}$ and $\overline{C}$ have images $B, C \in \mathcal{M}$. Note that $q'(Y)$ determines a morphism $v : [k'] \to [k]$ in $\mathbf{\Delta}$. Let $j = k - v(k')$. Unwinding the definitions, we see that $\mathrm{Map}_{\mathcal{N}_{g_0/}}(\overline{B}, \overline{C})$ can be identified with the total homotopy fiber of a diagram

$$
\begin{array}{ccc}
\mathrm{Map}_{\mathcal{M}}(R^{\otimes j} \otimes B, C) & \longrightarrow & \mathrm{Map}_{\mathcal{M}}(R^{\otimes j} \otimes A, C) \\
\downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle \theta_1} \\
\mathrm{Map}_{\mathcal{M}}(B, C) & \longrightarrow & \mathrm{Map}_{\mathcal{M}}(A, C)
\end{array}
$$

over a point having image $\psi \in \mathrm{Map}_{\mathcal{M}}(B, C)$. Our assumption that ϕ is cocentric relative to ψ guarantees that the centralizers $\mathfrak{Z}(\psi)$ and $\mathfrak{Z}(\psi \circ \phi)$ exist. Note that the homotopy fiber of θ_0 over ψ can be identified with the mapping space $\mathrm{Map}_{\mathcal{C}_{1/}}(R^{\otimes j}, \mathfrak{Z}(\psi))$, and the homotopy fiber of θ_1 over $\psi \circ \phi$ can be identified with the mapping space $\mathrm{Map}_{\mathcal{C}_{1/}}(R^{\otimes j}, \mathfrak{Z}(\psi \circ \phi))$. To complete the construction of $F_{\beta+1}$, it suffices to show that the canonical map $\mathfrak{Z}(\psi) \to \mathfrak{Z}(\psi \circ \phi)$ is an equivalence in $\mathcal{C}$; this follows from our assumption that ϕ is cocentric relative to ψ.

Since $\mathcal{E}_\alpha = N(\mathbf{\Delta})^{op} \times \Delta^n$, the morphism $F = F_\alpha$ determines a map $\overline{U} : \Delta^n \to \mathrm{Fun}_{N(\mathbf{\Delta})^{op}}(N(\mathbf{\Delta})^{op}, \mathcal{N})$ extending $\overline{U}_0$. To complete the proof, it suffices to show that $\overline{U}$ factors through the full subcategory $^{\Delta}\mathrm{LMod}_R(\mathcal{M}) \subseteq \mathrm{Fun}_{N(\mathbf{\Delta})^{op}}(N(\mathbf{\Delta})^{op}, \mathcal{N})$. This is automatic if $n > 1$. When $n = 1$, it follows from the fact that our construction satisfies condition $(*)$. $\qquad\square$

8.5.4   Étale Morphisms of $\mathbb{E}_k$-Rings

Let $2 \le k \le \infty$ and let A be an $\mathbb{E}_k$-ring. Theorem 8.5.0.6 asserts that every étale map of commutative rings $\pi_0 A \to B_0$ can be lifted (in an essentially unique way) to an étale map between $\mathbb{E}_k$-rings $\phi : A \to B$. Our goal in this section is to prove Theorem 8.5.0.6. In fact, we will prove a slightly stronger result, which characterizes the $\mathbb{E}_k$-ring B by a universal property. To state this result, we need to introduce a bit of terminology.

Notation 8.5.4.1. Let $2 \leq k \leq \infty$, let R be an $\mathbb{E}_{k+1}$-ring, and let A be an $\mathbb{E}_k$-algebra over R¿ We let $(\mathrm{Alg}_R^{(k)})_{A/}^{\mathrm{ét}}$ denote the full subcategory of $(\mathrm{Alg}_R^{(k)})_{A/}$ spanned by the étale morphisms $\phi : A \to B$ of $\mathbb{E}_k$-algebras over R. If we are given a morphism of $\mathbb{E}_k$-algebras $\psi : A \to C$, we let $(\mathrm{Alg}_R^{(k)})_{A//C}^{\mathrm{ét}}$ denote the full subcategory of $(\mathrm{Alg}_R^{(k)})_{A//C}$ spanned by those diagrams

$$
\begin{array}{ccc}
 & B & \\
 \phi \nearrow & & \searrow \\
 A & \xrightarrow{\quad \psi \quad} & C
\end{array}
$$
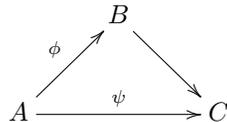
where ϕ is étale.

Theorem 8.5.4.2. *Fix* $2 \leq k \leq \infty$, *let* R *be an* $\mathbb{E}_{k+1}$-*ring, and let* $\psi : A \to C$ *be a morphism of* $\mathbb{E}_k$-*algebras over* R. *Then the construction* $B \mapsto \pi_0 B$ *induces an equivalence of* ∞-*categories*

$$
(\mathrm{Alg}_R^{(k)})_{A//C}^{\mathrm{ét}} \to \mathrm{N}(\mathrm{Ring}_{\pi_0 A/\ /\pi_0 C}^{\mathrm{ét}}).
$$

In the special case $C = 0$ of Theorem 8.5.4.2, we recover Theorem 8.5.0.6; for the reader's convenience we recall the statement:

Corollary 8.5.4.3. *Let* $2 \leq k \leq \infty$, *let* R *be an* $\mathbb{E}_{k+1}$-*ring, let* A *be an* $\mathbb{E}_k$-*algebra over* R. *Then the forgetful functor* $B \mapsto \pi_0 B$ *induces an equivalence from* $(\mathrm{Alg}_R^{(k)})_{A/}^{\mathrm{ét}}$ *to the nerve of the ordinary category of étale* $\pi_0 A$-*algebras.*

Theorem 8.5.4.2 has some other pleasant consequences.

Corollary 8.5.4.4. *Let* $k \geq 2$, *let* R *be an* $\mathbb{E}_{k+1}$-*ring, and let* A *be an* $\mathbb{E}_k$-*algebra over* R. *If* $\phi : A \to B$ *is an étale morphism in* $\mathrm{Alg}_R^{(k)}$, *then* ϕ *exhibits* B *as a compact object of* $(\mathrm{Alg}_R^{(k)})_{A/}$.

Corollary 8.5.4.5. *Let* $2 \leq k \leq \infty$, *let* R *be an* $\mathbb{E}_{k+1}$-*ring, and let* $f : A \to B$ *be an étale morphism between connective* $\mathbb{E}_k$-*algebras over* R. *Then the relative cotangent complex* $L_{B/A} \in \mathrm{Sp}((\mathrm{Alg}_R^{(k)})_{/B}) \simeq \mathrm{Mod}_B^{\mathbb{E}_k}(\mathrm{LMod}_R)$ *vanishes.*

Proof. Let $\mathcal{C} = \mathrm{Mod}_B^{\mathbb{E}_k}(\mathrm{LMod}_R)$. Fix an object $M \in \mathcal{C}$, and let $C = B \oplus M \in \mathrm{Alg}_R^{(k)}$ denote the corresponding square-zero extension. We wish to prove that $\mathrm{Map}_{\mathcal{C}}(L_{B/A}, M)$ is contractible. For this, it suffices to show that composition with f induces a homotopy equivalence $\mathrm{Map}_{(\mathrm{Alg}_R^{(k)})_{/B}}(B, C) \to \mathrm{Map}_{(\mathrm{Alg}_R^{(k)})_{/B}}(A, C)$. Using Theorem 8.5.4.2, we are reduced to proving that the map

$$
\mathrm{Hom}_{\mathrm{Ring}_{/\pi_0 B}}(\pi_0 B, \pi_0 B \oplus \pi_0 M) \to \mathrm{Hom}_{\mathrm{Ring}_{/\pi_0 B}}(\pi_0 A, \pi_0 B \oplus \pi_0 M)
$$

is bijective, which follows from our assumption that $\pi_0 B$ is an étale $\pi_0 A$-algebra (Remark 8.5.0.3). $\square$

Corollary 8.5.4.6. *Let* $2 \leq k \leq \infty$, *let* R *be an* $\mathbb{E}_{k+1}$-*ring, let* $\phi : A \to B$ *be an étale map of* $\mathbb{E}_k$-*algebras over* R, *and let* $C \in \mathrm{Alg}_R^{(k)}$ *be arbitrary. Then the canonical map*

$$
\mathrm{Map}_{(\mathrm{Alg}_R^{(k)})_{A/}}(B, C) \to \mathrm{Hom}_{\mathrm{Ring}_{\pi_0 A/}}(\pi_0 B, \pi_0 C)
$$

is a homotopy equivalence. In particular, $\mathrm{Map}_{(\mathrm{Alg}_R^{(k)})_{A/}}(B, C)$ *is homotopy equivalent to a discrete space.*

Remark 8.5.4.7. Let A be an $\mathbb{E}_k$-ring for $2 \leq k \leq \infty$, and suppose we are given a map $\phi_0 : \pi_0 A \to B_0$ in the category of ordinary commutative rings. One can then study the problem of *realizing* ϕ_0 be a map of $\mathbb{E}_k$-rings: that is, finding a map of $\mathbb{E}_k$-rings $\phi : A \to B$ such that the induced map $\pi_0 A \to \pi_0 B$ can be identified with ϕ_0. In general, there may exist many choices for B. There are (at least) two different ways to narrow our selection:

(*i*) If ϕ_0 is a flat map, then we can demand that B be *flat* over A. In this case, the homotopy groups of B are determined by the homotopy groups of A. Consequently, we have good understanding of mapping spaces $\mathrm{Map}_{(\mathrm{Alg}^{(k)})_{A/}}(C, B)$ with *codomain B*, at least in the case where C is well-understood (for example, if C is free).

(*ii*) We can demand that the canonical map

$$\mathrm{Map}_{\mathrm{Alg}_{A/}^{(k)}}(B, C) \to \mathrm{Hom}_{(\mathrm{Ring})_{\pi_0 A/}}(B_0, \pi_0 C)$$

be a homotopy equivalence for every $C \in \mathrm{Alg}_{A/}^{(k)}$. In this case, we have a good understanding of the mapping spaces $\mathrm{Map}_{\mathrm{Alg}_{A/}^{(k)}}(B, C)$ with *domain B*.

It is clear that property (*ii*) characterized B up to equivalence. If ϕ_0 is étale, then Theorem 8.5.4.2 asserts that (*i*) $\Rightarrow$ (*ii*). Moreover, Theorem 8.5.4.2 implies the existence of an A-algebra B satisfying (*i*). We therefore have an example satisfying both (*i*) and (*ii*); since property (*ii*) characterizes B up to equivalence, we conclude that (*i*) $\Rightarrow$ (*ii*). The equivalence of (*i*) and (*ii*) makes the theory of étale morphisms between $\mathbb{E}_k$-rings extremely well-behaved.

We now turn to the proof of Theorem 8.5.4.2. According to Corollary 8.5.3.13, the conclusion of Theorem 8.5.4.2 is valid in the case $k = 1$ provided that we assume that A and C are quasi-commutative. We will prove Theorem 8.5.4.2 by reducing to the case $k = 1$, using Theorem 5.1.2.2. Note that if R is an $\mathbb{E}_{k+1}$-ring, then LMod_R is an $\mathbb{E}_k$-monoidal ∞-category, so that $\mathrm{Alg}_R^{(1)}$ inherits an $\mathbb{E}_{k-1}$-monoidal structure. In particular, if $k \geq 2$, then $\mathrm{Alg}_R^{(1)}$ inherits a monoidal structure. We begin by showing that this monoidal structure is compatible with the notion of étale morphism introduced in Definition 8.5.0.4.

Lemma 8.5.4.8. *Let $k \geq 2$, let R be a connective $\mathbb{E}_{k+1}$-ring, and let $f : A \to A'$ be an étale morphism of quasi-commutative $\mathbb{E}_1$-algebras over R. Let B be another $\mathbb{E}_1$-algebra over R. Assume that B and $A \otimes_R B$ are quasi-commutative. Then the induced map $f' : A \otimes_R B \to A' \otimes_R B$ is étale (in particular, $A' \otimes_R B$ is quasi-commutative).*

Proof. We have an equivalence $A' \otimes_R B \simeq A' \otimes_A (A \otimes_R B)$. Since A' is flat over A, we obtain isomorphisms

$$\theta_n : \pi_n(A' \otimes_R B) \simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 A', \pi_n(A \otimes_R B)).$$

Let $\phi : \pi_0 A' \to \pi_0(A' \otimes_R B)$ and $\psi : \pi_0(A \otimes_R B) \to \pi_0(A' \otimes_R B)$ denote the canonical maps.

We first claim that $A' \otimes_R B$ is quasi-commutative. Let X be the collection of all elements of $\pi_0(A' \otimes_R B)$ which are central in $\pi_*(A' \otimes_R B)$. Then X is an additive subgroup of $\pi_0(A' \otimes_R B)$; to prove that $X = \pi_0(A' \otimes_R B)$ it suffices to show that it contains $\phi(a')\psi(b)$, where $a' \in \pi_0 A'$ and $b \in \pi_0(A \otimes_R B)$. This follows immediately by inspecting the isomorphisms θ_n, since A' and $A \otimes_R B$ are quasi-commutative by assumption.

Comparing the isomorphisms θ_n and θ_0, we deduce that the canonical map

$$\mathrm{Tor}_0^{\pi_0(A \otimes_R B)}(\pi_0(A' \otimes_R B), \pi_n(A \otimes_R B)) \to \pi_n(A' \otimes_R B)$$

is an isomorphism. Thus $A' \otimes_R B$ is flat over $A \otimes_R B$. The isomorphism θ_0 shows that $\pi_0(A \otimes_R B) \to \pi_0(A' \otimes_R B)$ is an étale homomorphism of commutative rings, so that f' is étale as desired. $\square$

Proof of Theorem 8.5.4.2. Since R is an $\mathbb{E}_{k+1}$-algebra, the ∞-category $\mathrm{Alg}_R^{(1)}$ is $\mathbb{E}_{k-1}$-monoidal. In what follows, we will abuse notation by identifying A and C with the corresponding $\mathbb{E}_{k-1}$-algebra objects of $\mathrm{Alg}_R^{(1)}$.

Let B be an $\mathbb{E}_1$-algebra over R. We will say that B is *good* if, for every element $x \in \pi_0 B$, left and right multiplication by x induce homotopic maps from B to itself (in the ∞-category LMod_R). We make the following elementary observations:

(*i*) If B admits the structure of an $\mathbb{E}_2$-algebra over R, then B is good (when regarded as an $\mathbb{E}_1$-algebra).

(ii) If B is good, then B is quasi-commutative.

(iii) The collection of good $\mathbb{E}_1$-algebras over R is closed under tensor products. (in particular, R itself is good).

Theorem 2.2.2.4 implies that $(\mathrm{Alg}_R^{(1)})_{/C}$ inherits an $\mathbb{E}_{k-1}$-monoidal structure. Let $\mathcal{C}$ denote the full subcategory of $(\mathrm{Alg}_R^{(1)})_{/C}$ spanned those maps $B \to C$ where B is good. Assertion (iii) guarantees $\mathcal{C}'$ is closed under tensor products, and therefore inherits an $\mathbb{E}_{k-1}$-monoidal structure.

Let $\mathcal{D}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, (\mathrm{Alg}_R^{(1)})_{/C})$ spanned by those maps $f : B \to B'$ in $(\mathrm{Alg}_R^{(1)})_{/C}$ such that B is good and f is étale. The functor ∞-category $\mathrm{Fun}(\Delta^1, (\mathrm{Alg}_R^{(1)})_{/C})$ is equipped with an $\mathbb{E}_{k-1}$-monoidal structure (given by pointwise tensor product; see Remark 2.1.3.4). We claim that $\mathcal{D}$ is closed under tensor products in $\mathrm{Fun}(\Delta^1, (\mathrm{Alg}_R^{(1)})_{/C})$ and therefore inherits an $\mathbb{E}_{k-1}$-monoidal structure. Since the unit object $\mathrm{id}_R : R \to R$ obviously belongs to $\mathcal{D}$, we are reduced to proving that if a pair of morphisms $f_0 : B_0 \to B_0'$ and $f_1 : B_1 \to B_1'$ belong to $\mathcal{D}$, then $(f_0 \otimes f_1) : B_0 \otimes_R B_1 \to B_0' \otimes_R B_1'$ belongs to $\mathcal{D}$. Since $B_0 \otimes_R B_1$ is good by (iii), it will suffice to show that $f \otimes f'$ is étale. Using (ii) and Lemma 8.5.4.8, we deduce that $f_0 \otimes \mathrm{id}_{B_1}$ is étale ; in particular, $B_0' \otimes B_1$ is quasi-commutative. Applying Lemma 8.5.4.8 again, we conclude that $\mathrm{id}_{B_0'} \otimes f_1$ is étale. Using Remark 8.5.1.6, we conclude that $f \otimes f'$ is étale, as desired.

Let $\mathrm{Fun}^{\text{ét}}(\Delta^1, (\mathrm{Alg}_R^{(k)})_{/C})$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, (\mathrm{Alg}_R^{(k)})_{/C})$ spanned by those morphisms $B \to B'$ which are étale. Using observation (i) and Proposition 5.1.2.2, we obtain equivalences of ∞-categories

$$(\mathrm{Alg}_R^{(k)})_{/C} \simeq \mathrm{Alg}_{/\mathbb{E}_{k-1}}(\mathcal{C})$$

$$\mathrm{Fun}^{\text{ét}}(\Delta^1, (\mathrm{Alg}_R^{(k)})_{/C}) \simeq \mathrm{Alg}_{/\mathbb{E}_{k-1}}(\mathcal{D}).$$

Let Ring' denote the full subcategory of $\mathrm{Fun}([1], \mathrm{Ring}_{/\pi_0 C})$ spanned by the etale maps of commutative rings $B \to B'$ over $\pi_0 C$. The ∞-categories $\mathrm{N}(\mathrm{Ring}_{/\pi_0 C})$ and $\mathrm{N}(\mathrm{Ring}')$ are also endowed with $\mathbb{E}_{k-1}$-monoidal structures, arising from the coCartesian symmetric monoidal structure given by tensor products of commutative rings. Since the ∞-operad $\mathbb{E}_{k-1}$ is unital, Proposition 2.4.3.9 provides equivalences

$$\mathrm{Alg}_{/\mathbb{E}_{k-1}}(\mathrm{N}(\mathrm{Ring}_{/\pi_0 B})) \to \mathrm{N}(\mathrm{Ring}_{/\pi_0 C})$$

$$\mathrm{Alg}_{/\mathbb{E}_{k-1}}(\mathrm{N}(\mathrm{Ring}')) \to \mathrm{N}(\mathrm{Ring}').$$

The construction $B \mapsto \pi_0 B$ determines a lax $\mathbb{E}_{k-1}$-monoidal functor $\mathcal{C}^{\otimes} \to \mathrm{N}(\mathrm{Ring}_{/\pi_0 C})^{\otimes}$. Let $\overline{\mathcal{C}}^{\otimes}$ denote the fiber product $\mathcal{C}^{\otimes} \times_{\mathrm{N}(\mathrm{Ring}_{/\pi_0 C})^{\otimes}} \mathrm{N}(\mathrm{Ring}')^{\otimes}$ and $\overline{\mathcal{C}} = \mathcal{C} \times_{\mathrm{N}(\mathrm{Ring}_{/\pi_0 C})} \mathrm{N}(\mathrm{Ring}')$ its underlying ∞-category. We can identify the objects of $\overline{\mathcal{C}}$ with pairs $(B, f : \pi_0 B \to T)$, where $B \in (\mathrm{Alg}_R^{(1)})_{/C}$ is good and f is an étale morphism in $\mathrm{Ring}_{/\pi_0 C}$.

The forgetful functor $\overline{\mathcal{C}}^{\otimes} \to \mathbb{E}_{k-1}^{\otimes}$ determines an $\mathbb{E}_{k-1}$-monoidal structure on $\overline{\mathcal{C}}$, where the tensor product of objects is given by the formula

$$(B_0, \pi_0 B_0 \to T_0) \otimes (B_1, \pi_0 B_1 \to T_1) \simeq (B_0 \otimes B_1, \pi_0(B_0 \otimes B_1) \to T_0 \otimes_{\pi_0 B_0} \pi_0(B_0 \otimes B_1) \otimes_{\pi_0 B_1} T_1).$$

The construction $(f : B \to B') \mapsto (B, (\pi_0 f) : \pi_0 B \to \pi_0 B')$ determines an $\mathbb{E}_{k-1}$-monoidal functor from $\theta : \mathcal{D} \to \overline{\mathcal{C}}$. We claim that θ is an equivalence of $\mathbb{E}_{k-1}$-monoidal ∞-categories. Using Remark 2.1.3.8, we are reduced to proving that θ induces an equivalence on underlying ∞-categories. We have a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\theta} & \overline{\mathcal{C}} \\ & {}_{p}\searrow \quad \swarrow_{q} & \\ & \mathcal{C} & \end{array}$$

where p and q are coCartesian fibrations and the functor θ carries p-coCartesian morphisms to q-coCartesian morphisms. It therefore suffices to show that θ induces an equivalence of ∞-categories after passing to the fiber over any object of $\mathcal{C}$ (Corollary T.2.4.4.4), which follows from Corollary 8.5.3.13.

Passing to $\mathbb{E}_{k-1}$-algebra objects, we obtain a chain of equivalences

$$
\begin{aligned}
\mathrm{Fun}^{\text{ét}}(\Delta^1, (\mathrm{Alg}_R^{(k)})_{/C}) \quad &\simeq \quad \mathrm{Alg}_{/\mathbb{E}_{k-1}}(\mathcal{D}) \\
&\simeq \quad \mathrm{Alg}_{/\mathbb{E}_{k-1}}(\overline{\mathcal{C}}) \\
&\simeq \quad \mathrm{Alg}_{/\mathbb{E}_{k-1}}(\mathcal{C}) \times_{\mathrm{Alg}_{/\mathbb{E}_{k-1}}(\mathrm{N}(\mathrm{Ring}_{/\pi_0 C}))} \mathrm{Alg}_{/\mathbb{E}_{k-1}}(\mathrm{N}(\mathrm{Ring}')) \\
&\simeq \quad (\mathrm{Alg}_R^{(k)})_{/C} \times_{\mathrm{N}(\mathrm{Ring}_{/\pi_0 C})} \mathrm{N}(\mathrm{Ring}').
\end{aligned}
$$

Taking homotopy fibers over the object $A \in (\mathrm{Alg}_R^{(k)})_{/C}$, we conclude that the forgetful functor

$$
(\mathrm{Alg}_R^{(k)})_{A//C}^{\text{ét}} \to \mathrm{N}(\mathrm{Ring}_{\pi_0 A/\ /\pi_0 C}^{\text{ét}})
$$

is an equivalence of ∞-categories. $\square$

Appendix A

Constructible Sheaves and Exit Paths

Let X be a topological space and let $\mathcal{F}$ be a locally constant sheaf of sets on X. For every point $x \in X$, we let $\mathcal{F}_x$ denote the stalk of the sheaf $\mathcal{F}$ at the point x. The construction $(x \in X) \mapsto \mathcal{F}_x$ determines a functor from the fundamental groupoid $\pi_{\leq 1} X$ of X to the category of sets: every path $p : [0,1] \to X$ from $x = p(0)$ to $y = p(1)$ determines a bijection of sets $\mathcal{F}_x \to \mathcal{F}_y$, depending on only the homotopy class of the path p. If the topological space X is sufficiently nice, then the converse holds: every functor from the fundamental groupoid of X into the category of sets arises via this construction, for some locally constant sheaf of sets $\mathcal{F}$. In fact, the category of functors $\mathrm{Fun}(\pi_{\leq 1} X, \mathrm{Set})$ is *equivalent* to the category of locally constant sheaves on X.

Our goal in this appendix is to describe some generalizations of the equivalence of categories sketched above. The situation we consider will be more general in two respects:

(a) Rather than working with sheaves of sets, we will consider arbitrary $\mathcal{S}$-valued sheaves on X.

(b) We will consider not only locally constant sheaves, but sheaves that are locally constant along the strata of some stratification of X.

We begin in §A.1 by introducing the notion of a *locally constant* sheaf on an ∞-topos $\mathcal{X}$. This is a poor notion in general, but behaves well if we make a technical assumption on $\mathcal{X}$ (namely, that $\mathcal{X}$ is *locally of constant shape*: see Definition A.1.5). Under this assumption, we prove an abstract version of the equivalence described above: namely, the ∞-category of locally constant sheaves on $\mathcal{X}$ can be identified with $\mathrm{Fun}(K, \mathcal{S}) \simeq \mathcal{S}_{/K}$ (see Theorem A.1.15), where K is a Kan complex called the *shape* of $\mathcal{X}$.

For the abstract result cited above to be useful in practice, we need an explicit description of the shape of an ∞-topos $\mathcal{X}$. Suppose, for example, that $\mathcal{X}$ is the ∞-category $\mathrm{Shv}(X)$ of $\mathcal{S}$-valued sheaves on a topological space X. In §A.2, we show that the shape of $\mathcal{X}$ is a homotopy invariant of X. In good cases, we can identify the shape of $\mathcal{X}$ with the singular complex $\mathrm{Sing}(X)$ of X. In §A.4, we will establish such an identification for a large class of topological spaces X (including, for example, all metric absolute neighborhood retracts); see Theorem A.4.19. The proof relies on a generalization of the Seifert-van Kampen theorem, which we describe in §A.3.

We can summarize the above discussion as follows: if X is a sufficiently nice topological space, then there is a fully faithful embedding $\Psi_X : \mathrm{Fun}(\mathrm{Sing}(X), \mathcal{S}) \to \mathrm{Shv}(X)$, whose essential image is the ∞-category of locally constant sheaves on X. The remainder of this appendix is devoted to explaining how to enlarge the ∞-category $\mathrm{Fun}(\mathrm{Sing}(X), \mathcal{S})$ to obtain a description of sheaves on X which are not assumed to be locally constant. Suppose that X is equipped with a stratification: that is, a partition of X into subsets X_α indexed by a partially ordered set A. In §A.5, we will study the notion of an A-*constructible* sheaf on X: that is, a sheaf on X whose restriction to each stratum X_α is locally constant. In §A.6, we will define a simplicial subset $\mathrm{Sing}^A(X) \subseteq \mathrm{Sing}(X)$. Under some mild assumptions, we will show that $\mathrm{Sing}^A(X)$ is an ∞-category (Theorem A.6.4), which we call the ∞-*category of exit paths in* X. Our main goal is to show that Ψ_X

extends to a fully faithful embedding $\mathrm{Fun}(\mathrm{Sing}^A(X), \mathcal{S}) \to \mathrm{Shv}(X)$, whose essential image is the ∞-category of A-constructible sheaves on X. We will prove a result of this type in §A.9 (Theorem A.9.3). The proof will require a further generalization of the Seifert-van Kampen theorem (which we explain in §A.7) together with a general mechanism for studying extensions of one ∞-category by another (which we explain in §A.8).

Most of the results in this appendix are not explicitly used in the body of the book (an exception is the version of the Seifert-van Kampen theorem given in §A.3, which we use several times in Chapter 5). However, the description of constructible sheaves in terms of exit path ∞-categories is indirectly relevant to our study of factorizable (co)sheaves in §5.3, and should prove useful in studying applications of the theory developed there.

A.1 Locally Constant Sheaves

Let X be a topological space. A sheaf of sets $\mathcal{F}$ on X is said to be *constant* if there exists a set A and a map $\eta : A \to \mathcal{F}(X)$ such that, for every point $x \in X$, the composite map $A \to \mathcal{F}(X) \to \mathcal{F}_x$ is a bijection from A to the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x. More generally, we say that a sheaf of sets $\mathcal{F}$ is *locally constant* if every point $x \in X$ has an open neighborhood U such that the restriction $\mathcal{F}|U$ is a constant sheaf on U. The category of locally constant sheaves of sets on X is equivalent to the category of covering spaces of X. If X is path connected and semi-locally simply connected, then the theory of covering spaces guarantees that this category is equivalent to the category of sets with an action of the fundamental group $\pi_1(X, x)$ (where x is an arbitrarily chosen point of X).

Our goal in this section is to obtain an ∞-categorical analogue of the above picture. More precisely, we will replace the topological space X by an ∞-topos $\mathcal{X}$. Our goal is to introduce a full subcategory of $\mathcal{X}$ consisting of "locally constant" objects (see Definition A.1.12). We will further show that if $\mathcal{X}$ is sufficiently well-behaved, then this full subcategory is itself an ∞-topos: more precisely, it is equivalent to an ∞-category of the form $\mathcal{S}_{/K}$, for some Kan complex K. In §A.4, we will show that if $\mathcal{X}$ is the ∞-category $\mathrm{Shv}(X)$ of sheaves on a well-behaved topological space X, then we can take K to be the Kan complex $\mathrm{Sing}(X)$.

The first step is to formulate a condition on an ∞-topos which is a counterpart to the hypothesis of semi-local simple connectivity in the usual theory of covering spaces.

Definition A.1.1. Let $\mathcal{X}$ be an ∞-topos, let $\pi_* : \mathcal{X} \to \mathcal{S}$ be a functor corepresented by the final object of $\mathcal{X}$, and let π^* be a right adjoint to π_*. We will say that $\mathcal{X}$ has *constant shape* if the composition $\pi_*\pi^* : \mathcal{S} \to \mathcal{S}$ is corepresentable.

Remark A.1.2. Recall that the *shape* of an ∞-topos $\mathcal{X}$ is the functor $\pi_*\pi^* : \mathcal{S} \to \mathcal{S}$, which can be regarded as a pro-object of the ∞-category $\mathcal{S}$ (see §T.7.1.6). The ∞-topos $\mathcal{X}$ has constant shape if this pro-object can be taken to be constant.

Remark A.1.3. According to Proposition T.5.5.2.7, an ∞-topos $\mathcal{X}$ has constant shape if and only if the functor $\pi_*\pi^*$ preserves small limits.

Remark A.1.4. Let X be a paracompact topological space, and let $\pi_* : \mathrm{Shv}(X) \to \mathrm{Shv}(*) \simeq \mathcal{S}$ be the global sections functor. It follows from the results of §T.7.1 that we can identify the composition $\pi_*\pi^*$ with the functor $K \mapsto \mathrm{Map}_{\mathrm{Top}}(X, |K|)$. Consequently, the ∞-topos $\mathrm{Shv}(X)$ has constant shape if and only if there exists a simplicial set K_0 and a continuous map $f : X \to |K_0|$ such that, for every Kan complex K, composition with f induces a homotopy equivalence $\mathrm{Map}_{\mathrm{Set}_\Delta}(K_0, K) \simeq \mathrm{Map}_{\mathrm{Top}}(|K_0|, |K|) \to \mathrm{Map}_{\mathrm{Top}}(X, |K|)$. This is guaranteed, for example, if f is a homotopy equivalence: in other words, if X is a paracompact topological space with the homotopy type of a CW complex, then X has constant shape.

Definition A.1.5. Let $\mathcal{X}$ be an ∞-topos. We will say that an object $U \in \mathcal{X}$ has *constant shape* if the ∞-topos $\mathcal{X}_{/U}$ has constant shape. We will say that $\mathcal{X}$ is *locally of constant shape* if every object $U \in \mathcal{X}$ has constant shape.

The following result guarantees that Definition A.1.5 is reasonable:

Proposition A.1.6. *Let $\mathfrak{X}$ be an ∞-topos, and let $\mathfrak{X}'$ be the full subcategory of $\mathfrak{X}$ spanned by those objects which have constant shape. Then $\mathfrak{X}'$ is stable under small colimits in $\mathfrak{X}$.*

Proof. For each $U \in \mathfrak{X}$, let $\chi_U : \mathfrak{X} \to \mathcal{S}$ be the functor corepresented by U, and let $\pi^* : \mathcal{S} \to \mathfrak{X}$ be a geometric morphism. Then U has constant shape if and only if the functor $\chi_U \circ \pi^*$ is corepresentable: in other words, if and only if $\chi_U \circ \pi^*$ preserves small limits (Remark A.1.3). Suppose that U is the colimit of a diagram $\{U_\alpha\}$. Then χ_U is the limit of the induced diagram of functors $\{\chi_{U_\alpha}\}$ (Proposition T.5.1.3.2), so that $\chi_U \circ \pi^*$ is a limit of the diagram of functors $\{\chi_{U_\alpha} \circ \pi^*\}$. If each U_α has constant shape, then each of the functors $\chi_{U_\alpha} \circ \pi^*$ preserves small limits, so that $\chi_U \circ \pi^*$ preserves small limits (Lemma T.5.5.2.3). $\qquad\square$

Corollary A.1.7. *Let $\mathfrak{X}$ be an ∞-topos. Suppose that there exists a collection of objects $U_\alpha \in \mathfrak{X}$ such that the projection $U = \coprod_\alpha U_\alpha \to \mathbf{1}$ is an effective epimorphism, where $\mathbf{1}$ denotes the final object of $\mathfrak{X}$. If each of the ∞-topoi $\mathfrak{X}_{/U_\alpha}$ is locally of constant shape, then $\mathfrak{X}$ is locally of constant shape.*

Proof. Let $V \in \mathfrak{X}$; we wish to show that V has constant shape. Let $V_0 = U \times V$, and let $V_\bullet$ be the Čech nerve of the effective epimorphism $V_0 \to V$. Since $\mathfrak{X}$ is an ∞-topos, V is equivalent to the geometric realization of the simplicial object $V_\bullet$. In view of Proposition A.1.6, it will suffice to show that each V_n has constant shape. We note that V_n is a coproduct of objects of the form $U_{\alpha_0} \times \ldots \times U_{\alpha_n} \times V$. Then $\mathfrak{X}_{/V_n}$ admits an étale geometric morphism to the ∞-topos $\mathfrak{X}_{/U_{\alpha_0}}$, which is locally of constant shape by assumption. It follows that $\mathfrak{X}_{/V_n}$ is of constant shape. $\qquad\square$

Proposition A.1.8. *Let $\mathfrak{X}$ be an ∞-topos and let $\pi^* : \mathcal{S} \to \mathfrak{X}$ be a geometric morphism. The following conditions are equivalent:*

(1) *The ∞-topos $\mathfrak{X}$ is locally of constant shape.*

(2) *The functor π^* admits a left adjoint $\pi_!$.*

Proof. According to Corollary T.5.5.2.9, condition (2) is equivalent to the requirement that π^* preserves small limits. In view of Proposition T.5.1.3.2, this is equivalent to the assertion that for each $U \in \mathfrak{X}$, the composition $\chi_U \circ \pi^* : \mathcal{S} \to \mathcal{S}$ preserves limits, where $\chi_U : \mathfrak{X} \to \mathcal{S}$ is the functor corepresented by U. $\qquad\square$

Let $\mathfrak{X}$ be an ∞-topos which is locally of constant shape, and let $\pi_!$ and π^* be the adjoint functors appearing in Proposition A.1.8. Let $X \to Y$ be a morphism in $\mathcal{S}$ and let $Z \to \pi^* Y$ be a morphism in $\mathfrak{X}$. Then we have a commutative diagram

$$
\begin{array}{ccc}
\pi_!(\pi^* X \times_{\pi^* Y} Z) & \longrightarrow & \pi_! Z \\
\downarrow & & \downarrow \\
\pi_! \pi^* X & \longrightarrow & \pi_! \pi^* Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y,
\end{array}
$$

and the outer square determines a canonical map $\pi_!(\pi^* X \times_{\pi^* Y} Z) \to X \times_Y \pi_! Z$.

Proposition A.1.9. *Let $\mathfrak{X}$ be an ∞-topos which is locally of constant shape, let $\pi^* : \mathcal{S} \to \mathfrak{X}$ be a geometric morphism and $\pi_!$ a left adjoint to π^* (so that $\mathfrak{X}$ is locally of constant shape). For every morphism $\alpha : X \to Y$ in $\mathcal{S}$ and every morphism $\beta : Z \to \pi^* Y$ in $\mathfrak{X}$, the associated push-pull morphism*

$$
\pi_!(\pi^* X \times_{\pi^* Y} Z) \to X \times_Y \pi_! Z
$$

is an equivalence.

Proof. Let us first regard the morphism α as fixed, and consider the full subcategory $\mathcal{Y} \subseteq \mathcal{X}_{/\pi^* Y}$ spanned by those objects Z for which the conclusion holds. Since both $\pi_!(\pi^* X \times_{\pi^* Y} Z)$ and $X \times_Y \pi_! Z$ are colimit-preserving functors of Z, the full subcategory $\mathcal{Y}$ is stable under colimits in $\mathcal{X}_{/\pi^* Y}$. Regard Y as a Kan complex, and let $\mathcal{C}$ be the category of simplices of Y, so that we can identify Y with the colimit $\varinjlim_{C \in \mathcal{C}}(\Delta^0)$ of the constant diagram $\mathcal{C} \to \mathcal{S}$ taking the value Δ^0. For every $Z \in \mathcal{X}_{/\pi^* Y}$, we have a canonical equivalence $Z \simeq \varinjlim_{C \in \mathcal{C}}(Z \times_{\pi^* Y} \pi^* \Delta^0)$. We may therefore replace Z by the fiber product $Z \times_{\pi^* Y} \pi^* \Delta^0$, and thereby reduce to the case where β factors through the map $\pi^* \Delta^0 \to \pi^* Y$ determined by a point of Y. Replacing Y by Δ^0 and X by $X \times_Y \Delta^0$, we can reduce to the case where $Y = \Delta^0$. In this case, we must show that the canonical map $\pi_!(\pi^* X \times Z) \to X \times \pi_! Z$ is an equivalence. Let us now regard Z as fixed and consider the full subcategory $\mathcal{Z} \subseteq \mathcal{S}$ spanned by those objects for which the conclusion holds. Since the functors $\pi_!(\pi^* X \times Z)$ and $X \times \pi_! Z$ both preserve colimits in X, the full subcategory $\mathcal{Z} \subseteq \mathcal{S}$ is stable under small colimits. It will therefore suffice to show that $\Delta^0 \in \mathcal{S}$, which is obvious. $\qquad\square$

Let $\mathcal{X}$ be an ∞-topos which is locally of constant shape. Let $\pi_!$ and π^* denote the adjoint functors appearing in Proposition A.1.8. Let $\mathbf{1}$ be a final object of $\mathcal{X}$. We have a canonical functor

$$\mathcal{X} \simeq \mathcal{X}_{/\mathbf{1}} \xrightarrow{\pi_!} \mathcal{S}_{/\pi_! \mathbf{1}},$$

which we will denote by $\psi_!$. The functor $\psi_!$ admits a right adjoint ψ^*, which can be described informally by the formula $\psi^* X = \pi^* X \times_{\pi^* \pi_! \mathbf{1}} \mathbf{1}$ (Proposition T.5.2.5.1). We observe that ψ^* preserves small colimits, and is therefore a geometric morphism of ∞-topoi.

Remark A.1.10. The object $\pi_! \mathbf{1} \in \mathcal{S}$ can be identified with the shape of the ∞-topos $\mathcal{X}$.

Proposition A.1.11. *Let $\mathcal{X}$ be an ∞-topos which is locally of constant shape, and let $\psi^* : \mathcal{S}_{/\pi_! \mathbf{1}} \to \mathcal{X}$ be defined as above. Then ψ^* is fully faithful.*

Proof. Fix an object $X \to \pi_! \mathbf{1}$ in $\mathcal{S}_{/\pi_! \mathbf{1}}$; we wish to show that the counit map $v : \psi_! \psi^* X \to X$ is an equivalence. Unwinding the definitions, we see that v can be identified with the push-pull transformation

$$\pi_!(\mathbf{1} \times_{\pi^* \pi_! \mathbf{1}} \pi^* X) \to \pi_! \mathbf{1} \times_{\pi_! \mathbf{1}} X \simeq X,$$

which is an equivalence by virtue of Proposition A.1.9. $\qquad\square$

We now describe the essential image of the fully faithful embedding ψ^*.

Definition A.1.12. Let $\mathcal{X}$ be an ∞-topos, and let $\mathcal{F}$ be an object of $\mathcal{X}$. We will say that $\mathcal{F}$ is *constant* if it lies in the essential image of a geometric morphism $\pi^* : \mathcal{S} \to \mathcal{X}$ (the geometric morphism π^* is unique up to equivalence, by virtue of Proposition T.6.3.4.1). We will say that $\mathcal{F}$ is *locally constant* if there exists a small collection of objects $\{U_\alpha \in \mathcal{X}\}_{\alpha \in S}$ such that the following conditions are satisfied:

(*i*) The objects U_α cover $\mathcal{X}$: that is, there is an effective epimorphism $\coprod U_\alpha \to \mathbf{1}$, where $\mathbf{1}$ denotes the final object of $\mathcal{X}$.

(*ii*) For each $\alpha \in S$, the product $\mathcal{F} \times U_\alpha$ is a constant object of the ∞-topos $\mathcal{X}_{/U_\alpha}$.

Remark A.1.13. Let $f^* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. Then f^* carries constant objects of $\mathcal{X}$ to constant objects of $\mathcal{Y}$ and locally constant objects of $\mathcal{X}$ to locally constant objects of $\mathcal{Y}$.

Remark A.1.14. Let $\mathcal{F}$ be a locally constant object of $\mathrm{Shv}(X)$, where X is a topological space. Then there exists an open covering $\{U_\alpha \subseteq X\}$ such that each $\mathcal{F}|U_\alpha$ is constant. Moreover, if X is paracompact, we can assume that each U_α is an open F_σ set.

We now come to the main result of this section, which provides an ∞-categorical version of the classical theory of covering spaces.

Theorem A.1.15. *Let $\mathfrak{X}$ be an ∞-topos which is locally of constant shape, and let $\psi^* : \mathcal{S}_{/\pi_!\mathbf{1}} \to \mathfrak{X}$ be the functor of Proposition A.1.11. Then ψ^* is a fully faithful embedding, whose essential image is the full subcategory of $\mathfrak{X}$ spanned by the locally constant objects.*

Proof. Suppose first that $X \to \pi_!\mathbf{1}$ is an object of $\mathcal{S}_{/\pi_!\mathbf{1}}$; we will prove that $\psi^*(X)$ is locally constant. Choose an effective epimorphism $\coprod_{\alpha \in A} K_\alpha \to \pi_!\mathbf{1}$ in $\mathcal{S}$, where each K_α is contractible. Then we obtain an effective epimorphism $\coprod_{\alpha \in A} \psi^* K_\alpha \to \mathbf{1}$; it will therefore suffice to show that each $\psi^* X \times \psi^* K_\alpha$ is a constant object of $\mathfrak{X}_{/\psi^* K_\alpha}$. The composite functor

$$\mathcal{S}_{/\pi_!\mathbf{1}} \xrightarrow{\psi^*} \mathfrak{X} \xrightarrow{\times \psi^* K_\alpha} \mathfrak{X}_{/\psi^* K_\alpha}$$

is equivalent to a composition of geometric morphisms

$$\mathcal{S}_{/\pi_!\mathbf{1}} \to \mathcal{S}_{/K_\alpha} \simeq \mathcal{S} \to \mathfrak{X}_{/\psi^* K_\alpha}$$

and so its essential image consists of constant objects.

For the converse, suppose that $\mathcal{F} \in \mathfrak{X}$ is a locally constant object; we wish to show that $\mathcal{F}$ belongs to the essential image of ψ^*. Since $\mathcal{F}$ is locally constant, there exists a diagram $\{U_\alpha\}$ in $\mathfrak{X}$ having colimit $\mathbf{1}$, such that each product $U_\alpha \times \mathcal{F}$ is a constant object of $\mathfrak{X}_{/U_\alpha}$. We observe that $\mathcal{S}_{/\pi_!\mathbf{1}}$ can be identified with the limit of the diagram of ∞-categories $\{\mathcal{S}_{/\pi_!U_\alpha}\}$, and that $\mathfrak{X}$ can be identified with the limit of the diagram of ∞-categories $\{\mathfrak{X}_{/U_\alpha}\}$ (Theorem T.6.1.3.9). Moreover, the fully faithful embedding ψ^* is the limit of fully faithful embeddings $\psi^*_\alpha : \mathcal{S}_{/\pi_!U_\alpha} \to \mathfrak{X}_{/U_\alpha}$. Consequently, $\mathcal{F}$ belongs to the essential image of ψ^* if and only if each product $\mathcal{F} \times U_\alpha$ belongs to the essential image of ψ^*_α. We may therefore replace $\mathfrak{X}$ by $\mathfrak{X}_{/U_\alpha}$ and thereby reduce to the case where $\mathcal{F}$ is constant. In this case, $\mathcal{F}$ belongs to the essential image of *any* geometric morphism $\phi^* : \mathcal{Y} \to \mathfrak{X}$, since we have a homotopy commutative diagram of geometric morphisms

$$
\begin{array}{ccc}
 & \mathcal{Y} & \\
 \nearrow & & \searrow{\scriptstyle \phi^*} \\
\mathcal{S} & \xrightarrow{\quad \pi^* \quad} & \mathfrak{X}.
\end{array}
$$

$\square$

Corollary A.1.16. *Let $\mathfrak{X}$ be an ∞-topos which is locally of constant shape. Then the collection of locally constant objects of $\mathfrak{X}$ is stable under small colimits.*

Corollary A.1.17. *Let $\mathfrak{X}$ be an ∞-topos which is locally of constant shape. Then for every locally constant object $X \in \mathfrak{X}$, the canonical map $X \to \varprojlim \tau_{\leq n} X$ is an equivalence; in particular, X is hypercomplete.*

Proof. Let $\pi_! : \mathfrak{X} \to \mathcal{S}$ and $\psi^* : \mathcal{S}_{/\pi_!\mathbf{1}} \to \mathfrak{X}$ be as in Proposition A.1.11. According to Theorem A.1.15, we can write $X = \psi^* X_0$ for some $X_0 \in \mathcal{S}_{/\pi_!\mathbf{1}}$. Since ψ^* commutes with truncations and preserves limits (being a right adjoint), we can replace $\mathfrak{X}$ by $\mathcal{S}_{/\pi_!\mathbf{1}}$. Since the result is local on $\mathfrak{X}$, we can reduce further to the case where $\mathfrak{X} = \mathcal{S}$, in which case there is nothing to prove. $\square$

A.2 Homotopy Invariance

Let X be a topological space, and let $\mathcal{F}$ be a locally constant sheaf of sets on X. If $p : [0,1] \to X$ is a continuous path from $x = p(0)$ to $y = p(1)$, then p induces a bijection between the stalks $\mathcal{F}_x$ and $\mathcal{F}_y$ of the sheaf $\mathcal{F}$, given by transport along p. More generally, if $h : Y \times [0,1] \to X$ is any homotopy from a continuous map $h_0 : Y \to X$ to a continuous map $h_1 : Y \to X$, then h induces an isomorphism of sheaves $h_0^* \mathcal{F} \simeq h_1^* \mathcal{F}$. Our goal in this section is to generalize these statements to the case where $\mathcal{F}$ is a sheaf of spaces.

Our first step is to study locally constant sheaves on the unit interval $[0,1]$. These are characterized by the following result:

Proposition A.2.1. *Let X be the unit interval $[0,1]$, and let $\mathcal{F} \in \mathrm{Shv}(X)$. Let $\pi_* : \mathrm{Shv}(X) \to \mathrm{Shv}(*) = \mathcal{S}$ be the global sections functor, and let π^* be a left adjoint to π_*. The following conditions are equivalent:*

(i) *The sheaf $\mathcal{F}$ is locally constant.*

(ii) *The sheaf $\mathcal{F}$ is constant.*

(iii) *The canonical map $\theta : \pi^*\pi_* \mathcal{F} \to \mathcal{F}$ is an equivalence.*

Before giving the proof, we need an easy lemma.

Lemma A.2.2. *Let X be a contractible paracompact topological space, let $\pi_* : \mathrm{Shv}(X) \to \mathrm{Shv}(*) \simeq \mathcal{S}$ be the global sections functor, and let π^* be a right adjoint to π_*. Then π^* is fully faithful.*

Proof. Let K be a Kan complex (regarded as an object of $\mathcal{S}$; we wish to prove that the unit map $u : K \to \pi_*\pi^*K$ is an equivalence. The results of §T.7.1 show that $\pi_*\pi^*K$ has the homotopy type of the Kan complex of maps $\mathrm{Map}_{\mathrm{Top}}(X, |K|)$. Under this identification, the map u corresponds to the diagonal inclusion $K \to \mathrm{Sing}\,|K| \simeq \mathrm{Map}_{\mathrm{Top}}(*, |K|) \to \mathrm{Map}_{\mathrm{Top}}(X, |K|)$. Since X is contractible, this inclusion is a homotopy equivalence. $\qquad\square$

Proof of Proposition A.2.1. The implications $(iii) \Rightarrow (ii) \Rightarrow (i)$ are obvious. We prove that $(ii) \Rightarrow (iii)$. Suppose that $\mathcal{F}$ is constant; then $\mathcal{F} \simeq \pi^*K$ for some $K \in \mathcal{S}$. Then θ admits a right homotopy inverse, given by applying π^* to the unit map $u : K \to \pi_*\pi^*K$. It follows from Lemma A.2.2 that u is an equivalence, so that θ is an equivalence as well.

We now prove that $(i) \Rightarrow (ii)$. Assume that $\mathcal{F}$ is locally constant. Let $S \subseteq [0,1]$ be the set of real numbers t such that $\mathcal{F}$ is constant in some neighborhood of the interval $[0,t] \subseteq [0,1]$. Let s be the supremum of the set S (since $\mathcal{F}$ is constant in a neighborhood of 0, we must have $s > 0$). We will show that $s \in S$. It will follow that $s = 1$ (otherwise, since $\mathcal{F}$ is locally constant on $[0, s+\epsilon]$ for ϵ sufficiently small, we would have $s + \frac{\epsilon}{2} \in S$) so that $\mathcal{F}$ is locally constant on $[0,1]$, as desired.

Since $\mathcal{F}$ is locally constant, it is constant when restricted to some open neighborhood U of $s \in [0,1]$. Since s is a limit point of S, we have $S \cap U \neq \emptyset$. Consequently, we can choose some point $t \in S \cap U$, so that $\mathcal{F}$ is constant on U and on $[0,t)$. We will prove that $\mathcal{F}$ is constant on the neighborhood $V = U \cup [0,t)$ of $[0,s]$, so that $s \in S$ as desired.

Since $\mathcal{F}$ is constant on $[0,t)$, we have an equivalence $\alpha : (\mathcal{F}|[0,t)) \simeq (\pi^*K|[0,t))$ for some object $K \in \mathcal{S}$. Similarly, we have an equivalence $\beta : (\mathcal{F}|U) \simeq (\pi^*K'|U)$ for some $K' \in \mathcal{S}$. Restricting to the intersection, we get an equivalence $\gamma : (\pi^*K|U \cap [0,t)) \simeq (\pi^*K'|U \cap [0,t))$. Since the intersection $U \cap [0,t)$ is contractible, Lemma A.2.2 guarantees that γ is induced by an equivalalance $\gamma_0 : K \simeq K'$ in the ∞-category $\mathcal{S}$. Identifying K with K' via γ_0, we can reduce to the case where $K = K'$ and γ' is homotopic to the identity. For every open subset $W \subseteq [0,1]$, let $\chi_W \in \mathrm{Shv}(X)$ denote the sheaf given by the formula

$$\chi_W(W') = \begin{cases} * & \text{if } W' \subseteq W \\ \emptyset & \text{otherwise.} \end{cases}$$

We then have a commutative diagram

$$
\begin{array}{ccc}
\pi^*K \times \chi_{U \cap [0,t)} & \longrightarrow & \pi^*K \times \chi_U \\
\downarrow & & \downarrow \\
\pi^*K \times \chi_{[0,t)} & \longrightarrow & \mathcal{F}.
\end{array}
$$

This diagram induced a map $\pi^*K \times \chi_V \to \mathcal{F}$, which determines the required equivalence $\pi^*K|V \simeq \mathcal{F}|V$. $\quad\square$

Remark A.2.3. Proposition A.2.1 remains valid (with essentially the same proof) if we replace the closed unit interval $[0,1]$ by an open interval $(0,1)$ or a half-open interval $[0,1)$.

Let $h_0, h_1 : X \to Y$ be a pair of continuous maps from a topological space X to another topological space Y. If h_0 is homotopic to h_1, then there exists a continuous map $h : X \times \mathbf{R} \to Y$ such that $h_0 = h|X \times \{0\}$ and $h_1 = h|X \times \{1\}$. In this case, we can attempt to understand the relationship between the pullbacks $h_0^* \mathcal{F}$ and $h_1^* \mathcal{F}$ of a sheaf $\mathcal{F}$ on Y by studying the pullback $h^* \mathcal{F} \in \mathrm{Shv}(Y \times \mathbf{R})$. If $\mathcal{F}$ is locally constant, then so is $h^* \mathcal{F}$. It will be convenient for us to consider a more general situation where $\mathcal{F}$ is only required to be locally constant *along* the paths $h|(\{y\} \times \mathbf{R})$ (and, for technical reasons, hypercomplete). The following definition axiomatizes the expected properties of the pullback $h^* \mathcal{F}$:

Definition A.2.4. Let X be a topological space and let $\mathcal{F} \in \mathrm{Shv}(X \times \mathbf{R})$. We will say that $\mathcal{F}$ is *foliated* if the following conditions are satisfied:

(i) The sheaf $\mathcal{F}$ is hypercomplete (see §T.6.5.2).

(ii) For every point $x \in X$, the restriction $\mathcal{F}|(\{x\} \times \mathbf{R})$ is constant.

The main result of this section is the following result, which should be regarded as a relative version of Proposition A.2.1 (where we have replaced the unit interval $[0, 1]$ with the entire real line):

Proposition A.2.5. *Let X be a topological space, let $\pi : X \times \mathbf{R} \to X$ denote the projection, and let $\mathcal{F} \in \mathrm{Shv}(X \times \mathbf{R})$. The following conditions are equivalent:*

(1) *The sheaf $\mathcal{F}$ is foliated.*

(2) *The pushforward $\pi_* \mathcal{F}$ is hypercomplete, and the counit map $v : \pi^* \pi_* \mathcal{F} \to \mathcal{F}$ is an equivalence.*

The proof of Proposition A.2.5 will require a few preliminaries.

Lemma A.2.6. *Let $f^* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. Assume that f^* admits a left adjoint $f_!$. Then f^* carries hypercomplete objects of $\mathcal{X}$ to hypercomplete objects of $\mathcal{Y}$.*

Proof. To show that f^* preserves hypercomplete objects, it will suffice to show that the left adjoint $f_!$ preserves ∞-connective morphisms. We will show that $f_!$ preserves n-connective morphisms for every non-negative integer n. This is equivalent to the assertion that f^* preserves $(n - 1)$-truncated morphisms, which follows from Proposition T.5.5.6.16. $\square$

Example A.2.7. Every étale map of ∞-topoi satisfies the hypothesis of Lemma A.2.6. Consequently, if X is a hypercomplete object of an ∞-topos $\mathcal{X}$, then $X \times U$ is a hypercomplete object of $\mathcal{X}_{/U}$ for each $U \in \mathcal{X}$.

Example A.2.8. Let X and Y be topological spaces, and let $\pi : X \times Y \to X$ be the projection. Assume that Y is locally compact and locally of constant shape. Then π^* satisfies the hypothesis of Lemma A.2.6, and therefore preserves hypercompletess. To prove this, we observe that $\mathrm{Shv}(X \times Y)$ can be identified with $\mathrm{Shv}(X) \otimes \mathrm{Shv}(Y)$, where $\otimes$ denotes the tensor product operation on presentable ∞-categories described in §6.3.1: this follows from Proposition T.7.3.1.11 and Example 6.3.1.18. The functor π^* can be identified with the tensor product $\mathrm{id}_{\mathrm{Shv}(X)} \otimes \pi'^*$, where $\pi' : Y \to *$ is the projection. Proposition A.1.8 guarantees that π'^* admits a left adjoint $\pi'_!$. It follows that $\mathrm{id}_{\mathrm{Shv}(X)} \otimes \pi'_!$ is a left adjoint to π^*. Moreover, if π'^* is fully faithful, then the counit map $v : \pi'_! \pi'^* \to \mathrm{id}$ is an equivalence, so the counit map $\pi_! \pi^* \to \mathrm{id}_{\mathrm{Shv}(X)}$ is also an equivalence: it follows that π^* is fully faithful.

Lemma A.2.9. *Let X be a topological space and let $\pi : X \times (0, 1) \to X$ denote the projection. Then the pullback functor $\pi^* : \mathrm{Shv}(X) \to \mathrm{Shv}(X \times (0, 1))$ is fully faithful (so that the unit map $\mathcal{F} \to \pi_* \pi^* \mathcal{F}$ is an equivalence for every $\mathcal{F} \in \mathrm{Shv}(X)$).*

Proof. Let $\psi : (0, 1) \to *$ denote the projection map, and let $\psi^* : \mathcal{S} \to \mathrm{Shv}((0, 1))$ be the associated geometric morphism. Then ψ^* admits a left adjoint $\psi_!$ (Proposition A.1.8) and the counit transformation $v : \psi_! \psi^* \to \mathrm{id}$

is an equivalence of functors from $\mathcal{S}$ to itself. As in Example A.2.8, we can identify $\mathrm{Shv}(X \times (0,1))$ with the tensor product $\mathrm{Shv}(X) \otimes \mathrm{Shv}((0,1))$, so that $\psi_!$ and ψ^* induce a pair of adjoint functors

$$\mathrm{Shv}(X \times (0,1)) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Shv}(X).$$

The functor G can be identified with π^*. Since the counit map v is an equivalence, the counit $F \circ G \to \mathrm{id}_{\mathrm{Shv}(X)}$ is likewise an equivalence, which proves that $G \simeq \pi^*$ is fully faithful. $\square$

Variant A.2.10. In the statement of Lemma A.2.9, we can replace $(0,1)$ by a closed or half-open interval.

Proof of Proposition A.2.5. Suppose first that (2) is satisfied, and let $\mathcal{G} = \pi_* \mathcal{F}$. Then $\mathcal{G}$ is hypercomplete, so $\pi^* \mathcal{G}$ is hypercomplete (Example A.2.8); since $v : \pi^* \mathcal{G} \to \mathcal{F}$ is an equivalence, it follows that $\mathcal{F}$ is hypercomplete. It is clear that $\mathcal{F} \simeq \pi^* \mathcal{G}$ is constant along $\{x\} \times \mathbf{R}$, for each $x \in X$.

Conversely, suppose that $\mathcal{F}$ is foliated. To prove that $\pi_* \mathcal{F}$ is hypercomplete, it suffices to show that $\pi_* \mathcal{F}$ is local with respect to every ∞-connective morphism α in $\mathrm{Shv}(X)$. This is equivalent to the requirement that $\mathcal{F}$ is local with respect to $\pi^*(\alpha)$. This follows from our assumption that $\mathcal{F}$ is hypercomplete, since $\pi^*(\alpha)$ is again ∞-connective. To complete the proof that $(1) \Rightarrow (2)$, it will suffice to show that the counit map $v : \pi^* \mathcal{G} \to \mathcal{F}$ is an equivalence.

For each positive integer n, let $\mathcal{F}_n = \mathcal{F}|(X \times (-n,n)) \in \mathrm{Shv}(X \times (-n,n))$, let $\pi_n : X \times (-n,n) \to X$ be the projection map, and let $\mathcal{G}_n = (\pi_n)_* \mathcal{F}$. We have a commutative diagram

$$
\begin{array}{ccc}
(\pi^* \mathcal{G})|(X \times (-n,n)) & \overset{v}{\longrightarrow} & \mathcal{F}|(X \times (-n,n)) \\
\downarrow & & \downarrow \\
\pi_n^* \mathcal{G}_n & \overset{v_n}{\longrightarrow} & \mathcal{F}_n .
\end{array}
$$

To prove that v is an equivalence, it will suffice to show that the left vertical and lower horizontal maps in this diagram are equivalences (for each n). This will follow from the following pair of assertions:

(a) For each $n > 0$, the restriction map $\mathcal{G}_{n+1} \to \mathcal{G}_n$ is an equivalence (so that $\mathcal{G} \simeq \varprojlim_n \mathcal{G}_n$ is equivalent to each $\mathcal{G}_n$).

(b) For each $n > 0$, the map $\pi_n^* \mathcal{G}_n \to \mathcal{F}_n$ is an equivalence.

Note that assertion (a) follows from (b): if we let $i : X \to X \times \mathbf{R}$ be the map induced by the inclusion $\{0\} \hookrightarrow \mathbf{R}$, then we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{G}_n & \longrightarrow & \mathcal{G}_{n+1} \\
\downarrow & & \downarrow \\
i^* \pi_n^* \mathcal{G}_n & \longrightarrow & i^* \pi_{n+1}^* \mathcal{G}_{n+1} \\
\downarrow & & \downarrow \\
i^* \mathcal{F}_n & \overset{s}{\longrightarrow} & i^* \mathcal{F}_{n+1}
\end{array}
$$

in which the upper vertical maps are equivalences, the lower horizontal maps are equivalences by (b), and the map s is an equivalence by construction.

To prove (b), let $\mathcal{F}_n^+ \in \mathrm{Shv}(X \times [-n,n])$ denote the hypercompletion of the restriction $\mathcal{F}|(X \times [-n,n])$, let $\pi^n : X \times [-n,n] \to X$ be the projection, and let $\mathcal{G}_n^+ = \pi_*^n \mathcal{F}_n^+$. Let $v' : (\pi^n)^* \mathcal{G}_n^+ \to \mathcal{F}_n^+$ be the counit map. We claim that v' is an equivalence. Since $\mathcal{F}_n^+$ is hypercomplete by assumption, $\mathcal{G}_n^+ \simeq \pi_*^n \mathcal{F}_n^+$ is likewise hypercomplete and so $(\pi^n)^* \mathcal{G}_n^+$ is hypercomplete by virtue of Example A.2.8. Consequently, to prove that

v' is an equivalence, it will suffice to show that v' is ∞-connective. To prove this, choose a point $x \in X$ and let $j : [-n, n] \to X \times [-n, n]$ be the map induced by the inclusion $j' : \{x\} \hookrightarrow X$. We will show that $j^*(v')$ is an equivalence. Consider the diagram of ∞-topoi

$$
\begin{array}{ccccc}
\mathrm{Shv}([-n,n]) & \xrightarrow{\ j_*\ } & \mathrm{Shv}(X \times [-n,n]) & \longrightarrow & \mathrm{Shv}([0,1]) \\
\downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle \pi^n_*} & & \downarrow{\scriptstyle \psi_*} \\
\mathrm{Shv}(*) & \xrightarrow{\ j'_*\ } & \mathrm{Shv}(X) & \longrightarrow & \mathrm{Shv}(*).
\end{array}
$$

The right square and the outer rectangle are pullback diagrams (Proposition T.7.3.1.11), so the left square is a pullback diagram as well. Moreover, the geometric morphism ψ_* is proper (Corollary T.7.3.4.11), so that π^n_* is likewise proper and the push-pull morphism $e : j'^* \pi^n_* \to \psi_* j^*$ is an equivalence. We have a commutative diagram

$$
\begin{array}{ccc}
\psi^* j'^* \pi^n_* \,\mathcal{F}^+_n & \longrightarrow & j^*(\pi^n)^* \pi^n_* \,\mathcal{F}^+_n \\
\downarrow{\scriptstyle e} & & \downarrow{\scriptstyle j^*(v')} \\
\psi^* \psi_* j^* \,\mathcal{F}^+_n & \xrightarrow{\ v'_x\ } & j^* \,\mathcal{F}^+_n \,.
\end{array}
$$

By virtue of the above diagram (and the fact that e is an equivalence), we are reduced to proving that v'_x is an equivalence. To prove this, it suffices to verify that $j^* \mathcal{F}^+_n \in \mathrm{Shv}([-n,n])$ is constant (Proposition A.2.1). We have an ∞-connective morphism $\theta : \mathcal{F}|(\{x\} \times [-n,n]) \to j^* \mathcal{F}^+_n$. Since every open subset of the topological space $[-n,n]$ has covering dimension ≤ 1, the ∞-topos $\mathrm{Shv}([-n,n])$ is locally of homotopy dimension ≤ 1 (Theorem T.7.2.3.6) and therefore hypercomplete. It follows that θ is an equivalence. Since $\mathcal{F}$ is foliated, the restriction $\mathcal{F}|(\{x\} \times [-n,n])$ is constant, from which it follows immediately that $j^* \mathcal{F}^+_n$ is constant as well.

The ∞-connective morphism $\mathcal{F}|(X \times [-n,n]) \to \mathcal{F}^+_n$ induces another ∞-connective morphism $\alpha : \mathcal{F}_n \to \mathcal{F}^+_n |(X \times (-n,n))$. Since the domain and codomain of α are both hypercomplete (Example A.2.7), we deduce that α is an equivalence. In particular, we have $\mathcal{F}_n \simeq ((\pi^n)^* \mathcal{G}^+_n)|(X \times (-n,n)) = \pi^*_n \mathcal{G}^+_n$. Thus $\mathcal{F}_n$ lies in the essential image of the functor π^*_n, which is fully faithful by virtue of Lemma A.2.9. It follows that that the counit map $\pi^*_n (\pi_n)_* \mathcal{F}_n \to \mathcal{F}_n$ is an equivalence as desired.   □

A.3   The Seifert-van Kampen Theorem

Let X be a topological space covered by a pair of open sets U and V, such that U, V, and $U \cap V$ are path-connected. The Seifert-van Kampen theorem asserts that, for any choice of base point $x \in U \cap V$, the diagram of groups

$$
\begin{array}{ccc}
\pi_1(U \cap V, x) & \longrightarrow & \pi_1(U, x) \\
\downarrow & & \downarrow \\
\pi_1(V, x) & \longrightarrow & \pi_1(X, x)
\end{array}
$$

is a pushout square. In this section, we will prove a generalization of the Seifert-van Kampen theorem, which describes the entire weak homotopy type of X in terms of any sufficiently nice covering of X by open sets:

Theorem A.3.1. *Let X be a topological space, let $\mathcal{U}(X)$ denote the collection of all open subsets of X (partially ordered by inclusion). Let $\mathcal{C}$ be a small category and let $\chi : \mathcal{C} \to \mathcal{U}(X)$ be a functor. For every $x \in X$, let $\mathcal{C}_x$ denote the full subcategory of $\mathcal{C}$ spanned by those objects $C \in \mathcal{C}$ such that $x \in \chi(C)$. Assume that χ satisfies the following condition:*

  (∗) *For every point x, the simplicial set $\mathrm{N}(\mathcal{C}_x)$ is weakly contractible.*

Then the canonical map $\varinjlim_{C \in \mathcal{C}} \mathrm{Sing}(\chi(C)) \to \mathrm{Sing}(X)$ exhibits the simplicial set $\mathrm{Sing}(X)$ as a homotopy colimit of the diagram $\{\mathrm{Sing}(\chi(C))\}_{C \in \mathcal{C}}$.

The proof of Theorem A.3.1 will occupy our attention throughout this section. The main step will be to establish the following somewhat weaker result:

Proposition A.3.2. *Let X be a topological space, let $\mathcal{U}(X)$ be the partially ordered set of all open subsets of X, and let $S \subseteq \mathcal{U}(X)$ be a covering sieve on X. Then the canonical map $\varinjlim_{U \in S} \mathrm{Sing}(U) \to \mathrm{Sing}(X)$ exhibits the simplicial set $\mathrm{Sing}(X)$ as the homotopy colimit of the diagram of simplicial sets $\{\mathrm{Sing}(U)\}_{U \in S}$.*

Proposition A.3.2 is itself a consequence of the following result, which guarantees that $\mathrm{Sing}(X)$ is weakly homotopy equivalent to the simplicial subset consisting of "small" simplices:

Lemma A.3.3. *Let X be a topological space, and let $\{U_\alpha\}$ be an open covering of X. Let $\mathrm{Sing}'(X)$ be the simplicial subset of $\mathrm{Sing}(X)$ spanned by those n-simplices $|\Delta^n| \to X$ which factor through some U_α. Then the inclusion $i : \mathrm{Sing}'(X) \subseteq \mathrm{Sing}(X)$ is a weak homotopy equivalence of simplicial sets.*

The proof of Lemma A.3.3 will require a few technical preliminaries.

Lemma A.3.4. *Let X be a compact topological space and let K be a simplicial set. Then every continuous map $f : X \to |K|$ factors through $|K_0|$, for some finite simplicial subset $K_0 \subseteq K$.*

Proof. Let K_0 be the simplicial subset of K spanned by those simplices σ such that the interior of $|\sigma|$ intersects $f(X)$. We claim that K_0 is finite. Otherwise, we can choose an infinite sequence of points $x_0, x_1, \ldots \in X$ such that each $f(x_i)$ belongs to the interior of a *different* simplex of $|K|$. Let $U = |K| - \{f(x_0), f(x_1), \ldots, \}$, and for each $i \geq 0$ let $U_i = U \cup \{f(x_i)\}$. Then the collection of open sets $\{U_i\}$ forms an open cover of K, so that $\{f^{-1}U_i\}$ forms an open covering of X. This open covering does not admit a finite subcovering, contradicting our assumption that X is compact. $\square$

Lemma A.3.5. *Let $i : K_0 \subseteq K$ be an inclusion of simplicial sets. Suppose that the following condition is satisfied:*

$(*)$ *For every finite simplicial subset $L \subseteq K$, there exists a homotopy $h : |L| \times [0,1] \to |K|$ such that $h|(|L| \times \{0\})$ is the inclusion, $h|(|L| \times \{1\}) \subseteq |K_0|$, and $h|(|L_0| \times [0,1]) \subseteq |K_0|$, where $L_0 = L \cap K_0$.*

Then the inclusion i is a weak homotopy equivalence.

Proof. We first show the following:

$(*')$ *Let X be a compact topological space, X_0 a closed subspace, and $f : X \to |K|$ a continuous map such that $f(X_0) \subseteq |K_0|$. Then there exists a homotopy $h : X \times [0,1] \to |K|$ such that $h|(X \times \{0\}) = f$, $h(X \times \{1\}) \subseteq |K_0|$, and $h|(X_0 \times [0,1]) \subseteq |K_0|$.*

To prove $(*')$, we note that since X is compact, the map f factors through $|L|$, where L is some finite simplicial subset of K. Then $f|X_0$ factors through $|L_0|$, where $L_0 = L \cap K_0$. We may therefore replace X and X_0 by $|L|$ and $|L_0|$, in which case $(*')$ is equivalent to our assumption $(*)$.

Applying $(*')$ in the case where X is a point and X_0 is empty, we deduce that the inclusion i is surjective on connected components. It will therefore suffice to show that i induces a bijection $\phi : \pi_n(|K_0|, v) \to \pi_n(|K|, v)$ for each $n \geq 0$ and each vertex v of K. To prove that ϕ is surjective, consider a homotopy class $\eta \in \pi_n(|K|, v)$. This homotopy class can be represented by a pointed map $f : (S^n, *) \to (|K|, v)$. Applying $(*')$, we deduce that f is homotopic to a another map $g : S^n \to |K_0|$, via a homotopy which, when restricted to the base point $* \in S^n$, determines a path p from v to another point $v' \in |K_0|$. Then g determines an element $\eta' \in \pi_n(|K_0|, v')$. The image of η' under the transport isomorphism $p_* : \pi_n(|K_0|, v') \simeq \pi_n(|K_0|, v)$ is a preimage of η under ϕ.

We now prove that ϕ is injective. Suppose we are given a continuous map $f_0 : S^n \to |K_0|$ which extends to a map $f : D^{n+1} \to |K|$; we wish to show that f_0 is nullhomotopic. Applying $(*')$, we deduce that f_0 is homotopic to a map which extends over the disk D^{n+1}, and is therefore itself nullhomotopic. $\square$

Before we can proceed with the proof of Lemma A.3.3, we need to recall some properties of the *barycentric subdivision* construction in the setting of simplicial sets.

Notation A.3.6. Let $[n]$ be an object of $\mathbf{\Delta}$. We let $P[n]$ denote the collection of all nonempty subsets of $[n]$, partially ordered by inclusion. We let $\overline{P}[n]$ denote the disjoint union $P[n] \coprod [n]$. We regard $\overline{P}[n]$ as endowed with a partial ordering which extends the partial orderings on $P[n]$ and $[n]$, where we let $i \not\leq \sigma$ for $i \in [n]$ and $\sigma \in P[n]$, while $\sigma \leq i$ if and only if each element of σ is $\leq i$.

The functors $[n] \mapsto \mathrm{N}\, P[n]$ and $[n] \mapsto \mathrm{N}\, \overline{P}[n]$ extend to colimit-preserving functors from the category of simplicial sets to itself. We will denote these functors by sd and $\overline{\mathrm{sd}}$, respectively.

Let us identify the topological n-simplex $|\Delta^n|$ which the set of all maps $t : [n] \to [0,1]$ such that $t(0) + \ldots + t(n) = 1$. For each $n \geq 0$, there is a homeomorphism $\eta_n : |\overline{P}[n]| \to |\Delta^n| \times [0,1]$ which is linear on each simplex, carries a vertex $i \in [n]$ to $(t_i, 0)$ where t_i is given by the formula $t_i(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$ and carries a vertex $\sigma \in P[n]$ to the pair $(t_\sigma, 1)$, where

$$t_\sigma(i) = \begin{cases} \frac{1}{m} & \text{if } i \in \sigma \\ 0 & \text{if } i \notin \sigma \end{cases}$$

where m is the cardinality of σ. This construction is functorial in $[n]$, and induces a homeomorphism $|\overline{\mathrm{sd}}K| \to |K| \times [0,1]$ for every simplicial set K. We observe that $\overline{\mathrm{sd}}K$ contains K and $\mathrm{sd}\, K$ as simplicial subsets, whose geometric realizations map homeomorphically to $|K| \times \{0\}$ and $|K| \times \{1\}$, respectively.

Proof of Lemma A.3.3. We will show that i satisfies the criterion of Lemma A.3.5. Let $L \subseteq \mathrm{Sing}(X)$ be a finite simplicial subset, and let $L_0 = L \cap \mathrm{Sing}'(X)$. Fix $n \geq 0$, let $\overline{L}$ denote the iterated pushout

$$\overline{\mathrm{sd}}\,\mathrm{sd}^{n-1} L \coprod_{\mathrm{sd}^{n-1} L} \overline{\mathrm{sd}}\,\mathrm{sd}^{n-2} L \coprod_{\mathrm{sd}^{n-2} L} \cdots \coprod_{\mathrm{sd}\, L} L,$$

and define $\overline{L}_0$ similarly. Using the homeomorphisms $|\overline{\mathrm{sd}}K| \simeq |K| \times [0,1]$ of Notation A.3.6 repeatedly, we obtain a homeomorphism $|\overline{L}| \simeq |L| \times [0,n]$ (which restricts to a homeomorphism $|\overline{L}_0| \simeq |L_0| \times [0,n]$).

The inclusion map $L \subseteq \mathrm{Sing}(X)$ is adjoint to a continuous map of topological spaces $f : |L| \to X$. Let $\overline{f}$ denote the composite map

$$|\overline{L}| \simeq |L| \times [0,n] \to |L| \xrightarrow{f} X.$$

Then $\overline{f}$ determines a map of simplicial sets $\overline{L} \to \mathrm{Sing}(X)$; we observe that this map carries $\overline{L}_0$ into $\mathrm{Sing}'(X)$. Passing to geometric realizations, we get a map $h : |L| \times [0,n] \simeq |\overline{L}| \to |\mathrm{Sing}(X)|$, which is a homotopy from the inclusion $|L| \subseteq |\mathrm{Sing}(X)|$ to the map $g = h|(|L| \times \{n\})$ (by construction, this homotopy carries $|L_0| \times [0,n]$ into $|\mathrm{Sing}'(X)|$). We note that g is the geometric realization of the map $\mathrm{sd}^n L \to \mathrm{Sing}'(X)$, which is adjoint to the composition $|\mathrm{sd}^n L| \simeq |L| \xrightarrow{f} X$. To complete the proof, it suffices to observe that for n sufficiently large, each simplex of the n-fold barycentric subdivision $|\mathrm{sd}^n L|$ will map into one of the open sets U_α, so that g factors through $|\mathrm{Sing}'(X)|$ as required. $\square$

Armed with Lemma A.3.3, it is easy to finish the proof of Proposition A.3.2.

Proof of Proposition A.3.2. Choose a collection of open sets $\{U_\alpha\}_{\alpha \in A}$ which generates the sieve S. Let $P(A)$ denote the collection of all nonempty subsets of A, partially ordered by reverse inclusion. Let $P_0(A)$ be the subset consisting of nonempty *finite* subsets of A. For each $A_0 \in P(A)$, let $U_{A_0} = \bigcap_{\alpha \in A_0} U_\alpha$ (if A_0 is finite, this is an open subset of X, though in general it need not be). The construction $A_0 \mapsto U_{A_0}$ determines a map of partially ordered sets $P_0(A) \to S$. Using Theorem T.4.1.3.1, we deduce that the map $\mathrm{N}(P_0(A)) \to \mathrm{N}(S)$ is left cofinal, so that (by virtue of Theorem T.4.2.4.1) it will suffice to show that $\mathrm{Sing}(X)$ is a homotopy colimit of the diagram $\{\mathrm{Sing}(U_{A_0})\}_{A_0 \in P_0(A)}$. A similar argument shows that the inclusion $\mathrm{N}(P_0(A)) \subseteq \mathrm{N}(P(A))$ is left cofinal, so we are reduced to showing that $\mathrm{Sing}(X)$ is a homotopy colimit of

the diagram $\psi = \{\mathrm{Sing}(U_{A_0})\}_{A_0 \in P(A)}$. The actual colimit of the diagram ψ is the simplicial set $\mathrm{Sing}'(X)$ which is weakly equivalent to $\mathrm{Sing}(X)$ by Lemma A.3.3. It will therefore suffice to show that the diagram ψ is projectively cofibrant. To prove this, we will show more generally that for any pair of simplicial subsets $K_0 \subseteq K \subseteq \mathrm{Sing}(X)$, the induced map

$$\phi : \{\mathrm{Sing}(U_{A_0}) \cap K_0\}_{A_0 \in P(A)} \hookrightarrow \{\mathrm{Sing}(U_{A_0}) \cap K\}_{A_0 \in P(A)}$$

is a projective cofibration of diagrams (taking $K_0 = \emptyset$ and $K = \mathrm{Sing}(X)$ will then yield the desired result). Working simplex by simplex, we may assume that K is obtained from K_0 by adjoining a single nondegenerate simplex $\sigma : |\Delta^n| \to X$ whose boundary already belongs to K_0. Let $A' = \{\alpha \in A : \sigma(|\Delta^n|) \subseteq U_\alpha\}$. If A' is empty, then ϕ is an isomorphism. Otherwise, ϕ is a pushout of the projective cofibration $F_0 \hookrightarrow F$, where

$$F_0(A_0) = \begin{cases} \partial\,\Delta^n & \text{if } A_0 \subseteq A' \\ \emptyset & \text{otherwise} \end{cases} \qquad F(A_0) = \begin{cases} \Delta^n & \text{if } A_0 \subseteq A' \\ \emptyset & \text{otherwise.} \end{cases}$$

$\square$

Variant A.3.7. If X is a paracompact topological space, we can replace $\mathcal{U}(X)$ with the collection of all open F_σ subsets of X in the statement of Proposition A.3.2; the proof remains the same.

Remark A.3.8. Let X be a topological space, and let $\mathcal{U}(X)$ denote the partially ordered set of all open subsets of X. The construction $U \mapsto \mathrm{Sing}(U)$ determines a functor between ∞-categories $\mathrm{N}(\mathcal{U}(X)) \to \mathcal{S}$. Theorem T.5.1.5.6 implies that this functor is equivalent to a composition

$$\mathrm{N}(\mathcal{U}(X)) \xrightarrow{j} \mathcal{P}(\mathcal{U}(X)) \xrightarrow{F} \mathcal{S},$$

where j denotes the Yoneda embedding and the functor F preserves small colimits (moreover, the functor F is determined uniquely up to equivalence). Proposition A.3.2 implies that F is equivalent to the composition

$$\mathcal{P}(\mathcal{U}(X)) \xrightarrow{L} \mathrm{Shv}(X) \xrightarrow{F} \mathcal{S},$$

where L denotes a left adjoint to the inclusion $\mathrm{Shv}(X) \subseteq \mathcal{P}(\mathcal{U}(X))$ and we identify F with its restriction to $\mathrm{Shv}(X)$. In particular, the functor $F : \mathrm{Shv}(X) \to \mathcal{S}$ preserves small colimits.

We now explain how to deduce Theorem A.3.1 from Proposition A.3.2. The main technical obstacle is that the ∞-topos $\mathrm{Shv}(X)$ need not be hypercomplete. We will address this problem by showing that the functor F of Remark A.3.8 factors through the hypercompletion of $\mathrm{Shv}(X)$: in other words, that F carries ∞-connected morphisms in $\mathrm{Shv}(X)$ to equivalences in $\mathcal{S}$ (Lemma A.3.10). We first note that ∞-connectedness is a condition which can be tested "stalkwise":

Lemma A.3.9. *Let X be a topological space, and let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a morphism in the ∞-category $\mathrm{Shv}(X)$. For each point $x \in X$, let $x^* : \mathrm{Shv}(X) \to \mathrm{Shv}(\{x\}) \simeq \mathcal{S}$ denote the pullback functor. The following conditions are equivalent:*

(1) *The morphism α is ∞-connective.*

(2) *For each $x \in X$, the morphism $x^*(\alpha)$ is an equivalence in $\mathcal{S}$.*

Proof. The implication $(1) \Rightarrow (2)$ is obvious, since the pullback functors x^* preserve ∞-connectivity and the ∞-topos $\mathcal{S}$ is hypercomplete. Conversely, suppose that (2) is satisfied. We will prove by induction on n that the morphism α is n-connective. Assume that $n > 0$. By virtue of Proposition T.6.5.1.18, it will suffice to show that the diagonal map $\mathcal{F} \times_{\mathcal{F}'} \mathcal{F}$ is $(n-1)$-connective, which follows from the inductive hypothesis. We may therefore reduce to the case $n = 0$: that is, we must show that α is an effective epimorphism. According to Proposition T.7.2.1.14, this is equivalent to the requirement that the induced map $\alpha' : \tau_{\leq 0}\,\mathcal{F} \to \tau_{\leq 0}\,\mathcal{F}'$ is an effective epimorphism. We may therefore replace α by α' and thereby reduce to the case where $\mathcal{F}, \mathcal{F}' \in \mathrm{Shv}_{\mathrm{Set}}(X)$ are sheaves of sets on X, in which case the result is obvious. $\square$

Lemma A.3.10. *Let X be a topological space, and let $F : \mathrm{Shv}(X) \to \mathcal{S}$ be as in Remark A.3.8. Then F carries ∞-connective morphisms of $\mathrm{Shv}(X)$ to equivalences in $\mathcal{S}$.*

Proof. Let α be an ∞-connectivemorphism in $\mathrm{Shv}(X)$. We will show that $F(\alpha)$ is an ∞-connective morphism in $\mathcal{S}$, hence an equivalence (since the ∞-topos $\mathcal{S}$ is hypercomplete). For this, it suffices to show that for each $n \geq 0$, the composite functor

$$\mathrm{Shv}(X) \xrightarrow{F} \mathcal{S} \xrightarrow{\tau_{\leq n}^{\mathcal{S}}} \tau_{\leq n} \mathcal{S}$$

carries α to an equivalence. Since $\tau_{\leq n} \mathcal{S}$ is an n-category, the functor $\tau_{\leq n}^{\mathcal{S}} \circ F$ is equivalent to a composition

$$\mathrm{Shv}(X) \xrightarrow{\tau_{\leq n}^{\mathrm{Shv}(X)}} \tau_{\leq n} \mathrm{Shv}(X) \xrightarrow{F_n} \tau_{\leq n} \mathcal{S}.$$

We now observe that $\tau_{\leq n}^{\mathrm{Shv}(X)}(\alpha)$ is an equivalence, since α is assumed to be ∞-connective. $\qquad\square$

We now have the tools in place to complete the proof of our main result.

Proof of Theorem A.3.1. Passing to nerves, we obtain a diagram of ∞-categories $\overline{p} : \mathrm{N}(\mathcal{C})^{\triangleright} \to \mathcal{S}$. In view of Theorem T.4.2.4.1, it will suffice to show that $\overline{p}$ is a colimit diagram. Note that $\overline{p}$ is equivalent to the composition

$$\mathrm{N}(\mathcal{C})^{\triangleright} \xrightarrow{\overline{\chi}} \mathrm{N}(\mathcal{U}(X)) \xrightarrow{j} \mathrm{Shv}(X)^{\wedge} \xrightarrow{F} \mathcal{S},$$

where $\mathrm{Shv}(X)^{\wedge}$ denotes the full subcategory of $\mathcal{P}(\mathcal{U}(X))$ spanned by the hypercomplete sheaves on X, j denotes the Yoneda embedding, and F is defined as in Remark A.3.8. Using Proposition A.3.2 and Lemma A.3.10, we deduce that F preserves small colimits. It therefore suffices to show that $j \circ \overline{\chi}$ is a colimit diagram. Since $\mathrm{Shv}(X)^{\wedge}$ is hypercomplete, it suffices to show that the composition $f^* \circ j \circ \overline{\chi}$ is a colimit diagram, where $f : \{x\} \hookrightarrow X$ is the inclusion of any point into X. This follows immediately from assumption $(*)$. $\qquad\square$

A.4 Singular Shape

In §A.1, we defined the notion of a *locally constant* object of an ∞-topos $\mathcal{X}$. Moreover, we proved that the ∞-topos $\mathcal{X}$ is locally of constant shape, then the ∞-category of locally constant objects of $\mathcal{X}$ is equivalent to the ∞-topos $\mathcal{S}_{/K}$ of spaces lying over some fixed object $K \in \mathcal{S}$ (Theorem A.1.15). This can be regarded as an analogue of the main result in the theory of covering spaces, which asserts that the category of covering spaces of a sufficiently nice topological space X can be identified with the category of sets acted on by the fundamental group of X. If we apply Theorem A.1.15 in the special case $\mathcal{X} = \mathrm{Shv}(X)$, then we deduce that the fundamental groups of X and K are isomorphic to one another. Our objective in this section is to strengthen this observation: we will show that if X is a sufficiently nice topological space, then the ∞-topos $\mathrm{Shv}(X)$ of sheaves on X is locally of constant shape, and the shape K of $\mathrm{Shv}(X)$ can be identified with the singular complex $\mathrm{Sing}(X)$.

Remark A.4.1. We refer the reader to [145] for a closely related discussion, at least in the case where X is a CW complex.

Our first step is to describe a class of topological spaces X for which the theory of locally constant sheaves on X is well-behaved. By definition, if $\mathcal{F}$ is a locally constant sheaf on X, then every point $x \in X$ has an open neighborhood U such that the restriction $\mathcal{F}|U$ is constant. Roughly speaking, we want a condition on X which guarantees that we can choose U to be independent of $\mathcal{F}$.

Definition A.4.2. Let $f^* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. We will say that f^* is a *shape equivalence* if it induces an equivalence of functors $\pi_* \pi^* \to \pi_* f_* f^* \pi^*$, where $\pi^* : \mathcal{S} \to \mathcal{X}$ is a geometric morphism.

Remark A.4.3. Let $\mathfrak{X}$ be an ∞-topos. Then $\mathfrak{X}$ has constant shape if and only if there exists a shape equivalence $f^* : \mathcal{S}_{/K} \to \mathfrak{X}$, for some Kan complex K. The "if" direction is obvious (since $\mathcal{S}_{/K}$ is of constant shape). Conversely, if $\mathfrak{X}$ is of constant shape, then $\pi_* \pi^*$ is corepresentable by some object $K \in \mathcal{S}$. In particular, there is a canonical map $\Delta^0 \to \pi_* \pi^* K$, which we can identify with a map $\alpha : \mathbf{1} \to \pi^* K$ in the ∞-topos $\mathfrak{X}$, where $\mathbf{1}$ denotes the final object of $\mathfrak{X}$. According to Proposition T.6.3.5.5, α determines a geometric morphism of ∞-topoi $f^* : \mathcal{S}_{/K} \to \mathfrak{X}$, which is easily verified to be a shape equivalence.

Definition A.4.4. Let $f : X \to Y$ be a continuous map of topological spaces. We will say that f is a *shape equivalence* if the associated geometric morphism $f^* \operatorname{Shv}(X) \to \operatorname{Shv}(Y)$ is a shape equivalence, in the sense of Definition A.4.2.

Example A.4.5. Let $f : X \to Y$ be a continuous map between paracompact topological spaces. Then f is a shape equivalence in the sense of Definition A.4.4 if and only if, for every CW complex Z, composition with f induces a homotopy equivalence of Kan complexes $\operatorname{Map}_{\operatorname{Top}}(Y, Z) \to \operatorname{Map}_{\operatorname{Top}}(X, Z)$.

Example A.4.6. If X is any topological space, then the projection map $\pi : X \times \mathbf{R} \to X$ is a shape equivalence. This follows immediately from the observation that π^* is fully faithful (Example A.2.8).

Remark A.4.7. It follows from Example A.4.6 that every homotopy equivalence of topological spaces is also a shape equivalence.

Warning A.4.8. For general topological spaces, Definition A.4.4 does *not* recover the classical notion of a shape equivalence (see, for example, [103]). However, if X and Y are both paracompact then we recover the usual notion of strong shape equivalence (Remark T.7.1.6.7).

Definition A.4.9. Let X be a topological space. We will say that X has *singular shape* if the counit map $|\operatorname{Sing}(X)| \to X$ is a shape equivalence.

Remark A.4.10. If X is a topological space with singular shape, then the ∞-topos $\operatorname{Shv}(X)$ has constant shape: indeed, $\operatorname{Shv}(X)$ is shape equivalent to $\operatorname{Shv}(|\operatorname{Sing}(X)|)$, and $|\operatorname{Sing}(X)|$ is a CW complex (Remark A.1.4).

Remark A.4.11. Let $f : X \to Y$ be a homotopy equivalence of topological spaces. Then X has singular shape if and only if Y has singular shape. This follows immediately from Remark A.4.7 by inspecting the diagram

$$
\begin{array}{ccc}
|\operatorname{Sing}(X)| & \longrightarrow & |\operatorname{Sing}(Y)| \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y.
\end{array}
$$

Example A.4.12. Let X be a paracompact topological space. Then X has singular shape if and only if, for every CW complex Y, the canonical map

$$
\operatorname{Map}_{\operatorname{Top}}(X, Y) \to \operatorname{Map}_{\mathcal{S}et_\Delta}(\operatorname{Sing}(X), \operatorname{Sing}(Y)) \simeq \operatorname{Map}_{\operatorname{Top}}(|\operatorname{Sing}(X)|, Y)
$$

is a homotopy equivalence of Kan complexes.

Remark A.4.13. Let X be a paracompact topological space. There are two different ways that we might try to assign to X a homotopy type. The first is to consider continuous maps from nice spaces (such as CW complexes) *into* the space X. Information about such maps is encoded in the Kan complex $\operatorname{Sing}(X) \in \mathcal{S}$, which controls the *weak homotopy type* of X. Alternatively, we can instead consider maps from X into CW complexes. These are controlled by the pro-object $\operatorname{Sh}(X)$ of $\mathcal{S}$ which corepresents the functor $K \mapsto \operatorname{Map}_{\operatorname{Top}}(X, |K|)$. There is a canonical map $\operatorname{Sing}(X) \to \operatorname{Sh}(X)$, and X has singular shape if and only if this map is an equivalence.

Lemma A.4.14. *Let X be a topological space, and let $\{U_\alpha \in \mathcal{U}(X)\}_{\alpha \in A}$ be an open covering of X. Assume that for every nonempty finite subset $A_0 \subseteq A$, the intersection $U_{A_0} = \bigcap_{\alpha \in A_0} U_\alpha$ has singular shape. Then X has singular shape.*

Proof. Let $\pi^* : \mathcal{S} \to \mathrm{Shv}(X)$ be a geometric morphism. For each open set $U \subseteq X$, let $F_U : \mathcal{S} \to \mathcal{S}$ be the functor given by composing π^* with evaluation at U, and let $G_U : \mathcal{S} \to \mathcal{S}$ be the functor given by $K \mapsto \mathrm{Fun}(\mathrm{Sing}(U), K)$. There is a natural transformation of functors $\gamma_U : F_U \to G_U$, and U has singular shape if and only if γ_U is an equivalence. We observe that F_X can be identified with a limit of the diagram $\{F_{U_{A_0}}\}$ where A_0 ranges over the finite subsets of A, and that G_X can be identified with a limit of the diagram $\{G_{U_{A_0}}\}$ (since $\mathrm{Sing}(X)$ is the homotopy colimit of $\{\mathrm{Sing}(U_{A_0})\}$ by Theorem A.3.1). Under these identifications, γ_X is a limit of the functors $\{\gamma_{U_{A_0}}\}$. Since each of these functors is assumed to be an equivalence, we deduce that γ_X is an equivalence. $\square$

Definition A.4.15. We will say that topological space X is *locally of singular shape* if every open set $U \subseteq X$ has singular shape.

Remark A.4.16. Let X be a topological space. Suppose that X admits a covering by open sets which are locally of singular shape. Then X is locally of singular shape (this follows immediately from Lemma A.4.14).

Let X be a topological space which is locally of singular shape. Then $\mathrm{Shv}(X)$ is locally of constant shape, and the shape of $\mathrm{Shv}(X)$ can be identified with the Kan complex $\mathrm{Sing}(X)$. It follows from Theorem A.1.15 that the ∞-category of locally constant objects of $\mathrm{Shv}(X)$ is equivalent to $\mathcal{S}_{/\mathrm{Sing}(X)}$. Our goal for the remainder of this section is to give a more explicit description of this equivalence.

Construction A.4.17. Let X be a topological space. We let $\mathbf{A}_X$ denote the category $(\mathrm{Set}_\Delta)_{/\mathrm{Sing}(X)}$, endowed with the usual model structure. Let $\mathbf{A}_X^\circ$ denote the full subcategory of $\mathbf{A}_X$ spanned by the fibrant-cofibrant objects (these are precisely the Kan fibrations $Y \to \mathrm{Sing}(X)$).

We define a functor $\theta : \mathcal{U}(X)^{op} \times \mathbf{A}_X \to \mathrm{Set}_\Delta$ by the formula $\theta(U, Y) = \mathrm{Fun}_{\mathrm{Sing}(X)}(\mathrm{Sing}(U), Y)$. Restricting to $\mathbf{A}_X^\circ$ and passing to nerves, we get a map of ∞-categories $\mathrm{N}(\mathcal{U}(X)^{op}) \times \mathrm{N}(\mathbf{A}_X^\circ) \to \mathcal{S}$, which we regard as a map of ∞-categories $\mathrm{N}(\mathbf{U}_X^\circ) \to \mathcal{P}(\mathcal{U}(X))$. It follows from Variant A.3.7 on Proposition A.3.2 that this functor factors through the full subcategory $\mathrm{Shv}(X) \subseteq \mathcal{P}(\mathcal{U}(X))$ spanned by the sheaves on X. We will denote the underlying functor $\mathrm{N}(\mathbf{A}_X^\circ) \to \mathrm{Shv}(X)$ by Ψ_X.

Example A.4.18. Let X be a topological space. The construction $K \mapsto K \times \mathrm{Sing}(X)$ determines a functor from $\mathrm{Set}_\Delta \simeq \mathbf{A}_*$ to $\mathbf{A}_X$, which restricts to a functor $\mathbf{A}_*^\circ \to \mathbf{A}_X^\circ$. Passing to nerves and composing with Ψ_X, we get a functor $\psi : \mathcal{S} \to \mathrm{Shv}(X)$, which carries a Kan complex K to the sheaf $U \mapsto \mathrm{Map}_{\mathrm{Set}_\Delta}(\mathrm{Sing}(U), K)$. Let $\pi_* : \mathrm{Shv}(X) \to \mathcal{S}$ be the functor given by evaluation on X. There is an evident natural transformation $\mathrm{id}_\mathcal{S} \to \pi_* \circ \psi$, which induces a natural transformation $\pi^* \to \psi$. The space X is locally of singular shape if and only if this natural transformation is an equivalence.

We note that the object $\psi \, \mathrm{Sing}(X) \in \mathrm{Shv}(X)$ has a canonical global section given by the identity map from $\mathrm{Sing}(X)$ to itself. If $Y \to \mathrm{Sing}(X)$ is any Kan fibration, then $\Psi_X(Y)$ can be identified with the (homotopy) fiber of the induced map $\psi(Y) \to \psi(\mathrm{Sing}(X))$. It follows that the functor Ψ_X is an explicit model for the fully faithful embedding described in Proposition A.1.11. Coupling this observation with Theorem A.1.15, we obtain the following:

Theorem A.4.19. *Let X be a topological space which is locally of singular shape. Then the functor $\Psi_X : \mathrm{N}(\mathbf{A}_X^\circ) \to \mathrm{Shv}(X)$ is a fully faithful embedding, whose essential image is the full subcategory of $\mathrm{Shv}(X)$ spanned by the locally constant sheaves on X.*

A.5 Constructible Sheaves

In §A.1 and §A.4, we studied the theory of *locally constant* sheaves on a topological space X. In many applications, one encounters sheaves $\mathcal{F} \in \mathrm{Shv}(X)$ which are not locally constant but are nevertheless *con-*

structible: that is, they are locally constant along each stratum of a suitable stratification of X. We begin by making this notion more precise.

Definition A.5.1. Let A be a partially ordered set. We will regard A as a topological space, where a subset $U \subseteq A$ is open if it is *closed upwards*: that is, if $x \leq y$ and $x \in U$ implies that $y \in U$.

Let X be a topological space. An *A-stratification* of X is a continuous map $f : X \to A$. Given an A-stratification of a space X and an element $a \in A$, we let X_a, $X_{\leq a}$, $X_{<a}$, $X_{\geq a}$, and $X_{>a}$ denote the subsets of X consisting of those points $x \in X$ such that $f(x) = a$, $f(x) \leq a$, $f(x) < a$, $f(x) \geq a$, and $f(x) > a$, respectively.

Definition A.5.2. Let A be a partially ordered set and let X be a topological space equipped with an A-stratification. We will say that an object $\mathcal{F} \in \mathrm{Shv}(X)$ is *A-constructible* if, for every element $a \in A$, the restriction $\mathcal{F}|X_a$ is a locally constant object of $\mathrm{Shv}(X_a)$. Here $\mathcal{F}|X_a$ denotes the image of $\mathcal{F}$ under the left adjoint to the pushforward functor $\mathrm{Shv}(X_a) \to \mathrm{Shv}(X)$.

We let $\mathrm{Shv}^A(X)$ denote the full subcategory of $\mathrm{Shv}(X)$ spanned by the A-constructible objects.

To ensure that the theory of A-constructible sheaves is well-behaved, we introduce a regularity condition on the stratification $X \to A$.

Definition A.5.3. Let A be a partially ordered set, and let $A^{\triangleleft}$ be the partially ordered set obtained by adjoining a new smallest element $-\infty$ to A. Let $f : X \to A$ be an A-stratified space. We define a new $A^{\triangleleft}$-stratified space $C(X)$ as follows:

(1) As a set $C(X)$ is given by the union $\{*\} \cup (X \times \mathbf{R}_{>0})$.

(2) A subset $U \subseteq C(X)$ is open if and only if $U \cap (X \times \mathbf{R}_{>0})$ is open, and if $* \in U$ then $X \times (0, \epsilon) \subseteq U$ for some positive real number ϵ.

(3) The $A^{\triangleleft}$-stratification of $C(X)$ is determined by the map $\overline{f} : C(X) \to A^{\triangleleft}$ such that $\overline{f}(*) = -\infty$ and $\overline{f}(x, t) = f(x)$ for $(x, t) \in X \times \mathbf{R}_{>0}$.

We will refer to $C(X)$ as the *open cone* on X.

Remark A.5.4. If the topological space X is compact and Hausdorff, then the open cone $C(X)$ is homeomorphic to the pushout $(X \times \mathbf{R}_{\geq 0}) \coprod_{X \times \{0\}} \{*\}$.

Definition A.5.5. Let A be a partially ordered set, let X be an A-stratified space, and let $x \in X_a \subseteq X$ be a point of X. We will say that X is *conically stratified at the point* x if there exists an $A_{>a}$-stratified topological space Y, a topological space Z, and an open embedding $Z \times C(Y) \hookrightarrow X$ of A-stratified spaces whose image U_x contains x. Here we regard $Z \times C(Y)$ as endowed with the A-stratification determined by the $A_{>a}^{\triangleleft} \simeq A_{\geq a}$-stratification of $C(Y)$.

We will say that X is *conically stratified* if it is conically stratified at every point $x \in X$.

Remark A.5.6. In Definition A.5.5, we do not require that the space Y itself be conically stratified.

Definition A.5.7. We will say that a partially ordered set A satisfies the *ascending chain condition* if every nonempty subset of A has a maximal element.

Remark A.5.8. Equivalently, A satisfies the ascending chain condition if there does not exist any infinite ascending sequence $a_0 < a_1 < \cdots$ of elements of A.

The main goal of this section is to prove the following somewhat technical convergence result concerning constructible sheaves:

Proposition A.5.9. *Let A be a partially ordered set, and let X be an A-stratified space. Assume that:*

(i) *The space X is paracompact and locally of singular shape.*

(ii) *The A-stratification of X is conical.*

(iii) *The partially ordered set A satisfies the ascending chain condition.*

Let $\mathcal{F} \in \mathrm{Shv}^A(X)$ be an A-constructible sheaf. Then the canonical map $\theta : \mathcal{F} \to \varprojlim \tau_{\leq n} \mathcal{F}$ is an equivalence. In particular, $\mathcal{F}$ is hypercomplete.

The proof of Proposition A.5.9 will require several preliminaries, and will be given at the end of this section. Our first step is to consider the case of a very simple stratification of X: namely, a decomposition of X into an open set and its closed complement. The following result is useful for working with constructible sheaves: it allows us to reduce global questions to questions which concern individual strata.

Lemma A.5.10. *Let $\mathcal{X}$ be an ∞-topos and U a (-1)-truncated object of $\mathcal{X}$. Let $i^* : \mathcal{X} \to \mathcal{X}/U$ and $j^* : \mathcal{X} \to \mathcal{X}_{/U}$ be the canonical geometric morphisms, j_* a right adjoint to j^*, and let $\overline{p} : K^{\triangleleft} \to \mathcal{X}$ be a small diagram in $\mathcal{X}$ indexed by a weakly contractible simplicial set K. Suppose that $i^*\overline{p}$, $j^*\overline{p}$, and $i^*j_*j^*\overline{p}$ are all limit diagrams. Then $\overline{p}$ is a limit diagram.*

Proof. Let $\mathcal{F}$ denote the image of the cone point of $K^{\triangleleft}$ under $\overline{p}$, let $p' : K \to \mathcal{X}$ be the constant diagram taking the value $\mathcal{F}$, and let $p = \overline{p}|K$. Then $\overline{p}$ determines a natural transformation of diagrams $\alpha : p' \to p$; we wish to prove that α induces an equivalence $\varprojlim(p') \to \varprojlim(p)$ in $\mathcal{X}$. For this, it suffices to show that for every object $V \in \mathcal{X}$, the induced map

$$\theta : \mathrm{Map}_{\mathcal{X}}(V, \varprojlim(p')) \to \mathrm{Map}_{\mathcal{X}}(V, \varprojlim(p))$$

is a homotopy equivalence. Replacing $\mathcal{X}$ by $\mathcal{X}_{/V}$, we can reduce to the case where V is the final object of $\mathcal{X}$. In this case, we let Γ denote the functor $\mathcal{X} \to \mathcal{S}$ corepresented by V (the functor of global sections).

Fix a point $\eta \in \Gamma(\varprojlim(p))$; we will show that the homotopy fiber of θ over η is contractible. Let j_* denote a right adjoint to j^*, let $q = j_* \circ j^* \circ p$, and let $q' = j_* \circ j^* \circ p'$. Then η determines a point $\eta_0 \in \Gamma(\varprojlim(q))$. Since $j^* \circ \overline{p}$ is a limit diagram (and the functor j_* preserves limits), the canonical map $\varprojlim(q') \to \varprojlim(q)$ is an equivalence, so we can lift η_0 to a point $\eta_1 \in \Gamma(\varprojlim(q'))$. This point determines a natural transformation from the constant diagram $c : K \to \mathcal{X}$ taking the value $V \simeq \mathbf{1}$ to the diagram q'. Let $p'_0 = c \times_{q'} p'$ and let $p_0 = c \times_q p$. We have a map of homotopy fiber sequences

$$\begin{array}{ccccc}
\Gamma(\varprojlim(p'_0)) & \longrightarrow & \Gamma(\varprojlim(p')) & \longrightarrow & \Gamma(\varprojlim(q')) \\
\downarrow{\scriptstyle\theta'} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta''} \\
\Gamma(\varprojlim(p_0)) & \longrightarrow & \Gamma(\varprojlim(p)) & \longrightarrow & \Gamma(\varprojlim(q)).
\end{array}$$

Here θ'' is a homotopy equivalence. Consequently, to prove that the homotopy fiber of θ is contractible, it will suffice to show that θ' is a homotopy equivalence.

By construction, the diagrams p'_0 and p_0 take values in the full subcategory $\mathcal{X}/U \subseteq \mathcal{X}$, so that the localization maps $p'_0 \to i^*p'_0$ and $p_0 \to i^*p_0$ are equivalences. It therefore suffices to show that the map $\Gamma(\varprojlim(i^*p'_0) \to \Gamma(\varprojlim(i^*p_0))$ is a homotopy equivalence. We have another map of homotopy fiber sequences

$$\begin{array}{ccccc}
\Gamma(\varprojlim(i^*p'_0)) & \longrightarrow & \Gamma(\varprojlim(i^*p')) & \longrightarrow & \Gamma(\varprojlim(i^*q')) \\
\downarrow{\scriptstyle\psi'} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi''} \\
\Gamma(\varprojlim(i^*p_0)) & \longrightarrow & \Gamma(i^* \varprojlim(p)) & \longrightarrow & \Gamma(\varprojlim(i^*q)).
\end{array}$$

The map ψ is a homotopy equivalence by virtue of our assumption that $i^*\overline{p}$ is a limit diagram, and the map ψ'' is a homotopy equivalence by virtue of our assumption that $i^*j_*j^*\overline{p}$ is a limit diagram. It follows that ψ' is also a homotopy equivalence, as desired. $\square$

Lemma A.5.11. *Let $\mathfrak{X}$ be an ∞-topos and U a (-1)-truncated object of $\mathfrak{X}$. Let $i^* : \mathfrak{X} \to \mathfrak{X}/U$ and $j^* : \mathfrak{X} \to \mathfrak{X}_{/U}$ be the canonical geometric morphisms, and let $\alpha : \mathfrak{F} \to \mathfrak{G}$ be a morphism in $\mathfrak{X}$. Suppose that $i^*(\alpha)$ and $j^*(\alpha)$ are equivalences. Then α is an equivalence.*

Proof. Apply Lemma A.5.10 in the special case where $K = \Delta^0$ (note that $i^* j_*$ automatically preserves j-indexed limits). $\qquad\square$

Lemma A.5.12. *Let X be a paracompact topological space, Y any topological space, V an open neighborhood of X in $X \times C(Y)$. Then there exists a continuous function $f : X \to (0, \infty)$ such that V contains*

$$V_f = \{(x, y, t) : t < f(x)\} \subseteq X \times Y \times (0, \infty) \subseteq X \times C(Y).$$

Proof. For each point $x \in X$, there exists a neighborhood U_x of x and a real number t_x such that $\{(x', y, t) : t < t_x \wedge x' \in U_x\} \subseteq V$. Since X is paracompact, we can choose a locally finite partition of unity $\{\psi_x\}_{x \in X}$ subordinate to the cover $\{U_x\}_{x \in X}$. We now define $f(y) = \sum_{x \in X} \psi_x(y) t_x$. $\qquad\square$

Remark A.5.13. In the situation of Lemma A.5.12, the collection of open sets of the form V_f is nonempty (take f to be a constant function) and stable under pairwise intersections ($V_f \cap V_g = V_{\inf\{f, g\}}$). The collection of such open sets is therefore cofinal in partially ordered set of all open subsets of $X \times C(Y)$ which contain X (ordered by reverse inclusion).

Lemma A.5.14. *Let X be a paracompact topological space. Let π denote the projection $X \times [0, \infty) \to X$, let j denote the inclusion $X \times (0, \infty) \hookrightarrow X \times [0, \infty)$, and let $\pi_0 = \pi \circ j$. Then the obvious equivalence $\pi_0^* \simeq j^* \pi^*$ is adjoint to an equivalence of functors $\alpha : \pi^* \to j_* \pi_0^*$ from $\mathrm{Shv}(X)$ to $\mathrm{Shv}(X \times [0, \infty))$.*

Proof. Let $\mathfrak{F} \in \mathrm{Shv}(X)$; we wish to prove that α induces an equivalence $\pi^* \mathfrak{F} \to j_* \pi_0^* \mathfrak{F}$. It is clear that this map is an equivalence when restricted to the open set $X \times (0, \infty)$. Let $i : X \to X \times [0, \infty)$ be the map induced by the inclusion $\{0\} \subseteq [0, \infty)$. By Corollary A.5.11, it will suffice to show that the map

$$\beta : \mathfrak{F} \simeq i^* \pi^* \mathfrak{F} \to i^* j_* \pi_0^* \mathfrak{F}$$

determined by α is an equivalence. Let U be an open F_σ subset of X; we will show that the map $\beta_U : \mathfrak{F}(U) \to (i^* j_* \pi_0^* \mathfrak{F})(U)$ is a homotopy equivalence. Replacing X by U, we can assume that $U = X$.

According to Corollary T.7.1.5.6, we can identify $(i^* j_* \pi_0^* \mathfrak{F})(X)$ with the colimit $\varinjlim_{V \in S}(j_* \pi_0^* \mathfrak{F})(V) \simeq \varinjlim_{V \in S}(\pi_0^* \mathfrak{F})(V - X)$, where V ranges over the collection S of all open neighborhoods of $X = X \times \{0\}$ in $X \times [0, \infty)$. Let $S' \subseteq S$ be the collection of all open neighborhoods of the form $V_f = \{(x, t) : t < f(x)\}$, where $f : X \to (0, \infty)$ is a continuous function (see Lemma A.5.12). In view of Remark A.5.13, we have an equivalence $\varinjlim_{V \in S}(\pi_0^* \mathfrak{F})(V - X) \simeq \varinjlim_{V \in S'}(\pi_0^* \mathfrak{F})(V - X)$. Since S' is a filtered partially ordered set (when ordered by reverse inclusion), to prove that β_X is an equivalence it suffices to show that the pullback map $\mathfrak{F}(X) \to (\pi_0^* \mathfrak{F})(V_f - X)$ is a homotopy equivalence, for every continuous map $f : X \to (0, \infty)$. Division by f determines a homeomorphism $V_f - X \to X \times (0, 1)$, and the desired result follows from Lemma A.2.9. $\quad\square$

Lemma A.5.15. *Let X be a paracompact topological space of the form $Z \times C(Y)$, and consider the (non-commuting) diagram*

$$
\begin{array}{ccccc}
Z \times Y \times (0, \infty) & \xrightarrow{\;j\;} & Z \times Y \times [0, \infty) & \xrightarrow{\;k\;} & X \\
& \searrow{\scriptstyle \pi_0} & \downarrow{\scriptstyle \pi} & & \uparrow{\scriptstyle i} \\
& & Z \times Y & \xrightarrow{\;\psi\;} & Z.
\end{array}
$$

Let i' denote the inclusion $Z \times Y \to Z \times Y \times [0, \infty)$ given by $\{0\} \hookrightarrow [0, \infty)$. Assume that X is paracompact. Then:

(i) *The canonical map $\alpha : \pi^* \to j_* \pi_0^*$ is an equivalence of functors from $\mathrm{Shv}(Z \times Y)$ to $\mathrm{Shv}(Z \times Y \times [0, \infty))$.*

(ii) *Let $\beta : \pi^* \to i'_*$ be the natural transformation adjoint to the equivalence $i'^*\pi^* \simeq \mathrm{id}_{\mathrm{Shv}(Z \times Y)}$. Then the natural transformation*

$$\gamma : i^* k_* \pi^* \xrightarrow{\beta} i^* k_* i'_* \simeq i^* i_* \psi_* \to \psi_*$$

is an equivalence of functors from $\mathrm{Shv}(Z \times Y)$ to $\mathrm{Shv}(Z)$.

(iii) *The functor $i^* j_* \pi_0^*$ is equivalent to ψ_*.*

Proof. Note that $Z \times Y \simeq Z \times Y \times \{1\}$ can be identified with a closed subset of X, and is therefore paracompact. Consequently, assertion (i) follows from Lemma A.5.14. Assertion (iii) follows immediately from (i) and (ii). It will therefore suffice to prove (ii).

Since Z can be identified with a closed subset of X, it is paracompact. Let $\mathcal{F} \in \mathrm{Shv}(Z \times Y)$, and let U be an open F_σ subset of Z. We will show that γ induces a homotopy equivalence $(i^* k_* \pi^* \mathcal{F})(U) \to (\psi_* \mathcal{F})(U)$. Shrinking Z if necessary, we may suppose that $Z = U$. The right hand side can be identified with $\mathcal{F}(Z \times Y)$, while the left hand side is given (by virtue of Corollary T.7.1.5.6) by the colimit $\varinjlim_{V \in S}(\pi^* \mathcal{F})(k^{-1} V)$, where V ranges over partially ordered set S of open subsets of $Z \times C(Y)$ which contain Z. By virtue of Remark A.5.13, we can replace S by the cofinal subset S' consisting of open sets of the form $V = V_f$, where $f : Z \to (0, \infty)$ is a continuous function (see Lemma A.5.12). Since S' is filtered, it will suffice to show that each of the maps $(\pi^* \mathcal{F})(k^{-1} V) \to \mathcal{F}(Z \times Y)$ is an equivalence. Division by f allows us to identify $(\pi^* \mathcal{F})(k^{-1} V)$ with $(\pi^* \mathcal{F})(Z \times Y \times [0, 1))$, and the desired result now follows from Variant A.2.10 on Lemma A.2.9. $\qquad\square$

Lemma A.5.16. *Let X be a paracompact space equipped with a conical A-stratification. Then every point $x \in X_a$ admits a open F_σ neighborhood V which is homeomorphic (as an A-stratified space) to $Z \times C(Y)$, where Y is some $A_{>a}$-stratified space.*

Proof. Since the stratification of X is conical, there exists an open neighborhood U of x which is homeomorphic (as an A-stratified space) to $Z \times C(Y)$, where Y is some $A_{>a}$-stratified space. The open set U need not be paracompact. However, there exists a smaller open set $U' \subseteq U$ containing x such that U' is an F_σ subset of X, and therefore paracompact. let $Z' = U' \cap Z$. Then Z' is a closed subset of the paracompact space U', and therefore paracompact. Replacing Z by Z', we can assume that Z is paracompact. Applying Lemma A.5.12, we deduce that there exists a continuous function $f : Z \to (0, \infty)$ such that $V_f \subseteq U$ (see Lemma A.5.12 for an explanation of this notation). The set V_f is the union of the closures in U' of the open sets $\{V_{\frac{n}{n+1}f}\}_{n>0}$. It is therefore an open F_σ subset of U' (and so also an F_σ subset of the space X). We conclude by observing that V_f is again homeomorphic to the product $Z \times C(Y)$. $\qquad\square$

Remark A.5.17. If A is a partially ordered set satisfying the ascending chain condition, then we can define an ordinal-valued rank function rk on A. The function rk is uniquely determined by the following requirement: for every element $a \in A$, the rank $\mathrm{rk}(a)$ is the smallest ordinal not of the form $\mathrm{rk}(b)$, where $b > a$. More generally, suppose that X is an A-stratified topological space. We define the *rank* of X to be the supremum of the set of ordinals $\{\mathrm{rk}(a) : X_a \neq \emptyset\}$.

Remark A.5.18. Let X be a paracompact topological space of the form $Z \times C(Y)$. Then Z is paracompact (since it is homeomorphic to a closed subset of X). Suppose that X has singular shape. Since the inclusion $Z \hookrightarrow X$ is a homotopy equivalence, we deduce also that Z has singular shape (Remark A.4.11). The same argument shows that if X is locally of singular shape, then Z is locally of singular shape.

Proof of Proposition A.5.9. The assertion that $\theta : \mathcal{F} \to \varprojlim \tau_{\leq n} \mathcal{F}$ is an equivalence is local on X. It will therefore suffice to prove that every point $x \in X_a$ admits an open F_σ neighborhood U such that θ is an equivalence over U. Since A satisfies the ascending chain condition, we may assume without loss of generality that the same result holds for every point $x' \in X_{>a}$. Using Lemma A.5.16, we may assume without loss of generality that U is a paracompact open set of the form $Z \times C(Y)$, where Y is some $A_{>a}$-stratified space.

Let $i : Z \to Z \times C(Y)$ and $j : Z \times Y \times (0, \infty) \to Z \times C(Y)$ denote the inclusion maps. According to Lemma A.5.10, it will suffice to verify the following:

(a) The canonical map $i^* \mathcal{F} \to \varprojlim i^* \tau_{\leq n} \mathcal{F} \simeq \varprojlim \tau_{\leq n} i^* \mathcal{F}$ is an equivalence.

(b) The canonical map $j^* \mathcal{F} \to \varprojlim j^* \tau_{\leq n} \mathcal{F} \simeq \varprojlim \tau_{\leq n} j^* \mathcal{F}$ is an equivalence.

(c) The canonical map $i^* j_* j^* \mathcal{F} \to \varprojlim i^* j_* j^* \tau_{\leq n} \mathcal{F}$ is an equivalence.

Assertion (a) follows from Corollary A.1.17 (note that Z is locally of singular shape by Remark A.5.18), and assertion (b) follows from the inductive hypothesis. To prove (c), let $\pi : Z \times Y \times (0, \infty)$ denote the projection. Using the inductive hypothesis, we deduce that $j^* \mathcal{F}$ is hypercomplete. Since each fiber $\{z\} \times \{y\} \times (0, \infty)$ is contained in a stratum of X, we deduce that $j^* \mathcal{F}$ is foliated, so that the counit map $\pi^* \pi_* j^* \mathcal{F} \to j^* \mathcal{F}$ is an equivalence. The same reasoning shows that $\pi^* \pi_* j^* \tau_{\leq n} \mathcal{F} \to j^* \tau_{\leq n} \mathcal{F}$ is an equivalence for each $n \geq 0$. Consequently, (c) is equivalent to the assertion that the canonical map

$$i^* j_* \pi^* \mathcal{G} \to \varprojlim i^* j_* \pi^* \mathcal{G}_n$$

is an equivalence, where $\mathcal{G} = \pi_* j^* \mathcal{F}$ and $\mathcal{G}_n = \pi_* j^* \tau_{\leq n} \mathcal{F}$. Since the functor π_* preserves limits, the canonical map $\mathcal{G} \to \varprojlim \mathcal{G}_n$ is an equivalence by virtue of (b). The desired result now follows from the fact that the functor $i^* j_* \pi^*$ is equivalent to π_*, and therefore preserves limits (Lemma A.5.15). □

Remark A.5.19. Let X be a paracompact topological space equipped with a conical A-stratification, where A is a partially ordered set which satisfies the ascending chain condition. Suppose that each stratum X_a is locally of singular shape. Then X is locally of singular shape. To prove this, it suffices to show that X has a covering by open sets which are locally of singular shape (Remark A.4.16). Using Lemma A.5.16, we may reduce to the case where $X = Z \times C(Y)$, where Y is some $A_{>a}$-stratified space and $Z \times C(Y)$ is endowed with the induced $A_{\geq a}$-stratification. Working by induction on a, we may suppose that $X - Z \simeq Z \times Y \times (0, \infty)$ is locally of singular shape. Let U be an open F_σ subset of X and let $U_0 = U \cap Z$. We wish to prove that U is locally of singular shape. Using Lemma A.5.12, we deduce that there exists a continuous map $f : U_0 \to (0, \infty)$ such that U contains the open set $V_f = U_0 \cup \{(z, y, t) \in U_0 \times Y \times (0, \infty) : t < f(z)\}$. Then U is covered by the open subsets V_f and $U - U_0$. According to Lemma A.4.14, it suffices to show that V_f, $U - U_0$, and $V_f \cap (U - U_0)$ are of singular shape. The open sets $U - U_0$ and $V_f \cap (U - U_0)$ belong to $X_{>a}$ and are therefore of singular shape by the inductive hypothesis. The open set V_f is homotopy equivalent to U_0, and thus has singular shape by virtue of our assumption that X_a is locally shapely (Remark A.4.11).

A.6 ∞-Categories of Exit Paths

If X is a sufficiently nice topological space, then Theorem A.4.19 guarantees that the ∞-category of locally constant sheaves on X can be identified with the ∞-category $\mathcal{S}_{/\operatorname{Sing}(X)} \simeq \operatorname{Fun}(\operatorname{Sing}(X), \mathcal{S})$. Roughly speaking, we can interpret a sheaf $\mathcal{F}$ on X as a functor which assigns to each $x \in X$ the stalk $\mathcal{F}_x \in \mathcal{S}$, and to each path $p : [0, 1] \to X$ joining $x = p(0)$ to $y = p(1)$ the homotopy equivalence $\mathcal{F}_x \simeq \mathcal{F}_y$ given by transport along p (see §A.2).

Suppose now that $\mathcal{F}$ is a sheaf on X which is *not* locally constant. In this case, a path $p : [0, 1] \to X$ from $x = p(0)$ to $y = p(1)$ does not necessarily define a transport map $\mathcal{F}_x \to \mathcal{F}_y$. However, every point η_0 in the stalk $\mathcal{F}_x$ can be lifted to a section of $\mathcal{F}$ over some neighborhood of x, which determines points $\eta_t \in \mathcal{F}_{p(t)}$ for t sufficiently small. If we assume that $p^* \mathcal{F}$ is locally constant on the half-open interval $(0, 1]$, then each η_t can be transported to a point in the stalk $\mathcal{F}_y$, and we should again expect to obtain a well-defined map $\mathcal{F}_x \to \mathcal{F}_y$. For example, suppose that $\mathcal{F}$ is a sheaf which is locally constant when restricted to some closed subset $X_0 \subseteq X$, and also when restricted to the open set $X - X_0$. In this case, the above analysis should apply whenever $p^{-1} X_0 = \{0\}$: that is, whenever p is a path which is *exiting* the closed subset $X_0 \subseteq X$. Following a proposal of MacPherson, we might try to identify $\mathcal{F}$ with an $\mathcal{S}$-valued functor defined on some subset of the Kan complex $\operatorname{Sing}(X)$, which allows paths to travel from X_0 to $X - X_0$ but not vice-versa.

Our objective in this section is to introduce a simplicial subset $\operatorname{Sing}^A(X)$ associated to any stratification $f : X \to A$ of a topological space X by a partially ordered set A. Our main result, Theorem A.6.4, asserts

that $\mathrm{Sing}^A(X)$ is an ∞-category provided that the stratification of X is conical (Definition A.5.5). In this case, we will refer to $\mathrm{Sing}^A(X)$ as the ∞-*category of exit paths in X with respect to the stratification $X \to A$*. In §A.9, we will show that (under suitable hypotheses) the ∞-category of A-constructible sheaves on X is equivalent to the ∞-category of functors $\mathrm{Fun}(\mathrm{Sing}^A(X), \mathcal{S})$.

Remark A.6.1. The exit path ∞-category $\mathrm{Sing}^A(X)$ can be regarded as an ∞-categorical generalization of the 2-category of exit paths constructed in [153].

Definition A.6.2. Let A be a partially ordered set, and let X be a topological space equipped with an A-stratification $f : X \to A$. We $\mathrm{Sing}^A(X) \subseteq \mathrm{Sing}(X)$ to be the simplicial subset consisting of those n-simplices $\sigma : |\Delta^n| \to X$ which satisfy the following condition:

(∗) Let $|\Delta^n| = \{(t_0, \dots, t_n) \in [0,1]^{n+1} : t_0 + \dots + t_n = 1\}$. Then there exists a chain $a_0 \leq \dots \leq a_n$ of elements of A such that for each point $(t_0, \dots, t_i, 0, \dots, 0) \in |\Delta^n|$ where $t_i \neq 0$, we have $f(\sigma(t_0, \dots, t_n)) = a_i$.

Remark A.6.3. Let A be a partially ordered set, regarded as a topological space as in Definition A.5.1. Then there is a natural map of simplicial sets $\mathrm{N}(A) \to \mathrm{Sing}(A)$, which carries an n-simplex $(a_0 \leq \dots \leq a_n)$ of $\mathrm{N}(A)$ to the map $\sigma : |\Delta^n| \to A$ characterized by the formula

$$\sigma(t_0, \dots, t_i, 0, \dots, 0) = a_i$$

whenever $t_i > 0$. For any A-stratified topological space X, the simplicial set $\mathrm{Sing}^A(X)$ can be described as the fiber product $\mathrm{Sing}(X) \times_{\mathrm{Sing}(A)} \mathrm{N}(A)$. In particular, there is a canonical map of simplicial sets $\mathrm{Sing}^A(X) \to \mathrm{N}(A)$.

We can now state our main result as follows:

Theorem A.6.4. *Let A be a partially ordered set, and let X be a conically A-stratified topological space. Then:*

(1) *The projection $\mathrm{Sing}^A(X) \to \mathrm{N}(A)$ is an inner fibration of simplicial sets.*

(2) *The simplicial set $\mathrm{Sing}^A(X)$ is an ∞-category.*

(3) *A morphism in $\mathrm{Sing}^A(X)$ is an equivalence if and only if its image in $\mathrm{N}(A)$ is degenerate (in other words, if and only if the underlying path $[0,1] \to X$ is contained in a single stratum).*

Remark A.6.5. In the situation of Theorem A.6.4, we will refer to the ∞-category $\mathrm{Sing}^A(X)$ as the ∞-*category of A-stratified exit paths in X* or simply as the ∞-*category of exit paths in X* if the stratification of X is clear from context.

Proof. The implication (1) $\Rightarrow$ (2) is obvious. The "only if" direction of (3) is clear (since any equivalence in $\mathrm{Sing}^A(X)$ must project to an equivalence in $\mathrm{N}(A)$), and the "if" direction follows from the observation that each fiber $\mathrm{Sing}^A(X) \times_{\mathrm{N}(A)} \{a\}$ is isomorphic to the Kan complex $\mathrm{Sing}(X_a)$. It will therefore suffice to prove (1). Fix $0 < i < n$; we wish to prove that every lifting problem of the form

$$
\begin{array}{ccc}
\Lambda^n_i & \xrightarrow{\sigma_0} & \mathrm{Sing}^A(X) \\
\downarrow & \overset{\sigma}{\nearrow} & \downarrow \\
\Delta^n & \longrightarrow & \mathrm{N}(A)
\end{array}
$$

admits a solution.

The map $\Delta^n \to \mathrm{N}(A)$ determines a chain of elements $a_0 \leq a_1 \leq \dots \leq a_n$. Without loss of generality, we may replace A by $A' = \{a_0, \dots, a_n\}$ and X by $X \times_A A'$. We may therefore assume that A is a finite nonempty linearly ordered set. We now work by induction on the number of elements of A. If A has only a single element, then $\mathrm{Sing}^A(X) = \mathrm{Sing}(X)$ is a Kan complex and there is nothing to prove. Otherwise, there exists some integer $p < n$ such that $a_p = a_0$ and $a_{p+1} \neq a_0$. There are two cases to consider.

(a) Suppose that $p < i < n$. Let $q = n - p - 1$ and let $j = i - p - 1$, so that we have isomorphisms of simplicial sets

$$\Delta^n \simeq \Delta^p \star \Delta^q \qquad \Lambda_i^n \simeq (\Delta^p \star \Lambda_{q'}^q) \coprod_{\partial \Delta^p \star \Lambda_j^q} (\partial \Delta^p \star \Delta^q).$$

We will use the first isomorphism to identify $|\Delta^n|$ with the pushout

$$|\Delta^p| \coprod_{|\Delta^p| \times |\Delta^q| \times \{0\}} (|\Delta^p| \times |\Delta^q| \times [0,1]) \coprod_{|\Delta^p| \times |\Delta^q| \times \{1\}} |\Delta^q|.$$

Let $K \subseteq |\Delta^p| \times |\Delta^q|$ be the union of the closed subsets $|\partial \Delta^p| \times |\Delta^q|$ and $|\Delta^p| \times |\Lambda_j^q|$, so that $|\Lambda_i^n|$ can be identified with the pushout

$$|\Delta^p| \coprod_{|\Delta^p| \times |\Delta^q| \times \{0\}} (K \times [0,1]) \coprod_{|\Delta^p| \times |\Delta^q| \times \{1\}} |\Delta^q|.$$

Let $K' \subseteq |\Delta^p| \times |\Delta^q| \times [0,1]$ be the union of $K \times [0,1]$ with $|\Delta^p| \times |\Delta^q| \times \{0,1\}$. Then σ_0 determines a continuous map $F_0 : K' \to X$. To construct the map σ, we must extend F_0 to a map $F : |\Delta^p| \times |\Delta^q| \times [0,1] \to X$ satisfying the following condition: for every point $s \in (|\Delta^p| \times |\Delta^q| \times [0,1]) - K'$, we have $F(s) \in X_{a_n}$.

Let $F_- : |\Delta^p| \to X_{a_0}$ be the map obtained by restricting F_0 to $|\Delta^p| \times |\Delta^q| \times \{0\}$. For every point $x \in X_{a_0}$, choose an open neighborhood $U_x \subseteq X$ as in Definition A.5.5. Choose a triangulation of the simplex $|\Delta^p|$ with the following property: for every simplex τ of the triangulation, the image $F_-(\tau)$ is contained in some U_x. Refining our triangulation if necessary, we may assume that $|\partial \Delta^p|$ is a subcomplex of $|\Delta^p|$. For every subcomplex L of $|\Delta^p|$ which contains $|\partial \Delta^p|$, we let $K_L \subseteq |\Delta^p| \times |\Delta^q|$ denote the union of the closed subsets $L \times |\Delta^q|$ and $|\Delta^p| \times |\Lambda_j^q|$ and $K_L' \subseteq |\Delta^p| \times |\Delta^q| \times [0,1]$ denote the union of the closed subsets $K_L \times [0,1]$ and $|\Delta^p| \times |\Delta^q| \times \{0,1\}$. We will show that F_0 can be extended to a continuous map $F_L : K_L' \to X$ (satisfying the condition that $F_L(s) \in X_{a_n}$ for $s \notin K'$), using induction on the number of simplices of L. If $L = |\partial \Delta^p|$, there is nothing to prove. Otherwise, we may assume without loss of generality that $L = L_0 \cup \tau$, where L_0 is another subcomplex of $|\Delta^p|$ containing $|\partial \Delta^p|$ and τ is a simplex of L such that $\tau \cap L_0 = \partial \tau$. The inductive hypothesis guarantees the existence of a map $F_{L_0} : K_{L_0}' \to X$ with the desired properties.

Let $K_\tau \subseteq \tau \times |\Delta^q|$ be the union of the closed subsets $\partial \tau \times |\Delta^q|$ and $\tau \times |\Lambda_j^q|$, and let $K_\tau' \subseteq \tau \times |\Delta^q| \times [0,1]$ be the union of the closed subsets $K_\tau \times [0,1]$ and $\tau \times |\Delta^q| \times \{0,1\}$. The map F_{L_0} restricts to a map $G_0 : K_\tau' \to X$. To construct F_L, it will suffice to extend G_0 to a continuous map $G : \tau \times |\Delta^q| \times [0,1] \to X$ (satisfying the condition that $G(s) \in X_{a_n}$ for $s \notin K_\tau'$).

By assumption, the map G_0 carries $\tau \times |\Delta^q| \times \{0\}$ into an open subset U_x, for some $x \in X_{a_0}$. Let $U = U_x$, and choose a homeomorphism $U \simeq Z \times C(Y)$, where Y is an $A_{>a_0}$-stratified space. Since $\tau \times |\Delta^q|$ is compact, we deduce that $G_0(\tau \times |\Delta^q| \times [0,r]) \subseteq U$ for some real number $0 < r < 1$. Let $X' = X - X_{a_0}$ and let $A' = A - \{a_0\}$, so that X' is an A'-stratified space. Let m be the dimension of the simplex τ. The restriction $G_0|(\tau \times |\Delta^q| \times \{1\})$ determines a map of simplicial sets $\overline{h}_1 : \Delta^m \times \Delta^q \to \mathrm{Sing}^{A'}(X')$. Let J denote the simplicial set $(\partial \Delta^m \times \Delta^q) \coprod_{\partial \Delta^m \times \Lambda_j^q} (\Delta^m \times \Lambda_j^q)$. The restriction of G_0 to $K_\tau \times [r,1]$ determines another map of simplicial sets $h : J \times \Delta^1 \to \mathrm{Sing}^{A'}(X')$, which is a natural transformation from $h_0 = h|(J \times \{0\})$ to $h_1 = h|(J \times \{1\}) = \overline{h}_1|J$. It follows from the inductive hypothesis that $\mathrm{Sing}^{A'}(X')$ is an ∞-category, and (using (3)) that natural transformation h is an equivalence. Consequently, we can lift h to an equivalence $\overline{h} : \overline{h}_0 \to \overline{h}_1$ in $\mathrm{Fun}(J, \mathrm{Sing}^{A'}(X'))$. This morphism determines a continuous map $G_+ : \tau \times |\Delta^q| \times [r,1] \to X$ which agrees with G_0 on $(\tau \times |\Delta^q| \times [r,1]) \cap K_\tau'$.

Let us identify $|\Delta^q|$ with the set of tuples of real numbers $\vec{t} = (t_0, t_1, \ldots, t_q)$ such that $0 \leq t_k \leq 1$ and $t_0 + \cdots + t_q = 1$. In this case, we let $d(\vec{t}) = \inf\{t_k : k \neq j\}$: note that $d(\vec{t}) = 0$ if and only if $\vec{t} \in |\Lambda_j^q|$.

If u is a real number satisfying $0 \leq u \leq d(\vec{t})$, we let $\vec{t}_u$ denote the tuple

$$(t_0 - u, t_1 - u, \ldots, t_{j-1} - u, t_j + qu, t_{j+1-u} - u, \ldots, t_q - u) \in |\Delta^q|.$$

Choose a continuous function $d' : \tau \to [0,1]$ which vanishes on $\partial \tau$ and is positive on the interior of $\mathbf{R}$. For every positive real number v, let $c_v : \tau \times |\Delta^q| \times [r,1] \to \tau \times |\Delta^q| \times [r,1]$ given by the formula

$$c_v(s, \vec{t}, r') = (s, \vec{t}_{d(\vec{t})} \frac{vd'(s)(1 - r')}{1 + vd'(s)(1 - r')}, r'),$$

and let G_+^v denote the composition $G_+ \circ c_v$. Since G_+ agrees with G_0 on $K_\tau \times \{r\}$, it carries $K_\tau \times \{r\}$ into U. By continuity, there exists a neighborhood V of K_τ in $\tau \times |\Delta^q|$ such that $G_+(V \times \{r\}) \subseteq U$. If the real number v is sufficiently large, then $c_v(\tau \times |\Delta^q| \times \{r\}) \subseteq V$, so that $G_+^v(\tau \times |\Delta^q| \times \{r\}) \subseteq U$. Replacing G_+ by G_+^v, we may assume that $G_+(\tau \times |\Delta^q| \times \{r\}) \subseteq U$ (here we invoke the assumption that $j < q$ to guarantee that G_+ continues to satisfy the requirement that $G_+(s, \vec{t}, r') \in X_{a_n}$ whenever $\vec{t} \notin |\Lambda_j^q|$).

Let $X'' = U - X_{a_0} \simeq Z \times Y \times \mathbf{R}_{>0}$. The A'-stratification of X' restricts to a (conical) A'-stratification of X''. Let $g : \tau \times |\Delta^q| \times \{r\} \to X''$ be the map obtained by restricting G_+. Then g determines a map of simplicial sets $\overline{\phi}_0 : \Delta^m \times \Delta^q \to \mathrm{Sing}^{A'}(X'')$. Let I denote the simplicial set

$$\Delta^{\{0,1\}} \coprod_{\{1\}} \Delta^{\{1,2\}} \coprod_{\{2\}} \Delta^{\{2,3\}} \coprod_{\{3\}} \cdots,$$

and identify the geometric realization $|I|$ with the open interval $(0, r]$. Then G_0 determines a map of simplicial sets $J \times I \to \mathrm{Sing}^{A'}(X'')$, which we can identify with a sequence of maps $\phi_0, \phi_1, \ldots \in \mathrm{Fun}(J, \mathrm{Sing}^{A'}(X''))$ together with natural transformations $\phi_0 \to \phi_1 \to \ldots$. We note that $\phi_0 = \overline{\phi}_0 | J$. The inductive hypothesis guarantees that $\mathrm{Sing}^{A'}(X'')$ is an ∞-category, and assertion (3) ensures that each of the natural transformations $\phi_k \to \phi_{k+1}$ is an equivalence. It follows that we can lift these natural transformations to obtain a sequence of equivalences

$$\overline{\phi}_0 \to \overline{\phi}_1 \to \overline{\phi}_2 \to \cdots$$

in the ∞-category $\mathrm{Fun}(\Delta^m \times \Delta^q, \mathrm{Sing}^{A'}(X''))$. This sequence of equivalences is given by a map of simplicial sets $\Delta^m \times \Delta^q \times I \to \mathrm{Sing}^{A'}(X'')$, which we can identify with a continuous map $\tau \times |\Delta^q| \times (0, r] \to Z \times Y \times \mathbf{R}_{>0}$. Let $y : \tau \times |\Delta^q| \times (0, r] \to Y$ be the projection of this map onto the second fiber. We observe that G_+ and G_0 together determine a map $(K_\tau \times [0, r]) \coprod_{K_\tau \times \{0,r\}} (\tau \times |\Delta^q| \times \{0, r\}) \to X'$. Let z denote the composition of this map with the projection $U \to Z \times \mathbf{R}_{\geq 0}$. Since the domain of z is a retract of $\tau \times |\Delta^q| \times [0, r]$, we can extend z to a continuous map $\overline{z} : \tau \times |\Delta^q| \times [0, r] \to Z \times \mathbf{R}_{\geq 0}$. Let $\overline{z}_1 : \tau \times |\Delta^q| \times [0, r] \to \mathbf{R}_{\geq 0}$ be obtained from $\overline{z}$ by projection onto the second factor. By adding to $\overline{z}_1$ a function which vanishes on $(K_\tau \times [0, r]) \coprod_{K_\tau \times \{0,r\}} (\tau \times |\Delta^q| \times \{0, r\})$ and is positive elsewhere, we can assume that $\overline{z}_1^{-1}\{0\} = \tau \times |\Delta^q| \times \{0\}$. Let $G_- : \tau \times |\Delta^q| \times [0, r] \to U \simeq Z \times C(Y)$ be the map which is given by $\overline{z}$ on $\tau \times |\Delta^q| \times \{0\}$ and by the pair $(\overline{z}, y)$ on $\tau \times |\Delta^q| \times (0, r]$. Then G_- and G_+ together determine an extension $G : \tau \times |\Delta^q| \times [0, 1] \to X$ of G_0 with the desired properties.

(b) Suppose now that $0 < i \leq p$. The proof proceeds as in case (a) with some minor changes. We let $q = n - p - 1$ as before, so that we have an identification of $|\Delta^n|$ with the pushout

$$|\Delta^p| \coprod_{|\Delta^p| \times |\Delta^q| \times \{0\}} (|\Delta^p| \times |\Delta^q| \times [0, 1]) \coprod_{|\Delta^p| \times |\Delta^q| \times \{1\}} |\Delta^q|.$$

Let $K \subseteq |\Delta^p| \times |\Delta^q|$ be the union of the closed subsets $|\Lambda_i^p| \times |\Delta^q|$ and $|\Delta^p| \times |\partial \Delta^q|$, so that $|\Lambda_i^n|$ can be identified with the pushout

$$|\Delta^p| \coprod_{|\Delta^p| \times |\Delta^q| \times \{0\}} (K \times [0, 1]) \coprod_{|\Delta^p| \times |\Delta^q| \times \{1\}} |\Delta^q|.$$

Let $K' \subseteq |\Delta^p| \times |\Delta^q| \times [0,1]$ be the union of $K \times [0,1]$ with $|\Delta^p| \times |\Delta^q| \times \{0,1\}$. Then σ_0 determines a continuous map $F_0 : K' \to X$. To construct the map σ, we must extend F_0 to a map $F : |\Delta^p| \times |\Delta^q| \times [0,1] \to X$ satisfying the following condition: for every point $s \in (|\Delta^p| \times |\Delta^q| \times [0,1]) - K'$, we have $F(s) \in X_{a_n}$.

We observe that there is a homeomorphism of $|\Delta^p|$ with $|\Delta^{p-1}| \times [0,1]$ which carries $|\Lambda_i^p|$ to $|\Delta^{p-1}| \times \{0\}$. Let $F_- : |\Delta^{p-1}| \times [0,1] \to X_{a_0}$ be the map determined by σ_0 together with this homeomorphism. For every point $x \in X_{a_0}$, choose an open neighborhood $U_x \subseteq X$ as in Definition A.5.5. Choose a triangulation of the simplex $|\Delta^{p-1}|$ and a large positive integer N so that the following condition is satisfied: for every simplex τ of $|\Delta^{p-1}|$ and every nonnegative integer $k < N$, the map F_- carries $\tau \times [\frac{k}{N}, \frac{k+1}{N}]$ into some U_x. For every subcomplex L of $|\Delta^{p-1}|$, we let $K_L \subseteq |\Delta^p| \times |\Delta^q|$ denote the union of the closed subsets $L \times [0,1] \times |\Delta^q|$, $|\Delta^{p-1}| \times \{0\} \times |\Delta^q|$, and $|\Delta^{p-1}| \times [0,1] \times |\partial \Delta^q|$. Let $K_L' \subseteq |\Delta^p| \times |\Delta^q| \times [0,1]$ denote the union of the closed subsets $K_L \times [0,1]$ and $|\Delta^p| \times |\Delta^q| \times \{0,1\}$. We will show that F_0 can be extended to a continuous map $F_L : K_L' \to X$ (satisfying the condition that $F_L(s) \in X_{a_n}$ for $s \notin K'$), using induction on the number of simplices of L. If L is empty there is nothing to prove. Otherwise, we may assume without loss of generality that $L = L_0 \cup \tau$, where τ is a simplex of $|\Delta^{p-1}|$ such that $\tau \cap L_0 = \partial \tau$. The inductive hypothesis guarantees the existence of a map $F_{L_0} : K_{L_0}' \to X$ with the desired properties.

For $0 \leq k \leq N$, let $K_{\tau,k} \subseteq \tau \times [0,1] \times |\Delta^q|$ be the union of the closed subsets $\partial \tau \times [0,1] \times |\Delta^q|$, $\tau \times [0, \frac{k}{N}] \times |\Delta^q|$, and $\tau \times [0,1] \times |\partial \Delta^q|$. Let $K_{\tau,k}' \subseteq \tau \times [0,1] \times |\Delta^q| \times [0,1]$ be the union of the closed subsets $K_{\tau,k} \times [0,1]$ and $\tau \times |\Delta^q| \times \{0,1\}$. The map F_{L_0} restricts to a map $F[0] : K_{\tau,0}' \to X$. To construct F_L, it will suffice to extend G_0 to a continuous map $F[N] : K_{\tau,N} \times [0,1] \to X$ (satisfying the condition that $F[n](s) \in X_{a_n}$ for $s \notin K_\tau'$). We again proceed by induction, constructing maps $F[k] : K_{\tau,k}' \to X$ for $k \leq N$ using recursion on k. Assume that $k > 0$ and that $F[k-1]$ has already been constructed.

Let $\overline{\tau}$ denote the prism $\tau \times [\frac{k-1}{N}, \frac{k}{N}]$, and let $\overline{\tau}_0$ denote the closed subset of $\overline{\tau}$ which is the union of $\partial \tau \times [\frac{k-1}{N}, \frac{k}{N}]$ with $\tau \times \{\frac{k-1}{N}\}$. Let $K_{\overline{\tau}} \subseteq \overline{\tau} \times |\Delta^q|$ denote the union of the closed subsets $\overline{\tau} \times |\partial \Delta^q|$ and $\overline{\tau}_0 \times |\Delta^q|$. Let $K_{\overline{\tau}}' \subseteq \overline{\tau} \times |\Delta^q| \times [0,1]$ be the union of the closed subsets $K_{\overline{\tau}} \times [0,1]$ with $\overline{\tau} \times |\Delta^q| \times \{0,1\}$. Then $F[k-1]$ determines a map $G_0 : K_{\overline{\tau}}' \to X$. To find the desired extension $F[k]$ of $F[k-1]$, it will suffice to prove that G_0 admits a continuous extension $G : \overline{\tau} \times |\Delta^q| \times [0,1]$ (again satisfying the condition that $G(s) \in X_{a_n}$ whenever $s \notin K_{\overline{\tau}}'$).

By assumption, the map G_0 carries $\overline{\tau} \times |\Delta^q| \times \{0\}$ into an open subset U_x, for some $x \in X_{a_0}$. Let $U = U_x$, and choose a homeomorphism $U \simeq Z \times C(Y)$, where Y is an $A_{>a_0}$-stratified space. Since $\tau \times |\Delta^q|$ is compact, we deduce that $G_0(\overline{\tau} \times |\Delta^q| \times [0,r]) \subseteq U$ for some real number $0 < r < 1$. Let $X' = X - X_{a_0}$ and let $A' = A - \{a_0\}$, so that X' is an A'-stratified space. Let m be the dimension of the simplex τ. The restriction $G_0|(\overline{\tau} \times |\Delta^q| \times \{1\})$ determines a map of simplicial sets $\overline{h}_1 : \Delta^m \times \Delta^1 \times \Delta^q \to \mathrm{Sing}^{A'}(X')$. Let J denote the simplicial subset of $\Delta^m \times \Delta^1 \times \Delta^q$ spanned by $\Delta^m \times \{0\} \times \Delta^q$, $\Delta^m \times \Delta^1 \times \partial \Delta^q$, and $\partial \Delta^m \times \Delta^1 \times \Delta^q$. The restriction of G_0 to $K_\tau \times [r,1]$ determines another map of simplicial sets $h : J \times \Delta^1 \to \mathrm{Sing}^{A'}(X')$, which is a natural transformation from $h_0 = h|(J \times \{0\})$ to $h_1 = h|(J \times \{1\}) = \overline{h}_1|J$. It follows from the inductive hypothesis that $\mathrm{Sing}^{A'}(X')$ is an ∞-category, and (using (3)) that natural transformation h is an equivalence. Consequently, we can lift h to an equivalence $\overline{h} : \overline{h}_0 \to \overline{h}_1$ in $\mathrm{Fun}(J, \mathrm{Sing}^{A'}(X'))$. This morphism determines a continuous map $G_+ : \overline{\tau} \times |\Delta^q| \times [r,1] \to X$ which agrees with G_0 on $(\overline{\tau} \times |\Delta^q| \times [r,1]) \cap K_\tau'$.

Let $d : |\Delta^q| \to [0,1]$ be a continuous function which vanishes precisely on $|\partial \Delta^q|$, and choose $d' : \tau \to [0,1]$ similarly. For every nonnegative real number v, let c_v be the map from $\tau \times [\frac{k-1}{N}, \frac{k}{N}] \times |\Delta^q| \times [r,1] \to \tau \times |\Delta^q| \times [r,1]$ to itself which is given by the formula

$$c_v(x, \frac{k-1}{N} + t, y, r') = (x, \frac{k-1}{N} + \frac{t}{1 + vd'(x)d(y)(1-r')}, y, r')$$

and let G_+^v denote the composition $G_+ \circ c_v$. Since G_+ agrees with G_0 on $K_{\overline{\tau}} \times \{r\}$, it carries $K_\tau \times \{r\}$

into U. By continuity, there exists a neighborhood V of $K_{\overline{\tau}}$ in $\overline{\tau} \times |\Delta^q|$ such that $G_+(V \times \{r\}) \subseteq U$. If the real number v is sufficiently large, then $c_v(\overline{\tau} \times |\Delta^q| \times \{r\}) \subseteq V$, so that $G_+^v(\overline{\tau} \times |\Delta^q| \times \{r\}) \subseteq U$. Replacing G_+ by G_+^v, we may assume that $G_+(\overline{\tau} \times |\Delta^q| \times \{r\}) \subseteq U$.

Let $X'' = U - X_{a_0} \simeq Z \times Y \times \mathbf{R}_{>0}$. The A'-stratification of X' restricts to a (conical) A'-stratification of X''. Let $g : \overline{\tau} \times |\Delta^q| \times \{r\} \to X''$ be the map obtained by restricting G_+. Then g determines a map of simplicial sets $\overline{\phi}_0 : \Delta^m \times \Delta^1 \times \Delta^q \to \mathrm{Sing}^{A'}(X'')$. Let I denote the simplicial set

$$\Delta^{\{0,1\}} \coprod_{\{1\}} \Delta^{\{1,2\}} \coprod_{\{2\}} \Delta^{\{2,3\}} \coprod_{\{3\}} \cdots,$$

and identify the geometric realization $|I|$ with the open interval $(0, r]$. Then G_0 determines a map of simplicial sets $J \times I \to \mathrm{Sing}^{A'}(X'')$, which we can identify with a sequence of maps $\phi_0, \phi_1, \ldots \in \mathrm{Fun}(J, \mathrm{Sing}^{A'}(X''))$ together with natural transformations $\phi_0 \to \phi_1 \to \ldots$. We note that $\phi_0 = \overline{\phi}_0|J$. The inductive hypothesis guarantees that $\mathrm{Sing}^{A'}(X'')$ is an ∞-category, and assertion (3) ensures that each of the natural transformations $\phi_k \to \phi_{k+1}$ is an equivalence. It follows that we can lift these natural transformations to obtain a sequence of equivalences

$$\overline{\phi}_0 \to \overline{\phi}_1 \to \overline{\phi}_2 \to \cdots$$

in the ∞-category $\mathrm{Fun}(\Delta^m \times \Delta^1 \times \Delta^q, \mathrm{Sing}^{A'}(X''))$. This sequence of equivalences is given by a map of simplicial sets $\Delta^m \times \Delta^1 \times \Delta^q \times I \to \mathrm{Sing}^{A'}(X'')$, which we can identify with a continuous map $\overline{\tau} \times |\Delta^q| \times (0, r] \to Z \times Y \times \mathbf{R}_{>0}$. Let $y : \overline{\tau} \times |\Delta^q| \times (0, r] \to Y$ be the projection of this map onto the second fiber.

We observe that G_+ and G_0 together determine a map $(K_{\overline{\tau}} \times [0, r]) \coprod_{\overline{K_\tau} \times \{0, r\}} (\overline{\tau} \times |\Delta^q| \times \{0, r\}) \to X'$. Let z denote the composition of this map with the projection $U \to Z \times \mathbf{R}_{\geq 0}$. Since the domain of z is a retract of $\overline{\tau} \times |\Delta^q| \times [0, r]$, we can extend z to a continuous map $\overline{z} : \overline{\tau} \times |\Delta^q| \times [0, r] \to Z \times \mathbf{R}_{\geq 0}$. Let $\overline{z}_1 : \overline{\tau} \times |\Delta^q| \times [0, r] \to \mathbf{R}_{\geq 0}$ be obtained from $\overline{z}$ by projection onto the second factor. By adding to $\overline{z}_1$ a function which vanishes on $(K_{\overline{\tau}} \times [0, r]) \coprod_{K_{\overline{\tau}} \times \{0, r\}} (\overline{\tau} \times |\Delta^q| \times \{0, r\})$ and is positive elsewhere, we can assume that $\overline{z}_1^{-1}\{0\} = \overline{\tau} \times |\Delta^q| \times \{0\}$. Let $G_- : \overline{\tau} \times |\Delta^q| \times [0, r] \to U \simeq Z \times C(Y)$ be the map which is given by $\overline{z}$ on $\overline{\tau} \times |\Delta^q| \times \{0\}$ and by the pair $(\overline{z}, y)$ on $\overline{\tau} \times |\Delta^q| \times (0, r]$. Then G_- and G_+ together determine an extension $G : \overline{\tau} \times |\Delta^q| \times [0, 1] \to X$ of G_0 with the desired properties.

$\square$

We conclude this section by describing the ∞-category of exit paths for a particularly simple class of stratified spaces: namely, the collection of simplicial complexes. We begin by reviewing some definitions.

Definition A.6.6. An *abstract simplicial complex* consists of the following data:

(1) A set V (the set of *vertices* of the complex).

(2) A collection S of nonempty finite subsets of V satisfying the following condition:

   (∗) If $\emptyset \neq \sigma \subseteq \sigma' \subseteq V$ and $\sigma' \in S$, then $\sigma \in S$.

We will say that (V, S) is *locally finite* if each element $\sigma \in S$ is contained in only finitely many other elements of S.

Let (V, S) be an abstract simplicial complex, and choose a linear ordering on V. We let $\Delta^{(V,S)}$ denote the simplicial subset of Δ^V spanned by those simplices σ of Δ^V such that the set of vertices of σ belongs to S. Let $|\Delta^{(V,S)}|$ denote the geometric realization of $\Delta^{(V,S)}$. This topological space is independent of the choice of linear ordering on S, up to canonical homeomorphism. As a set, $|\Delta^{V,S}|$ can be identified with the collection of maps $w : V \to [0, 1]$ such that $\mathrm{Supp}(w) = \{v \in V : w(v) \neq 0\} \in S$ and $\sum_{v \in V} w(v) = 1$.

Definition A.6.7. Let (V, S) be an abstract simplicial complex. We regard S as a partially ordered set with respect to inclusions. Then $|\Delta^{(V,S)}|$ is equipped with a natural S-stratification, given by the map

$$(t \in |\Delta^{(V,S)}|) \mapsto (\mathrm{Supp}(t) \in S).$$

Proposition A.6.8. *Let (V, S) be a locally finite abstract simplicial complex. Then the S-stratification of $|\Delta^{(V,S)}|$ is conical.*

Proof. Consider an arbitrary $\sigma \in S$. Let $V' = V - \sigma$, and let $S' = \{\sigma' - \sigma : \sigma \subset \sigma' \in S\}$. Then (V', S') is another abstract simplicial complex. Let $Z = |\Delta^{(V,S)}|_\sigma$ and let $Y = |\Delta^{(V',S')}|$. Then the inclusion $Z \hookrightarrow |\Delta^{(V,S)}|$ extends to an open embedding $h : Z \times C(Y) \to |\Delta^{(V,S)}|$, which is given on $Z \times Y \times (0, \infty)$ by the formula

$$h(w_Z, w_Y, t)(v) = \begin{cases} \frac{w_Z(v)}{t+1} & \text{if } v \in \sigma \\ \frac{t w_Y(v)}{t+1} & \text{if } v \notin \sigma \end{cases}$$

If (V, S) is locally finite, then h is an open embedding whose image is $|\Delta^{(V,S)}|_{>\sigma}$, which proves that the S-stratification of $|\Delta^{(V,S)}|$ is conical. $\qquad\square$

Corollary A.6.9. *Let (V, S) be an abstract simplicial complex. Then the simplicial set $\mathrm{Sing}^S |\Delta^{(V,S)}|$ is an ∞-category.*

Proof. For every subset $V_0 \subseteq V$, let $S_0 = \{\sigma \in S : \sigma \subseteq V_0\}$. Then $\mathrm{Sing}^S |\Delta^{(V,S)}|$ is equivalent to the filtered colimit $\varinjlim_{V_0} \mathrm{Sing}^{S_0} |\Delta^{(V_0,S_0)}|$, where the colimit is taken over all finite subsets $V_0 \subseteq V$. It will therefore suffice to prove that each $\mathrm{Sing}^{S_0} |\Delta^{(V_0,S_0)}|$ is an ∞-category. Replacing V by V_0, we may assume that V is finite so that (V, S) is locally finite. In this case, the desired result follows immediately from Proposition A.6.8 and Theorem A.6.4. $\qquad\square$

Theorem A.6.10. *Let (V, S) be an abstract simplicial complex. Then the projection $q : \mathrm{Sing}^S |\Delta^{(V,S)}| \to \mathrm{N}(S)$ is an equivalence of ∞-categories.*

Proof. Since each stratum of $|\Delta^{(V,S)}|$ is nonempty, the map q is essentially surjective. To prove that q is fully faithful, fix points $x \in |\Delta^{(V,S)}|_\sigma$ and $y \in |\Delta^{(V,S)}|_{\sigma'}$. It is clear that $M = \mathrm{Map}_{\mathrm{Sing}^S |\Delta^{(V,S)}|}(x, y)$ is empty unless $\sigma \subseteq \sigma'$. We wish to prove that M is contractible if $\sigma \subseteq \sigma'$. We can identify M with $\mathrm{Sing}\, P$, where P is the space of paths $p : [0, 1] \to |\Delta^{(V,S)}|$ such that $p(0) = x$, $p(1) = y$, and $p(t) \in |\Delta(V,S)|_{\sigma'}$ for $t > 0$. It now suffices to observe that there is a contracting homotopy $h : P \times [0, 1] \to P$, given by the formula

$$h(p, s)(t) = (1 - s)p(t) + s(1 - t)x + sty.$$

$$\square$$

Remark A.6.11. Let (V, S) be an abstract simplicial complex. It is possible to construct an explicit homotopy inverse to the equivalence of ∞-categories $q : \mathrm{Sing}^S |\Delta^{(V,S)}| \to \mathrm{N}(S)$ of Theorem A.6.10. For each $\sigma \in S$ having cardinality n, we let $w_\sigma \in |\Delta^{(V,S)}|$ be the point described by the formula

$$w_\sigma(v) = \begin{cases} \frac{1}{n} & \text{if } v \in \sigma \\ 0 & \text{if } v \notin \sigma. \end{cases}$$

For every chain of subsets $\emptyset \neq \sigma_0 \subseteq \sigma_1 \subseteq \ldots \subseteq \sigma_k \in S$, we define a map $|\Delta^k| \to |\Delta^{(V,S)}|$ by the formula

$$(t_0, \ldots, t_k) \mapsto t_0 w_{\sigma_0} + \cdots + t_k w_{\sigma_k}.$$

This construction determines section $\phi : \mathrm{N}(S) \to \mathrm{Sing}^S |\Delta^{(V,S)}|$ of q, and is therefore an equivalence of ∞-categories. The induced map of topological spaces $|\mathrm{N}(S)| \to |\Delta^{(V,S)}|$ is a homeomorphism: it is given by the classical process of *barycentric subdivision* of the simplicial complex $|\Delta^{(V,S)}|$.

A.7   A Seifert-van Kampen Theorem for Exit Paths

Our goal in this section is to prove the following generalization of Theorem A.3.1:

Theorem A.7.1. *Let A be a partially ordered set, let X be an A-stratified topological space, and let $\mathcal{C}$ be a category equipped with a functor $U : \mathcal{C} \to \mathcal{U}(X)$, where $\mathcal{U}(X)$ denotes the partially ordered set of all open subsets of X. Assume that the following conditions are satisfied:*

(*i*) *The A-stratification of X is conical.*

(*ii*) *For every point $x \in X$, the full subcategory $\mathcal{C}_x \subseteq \mathcal{C}$ spanned by those objects $C \in \mathcal{C}$ such that $x \in U(C)$ has weakly contractible nerve.*

Then U exhibits the ∞-category $\mathrm{Sing}^A(X)$ as the colimit (in the ∞-category Cat_∞) of the diagram

$$\{\mathrm{Sing}^A(U(C))\}_{C \in \mathcal{C}}.$$

Remark A.7.2. Theorem A.7.1 reduces to Theorem A.3.1 in the special case where A has only a single element.

The proof of Theorem A.7.1 will occupy our attention throughout this section. We begin by establishing some notation.

Definition A.7.3. Let A be a partially ordered set and X an A-stratified topological space. Given a chain of elements $a_0 \leq \ldots \leq a_n$ in A (which we can identify with an n-simplex $\vec{a}$ in $\mathrm{N}(A)$), we let $\mathrm{Sing}^A(X)[\vec{a}]$ denote the fiber product $\mathrm{Fun}(\Delta^n, \mathrm{Sing}^A(X)) \times_{\mathrm{Fun}(\Delta^n, \mathrm{N}(A))} \{\vec{a}\}$.

Remark A.7.4. Suppose that X is a conically A-stratified topological space. It follows immediately from Theorem A.6.4 that for every n-simplex $\vec{a}$ of $\mathrm{N}(A)$, the simplicial set $\mathrm{Sing}^A(X)[\vec{a}]$ is a Kan complex.

Example A.7.5. Let $a \in A$ be a 0-simplex of $\mathrm{N}(A)$, and let X be an A-stratified topological space. Then $\mathrm{Sing}^A(X)[a]$ can be identified with the Kan complex $\mathrm{Sing}(X_a)$.

In the special case where $\vec{a} = (a_0 \leq a_1)$ is an edge of $\mathrm{N}(A)$, the simplicial set $\mathrm{Sing}^A(X)[\vec{a}]$ can be viewed as the space of paths $p : [0,1] \to X$ such that $p(0) \in X_{a_0}$ and $p(t) \in X_{a_1}$ for $t \neq 0$. The essential information is encoded in the behavior of the path $p(t)$ where t is close to zero. To make this more precise, we need to introduce a bit of notation.

Definition A.7.6. Let A be a partially ordered set, let X be an A-stratified topological space, and let $a \leq b$ be elements of A. We define a simplicial set $\mathrm{Sing}^A_{a \leq b}(X)$ as follows:

(∗) An n-simplex of $\mathrm{Sing}^A_{a \leq b}(X)$ consists of an equivalence class of pairs (ϵ, σ), where ϵ is a positive real number and $\sigma : |\Delta^n| \times [0, \epsilon] \to X$ is a continuous map such that $\sigma(|\Delta^n| \times \{0\}) \subseteq X_a$ and $\sigma(|\Delta^n| \times (0, \epsilon]) \subseteq X_b$. Here we regard (ϵ, σ) and (ϵ', σ') as equivalent if there exists a positive real number $\epsilon'' < \epsilon, \epsilon'$ such that $\sigma|(|\Delta^n| \times [0, \epsilon'']) = \sigma'|(|\Delta^n| \times [0, \epsilon''])$.

More informally, we can think of $\mathrm{Sing}^A_{a \leq b}(X)$ as the space of germs of paths in X which begin in X_a and then pass immediately into X_b. There is an evident map $\mathrm{Sing}^A(X)[a \leq b] \to \mathrm{Sing}^A_{a \leq b}(X)$, which is given by passing from paths to germs of paths.

Lemma A.7.7. *Let A be a partially ordered set, X an A-stratified topological space, and $a \leq b$ elements of A. Then the map $\phi : \mathrm{Sing}^A(X)[a \leq b] \to \mathrm{Sing}^A_{a \leq b}(X)$ is a weak homotopy equivalence of simplicial sets.*

Proof. For every positive real number ϵ, let $S[\epsilon]$ denote the simplicial set whose n-simplces are maps $\sigma : |\Delta^n| \times [0, \epsilon] \to X$ such that $\sigma(|\Delta^n| \times \{0\}) \subseteq X_a$ and $\sigma(|\Delta^n| \times (0, \epsilon]) \subseteq X_b$. There are evident restriction maps

$$\mathrm{Sing}^A(X)[a \leq b] = S[1] \to S[\tfrac{1}{2}] \to S[\tfrac{1}{4}] \to \cdots$$

and the colimit of this sequence can be identified with $\mathrm{Sing}_{a \leq b}^{A}(X)$. Consequently, to prove that ϕ is a weak homotopy equivalence, it will suffice to show that each of the restriction maps $\psi : S[\epsilon] \to S[\frac{\epsilon}{2}]$ is a weak homotopy equivalence. It now suffices to observe that ψ is a pullback of the trivial Kan fibration $\mathrm{Fun}(\Delta^1, \mathrm{Sing}(X_b)) \to \mathrm{Fun}(\{0\}, \mathrm{Sing}(X_b))$. $\square$

The space of germs $\mathrm{Sing}_{a \leq b}^{A}(X)$ enjoys a formal advantage over the space of paths of fixed length:

Lemma A.7.8. *Let A be a partially ordered set, X a conically A-stratified topological space, and $a \leq b$ elements of A. Then the restriction map $\mathrm{Sing}_{a \leq b}^{A}(X) \to \mathrm{Sing}(X_a)$ is a Kan fibration.*

Proof. We must show that every lifting problem of the form

$$
\begin{array}{ccc}
\Lambda_i^{n+1} & \xrightarrow{F_+^0} & \mathrm{Sing}_{a \leq b}^{A}(X) \\
\downarrow & \nearrow & \downarrow \\
\Delta^{n+1} & \xrightarrow{F_-^0} & \mathrm{Sing}(X_a)
\end{array}
$$

admits a solution. Let us identify $|\Delta^{n+1}|$ with a product $|\Delta^n| \times [0,1]$ in such a way that the closed subset $|\Lambda_i^{n+1}|$ is identified with $|\Delta^n| \times \{0\}$. We can identify F_+^0 with a continuous map $|\Delta^n| \times \{0\} \times [0,\epsilon] \to X$ for some positive real number ϵ, and F_-^0 with a continuous map $|\Delta^n| \times [0,1] \times \{0\} \to X_a$. To solve the lifting problem, we must construct a positive real number $\epsilon' \leq \epsilon$ and a map $F : |\Delta^n| \times [0,1] \times [0,\epsilon'] \to X$ compatible with F_-^0 and F_+^0 with the following additional property:

($*$) For $0 < t$, we have $F(v, s, t) \in X_b$.

For each point $x \in X_a$, choose a neighborhood U_x of x as in Definition A.5.5. Choose a triangulation of $|\Delta^n|$ and a nonnegative integer $N \gg 0$ with the property that for each simplex τ of $|\Delta^n|$ and $0 \leq k < N$, the map F_-^0 carries $\tau \times [\frac{k}{N}, \frac{k+1}{N}]$ into some U_x for some point $x \in X_a$. For each subcomplex L of $|\Delta^n|$, we will prove that there exists a map $F_L : L \times [0,1] \times [0,\epsilon] \to X$ (possibly after shrinking ϵ) compatible with F_-^0 and F_+^0 and satisfying condition $*$. Taking $L = |\Delta^n|$ we will obtain a proof of the desired result.

The proof now proceeds by induction on the number of simplices of L. If $L = \emptyset$ there is nothing to prove. Otherwise, we can write $L = L_0 \cup \tau$, where τ is a simplex of $|\Delta^n|$ such that $L_0 \cap \tau = \partial \tau$. By the inductive hypothesis, we may assume that the map F_{L_0} has already been supplied; let $F_{\partial \tau}$ be its restriction to $\partial \tau \times [0,1] \times [0,\epsilon]$. To complete the proof, it will suffice to show that we can extend $F_{\partial \tau}$ to a map $F_\tau : \tau \times [0,1] \times [0,\epsilon] \to X$ compatible with F_-^0 and F_+^0 and satisfying ($*$) (possibly after shrinking the real number ϵ). We again proceed in stages by defining a compatible sequence of maps $F_\tau^k : \tau \times [0, \frac{k}{N}] \times [0,\epsilon] \to X$ using induction on $k \leq N$. The map F_τ^0 is determined by F_+^0. Assume that F_τ^{k-1} has already been constructed. Let $K = \tau \times [\frac{k-1}{N}, \frac{k}{N}]$ and let K_0 be the closed subset of K given by the union of $\partial \tau \times [\frac{k-1}{N}, \frac{k}{N}]$ and $\tau \times \{\frac{k-1}{N}\}$. Then F_τ^{k-1} determines a continuous map

$$
g_0 : (K \times \{0\}) \coprod_{K_0 \times \{0\}} (K_0 \times [0,\epsilon]) \to X.
$$

To construct F_τ^k, it will suffice to extend g_0 to a continuous map $g : K \times [0,\epsilon] \to X$ satisfying ($*$) (possibly after shrinking ϵ).

By assumption, the map g_0 carries $K \times \{0\}$ into some open set $U = U_x$ of the form $Z \times C(Y)$ described in Definition A.5.5. Shrinking ϵ if necessary, we may assume that g_0 also carries $K_0 \times [0,\epsilon]$ into U. Let $g_0' : (K \times \{0\}) \coprod_{K_0 \times \{0\}} (K_0 \times [0,\epsilon]) \to C(Y)$ be the composition of g_0 with the projection to $C(Y)$, and let $g_0'' : (K \times \{0\}) \coprod_{K_0 \times \{0\}} (K_0 \times [0,\epsilon]) \to Z$ be defined similarly. Let r be a retraction of K onto K_0, and let g' be the composition $K \times [0,\epsilon] \to K_0 \times [0,\epsilon] \xrightarrow{g_0'} C(Y)$; we observe that g' is an extension of g_0' (since g_0' is constant on $K \times \{0\}$). Let r' be a retraction of $K \times [0,\epsilon]$ onto $(K \times \{0\}) \coprod_{K_0 \times \{0\}} (K_0 \times [0,\epsilon])$, and let g'' be the composition $g_0'' \circ r'$. The pair (g', g'') determines a map $g : K \times [0,\epsilon] \to X$ with the desired properties. $\square$

Proposition A.7.9. *Let A be a partially ordered set, let X be a conically A-stratified space, let U be an open subset of X (which inherits the structure of a conically A-stratified space), and let $\vec{a} = (a_0 \leq a_1 \leq \ldots \leq a_n)$ be an n-simplex of $\mathrm{N}(A)$. Then the diagram of Kan complexes*

$$
\begin{array}{ccc}
\mathrm{Sing}^A(U)[\vec{a}] & \longrightarrow & \mathrm{Sing}^A(X)[\vec{a}] \\
\downarrow & & \downarrow \\
\mathrm{Sing}(U_{a_0}) & \longrightarrow & \mathrm{Sing}(X_{a_0})
\end{array}
$$

is a homotopy pullback square.

Proof. The proof proceeds by induction on n. If $n = 0$ the result is obvious. If $n > 1$, then let $\vec{a}'$ denote the truncated chain $(a_0 \leq a_1)$ and $\vec{a}''$ the chain $(a_1 \leq \ldots \leq a_{n-1} \leq a_n)$. We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Sing}^A(U)[\vec{a}] & \longrightarrow & \mathrm{Sing}^A(X)[\vec{a}] \\
\downarrow & & \downarrow \\
\mathrm{Sing}^A(U)[\vec{a}'] \times_{\mathrm{Sing}(U_{a_1})} \mathrm{Sing}^A(U)[\vec{a}''] & \longrightarrow & \mathrm{Sing}^A(X)[\vec{a}'] \times_{\mathrm{Sing}(X_{a_1})} \mathrm{Sing}^A(X)[\vec{a}''] \\
\downarrow & & \downarrow \\
\mathrm{Sing}^A(U)[\vec{a}'] & \longrightarrow & \mathrm{Sing}^A(X)[\vec{a}'] \\
\downarrow & & \downarrow \\
\mathrm{Sing}(U_{a_0}) & \longrightarrow & \mathrm{Sing}(X_{a_0}).
\end{array}
$$

The upper square is a homotopy pullback because the vertical maps are weak homotopy equivalences (since $\mathrm{Sing}^A(U)$ and $\mathrm{Sing}^A(X)$ are ∞-categories, by virtue of Theorem A.6.4). The lower square is a homotopy pullback by the inductive hypothesis. The middle square is a (homotopy) pullback of the diagram

$$
\begin{array}{ccc}
\mathrm{Sing}^A(U)[\vec{a}''] & \longrightarrow & \mathrm{Sing}^A(X)[\vec{a}''] \\
\downarrow & & \downarrow \\
\mathrm{Sing}(U_{a_1}) & \longrightarrow & \mathrm{Sing}(X_{a_1}),
\end{array}
$$

and therefore also a homotopy pullback square by the inductive hypothesis. It follows that the outer rectangle is a homotopy pullback as required.

It remains to treat the case $n = 1$. We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Sing}^A(U)[a_0 \leq a_1] & \longrightarrow & \mathrm{Sing}^A(X)[a_0 \leq a_1] \\
\downarrow & & \downarrow \\
\mathrm{Sing}^A_{a_0 \leq a_1}(U) & \longrightarrow & \mathrm{Sing}^A_{a_0 \leq a_1}(X) \\
\downarrow & & \downarrow \\
\mathrm{Sing}(U_{a_0}) & \longrightarrow & \mathrm{Sing}(X_{a_0}).
\end{array}
$$

The lower square is a homotopy pullback since it is a pullback square in which the vertical maps are Kan fibrations (Lemma A.7.8). The upper square is a homotopy pullback since the upper vertical maps are weak homotopy equivalences (Lemma A.7.7). It follows that the outer square is also a homotopy pullback, as desired. $\square$

Proposition A.7.10. *Let $G : \mathrm{Cat}_\infty \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{S})$ be the functor given by the formula*

$$G(\mathcal{C})([n]) = \mathrm{Map}_{\mathrm{Cat}_\infty}(\Delta^n, \mathcal{C}).$$

Then G is a fully faithful embedding.

Remark A.7.11. In fact, one can be more precise: the essential image of the functor $G : \mathrm{Cat}_\infty \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{S})$ can be identified with the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{S})$ spanned by the *complete Segal spaces*. We refer the reader to [82] for a proof of this statement (in the language of model categories).

Proof of Proposition A.7.10. Let $f : \mathrm{N}(\mathbf{\Delta}) \to \mathrm{Cat}_\infty$ be the functor given by $[n] \mapsto \Delta^n$, so that f extends (in an essentially unique way) to a colimit-preserving functor $F : \mathcal{P}(\mathrm{N}(\mathbf{\Delta})) \to \mathrm{Cat}_\infty$ which is left adjoint to G. We will show that the counit map $F \circ G \to \mathrm{id}$ is an equivalence from Cat_∞ to itself.

We now reformulate the desired conclusion in the language of model categories. We can identify Cat_∞ with the underlying ∞-category $\mathbf{A}^\circ$ of the simplicial model category $\mathbf{A} = \mathrm{Set}_\Delta^+$ of marked simplicial sets, with the Cartesian model structure described in §T.3.1. The diagram f is then obtained from a diagram $\overline{f} : \mathbf{\Delta} \to \mathbf{A}$, given by the cosimplicial object $[n] \mapsto (\Delta^n)^\flat$. given by the formula $X \otimes K = X \times K^\sharp$), we can extend $\overline{f}$ to a colimit preserving functor

$$\overline{F} : \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathrm{Set}_\Delta) \to \mathbf{A}.$$

Here $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathrm{Set}_\Delta)$ can be identified with the category of bisimplicial sets. Since the cosimplicial object $\overline{f} \in \mathrm{Fun}(\mathbf{\Delta}, \mathbf{A})$ is Reedy cofibrant (see §T.A.2.9), the functor $\overline{F}$ is a left Quillen functor if we endow $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathrm{Set}_\Delta)$ with the injective model structure (Example T.A.2.9.28). The functor $\overline{F}$ has a right adjoint $\overline{G}$, given by the formula

$$\overline{G}(X)^{m,n} = \mathrm{Hom}_{\mathbf{A}}((\Delta^m)^\flat \times (\Delta^n)^\sharp, X).$$

This right adjoint induces a functor from $\mathbf{A}^\circ$ to $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathrm{Set}_\Delta)^\circ$, which (after passing to the simplicial nerve) is equivalent to the functor $G : \mathrm{Cat}_\infty \to \mathcal{P}(\mathrm{N}(\mathbf{\Delta}))$ considered above. Consequently, it will suffice to show that the counit map $L\overline{F} \circ R\overline{G} \to \mathrm{id}_{\mathrm{h}\mathbf{A}}$ is an equivalence of functors, where $L\overline{F}$ and $R\overline{G}$ denote the left and right derived functors of $\overline{F}$ and $\overline{G}$, respectively. Since every object of $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathrm{Set}_\Delta)$ is cofibrant, we can identify $\overline{F}$ with its left derived functor. We are therefore reduced to proving the following:

(∗) Let $\overline{X} = (X, M)$ be a fibrant object of the category $\mathbf{A}$ of marked simplicial sets. Then the counit map $\eta_{\overline{X}} : \overline{FGX} \to \overline{X}$ is a weak equivalence in $\mathbf{A}$.

Since $\overline{X}$ is fibrant, the simplicial set X is an ∞-category and M is the collection of all equivalences in X. Unwinding the definitions, we can identify $\overline{FGX}$ with the marked simplicial set (Y, N) described as follows:

(a) An n-simplex of Y is a map of simplicial sets $\Delta^n \times \Delta^n \to X$, which carries every morphism of $\{i\} \times \Delta^n$ to an equivalence in $\mathcal{C}$, for $0 \leq i \leq n$.

(b) An edge $\Delta^1 \to Y$ belongs to N if and only if the corresponding map $\Delta^1 \times \Delta^1 \to X$ factors through the projection onto the second factor.

In terms of this identification, the map $\eta_{\overline{X}} : (Y, N) \to (X, M)$ is defined on n-simplices by composing with the diagonal map $\Delta^n \to \Delta^n \times \Delta^n$.

Let N' denote the collection of all edges of Y which correspond to maps from $(\Delta^1 \times \Delta^1)^\sharp$ into $\overline{X}$. The map $\eta_{\overline{X}}$ factors as a composition

$$(Y, N) \xrightarrow{i} (Y, N') \xrightarrow{\eta'_{\overline{X}}} (X, M).$$

We claim that the map i is a weak equivalence of marked simplicial sets. To prove this, it will suffice to show that for every edge α which belongs to N', there exists a 2-simplex σ :
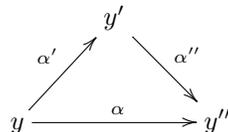
in Y, where α' and α'' belong to N. To see this, let us suppose that α classifies a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\;q\;} & A' \\
\downarrow{\scriptstyle p} & {\scriptstyle r}\searrow & \downarrow{\scriptstyle p'} \\
B & \xrightarrow{\;q'\;} & B'
\end{array}
$$

in the ∞-category X. We wish to construct an appropriate 2-simplex σ in Y, corresponding to a map $\widetilde{\sigma} : \Delta^2 \times \Delta^2 \to X^{\simeq}$ (here $X^{\simeq}$ denotes the largest Kan complex contained in X). Let T denote the full subcategory of $\Delta^2 \times \Delta^2$ spanned by all vertices except for $(0, 2)$, and let $\widetilde{\sigma}_0 : T \to X^0$ be the map described by the diagram

$$
\begin{array}{ccccc}
A & \xrightarrow{\;\text{id}\;} & A & \xrightarrow{\;q\;} & A' \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \text{id}} \\
A' & \xrightarrow{\;\text{id}\;} & A' & \xrightarrow{\;\text{id}\;} & A' \\
& & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p'} \\
& & B' & \xrightarrow{\;\text{id}\;} & B'.
\end{array}
$$

To prove that $\widetilde{\sigma}_0$ can be extended to a map $\widetilde{\sigma}$ with the desired properties, it suffices to solve an extension problem of the form

$$
\begin{array}{ccc}
T \coprod_{\Delta^{\{0,2\}}} \Delta^2 & \longrightarrow & X^0 \\
\downarrow & \nearrow & \\
\Delta^2 \times \Delta^2. & &
\end{array}
$$

This is possible because X^0 is a Kan complex and the left vertical map is a weak homotopy equivalence. This completes the proof that i is a weak equivalence. By the two-out-of-three property, it will now suffice to show that $\eta'_{\overline{X}} : (Y, N') \to (X, M)$ is an equivalence of marked simplicial sets.

We now define maps $R_{\leq}, R_{\geq} : \Delta^1 \times Y \to Y$ as follows. Consider a map $g : \Delta^n \to \Delta^1 \times Y$, corresponding to a partition $[n] = [n]_- \cup [n]_+$ and a map $\widetilde{g} : \Delta^n \times \Delta^n \to X$. We then define $R_{\leq} \circ g$ to be the n-simplex of Y corresponding to the map $\widetilde{g} \circ \tau : \Delta^n \times \Delta^n \to X$, where $\tau : \Delta^n \times \Delta^n \to \Delta^n \times \Delta^n$ is defined on vertices by the formula

$$
\tau(i, j) = \begin{cases} (i, j) & \text{if } i \leq j \\ (i, j) & \text{if } j \in [n]_- \\ (i, i) & \text{otherwise.} \end{cases}
$$

Similarly, we let $R_{\geq} \circ g$ correspond to the map $\widetilde{g} \circ \tau'$, where τ' is given on vertices by the formula

$$
\tau_{i,j} = \begin{cases} (i, j) & \text{if } i \geq j \\ (i, j) & \text{if } j \in [n]_+ \\ (i, I) & \text{otherwise.} \end{cases}
$$

The map $R_{\leq}$ defines a homotopy from id_Y to an idempotent map $r_{\leq} : Y \to Y$. Similarly, $R_{\geq}$ defines a homotopy from an idempotent map $r_{\geq} : Y \to Y$ to the identity map id_Y. Let $Y_{\leq}, Y_{\geq} \subseteq Y$ denote the images of the maps $r_{\leq}$ and $r_{\geq}$, respectively. Let $N'_{\leq}$ denote the collection of all edges of Y which belong to N', and define $N'_{\geq}$ similarly. The map $R_{\leq}$ determines a map $(Y, N') \times (\Delta^1)^{\sharp} \to (Y, N')$, which exhibits $(Y_{\leq}, N'_{\leq})$ as a deformation retract of (Y, N') in the category of marked simplicial sets. Similarly, the map $R_{\geq}$ exhibits $(Y_{\leq} \cap Y_{\geq}, N'_{\leq} \cap N'_{\geq})$ as a deformation retract of $(Y_{\leq}, N'_{\leq})$. It will therefore suffice to show that the composite map

$$
(Y_{\leq} \cap Y_{\geq}, N'_{\leq} \cap N'_{\geq}) \subseteq (Y, N') \to (X, M)
$$

is a weak equivalence of marked simplicial sets. We now complete the proof by observing that this composite map is an isomorphism. $\square$

We are now ready to establish our main result.

Proof of Theorem A.7.1. Let $G : \mathrm{Cat}_\infty \to \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{S})$ be the functor described in Proposition A.7.10. Since G is fully faithful, it will suffice to prove that the composite functor

$$\theta : \mathrm{N}(\mathcal{C})^{\triangleright} \to \mathrm{Cat}_\infty \to \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{S})$$

is a colimit diagram. Since colimits in $\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{S})$ are computed pointwise, it will suffice to show that θ determines a colimit diagram in $\mathcal{S}$ after evaluation at each object $[n] \in \boldsymbol{\Delta}$. Unwinding the definitions, we see that this diagram is given by the formula

$$C \mapsto \coprod_{a_0 \leq a_1 \leq \cdots \leq a_n} \mathrm{Sing}^A(U(C))[a_0 \leq \ldots \leq a_n].$$

Since the collection of colimit diagrams is stable under coproducts (Lemma T.5.5.2.3), it will suffice to show that for every n-simplex $\vec{a} = (a_0 \leq \ldots \leq a_n)$ of $\mathrm{N}(A)$, the functor

$$\theta_{\vec{a}} : \mathrm{N}(\mathcal{C})^{\triangleright} \to \mathcal{S}$$

given by the formula $C \mapsto \mathrm{Sing}^A(U(C))[\vec{a}]$ is a colimit diagram in $\mathcal{S}$.

We have an evident natural transformation $\alpha : \theta_{\vec{a}} \to \theta_{a_0}$. The functor θ_{a_0} is a colimit diagram in $\mathcal{S}$: this follows by applying Theorem A.3.1 to the stratum X_a. Proposition A.7.9 guarantees that α is a Cartesian natural transformation. Since $\mathcal{S}$ is an ∞-topos, Theorem T.6.1.0.6 guarantees that $\theta_{\vec{a}}$ is also a colimit diagram, as desired. $\square$

A.8 Complementary Localizations

Let X be a topological space, let U be an open subset of X, and let $Y = X - U$. Let $i : Y \to X$ and $j : U \to X$ denote the inclusion maps. If $\mathcal{F}$ is a sheaf (of sets, say) on X, then $\mathcal{F}$ determines sheaves $\mathcal{F}_Y = i^* \mathcal{F}$ and $\mathcal{F}_U = j^* \mathcal{F}$ on Y and U, respectively. Moreover, there is a canonical map

$$u : \mathcal{F}_Y = i^* \mathcal{F} \to i^*(j_* j^* \mathcal{F}) = (i^* j_*) \mathcal{F}_U \,.$$

Moreover, we can recover $\mathcal{F}$ from the sheaves $\mathcal{F}_Y$ and $\mathcal{F}_U$, together with the map u: we have a pullback diagram of sheaves

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & i_* \mathcal{F}_Y \\ \downarrow & & \downarrow {\scriptstyle i_*(u)} \\ j_* \mathcal{F}_U & \longrightarrow & i_* i^* j_* \mathcal{F}_U \,. \end{array}$$

In fact, something even stronger is true: we can reconstruct the category of sheaves on X from the categories of sheaves on U and Y, respectively, together with the functor $i^* j_*$. Our goal in this section is to give a general formalization of this reconstruction procedure.

Our first step is to extract the essence of the above situation. If $i : Y \to X$ and $j : U \to X$ are closed and open embeddings of topological spaces, respectively, then the pushforward functors i_* and j_* are fully faithful. Moreover, these fully faithful functors admit left adjoints i^* and j^*, respectively, both of which are left exact.

Definition A.8.1. Let $\mathcal{C}$ be an ∞-category which admits pullbacks, and suppose we are given full subcategories $\mathcal{C}_0, \mathcal{C}_1 \subseteq \mathcal{C}$. We will say that $\mathcal{C}_0$ is *complementary* to $\mathcal{C}_1$ if the following conditions are satisfied:

(a) The full subcategories $\mathcal{C}_0$ and $\mathcal{C}_1$ are stable under equivalence (that is, if $X \in \mathcal{C}$ is equivalent to an object $Y \in \mathcal{C}_i$, then $X \in \mathcal{C}_i$).

(b) The inclusion functors $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ and $\mathcal{C}_1 \hookrightarrow \mathcal{C}$ admit left adjoints L_0 and L_1. It follows that $\mathcal{C}_0$ and $\mathcal{C}_1$ are closed under limits in $\mathcal{C}$, and therefore admit finite limits.

(c) The functors L_0 and L_1 are left exact.

(d) The functor L_1 carries every object of $\mathcal{C}_0$ to the final object of $\mathcal{C}$.

(e) If α is a morphism in $\mathcal{C}$ such that $L_0(\alpha)$ and $L_1(\alpha)$ are equivalences, then α is an equivalence.

In this case, we will also say that the localization functor L_0 is *complementary* to the localization functor L_1.

Warning A.8.2. The relation of complementarity is not symmetric: if a localization functor L_0 is complementary to L_1, then L_1 is generally not complementary to L_0.

Remark A.8.3. There is an evident dual version of Definition A.8.1, which we will need in §A.9. If $\mathcal{C}$ is an ∞-category which admits finite colimits and we are given colocalization functors $L_0, L_1 : \mathcal{C} \to \mathcal{C}$, then we will say that L_0 is *complementary* to L_1 if the full subcategories $L_0 \, \mathcal{C}^{op}$ is complementary to $L_1 \, \mathcal{C}^{op}$ in $\mathcal{C}^{op}$, in the sense of Definition A.8.1.

Example A.8.4. Let X be a topological space and suppose we are given a closed embedding $i : Y \to X$ and an open embedding $j : U \to X$. Let $\mathrm{Shv}_{\mathrm{Set}}(X)$ denote the nerve of the category of sheaves of sets on X, and define $\mathrm{Shv}_{\mathrm{Set}}(U)$ and $\mathrm{Shv}_{\mathrm{Set}}(Y)$ similarly. Let $\mathcal{C}_0 \subseteq \mathrm{Shv}_{\mathrm{Set}}(X)$ be the essential image of the pushforward functor i_*, and let $\mathcal{C}_1 \subseteq \mathrm{Shv}_{\mathrm{Set}}(X)$ be the essential image of the pushforward functor j_*. Then $\mathcal{C}_0, \mathcal{C}_1 \subseteq \mathrm{Shv}_{\mathrm{Set}}(X)$ automatically satisfy conditions (a), (b), and (c) of Definition A.8.1. Condition (d) is satisfied if $Y \cap U = \emptyset$, and the condition (e) is satisfied if $Y \cup U = X$. In particular, $\mathcal{C}_0$ is complementary to $\mathcal{C}_1$ if $Y = X - U$.

Remark A.8.5. Let $\mathcal{C}_0, \mathcal{C}_1 \subseteq \mathcal{C}$ be as in Definition A.8.1. If $\mathcal{C}_0$ is complementary to $\mathcal{C}_1$, then $\mathcal{C}_0$ is uniquely determined by $\mathcal{C}_1$: namely, it is the full subcategory of $\mathcal{C}$ spanned by those objects C such that $L_1(C)$ is a final object of $\mathcal{C}$. It follows from (d) that every object of $\mathcal{C}_0$ has this property. Conversely, suppose that $C \in \mathcal{C}$ is such that $L_1(C)$ is final. Let $u : C \to L_0(C)$ be the canonical map. Then $L_0(u)$ is tautologically an equivalence, and $L_1(u)$ is an equivalence since it is a map between final objects of $\mathcal{C}$. It follows from (e) that u is an equivalence, so that $C \simeq L_0(C)$ belongs to $\mathcal{C}_0$.

We next describe a large class of examples of complementary localizations.

Definition A.8.6. Let $p : \mathcal{M} \to \Delta^1$ be a correspondence from an ∞-category $\mathcal{M}_0 = p^{-1}\{0\}$ to an ∞-category $\mathcal{M}_1 = p^{-1}\{1\}$. We will say that p is a *left exact correspondence* if the following conditions are satisfied:

(i) The ∞-categories $\mathcal{M}_0$ and $\mathcal{M}_1$ admit finite limits.

(ii) The map p is a Cartesian fibration.

(iii) The functor $\mathcal{M}_1 \to \mathcal{M}_0$ determined by p is left exact.

Proposition A.8.7. *Let $p : \mathcal{M} \to \Delta^1$ be a left exact correspondence, let $\mathcal{C} = \mathrm{Fun}_{\Delta^1}(\Delta^1, \mathcal{M})$ be the ∞-category of sections of p. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory of $\mathcal{C}$ spanned by those sections $s : \Delta^1 \to \mathcal{M}$ such that $s(1)$ is a final object of $\mathcal{M}_1$, and let $\mathcal{C}_1 \subseteq \mathcal{C}$ be the full subcategory of $\mathcal{C}$ spanned by the p-Cartesian morphisms in $\mathcal{M}$. Then $\mathcal{C}_0$ is complementary to $\mathcal{C}_1$.*

Proof. We will verify that $\mathcal{C}_0$ and $\mathcal{C}_1$ satisfy the conditions of Definition A.8.1. Condition (a) is obvious. It follows from Proposition T.4.3.2.15 that the evaluation functors

$$\mathcal{C}_0 \xrightarrow{e_0} \mathcal{M}_0 \qquad \mathcal{C}_1 \xrightarrow{e_1} \mathcal{M}_1$$

are equivalences of ∞-categories. Let e_0^{-1} and e_1^{-1} denote homotopy inverses to e_0 and e_1, respectively. Then the composite functors

$$L_0 : \mathcal{C} \to \mathcal{M}_0 \xrightarrow{e_0^{-1}} \mathcal{C}_0 \qquad L_1 : \mathcal{C} \to \mathcal{M}_1 \xrightarrow{e_1^{-1}} \mathcal{C}_1$$

are left adjoints to the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C} \hookleftarrow \mathcal{C}_1$, which are evidently left exact. This proves (b) and (c). Assertion (d) follows from the description of L_1 given above, together with the definition of $\mathcal{C}_0$. Assertion (e) follows from the observation that a morphism in $\mathcal{C}$ is an equivalence if and only if its images in $\mathcal{M}_0$ and $\mathcal{M}_1$ are equivalences. $\square$

Proposition A.8.7 implies that every left exact correspondence determines an ∞-category $\mathcal{C}$ and a complementary pair of localizations of $\mathcal{C}$. The main result of this section asserts that every complementary pair of localizations arises in this way, for an essentially unique left exact correspondence. The uniqueness is a consequence of the following:

Proposition A.8.8. *Let* $p : \mathcal{M} \to \Delta^1$ *and* $q : \mathcal{N} \to \Delta^1$ *be left exact correspondences. Define* $\mathcal{C} \subseteq \mathrm{Fun}_{\Delta^1}(\Delta^1, \mathcal{M})$, $\mathcal{D} \subseteq \mathrm{Fun}_{\Delta^1}(\Delta^1, \mathcal{N})$ *and full subcategories*

$$\mathcal{C}_0, \mathcal{C}_1 \subseteq \mathcal{C} \qquad \mathcal{D}_0, \mathcal{D}_0 \subseteq \mathcal{D}$$

as in Proposition A.8.7, so that the ∞-categories $\mathcal{C}_0$ and $\mathcal{C}_1$ are the essential images of localization functors $L_0, L_1 : \mathcal{C} \to \mathcal{C}$ and the ∞-categories $\mathcal{D}_0$ and $\mathcal{D}_1$ are the essential images of localization functors L'_0 and L'_1. Then the canonical map $\mathrm{Fun}_{\Delta^1}(\mathcal{M}, \mathcal{N}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is a fully faithful embedding, whose essential image is spanned by those functors $F : \mathcal{C} \to \mathcal{D}$ which carry L_0 equivalences to L'_0 equivalences and L_1-equivalences to L'_1-equivalences.

Remark A.8.9. Proposition A.8.8 is valid under much weaker hypotheses than the ones we have given: it is not necessary that p and q be left exact correspondences, only that p is a Cartesian fibration and that $\mathcal{M}_1$ admits a final object.

Proof. Choose a right adjoint G to the evaluation functor $\mathcal{C} \to \mathcal{M}_1$. Then G determines a map $\Delta^1 \times \mathcal{M}_1 \to \mathcal{M}$, whose restriction to $\{0\} \times \mathcal{M}_1$ is a functor $\phi : \mathcal{M}_1 \to \mathcal{M}_0$ associated to the Cartesian fibration p. It follows from Let $\mathrm{Fun}'(\mathcal{C}, \mathcal{D})$ be the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which carry L_1-equivalences in $\mathcal{C}$ to L'_1-equivalence in $\mathcal{D}$. We have a commutative diagram σ :

$$
\begin{array}{ccc}
\mathrm{Fun}_{\Delta^1}(\mathcal{M}, \mathcal{N}) & \longrightarrow & \mathrm{Fun}(\mathcal{M}_0, \mathcal{N}_0) \\
\downarrow & & \downarrow \\
\mathrm{Fun}'(\mathcal{C}, \mathcal{D}) & \longrightarrow & \mathrm{Fun}(\mathcal{C}, \mathcal{N}_0) \\
\downarrow & & \downarrow \\
\mathrm{Fun}(\mathcal{M}_1, \mathcal{D}) & \longrightarrow & \mathrm{Fun}(\mathcal{M}_1, \mathcal{N}_0).
\end{array}
$$

Proposition T.3.2.2.7 that the induced map $\mathcal{M}_0 \coprod_{\{0\} \times \mathcal{M}_1} (\Delta^1 \times \mathcal{M}_1) \to \mathcal{D}$ is a categorical equivalence of simplicial sets, so that the outer rectangle in the diagram σ is a homotopy Cartesian. We claim that the lower square in σ is also homotopy Cartesian. To prove this, it suffices to show that the canonical map

$$\theta : \mathrm{Fun}'(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{N}_0) \times_{\mathrm{Fun}(\mathcal{C}_1, \mathcal{N}_0)} \mathrm{Fun}(\mathcal{C}_1, \mathcal{D})$$

is a trivial Kan fibration, where $\mathcal{C}_1 \subseteq \mathcal{C}$ denotes the essential image of G. Let $\mathcal{E}$ denote the full subcategory of $\mathcal{C} \times \Delta^1$ spanned by those pairs (C, i), where $C \in \mathcal{C}_1$ if $i = 1$. Proposition T.3.2.2.7 implies that the inclusion $\mathcal{C} \coprod_{\mathcal{C}_1 \times \{0\}} (\mathcal{C}_1 \times \Delta^1) \hookrightarrow \mathcal{E}$ is a categorical equivalence, so that the map $\mathrm{Fun}_{\Delta^1}(\mathcal{E}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{N}_0) \times_{\mathrm{Fun}(\mathcal{C}_1, \mathcal{N}_0)} \mathrm{Fun}(\mathcal{C}_1, \mathcal{D})$ is a trivial Kan fibration. It will therefore suffice to show that the restriction map

$$\theta' : \mathrm{Fun}'(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}_{\Delta^1}(\mathcal{C} \times \Delta^1, \mathcal{D}) \to \mathrm{Fun}_{\mathcal{D}^1}(\mathcal{E}, \mathcal{D})$$

is a trivial Kan fibration. Note that a functor $F : \mathcal{C} \to \mathcal{D}$ belongs to $\mathrm{Fun}'(\mathcal{C}, \mathcal{D})$ if and only if the induced map $\mathcal{C} \times \Delta^1 \to \mathcal{D}$ is a q-right Kan extension of its restriction to $\mathcal{E}$. It now follows from Proposition T.4.3.2.15 that θ' is a trivial Kan fibration. This completes the proof that the lower square in the diagram σ is homotopy Cartesian. It follows that the upper square is also homotopy Cartesian.

Note that the evaluation map $e_0 : \mathcal{C} \to \mathcal{M}_0$ admits a fully faithful right adjoint. It follows that composition with e_0 induces a fully faithful embedding $\mathrm{Fun}(\mathcal{M}_0, \mathcal{N}_0) \to \mathrm{Fun}(\mathcal{C}, \mathcal{N}_0)$, whose essential image is the collection of functors which carry every L_0-equivalence in $\mathcal{C}$ to an equivalence in $\mathcal{N}_0$. Since the diagram

$$
\begin{array}{ccc}
\mathrm{Fun}_{\Delta^1}(\mathcal{M}, \mathcal{N}) & \longrightarrow & \mathrm{Fun}(\mathcal{M}_0, \mathcal{N}_0) \\
\downarrow & & \downarrow \\
\mathrm{Fun}'(\mathcal{C}, \mathcal{D}) & \longrightarrow & \mathrm{Fun}(\mathcal{C}, \mathcal{N}_0)
\end{array}
$$

is homotopy Cartesian, we conclude that the functor $\mathrm{Fun}_{\Delta^1}(\mathcal{M}, \mathcal{N}) \to \mathrm{Fun}'(\mathcal{C}, \mathcal{D})$ is fully faithful, and its essential image is spanned by those functors $F \in \mathrm{Fun}'(\mathcal{C}, \mathcal{D})$ which carry each L_0-equivalence in $\mathcal{C}$ to an L_0'-equivalence in $\mathcal{D}$. $\qquad\square$

Remark A.8.10. In the situation of Proposition A.8.8, a functor $f : \mathcal{M} \to \mathcal{N}$ induces a left exact functor $F : \mathcal{C} \to \mathcal{D}$ if and only if the underlying maps $f_0 : \mathcal{M}_0 \to \mathcal{N}_0$ and $f_1 : \mathcal{M}_1 \to \mathcal{N}_1$ are left exact.

We now prove a converse to Proposition A.8.7:

Proposition A.8.11. *Let $\mathcal{C}$ be an ∞-category which admits finite limits, and let $\mathcal{C}_0, \mathcal{C}_1 \subseteq \mathcal{C}$ be full subcategories such that $\mathcal{C}_0$ is complementary to $\mathcal{C}_1$. Then there exists a left exact correspondence $p : \mathcal{M} \to \Delta^1$ and an equivalence of ∞-categories $\mathcal{C} \to \mathrm{Fun}_{\Delta^1}(\Delta^1, \mathcal{M})$, such that $\mathcal{C}_0$ and $\mathcal{C}_1$ are the essential images of left adjoints to the induced localization functors $\mathcal{C} \to \mathcal{M}_0$ and $\mathcal{C} \to \mathcal{M}_1$.*

Proof. Let $\mathcal{M}_0 = \mathcal{C}_0$, and let $\mathcal{M}_1$ be the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by those morphisms $f : C \to C'$ such that $C \in \mathcal{C}_1$ and f exhibits C' as a $\mathcal{C}_0$-localization of C. Evaluation at $\{1\} \subseteq \Delta^1$ determines a map $\mathcal{M}_1^{op} \to \mathcal{M}_0^{op}$, which we can view as a functor $\theta : [1] \to \mathcal{S}et_\Delta$. Let $\mathcal{M} = \mathrm{N}_\theta([1])^{op}$ denote the (opposite of the) nerve of the category $[1]$ relative to θ (see Definition T.3.2.5.2), so that we have a Cartesian fibration $p : \mathcal{M} \to \Delta^1$ together with isomorphisms

$$\mathcal{M} \times_{\Delta^1} \{0\} \simeq \mathcal{M}_0 \qquad \mathcal{M} \times_{\Delta^1} \{1\} \simeq \mathcal{M}_1,$$

such that the associated functor $\mathcal{M}_1 \to \mathcal{M}_0$ is given by the evaluation.

Unwinding the definitions, we see that $\mathcal{D} = \mathrm{Fun}_{\Delta^1}(\Delta^1, \mathcal{M})$ is isomorphic to the full subcategory of $\mathrm{Fun}(\Lambda_2^2, \mathcal{C})$ spanned by those diagrams $C_0 \to C_{01} \xleftarrow{g} C_1$ satisfying the following conditions:

(*i*) The object C_0 belongs to $\mathcal{C}_0$ and the object C_1 belongs to $\mathcal{C}_1$.

(*ii*) The morphism g exhibits C_{01} as a $\mathcal{C}_0$-localization of C_1.

Let $\overline{\mathcal{D}}$ be the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ spanned by those diagrams τ:

$$
\begin{array}{ccc}
C & \longrightarrow & C_1 \\
\downarrow & & \downarrow{\scriptstyle g} \\
C_0 & \longrightarrow & C_{01}
\end{array}
$$

satisfying (*i*) and (*ii*) together with the following:

(iii) The diagram τ is a pullback square in $\mathcal{C}$.

Using Proposition T.4.3.2.15, we deduce that the evident restriction functor $\overline{\mathcal{D}} \to \mathcal{D}$ is a trivial Kan fibration. We will need the following fact:

$(*)$ Evaluation at $(0,0) \in \Delta^1 \times \Delta^1$ induces a trivial Kan fibration $e : \overline{\mathcal{D}} \to \mathcal{C}$.

To prove this, we let $\overline{\mathcal{D}}' \subseteq \overline{\mathcal{D}}$ denote the full subcategory spanned by those diagrams τ which satisfy the following additional conditions:

(iv) The map $C \to C_0$ exhibits C_0 is a $\mathcal{C}_0$-localization of C.

(v) The map $C \to C_1$ exhibits C_1 as a $\mathcal{C}_1$-localization of C.

Let $\overline{\mathcal{D}}''$ denote the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ spanned by those functors satisfying conditions (i), (ii), (iv), and (v). Let $\overline{\mathcal{C}}$ denote the full subcategory of $\mathcal{C} \times \Delta^1 \times \Delta^1$ spanned by those objects (C, i, j) such that $C \in \mathcal{C}_0$ if $i = 1$ and $C \in \mathcal{C}_1$ if $0 = i < j = 1$. Let $q : \overline{\mathcal{C}} \to \Delta^1 \times \Delta^1$ denote the projection map, and $q_0 : \overline{\mathcal{C}} \times_{\Delta^1 \times \Delta^1} (\{0\} \times \Delta^1) \to \{0\} \times \Delta^1$ the restriction of q. Note that $\overline{\mathcal{D}}''$ can be identified with the ∞-category of functors $F \in \mathrm{Fun}_{\Delta^1 \times \Delta^1}(\Delta^1 \times \Delta^1, \overline{\mathcal{C}})$ such that F is a q-left Kan extension of $F_0 = F|(\{0\} \times \Delta^1)$ and F_0 is a q_0-left Kan extension of $F|\{(0,0)\}$. It follows from Proposition T.4.3.2.15 that the evaluation map $\overline{\mathcal{D}}'' \to \mathcal{C}$ is a trivial Kan fibration. We will prove $(*)$ by verifying that $\overline{\mathcal{D}} = \overline{\mathcal{D}}' = \overline{\mathcal{D}}''$.

To prove that $\overline{\mathcal{D}}' = \overline{\mathcal{D}}''$, consider a diagram σ :

$$
\begin{array}{ccc}
C & \longrightarrow & L_1(C) \\
\downarrow & & \downarrow \\
L_0(C) & \longrightarrow & (L_0 L_1)(C)
\end{array}
$$

belonging to $\overline{\mathcal{D}}''$. This diagram induces a map $\alpha : C \to L_0(C) \times_{(L_0 L_1)(C)} L_1(C)$. To prove that $\sigma \in \overline{\mathcal{D}}'$, we must show that α is an equivalence. For this, it suffices to show that both $L_0(\alpha)$ and $L_1(\alpha)$ are equivalences. Since L_0 and L_1 are left exact, we are reduced to proving that the diagrams $L_0(\sigma)$ and $L_1(\sigma)$ are pullback squares. This is clear: in the diagram $L_0(\sigma)$, the vertical maps are both equivalences; in the diagram $L_1(\sigma)$, the horizontal maps are both equivalences.

To show that $\overline{\mathcal{D}} = \overline{\mathcal{D}}'$, consider an arbitrary diagram σ :

$$
\begin{array}{ccc}
C & \longrightarrow & C_1 \\
\downarrow & & \downarrow \\
C_0 & \longrightarrow & C_{01}
\end{array}
$$

satisfying conditions (i) through (iii). Since L_0 is left exact, we obtain a pullback diagram

$$
\begin{array}{ccc}
L_0(C) & \longrightarrow & L_0(C_1) \\
\downarrow & & \downarrow \\
L_0(C_0) & \longrightarrow & L_0(C_{01}).
\end{array}
$$

The right vertical map is an equivalence by assumption (ii), so the left vertical map is also an equivalence. Since $C_0 \in \mathcal{C}_0$ by (i), σ satisfies (iv). Similarly, since the functor L_1 is left exact, we have a pullback diagram

$$
\begin{array}{ccc}
L_1(C) & \longrightarrow & L_1(C_1) \\
\downarrow & & \downarrow \\
L_1(C_0) & \longrightarrow & L_1(C_{01}).
\end{array}
$$

Since $C_0, C_{01} \in \mathcal{C}_0$, the lower horizontal map is a morphism between final objects of $\mathcal{C}$ and therefore an equivalence. It follows that the upper horizontal map is an equivalence. Since $C_1 \in \mathcal{C}_1$, we conclude that σ satisfies (v). This completes the proof of $(*)$.

Choose a section $s : \mathcal{C} \to \overline{\mathcal{D}}$ of the projection map e, and let ψ denote the composite map $\mathcal{C} \xrightarrow{s} \overline{\mathcal{D}} \to \mathcal{D}$, where s is a section of e. Then ψ is an equivalence of ∞-categories, which carries an object $C \in \mathcal{C}$ to the diagram

$$L_0(C) \to (L_0 \circ L_1)(C) \leftarrow L_1(C).$$

We claim that ψ has the desired properties. To verify this, consider full subcategories $\mathcal{D}_0, \mathcal{D}_1 \subseteq \mathcal{D}$ as in Proposition A.8.7. We must show that $\mathcal{C}_0 = \psi^{-1} \mathcal{D}_0$ and $\mathcal{C}_1 = \psi^{-1} \mathcal{D}_1$. The equality $\mathcal{C}_0 = \psi^{-1} \mathcal{D}_0$ follows from the observation that $C \in \mathcal{C}_0$ if and only if $L_1(C)$ is a final object of $\mathcal{C}_1$ (Remark A.8.5). To prove that $\mathcal{C}_1 = \psi^{-1} \mathcal{D}_1$, we must show that an object $C \in \mathcal{C}$ belongs to $\mathcal{C}_1$ if and only if the map $\alpha : L_0(C) \to (L_0 \circ L_1)(C)$ is an equivalence. The "only if" direction is obvious. Conversely, suppose that α is satisfied. The proof of $(*)$ shows that the diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\alpha'} & L_1(C) \\
\downarrow & & \downarrow \\
L_0(C) & \xrightarrow{\alpha} & (L_0 L_1)(C)
\end{array}
$$

is a pullback square, so that α' is an equivalence and $C \simeq L_1(C)$ belongs to $\mathcal{C}_1$. $\qquad\square$

Remark A.8.12. Propositions A.8.7, A.8.8, and A.8.11 can be informally summarized by saying that, for every pair of ∞-categories $\mathcal{C}_0$ and $\mathcal{C}_1$, the following types of data are equivalent:

(a) An ∞-category $\mathcal{C}$ containing $\mathcal{C}_0$ and $\mathcal{C}_1$ as full subcategories, such that $\mathcal{C}_0$ is complementary to $\mathcal{C}_1$.

(b) A left exact correspondence from $\mathcal{C}_0$ to $\mathcal{C}_1$.

Proposition A.8.13. *Let $\mathcal{C}$ and $\mathcal{C}'$ be ∞-categories which admit finite limits. Suppose that we are given inclusions of full subcategories*

$$\mathcal{C}_0, \mathcal{C}_1 \subseteq \mathcal{C} \qquad \mathcal{C}_0', \mathcal{C}_1' \subseteq \mathcal{C}'$$

which admit left adjoints L_0, L_1, L_0', and L_1'. Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor satisfying the following conditions:

(1) *The localizations L_0 and L_1 are complementary in $\mathcal{C}$, and the localizations L_0' and L_1' are complementary in $\mathcal{C}'$.*

(2) *The functor F restricts to equivalences $\mathcal{C}_0 \to \mathcal{C}_0'$ and $\mathcal{C}_1 \to \mathcal{C}_1'$.*

(3) *The functor F is left exact.*

(4) *Let $C \in \mathcal{C}_1$ and $\alpha : C \to C'$ be a morphism in $\mathcal{C}$ which exhibits C' as a $\mathcal{C}_0$-localization of C. Then $F(\alpha)$ exhibits $F(C') \in \mathcal{C}_0'$ as a $\mathcal{C}_0'$-localization of $F(C) \in \mathcal{C}_1' \subseteq \mathcal{C}'$.*

Then F is an equivalence of ∞-categories.

Proof. Let $\psi : \mathcal{C} \to \mathcal{D}$ be defined as in the proof of Proposition A.8.11 and let $\psi' : \mathcal{C}' \to \mathcal{D}'$ be defined similarly, so that we have a commutative diagram of ∞-categories

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{C}' \\
\downarrow{\psi} & & \downarrow{\psi'} \\
\mathcal{D} & \xrightarrow{F_0} & \mathcal{D}'.
\end{array}
$$

The proof of Proposition A.8.11 shows that ψ and ψ' are equivalences of ∞-categories. It will therefore suffice to show that F_0 is an equivalence of $|infty$-categories. The map F_0 extends to a map of (homotopy) pullback diagrams

$$
\begin{array}{ccc}
\mathcal{D} \longrightarrow \operatorname{Fun}(\Delta^1, \mathcal{C}_0) & \qquad & \mathcal{D}' \longrightarrow \operatorname{Fun}(\Delta^1, \mathcal{C}'_0) \\
\downarrow \qquad\qquad \downarrow & & \downarrow \qquad\qquad \downarrow \\
\mathcal{M} \longrightarrow \operatorname{Fun}(\{0\}, \mathcal{C}_0) & & \mathcal{M}' \longrightarrow \operatorname{Fun}(\{0\}, \mathcal{C}'_0),
\end{array}
$$

where $\mathcal{M}$ is denotes the full subcategory of $\mathcal{C}$ spanned by those morphisms $f : C_1 \to C_{01}$ such that $C_1 \in \mathcal{C}_1$ and f exhibits C_{01} as a $\mathcal{C}$-colocalization of C_1, and $\mathcal{M}'$ is defined similarly. Since F induces an equivalence $\mathcal{C}_0 \to \mathcal{C}'_0$ by assumption, it suffices to show that the map $\mathcal{M} \to \mathcal{M}'$ (which is well-defined by virtue of (3)) is an equivalence of ∞-categories. This follows from the assumption that F restricts to an equivalence $\mathcal{C}_1 \to \mathcal{C}'_1$, since we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{M} & \longrightarrow & \mathcal{M}' \\
\downarrow & & \downarrow \\
\mathcal{C}_1 & \longrightarrow & \mathcal{C}'_1
\end{array}
$$

in which the vertical maps are trivial Kan fibrations. $\qquad\square$

In order to apply Proposition A.8.13, we will need to be able to recognize the existence of complementary localizations. The following result provides a useful criterion:

Proposition A.8.14. *Let $p : \mathcal{M} \to \Delta^1$ be a correspondence between ∞-categories. Assume that there exists a retraction r from $\mathcal{M}$ onto the full subcategory $\mathcal{M}_1$. Let $\mathcal{A}$ be an ∞-category which admits finite limits, and let $\mathcal{C} = \operatorname{Fun}(\mathcal{M}, \mathcal{A})$. We define full subcategories $\mathcal{C}_0, \mathcal{C}_1 \subseteq \mathcal{C}$ as follows:*

(a) A functor $f : \mathcal{M} \to \mathcal{A}$ belongs to $\mathcal{C}_0$ if f is a right Kan extension of $f | \mathcal{M}_0$ (that is, if $f(M)$ is a final object of $\mathcal{A}$, for each $M \in \mathcal{M}_1$).

(b) A functor $f : \mathcal{M} \to \mathcal{A}$ belongs to $\mathcal{C}_1$ if $f(\alpha)$ is an equivalence, for every p-Cartesian morphism in $\mathcal{M}$.

Then $\mathcal{C}_0$ is complementary to $\mathcal{C}_1$ (in the sense of Definition A.8.1.

Proof. We verify that $\mathcal{C}_0$ and $\mathcal{C}_1$ satisfy the requirements of Definition A.8.1. Condition (a) is obvious. We next prove that the inclusions $\mathcal{C}_0 \subseteq \mathcal{C}$, $\mathcal{C}_1 \subseteq \mathcal{C}$ admit left adjoints, by explicit construction. The functor L_0 is given by composing the restriction functor $\operatorname{Fun}(\mathcal{M}, \mathcal{A}) \to \operatorname{Fun}(\mathcal{M}_0, \mathcal{A})$ with a section of the trivial Kan fibration $\mathcal{C}_0 \to \operatorname{Fun}(\mathcal{M}_0, \mathcal{A})$. The functor L_1 is given by composing the restriction functor $\operatorname{Fun}(\mathcal{M}, \mathcal{A}) \to \operatorname{Fun}(\mathcal{M}_1, \mathcal{A})$ with the retraction $r : \mathcal{M} \to \mathcal{M}_1$. This verifies condition (b) of Definition A.8.1. It is clear that the ∞-category $\mathcal{C}$ admits finite limits (which are computed pointwise), and the explicit construction given above shows that L_0 and L_1 are left exact, so that (c) is satisfied.

The restriction $L_1 | \mathcal{C}_0$ factors through $\operatorname{Fun}(\mathcal{M}_1, \mathcal{A}')$, where $\mathcal{A}' \subseteq \mathcal{A}$ is the contractible Kan complex spanned by the final objects of $\mathcal{A}$, which implies (d). To prove (e), we note that if α is a morphism in $\mathcal{C}$ such that $L_0(\alpha)$ and $L_1(\alpha)$ are both equivalences, then α is a natural transformation of functors from $\mathcal{M}$ to $\mathcal{A}$ which induces an equivalence after evaluation at every object $\mathcal{M}_1$ and every object of $\mathcal{M}_0$. Since every object of $\mathcal{M}$ belongs to $\mathcal{M}_0$ or $\mathcal{M}_1$, we conclude that α is an equivalence. $\qquad\square$

We conclude this section with a discussion of complementary localizations of a topological nature.

Proposition A.8.15. *Let $\mathcal{X}$ be an ∞-category which admits finite limits, and suppose we are given localization functors $L_0, L_1 : \mathcal{X} \to \mathcal{X}$ such that L_0 is complementary to L_1. The following conditions are equivalent:*

(1) The ∞-category $\mathcal{X}$ is an ∞-topos.

(2) *The ∞-categories $\mathfrak{X}_0 = L_0 \mathfrak{X}$ and $\mathfrak{X}_1 = L_1 \mathfrak{X}$ are ∞-topoi, and the functor $(L_0 | \mathfrak{X}_1) : \mathfrak{X}_1 \to \mathfrak{X}_0$ is accessible.*

If these conditions are satisfied, then there exists a (-1)-truncated object $U \in \mathfrak{X}$ with the following properties:

(i) *A morphism $f : X \to Y$ in $\mathfrak{X}$ is an L_1-equivalence if and only if $X \times U \to Y \times U$ is an equivalence (consequently, $\mathfrak{X}_1$ is equivalent to the ∞-topos $\mathfrak{X}_{/U}$).*

(ii) *The ∞-category $\mathfrak{X}_0$ is the closed subtopos $\mathfrak{X}/U \subseteq \mathfrak{X}$.*

Remark A.8.16. Fix a pair of ∞-topoi $\mathcal{U}$ and $\mathcal{Y}$. Suppose we are given an ∞-topos $\mathfrak{X}$ equipped with a (-1)-truncated object $U \in \mathfrak{X}$, together with equivalences of ∞-topoi $\mathcal{U} \simeq \mathfrak{X}_{/U}$ and $\mathcal{Y} \simeq \mathfrak{X}/U$. This data determines geometric morphisms of ∞-topoi

$$\mathcal{U} \xrightarrow{j_*} \mathfrak{X} \xleftarrow{i_*} \mathcal{Y},$$

so that $i^* j_*$ is an accessible left exact functor from $\mathcal{U}$ to $\mathcal{Y}$. Combined with our earlier results on complementary localization, Proposition A.8.15 provides a converse: a left exact accessible functor $F : \mathcal{U} \to \mathcal{Y}$ determines an ∞-topos $\mathfrak{X}$ which is "glued" from $\mathcal{U}$ and $\mathcal{Y}$. Moreover, the data of F is equivalent to the data of diagram of geometric morphisms $\mathcal{U} \xrightarrow{j_*} \mathfrak{X} \xleftarrow{i_*} \mathcal{Y}$.

Proof of Proposition A.8.15. Suppose first that condition (1) is satisfied. Let $\emptyset$ denote an initial object of $\mathfrak{X}$ and $\mathbf{1}$ a final object of $\mathfrak{X}$, and set $U = L_0(\emptyset)$. Since $\mathfrak{X}$ is an ∞-topos, the morphism $\emptyset \to \mathbf{1}$ is (-1)-truncated. We will prove that U satisfies conditions (i) and (ii). It follows that $L_0 | \mathfrak{X}_1$ can be identified with the composition

$$\mathfrak{X}_{/U} \xrightarrow{j_*} \mathfrak{X} \xrightarrow{i^*} \mathfrak{X}/U,$$

and is therefore an accessible functor.

Because L_0 is left exact, we deduce that the canonical map $U = L_0(\emptyset) \to L_0(\mathbf{1}) \simeq \mathbf{1}$ is (-1)-truncated: that is, U is a (-1)-truncated object of $\mathfrak{X}$. Note that if $X \in \mathfrak{X}$ is an object which admits a morphism $f : X \to U$, then

$$L_0(X) \simeq L_0(X) \times_{L_0(U)} L_0(\emptyset) \simeq L_0(X \times_U \emptyset) \simeq L_0(\emptyset) = U$$

is an initial object of $L_0 \mathfrak{X}$.

Let X be an object of $\mathfrak{X}$, and consider the map $u_X : X \times_{L_0(X)} U \to X \times U$. Using condition (d) of Definition A.8.1, we obtain $L_1(L_0(X)) \simeq \mathbf{1}$. Since L_1 is left exact, we conclude that $L_1(u_X)$ is an equivalence. Since both $X \times_{L_0(X)} U$ and $X \times U$ admit a map to U, $L_0(u_X)$ is a map between initial objects of $L_0 \mathfrak{X}$ and therefore an equivalence. Using condition (e) of Definition A.8.1, we conclude that u_X is an equivalence.

We now verify (i). Suppose that $f : X \to Y$ is a morphism in $\mathfrak{X}$ such that $f \times \mathrm{id}_U$ is an equivalence. Then the composition

$$
\begin{array}{rcl}
L_1(X) & \simeq & L_1(X) \times \mathbf{1} \\
& \simeq & L_1(X) \times L_1(U) \\
& \simeq & L_1(X \times U) \\
& \to & L_1(Y \times U) \\
& \simeq & L_1(Y) \times L_1(U) \\
& \simeq & L_1(Y) \times \mathbf{1} \\
& \simeq & L_1(Y)
\end{array}
$$

is an equivalence. Conversely, suppose that $L_1(f)$ is an equivalence. Consider the diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \longrightarrow & L_1(Y) \\
\downarrow & & \downarrow & & \downarrow \\
L_0(X) & \longrightarrow & L_0(Y) & \longrightarrow & (L_0 L_1)(Y).
\end{array}
$$

The proof of Proposition A.8.11 shows that the outer rectangle and the right square are pullbacks, so that the left square is also a pullback. It follows that the map $X \times_{L_0 X} Y \to Y \times_{L_0 Y} U$ is an equivalence, so that by the above argument we conclude that $X \times U \to Y \times U$ is an equivalence. This completes the proof of (i). Assertion (ii) now follows from Remark A.8.5 and the definition of the ∞-topos $\mathfrak{X}/U$.

We now complete the proof by showing that $(2) \Rightarrow (1)$. Assume that $\mathfrak{X}_0$ and $\mathfrak{X}_1$ are ∞-topoi and that $L_0 | \mathfrak{X}_1$ is an accessible functor from $\mathfrak{X}_1$ to $\mathfrak{X}_0$. We will prove that $\mathfrak{X}$ is an ∞-topos. Using Proposition A.8.11, we may assume without loss of generality that $\mathfrak{X}$ is the ∞-category of sections of a left exact correspondence $\mathcal{M} \to \Delta^1$ with $\mathcal{M}_0 \simeq \mathfrak{X}_0$, $\mathcal{M}_1 \simeq \mathfrak{X}_1$, associated to a left exact accessible functor $F : \mathcal{M}_1 \to \mathcal{M}_0$. Since the fibers of p admit small colimits, we deduce that $\mathfrak{X}$ admits small colimits (and is therefore a presentable ∞-category) and the evaluation functors $e_0 : \mathfrak{X} \to \mathcal{M}_0$ and $e_1 : \mathfrak{X} \to \mathcal{M}_1$ preserve small colimits. Since F is left exact, finite limits in $\mathfrak{X}$ are computed pointwise: that is, the evaluation functors e_0 and e_1 are left exact. We now prove that $\mathfrak{X}$ is an ∞-topos by verifying the ∞-categorical versions of Giraud's axioms (see Theorem T.6.1.0.6):

(i) The ∞-category $\mathfrak{X}$ is presentable. Since $\mathfrak{X}$ admits small limits, it will suffice to show that $\mathfrak{X}$ is accessible. This follows from Corollary T.5.4.7.17, since $\mathcal{M}_0$ and $\mathcal{M}_1$ are both accessible and the functor F is accessible.

(ii) Colimits in $\mathfrak{X}$ are universal. Suppose we are given a diagram $\{X_\alpha\}$ in $\mathfrak{X}$ having a colimit X and a morphism $Y \to X$ in $\mathfrak{X}$; we wish to show that the canonical map $v : \varinjlim(Y \times_X X_\alpha) \to Y$ is an equivalence. For this, it suffices to show that $e_i(v)$ is an equivalence in $\mathcal{M}_i$ for $i \in \{0, 1\}$. Since e_i is left exact and commutes with small colimits, we can identify $e_i(v)$ with the map $\varinjlim e_i(Y) \times_{e_i(X)} e_i(X_\alpha) \to e_i(Y)$, which is an equivalence since colimits are universal in the ∞-topos $\mathcal{M}_i$.

(iii) Coproducts in $\mathfrak{X}$ are disjoint. Suppose we are given objects $X, Y \in \mathfrak{X}$; we wish to show that the fiber product $X \times_{X \coprod Y} Y$ is an initial object of $\mathfrak{X}$. For this, it suffices to show that $e_i(X \times_{X \coprod Y} Y)$ is an initial object of $\mathcal{M}_i$ for $i \in \{0, 1\}$. Since e_i is left exact and commutes with coproducts, we are reduced to proving that $e_i(X) \times_{e_i(X) \coprod e_i(Y)} e_i(Y)$ is an initial object of $\mathcal{M}_i$, which follows from the fact that coproducts are disjoint in the ∞-topos $\mathcal{M}_i$.

(iv) Every groupoid object of $\mathfrak{X}$ is effective. Let $X_\bullet$ be a groupoid object of $\mathfrak{X}$ having geometric realization $X \in \mathfrak{X}$. We wish to show that the canonical map $w : X_1 \to X_0 \times_X X_0$ is an equivalence in $\mathfrak{X}$. For this, it suffices to show that $e_i(w)$ is an equivalence for $i \in \{0, 1\}$. Since e_i is left exact and commutes with geometric realizaiton, we are reduced to proving that $e_i(X_1) \to e_i(X_0) \times_{|e_i(X_\bullet)|} e_i(X_0)$ is an equivalence. Since e_i is left exact, $e_i(X_\bullet)$ is a groupoid object of $\mathcal{M}_i$, which is effective by virtue of the fact that $\mathcal{M}_i$ is an ∞-topos.

$\square$

A.9   Exit Paths and Constructible Sheaves

Let A be a partially ordered set and let X be a space equipped with an A-stratificatin $f : X \to A$. Our goal in this section is to prove that, if X is sufficiently well-behaved, then the ∞-category of A-constructible objects of $\mathcal{S}hv(X)$ can be identified with the ∞-category $\mathrm{Fun}(\mathrm{Sing}^A(X), \mathcal{S})$, where $\mathrm{Sing}^A(X)$ is the ∞-category of exit paths defined in §A.6. In fact, we will give an explicit construction of this equivalence, generalizing the analysis we carried out for locally constant sheaves in §A.4. First, we need to establish a bit of terminology.

Notation A.9.1. Let A be a partially ordered set and let X be a paracompact A-stratified space. We let $\mathbf{A}_X$ denote the category $(\mathcal{S}et_\Delta)_{/\mathrm{Sing}^A(X)}$, which we regard as endowed with the covariant model structure described in §T.2.1.4. Let $\mathcal{B}(X)$ denote the partially ordered collection of all open F_σ subsets of X. We let $\mathcal{S}hv(X)$ denote the full subcategory of $\mathcal{P}(\mathcal{B}(X))$ spanned by those objects which are sheaves with respect to the natural Grothendieck topology on $\mathcal{B}(X)$.

Proposition T.4.2.4.4 and Theorem T.2.2.1.2 furnish a chain of equivalences of ∞-categories

$$\mathrm{Fun}(\mathrm{Sing}^A(X), \mathcal{S}) \leftarrow \mathrm{N}((\mathcal{S}\mathrm{et}_\Delta^{\mathfrak{C}[\mathrm{Sing}^A(X)]})^\circ) \to \mathrm{N}(\mathbf{A}_X^\circ).$$

Construction A.9.2. We define a functor $\theta : \mathcal{B}(X)^{op} \times \mathbf{A}_X \to \mathcal{S}\mathrm{et}_\Delta$ by the formula

$$\theta(U, Y) = \mathrm{Fun}_{\mathrm{Sing}^A(X)}(\mathrm{Sing}^A(U), Y).$$

Note that if $Y \in \mathbf{A}_X$ is fibrant, then $Y \to \mathrm{Sing}^A(X)$ is a left fibration so that each of the simplicial sets $\theta(U, Y)$ is a Kan complex. Passing to the nerve, θ induces a map of ∞-categories $\mathrm{N}(\mathcal{B}(X)^{op}) \times \mathrm{N}(\mathbf{A}_X^\circ) \to \mathcal{S}$, which we will identify with a map of ∞-categories

$$\Psi_X : \mathrm{N}(\mathbf{A}_X^\circ) \to \mathcal{P}(\mathcal{B}(X)).$$

We are now ready to state the main result of this section.

Theorem A.9.3. *Let X be a paracompact topological space which is locally of singular shape and is equipped with a conical A-stratification, where A is a partially ordered set satisfying the ascending chain condition. Then the functor Ψ_X induces an equivalence $\mathrm{N}(\mathbf{A}_X^\circ) \to \mathcal{S}\mathrm{hv}^A(X)$.*

The proof of Theorem A.9.3 will be given at the end of this section, after we have developed a number of preliminary ideas. For later use, we record the following easy consequence of Theorem A.9.3:

Corollary A.9.4. *Let X be a paracompact topological space which is locally of singular shape and is equipped with a conical A-stratification, where A is a partially ordered set satisfying the ascending chain condition. Then the inclusion $i : \mathrm{Sing}^A(X) \hookrightarrow \mathrm{Sing}(X)$ is a weak homotopy equivalence of simplicial sets.*

Proof. Let X' denote the topological space X equipped with the trivial stratification. The inclusion i induces a pullback functor $i^* : \mathrm{N}(\mathbf{A}_X'^\circ) \to \mathrm{N}(\mathbf{A}_X^\circ)$, and we have an evident natural transformation $\alpha : \Psi_X \circ i^* \to \Psi_{X'}$ from $\mathrm{N}(\mathbf{A}_X'^\circ)$ to $\mathcal{S}\mathrm{hv}(X)$. We claim that α is an equivalence. Since both functors take values in the full subcategory of hypercomplete objects of $\mathcal{S}\mathrm{hv}(X)$ (Lemma A.9.10 and Proposition A.5.9), it suffices to show that $\alpha(Y)$ is ∞-connective for each $Y \in \mathrm{N}(\mathbf{A}_X'^\circ)$. For this, it suffices to show that $x^*\alpha(Y)$ is an equivalence for every point $x \in X$ (Lemma A.3.9). Using Proposition A.9.16, we can reduce to the case $X = \{x\}$ where the result is obvious. Applying the functor of global sections to α, we deduce that for every Kan fibration $Y \to \mathrm{Sing}(X)$ the restriction map

$$\mathrm{Fun}_{\mathrm{Sing}(X)}(\mathrm{Sing}(X), Y) \to \mathrm{Fun}_{\mathrm{Sing}(X)}(\mathrm{Sing}^A(X), Y)$$

is a homotopy equivalence of Kan complexes, which is equivalent to the assertion that i is a weak homotopy equivalence. $\square$

We now turn to the proof of Theorem A.9.3 itself. Our first objective is to show that the functor Ψ_X takes values in the the full subcategory $\mathcal{S}\mathrm{hv}(X) \subseteq \mathcal{P}(\mathcal{B}(X))$.

Lemma A.9.5. *Let A be a partially ordered set, let X be a paracompact topological space equipped with a conical A-stratification. The functor $\Psi_X : \mathrm{N}(\mathbf{A}_X^\circ) \to \mathcal{P}(\mathcal{B}(X))$ factors through the full subcategory $\mathcal{S}\mathrm{hv}(X) \subseteq \mathcal{P}(\mathcal{B}(X))$.*

Proof. Let $U \in \mathcal{B}(X)$, and let $S \subseteq \mathcal{B}(U)$ be a covering sieve on U. In view of Theorem T.4.2.4.1, it will suffice to show that for every left fibration $Y \to \mathrm{Sing}^A(X)$, the canonical map

$$\mathrm{Fun}_{\mathrm{Sing}^A(X)}(\mathrm{Sing}^A(U), Y) \to \varprojlim_{V \in S} \mathrm{Fun}_{\mathrm{Sing}^A(X)}(\mathrm{Sing}^A(V), Y)$$

exhibits the Kan complexes $\mathrm{Fun}_{\mathrm{Sing}^A(X)}(\mathrm{Sing}^A(U), Y)$ as a homotopy limit of the diagram of Kan complexes $\{\mathrm{Fun}_{\mathrm{Sing}^A(X)}(\mathrm{Sing}^A(V), Y)\}_{V \in S}$. For this, it suffices to show that $\mathrm{Sing}^A(U)$ is a homotopy colimit of the

simplicial sets $\{\mathrm{Sing}^A(V)\}_{V \in S}$ in the category $(\mathcal{S}\mathrm{et}_\Delta)_{/\mathrm{Sing}^A(X)}$, endowed with the covariant model structure. This follows from the observation that the covariant model structure on $(\mathcal{S}\mathrm{et}_\Delta)_{/\mathrm{Sing}^A(X)}$ is a localization of the Joyal model structure, and $\mathrm{Sing}^A(U)$ is a homotopy colimit of $\{\mathrm{Sing}^A(V)\}_{V \in S}$ with respect to the Joyal model structure (by Theorems A.7.1 and T.4.2.4.1). $\qquad \square$

Remark A.9.6. Let X be a paracompact space equipped with an A-stratification. For each open F_σ subset U of X, the composition of $\Psi_X : \mathrm{N}(\mathbf{A}^\circ) \to \mathcal{S}\mathrm{hv}(X)$ with the evaluation functor $\mathcal{F} \mapsto \mathcal{F}(U)$ from $\mathcal{S}\mathrm{hv}(X)$ to $\mathcal{S}$ is equivalent to the functor $\mathrm{N}(\mathbf{A}_X^\circ) \to \mathcal{S}$ corepresented by (a fibrant replacement for) the object $\mathrm{Sing}^A(U) \in \mathbf{A}_X$. It follows that Ψ_X preserves small limits.

Remark A.9.7. Combining Remark A.9.6 with Proposition T.5.5.6.16, we deduce that the functor

$$\Psi_X : \mathrm{Fun}(\mathrm{Sing}^A(X), \mathcal{S}) \simeq \mathrm{N}(\mathbf{A}_X^\circ) \to \mathcal{S}\mathrm{hv}(X)$$

preserves n-truncated objects for each $n \geq -1$. Since every object $F \in \mathrm{Fun}(\mathrm{Sing}^A(X), \mathcal{S})$ equivalent to a limit of truncated objects (since Postnikov towers in $\mathcal{S}$ are convergent), we deduce from Remark A.9.6 that $\Psi_X(F)$ is also equivalent to a limit of truncated objects, and therefore hypercomplete.

We now discuss the functorial behavior of the map Ψ_X. Let $f : X' \to X$ be a continuous map of paracompact spaces. Let A be a partially ordered set such that X is endowed with an A-stratification. Then X' inherits an A-stratification. The map f determines a morphism of simplicial sets $\mathrm{Sing}^A(X') \to \mathrm{Sing}^A(X)$; let $r : \mathbf{A}_X \to \mathbf{A}_{X'}$ be the associated pullback functor and $R : \mathrm{N}(\mathbf{A}_X^\circ) \to \mathrm{N}(\mathbf{A}_X'^\circ)$ the induced map of ∞-categories. For each $U \in \mathcal{B}(X)$, we have $f^{-1}U \in \mathcal{B}(X')$. The canonical map $\mathrm{Sing}^A(f^{-1}U) \to \mathrm{Sing}^A(U)$ induces a map $\theta_X(U, Y) \to \theta_{X'}(f^{-1}U, r(Y))$. These maps together determine a natural transformation of functors $\Psi_X \to f_* \Psi_{X'} R$ from $\mathrm{N}(\mathbf{A}_X^\circ)$ to $\mathcal{S}\mathrm{hv}(X)$. We let $\phi_{X',X} : f^* \Psi_X \to \Psi_{X'} R$ denote the adjoint transformation (which is well-defined up to homotopy).

Example A.9.8. If X' is an open F_σ subset of X, then the pullback functor $f^* : \mathcal{S}\mathrm{hv}(X) \to \mathcal{S}\mathrm{hv}(X')$ can be described as the restriction along the inclusion of partially ordered sets $\mathcal{B}(X') \subseteq \mathcal{B}(X)$. In this case, the natural transformation $\phi_{X',X}$ can be chosen to be an isomorphism of simplicial sets, since the maps $\theta_X(U, Y) \to \theta_{X'}(U, r(Y))$ are isomorphisms for $U \subseteq X'$.

Lemma A.9.9. *Let X be a paracompact topological space equipped with an A-stratification. Let $a \in A$, let $X' = X_a$, and let $f : X' \to X$ denote the inclusion map. Assume that X_a is paracompact. Then the natural transformation $\phi_{X',X}$ defined above is an equivalence.*

Proof. Fix a left fibration $M \to \mathrm{Sing}^A(X)$, and let $M' = M \times_{\mathrm{Sing}^A(X)} \mathrm{Sing}(X_a)$. We wish to show that $\phi_{X',X}$ induces an equivalence of sheaves $f^* \Psi_X(M) \to \Psi_{X_a}(M')$. This assertion is local on X_a. We may therefore use Lemma A.5.16 (and Example A.9.8) to reduce to the case where X has the form $Z \times C(Y)$, where Y is an $A_{>a}$-stratified space. Corollary T.7.1.5.6 implies that the left hand side can be identified with the (filtered) colimit $\varinjlim_V (\Psi_X(Y))(V)$, where V ranges over the collection of all open neighborhoods of Z in $Z \times C(Y)$. In view of Lemma A.5.12, it suffices to take the same limit indexed by those open neighborhoods of the form V_g, where $g : Z \to (0, \infty)$ is a continuous function. It will therefore suffice to show that each of the maps $\Psi_X(Y)(V_g) \to \Psi_{X_a}(Y')(Z)$ is a homotopy equivalence. This map is given by the restriction

$$\mathrm{Fun}_{\mathrm{Sing}^A(X)}(\mathrm{Sing}^A(V_g), Y) \to \mathrm{Fun}_{\mathrm{Sing}^A(X)}(\mathrm{Sing}(Z), Y).$$

To show that this map is a homotopy equivalence, it suffices to show that the inclusion $i : \mathrm{Sing}(Z) \hookrightarrow \mathrm{Sing}^A(V_g)$ is a covariant equivalence in $\mathrm{Sing}^A(X)$. We will show that i is left anodyne. Let $h : C(Y) \times [0, 1] \to C(Y)$ be the map which carries points $(y, s, t) \in Y \times (0, \infty) \times (0, 1]$ to $(y, st) \in Y \times (0, \infty)$, and every other point to the cone point of $C(Y)$. Then h induces a homotopy $H : V_g \times [0, 1] \to V_g$ from the projection $V_g \to Z \subseteq V_g$ to the identity map on V_g. The homotopy H determines a natural transformation from the

projection $\mathrm{Sing}^A(V_g) \to \mathrm{Sing}(Z)$ to the identity map from $\mathrm{Sing}^A(V_g)$ to itself, which exhibits the map i as a retract of the left anodyne inclusion

$$(\mathrm{Sing}(Z) \times \Delta^1) \coprod_{\mathrm{Sing}(Z) \times \{0\}} (\mathrm{Sing}^A(V_g) \times \{0\}) \subseteq \mathrm{Sing}^A(V_g) \times \Delta^1.$$

$\square$

Lemma A.9.10. *Let X be a paracompact topological space which is locally of singular shape and is equipped with a conical A-stratification. Then the functor $\Psi_X : \mathrm{N}(\mathbf{A}_X^\circ) \to \mathrm{Shv}(X)$ factors through the full subcategory $\mathrm{Shv}^A(X) \subseteq \mathrm{Shv}(X)$ spanned by the A-constructible sheaves on X.*

Proof. Choose a left fibration $Y \to \mathrm{Sing}^A(X)$ and an element $a \in A$; we wish to prove that $(\Psi_X(Y)|X_a) \in \mathrm{Shv}(X_a)$ is locally constant. The assertion is local on X, so we may assume without loss of generality that X has the form $Z \times C(Y)$ (Lemma A.5.16), so that $X_a \simeq Z$ is locally of singular shape (Remark A.5.18). Using Lemma A.9.9, we can replace X by Z, and thereby reduce to the case where X consists of only one stratum. In this case, the desired result follows from Theorem A.4.19. $\square$

Lemma A.9.11. *Let X be a paracompact topological space of the form $Z \times C(Y)$, and let $\pi : X \to Z$ denote the projection map. Then the pullback functor $\pi^* : \mathrm{Shv}(Z) \to \mathrm{Shv}(X)$ is fully faithful.*

Proof. Fix an object $\mathcal{F} \in \mathrm{Shv}(Z)$; we will show that the unit map $\mathcal{F} \to \pi_* \pi^* \mathcal{F}$ is an equivalence. In view of Corollary T.7.1.4.4, we may suppose that there exists a map of topological spaces $Z' \to Z$ such that $\mathcal{F}$ is given by the formula $U \mapsto \mathrm{Map}_{\mathrm{Top}_{/Z}}(U, Z')$. Using the results of §T.7.1.5, we may suppose also that $\pi^* \mathcal{F}$ is given by the formula $V \mapsto \mathrm{Map}_{\mathrm{Top}_{/X}}(V, Z' \times_Z X)$. It will suffice to show that the induced map $\mathcal{F}(U) \to (\pi^* \mathcal{F})(\pi^{-1}U)$ is a homotopy equivalence for each $U \in \mathcal{B}(Z)$. Replacing Z by U, we may assume that $U = Z$. In other words, we are reduced to proving that the map

$$\mathrm{Map}_{\mathrm{Top}_{/Z}}(Z, Z') \to \mathrm{Map}_{\mathrm{Top}_{/Z}}(X, Z')$$

is a homotopy equivalence of Kan complexes. This follows from the observation that there is a deformation retraction from X onto Z (in the category $\mathrm{Top}_{/Z}$ of topological spaces over Z). $\square$

Lemma A.9.12. *Let X be a paracompact space of the form $Z \times C(Y)$, let $\pi : X \to Z$ denote the projection map, and let $i : Z \to X$ be the inclusion. Let $\mathcal{F} \in \mathrm{Shv}(X)$ be a sheaf whose restriction to $Z \times Y \times (0, \infty)$ is foliated. Then the canonical map $\pi_* \mathcal{F} \to i^* \mathcal{F}$ is an equivalence.*

Proof. It will suffice to show that for every $U \in \mathcal{B}(Z)$, the induced map $\mathcal{F}(\pi^{-1}(U)) \to (i^* \mathcal{F})(U)$ is a homotopy equivalence. Replacing Z by U, we can assume $U = Z$. Using Corollary T.7.1.5.6, we can identify $(i^* \mathcal{F})(Z)$ with the filtered colimit $\varinjlim_V \mathcal{F}(V)$, where V ranges over all open neighborhoods of Z in X. In view of Lemma A.5.12, it suffices to take the colimit over the cofinal collection of open sets of the form V_f, where $f : Z \to (0, \infty)$ is a continuous map. To prove this, it suffices to show that each of the restriction maps $\theta : \mathcal{F}(X) \to \mathcal{F}(V_f)$ is an equivalence. Let $W \subseteq Z \times Y \times (0, \infty)$ be the set of triples (z, y, t) such that $t > \frac{f(z)}{2}$, so that we have a pullback diagram

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\theta} & \mathcal{F}(V_f) \\ \downarrow & & \downarrow \\ \mathcal{F}(W) & \xrightarrow{\theta'} & \mathcal{F}(W \cap V_f). \end{array}$$

To prove that θ is a homotopy equivalence, it suffices to show that θ' is a homotopy equivalence. The map θ' fits into a commutative diagram

$$\mathcal{F}(W) \xrightarrow{\theta'} \mathcal{F}(W \cap V_f)$$
$$(s^* \mathcal{F})(Z \times Y),$$

where $s : Z \times Y \to W \cap V_f$ is the section given by the continuous map $\frac{3}{4} f : Z \to (0, \infty)$. Since $\mathcal{F}$ is foliated, Proposition A.2.5 and Lemma A.2.9 guarantee that the vertical maps in this diagram are both equivalences, so that θ' is an equivalence as well. $\qquad\square$

Lemma A.9.13. *Let A be a partially ordered set containing an element a. Let X be a paracompact $A_{\geq a}$-stratified topological space of the form $Z \times C(Y)$, where Y is an $A_{>a}$-stratified space. Let $\mathcal{C} = \mathrm{Shv}^A(X)$. Let $j : Z \times Y \times (0, \infty) \to X$ denote the inclusion and let $\mathcal{C}_0$ denote the intersection of $\mathcal{C}$ with the essential image of the left adjoint $j_! : \mathrm{Shv}(Z \times Y \times (0, \infty)) \to \mathrm{Shv}(X)$ to the pullback functor j^*. Let $\pi : X \to Z$ be the projection map, and let $\mathcal{C}_1$ denote the intersection of $\mathcal{C}$ with the essential image of π^* (which is fully faithful by Lemma A.9.11). Then:*

(1) *The inclusion functors $\mathcal{C}_0 \subseteq \mathcal{C}$ and $\mathcal{C}_1 \subseteq \mathcal{C}$ admit right adjoints L_0 and L_1.*

(2) *The functor L_0 is complementary to L_1.*

Proof. Let $i : Z \to X$ be the inclusion map. The functor L_0 is given by the composition $j_! j^*$, and the functor L_1 is given by the composition $\pi^* \pi_*$ (which is equivalent to $\pi^* i^*$ by Lemma A.9.12, and therefore preserves constructibility and pushout diagrams). Since the composition $i^* j_!$ is equivalent to the constant functor $\mathrm{Shv}(Z \times Y \times (0, \infty)) \to \mathrm{Shv}(Z)$ (taking value equal to the initial object of $\mathrm{Shv}(Z)$), the functor L_1 carries every morphism in $\mathcal{C}_0$ to an equivalence. Finally, suppose that α is a morphism in $\mathcal{C}$ such that $L_0(\alpha)$ and $L_1(\alpha)$ are equivalences. Since $j_!$ and π^* are fully faithful, we conclude that $j^*(\alpha)$ and $i^*(\alpha)$ are equivalences, so that α is an equivalence (Corollary A.5.11). $\qquad\square$

Lemma A.9.14. *Let X be a paracompact topological space which is locally of singular shape and is equipped with a conical A-stratification. Then the full subcategory $\mathrm{Shv}^A(X) \subseteq \mathrm{Shv}(X)$ is stable under finite colimits in $\mathrm{Shv}(X)$.*

Proof. Let $\mathcal{F} \in \mathrm{Shv}(X)$ be a finite colimit of A-constructible sheaves; we wish to show that $\mathcal{F}|X_a$ is constructible for each $a \in A$. The assertion is local; we may therefore assume that X has the form $Z \times C(Y)$ (Lemma A.5.16). Then $X_a \simeq Z$ is paracompact and locally of singular shape (Remark A.5.18) so the desired result follows from Corollary A.1.16. $\qquad\square$

Lemma A.9.15. *Let X be a paracompact topological space which is locally of singular shape and equipped with a conical A-stratification, where A satisfies the ascending chain condition. Then the functor $\Psi_X : \mathrm{N}(\mathbf{A}_X^o) \to \mathrm{Shv}(X)$ preserves finite colimits.*

Proof. Fix a diagram $p : K \to \mathrm{N}(\mathbf{A}_X^o)$ having a colimit Y, where K is finite. We wish to prove that the induced map $\alpha : \varinjlim(\Psi_X \circ p) \to \Psi_X(Y)$ is an equivalence. Lemma A.9.10 implies that $\Psi_X(Y) \in \mathrm{Shv}^A(X)$, and is therefore hypercomplete (Proposition A.5.9). Similarly, $\varinjlim(\Psi_X \circ p)$ is a finite colimit in $\mathrm{Shv}(X)$ of A-constructible sheaves, hence A-constructible (Lemma A.9.14) and therefore hypercomplete. Consequently, to prove that α is an equivalence, it will suffice to show that α is ∞-connective. This condition can be tested pointwise (Lemma A.3.9); we may therefore reduce to the problem of showing that α is an equivalence when restricted to each stratum X_a. Shrinking X if necessary, we may suppose that X has the form $Z \times C(Y)$ (Lemma A.5.16) so that $X_a \simeq Z$ is paracompact and locally of singular shape (Remark A.5.18). Using Lemma A.9.9 we can replace X by X_a and thereby reduce to the case of a trivial stratification. In this case, the functor Ψ_X is a fully faithful embedding (Theorem A.4.19) those essential image is stable under finite colimits (Corollary A.1.16), and therefore preserves finite colimits. $\qquad\square$

We can use the same argument to prove a sharpened version of Lemma A.9.9 (at least in case where A satisfies the ascending chain condition):

Proposition A.9.16. *Let A be a partially ordered set which satisfies the ascending chain condition, and let $f : X' \to X$ be a continuous map between paracompact topological spaces which are locally of singular shape. Suppose that X is endowed with a conical A-stratification, and that the induced A-stratification of X' is also conical. Then the natural transformation $\phi_{X',X}$ is an equivalence of functors from $N(\mathbf{A}_X^o)$ to $Shv(X')$.*

Lemma A.9.17. *Let X be a topological space of singular shape. For every point $x \in X$, there exists an open neighborhood U of x such that the inclusion of Kan complexes $Sing(U) \to Sing(X)$ is nullhomotopic.*

Proof. Let $K = Sing(X) \in \mathcal{S}$, and let $\pi : X \to *$ denote the projection map. Since X is of singular shape, there exists a morphism $\mathbf{1} \to \pi^* K$ in $Shv(X)$ The geometric realization $|Sing(X)|$ is a CW complex. Since X is of singular shape, composition with the counit map $v : |Sing(X)| \to X$ induces a homotopy equivalence of Kan complexes $Map_{Top}(X, |Sing(X)|) \to Map_{Top}(|Sing(X)|, |Sing(X)|)$. In particular, there exists a continuous map $s : X \to |Sing(X)|$ such that $s \circ v$ is homotopic to the identity. Choose a contractible open subset $V \subseteq |Sing(X)|$ containing $s(x)$, and let $U = s^{-1}(V)$. We claim that the inclusion $i : Sing(U) \to Sing(X)$ is nullhomotopic. To prove this, it suffices to show that $|i| : |Sing(U)| \to |Sing(X)|$ is nullhomotopic. This map is homotopic to the composition $s \circ v \circ |i|$, which factors through the contractible open subset $V \subseteq |Sing(X)|$. $\qquad\square$

Proof of Proposition A.9.16. Let $Y \in N(\mathbf{A}_X^o)$, and let $Y' = Y \times_{Sing^A(X)} Sing^A(X')$. We wish to prove that the map $\alpha : f^* \Psi_X(Y) \to \Psi_{X'}(Y')$ is an equivalence in $Shv(X')$. Lemma A.9.10 implies that $\Psi_X(Y) \in Shv^A(X)$, so that $f^* \Psi_X(Y) \in Shv^A(X')$. Similarly, $\Psi_{X'}(Y') \in Shv^A(X')$, so that both $f^* \Psi_X(Y)$ and $\Psi_{X'}(Y')$ are hypercomplete (Proposition A.5.9). To prove that α is an equivalence, it will suffice to show that α is ∞-connective. Since this condition can be tested pointwise, it will suffice to show that α induces an equivalence after restricting to each stratum X_a' of X'. Using Lemma A.5.16 and Remark A.5.18, we can shrink X and X' so that X_a and X_a' are again paracompact and locally of singular shape. Applying Lemma A.9.9, we can reduce to the case where $X = X_a$ and $X' = X_a'$. Shrinking X further (using Lemma A.9.17), we may assume that $Y \simeq Sing(X) \times K$ for some Kan complex $K \in \mathcal{S}$. In this case, Example A.4.18 allows us to identify $\Psi_X(Y)$ with the pullback $\pi^* K$ and $\Psi_{X'}(Y')$ with $\pi'^* K$, where $\pi : X \to *$ and $\pi' : X' \to *$ denote the projection maps. Under these identifications, the natural transformation $\phi_{X',X}(Y)$ is induced by the canonical equivalence $f^* \circ \pi^* \simeq (\pi \circ f)^* = \pi'^*$. $\qquad\square$

Proof of Theorem A.9.3. We will prove more generally that for every $U \in \mathcal{B}(X)$, the functor $\Psi_U : N(\mathbf{A}_U^o) \to Shv^A(U)$ is an equivalence of ∞-categories. The proof proceeds by induction on $rk(U)$, where the rank functor rk is defined in Remark A.5.17.

Let S denote the partially ordered set of all open sets $V \in \mathcal{B}(U)$ which are homeomorphic to a product $Z \times C(Y)$, where Y is an $A_{>a}$-stratified space, and $Z \times C(Y)$ is endowed with the induced $A_{\geq a}$-stratification. For every such open set V, let $\chi_V \in Shv(X)$ be the sheaf determined by the formula

$$\chi_V(W) = \begin{cases} * & \text{if } W \subseteq V \\ \emptyset & \text{otherwise.} \end{cases}$$

Let α denote the canonical map $\varinjlim_V \chi_V \to \chi_U$. For each point $x \in U$, the stalk of the colimit $\varinjlim_V \chi_V$ at x is homotopy equivalent to the nerve of the partially ordered set $S_x = \{V \in S : x \in V\}$. It follows from Lemma A.5.16 that the partially ordered set S_x^{op} is filtered, so that $|S_x|$ is contractible: consequently, the map α is ∞-connective. Consequently, α induces an equivalence $\varinjlim_V \chi_V \to \chi_U$ in the hypercomplete ∞-topos $Shv(X)^\wedge$. Applying Theorem T.6.1.3.9 to the ∞-topos $Shv(X)^\wedge$, we conclude that $Shv(U)^\wedge \simeq Shv(X)^\wedge_{/\chi_U}$ is equivalent to the homotopy limit of the diagram of ∞-categories $\{Shv(V)^\wedge \simeq Shv(X)^\wedge_{/\chi_V}\}_{V \in S}$. Proposition A.5.9 guarantees that $Shv^A(U) \subseteq Shv(U)^\wedge$ (and similarly $Shv^A(V) \subseteq Shv(V)^\wedge$) for each $V \in S$). Since the property of being constructible can be tested locally, we obtain an equivalence

$$Shv^A(U) \simeq \varprojlim \{Shv^A(V)\}_{V \in S}.$$

We next show that the restriction maps $\mathrm{N}(\mathbf{A}_U^\circ) \to \mathrm{N}(\mathbf{A}_V^\circ)$ exhibit $\mathrm{N}(\mathbf{A}_U^\circ)$ as the homotopy limit of the diagram of ∞-categories $\{\mathrm{N}(\mathbf{A}_V^\circ)\}_{V \in S}$. In view of the natural equivalences

$$\mathrm{Fun}(\mathrm{Sing}^A(V), \mathcal{S}) \leftarrow \mathrm{N}((\mathcal{S}et_\Delta^{\mathfrak{C}[\mathrm{Sing}^A(V)]})^\circ) \to \mathrm{N}(\mathbf{A}_V^\circ),$$

it will suffice to show that the canonical map

$$\mathrm{Fun}(\mathrm{Sing}^A(U), \mathcal{S}) \to \varprojlim \{\mathrm{Fun}(\mathrm{Sing}^A(V), \mathcal{S})\}_{V \in S}$$

is an equivalence. This follows immediately from Theorem A.7.1.

We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{N}(\mathbf{A}_U^\circ) & \longrightarrow & \varprojlim_{V \in S} \mathrm{N}(\mathbf{A}_V^\circ) \\
\downarrow & & \downarrow \\
\mathrm{Shv}^A(U) & \longrightarrow & \varprojlim_{V \in S} \mathrm{Shv}^A(V)
\end{array}
$$

where the vertical maps are equivalences. Consequently, to prove that Ψ_U is an equivalence, it will suffice to show that Ψ_V is an equivalence for each $V \in S$. Replacing U by V, we can assume that U has the form $Z \times C(Y)$. We will also assume that Z is nonempty (otherwise there is nothing to prove).

Let $U' = Z \times Y \times (0, \infty)$, which we regard as an open subset of U. Let $\mathcal{C}_0 \subseteq \mathrm{N}(\mathbf{A}_U^\circ)$ be the full subcategory spanned by the left fibrations $Y \to \mathrm{Sing}^A(U)$ which factor through $\mathrm{Sing}^A(U')$, and let $\mathcal{C}_1 \subseteq \mathrm{N}(\mathbf{A}_U^\circ)$ be the full subcategory spanned by the Kan fibrations $Y \to \mathrm{Sing}^A(U)$. Under the equivalence $\mathrm{N}(\mathbf{A}_U^\circ) \simeq \mathrm{Fun}(\mathrm{Sing}^A(U), \mathcal{S})$, these correspond to the full subcategories described in the dual of Proposition A.8.14 (where $\mathcal{A} = \mathcal{S}$ and $p : \mathrm{Sing}^A(U) \to \Delta^1$ is characterized by the requirements that $p^{-1}\{0\} = \mathrm{Sing}(U_a)$ and $p^{-1}\{1\} = \mathrm{Sing}^A(U')$). It follows that the inclusions $\mathcal{C}_0, \mathcal{C}_1 \subseteq \mathrm{N}(\mathbf{A}_U^\circ)$ admit right adjoints L_0 and L_1, and that L_0 is complementary to L_1. Let $\mathcal{C}_0', \mathcal{C}_1' \subseteq \mathrm{Shv}^A(U)$ be defined as in Lemma A.9.13, so that we again have right adjoints $L_0' : \mathrm{Shv}^A(U) \to \mathcal{C}_0'$ and $L_1' : \mathrm{Shv}^A(U) \to \mathcal{C}_1'$ which are complementary. We will prove that the functor Ψ_U is an equivalence of ∞-categories by verifying the hypotheses of (the dual version of) Proposition A.8.13:

(2) The functor Ψ_U restricts to an equivalence $\mathcal{C}_0 \to \mathcal{C}_0'$. Let $Y \to \mathrm{Sing}^A(U')$ be an object of $\mathcal{C}_0$. Then $(\Psi_U(Y))(W)$ is empty if W is not contained in U', so that $\Psi_U(Y) \in \mathcal{C}_0'$. Moreover, the composition of $\Psi_U | \mathcal{C}_0$ with the equivalence $\mathcal{C}_0' \simeq \mathrm{Shv}^A(U')$ coincides with the functor $\Psi_{U'}$. Since the strata U_b' are empty unless $b > a$, while U_a is nonempty (since $Z \neq \emptyset$), we have $\mathrm{rk}(U') < \mathrm{rk}(U)$ so that $\Psi_{U'}$ is an equivalence of ∞-categories by the inductive hypothesis.

(2') We must show that the functor Ψ_U restricts to an equivalence $\mathcal{C}_1 \to \mathcal{C}_1'$. Let $\pi : U \to Z$ denote the projection. We have a diagram of ∞-categories

$$
\begin{array}{ccc}
\mathrm{N}(\mathbf{A}_Z^\circ) & \longrightarrow & \mathrm{N}(\mathbf{A}_U^\circ) \\
\downarrow {\scriptstyle \Psi_Z} & & \downarrow {\scriptstyle \Psi_U} \\
\mathrm{Shv}(Z) & \xrightarrow{\ \pi^*\ } & \mathrm{Shv}(U)
\end{array}
$$

which commutes up to homotopy (Proposition A.9.16). The upper horizontal arrow is fully faithful, and its essential image is precisely the ∞-category $\mathcal{C}_1$. Consequently, it suffices to show that the composite map $\pi^* \Psi_Z$ is a fully faithful embedding whose essential image is precisely $\mathcal{C}_1'$. Theorem A.4.19 implies that Ψ_Z is fully faithful, and that its essential image is the full subcategory of $\mathrm{Shv}(Z)$ spanned by the locally constant sheaves. The desired result now follows from the definition of $\mathcal{C}_1'$.

(3) The functor Ψ_U preserves pushouts. This follows from Lemma A.9.15.

(4) If $\alpha : Y_0 \to Y$ is a morphism which exhibits $Y_0 \in \mathcal{C}_0$ as a $\mathcal{C}_0$-colocalization of $Y \in \mathcal{C}_1$, then $\Psi_U(\alpha)$ exhibits $\Psi_U(Y_0)$ as a $\mathcal{C}'_0$-localization of $\Psi_U(Y_1)$. Unwinding the definitions, α induces an equivalence of left fibrations $Y_0 \to Y \times_{\mathrm{Sing}^A(U)} \mathrm{Sing}^A(U')$, and we must show that for each $W \in \mathcal{B}(U')$ that the induced map $\mathrm{Fun}_{\mathrm{Sing}^A(U)}(\mathrm{Sing}^A(W), Y_0) \to \mathrm{Fun}_{\mathrm{Sing}^A(U)}(\mathrm{Sing}^A(W), Y)$ is a homotopy equivalence. This is clear, since the condition that $W \subseteq U'$ guarantees that any map $\mathrm{Sing}^A(W) \to Y$ factors uniquely through the fiber product $Y \times_{\mathrm{Sing}^A(U)} \mathrm{Sing}^A(U')$.

$\square$

Appendix B

Categorical Patterns

Let S be a simplicial set, and let $(\mathcal{S}et_\Delta^+)_{/S}$ denote the category of marked simplicial sets $\overline{X} = (X, M)$ equipped with a map $X \to S$. According to Proposition T.3.1.3.7, there is a simplicial model structure on the category $(\mathcal{S}et_\Delta^+)_{/S}$ (the *coCartesian model structure*) whose fibrant objects can be identified with coCartesian fibrations of simplicial sets $X \to S$. In practice, there are a variety of related conditions that a map $p : X \to S$ might be required to satisfy:

(a) A coCartesian fibration $p : X \to S$ is classified by a functor $\chi : S \to \mathcal{C}at_\infty$. In various contexts it is natural to demand that the functor χ carry certain diagrams in S to limit diagrams in $\mathcal{C}at_\infty$.

(b) If $p : X \to S$ is a locally coCartesian fibration, then every edge $\phi : s \to s'$ in S determines a functor $\phi_! : X_s \to X_{s'}$. If we are given a 2-simplex σ :

$$\begin{array}{ccc} & s' & \\ {}^{\phi}\nearrow & & \searrow^{\psi} \\ s & \xrightarrow{\ \ \theta\ \ } & s'' \end{array}$$
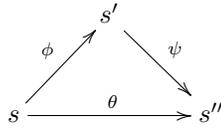
then we obtain a natural transformation $u_\sigma : \theta_! \to \psi_! \circ \phi_!$. Moreover, p is a coCartesian fibration if and only if each of these natural transformations is an equivalence (Remark T.2.4.2.9). In general, we might demand that u_σ be an equivalence only for some specified collection of 2-simplices σ of S.

(c) Let $p : X \to S$ be an inner fibration of simplicial sets. Then p is a locally coCartesian fibration if and only if, for every edge $e : s \to s'$ and every vertex $x \in X_s$, there exists a locally coCartesian edge $\overline{e} : x \to x'$ with $p(\overline{e}) = e$. In general, we might demand that $\overline{e}$ exists only for a specific class of edges e of S.

In specific situations, we might be interested in studying maps $p : X \to S$ which satisfy some combination of the conditions suggested in (a), (b) and (c). For example, the notion of ∞-operad introduced in Chapter 2 can be described in this way (see Definition 2.1.1.10). Our goal in this appendix is to develop a variant of the coCartesian model structure on $(\mathcal{S}et_\Delta^+)_{/S}$, which is adapted to these types of applications. We begin by introducing some terminology.

Definition B.0.18. Let S be a simplicial set. A *categorical pattern* on S is a triple $(M_S, T, \{p_\alpha : K_\alpha^\triangleleft \to S'\}_{\alpha \in A})$, where M_S is a collection of edges of S which contains all degenerate edges, T is a collection of 2-simplices of S which contains all degenerate 2-simplices, and $\{p_\alpha : K_\alpha^\triangleleft \to S\}_{\alpha \in A}$ is a collection of maps of simplicial sets which carry each edge of $K_\alpha^\triangleleft$ into M_S and each 2-simplex of $K_\alpha^\triangleleft$ into T.

Suppose we are given a categorical pattern $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^\triangleleft \to S\}_{\alpha \in A})$ on S. A *marked simplicial set* over $\mathfrak{P}$ is a marked simplicial set $\overline{X} = (X, M)$ equipped with a map $f : X \to S$ satisfying the following

condition: for every edge e of X which belongs to M, $f(e)$ belongs to M_S. We let $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ denote the category of marked simplicial sets over $\mathfrak{P}$.

We will say that an object $\overline{X} \in (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ is $\mathfrak{P}$-*fibered* if the following conditions are satisfied:

(1) The underlying map of simplicial sets $f : X \to S$ is an inner fibration.

(2) For each edge $\Delta^1 \to S$ belonging to M_S, the induced map $f' : X \times_S \Delta^1 \to \Delta^1$ is a coCartesian fibration.

(3) An edge e of X belongs to M if and only if $f(e)$ belongs to M_S and e is an f'-coCartesian edge of $X \times_S \Delta^1$.

(4) Given a commutative diagram

$$
\begin{array}{ccc}
\Delta^{\{0,1\}} & \xrightarrow{\ e\ } & X \\
\downarrow & & \downarrow \\
\Delta^2 & \xrightarrow{\ \sigma\ } & S,
\end{array}
$$

if $e \in M$ and $\sigma \in T$, then e determines an f'-coCartesian edge of $X \times_S \Delta^2$, where $f' : X \times_S \Delta^2 \to \Delta^2$ denotes the projection map.

(5) For every index $\alpha \in A$, the induced coCartesian fibration $f_\alpha : X \times_S K_\alpha^\triangleleft \to K_\alpha^\triangleleft$ is classified by a limit diagram $K_\alpha^\triangleleft \to \mathcal{C}at_\infty$.

(6) For every index $\alpha \in A$ and every coCartesian section s of the map f_α, s is an f-limit diagram in X.

We can now state the main result of this appendix:

Theorem B.0.19. *Let $\mathfrak{P}$ be a categorical pattern on a simplicial set S. Then there exists a left proper combinatorial simplicial model structure on $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$, which is uniquely characterized by the following properties:*

(C) *A morphism $f : \overline{X} \to \overline{Y}$ in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ is a cofibration if and only if f induces a monomorphism between the underlying simplicial sets.*

(F) *An object $\overline{X} \in (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ is fibrant if and only if $\overline{X}$ is $\mathfrak{P}$-fibered.*

Example B.0.20. Let S be a simplicial set. The *canonical categorical pattern* on S is the categorical pattern $\mathfrak{P} = (M_S, T, \emptyset)$, where M_S consists of all edges of S and T consists of all 2-simplices of S. Then $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ admits a unique model structure satisfying the conditions of Theorem B.0.19: the *coCartesian model structure* described in §T.3.1.3.

We will give the proof of Theorem B.0.19 in §B.2. Our proof will rely on the construction of a large class of trivial cofibrations in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$, which we carry out in §B.1. In §B.4, we will prove that the model structure on $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ is functorial with respect to the categorical pattern in $\mathfrak{P}$ in a very robust way (Theorem B.4.2). The exact formulation of this functoriality result depends on the notion of a *flat inner fibration* between simplicial sets, which we explain in §B.3.

Warning B.0.21. Our notion of *categorical pattern* is of a somewhat ad-hoc nature. More general results along the lines of Theorem B.0.19 are possible, and there are less general results that would suffice for the applications in this book. Moreover, the proofs the results presented here are somewhat dry and technical. We recommend that most of this appendix be treated as a "black box" by most readers of this book. The exception is §B.3, which can be read independently of the other sections: the notion of flat inner fibration is of some independent interest for its own sake, and plays an important role in the main part of the book.

We close this introduction with a few simple observations about Definition B.0.18 and Theorem B.0.19.

Remark B.0.22. Let $\mathfrak{P}$ be a categorical pattern on a simplicial set S. We will sometimes abuse terminology by saying that a map of simplicial sets $X \to S$ is $\mathfrak{P}$-*fibered* if there exists a collection of edges M in X such that $\overline{X} = (X, M)$ is a $\mathfrak{P}$-fibered object of $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$. In this case, the set M is uniquely determined (requirement (3) of Definition B.0.18).

Remark B.0.23. In the situation of Definition B.0.18, conditions (5) and (6) are automatic whenever the simplicial set K_α is weakly contractible and the diagram p_α is constant.

Remark B.0.24. Let $\mathfrak{P}$ be a categorical pattern on a simplicial set S. For every pair of objects $\overline{X}, \overline{Y} \in (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$, there exists a simplicial set $\mathrm{Map}_S^\sharp(\overline{X}, \overline{Y})$ with the following universal property: for every simplicial set K, we have a canonical bijection

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(K, \mathrm{Map}_S^\sharp(\overline{X}, \overline{Y})) \simeq \mathrm{Hom}_{(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}}(K^\sharp \times \overline{X}, \overline{Y}).$$

This definition of mapping spaces endows $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ with the structure of a simplicial category.

Remark B.0.25. Let $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^\triangleleft \to S'\}_{\alpha \in A})$ be a categorical pattern on a simplicial set S and let $\overline{X} = (X, M)$ be an object of $\mathcal{S}et_\Delta^+$ satisfying conditions (1) through (4) of Definition B.0.18. For each index $\alpha \in A$, let $X_\alpha = X \times_S K_\alpha^\triangleleft$. Then the projection map $q : X_\alpha \to K_\alpha^\triangleleft$ is a coCartesian fibration, classified by a functor $\chi : K_\alpha^\triangleleft \to \mathcal{C}at_\infty$. According to Proposition T.3.3.3.1, the map χ is a limit diagram if and only if the restriction functor $r : Z \to Z_0$ is an equivalence of ∞-categories, where Z denotes the ∞-category of coCartesian sections of q and Z_0 the ∞-category of coCartesian sections of the restriction $X \times_S K_\alpha \to K_\alpha$.

Now suppose that $\overline{X}$ also satisfies condition (6) of Definition B.0.18. In this case, every coCartesian section s of q is a q-limit diagram, so that the map $\mathrm{Map}_Z(s', s) \to \mathrm{Map}_{Z_0}(s'|K, s|K)$ is a homotopy equivalence for any $s' \in Z$ (in fact, the analogous statement is true for *any* section of q). It follows that the functor r is automatically fully faithful. Now r is an equivalence of ∞-categories if and only if it is essentially surjective, which (since r is evidently a categorical fibration) is equivalent to the requirement that r be surjective on vertices. Consequently, in the definition of a $\mathfrak{P}$-fibered object of $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$, we are free to replace assumption (5) by the following apparently weaker condition:

(5′) For each $\alpha \in A$ and every coCartesian section s_0 of the projection $X \times_S K_\alpha \to K_\alpha$, there exists a coCartesian section s of $X \times_S K_\alpha^\triangleleft \to K_\alpha^\triangleleft$ extending s_0.

Remark B.0.26. Let $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^\triangleleft \to S\}_{\alpha \in A})$ be a categorical pattern on a simplicial set S, and suppose we are given a $\mathfrak{P}$-fibered object $\overline{X} = (X, M) \in (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$. Let $\pi : X \to S$ denote the underlying map of simplicial sets. We can define a categorical pattern $\pi^* \mathfrak{P} = (M, T', \{q_\beta : K_\beta^\triangleleft \to X\}_{\beta \in B})$ on X as follows:

(1) The set M is the collection of marked edges of X (in other words, the collection of locally π-coCartesian edges e of X such that $\pi(e) \in M_S$).

(2) The set $T' = \pi^{-1}T$ is the collection of all 2-simplices of X whose images in S belong to T.

(3) We let $\{q_\beta : K_\beta^\triangleleft \to X\}_{\beta \in B}$ be the collection of those diagrams $q : K^\triangleleft \to X$ such that q carries each edge of $K^\triangleleft$ into M, and $\pi \circ q$ belongs to $\{p_\alpha : K_\alpha^\triangleleft \to S\}_{\alpha \in A}$.

Remark B.0.27. Let S be a simplicial set, and suppose we are given a categorical pattern $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^\triangleleft \to S\}_{\alpha \in A})$, where M_S consists of all edges of S, T consists of all 2-simplices of S, each of the simplicial sets K_α is weakly contractible and each of the maps p_α is constant. Then the model structure on $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ described by Theorem B.0.19 coincides with the coCartesian model structure of Example B.0.20: this follows immediately from Remark B.0.23.

B.1 $\mathfrak{P}$-Anodyne Morphisms

Let $\mathfrak{P}$ be a categorical pattern on a simplicial set S. The main step in proving Theorem B.0.19 is to show that there is a sufficiently large supply of trivial cofibrations in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$. To this end, we introduce the following definition:

Definition B.1.1. Let $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^\triangleleft \to S\}_{\alpha \in A})$ be a categorical pattern on a simplicial set S. The collection of $\mathfrak{P}$-*anodyne* morphisms in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ is the smallest weakly saturated (see Definition T.A.1.2.2) class of morphisms which contain all morphisms of the following types:

(A_0) The inclusion $(\Lambda_1^2)^\sharp \coprod_{(\Lambda_1^2)^\flat} (\Delta^2)^\flat \subseteq (\Delta^2)^\sharp$, for every map $\Delta^2 \to S$ belonging to T which carries every edge into M_S.

(A_1) The inclusion $Q^\flat \subseteq Q^\sharp$, where $Q = \Delta^0 \coprod_{\Delta^{\{0,2\}}} \Delta^3 \coprod_{\Delta^{\{1,3\}}} \Delta^0$ and the map $Q \to S$ carries every edge of Q into M_S and every 2-simplex of Q into T.

(B_0) The inclusion $\{0\}^\sharp \subseteq (\Delta^1)^\sharp$ lying over an edge of M_S.

(B_1) For each $\alpha \in A$, the inclusion $K_\alpha^\sharp \subseteq (K_\alpha^\triangleleft)^\sharp$ (where $K_\alpha^\triangleleft$ maps to S via p_α).

(C_0) The inclusion
$$(\Lambda_0^n)^\flat \coprod_{(\Delta^{\{0,1\}})^\flat} (\Delta^{\{0,1\}})^\sharp \subseteq (\Delta^n)^\flat \coprod_{(\Delta^{\{0,1\}})^\flat} (\Delta^{\{0,1\}})^\sharp,$$
for every $n > 1$ and every map $\Delta^n \to S$ whose restriction to $\Delta^{\{0,1,n\}}$ belongs to T.

(C_1) The inclusion $(\Lambda_i^n)^\flat \subseteq (\Delta^n)^\flat$, for every $0 < i < n$ and every map $\Delta^n \to S$.

(C_2) For each $n \geq 1$, $\alpha \in A$, and map $f : \Delta^n \star K_\alpha \to S$ extending $p_\alpha : \{n\} \star K_\alpha \to S$, the inclusion
$$(\partial \Delta^n \star K_\alpha)^\flat \coprod_{(\{n\}\star K_\alpha)^\flat} (\{n\} \star K_\alpha)^\sharp \subseteq (\Delta^n \star K_\alpha)^\flat \coprod_{(\{n\}\star K_\alpha)^\flat} (\{n\} \star K_\alpha)^\sharp).$$

Example B.1.2. Let $\mathfrak{P}$ be a categorical pattern on a simplicial set S, and suppose we are given maps of simplicial sets $A \xrightarrow{i} B \to S$. If i is inner anodyne, then the induced map $A^\flat \to B^\flat$ is a $\mathfrak{P}$-anodyne morphism in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$.

Example B.1.3. Let $\mathfrak{P}$ be a categorical pattern on a simplicial set S, and let $e : \Delta^1 \to S$ be a marked edge of S. For every simplicial set A, let $\overline{A}^\triangleleft$ denote the marked simplicial set $(A^\triangleleft, M_A)$, where M_A is the collection of all edges of $A^\triangleleft$ which are either degenerate or contain the cone point. We regard $\overline{A}^\triangleleft$ as an object of $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ via the map $A^\triangleleft \to (\Delta^0)^\triangleleft \simeq \Delta^1 \xrightarrow{e} S$. For any cofibration of simplicial sets $i : A \to B$, the induced map $j : \overline{A}^\triangleleft \to \overline{B}^\triangleleft$ is $\mathfrak{P}$-anodyne. To prove this, it suffices to treat the basic case where $B = \Delta^n$ and $A = \partial \Delta^n$, in which case the map j is a generating $\mathfrak{P}$-anodyne map which is either of type (B_0) (if $n = 0$) or (C_0) (if $n > 0$).

Example B.1.4. Let $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^\triangleleft \to S\}_{\alpha \in A})$ be a categorical pattern on a simplicial set S. Let $B_0 \subseteq B$ be a simplicial sets containing a vertex b, and let $f : B \star K_\alpha \to S$ be a map whose restriction to $\{b\} \star K_\alpha \simeq K_\alpha^\triangleleft$ is given by p_α. Suppose that every simplex of B either belongs to B_0 or contains b as a final vertex. Then the inclusion
$$(B_0 \star K_\alpha)^\flat \coprod_{(\{b\}\star K_\alpha)^\flat} (\{b\} \star K_\alpha)^\sharp \subseteq (B \star K_\alpha)^\flat \coprod_{(\{b\}\star K_\alpha)^\flat} (\{b\} \star K_\alpha)^\sharp)$$

is $\mathfrak{P}$-anodyne, because it can be obtained as an iterated pushout of $\mathfrak{P}$-anodyne inclusions of type (C_2).

Remark B.1.5. Let $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^\triangleleft \to S\}_{\alpha \in A})$ be a categorical pattern on a simplicial set S, and let $\overline{X} = (X, M)$ be an object of $(Set_\Delta^+)_{/\mathfrak{P}}$. Let T' denote the inverse image of T in $\mathrm{Hom}_{Set_\Delta}(\Delta^2, X)$, and let B denote the set of pairs $\beta = (\alpha, \overline{p}_\beta)$ where $\alpha \in A$ and $\overline{p}_\beta : K_\alpha^\triangleleft \to X$ is a map lifting p_α. Then $\mathfrak{P}_{\overline{X}} = (M, T', \{\overline{p}_\beta\}_{\beta \in B})$ is a categorical pattern on X. Unwinding the definitions, we deduce that a morphism in $(Set_\Delta^+)_{/\mathfrak{P}_{\overline{X}}}$ is $\mathfrak{P}_{\overline{X}}$-anodyne if and only if it is $\mathfrak{P}$-anodyne.

Our next result highlights the relevance of Definition B.1.1 to the proof of Theorem B.0.19:

Proposition B.1.6. *Let $\mathfrak{P}$ be a categorical pattern on a simplicial set S, and let $\overline{X} \in (Set_\Delta^+)_{/\mathfrak{P}}$. The following conditions are equivalent:*

(1) *The object $\overline{X}$ has the extension property with respect to every $\mathfrak{P}$-anodyne morphism in $(Set_\Delta^+)_{/\mathfrak{P}}$.*

(2) *The object $\overline{X}$ is $\mathfrak{P}$-fibered.*

The proof of Proposition B.1.6 will require the following preliminary result:

Lemma B.1.7. *Let $n \geq 2$, and let $p : X \to \Delta^n$ be an inner fibration of simplicial sets. Consider a commutative diagram*

$$
\begin{array}{ccc}
\Lambda_0^n & \xrightarrow{f_0} & X \\
\downarrow & \nearrow^{\!\!f} & \downarrow{\scriptstyle p} \\
\Delta^n & \xrightarrow{\mathrm{id}} & \Delta^n,
\end{array}
$$

where f_0 carries $\Delta^{\{0,1\}} \subseteq \Lambda_0^n$ to a locally p'-coCartesian edge of $X \times_{\Delta^n} \Delta^{\{0,1,n\}}$, where p' denotes the projection $X \times_{\Delta^n} \Delta^{\{0,1,n\}} \to \Delta^{\{0,1,n\}}$. Then there exists a map $f : \Delta^n \to X$ as indicated, rendering the diagram commutative.

Proof. To prove the assertion, it will suffice to show that f_0 extends to an n-simplex of X (the compatibility with the projection p is automatic, since Λ_0^n contains every vertex of Δ^n). Choose a categorical equivalence $X \to \mathrm{N}(\mathcal{C})$, where $\mathcal{C}$ is a topological category (for example, we could take $\mathcal{C} = |\mathfrak{C}[X]|$). Note that the projection p factors (uniquely) through some projection map $\mathrm{N}(\mathcal{C}) \to \Delta^n$. Since p is an inner fibration, the simplicial set X is an ∞-category, and therefore fibrant with respect to the Joyal model structure. Consequently, it will suffice to prove the existence of the desired extension after replacing X by $\mathrm{N}(\mathcal{C})$. We may therefore assume that X is the nerve of a topological category $\mathcal{C}$.

The functor f_0 determines the following data in the topological category $\mathcal{C}$:

(1) A collection of objects $C_i = f_0(\{i\})$.

(2) A morphism $\alpha : C_0 \to C_1$ in $\mathcal{C}$, given by evaluating f_0 on the edge $\Delta^{\{0,1\}} \subseteq \Lambda_0^n$. Let $q : \mathrm{Map}_{\mathcal{C}}(C_1, C_n) \to \mathrm{Map}_{\mathcal{C}}(C_0, C_n)$ be the map induced by composition with α. Since α is locally p-coCartesian, it is coCartesian with respect to the projection $X \times_{\Delta^n} \Delta^{\{0,1,n\}} \to \Delta^{\{0,1,n\}}$, so that q is a weak homotopy equivalence.

(3) A continuous map $g_0 : \partial[0,1]^{n-2} \to \mathrm{Map}_{\mathcal{C}}(C_1, C_n)$, given by evaluating f_0 on $\partial \Delta^{\{1,2,\ldots,n\}}$.

(4) Another continuous map

$$
H_0 : ((\partial[0,1]^{n-2}) \times [0,1]) \coprod_{\partial[0,1]^{n-2} \times \{0\}} ([0,1]^{n-2} \times \{0\}) \to \mathrm{Map}_{\mathcal{C}}(C_0, C_n)
$$

such that the restriction $H_0|(\partial[0,1]^{n-2} \times \{1\})$ coincides with the composition

$$
\partial[0,1]^{n-2} \xrightarrow{g_0} \mathrm{Map}_{\mathcal{C}}(C_1, C_n) \xrightarrow{q} \mathrm{Map}_{\mathcal{C}}(C_0, C_n).
$$

Let $h_1 = H_0|([0, 1]^{n-2} \times \{0\})$. We can regard the restriction $H_0|(\partial[0, 1]^{n-2} \times [0, 1])$ as a homotopy from $q \circ g_0$ to $h_1|\partial[0, 1]^{n-2}$. Unwinding the definitions, we see that producing the desired extension f is equivalent to extending H_0 to a homotopy from $q \circ g$ to h_1, for some continuous map $g : [0, 1]^{n-2} \to \mathrm{Map}_{\mathcal{C}}(C_1, C_n)$. The existence of H (and g) now follows easily from the fact that q is a weak homotopy equivalence.   □

Proof of Proposition B.1.6. Let $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^\lhd \to S\}_{\alpha \in A})$ be a categorical pattern on the simplicial set S, and let $\overline{X}$ be an object of $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$. We wish to show that $\overline{X}$ is $\mathfrak{P}$-fibered if and only if it has the extension property with respect to every $\mathfrak{P}$-anodyne morphism. We begin by proving the "if" direction. Let $\overline{X} = (X, M)$, and let $q : X \to S$ denote the underlying map of simplicial sets. We will show that $\overline{X}$ satisfies conditions (1), (2), (3), (4) and (6) of Definition B.0.18, together with condition $(5')$ of Remark B.0.25:

(1) We must show that $q : X \to S$ is an inner fibration. This is equivalent to our assumption that $\overline{X}$ has the unique extension property with respect to every morphism of type (C_1) appearing in Definition B.1.1.

(2) Let $\overline{e} : \Delta^1 \to S$ belong to M_S, let $X' = X \times_S \Delta^1$, and let $q' : X' \to \Delta^1$ denote the projection map. We wish to prove that q' is a coCartesian fibration. Let M' denote the collection of edges in X' whose image in X belongs to M. Since $\overline{X}$ has the extension property with respect to morphisms of the type (C_0) appearing in Definition B.1.1, we deduce that every edge of M' is q'-coCartesian. The existence of a sufficient supply of such edges follows from the assumption that q has the extension property with respect to morphisms of type (B_0).

(3) Let $\overline{e}$, X', and q' be as in (2). We claim that an edge $e : x \to y$ of X' lifting $\overline{e}$ is q'-coCartesian if and only if $e \in M'$. The "if" direction follows from the above arguments. To prove the converse, we first treat the case where the edge $\overline{e}$ is degenerate, corresponding to a vertex $s \in S$. Let X_s denote the ∞-category $X \times_S \{s\}$, so that e is an equivalence in X_s and therefore belongs to $X_s^\simeq$. Let $Q = \Delta^0 \coprod_{\Delta^{\{0,2\}}} \Delta^3 \coprod_{\Delta^{\{1,3\}}} \Delta^0$, and let Q' denote the image of $\Delta^{\{1,2\}} \subseteq \Delta^3$ in K. The inclusion $Q' \subseteq Q$ is a weak homotopy equivalence. Consequently, the map $Q' \to Y$ determined by the edge e extends to a map $Q \to X_s^\simeq$. Since $\overline{X}$ has the extension property with respect to morphisms of type (A_1) appearing in Definition B.1.1, we deduce that the induced map $Q \to X$ carries each edge of Q into M, so that $e \in M$.

We now treat the general case where $\overline{e}$ is not assumed to be degenerate. Using the extension property with respect to morphisms of type (B_0), we can choose an edge $e' : x \to y'$ in M' which lies over $\overline{e}$. Since e' is q'-coCartesian, we can choose a 2-simplex

$$
\begin{array}{ccc}
 & y' & \\
e' \nearrow & & \searrow e'' \\
x & \xrightarrow{\quad e \quad} & y
\end{array}
$$
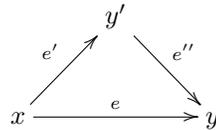
lying over the edge $\overline{e}$ in S', where e'' is an edge of the fiber $X'_{q'(y)}$. Since e is also q'-coCartesian, we deduce that e'' is an equivalence in $X'_{q'(y)}$, so that $e'' \in M$ by the above argument. Invoking our assumption that $\overline{X}$ has the extension property with respect to morphisms of the type (A_0), we deduce that $e \in M'$, as desired.

(4) Let $\Delta^2 \to S$ be a 2-simplex which belongs to T, let $X' = X \times_S \Delta^2$, and let e be an edge of X' lying over $\Delta^{\{0,1\}}$ whose image in X belongs to M. We wish to prove that e is q'-coCartesian, where q' denotes the projection map $X' \to \Delta^2$. This follows immediately from our assumption that $\overline{X}$ has the extension property with respect to morphisms of the type (C_0).

$(5')$ Fix an index $\alpha \in A$. Let $q_\alpha : X \times_S K_\alpha^\lhd \to K_\alpha^\lhd$ denote the projection map, and let $q_\alpha^0 : K \times_S K_\alpha \to K_\alpha$ its restriction. We must show that every coCartesian section of q_α^0 can be extended to a coCartesian section of q_α. In view of (3), this is equivalent to the requirement that $\overline{X}$ have the extension property with respect to morphisms of type (B_1) in Definition B.1.1.

(6) Let α and q_α be as in $(4')$; we must show that every coCartesian section of q_α is a q-limit diagram. In view of (3), this is equivalent the requirement that $\overline{X}$ has the extension property with respect to all morphisms of type (C_2) appearing in Definition B.1.1.

We now prove the "only if" direction. Assume that $\overline{X}$ is $\mathfrak{P}$-fibered. We will show that $\overline{X}$ has the extension property with respect to every $\mathfrak{P}$-anodyne morphism $f : A \to B$ in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$. It will suffice to treat the case where f is one of the generating $\mathfrak{P}$-anodyne morphisms appearing in Definition B.1.1. For morphisms of the types (B_1), (C_1), and (C_2), the relevant assertion follows from the arguments given above in cases $(5')$, (1), and (6), respectively. There are several more cases to consider:

(A_0) The map f is an inclusion $(\Lambda_1^2)^\sharp \coprod_{(\Lambda_1^2)^\flat} (\Delta^2)^\flat \subseteq (\Delta^2)^\sharp$, for some 2-simplex $\Delta^2 \to S$ belonging to T. Let $X' = X \times_S \Delta^2$, and let $q' : X' \to \Delta^2$ denote the projection. To prove that $\overline{X}$ has the extension property with respect to f, we must show that if we are given a 2-simplex
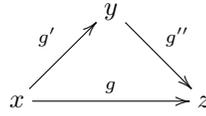
in X' such that g' and g'' are locally q'-coCartesian, then g is locally q'-coCartesian. We observe that g'' is automatically q'-coCartesian, and the hypothesis that $\overline{X}$ is $\mathfrak{P}$-fibered guarantees that g' is q'-coCartesian. It follows from Proposition T.2.4.1.7 that g is q'-coCartesian.

(A_1) The map f is an inclusion $Q^\flat \subseteq Q^\sharp$, where $Q = \Delta^0 \coprod_{\Delta^{\{0,2\}}} \Delta^3 \coprod_{\Delta^{\{1,3\}}} \Delta^0$, and the map $Q \to S$ carries each edge of Q into M_S and each 2-simplex of Q into T. Let $X' = X \times_S Q$ and let $q' : X' \to Q$ denote the projection map. It follows from Corollary T.2.4.2.10 that q' is a coCartesian fibration, classified by some functor $\chi : Q \to \mathcal{C}at_\infty$. Since the projection $Q \to \Delta^0$ is a categorical equivalence, the functor χ is equivalent to a constant functor; it follows that X' is equivalent to a product $Q \times \mathcal{C}$, for some ∞-category $\mathcal{C}$. To show that $\overline{X}$ has the extension property with respect to f, it suffices to show that every section of q' is coCartesian. Replacing X' by $Q \times \mathcal{C}$, we are reduced to proving that every diagram $Q \to \mathcal{C}$ carries each edge of Q to an equivalence in $\mathcal{C}$, which follows from a simple diagram chase.

(B_0) The map f is an inclusion $\{0\}^\sharp \subseteq (\Delta^1)^\sharp$ lying over an edge of M_S. Since the induced map $X \times_S \Delta^1 \to \Delta^1$ is a coCartesian fibration, the object $\overline{X}$ has the extension property with respect to f.

(C_0) The map f is an inclusion

$$(\Lambda_0^n)^\flat \coprod_{(\Delta^{\{0,1\}})^\flat} (\Delta^{\{0,1\}})^\sharp \subseteq (\Delta^n)^\flat \coprod_{(\Delta^{\{0,1\}})^\flat} (\Delta^{\{0,1\}})^\sharp,$$

for every $n > 1$ and every map $\Delta^n \to S$ which carries $\Delta^{\{0,1,n\}}$ into S'. The desired result in this case is a reformulation of Lemma B.1.7.

$\square$

We next show that the class of $\mathfrak{P}$-anodyne morphisms behaves well with respect to products. For a more precise statement, we need to introduce a bit of notation.

Definition B.1.8. Let S and S' be simplicial sets, and let $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^\triangleleft \to S\}_{\alpha \in A})$ and $\mathfrak{P}' = (M'_{S'}, T', \{q_\beta : L_\beta^\triangleleft \to S'\}_{\beta \in B})$ be categorical patterns on S and S', respectively. We let $\mathfrak{P} \times \mathfrak{P}'$ denote the categorical pattern

$$(M_S \times M'_{S'}, T \times T', \{K_\alpha^\triangleleft \xrightarrow{p_\alpha} S \times \{s'\} \to S'\}_{\alpha \in A, s' \in S'} \cup \{L_\beta^\triangleleft \xrightarrow{q_\beta} \{s\} \times S' \to S \times S'\}_{s \in S, \beta \in B})$$

on the product $S \times S'$.

Proposition B.1.9. *Let $\mathfrak{P}$ and $\mathfrak{P}'$ be categorical patterns on simplicial sets S and S'. Let $f : \overline{X} \to \overline{Y}$ be a cofibration in $(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$, and let $f' : \overline{X}' \to \overline{Y}'$ be a cofibration in $(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}'}$. If f is $\mathfrak{P}$-anodyne or f' is $\mathfrak{P}'$-anodyne, then the induced map*

$$f \wedge f' : (\overline{Y} \times \overline{X}') \coprod_{\overline{X} \times \overline{X}'} (\overline{X} \times \overline{Y}') \to \overline{Y} \times \overline{Y}'$$

is $\mathfrak{P} \times \mathfrak{P}'$-anodyne.

Lemma B.1.10. *Let $\mathfrak{P} = (M_S, T, \{K_{\alpha}^{\triangleleft} \to S\}_{\alpha \in A})$ be a categorical pattern on a simplicial set S. Let $B_0 \subseteq B$ be an inclusion of simplicial sets, and let $f : \Delta^1 \times B \to S$ be a map with the following properties:*

- *For every simplex $\sigma : \Delta^n \to B$ which does not belong to B_0, let τ be the 2-simplex of $\Delta^1 \times \Delta^n$ spanned by $(0,0)$, $(1,0)$ and $(1,n)$. Then the induced map*

$$\Delta^2 \xrightarrow{\tau} \Delta^1 \times \Delta^n \xrightarrow{\sigma} \Delta^1 \times B \xrightarrow{f} S$$

  belongs to T.

- *For every vertex b of B, the map f carries $\Delta^1 \times \{b\}$ into M_S.*

  Then the inclusion

$$((\Delta^1)^{\sharp} \times B_0^{\flat}) \coprod_{\{0\}^{\sharp} \times B_0^{\flat}} (\{0\}^{\sharp} \times B^{\flat}) \subseteq (\Delta^1)^{\sharp} \times B^{\flat}$$

is $\mathfrak{P}$-anodyne.

Proof. Working simplex-by-simplex, we can reduce to the case where $B = \Delta^n$ and $B_0 = \partial \Delta^n$. The simplicial set $\Delta^1 \times \Delta^n$ admits a filtration

$$(\{0\} \times \Delta^n) \coprod_{\{0\} \times \partial \Delta^n} (\Delta^1 \times \partial \Delta^n) = Z_0 \subset Z_1 \subset \ldots \subset Z_n \subseteq Z_{n+1} = \Delta^1 \times \Delta^n,$$

where each Z_{i+1} is obtained from Z_i by adjoining the $(n+1)$-simplex of $\Delta^1 \times \Delta^n$ corresponding to the map

$$\sigma_i : [n+1] \to [1] \times [n]$$

$$\sigma_i(j) = \begin{cases} (0, j) & \text{if } j \leq n - i \\ (1, j - 1) & \text{if } j > n - i. \end{cases}$$

Let $\overline{Z}_i = (Z_i, M_i)$ denote the marked simplicial set whose marked edges are precisely those edges which are marked in $(\Delta^1)^{\sharp} \times (\Delta^1)^{\flat}$. We wish to show that the inclusion $\overline{Z}_0 \subseteq \overline{Z}_{n+1}$ is $\mathfrak{P}$-anodyne. For this, it will suffice to show that each of the inclusions $h_i : \overline{Z}_i \subseteq \overline{Z}_{i+1}$ is $\mathfrak{P}$-anodyne. If $i = n = 0$, then h_i is a generating $\mathfrak{P}$-anodyne morphism of type (B_0). If $0 \leq i < n$, then h_i is a pushout of a generating $\mathfrak{P}$-anodyne morphism of type (C_1). If If $i = n > 0$, then h_i is a pushout of a generating $\mathfrak{P}$-anodyne morphism of type (C_0). $\square$

Proof of Proposition B.1.9. Let $\mathfrak{P} = (M_S, T, \{p_{\alpha} : K_{\alpha}^{\triangleleft} \to S\}_{\alpha \in A})$ and $\mathfrak{P}' = (M_{S'}', T', \{q_{\beta} : L_{\beta}^{\triangleleft} \to S'\}_{\beta \in B})$ be categorical patterns on simplicial sets S and S', respectively. Let $f : \overline{X} \to \overline{Y}$ be a $\mathfrak{P}$-anodyne morphism in $(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$, and let $f' : \overline{X}' \to \overline{Y}'$ be an arbitrary cofibration in $(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}'}$. We wish to show that $f \wedge f'$ is $\mathfrak{P} \times \mathfrak{P}'$-anodyne. Without loss of generality, we may assume that f' is a generator for the class of cofibrations in $(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}'}$, having either the form $(\Delta^1)^{\flat} \subseteq (\Delta^1)^{\sharp}$ or $(\partial \Delta^m)^{\flat} \subseteq (\Delta^m)^{\flat}$. Similarly, we may assume that f is one of the generating $\mathfrak{P}$-anodyne morphisms described in Definition B.1.1. There are fourteen cases to consider:

(A_0) The map f is an inclusion $(\Lambda_1^2)^\sharp \coprod_{(\Lambda_1^2)^\flat} (\Delta^2)^\flat \subseteq (\Delta^2)^\sharp$ where $\Delta^2 \to S$ belongs to T and carries every edge into M_S, and f' is an inclusion $(\Delta^1)^\flat \subseteq (\Delta^1)^\sharp$. In this case, $f \wedge f'$ can be obtained as a composition of two morphisms, each of which is a pushout of a morphism having type (A_0).

(A_1) The map f is an inclusion $Q^\flat \subseteq Q^\sharp$, where $Q = \Delta^0 \coprod_{\Delta^{\{0,2\}}} \Delta^3 \coprod_{\Delta^{\{1,3\}}} \Delta^0$ and the map $Q \to S$ carries every edge of Q into M_S and every 2-simplex of Q into T, and f' is an inclusion $(\Delta^1)^\flat \subseteq (\Delta^1)^\sharp$. In this case, $f \wedge f'$ can be obtained as a successive pushout of two morphisms of type (A_0).

(B_0) The map f is an inclusion $\{0\}^\sharp \subseteq (\Delta^1)^\sharp$, for some edge $\Delta^1 \to S$ belonging to M_S, and f' is an inclusion $(\Delta^1)^\flat \subseteq (\Delta^1)^\sharp$. In this case, $f \wedge f'$ can be obtained as a composition of two morphisms which are pushouts of maps of type (A_0) and the 𝔓-anodyne morphism of Lemma B.1.11.

(B_1) For some $\alpha \in A$, the map f is an inclusion $K_\alpha^\sharp \subseteq (K_\alpha^\triangleleft)^\sharp$ (where $K_\alpha^\triangleleft$ maps to S via p_α), and f' is an inclusion $(\Delta^1)^\flat \subseteq (\Delta^1)^\sharp$. We can factor the morphism $f \wedge f'$ as a composition

$$(K_\alpha^\triangleleft \times \Delta^1, M) \xrightarrow{g} (K_\alpha^\triangleleft \times \Delta^1, M') \xrightarrow{g'} (K_\alpha^\triangleleft \times \Delta^1)^\sharp,$$

where M' is the collection of all edges of $K_\alpha^\triangleleft \times \Delta^1$ except for $\{v\} \times \Delta^1$, where v is the cone point of $K_\alpha^\triangleleft$, and $M \subseteq M'$ is the collection of all those edges which do not join $(v, 0)$ to a vertex of $K_\alpha^\triangleleft \times \{1\}$. We begin by observing that g is a pushout of a coproduct of morphisms of type (A_0), indexed by the collection of vertices of K_α. It will therefore suffice to show that g' is $(\mathfrak{P} \times \mathfrak{P}')$-anodyne, which follows from the observation that g' is a pushout of a morphism of the type described in Lemma B.1.12.

(C) The map f is a generating 𝔓-anodyne morphism of one of the types (C_0), (C_1), or (C_2) described in Definition B.1.1, and f' is an inclusion $(\Delta^1)^\flat \subseteq (\Delta^1)^\sharp$. In this case, $f \wedge f'$ is an isomorphism and there is nothing to prove.

(A_0') The map f is an inclusion $(\Lambda_1^2)^\sharp \coprod_{(\Lambda_1^2)^\flat} (\Delta^2)^\flat \subseteq (\Delta^2)^\sharp$ where $\Delta^2 \to S$ belongs to T and carries every edge into M_S, and f' is an inclusion $(\partial \Delta^m)^\flat \subseteq (\Delta^m)^\flat$. If $m = 0$, then $f \wedge f'$ is a generating $(\mathfrak{P} \times \mathfrak{P}')$-anodyne morphism of type (A_0). If $m > 0$, then $f \wedge f'$ is an isomorphism.

(A_1') The map f is an inclusion $Q^\flat \subseteq Q^\sharp$, where $Q = \Delta^0 \coprod_{\Delta^{\{0,2\}}} \Delta^3 \coprod_{\Delta^{\{1,3\}}} \Delta^0$ and the map $Q \to S$ carries every edge of Q into M_S and every 2-simplex of S into T, and f' is an inclusion $(\partial \Delta^m)^\flat \subseteq (\Delta^m)^\flat$. If $m = 0$ then $f \wedge f'$ is a generating $(\mathfrak{P} \times \mathfrak{P}')$-anodyne morphism of type (A_1), and if $m > 0$ then $f \wedge f'$ is an isomorphism.

(B_0') The map f is an inclusion $\{0\}^\sharp \subseteq (\Delta^1)^\sharp$, for some edge $\Delta^1 \to S$ belonging to M_S, and f' is an inclusion $(\partial \Delta^m)^\flat \subseteq (\Delta^m)^\flat$. If $m = 0$, then $f \wedge f'$ is a generating $(\mathfrak{P} \times \mathfrak{P}')$-anodyne morphism of type (B_0). Let us assume therefore that $m > 0$. For $0 \leq k \leq m$, let $\sigma_k : \Delta^{m+1} \to \Delta^1 \times \Delta^m$ denote the simplex determined by the map of partially ordered sets $[m+1] \to [1] \times [m]$ given by the formula

$$j \mapsto \begin{cases} (0, j) & \text{if } j \leq m - k \\ (1, j - 1) & \text{otherwise.} \end{cases}$$

We have a sequence of simplicial sets

$$Z_0 \subseteq Z_1 \subseteq \ldots \subseteq Z_{m+1} = \Delta^1 \times \Delta^m$$

where Z_i is the simplicial subset of $\Delta^1 \times \Delta^m$ generated by $\Delta^1 \times (\partial \Delta^m)$, $\{0\} \times \Delta^m$, and $\{\sigma_j\}_{j<i}$. Let M denote the collection of edges of $\Delta^1 \times \Delta^m$ whose image in Δ^m is degenerate, and let $\overline{Z}_i = (Z_i, M)$. To prove that $f \wedge f'$ is $(\mathfrak{P} \times \mathfrak{P}')$-anodyne, it will suffice to show that each of the inclusions $g_i : \overline{Z}_i \subseteq \overline{Z}_{i+1}$ is 𝔓-anodyne. For $0 \leq i < m$, we observe that g_i is a pushout of a generating $(\mathfrak{P} \times \mathfrak{P}')$-anodyne morphism of type (C_1). For $i = m$, we note that g_i is a pushout of a generating $(\mathfrak{P} \times \mathfrak{P}')$-anodyme morphism of type (C_0).

(B_1') For some $\alpha \in A$, the map f is an inclusion $K_\alpha^\sharp \subseteq (K_\alpha^\triangleleft)^\sharp$ (where $K_\alpha^\triangleleft$ maps to S via p_α), and f' is an inclusion $(\partial \Delta^m)^\flat \subseteq (\Delta^m)^\flat$. If $m = 0$, then $f \wedge f'$ is a generating $\mathfrak{P} \times \mathfrak{P}'$-anodyne morphism of type (B_1) and there is nothing to prove. Let us assume therefore that $m > 0$. Let v denote the cone point of $K_\alpha^\triangleleft$ We define a filtration

$$Z_0 \subseteq Z_1 \subseteq \ldots \subseteq Z_m \subseteq Z_{m+1} = K_\alpha^\triangleleft \times \Delta^m$$

as follows. For each $i \leq m$, let Z_i denote the simplicial subset of $K_\alpha^\triangleleft \times \Delta^m$ generated by those simplices σ such that either $\sigma \cap (\{v\} \times \Delta^m) \subseteq \{v\} \times \Delta^{\{0,\ldots,i-1\}}$ or the projection map $\sigma \to \Delta^m$ is not surjective. Let $\overline{Z}_i$ denote the marked simplicial set (Z_i, M_i), where M_i is the collection of those edges of Z_i whose image in Δ^m is degenerate. The map $f \wedge f'$ can be identified with the inclusion $\overline{Z}_0 \subseteq \overline{Z}_{m+1}$. It will therefore suffice to show that each of the inclusions $g_i : \overline{Z}_i \subseteq \overline{Z}_{i+1}$ is $(\mathfrak{P} \times \mathfrak{P}')$-anodyne. If $i < m$, then g_i is a pushout of the inclusion $B^\flat \subseteq (\Delta^i \star (K_\alpha \times \Delta^{m-i}))^\flat$, where B denotes the pushout

$$(\partial \Delta^i \star (K_\alpha \times \Delta^{m-i})) \coprod_{\partial \Delta^i \star (K_\alpha \times \Lambda_0^{m-i})} (\Delta^i \star (K_\alpha \times \Delta^{m-i})).$$

In view of Example B.1.2, it will suffice to show that the inclusion of simplicial sets $B \subseteq \Delta^i \star (K_\alpha \times \Delta^{m-i})$ is inner anodyne. This follows from Lemma T.2.1.2.3, since the inclusion $K_\alpha \times \Lambda_0^{m-i} \subseteq K_\alpha \times \Delta^{m-i}$ is left anodyne (Corollary T.2.1.2.7).

In the case $i = m$, we observe that g_i is a pushout of the inclusion

$$((\partial \Delta^m) \star K_\alpha)^\flat \coprod_{(\{m\} \star K_\alpha)^\flat} (\{m\} \star K_\alpha)^\sharp \subseteq (\Delta^m \star K_\alpha)^\flat \coprod_{(\{m\} \star K_\alpha)^\flat} (\{m\} \star K_\alpha)^\sharp,$$

which is a $(\mathfrak{P} \times \mathfrak{P}')$-anodyne morphism of type (C_2).

(C_0') The map f is an inclusion

$$(\Lambda_0^n)^\flat \coprod_{(\Delta^{\{0,1\}})^\flat} (\Delta^{\{0,1\}})^\sharp \subseteq (\Delta^n)^\flat \coprod_{(\Delta^{\{0,1\}})^\flat} (\Delta^{\{0,1\}})^\sharp,$$

for some $n > 1$ such that the map $\Delta^n \to S$ carries $\Delta^{\{0,1,n\}}$ to a 2-simplex belonging to T, and f' is an inclusion $(\partial \Delta^m)^\flat \subseteq (\Delta^m)^\flat$. If $m = 0$, then $f \wedge f'$ is a $(\mathfrak{P} \times \mathfrak{P}')$-anodyne morphism of type (C_0). We may therefore assume without loss of generality that $m > 0$. We define maps

$$\Delta^n \xrightarrow{s} \Delta^1 \times \Delta^n \xrightarrow{r} \Delta^n$$

by the formulae

$$s(i) = (1, i)$$

$$r(i,j) = \begin{cases} 0 & \text{if } i = 0, j = 1 \\ j & \text{otherwise.} \end{cases}$$

These maps exhibit f as a retract of the inclusion

$$g : ((\Delta^1)^\sharp \times (\Lambda_0^n)^\flat) \coprod_{\{0\}^\sharp \times (\Lambda_0^n)^\flat} (\{0\}^\sharp \times (\Delta^n)^\flat) \subseteq (\Delta^1)^\sharp \times (\Delta^n)^\flat.$$

We regard $(\Delta^1)^\sharp \times (\Delta^n)^\flat$ as an object of $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ via the composition

$$\Delta^1 \times \Delta^n \xrightarrow{r} \Delta^n \to S.$$

Since f is a retract of g, it will suffice to show that $g \wedge f'$ is $(\mathfrak{P} \times \mathfrak{P}')$-anodyne, which follows immediately from Lemma B.1.10.

(C_1') The map f is an inclusion $(\Lambda_i^n)^\flat \subseteq (\Delta^n)^\flat$, for where $0 < i < n$, and f' is an inclusion $(\partial \Delta^m)^\flat \subseteq (\Delta^m)^\flat$. In this case, $f \wedge f'$ is a morphism of the form $B_0^\flat \subseteq B^\flat$, where $B_0 \subseteq B$ is an inner anodyne inclusion of simplicial sets (Corollary T.2.3.2.4). It follows from Example B.1.2 that $f \wedge f'$ is $(\mathfrak{P} \times \mathfrak{P}')$-anodyne.

(C_2') The map f has the form

$$(\partial \Delta^n \star K_\alpha)^\flat \coprod_{(\{n\} \star K_\alpha)^\flat} (\{n\} \star K_\alpha)^\sharp \subseteq (\Delta^n \star K_\alpha)^\flat \coprod_{(\{n\} \star K_\alpha)^\flat} (\{n\} \star K_\alpha)^\sharp)$$

for some $\alpha \in A$ and $n > 0$, where $\Delta^n \star K_\alpha \to S$ extends p_α, and f' is an inclusion of the form $(\partial \Delta^m)^\flat \subseteq (\Delta^m)^\flat$. The treatment of this case is similar to that of (B_1'). If $m = 0$, then $f \wedge f'$ is a generating $\mathfrak{P} \times \mathfrak{P}'$-anodyne morphism of type (C_2) and there is nothing to prove. Let us assume therefore that $m > 0$. We define a filtration

$$Z_0 \subseteq Z_1 \subseteq \ldots \subseteq Z_m \subseteq Z_{m+1} = (\Delta^n \star K_\alpha) \times \Delta^m$$

as follows. For each $i \leq m$, let Z_i denote the simplicial subset of $(\Delta^n \star K_\alpha) \times \Delta^m$ generated by those simplices σ such that either $\sigma \cap (\Delta^n \times \Delta^m) \subseteq \Delta^n \times \Delta^{\{0,\ldots,i-1\}}$ or the projection map $\sigma \to \Delta^m$ is not surjective. Let $\overline{Z}_i$ denote the marked simplicial set (Z_i, M_i), where M_i is the collection of those edges of Z_i which are marked in the product

$$((\Delta^n \star K_\alpha)^\flat \coprod_{(\{n\} \star K_\alpha)^\flat} (\{n\} \star K_\alpha)^\sharp)) \times (\Delta^m)^\flat.$$

The map $f \wedge f'$ can be identified with the inclusion $\overline{Z}_0 \subseteq \overline{Z}_{m+1}$. It will therefore suffice to show that each of the inclusions $g_i : \overline{Z}_i \subseteq \overline{Z}_{i+1}$ is $(\mathfrak{P} \times \mathfrak{P}')$-anodyne. If $i < m$, then g_i is a pushout of the inclusion $B^\flat \subseteq ((\Delta^n \times \Delta^i) \star (K_\alpha \times \Delta^{m-i})^\flat$, where B denotes the pushout

$$(\partial(\Delta^n \times \Delta^i) \star (K_\alpha \times \Delta^{m-i}) \coprod_{\partial(\Delta^n \times \Delta^i) \star (K_\alpha \times \Lambda_0^{m-i})} ((\Delta^n \times \Delta^i) \star (K_\alpha \times \Delta^{m-i})).$$

In view of Example B.1.2, it will suffice to show that the inclusion of simplicial sets $B \subseteq (\Delta^n \times \Delta^i) \star (K_\alpha \times \Delta^{m-i})$ is inner anodyne. This follows from Lemma T.2.1.2.3, since the inclusion $K_\alpha \times \Lambda_0^{m-i} \subseteq K_\alpha \times \Delta^{m-i}$ is left anodyne (Corollary T.2.1.2.7).

In the case $i = m$, we observe that g_i is a pushout of the inclusion

$$(\partial(\Delta^n \times \Delta^m) \star K_\alpha)^\flat \coprod_{(\{(n,m)\} \star K_\alpha)^\flat} (\{(n, m)\} \star K_\alpha)^\sharp \subseteq ((\Delta^n \times \Delta^m) \star K_\alpha)^\flat \coprod_{(\{(n,m)\} \star K_\alpha)^\flat} (\{(n, m)\} \star K_\alpha)^\sharp,$$

which is $(\mathfrak{P} \times \mathfrak{P}')$-anodyne (Example B.1.4).

$\square$

Lemma B.1.11. *Let $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^\triangleleft \to S\}_{\alpha \in A})$ be a categorical pattern on a simplicial set S, and let $\Delta^2 \to S$ be a 2-simplex which belongs to T. Then the inclusion $i : (\Lambda_0^2)^\sharp \coprod_{(\Lambda_0^2)^\flat} (\Delta^2)^\flat \subseteq (\Delta^2)^\sharp$ is a $\mathfrak{P}$-anodyne morphism in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$.*

Proof. We must show that i has the left lifting property with respect to every morphism morphism $f : \overline{X} \to \overline{Y}$ in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$, provided that f has the right lifting property with respect to every $\mathfrak{P}$-anodyne morphism in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$. Replacing $\mathfrak{P}$ by $\mathfrak{P}_{\overline{Y}}$ (and invoking Remark B.1.5), we are reduced to showing that $\overline{X}$ has the extension property with respect to i, provided that $\overline{X}$ has the extension property with respect to every $\mathfrak{P}$-anodyne morphism. In view of Proposition B.1.6, we may assume that $\overline{X}$ is $\mathfrak{P}$-fibered. The desired result is now an immediate consequence of Proposition T.2.4.1.7. $\square$

Lemma B.1.12. *Let $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^\lhd \to S\}_{\alpha \in A})$ be a categorical pattern on a simplicial set S. Fix $\alpha \in A$, let M be the collection of all edges of $\Delta^1 \star K_\alpha$ except for the initial edge $\Delta^1 \subseteq \Delta^1 \star K_\alpha$. Let $f : \Delta^1 \star K_\alpha \to S$ be a map such which carries each edge into M_S, each 2-simplex into T, and such that $f|(\{1\} \star K_\alpha)$ agrees with p_α. Then the inclusion $i : (\Delta^1 \star K_\alpha, M) \subseteq (\Delta^1 \star K_\alpha)^\sharp$ is a $\mathfrak{P}$-anodyne morphism in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$.*

Proof. Let $g : \overline{X} \to \overline{Y}$ be a morphism in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ which has the right lifting property with respect to every $\mathfrak{P}$-anodyne morphism; we will show that g has the right lifting property with respect to i. Replacing $\mathfrak{P}$ by $\mathfrak{P}_{\overline{Y}}$ (and invoking Remark B.1.5), we may assume that $\overline{Y}$ is a final object of $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$. Proposition B.1.6 now guarantees that $\overline{X}$ is $\mathfrak{P}$-fibered. Let X' denote the fiber product $X \times_S (\Delta^1 \star K_\alpha)$, so that the projection map $q : X' \to (\Delta^1 \star K_\alpha)$ is a coCartesian fibration. Unwinding the definitions, we must show the following:

($*$) Let s be a section of q. If s carries each edge of M to a q-coCartesian edge of X', then s carries every edge of $\Delta^1 \star K_\alpha$ to a q-coCartesian edge of X'.

To prove ($*$), let us write rewrite the domain of s as $\{x\} \star \{z\} \star K_\alpha$. Choose a q-coCartesian edge $e : s(x) \to \overline{y}$ in X' covering the initial edge $\Delta^1 \subseteq \Delta^1 \star K_\alpha$. Since e is q-coCartesian, we can extend s to a map $s' : \{x\} \star \{y\} \star \{z\} \star K_\alpha \to X'$ carrying $\{x\} \star \{y\}$ to e. It follows from Proposition T.2.4.1.7 that, for every vertex k of K_α, s' carries the edge $\{y\} \star \{k\}$ to a q-coCartesian edge of X'. Using the fact that $\overline{X}$ is $\mathfrak{P}$-fibered, we deduce that $s'|\{y\} \star K_\alpha$ and $s'|\{z\} \star K_\alpha$ are q-limit diagrams, so that s' carries $\{y\} \star \{z\}$ to an equivalence in X'_y. It follows that s carries the edge $\{x\} \star \{z\}$ into a composition of q-coCartesian edges $s'(\{x\} \star \{y\})$ and $s'(\{y\} \star \{z\})$, which is again a q-coCartesian edge (Proposition T.2.4.1.7). □

We conclude this section with a few miscellaneous results concerning $\mathfrak{P}$-anodyne morphisms which will be needed later.

Lemma B.1.13. *Let $\mathcal{P}_0$ denote the categorical pattern $(\Delta^0, \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^1, \Delta^0), \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^2, \Delta^0), \emptyset)$, so that $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}_0}$ is equivalent to $\mathcal{S}et_\Delta^+$. For every left anodyne inclusion of simplicial sets $A \subseteq B$, the induced map $j : A^\sharp \subseteq B^\sharp$ is $\mathcal{P}_0$-anodyne.*

Proof. Without loss of generality, we may assume that $B = \Delta^n$ and $A = \Lambda_i^n$, for some $0 \le i < n$, where $n > 0$. Suppose first that $n > 2$. If $0 < i < n$, then j is a pushout of the inclusion $j_0 : (\Lambda_i^n)^\flat \to (\Delta^n)^\flat$, and therefore $\mathcal{P}_0$-anodyne (case (C_1) of Definition B.1.1). If $i = 0$, then j is a pushout of the inclusion

$$j_0 : (\Lambda_0^n)^\flat \coprod_{(\Delta^{\{0,1\}})^\flat} (\Delta^{\{0,1\}})^\sharp \to (\Delta^n)^\flat \coprod_{(\Delta^{\{0,1\}})^\flat} (\Delta^{\{0,1\}})^\sharp$$

which is $\mathcal{P}_0$-anodyne (case (C_0) of Definition B.1.1).

Now suppose that $n = 2$. We observe that j can be obtained as a composite $j'' \circ j'$, where j' is a pushout of the morphism j_0 considered above, and j'' is either a generating $\mathfrak{P}$-anodyne morphism of type (A_0) or the $\mathfrak{P}$-anodyne morphism described in Lemma B.1.11.

Finally, in the case $n = 1$, j is itself a morphism of type (B_0) appearing in Definition B.1.1. □

Proposition B.1.14. *Let $\mathfrak{P}$ be a categorical pattern on a simplicial set S. Let $f : \overline{X} \to \overline{Y}$ be a cofibration in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$, and let $\overline{Z}$ be a $\mathfrak{P}$-fibered object of $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$. Then the induced map*

$$q : \mathrm{Map}_S^\sharp(\overline{Y}, \overline{Z}) \to \mathrm{Map}_S^\sharp(\overline{X}, \overline{Z})$$

is a Kan fibration between Kan complexes. If f is $\mathfrak{P}_0$-anodyne (where $\mathfrak{P}_0$ is defined as in Lemma B.1.13), then q is a trivial Kan fibration.

Proof. We first show that q is a left fibration by showing that q has the right lifting property with respect to every left anodyne inclusion of simplicial sets $A \subseteq B$ (or *every* inclusion of simplicial sets, in the case where

f is $\mathfrak{P}$-anodyne). Unwinding the definitions, this is equivalent to the assertion that $\overline{Z}$ has the extension property with respect to the induced inclusion

$$f' : (B^\sharp \times \overline{X}) \coprod_{A^\sharp \times \overline{X}} (A^\sharp \times \overline{Y}) \to B^\sharp \times \overline{Y}.$$

It follows from Proposition B.1.9 and Lemma B.1.13 that f' is $\mathfrak{P}$-anodyne, so that the desired result is a consequence of Proposition B.1.6.

Applying the above result to the inclusion $\emptyset \subseteq \overline{X}$, we deduce that the projection map $\mathrm{Map}^\sharp_S(\overline{X}, \overline{Z}) \to \Delta^0$ is a left fibration, so that $\mathrm{Map}^\sharp_S(\overline{X}, \overline{Z})$ is a Kan complex. It follows that q is a Kan fibration as desired (Lemma T.2.1.3.3). $\qquad\square$

B.2 The Model Structure on $(\mathcal{S}et^+_\Delta)_{/\mathfrak{P}}$

Our first main goal in this section is to prove Theorem B.0.19. Fix a categorical pattern $\mathfrak{P}$ on a simplicial set S. We wish to construct a model structure on the category $(\mathcal{S}et^+_\Delta)_{/\mathfrak{P}}$ such that the fibrant objects are precisely the $\mathfrak{P}$-fibered objects of $(\mathcal{S}et^+_\Delta)_{/\mathfrak{P}}$. Our first step will be to describe the class of weak equivalences in $(\mathcal{S}et^+_\Delta)_{/\mathfrak{P}}$.

Definition B.2.1. Let $\mathfrak{P}$ be a categorical pattern on a simplicial set S. We will say that a morphism $f : \overline{X} \to \overline{Y}$ in $(\mathcal{S}et^+_\Delta)_{/\mathfrak{P}}$ is a $\mathfrak{P}$-*equivalence* if, for every $\mathfrak{P}$-fibered object $\overline{Z} \in (\mathcal{S}et^+_\Delta)_{/\mathfrak{P}}$, the induced map

$$\mathrm{Map}^\sharp_S(\overline{Y}, \overline{Z}) \to \mathrm{Map}^\sharp_S(\overline{X}, \overline{Z})$$

is a homotopy equivalence of Kan complexes.

Example B.2.2. Any $\mathfrak{P}$-anodyne morphism is a $\mathfrak{P}$-equivalence; this follows immediately from Proposition B.1.14.

We now establish some properties of $\mathfrak{P}$-equivalences.

Lemma B.2.3. *Let $\mathfrak{P}$ be a categorical pattern on a simplicial set S, and suppose we are given a pushout diagram*

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\ f\ } & \overline{Y} \\ \downarrow & & \downarrow \\ \overline{X}' & \xrightarrow{\ f'\ } & \overline{Y}' \end{array}$$

in $(\mathcal{S}et^+_\Delta)_{/\mathfrak{P}}$. Assume that the vertical maps are cofibrations. If f is a $\mathfrak{P}$-equivalence, then f' is a $\mathfrak{P}$-equivalence.

Proof. Let $\overline{Z} \in (\mathcal{S}et^+_\Delta)_{/\mathfrak{P}}$ be $\mathfrak{P}$-fibered. We have a pullback diagram of simplicial sets

$$\begin{array}{ccc} \mathrm{Map}^\sharp_S(\overline{X}, \overline{Z}) & \longleftarrow & \mathrm{Map}^\sharp_S(\overline{Y}, \overline{Z}) \\ \uparrow & & \uparrow \\ \mathrm{Map}^\sharp_S(\overline{X}', \overline{Z}) & \longleftarrow & \mathrm{Map}^\sharp_S(\overline{Y}', \overline{Z}) \end{array}$$

Proposition B.1.14 implies that the vertical maps are Kan fibrations, so that the diagram is also a homotopy pullback square. Since f is a $\mathfrak{P}$-equivalence, the upper horizontal maps is a homotopy equivalence of Kan complexes. It follows that the lower horizontal map is also a homotopy equivalence of Kan complexes, as desired. $\qquad\square$

Lemma B.2.4. *Let $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^\lhd \to S\}_{\alpha \in A})$ be a categorical pattern on a simplicial set S, and let $f : \overline{X} \to \overline{Y}$ be a map between $\mathfrak{P}$-fibered objects $\overline{X} = (X, M), \overline{Y} = (Y, M')$ of $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$. The following conditions are equivalent:*

(1) *The map f is a $\mathfrak{P}$-equivalence.*

(2) *The map f admits a homotopy inverse; that is, there exists a map $g : \overline{Y} \to \overline{X}$ in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ and homotopies*
$$h : (\Delta^1)^\sharp \times \overline{X} \to \overline{X} \qquad h' : (\Delta^1)^\sharp \times \overline{Y} \to \overline{Y}$$
connecting $g \circ f$ and $f \circ g$ to $\mathrm{id}_{\overline{X}}$ and $\mathrm{id}_{\overline{Y}}$, respectively.

(3) *For every edge $\Delta^1 \to S$, the induced map $X \times_S \Delta^1 \to Y \times_S \Delta^1$ is an equivalence of ∞-categories.*

If every edge of S belongs to M_S, then (3) *can be replaced by the following apparently weaker condition:*

(3′) *For every vertex $s \in S$, the induced map $X_s \to Y_s$ is an equivalence of ∞-categories.*

Proof. The equivalence of (1) and (2) is formal, and the implications (2) $\Rightarrow$ (3) $\Rightarrow$ (3′) are clear. If every edge of S belongs to M_S, then the implication (3′) $\Rightarrow$ (3) follows from Corollary T.2.4.4.4. To complete the proof, let us suppose that f satisfies (3). We will say that an object $\overline{W} = (W, M'') \in (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ is *good* if composition with f induces a homotopy equivalence
$$\mathrm{Map}_S^\sharp(\overline{W}, \overline{X}) \to \mathrm{Map}_S^\sharp(\overline{W}, \overline{Y}).$$

Our goal is to prove that every object $\overline{W} \in (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ is good. The proof proceeds in several steps:

(a) We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Map}_S^\sharp(\overline{W}, \overline{X}) & \longrightarrow & \mathrm{Map}_S^\sharp(\overline{W}, \overline{Y}) \\
\downarrow & & \downarrow \\
\mathrm{Map}_S^\sharp(W^\flat, \overline{X}) & \longrightarrow & \mathrm{Map}_S^\sharp(W^\flat, \overline{Y}).
\end{array}
$$

The left vertical map exhibits $\mathrm{Map}_S^\sharp(\overline{W}, \overline{X})$ as the full simplicial subset of $\mathrm{Map}_S^\sharp(W^\flat, \overline{X})$ spanned by those maps $W \to X$ which carry every edge in M'' to a locally p-coCartesian edge of X, where $p : X \to S$ denotes the projection, and the right vertical map admits a similar description in terms of the projection $q : Y \to S$. Assumption (3) implies that an edge of X is locally p-coCartesian if and only if its image in Y is locally q-coCartesian. Consequently, to prove that $\overline{W}$ is good, it will suffice to show that $W^\flat$ is good.

(b) Suppose given a pushout diagram

$$
\begin{array}{ccc}
V & \longrightarrow & V' \\
\downarrow & & \downarrow \\
W & \longrightarrow & W'
\end{array}
$$

in the category of simplicial sets over S, where the vertical maps are cofibrations. We then obtain pullback diagram

$$
\begin{array}{ccc}
\mathrm{Map}_S^\sharp(V^\flat, \overline{X}) & \longleftarrow & \mathrm{Map}_S^\sharp(V'^\flat, \overline{X}) \\
\uparrow & & \uparrow \\
\mathrm{Map}_S^\sharp(W^\flat, \overline{X}) & \longleftarrow & \mathrm{Map}_S^\sharp(W'^\flat, \overline{X})
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Map}_S^\sharp(V^\flat, \overline{Y}) & \longleftarrow & \mathrm{Map}_S^\sharp(V'^\flat, \overline{Y}) \\
\uparrow & & \uparrow \\
\mathrm{Map}_S^\sharp(W^\flat, \overline{Y}) & \longleftarrow & \mathrm{Map}_S^\sharp(W'^\flat, \overline{Y})
\end{array}
$$

Proposition B.1.14 implies that the vertical maps are Kan fibrations, so both diagrams are homotopy pullback squares. It follows that if $V^\flat$, $V'^\flat$, and $W^\flat$ are good, then $W'^\flat$ is good.

(c) Let $\Delta^n \to S$ be a map; then $(\Delta^n)^{\flat}$ is good for $n \le 1$; this follows immediately from (3).

(d) For any map $\Delta^n \to S$, the object $(\Delta^{\{0,1\}} \coprod_{\{0\}} \cdots \coprod_{\{n-1\}} \Delta^{\{n-1,n\}})^{\flat} \in (\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$ is good; this follows from (b) and (c).

(e) Let $u : \overline{W} \to \overline{W}'$ be a $\mathfrak{P}$-equivalence (for example, any $\mathfrak{P}$-anodyne map). Then $\overline{W}$ is good if and only if $\overline{W}'$ is good.

(f) For any map $\Delta^n \to S$, the resulting object $(\Delta^n)^{\flat} \in (\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$ is good. This follows from (e) and (d), since the inclusion $(\Delta^{\{0,1\}} \coprod_{\{0\}} \cdots \coprod_{\{n-1\}} \Delta^{\{n-1,n\}})^{\flat} \subseteq (\Delta^n)^{\flat}$ is $\mathfrak{P}$-anodyne (Example B.2.2).

(g) The collection of good objects in $(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$ is closed under coproducts (since a product of homotopy equivalences between Kan complexes is again a homotopy equivalence).

(h) If the simplicial set W is finite-dimensional, then $W^{\flat} \in (\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$ is good. The proof goes by induction on the dimension $n \ge 0$ of W. If W is empty, then the result is obvious. Otherwise, let K denote the set of nondegenerate n-simplices of W, and let W' denote the $(n-1)$-skeleton of W. We have a pushout diagram

$$
\begin{array}{ccc}
K \times \partial \Delta^n & \longrightarrow & W' \\
\downarrow & & \downarrow \\
K \times \Delta^n & \longrightarrow & W.
\end{array}
$$

The inductive hypothesis guarantees that $(K \times \partial \Delta^n)^{\flat}$ and $W'^{\flat}$ are good, and $(K \times \Delta^n)^{\flat}$ is good by virtue of (g) and (f). It follows from (b) that $W^{\flat}$ is good.

(i) Suppose that W is obtained as the direct limit of a sequence of inclusions

$$W(0) \to W(1) \to W(2) \to \dots$$

Then $\mathrm{Map}_S^{\sharp}(W^{\flat}, \overline{X})$ can be obtained as the homotopy inverse limit of the tower $\{\mathrm{Map}_S^{\sharp}(W(n)^{\flat}, \overline{X})\}_{n \ge 0}$, and $\mathrm{Map}_S^{\sharp}(W^{\flat}, \overline{Y})$ can be described similarly. It follows that if each $W(n)^{\flat}$ is good, then $W^{\flat}$ is good.

(j) For every map of simplicial sets $W \to S$, the object $W^{\flat} \in (\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$ is good. This follows from (h) and (i), if we take $W(n)$ to be the n-skeleton of W.

$\square$

We now come to the proof of our main result:

Proof of Theorem B.0.19. Let $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^\lhd \to S\}_{\alpha \in A})$ be a categorical pattern on a simplicial set S. We wish to prove that the category $(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$ admits a combinatorial simplicial model structure in which the cofibrations are given by monomorphisms and the fibrant objects are precisely the $\mathfrak{P}$-fibered objects. Assume for the moment that each of the simplicial sets K_α is finite. It follows from the small object argument that there exists a functor $E : (\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}} \to (\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$ and a natural transformation $\alpha : \mathrm{id} \to T$ with the following properties:

(a) The functor E commutes with filtered colimits.

(b) For every object $\overline{X} \in (\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$, the object $E\overline{X} \in (\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$ has the extension property with respect to every $\mathfrak{P}$-anodyne map (and is therefore $\mathfrak{P}$-fibered, by virtue of Proposition B.1.6.

(c) For every object $\overline{X} \in (\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$, the map $\overline{X} \to E\overline{X}$ is $\mathfrak{P}$-anodyne.

Let $f : \overline{X} \to \overline{Y}$ be a morphism in $(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$. It follows from (c) and Example B.2.2 that f is a $\mathfrak{P}$-equivalence if and only if $E(f)$ is a $\mathfrak{P}$-equivalence. Using (b) and Lemma B.2.4, we deduce that f is an equivalence if and only if for each edge $e : \Delta^1 \to S$, the map $E(f)$ induces a categorical equivalence of simplicial sets after pulling back along e. Using (a) and Corollary T.A.2.6.12, we deduce that the collection of $\mathfrak{P}$-equivalences in $(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$ is perfect, in the sense of Definition T.A.2.6.10.

We now wish to deduce the existence of a left proper, combinatorial model structure on $(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$ such that the cofibrations are the monomorphisms and the weak equivalences are given by the $\mathfrak{P}$-equivalences. It will suffice to show that $(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$ satisfies the hypotheses of Proposition T.A.2.6.13:

(1) The collection of $\mathfrak{P}$-equivalences is perfect: this follows from the above arguments.

(2) The collection of $\mathfrak{P}$-equivalences is stable under pushouts by cofibrations: this follows from Lemma B.2.3.

(3) Let $f : \overline{X} \to \overline{Y}$ be a morphism in $(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$ which has the right lifting property with respect to every cofibration; we wish to show that f is a $\mathfrak{P}$-equivalence. To prove this, it suffices to observe that f admits a section s and that the composition $s \circ f : \overline{X} \to \overline{X}$ is homotopic to the identity (that is, there exists a homotopy $h : \overline{X} \times (\Delta^1)^{\sharp} \to \overline{X}$ from $\mathrm{id}_{\overline{X}}$ to $s \circ f$ in the category $(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$).

We next claim that the simplicial structure on $(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$ is compatible with its model structure. In view of Proposition T.A.3.1.7, it will suffice to prove that for every object $\overline{X} \in (\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$ and each $n \geq 0$, the projection map $p : \overline{X} \times (\Delta^n)^{\sharp} \to \overline{X}$ is a $\mathfrak{P}$-equivalence. The inclusion $i : \{0\}^{\sharp} \subseteq (\Delta^n)^{\sharp}$ determines a section s of p; it will therefore suffice to show that s is a $\mathfrak{P}$-equivalence. Lemma B.1.13 implies that i is $\mathfrak{P}_0$-anodyne (where $\mathfrak{P}_0$ is defined as in the statement of Lemma B.1.13). Using Proposition B.1.9, we conclude that s is $\mathfrak{P}$-anodyne, so that s is a $\mathfrak{P}$-equivalence by Example B.2.2.

We now discuss the case of a general categorical pattern $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^{\triangleleft} \to S'\}_{\alpha \in A})$ on S. Let $\mathfrak{P}' = (M_S, T, \emptyset)$. We have already shown that $(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}'}$ has the structure of a left proper combinatorial simplicial model category. We may therefore define a model structure on the category $(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$ to be the localization of $(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$ with respect to the generating $\mathfrak{P}$-anodyne maps appearing in Definition B.1.1. It follows from Proposition T.A.3.7.3 that $(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$ is again a left proper combinatorial simplicial model category.

To complete the proof, it will suffice to show that an object $\overline{X} \in (\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}}$ is fibrant if and only if it is $\mathfrak{P}$-fibered. It follows from Proposition T.A.3.7.3 that $\overline{X}$ is fibrant if and only if the following conditions are satisfied:

(i) The object $\overline{X}$ is fibrant in $(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}'}$: that is, $\overline{X}$ has the extension property with respect to every cofibration $f : \overline{Y} \to \overline{Y}'$ which is a $\mathfrak{P}'$-equivalence.

(ii) For every generating $\mathfrak{P}$-anodyne map $f : \overline{Y} \to \overline{Y}'$, the induced map $q : \mathrm{Map}_S^{\sharp}(\overline{Y}', \overline{X}) \to \mathrm{Map}_S^{\sharp}(\overline{Y}, \overline{X})$ is a homotopy equivalence of Kan complexes.

Suppose that $\overline{X}$ satisfies (ii). Note that for every $\mathfrak{P}$-anodyne map f, the map q is a Kan fibration (Proposition B.1.14), and therefore a trivial Kan fibration. It follows that q is surjective on vertices, so that $\overline{X}$ has the extension property with respect to every $\mathfrak{P}$-anodyne map and is therefore $\mathfrak{P}$-fibered by virtue of Proposition B.1.6.

Conversely, suppose that $\overline{X}$ is $\mathfrak{P}$-fibered; we wish to show that $\overline{X}$ satisfies conditions (i) and (ii). To prove (i), consider the map $q : \mathrm{Map}_S^{\sharp}(\overline{Y}', \overline{X}) \to \mathrm{Map}_S^{\sharp}(\overline{Y}, \overline{X})$. This map is a Kan fibration (Proposition B.1.14) and a homotopy equivalence by virtue of our assumption that f is a $\mathfrak{P}'$-equivalence (since $\overline{X}$ is $\mathfrak{P}'$-fibered). It follows that q is a trivial Kan fibration and therefore surjective on vertices, which proves (i). To prove (ii), it will suffice to show that q is a trivial Kan fibration whenever f is $\mathfrak{P}$-anodyne. To see that

q has the right lifting property with respect to the inclusion $\partial\,\Delta^n \subseteq \Delta^n$, we need to show that $\overline{X}$ has the extension property with respect to the induced inclusion

$$f' = (\overline{Y} \times (\Delta^n)^\sharp) \coprod_{\overline{Y} \times (\partial\,\Delta^n)^\sharp} (\overline{Y}' \times (\partial\,\Delta^n)^\sharp) \subseteq \overline{Y}' \times (\Delta^n)^\sharp.$$

This follows from Proposition B.1.6, since f' is $\mathfrak{P}$-anodyne by virtue of Proposition B.1.9. $\qquad\square$

Remark B.2.5. Let $\mathfrak{P}$ and $\mathfrak{P}'$ be categorical patterns, and let $\mathfrak{P} \times \mathfrak{P}'$ be defined as in Definition B.1.8. The formation of Cartesian products induces a functor

$$F : (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}} \times (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}'} \to (\mathcal{S}et_\Delta^+)_{/\mathfrak{P} \times \mathfrak{P}'}.$$

With respect to the model structures of Theorem B.0.19, the map F is a left Quillen bifunctor. To prove this, we must show that if $f : \overline{X} \to \overline{X}'$ is a cofibration in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ and $g : \overline{Y} \to \overline{Y}'$ is a cofibration in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}'}$, then the induced map

$$f \wedge g : (\overline{X}' \times \overline{Y}) \coprod_{\overline{X} \times \overline{Y}} (\overline{X} \times \overline{Y}') \to \overline{X}' \times \overline{Y}'$$

is a cofibration, which is trivial if either f or g is trivial. The first claim is obvious, and the second is equivalent to the requirement that the diagram

$$\begin{array}{ccc} \overline{X} \times \overline{Y} & \xrightarrow{\ i\ } & \overline{X}' \times \overline{Y} \\ \downarrow & & \downarrow \\ \overline{X} \times \overline{Y}' & \xrightarrow{\ j\ } & \overline{X}' \times \overline{Y}' \end{array}$$

is a homotopy pushout square. For this, it suffices to show that the horizontal maps are weak equivalences. We will prove that i is a weak equivalence; the proof that j is a weak equivalence is similar. Choose a $\mathfrak{P}$-anodyne map $f' : \overline{X}' \to \overline{X}''$, where $\overline{X}''$ is $\mathfrak{P}$-fibered. Proposition B.1.9 guarantees that the map $\overline{X}' \times \overline{Y} \to \overline{X}'' \to \overline{Y}$ is $(\mathfrak{P} \times \mathfrak{P}')$-anodyne. It therefore suffices to show that the composite map $\overline{X} \times \overline{Y} \to \overline{X}'' \times \overline{Y}$ is a $(\mathfrak{P} \times \mathfrak{P}')$-equivalence. We may therefore replace $\overline{X}'$ by $\overline{X}''$ and thereby reduce to the case where $\overline{X}'$ is $\mathfrak{P}$-fibered. By a similar argument, we can assume that the map $\overline{X} \to \overline{X}'$ has the right lifting property with respect to all $\mathfrak{P}$-anodyne morphisms, so that $\overline{X}$ is $\mathfrak{P}$-fibered as well. The $\mathfrak{P}$-equivalence f now admits a homotopy inverse, so that the induced map $\overline{X} \times \overline{Y} \to \overline{X}' \times \overline{Y}$ admits a homotopy inverse as well.

Remark B.2.6. Let $\mathfrak{P}$ be a categorical pattern, and let $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ be endowed with the model structure of Theorem B.0.19. Then the weak equivalences in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ are precisely the $\mathfrak{P}$-equivalences.

Let $\mathfrak{P}$ be a categorical pattern on a simplicial set S, and regard $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ as endowed with the model structure of Theorem B.0.19. An object of $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ is fibrant if and only if it is $\mathfrak{P}$-fibered. Under some mild assumptions on $\mathfrak{P}$, we can explicitly describe all *fibrations* between fibrant objects of $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$:

Proposition B.2.7. *Let* $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^\triangleleft \to S\}_{\alpha\in A})$ *be a categorical pattern on an ∞-category S. Suppose that M_S contains all equivalences in S and that T contains all 2-simplices $\Delta^2 \to S$ whose restriction to $\Delta^{\{0,1\}}$ in an equivalence in S. Let $\overline{Y} = (Y, M_Y)$ be a $\mathfrak{P}$-fibered object of $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$, and let $\pi : Y \to S$ denote the underlying map of simplicial sets. Let $\overline{X} = (X, M_X)$ be another object of $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$, and let $f : \overline{X} \to \overline{Y}$ be a morphism. The following conditions are equivalent:*

(a) *The map p is a fibration in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$.*

(b) *The object $\overline{X}$ is $\mathfrak{P}$-fibered, and the underlying map of simplicial sets $X \to Y$ is a categorical fibration.*

(c) *The map p exhibits $\overline{X}$ as a $\pi^* \mathfrak{P}$-fibered object of $(\mathcal{S}et_\Delta^+)_{/\pi^* \mathfrak{P}}$.*

The proof of Proposition B.2.7 will require some preliminaries. We begin with some remarks on the functoriality of the construction $\mathfrak{P} \mapsto (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ (for a generalization, see Theorem B.4.2).

Definition B.2.8. Let $f : S \to S'$ be a map of simplicial sets. Suppose we are given categorical patterns $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^\triangleleft \to S\}_{\alpha \in A})$ and $\mathfrak{P}' = (M'_S, T', \{p'_\alpha : K'_\alpha{}^\triangleleft \to S'\}_{\alpha \in A'})$ on S and S', respectively. We will say that f is *compatible* with $\mathfrak{P}$ and $\mathfrak{P}'$ if the following conditions are satisfied:

- The map f carries M_S into $M_{S'}$.

- The map f carries T into T'.

- For each $\alpha \in A$, the composition

$$K_\alpha^\triangleleft \overset{p_\alpha}{\to} S \overset{f}{\to} S'$$

belongs to $\{p'_\alpha : K'_\alpha{}^\triangleleft \to S'\}_{\alpha \in A'})$.

Proposition B.2.9. *Let $f : S \to S'$ be a map of simplicial sets, and suppose that f is compatible with categorical patterns $\mathfrak{P}$ and $\mathfrak{P}'$ on S and S', respectively. Then composition with f induces a left Quillen functor $f_! : (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}} \to (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}'}$.*

Proof. It is clear that $f_!$ preserves cofibrations. It also admits a right adjoint, given by the pullback functor f^* described by the formula $f^*\overline{X} \simeq \overline{X} \times_{(S', M_{S'})} (S, M_S)$. To complete the proof, it will suffice to show that $f_!$ preserves $\mathfrak{P}$-equivalences. Let $\overline{X}, \overline{Y} \in (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$, and let $\alpha : \overline{X} \to \overline{Y}$ be a $\mathfrak{P}$-equivalence. We wish to show that $f_!(\alpha)$ is a $\mathfrak{P}'$-equivalence. For this, it suffices to show that for every $\mathfrak{P}'$-fibered object $\overline{Z} \in (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}'}$, the induced map

$$\mathrm{Map}_{S'}^\sharp(f_!\overline{Y}, \overline{Z}) \to \mathrm{Map}_{S'}^\sharp(f_!\overline{X}, \overline{Z})$$

is a homotopy equivalence. The left hand side can be identified with $\mathrm{Map}_S^\sharp(\overline{Y}, f^*\overline{Z})$, and the right hand side with $\mathrm{Map}_S^\sharp(\overline{X}, f^*\overline{Z})$. The desired result now follows from the assumption that α is a $\mathfrak{P}$-equivalence, and the observation that $f^*\overline{Z}$ is $\mathfrak{P}$-fibered. $\qquad\square$

Example B.2.10. For any categorical pattern $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^\triangleleft \to S\}_{\alpha \in A})$ on any simplicial set S, the forgetful functor $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}} \to \mathcal{S}et_\Delta^+$ is a left Quillen functor, where we endow $\mathcal{S}et_\Delta^+$ with the model structure determined by Theorem B.0.19 for the categorical pattern $\mathfrak{P}_0 = (M_0, T_0, \{K_\alpha^\triangleleft \to \Delta^0\}_{\alpha \in A})$ on Δ^0 (here M_0 and T_0 consist of all edges and 2-simplices of Δ^0, respectively). If each of the simplicial sets K_α is contractible, then this coincides with the usual model structure on $\mathcal{S}et_\Delta^+$ (Remark B.0.27).

Lemma B.2.11. *Let $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^\triangleleft \to S\}_{\alpha \in A})$ be a categorical pattern on an ∞-category S, and let $\overline{X} = (X, M) \in (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ be $\mathfrak{P}$-fibered. Assume that M_S is the collection of all equivalences in S and that T contains all 2-simplices $\Delta^2 \to S$ whose restriction to $\Delta^{\{0,1\}}$ in an equivalence in S. Then M is the collection of all equivalences in X.*

Proof. Let $p : X \to S$ denote the underlying map of simplicial sets. The set M consists of all locally p-coCartesian morphisms f in X such that $p(f)$ is an equivalence in S. In view of Proposition T.2.4.1.5, it will suffice to show that every such morphism is p-coCartesian. This follows from Lemma B.1.7 together with our assumption on the set of 2-simplices T. $\qquad\square$

Proof of Proposition B.2.7. Fix a categorical pattern $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^\triangleleft \to S\}_{\alpha \in A})$ on an ∞-category S, where M_S contains all equivalences in S and T contains all 2-simplices $\Delta^2 \to S$ whose restriction to $\Delta^{\{0,1\}}$ in an equivalence in S. Let $p : \overline{X} \to \overline{Y}$ be a morphism in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$, where $\overline{Y}$ is $\mathfrak{P}$-fibered. We wish to prove that conditions (a), (b) and (c) of Proposition B.2.7 are equivalent.

We first prove that $(a) \Rightarrow (b)$. Assume that p is a fibration in $(\mathcal{S}et^+_\Delta)_{/\mathfrak{P}}$: we wish to prove that $\overline{X}$ is $\mathfrak{P}$-fibered and that the underlying map of simplicial sets $X \to Y$ is a categorical fibration. The first assertion is obvious; to prove the second, we must show that every lifting problem of the form

$$\begin{array}{ccc} A^\flat & \longrightarrow & \overline{X} \\ \downarrow{\scriptstyle i} & \nearrow & \downarrow \\ B^\flat & \longrightarrow & \overline{Y} \end{array}$$

admits a solution, provided that the underlying map of simplicial sets $A \to B$ is a trivial cofibration with respect to the Joyal model structure. To prove this, it will suffice to show that the map i is a $\mathfrak{P}$-equivalence. By virtue of Proposition B.2.9, it will suffice to prove this after replacing $\mathfrak{P}$ by the categorical pattern $\mathfrak{P}' = (M'_S, T, \emptyset)$, where M'_S is the collection of all equivalences in S. We must now show that for every $\mathfrak{P}$-fibered object $\overline{Z} = (Z, M) \in (\mathcal{S}et^+_\Delta)_{/\mathfrak{P}}$, the induced map $\theta : \mathrm{Map}^\sharp_S(B^\flat, \overline{Z}) \to \mathrm{Map}^\sharp_S(A^\flat, \overline{Z})$ is a weak homotopy equivalence. We observe that Z is an ∞-category and M_Z can be identified with the collection of all equivalences in Z (Lemma B.2.11). For every simplicial set K and every ∞-category $\mathcal{C}$, we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{Map}^\sharp_S(B^\flat, \overline{Z}) & \longrightarrow & \mathrm{Fun}(B, Z)^\simeq & \longrightarrow & \mathrm{Fun}(B, S)^0 \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \theta''} \\ \mathrm{Map}^\sharp_S(A^\flat, \overline{Z}) & \longrightarrow & \mathrm{Fun}(A, Z)^0 & \longrightarrow & \mathrm{Fun}(A, S)^\simeq. \end{array}$$

where the rows are homotopy fiber sequences. Consequently, to prove that θ is a homotopy equivalence, it suffices to show that θ' and θ'' are homotopy equivalences. This follows from the observation that the maps

$$\mathrm{Fun}(B, Z) \to \mathrm{Fun}(A, Z) \qquad \mathrm{Fun}(B, S) \to \mathrm{Fun}(A, S)$$

are categorical equivalences (in fact, trivial Kan fibrations), since $A \to B$ is a trivial cofibration and the simplicial sets S and Z are fibrant (with respect to the Joyal model structure).

We now show that $(b) \Rightarrow (a)$. Assume that $\overline{X}$ is $\mathfrak{P}$-fibered and that the underlying map $X \to Y$ is a categorical fibration; we wish to show that $p : \overline{X} \to \overline{Y}$ is a fibration in $(\mathcal{S}et^+_\Delta)_{/\mathfrak{P}}$. We must prove that every lifting problem of the form

$$\begin{array}{ccc} \overline{A} & \xrightarrow{f_0} & \overline{X} \\ \downarrow{\scriptstyle i} & \nearrow & \downarrow{\scriptstyle p} \\ \overline{B} & \xrightarrow{g} & \overline{Y} \end{array}$$

admits a solution, provided that i is a monomorphism and a $\mathfrak{P}$-equivalence. Since $\overline{X}$ is $\mathfrak{P}$-fibered, the lifting problem

$$\begin{array}{ccc} \overline{A} & \xrightarrow{f_0} & \overline{X} \\ \downarrow{\scriptstyle i} & \overset{f'}{\nearrow} & \downarrow \\ \overline{B} & \longrightarrow & (S, M_S). \end{array}$$

admits a solution. The map $g' = p \circ f'$ does not necessarily coincide with g. However, g and g' agree on $\overline{A}$ and therefore determine a map

$$G_0 : (\overline{A} \times (\Delta^1)^\sharp) \coprod_{\overline{A} \times (\partial \Delta^1)^\sharp} (\overline{B} \times (\partial \Delta^1)^\sharp) \to \overline{Y}.$$

Consider the diagram

$$
\begin{CD}
(\overline{A} \times (\Delta^1)^\sharp) \coprod_{\overline{A} \times (\partial\,\Delta^1)^\sharp} (\overline{B} \times (\partial\,\Delta^1)^\sharp) @>{G_0}>> \overline{Y} \\
@V{j}VV @VVV \\
\overline{B} \times (\Delta^1)^\sharp @>>{\overline{B} \xrightarrow{\pi\circ g}}> (S, M_S).
\end{CD}
$$

with diagonal dashed arrow G.

Since the map j is a $\mathfrak{P}$-equivalence (Proposition B.1.9) and $\overline{X}$ is $\mathfrak{P}$-fibered, there exists a map G rendering this diagram commutative. We regard G as an equivalence from g to $p \circ f'$ in $\mathrm{Fun}(B, Y)$. Since p is a categorical fibration, it induces a fibration $X^\sharp \to Y^\sharp$ in the category of marked simplicial sets; here $X^\sharp = (X, E_X)$ where E_X is the collection of all equivalences in X and $Y^\sharp$ is defined similarly. It follows that the lifting problem

$$
\begin{CD}
(A^\flat \times (\Delta^1)^\sharp) \coprod_{A^\flat \times \{1\}^\sharp} (B^\flat \times \{1\}^\sharp) @>>> A^\flat @>{f'}>> X^\sharp \\
@VVV @VVV \\
B^\flat \times (\Delta^1)^\sharp @>>{G}> Y^\sharp
\end{CD}
$$

with diagonal dashed arrow F.

admits a solution. We can regard F as an equivalence from f to f' in $\mathrm{Fun}(B, X)$, where f is an extension of f_0 lifting g. Since f is equivalent to f', it carries marked edges of $\overline{A}$ to marked edges of $\overline{X}$, and therefore constitutes a solution to the original lifting problem.

We next show that $(a) \Rightarrow (c)$. Assume that f is a fibration in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$. We must prove that every lifting problem of the form

$$
\begin{CD}
\overline{A} @>{f_0}>> \overline{X} \\
@V{i}VV @VV{p}V \\
\overline{B} @>>{g}> \overline{Y}
\end{CD}
$$

admits a solution, provided that i is a trivial cofibration in $(\mathcal{S}et_\Delta^+)_{/\pi^* \mathfrak{P}}$; here π denotes the projection $Y \to S$. Since p is assumed to be a fibration in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$, it suffices to show that i is a trivial cofibration in $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$, which follows from Proposition B.2.9.

Finally, we show that $(c) \Rightarrow (b)$. Assume that $\overline{X}$ is $(\pi^* \mathfrak{P})$-fibrered. Replacing $\mathfrak{P}$ by $\pi^* \mathfrak{P}$ and invoking the implication $(a) \Rightarrow (b)$, we deduce that $X \to Y$ is a categorical fibration. It will therefore suffice to show that $\overline{X}$ is $\mathfrak{P}$-fibered. We will show that $\overline{X}$ satisfies conditions (1), (2), (3), (4), and (6) of Definition B.0.18, together with condition $(5')$ of Remark B.0.25:

(1) The underlying map of simplicial sets $q : X \to S$ is an inner fibration. This is clear, since $q = \pi \circ p$, where both π and p are inner fibrations.

(2) For each edge $\Delta^1 \to S$ belonging to M_S, the induced map $q_{\Delta^1} : X \times_S \Delta^1 \to \Delta^1$ is a coCartesian fibration. In other words, we must show that for every object $x \in X$ and every edge $e : q(x) \to s$ belonging to M_S, there exists a locally q-coCartesian edge $\overline{e} : x \to \overline{s}$ with $q(\overline{e}) = e$. Since $\overline{Y}$ is $\mathfrak{P}$-fibered, we can choose a locally π-coCartesian edge $\widetilde{e} : p(x) \to \widetilde{s}$ with $\pi(\widetilde{e}) = e$. Moreover, the edge $\widetilde{e}$ belongs to M_Y, so we can choose a locally p-coCartesian edge $\overline{e}$ with $p(\overline{e}) = \widetilde{e}$ (note that $\overline{e}$ belongs to M_X). To complete the proof, it will suffice to show that $\overline{e}$ is locally q-coCartesian: in other words, that it determines a q_{Δ^1}-coCartesian edge $\overline{e}'$ of $X \times_S \Delta^1$. We note that q_{Δ^1} factors as a composition

$$
X \times_S \Delta^1 \xrightarrow{q'_{\Delta^1}} Y \times_S \Delta^1 \xrightarrow{\pi_{\Delta^1}} \Delta^1,
$$

and that $q'_{\Delta^1}(\overline{e}')$ is π_{Δ^1}-coCartesian by construction. In view of Proposition T.2.4.1.3, it suffices to show that $\overline{e}'$ is q'_{Δ^1}-coCartesian. This follows from Lemma B.1.7, since the image of every 2-simplex σ of $X \times_S \Delta^1$ in Y is a thin 2-simplex with respect to $\pi^* \mathfrak{P}$ (since the image of σ in S is degenerate).

(3) An edge $\overline{e} : x \to x'$ of X belongs to M_X if and only if $e = q(\overline{e})$ belongs to M_S and e locally q-coCartesian. The "if" direction follows from the proof of (2). For the converse, we observe that if $e \in M_S$ then we can apply the construction of (2) to produce a locally q-coCartesian edge $\overline{e}' : x \to x''$ of X covering e, where $\overline{e}' \in M_X$. If $\overline{e}$ is also locally q-coCartesian, then $\overline{e}$ and $\overline{e}'$ are equivalent, so $\overline{e}$ also belongs to M_X.

(4) Given a commutative diagram

$$
\begin{array}{ccc}
\Delta^{\{0,1\}} & \xrightarrow{\ e\ } & X \\
\downarrow & & \downarrow \\
\Delta^2 & \xrightarrow{\ \sigma\ } & S,
\end{array}
$$

if $e \in M_X$ and $\sigma \in T$, then e determines an q_{Δ^2}-coCartesian edge of $X \times_S \Delta^2$, where $q_{\Delta^2} : X \times_S \Delta^2 \to \Delta^2$ denotes the projection map. To prove this, we factor q_{Δ^2} as a composition

$$
X \times_S \Delta^2 \xrightarrow{p_{\Delta^2}} Y \times_S \Delta^2 \xrightarrow{\pi_{\Delta^2}} \Delta^2.
$$

Since $\overline{Y}$ is $\mathfrak{P}$-fibered and $p(e) \in M_Y$, we conclude that the image of e in $Y \times_S \Delta^2$ is π_{Δ^2}-coCartesian. In view of Proposition T.2.4.1.3, it will suffice to show that e determines a p_{Δ^2}-coCartesian edge of $X \times_S \Delta^2$. This follows from Lemma B.1.7, since e determines a locally p_{Δ^2}-coCartesian edge of $X \times_S \Delta^2$ and the image of every 2-simplex of $X \times_S \Delta^2$ in Y is thin with respect to $\pi^* \mathfrak{P}$.

(5′) For each $\alpha \in A$, every lifting problem of the form

$$
\begin{array}{ccc}
K^\sharp_\alpha & \longrightarrow & \overline{X} \\
\downarrow & \nearrow & \downarrow \\
(K^\triangleleft_\alpha)^\sharp & \xrightarrow{\ p_\alpha\ } & (S, M_S)
\end{array}
$$

admits a solution. We first invoke the fact that $\overline{Y}$ is $\mathfrak{P}$-fibered to solve the induced lifting problem

$$
\begin{array}{ccc}
K^\sharp_\alpha & \longrightarrow & \overline{Y} \\
\downarrow & \overset{f}{\nearrow} & \downarrow \\
(K^\triangleleft_\alpha)^\sharp & \xrightarrow{\ p_\alpha\ } & (S, M_S).
\end{array}
$$

We then invoke the assumption that $\overline{X}$ is $\pi^* \mathfrak{P}$ fibered to solve the lifting problem

$$
\begin{array}{ccc}
K^\sharp_\alpha & \longrightarrow & \overline{X} \\
\downarrow & \nearrow & \downarrow \\
(K^\triangleleft_\alpha)^\sharp & \xrightarrow{\ f\ } & \overline{Y}.
\end{array}
$$

(6) For every index $\alpha \in A$, every map $\overline{p}_\alpha : (K^\triangleleft_\alpha)^\sharp \to \overline{X}$ lifting $p_\alpha : (K^\triangleleft_\alpha)^\sharp \to (X, S)$ is a q-limit diagram. Invoking the assumption that $\overline{Y}$ is $\mathfrak{P}$-fibered, we deduce that $\widetilde{p}_\alpha = p \circ \overline{p}_\alpha$ is a π-limit diagram in Y. Moreover, $\widetilde{p}_\alpha$ is one of the diagrams defining the categorical pattern $\pi^* \mathfrak{P}$, so our assumption that $\overline{X}$ is $\mathfrak{P}$-fibered ensures that $\overline{p}_\alpha$ is a p-limit diagram. Since $q = \pi \circ p$, the desired result now follows from Proposition T.4.3.1.5.

$\square$

B.3   Flat Inner Fibrations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *distributor* from $\mathcal{C}$ to $\mathcal{D}$ is a functor $M : \mathcal{C}^{op} \times \mathcal{D} \to \mathcal{S}et$. Any functor $F : \mathcal{C} \to \mathcal{D}$ determines a distributor M_F from $\mathcal{C}$ to $\mathcal{D}$, given by the formula $M_F(C, D) = \operatorname{Hom}_{\mathcal{D}}(FC, D)$. Consequently, we can think of a distributor as a kind of generalized functor. As with ordinary functors, distributors can be composed: if we are given distributors $M : \mathcal{C}^{op} \times \mathcal{D} \to \mathcal{S}et$ and $N : \mathcal{D}^{op} \times \mathcal{E} \to \mathcal{S}et$, then the composition $N \circ M : \mathcal{C}^{op} \times \mathcal{E} \to \mathcal{S}et$ is given by the formula

$$(N \circ M)(C, E) = \int_{D \in \mathcal{D}} M(C, D) \times N(D, E),$$

(the right hand side indicates the coend of the functor $M(C, \bullet) \times N(\bullet, E)$ along $\mathcal{D}$).

  The above ideas can be reformulated using the language of correspondences. Recall that a *correspondence* from a category $\mathcal{C}$ to another category $\mathcal{D}$ is a category category $\mathcal{M}$ containing $\mathcal{C}$ and $\mathcal{D}$ as full subcategories, equipped with a functor $p : \mathcal{M} \to [1]$ such that $\mathcal{C} = p^{-1}\{0\}$ and $\mathcal{D} = p^{-1}\{1\}$. Every correspondence $\mathcal{M}$ from $\mathcal{C}$ to $\mathcal{D}$ determines a distributor M, given by the formula $M(C, D) = \operatorname{Hom}_{\mathcal{M}}(C, D)$. Conversely, if we are given a distributor M, we can construct a correspondence $\mathcal{M}$ as follows:

- An object of $\mathcal{M}$ is either an object of $\mathcal{C}$ or an object of $\mathcal{D}$.

- Morphisms in $\mathcal{M}$ are given by the formula

$$\operatorname{Hom}_{\mathcal{M}}(X, Y) = \begin{cases} \operatorname{Hom}_{\mathcal{C}}(X, Y) & \text{if } X, Y \in \mathcal{C} \\ \operatorname{Hom}_{\mathcal{D}}(X, Y) & \text{if } X, Y \in \mathcal{D} \\ M(X, Y) & \text{if } X \in \mathcal{C}, Y \in \mathcal{D} \\ \emptyset & \text{if } X \in \mathcal{D}, Y \in \mathcal{C}. \end{cases}$$

We can summarize the above discussion informally as follows: given a pair of categories $\mathcal{C}$ and $\mathcal{D}$, giving a distributor from $\mathcal{C}$ to $\mathcal{D}$ is equivalent to giving a correspondence from $\mathcal{C}$ to $\mathcal{D}$.

  The composition of distributors has a natural interpretation in the language of correspondences. To see this, suppose we are given categories $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$. Consider the problem of constructing a category $\mathcal{X}$ equipped with a functor $p : \mathcal{X} \to [2]$ such that $\mathcal{C} = p^{-1}\{0\}$, $\mathcal{D} = p^{-1}\{1\}$, and $\mathcal{E} = p^{-1}\{2\}$. Any such category $\mathcal{X}$ determines distributors $M : \mathcal{C}^{op} \times \mathcal{D} \to \mathcal{S}et$, $N : \mathcal{D}^{op} \times \mathcal{E} \to \mathcal{S}et$, and $P : \mathcal{C}^{op} \times \mathcal{E} \to \mathcal{S}et$, given by the formulas

$$M(C, D) = \operatorname{Hom}_{\mathcal{X}}(C, D) \qquad N(D, E) = \operatorname{Hom}_{\mathcal{X}}(D, E) \qquad P(C, E) = \operatorname{Hom}_{\mathcal{X}}(C, E).$$

The composition law on $\mathcal{X}$ determines (and is determined by) a natural transformation of distributors $(N \circ M) \to P$, where $N \circ M$ is defined as above.

  Now suppose that we are not given $\mathcal{X}$: instead, we are given a correspondence $\mathcal{M}$ from $\mathcal{C}$ to $\mathcal{D}$ and another correspondence $\mathcal{N}$ from $\mathcal{D}$ to $\mathcal{E}$. Then $\mathcal{M}$ and $\mathcal{N}$ determine distributors $M : \mathcal{C}^{op} \times \mathcal{D} \to \mathcal{S}et$ and $N : \mathcal{D}^{op} \times \mathcal{E} \to \mathcal{S}et$. From the above analysis, we see that the following data are equivalent:

(a) A category $\mathcal{X}$ equipped with a functor $p : \mathcal{X} \to [2]$ such that $p^{-1}\{0 < 1\} = \mathcal{M}$ and $p^{-1}\{1 < 2\} = \mathcal{N}$.

(b) A distributor $P : \mathcal{C}^{op} \times \mathcal{E} \to \mathcal{S}et$ together with a natural transformation $\alpha : (N \circ M) \to P$.

Neither type of data is uniquely determined by $\mathcal{M}$ and $\mathcal{N}$, even up to isomorphism. However, there is always a canonical choice for the data of type (b): namely, we can take $P = N \circ M$ and α to be the identity map. The equivalence between (b) and (a) then shows that there is a canonical choice for the category $\mathcal{X}$. We will refer to this canonical choice as the *composition* of the correspondences $\mathcal{M}$ and $\mathcal{N}$. Concretely, it can be described as the pushout $\mathcal{M} \coprod_{\mathcal{D}} \mathcal{N}$.

  Our goal in this section is to explain how some of the above ideas can be carried over to the ∞-categorical setting. Motivated by the preceding discussion, we introduce the following definition:

Definition B.3.1. Suppose we are given a functor of ∞-categories $p : \mathfrak{X} \to \Delta^2$. We will say that p is *flat* if the inclusion $\mathfrak{X} \times_{\Delta^2} \Lambda_1^2 \hookrightarrow \mathfrak{X}$ is a categorical equivalence.

In other words, a functor $p : \mathfrak{X} \to \Delta^2$ is flat if the diagram

$$
\begin{array}{ccc}
\mathfrak{X} \times_{\Delta^2} \{1\} & \longrightarrow & \mathfrak{X} \times_{\Delta^2} \Delta^{\{0,1\}} \\
\downarrow & & \downarrow \\
\mathfrak{X} \times_{\Delta^2} \Delta^{\{1,2\}} & \longrightarrow & \mathfrak{X}
\end{array}
$$

is a homotopy pushout diagram (with respect to the Joyal model structure). More informally, p is flat if it exhibits the correspondence $\mathfrak{X} \times_{\Delta^2} \Delta^{\{0,2\}}$ as the composition of the correspondences $\mathfrak{X} \times_{\Delta^2} \Delta^{\{0,1\}}$ and $\mathfrak{X} \times_{\Delta^2} \Delta^{\{1,2\}}$.

Our first goal in this section is to establish the following recognition criterion for flat maps:

Proposition B.3.2. *Let $\mathfrak{X}$ be an ∞-category equipped with a functor $p : \mathfrak{X} \to \Delta^2$. Let $\mathcal{C} = p^{-1}\{0\}$, let $\mathcal{D} = p^{-1}\{1\}$, and let $\mathcal{E} = p^{-1}\{2\}$. The following conditions are equivalent:*

(1) *The map p is flat.*

(2) *For every morphism $f : C \to E$ in $\mathfrak{X}$ from an object $C \in \mathcal{C}$ to an object $E \in \mathcal{E}$, the ∞-category $\mathcal{D}_{C//E} = \mathcal{D} \times_{\mathfrak{X}} \mathfrak{X}_{C//E}$ is weakly contractible.*

Remark B.3.3. Criterion (2) of Proposition B.3.2 can be regarded as a version of the formula

$$
(N \circ M)(C, E) = \int_{D \in \mathcal{D}} M(C, D) \times N(D, E)
$$

describing the composition of a pair of distributors.

Example B.3.4. Let $p : \mathfrak{X} \to \Delta^2$ be an inner fibration of simplicial sets. Let $\mathcal{C} = p^{-1}\{0\}$, $\mathcal{D} = p^{-1}\{1\}$, and $\mathcal{E} = p^{-1}\{2\}$. Suppose that for every object $C \in \mathcal{C}$, there exists a p-coCartesian morphism $f : C \to D$, where $D \in \mathcal{D}$. Then p is flat.

To prove this, consider an arbitrary morphism $g : C \to E$ in $\mathcal{M}$, where $C \in \mathcal{C}$ and $E \in \mathcal{E}$. Choose a p-coCartesian morphism $f : C \to D$ in $\mathcal{M}$ for $D \in \mathcal{D}$. Using the assumption that f is p-coCartesian, we can find a commutative diagram

$$
\begin{array}{ccc}
 & D & \\
{}^{f}\nearrow & & \searrow^{h} \\
C & \xrightarrow{\quad g \quad} & E
\end{array}
$$

which we can identify with an object $\overline{D} \in \mathcal{D}_{C//E}$ lifting D. To show that $\mathcal{D}_{C//E}$ is weakly contractible, it suffices to show that $\overline{D}$ is an initial object of $\mathcal{D}_{C//E}$. In view of Proposition T.1.2.13.8, it will suffice to show that $\overline{D}$ is an initial object of $\mathcal{D}_{C/}$, which is equivalent to the assertion that f is locally p-coCartesian.

Example B.3.5. Let $p : \mathfrak{X} \to \Delta^2$ be an inner fibration of simplicial sets. Let $\mathcal{C} = p^{-1}\{0\}$, $\mathcal{D} = p^{-1}\{1\}$, and $\mathcal{E} = p^{-1}\{2\}$. Suppose that for every object $E \in \mathcal{E}$, there exists a p-Cartesian morphism $f : D \to E$, where $D \in \mathcal{D}$. Then p is flat. The proof is identical to that of Example B.3.4.

The proof of Proposition B.3.2 will require some preliminaries.

Lemma B.3.6. *Let $\mathcal{C}$ be a simplicial category equipped with a functor $\mathcal{C} \to [1]$, where $[1]$ denotes the (discrete) category $\{0 < 1\}$. Suppose that the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ is a cofibration of simplicial categories. Then, for every object $D \in \mathcal{C}_1$, the functor $C \mapsto \mathrm{Map}_{\mathcal{C}}(C, D)$ is a projectively cofibrant object of $F \in (\mathrm{Set}_\Delta)^{\mathcal{C}_0^{op}}$.*

Proof. We must show that every trivial projective fibration $\alpha : G \to G'$ in $(\mathcal{S}et_\Delta)^{\mathcal{C}_0^{op}}$ has the right lifting property with respect to F. Define a new simplicial category $\mathcal{C}[G]$ as follows:

(i) The objects of $\mathcal{C}[G]$ are the objects of $\mathcal{C}$.

(ii) For $C, C' \in \mathcal{C}$, we have

$$\mathrm{Map}_{\mathcal{C}[G]}(C, C') = \begin{cases} \emptyset & \text{if } C \in \mathcal{C}_1, C' \in \mathcal{C}_0 \\ \mathrm{Map}_{\mathcal{C}}(C, C') \times G(C)^{\mathrm{Map}_{\mathcal{C}}(C', D)} & \text{if } C \in \mathcal{C}_0, C' \in \mathcal{C}_1 \\ \mathrm{Map}_{\mathcal{C}}(C, C') & \text{otherwise} \end{cases}$$

Let $\mathcal{C}[G']$ be defined similarly. Unwinding the definitions, we see that α has the right lifting property with respect to F if and only if the induced map $\overline{\alpha} : \mathcal{C}[G] \to \mathcal{C}[G']$ has the right lifting property with respect to the inclusion $i : \mathcal{C}_0 \subseteq \mathcal{C}$. Since i is a cofibration, this follows from the observation that $\overline{\alpha}$ is a trivial fibration of simplicial categories. $\square$

Lemma B.3.7. *Suppose we are given an inner fibration of simplicial sets $p : X \to \Lambda_1^2$. Let C be an initial object of $\mathcal{M} = p^{-1}\Delta^{\{0,1\}}$, let E be a final object of $\mathcal{N} = p^{-1}\Delta^{\{1,2\}}$, let $\mathcal{D} = \mathcal{M} \cap \mathcal{N} = p^{-1}\{1\}$, and let $f : X \to \mathcal{M}$ be a categorical equivalence from X to an ∞-category $\mathcal{M}$. Then there is a canonical isomorphism $\mathrm{Map}_{\mathcal{M}}(f(C), f(E)) \simeq [\mathcal{D}]$ in the homotopy category $\mathcal{H}$ of spaces.*

Proof. We can identify $\mathrm{Map}_{\mathcal{M}}(f(C), f(E))$ with the simplicial set $\mathrm{Map}_{\mathfrak{C}[X]}(C, E)$. Let $F : \mathfrak{C}[\mathcal{D}] \to \mathcal{S}et_\Delta$ be the functor given by the formula $F(D) = \mathrm{Map}_{\mathfrak{C}[\mathcal{M}]}(C, D)$, and let $G : \mathfrak{C}[\mathcal{D}]^{op} \to \mathcal{S}et_\Delta$ be given by the formula $G(D) = \mathrm{Map}_{\mathfrak{C}[\mathcal{N}]}(D, E)$. Since $\mathfrak{C}[X]$ is isomorphic to the pushout $\mathfrak{C}[\mathcal{M}] \coprod_{\mathfrak{C}[\mathcal{D}]} \mathfrak{C}[\mathcal{N}]$, the simplicial set $\mathrm{Map}_{\mathfrak{C}[X]}(C, E)$ can be computed as the coend

$$\int_{D \in \mathfrak{C}[\mathcal{D}]} F(D) \times G(D).$$

Lemma B.3.6 guarantees that the functor G is projectively cofibrant, so the construction

$$H \mapsto \int_{D \in \mathfrak{C}[\mathcal{D}]} H(D) \times G(D)$$

carries weak equivalences between injectively cofibrant objects of $(\mathcal{S}et_\Delta)^{\mathfrak{C}[\mathcal{D}]}$ to weak homotopy equivalences of simplicial sets (Remark T.A.2.9.27). Since C is an initial object of $\mathcal{M}$, the canonical map $F \to F_0$ is a weak equivalence, where $F_0 : \mathfrak{C}[\mathcal{D}] \to \mathcal{S}et_\Delta$ is the constant functor taking the value Δ^0. It follows that α induces a homotopy equivalence

$$\mathrm{Map}_{\mathfrak{C}[X]}(C, E) \to \varinjlim G.$$

Since $E \in \mathcal{N}$ is final, we also have a weak equivalence $G \to G_0$, where $G_0 : \mathfrak{C}[\mathcal{D}]^{op} \to \mathcal{S}et_\Delta$ denotes the constant functor taking the value 0. It follows that G is a cofibrant replacement for G_0 with respect to the projective model structure on $(\mathcal{S}et_\Delta)^{\mathfrak{C}[\mathcal{D}]^{op}}$, so we can identify $\varinjlim G$ with a homotopy colimit of the diagram G_0. Applying Theorem T.4.2.4.1, we can identify this homotopy colimit with a colimit of the constant diagram $\mathcal{D}^{op} \to \mathcal{S}$ taking the value Δ^0. This colimit is represented by the simplicial set $\mathcal{D}$ in the homotopy category $\mathcal{H}$ (Corollary T.3.3.4.6). $\square$

Proof of Proposition B.3.2. Let $X = \mathfrak{X} \times_{\Delta^2} \Lambda_1^2$. Using the small object argument, we can factor the inclusion $X \hookrightarrow \mathfrak{X}$ as a composition

$$X \overset{i}{\hookrightarrow} \mathfrak{X}' \overset{q}{\to} \mathfrak{X}$$

where i is inner anodyne, the map q is an inner fibration, and i induces an isomorphism $X \to \mathfrak{X}' \times_{\Delta^2} \Lambda_1^2$. We will abuse notation by identifying X (and therefore also the ∞-categories $\mathcal{C}, \mathcal{D}, \mathcal{E} \subseteq X$) with a simplicial subset of $\mathfrak{X}'$ via the map i.

Condition (1) is equivalent to the assertion that q is an equivalence of ∞-categories. Since q is bijective on vertices, this is equivalent to the assertion that q induces a homotopy equivalence $\theta : \mathrm{Map}_{\mathcal{X}'}(C, E) \to \mathrm{Map}_{\mathcal{X}}(C, E)$ for every pair of objects $C, E \in \mathcal{X}'$. This condition is obvious unless $C \in \mathcal{C}$ and $E \in \mathcal{E}$. In the latter case, it is equivalent to the requirement that for every morphism $f : C \to E$ in $\mathcal{X}$, the homotopy fiber of the map θ (taken over the point $f \in \mathrm{Map}_{\mathcal{X}}(C, E)$) is contractible. It will therefore suffice to prove the equivalence of the following conditions:

(1′) The homotopy fiber of θ over $\{f\}$ is contractible.

(2′) The ∞-category $\mathcal{D}_{C//E}$ is weakly contractible.

Suppose we are given a right fibration $\overline{\mathcal{X}} \to \mathcal{X}$, and that we can lift f to a morphism $\overline{f} : \overline{C} \to \overline{E}$ in $\overline{\mathcal{X}}$. Let $\overline{\mathcal{X}}' = \mathcal{X}' \times_{\mathcal{X}} \overline{\mathcal{X}}$; it follows from Proposition T.3.3.1.3 that the inclusion $\overline{\mathcal{X}} \times_{\Delta^2} \Lambda_1^2 \hookrightarrow \overline{\mathcal{X}}'$ remains a categorical equivalence. Using Proposition T.2.4.4.2, we deduce the existence of a homotopy pullback diagram

$$
\begin{array}{ccc}
\mathrm{Map}_{\overline{\mathcal{X}}'}(\overline{C}, \overline{E}) & \xrightarrow{\ \overline{\theta}\ } & \mathrm{Map}_{\overline{\mathcal{X}}}(\overline{C}, \overline{E}) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathcal{X}'}(C, E) & \xrightarrow{\ \theta\ } & \mathrm{Map}_{\mathcal{X}}(C, E).
\end{array}
$$

It follows that (1′) is satisfied by the morphism f of $\mathcal{X}$ if and only if it is satisfied by the morphism $\overline{f}$ over $\overline{\mathcal{X}}$. Proposition T.2.1.2.5 guarantees that the map $\overline{\mathcal{X}}_{\overline{C}//\overline{E}} \to \mathcal{X}_{C//E}$ is a trivial Kan fibration, so that (2′) is satisfied by f if and only if it is satisfied by $\overline{f}$. It follows that we are free to replace $\mathcal{M}$ by $\overline{\mathcal{X}} = \mathcal{X}_{/E}$, and thereby reduce to the case where E is a final object of $\mathcal{X}$. A similar argument shows that we can assume that C is an initial object of $\mathcal{X}$. In this special case, the space $\mathrm{Map}_{\mathcal{X}}(C, E)$ is contractible, so we can reformulate (1′) as follows:

(1″) The space $\mathrm{Map}_{\mathcal{X}'}(C, E)$ is contractible.

If C is an initial object of $\mathcal{X}$, then $\mathcal{X}_{C/} \to \mathcal{X}$ is a trivial Kan fibration. Moreover, if E is a final object of $\mathcal{X}$ then it is a final object of $\mathcal{X}_{C/}$ (Proposition T.1.2.13.8), so the projection $\mathcal{X}_{C//E} \to \mathcal{X}_{C/}$ is also a trivial Kan fibration. We therefore obtain the following reformulation of condition (2′):

(2″) The ∞-category $\mathcal{D}$ is weakly contractible.

The equivalence of (1″) and (2″) now follows from Lemma B.3.7. $\qquad\square$

For many applications, it is useful to generalize Definition B.3.1 to the case of an arbitrary base simplicial set S.

Definition B.3.8. Let $p : X \to S$ be an inner fibration of simplicial sets, and let σ be a 2-simplex of S. We will say that p is *flat over* σ if the induced inner fibration $X \times_S \Delta^2 \to \Delta^2$ is flat, in the sense of Definition B.3.1. We will say that p is *flat* if it is flat over every 2-simplex of S.

Remark B.3.9. Let $p : X \to S$ be an inner fibration of simplicial sets. Using Proposition B.3.2, we see that p is flat if and only if, for every 2-simplex

$$
\begin{array}{ccc}
 & s' & \\
\nearrow & & \searrow \\
s & \xrightarrow{\ \ f\ \ } & s''
\end{array}
$$
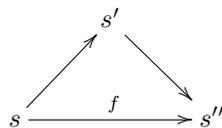
in S and every edge $\overline{f} : x \to y$ in X lifting f, the ∞-category $X_{x//y} \times_{S_{s//s''}} \{s'\}$ is weakly contractible.

Example B.3.10. Let $p : X \to S$ be an inner fibration of simplicial sets. Then p is flat over any degenerate 2-simplex of S, since the induced functor $X \times_S \Delta^2 \to \Delta^2$ satisfies the hypotheses of either Example B.3.4 or Example B.3.5. It follows that an inner fibration $p : X \to \Delta^2$ is flat in the sense of Definition B.3.8 if and only if it is flat in the sense of Definition B.3.1.

Example B.3.11. Let $p : X \to S$ be a coCartesian fibration of simplicial sets. Then p is a flat categorical fibration: this is an immediate consequence of Example B.3.4. Similarly, if p is a Cartesian fibration, then p is flat.

Remark B.3.12. Suppose given a pullback diagram of simplicial sets

$$
\begin{array}{ccc}
X' & \xrightarrow{\ q\ } & X \\
\downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\
S' & \longrightarrow & S.
\end{array}
$$

If p is a flat inner fibration, then so is p'.

Proposition B.3.13. *Let $q : X \to S$ be a flat inner fibration of simplicial sets, and let $x \in X$ be a vertex. Then the induced map $X_{x/} \to S_{q(x)/}$ is a flat inner fibration.*

Proof. Fix a 2-simplex

$$
\begin{array}{ccc}
 & s' & \\
\nearrow & & \searrow \\
s & \xrightarrow{\ \ f\ \ } & s''
\end{array}
$$

in $S_{q(x)/}$, and let $\overline{f} : y \to z$ be an edge of $X_{x/}$ lifting f. We wish to prove that the ∞-category

$$
\mathcal{C} = (X_{x/})_{y//z} \times_{(S_{q(x)/})_{s//s''}} \{s'\}
$$

is weakly contractible. Let $p : X_{x/} \to X$ be the projection map; then p induces a trivial Kan fibration

$$
\mathcal{C} \to \mathcal{C}_0 = X_{p(y)//p(z)} \times_{S_{qp(y)//qp(z)}} \Delta^0.
$$

Since q is flat, the ∞-category $\mathcal{C}_0$ is weakly contractible, so that $\mathcal{C}$ is weakly contractible as desired. $\qquad\square$

Proposition B.3.2 admits the following generalization:

Proposition B.3.14. *Let $p : X \to S$ be an inner fibration of simplicial sets. The following conditions are equivalent:*

(1) *For every inner anodyne map $A \hookrightarrow B$ of simplicial sets and every map $B \to S$, the induced map $X \times_S A \to X \times_S B$ is a categorical equivalence.*

(2) *The inner fibration p is flat.*

The remainder of this section is devoted to the proof of Proposition B.3.14. We begin by noting some of its consequences.

Corollary B.3.15. *Let $p : \mathcal{C} \to \mathcal{D}$ be a flat categorical fibration between ∞-categories. Then, for every categorical equivalence of simplicial sets $A \to B$ and every diagram $B \to \mathcal{D}$, the induced map $\theta : A \times_{\mathcal{D}} \mathcal{C} \to B \times_{\mathcal{D}} \mathcal{C}$ is an equivalence of ∞-categories.*

Proof. Every map $f : B \to \mathcal{D}$ factors as a composition

$$B \xrightarrow{f'} B' \xrightarrow{f''} \mathcal{D},$$

where f' is inner anodyne and f'' is an inner fibration (so that B' is an ∞-category). We obtain a commutative diagram

Proposition B.3.14 implies that α is a categorical equivalence. By the two-out-of-three property, it suffices to show that β is a categorical equivalence. We may therefore replace B by B' and thereby reduce to the case where B is an ∞-category.

The map $g : A \to B$ factors as a composition

$$A \xrightarrow{g'} A' \xrightarrow{g''} B$$

where g' is inner anodyne and g'' is an inner fibration (so that A' is an ∞-category). We obtain a commutative diagram

Proposition B.3.14 implies that γ is a categorical equivalence. Using the two-out-of-three property, we are reduced to proving that δ is a categorical equivalence. We may therefore replace A by A' and thereby reduce to the case where A is an ∞-category.

Consider the pullback diagram

Since the vertical maps in this diagram are categorical fibrations and the simplicial sets A and B are ∞-categories, Proposition T.A.2.4.4 guarantees that this diagram is homotopy Cartesian (with respect to the Joyal model structure). Since the g is a categorical equivalence, it follows that θ is a categorical equivalence as well. $\square$

Corollary B.3.16. *Let $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{E}$ be flat categorical fibrations between ∞-categories. Then $g \circ f$ is a flat categorical fibration.*

Proof. Since $g \circ f$ is evidently a categorical fibration, it will suffice to show that $g \circ f$ is flat. Choose a 2-simplex $\sigma : \Delta^2 \to \mathcal{E}$; we wish to show that the inclusion $\mathcal{C} \times_{\mathcal{E}} \Lambda_1^2 \subseteq \mathcal{C} \times_{\mathcal{E}} \Delta^2$ is a categorical equivalence. Let $\mathcal{D}' = \mathcal{D} \times_{\mathcal{E}} \Delta^2$ and $\mathcal{D}'' = \mathcal{D} \times_{\mathcal{E}} \Lambda_1^2$. Since g is flat, the inclusion $\mathcal{D}'' \subseteq \mathcal{D}'$ is a categorical equivalence. Since f is flat, Corollary B.3.15 guarantees that the inclusion

$$\mathcal{C} \times_{\mathcal{E}} \Lambda_1^2 \simeq \mathcal{C} \times_{\mathcal{D}} \mathcal{D}'' \subseteq \mathcal{C} \times_{\mathcal{D}} \mathcal{D}' \simeq \mathcal{C} \times_{\mathcal{E}} \Delta^2$$

is a categorical equivalence, as desired. $\square$

The proof of Proposition B.3.14 will require some preliminaries.

Proposition B.3.17. *Let $p : \mathcal{M} \to \Delta^1$ be a correspondence from an ∞-category $\mathcal{C} = \mathcal{M} \times_{\Delta^1} \{0\}$ to $\mathcal{D} = \mathcal{M} \times_{\Delta^1} \{1\}$. Let $\mathcal{X} = \mathrm{Map}_{\Delta^1}(\Delta^1, \mathcal{M})$ be the ∞-category of sections of the map p. Then the canonical map*

$$\mathcal{C} \coprod_{\mathcal{X} \times \{0\}} (\mathcal{X} \times \Delta^1) \coprod_{\mathcal{X} \times \{1\}} \mathcal{D} \to \mathcal{M}$$

is a categorical equivalence.

Proof. For every ∞-category $\mathcal{A}$, we let $\mathcal{A}^\natural$ denote the marked simplicial set $(\mathcal{A}, M_\mathcal{A})$, where $\mathcal{A}$ is the collection of all equivalences in $\mathcal{A}$. Since the category of marked simplicial sets is Quillen equivalent to the category of simplicial sets (with the Joyal model structure), it will suffice to prove the following:

(A) The diagram

$$
\begin{array}{ccc}
\mathcal{X}^\natural \times (\partial \Delta^{\{1,2\}})^\flat & \longrightarrow & \mathcal{C}^\natural \times \mathcal{D}^\natural \\
\downarrow & & \downarrow \\
\mathcal{X}^\natural \times (\Delta^{\{1,2\}})^\flat & \longrightarrow & \mathcal{M}^\natural
\end{array}
$$

is a homotopy pushout square of marked simplicial sets.

To prove this, we let $\mathcal{Y}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{M}) \times \Delta^3$ spanned by those pairs $(f : A \to A', i)$ satisfying one of the following conditions:

- We have $i = 0$ and f is an equivalence in $\mathcal{C}$.

- We have $i = 1$ or $i = 2$ and f belongs to $\mathcal{X}$.

- We have $i = 3$ and f is an equivalence in $\mathcal{D}$.

For each simplicial subset $K \subseteq \Delta^3$, we let $\mathcal{Y}_K = \mathcal{Y} \times_{\Delta^3} K$, and let $\overline{\mathcal{Y}}_K$ denote the marked simplicial set $(\mathcal{Y}_K, M_K)$, where M_K is the collection of all edges $\alpha : (f, i) \to (f', i')$ in $\mathcal{Y}_K$ satisfying one of the following three conditions:

- The map α is an equivalence in $\mathcal{Y}$.

- We have $i = 0$, $i' = 1$, and α corresponds to a commutative diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\ g\ } & C'' \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\
C' & \longrightarrow & D
\end{array}
$$

 for which g is an equivalence.

- We have $i = 2$, $i' = 3$, and α corresponds to a commutative diagram

$$
\begin{array}{ccc}
C & \longrightarrow & D \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\
D'' & \xrightarrow{\ g\ } & D'
\end{array}
$$

 for which g is an equivalence.

We observe that there is a retraction r of $\mathcal{Y}$ onto the full subcategory $\mathcal{Y}_{\Delta^{\{0,2,3\}}}$, which carries an object $f : C \to D$ of $\mathcal{Y}_{\{1\}}$ to the object $\mathrm{id}_C \in \mathcal{Y}_{\{0\}}$. This retraction is equipped with a natural transformation $r \to \mathrm{id}_\mathcal{Y}$, which determines a map of marked simplicial sets $\overline{\mathcal{Y}}_{\Delta^3} \times (\Delta^1)^\sharp \to \overline{\mathcal{Y}}_{\Delta^3}$. Using this deformation retraction, we deduce the following:

($*$) Let S be a subset of $\{0, 2, 3\}$ containing $\{0\}$. Then the inclusion $\overline{\mathcal{Y}}_{\Delta^S} \subseteq \overline{\mathcal{Y}}_{\Delta^{S \cup \{1\}}}$ is a weak equivalence of marked simplicial sets.

A similar argument proves:

($*'$) Let S be a subset of $\{0, 1, 3\}$ containing $\{3\}$. Then the inclusion $\overline{\mathcal{Y}}_{\Delta^S} \subseteq \overline{\mathcal{Y}}_{\Delta^{S \cup \{2\}}}$ is a weak equivalence of marked simplicial sets.

Let $\phi : \Delta^3 \to \Delta^1$ be the map characterized by $\phi^{-1}\{0\} = \Delta^{\{0,1\}} \subseteq \Delta^3$, and consider the map

$$\theta : \mathcal{Y} \subseteq \mathrm{Fun}(\Delta^1, \mathcal{M}) \times \Delta^3 \xrightarrow{\mathrm{id} \times \phi} \mathrm{Fun}(\Delta^1, \mathcal{M}) \times \Delta^1 \to \mathcal{M}.$$

Consider the diagram

Using ($*$) and the observation that the diagonal inclusion $\mathcal{C} \to \mathcal{Y}_{\{0\}}$ is an equivalence of ∞-categories, we deduce that θ_0 is a weak equivalence of marked simplicial sets. A similar argument gives an equivalence of marked simplicial sets $\overline{\mathcal{Y}}_{\Delta^{\{2,3\}}} \to \mathcal{D}^\natural$. Using this observation, we can reformulate (A) as follows:

(B) The diagram

is a homotopy pushout square of marked simplicial sets.

We have a commutative diagram of marked simplicial sets

where $K = \Delta^{\{0,1\}} \coprod_{\{1\}} \Delta^{\{1,2\}} \coprod_{\{2\}} \Delta^{\{2,3\}} \subseteq \Delta^3$. We wish to prove that β_3 is a weak equivalence of marked simplicial sets. Since β_0 is an isomorphism of marked simplicial sets, it suffices to show that β_1 and β_2 are weak equivalences of marked simplicial sets.

To prove that β_1 is a weak equivalence, we factor β_1 as a composition

$$\overline{\mathcal{Y}}_{\Delta^{\{0,3\}}} \xrightarrow{\beta'_1} \overline{\mathcal{Y}}_{\Delta^{\{0,1,3\}}} \xrightarrow{\beta''_1} \overline{\mathcal{Y}}_{\Delta^3}.$$

Assertion ($*$) implies that β'_1 is a weak equivalence, and assertion ($*'$) implies that β''_1 is a weak equivalence.

To prove that β_2 is a weak equivalence, we factor β_2 as a composition

$$\overline{\mathcal{Y}}_K \xrightarrow{\beta'_2} \overline{\mathcal{Y}}_{\Delta^{\{0,1,2\}} \coprod_{\{2\}} \Delta^{\{2,3\}}} \xrightarrow{\beta''_2} \overline{\mathcal{Y}}_{\Delta^{\{0,1,2\}} \coprod_{\Delta^{\{1,2\}}} \Delta^{\{1,2,3\}}} \xrightarrow{\beta'''_2} \overline{\mathcal{Y}}_{\Delta^3}.$$

The map β_2' is a pushout of the inclusion

$$\mathcal{Y}^{\natural}_{\Delta^{\{0,1\}}} \coprod_{\mathcal{Y}^{\natural}_{\{1\}}} \mathcal{Y}^{\natural}_{\Delta^{\{1,2\}}} \to \mathcal{Y}^{\natural}_{\Delta^{\{0,1,2\}}} \ .$$

Consequently, to prove β_2' it suffices to show that the map $\mathcal{Y}_{\Delta^{\{0,1,2\}}} \to \Delta^{\{0,1,2\}}$ is a flat inner fibration, which follows from Example B.3.5. The same argument shows that β_2'' is a weak equivalence. The map β_2''' is a pushout of the inclusion

$$\mathcal{Y}^{\natural}_{\Delta^{\{0,1,2\}}} \amalg_{\Delta^{\{1,2\}}} \Delta^{\{1,2,3\}} \to \mathcal{Y}^{\natural} \ .$$

To complete the proof, it will suffice to show that this map is a weak equivalence of marked simplicial sets, which is equivalent to the requirement that the composite map

$$\mathcal{Y} \to \Delta^3 \xrightarrow{\phi'} \Delta^2$$

is a flat inner fibration (here ϕ' is the map characterized by the property that $\phi'^{-1}\{1\} = \Delta^{\{1,2\}} \subseteq \Delta^3$).

In view of Proposition B.3.2, we must show that for every object $\overline{C} : C \to C'$ of $\mathrm{Fun}(\Delta^1, \mathcal{C}) \simeq \mathcal{Y}_{\{0\}}$ and every object $\overline{D} : D \to D'$ of $\mathrm{Fun}(\Delta^1, \mathcal{D}) \simeq \mathcal{Y}_{\{3\}}$, the simplicial set $\mathcal{Y}_{\overline{C}//\overline{D}} \times_{\Delta^3} \Delta^{\{1,2\}}$ is weakly contractible. This simplicial set can be identified with the product $\Delta^1 \times \mathcal{E}$, where $\mathcal{E} = \mathrm{Fun}(\Delta^1, \mathcal{M})_{\overline{C}//\overline{D}} \times_{\mathrm{Fun}(\Delta^1, \mathcal{M})} \mathcal{X}$. To complete the proof, we will show that the ∞-category $\mathcal{E}$ is weakly contractible.

We observe that an object of $\mathcal{E}$ can be identified with a commutative diagram

in $\mathcal{M}$, where $C'' \in \mathcal{C}$ and $D'' \in \mathcal{D}$. Let $\mathcal{E}_0$ denote the full subcategory of $\mathcal{E}$ spanned by those objects for which γ is an equivalence. The inclusion $\mathcal{E}_0 \subseteq \mathcal{E}$ admits a right adjoint, and is therefore a weak homotopy equivalence. It will therefore suffice to show that $\mathcal{E}_0$ is weakly contractible. Let $\mathcal{E}_1$ denote the full subcategory of $\mathcal{E}_0$ spanned by those diagrams for which γ' is an equivalence. The inclusion $\mathcal{E}_1 \subseteq \mathcal{E}_0$ admits a left adjoint, and is therefore a weak homotopy equivalence. It therefore suffices to show that $\mathcal{E}_1$ is weakly contractible. We complete the proof by observing that $\mathcal{E}_1$ is a contractible Kan complex. $\qquad\square$

Lemma B.3.18. *Let $p : \mathcal{M} \to \Delta^3$ be a flat inner fibration. Let $f : C \to D$ be a morphism in $\mathcal{M}$, where $C \in \mathcal{M}_0$ and $D \in \mathcal{M}_3$. Then the ∞-category $\mathcal{N} = \mathcal{M}_{C//D} \times_{\Delta^3} \Delta^{\{1,2\}}$ is weakly contractible.*

Proof. Let $\mathcal{X}$ denote the ∞-category $\mathrm{Fun}_{\Delta^{\{1,2\}}}(\Delta^{\{1,2\}}, \mathcal{N})$. According to Proposition B.3.17, we have a categorical equivalence

$$\mathcal{N}_1 \coprod_{\mathcal{X} \times \{1\}} (\mathcal{X} \times \Delta^1) \coprod_{\mathcal{X} \times \{2\}} \mathcal{N}_2 \to \mathcal{N}.$$

Since $\mathcal{N}_1$ and $\mathcal{N}_2$ are weakly contractible (by virtue of the assumption that p is flat over $\Delta^{\{0,1,3\}}$ and $\Delta^{\{0,2,3\}}$), it will suffice to show that $\mathcal{X}$ is weakly contractible. Let $q : \mathcal{X} \to \mathcal{N}_2$ be the map given by evaluation at $\{2\}$. Using Corollary T.2.4.7.12, we deduce that q is a coCartesian fibration. Since $\mathcal{N}_2$ is weakly contractible, it will suffice to show that the fiber $q^{-1}E$ is weakly contractible, for each $E \in \mathcal{N}_2$ (Lemma T.4.1.3.2). This fiber can be identified with the fiber product $\{1\} \times_{\Delta^3} (\mathcal{M}_{C//D})^{/E}$, which is categorically equivalent to $\{1\} \times_{\Delta^3} (\mathcal{M}_{C//D})_{/E}$ (Proposition T.4.2.1.5). Let E_0 denote the image of E in $\mathcal{M}$. We have a trivial Kan fibration $(\mathcal{M}_{C//D})_{/E} \to \mathcal{M}_{C//E_0}$. It therefore suffices to show that $\{1\} \times_{\Delta^3} \mathcal{M}_{C//E_0}$ is weakly contractible, which follows from the assumption that p is flat over the 2-simplex $\Delta^{\{0,1,2\}}$. $\qquad\square$

Lemma B.3.19. *Let $p : \mathcal{M} \to \Delta^n$ be a flat inner fibration, and let $q : \Delta^n \to \Delta^m$ be a map of simplices which is surjective on vertices. Then the composite map $q \circ p$ is a flat inner fibration.*

Proof. If $n - m > 1$, then we can factor q as a composition

$$\Delta^n \xrightarrow{q'} \Delta^{n-1} \xrightarrow{q''} \Delta^m$$

where q' and q'' are surjective on vertices. Using descending induction on $n - m$, we can assume that $n - m \leq 1$. If $n = m$ there is nothing to prove, so we may suppose that $n = m + 1$. To prove that $q \circ p$ is flat, it suffices to show that it is flat over every nondegenerate 2-simplex of Δ^m. Replacing $\mathcal{M}$ by the pullback $\mathcal{M} \times_{\Delta^m} \Delta^2$, we can reduce to the case where $m = 2$ and $n = 3$.

Fix objects $C \in (q \circ p)^{-1}\{0\}$ and $D \in (q \circ p)^{-1}\{2\}$ and a morphism $f : C \to D$ in $\mathcal{M}$; we wish to prove that the ∞-category $\mathcal{M}_{C//D} \times_{\Delta^2} \{1\}$ is weakly contractible. Let $i \in [2]$ be the unique integer such that $q^{-1}\{i\}$ is a 1-simplex of Δ^3. If $i = 0$, then the weak contractibility follows from the assumption that p is flat over $\Delta^{\{0,2,3\}} \subseteq \Delta^3$. If $i = 2$, then the weak contractibility follows from our assumption that p is flat over $\Delta^{\{0,1,3\}} \subseteq \Delta^3$. If $i = 1$, then the desired result follows from Lemma B.3.18. $\square$

Lemma B.3.20. *Let* $p : \mathcal{M} \to \Delta^n$ *be a flat inner fibration. Let* $f : C \to D$ *be a morphism in* $\mathcal{M}$, *where* $C \in \mathcal{M}_0$ *and* $D \in \mathcal{M}_n$. *Then the* ∞-*category* $\mathcal{N} = \mathcal{M}_{C//D} \times_{\Delta^n} \Delta^{\{1,\ldots,n-1\}}$ *is weakly contractible.*

Proof. Apply Lemma B.3.19 to the map $q : \Delta^n \to \Delta^2$ characterized by the requirement that $q^{-1}\{1\} = \{1, \ldots, n-1\}$. $\square$

Lemma B.3.21. *Let* $p : \mathcal{M} \to \Delta^n \times \Delta^m$ *be a flat inner fibration. Then the induced map* $p' : \mathcal{M} \to \Delta^m$ *is a flat inner fibration.*

Proof. It suffices to show that p' is flat over every nondegenerate 2-simplex of Δ^m. Replacing $\mathcal{M}$ by $\mathcal{M} \times_{\Delta^m} \Delta^2$, we can reduce to the case $m = 2$. Fix a morphism $f : C \to D$ in $\mathcal{M}$, where $C \in p'^{-1}\{0\}$ and $D \in p'^{-1}\{2\}$; we wish to show that the ∞-category $\mathcal{M}_{C//D} \times_{\Delta^2} \{1\}$ is weakly contractible. Let i and j denote the images of C and D in Δ^n, and let $\phi : \Delta^{2+j-i} \to \Delta^n \times \Delta^2$ be the map given on vertices by the formula

$$\phi(k) = \begin{cases} (i, 0) & \text{if } k = 0 \\ (i + k - 1, 1) & \text{if } 0 < k < 2 + j - i \\ (j, 2) & \text{if } k = 2 + j - i. \end{cases}$$

The desired result now follows after applying Lemma B.3.20 to the flat inner fibration $\mathcal{M} \times_{\Delta^n \times \Delta^2} \Delta^{2+j-i} \to \Delta^{2+j-i}$. $\square$

Proof of Proposition B.3.14. Fix an inner fibration of simplicial sets $p : X \to S$. By definition, the map p is flat if it induces a categorical equivalence $X \times_S \Lambda_1^2 \to X \times_S \Delta^2$, for every 2-simplex of S. This proves the implication (2) $\Rightarrow$ (1). For the converse, let us say that a monomorphism of simplicial sets $A \to B$ is *good* if it satisfies the following condition:

($*$) For every map of simplicial sets $B \to S$, the induced map $X \times_S A \to X \times_S B$ is a categorical equivalence.

Since the collection of trivial cofibrations with respect to the Joyal model structure is weakly saturated (in the sense of Definition T.A.1.2.2), we deduce that the collection of good morphisms in $\mathcal{S}et_\Delta$ is also weakly saturated. We wish to prove that every inner anodyne morphism is good. In view of Proposition T.2.3.2.1, it will suffice to show that for every monomorphism of simplicial sets $A \to B$ having only finitely many nondegenerate simplices, the induced map

$$(A \times \Delta^2) \coprod_{A \times \Lambda_1^2} (B \times \Lambda_1^2) \to B \times \Delta^2$$

is good. In other words, we must show that for every map $B \times \Delta^2 \to S$, the induced diagram

$$\begin{array}{ccc} X \times_S (A \times \Lambda_1^2) & \longrightarrow & X \times_S (B \times \Lambda_1^2) \\ \downarrow & & \downarrow \\ X \times_S (A \times \Delta^2) & \longrightarrow & X \times_S (B \times \Delta^2) \end{array}$$

is a homotopy pushout square (with respect to the Joyal model structure). To prove this, it suffices to show that the vertical maps are categorical equivalences. In other words, we are reduced to proving that the following assertion holds, for every simplicial set K having only finitely many nondegenerate simplices:

($*'$) For every map $K \times \Delta^2 \to S$, the inclusion $X \times_S (K \times \Lambda_1^2) \to X \times_S (K \times \Delta^2)$ is a categorical equivalence.

We now prove ($*'$) by induction on the dimension n of K and the number of nondegenerate n-simplices of K. If K is empty, there is nothing to prove. Otherwise, we have a pushout diagram

$$\begin{CD} \partial \Delta^n @>>> \Delta^n \\ @VVV @VVV \\ K' @>>> K. \end{CD}$$

Using the left properness of the Joyal model structure, we see that K will satisfy ($*'$) provided that K', $\partial \Delta^n$, and Δ^n satisfy ($*'$). In the first two cases, this follows from the inductive hypothesis. We are therefore reduced to the case $K = \Delta^n$. Fix a map $\Delta^n \times \Delta^2 \to S$, and consider the flat inner fibration $q : X \times_S (\Delta^n \times \Delta^2) \to \Delta^n \times \Delta^2$. To prove that ($*'$) is satisfied, we must show that the composition

$$X \times_S (\Delta^n \times \Delta^2) \xrightarrow{q} \Delta^n \times \Delta^2 \to \Delta^2$$

is flat, which follows from Lemma B.3.21.   □

B.4 Functoriality

Proposition B.2.9 can be interpreted roughly as saying that the model structure of Theorem B.0.19 defines a covariant functor of the underlying categorical pattern $\mathfrak{P}$. The remainder of this section is devoted to studying the behavior of this model structure as a *contravariant* functor of $\mathfrak{P}$. Our main result can be stated as follows:

Proposition B.4.1. *Suppose we are given categorical patterns* $\mathfrak{P} = (M_S, T, \{p_\alpha : K_\alpha^\triangleleft \to S\}_{\alpha \in A})$ *and* $\mathfrak{P}' = (M_{S'}, T', \{p'_\alpha : K_\alpha'^{\triangleleft} \to S'\}_{\alpha \in A'})$ *on ∞-categories S and S'. Let $\pi : S' \to S$ be a map satisfying the following conditions:*

(i) *For every vertex $s' \in S'$ and every morphism $f : s \to \pi(s')$ in S which belongs to M_S, there exists a locally π-Cartesian morphism $\overline{f} : \overline{s} \to s'$ in S' such that $\pi(\overline{f}) = f$.*

(ii) *The map π is a flat categorical fibration.*

(iii) *The map π carries $M_{S'}$ into M_S.*

(iv) *The collections of morphisms M_S and $M_{S'}$ contain all equivalences and are stable under composition (and are therefore stable under equivalence).*

(v) *Suppose given a commutative diagram*

$$\begin{array}{ccc} & s' & \\ {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} \\ s & \xrightarrow{\ h\ } & s'' \end{array}$$

in S', where g is locally π-Cartesian, $\pi(g) \in M_S$, and $\pi(f)$ is an equivalence. Then $f \in M_{S'}$ if and only if $h \in M_{S'}$. In particular (taking $f = \mathrm{id}_s$), we deduce that every locally π-Cartesian morphism g such that $\pi(g) \in M_S$ belongs to $M_{S'}$.

(*vi*) *The set of 2-simplices T' contains $\pi^{-1}(T)$, and T contains all 2-simplices $\Delta^2 \to S$ whose restriction to $\Delta^{\{0,1\}}$ is an equivalence in S.*

(*vii*) *Each of the simplicial sets K_α is an ∞-category, and each of the induced maps $\pi_\alpha : K_\alpha^\triangleleft \times_S S' \to K_\alpha^\triangleleft$ is a coCartesian fibration.*

(*viii*) *Suppose we are given $\alpha \in A$ and a commutative diagram*

$$
\begin{array}{ccc}
 & s' & \\
{\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} \\
s & \xrightarrow{\ h\ } & s''
\end{array}
$$
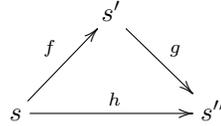
in $K_\alpha^\triangleleft \times_S S'$, where f is π_α-coCartesian and $\pi_\alpha(g)$ is an equivalence. Then the image of g in S' belongs to $M_{S'}$ if and only if the image of h in S' belongs to $M_{S'}$. In particular, the image in S' of any π_α-coCartesian morphism of $K_\alpha^\triangleleft$ belongs to $M_{S'}$.

(*ix*) *Let $\alpha \in A$, and suppose we are given a map $\overline{p}_\alpha : K_\alpha^\triangleleft \to S'$ lifting p_α, such that the corresponding section of π_α is π_α-coCartesian. Then $\overline{p}_\alpha \simeq p'_\beta$ for some $\beta \in A'$.*

Let $\pi^* : (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}} \to (\mathcal{S}et_\Delta^+)_{/\mathfrak{P}'}$ denote the functor $\overline{X} \mapsto \overline{X} \times_{(S,M_S)} (S', M_{S'})$. Then π^* is a left Quillen functor (with respect to the model structures described in Theorem B.0.19).

Using Propositions B.4.1 and B.2.9 in combination, we can obtain even more functoriality:

Theorem B.4.2. *Suppose we are given categorical patterns $\mathfrak{P} = (M_\mathcal{C}, T, \{p_\alpha : K_\alpha^\triangleleft \to \mathcal{C}\}_{\alpha \in A})$ and $\mathfrak{P}' = (M_{\mathcal{C}'}, T', \{p'_\alpha : K'^\triangleleft_\alpha \to \mathcal{C}'\}_{\alpha \in A'})$ on ∞-categories $\mathcal{C}$ and $\mathcal{C}'$. Suppose we are given a diagram of marked simplicial sets*

$$(\mathcal{C}, M_\mathcal{C}) \xleftarrow{\pi} (\mathcal{K}, M) \xrightarrow{\pi'} (\mathcal{C}', M_{\mathcal{C}'}).$$

Then the construction $\overline{X} \mapsto \overline{X} \times_{(\mathcal{C}, M_\mathcal{C})} (\mathcal{K}, M)$ determines a left Quillen functor from $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}}$ to $(\mathcal{S}et_\Delta^+)_{/\mathfrak{P}'}$ provided that the following conditions are satisfied:

(1) *The map $\pi : \mathcal{K} \to \mathcal{C}$ is a flat categorical fibration.*

(2) *The collections of morphisms M_S and M contain all equivalences in $\mathcal{C}$ and $\mathcal{K}$, respectively, and are closed under composition.*

(3) *For every 2-simplex σ of $\mathcal{K}$ such that $\pi(\sigma) \in T$, we have $\pi'(\sigma) \in T'$. Moreover, T contains all 2-simplices $\Delta^2 \to \mathcal{C}$ whose restriction to $\Delta^{\{0,1\}}$ is an equivalence in $\mathcal{C}$.*

(4) *For every edge $\Delta^1 \to \mathcal{C}$ belonging to $M_\mathcal{C}$, the induced map $\mathcal{K} \times_\mathcal{C} \Delta^1 \to \Delta^1$ is a Cartesian fibration.*

(5) *Each of the simplicial sets K_α is an ∞-category, and each of the induced maps $\pi_\alpha : K_\alpha^\triangleleft \times_\mathcal{C} \mathcal{K} \to K_\alpha^\triangleleft$ is a coCartesian fibration.*

(6) *For $\alpha \in A$ and every coCartesian section s of π_α, the composite map*

$$K_\alpha^\triangleleft \to K_\alpha^\triangleleft \times_\mathcal{C} \mathcal{K} \to \mathcal{K} \xrightarrow{\pi'} \mathcal{C}'$$

can be identified with p'_β, for some $\beta \in A'$.

(7) *Suppose we are given a commutative diagram*

$$
\begin{array}{ccc}
 & Y & \\
{\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} \\
X & \xrightarrow{\ h\ } & Z
\end{array}
$$
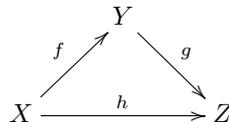
in $\mathcal{K}$, where g is locally π-Cartesian, $\pi(g) \in M_{\mathcal{C}}$, and $\pi(f)$ is an equivalence. Then $f \in M$ if and only if $h \in M$. (In particular, a locally π-Cartesian morphism g of $\mathcal{K}$ belongs to M if and only if $\pi(g) \in M_{\mathcal{C}}$.

(8) *Suppose we are given $\alpha \in A$ and a commutative diagram*

$$
\begin{array}{ccc}
 & Y & \\
 \;{}^{f}\nearrow & & \searrow {}^{g}\; \\
 X & \xrightarrow{\;\;\;h\;\;\;} & Z
\end{array}
$$

in $K_{\alpha}^{\triangleleft} \times_{\mathcal{C}} \mathcal{K}$, where f is π_{α}-coCartesian and $\pi_{\alpha}(g)$ is an equivalence. Then the image of g in $\mathcal{K}$ belongs to M if and only if the image of h in $\mathcal{K}$ belongs to M.

Remark B.4.3. In the situation of Theorem B.4.2, suppose that $M_{\mathcal{C}}$ is the collection of all equivalences in $\mathcal{C}$, that A is empty, and that T and T' are the collections of all simplices in $\mathcal{C}$ and $\mathcal{C}'$, respectively. Then conditions (3), (5), (6), and (8) are automatic; condition (4) follows from (1), and condition (7) follows from (2) (if g is locally π-Cartesian and $\pi(g) \in M_{\mathcal{C}}$, then g is an equivalence so that f and h are equivalent). It therefore suffices to verify conditions (1) and (2).

Proof of Theorem B.4.2. Consider the categorical pattern $\mathfrak{P}'' = (M, \pi^{-1}(T), \{p''_{\alpha,s} : K_{\alpha}^{\triangleleft} \to \mathcal{K}\}_{(\alpha,s) \in A''})$ on $\mathcal{K}$, where A'' consists of all pairs (α, s) such $\alpha \in A$ and s is a coCartesian section of π_{α}, and $p''_{\alpha,s}$ is the composition

$$
K_{\alpha}^{\triangleleft} \xrightarrow{s} \mathcal{K} \times_{\mathcal{C}} K_{\alpha}^{\triangleleft} \to \mathcal{K}.
$$

The functor in question admits a factorization

$$
(\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}} \xrightarrow{\pi^{*}} (\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}''} \xrightarrow{\pi'_{!}} (\mathcal{S}et_{\Delta}^{+})_{/\mathfrak{P}},
$$

where π^{*} and $\pi'_{!}$ are left Quillen functors by virtue of Propositions B.4.1 and B.2.9. □

The proof of Proposition B.4.1 will require a long digression.

Notation B.4.4. Suppose we are given maps of simplicial sets $X \xrightarrow{\phi} Y \xrightarrow{\pi} Z$. We let $\pi_{*}(X)$ denote a simplicial set equipped with a map $\pi_{*}X \to Z$ with the following universal property: for every map of simplicial sets $K \to Z$, we have a canonical bijection

$$
\operatorname{Hom}_{Z}(K, \pi_{*}(X)) \simeq \operatorname{Hom}_{Y}(K \times_{Z} Y, X).
$$

In the situation of Notation B.4.4, suppose that π is a Cartesian fibration and the map ϕ is a coCartesian fibration. Corollary T.3.2.2.12 implies that the map $\pi_{*}X \to Z$ is a coCartesian fibration. We will need some refinements of this result.

Proposition B.4.5. *Let $\pi : Y \to Z$ be a flat categorical fibration of simplicial sets. Then the functor $\pi_{*} : (\mathcal{S}et_{\Delta})_{/Y} \to (\mathcal{S}et_{\Delta})_{/Z}$ is a right Quillen functor (with respect to the Joyal model structures). In particular, if $X \to Y$ is a categorical fibration, then the induced map $\pi_{*}X \to Z$ is a categorical fibration.*

Proof. The functor π_{*} admits a left adjoint π^{*}, given by the formula $\pi^{*}A = A \times_{Z} Y$. To prove that π_{*} is a right Quillen functor, it suffices to show that π^{*} preserves cofibrations and weak equivalences. The case of cofibrations is clear, and the case of weak equivalences follows from Corollary B.3.15. □

Example B.4.6. Suppose we are given a diagram of simplicial sets $X \xrightarrow{\phi} Y \xrightarrow{\pi} Z$. We observe that there is a canonical map $\theta : X \to \pi_{*}X$. If the map π is a trivial Kan fibration, then θ is a categorical equivalence. To prove this, we first choose a section $s : Z \to Y$ of π. Composition with s yields a map $r : \pi_{*}X \to X$ such that $r \circ \theta = \operatorname{id}_{X}$. Moreover, since $s \circ \pi$ is homotopic (over Z) to the map id_{Y}. It follows that there

exists a contractible Kan complex K containing a pair of distinct points x and y and a map $h : K \times Y \to Y$ compatible with the projection map π such that $h|(\{x\} \times Y) = \mathrm{id}_Y$ and $h|(\{y\} \times Y) = s \circ \pi$. The map h induces a map $h' : K \times \pi_* X \to \pi_* X$ such that $h|(\{x\} \times \pi_* X) = \mathrm{id}_{\pi_* X}$ and $h|(\{y\} \times \pi_* X) = \theta \circ r$. It follows that r is a right homotopy inverse to θ (as well as being a strict left inverse) with respect to the Joyal model structure, so that θ is a categorical equivalence as desired.

Remark B.4.7. Suppose we are given a diagram of simplicial sets $X \xrightarrow{\phi} Y \xrightarrow{\pi} Z$, where ϕ is a categorical fibration and π is a flat categorical fibration. Let $\psi : Y' \to Y$ be a trivial Kan fibration, let $\pi' = \pi \circ \psi$, and let $X' = X \times_Y Y'$. Then the canonical map $f : \pi_* X \to \pi'_* X'$ is a categorical equivalence. To prove this, we observe that $\pi'_* X' \simeq \pi_* \psi_* X'$, and f is induced by applying π_* to a map $g : X \to \psi_* X'$. Example B.4.6 shows that g is a categorical equivalence. Since π_* is a right Quillen functor (Proposition B.4.5), it preserves categorical equivalences between fibrant objects of $(\mathrm{Set}_\Delta)_{/Y}$, so f is a categorical equivalence.

Lemma B.4.8. *Let $q : \mathcal{C} \to \Delta^n$ and $p : \mathcal{D} \to \mathcal{E}$ be categorical fibrations of ∞-categories, where $n \geq 2$. Let $\mathcal{C}^0$ be a full subcategory of $\mathcal{C}$ with the following properties:*

(i) The subcategory $\mathcal{C}^0 \times_{\Delta^n} \Delta^{\{n-1,n\}}$ is a cosieve on $\mathcal{C}$: that is, for every morphism $f : x \to y$ in $\mathcal{C} \times_{\Delta^n} \Delta^{\{n-1,n\}}$, if $x \in \mathcal{C}^0$, then $y \in \mathcal{C}^0$.

(ii) For every object $y \in \mathcal{C}^0 \times_{\Delta^n} \{n-1\}$ and each $i < n - 1$, there exists an object $x \in \mathcal{C}^0 \times_{\Delta^n} \{i\}$ and a q-Cartesian morphism $x \to y$ in $\mathcal{C}$.

Suppose we are given a lifting problem

$$
\begin{array}{ccc}
(\mathcal{C} \times_{\Delta^n} \Lambda^n_n) \coprod_{\mathcal{C}^0 \times_{\Delta^n} \Lambda^n_n} \mathcal{C}^0 & \xrightarrow{\;f_0\;} & \mathcal{D} \\[4pt]
\downarrow & \overset{f}{\nearrow} & \downarrow{\scriptstyle p} \\[4pt]
\mathcal{C} & \xrightarrow{\;g\;} & \mathcal{E}.
\end{array}
$$

Let $X = (\mathcal{C} \times_{\Delta^n} \{n\}) \coprod_{\mathcal{C}^0 \times_{\Delta^n} \{n\}} (\mathcal{C}^0 \times_{\Delta^n} \Delta^{\{n-1,n\}})$; condition (i) guarantees that X can be identified with a full subcategory of $\mathcal{C}$. Assume further that

(iii) The functor $f_0|(\mathcal{C} \times_{\Delta^n} \Delta^{\{n-1,n\}})$ is a p-right Kan extension of $f_0|X$.

Then there exists a dotted arrow f rendering the diagram commutative.

Proof. Let $\mathcal{C}^1$ denote the simplicial subset of $\mathcal{C}$ consisting of all those simplices σ satisfying one of the following conditions:

- The image of σ in Δ^n does not contain $\Delta^{\{0,1,\ldots,n-1\}}$.

- The intersection of σ with $\mathcal{C} \times_{\Delta^n} \{n-1\}$ is contained in $\mathcal{C}^0$.

We first extend f_0 to a map $f_1 : \mathcal{C}^1 \to \mathcal{D}$. Let A be the collection of simplices $\sigma : \Delta^m \to X$. For each $\sigma \in A$, let $d(\sigma)$ denote the dimension of the simplex σ. Choose a well-ordering of A such that if $d(\sigma) < d(\tau)$, then $\sigma < \tau$. For every nondegenerate simplex $\sigma : \Delta^m \to \mathcal{C}^1$, we let $r(\sigma)$ denote the induced map $\Delta^m \times_{\Delta^n} \Delta^{\{n-1,n\}} \to X$. Let α be the order type of A, so that we have an order-preserving bijection $\beta \mapsto \sigma_\beta$ where β ranges over the set of ordinals $\{\beta : \beta < \alpha\}$. For each $\beta \leq \alpha$, we let $\mathcal{C}^1_\beta$ denote the simplicial subset of $\mathcal{C}^1$ given by the union of $\mathcal{C}^0$, $\mathcal{C} \times_{\Delta^n} \Lambda^n_n$, and those simplices σ such that $r(\sigma) = \sigma_\gamma$ for some $\gamma < \beta$. Then f_0 can be identified with a map $F_0 : \mathcal{C}^1_0 \to \mathcal{D}$. We will show that F_0 can be extended to a compatible family of maps

$$F_\beta : \mathcal{C}^1_\beta \to \mathcal{D}$$

such that $p \circ F_\beta = g| \mathcal{C}^1_\beta$. Taking $\beta = \alpha$, we will obtain the desired extension $f_1 : \mathcal{C}^1 \to \mathcal{D}$ of f_0.

The construction of F_β proceeds by induction on β. If β is a nonzero limit ordinal, we set $F_\beta = \bigcup_{\gamma < \beta} F_\gamma$. To handle successor stages, let us assume that F_β has already been defined for some $\beta < \alpha$, and let $\sigma = \sigma_\beta : \Delta^m \to X$ be the corresponding simplex. There are two cases to consider. Assume first that σ is nondegenerate. Let $\mathcal{A} = \mathcal{C}_{/\sigma} \times_{\Delta^n} \Delta^{\{0,\dots,n-2\}}$. For every simplicial subset $K \subseteq \Delta^{\{0,\dots,n-2\}}$, we let $\mathcal{A}_K$ denote the simplicial subset of $\mathcal{A}$ given by the union of $\mathcal{A} \times_{\mathcal{C}} \mathcal{C}^0$ and $\mathcal{A} \times_{\Delta^{\{0,\dots,n-2\}}} K$, Let $\mathcal{A}_\partial$ denote $\mathcal{A}_{\partial \Delta^{\{0,\dots,n-2\}}}$. We have a diagram

$$
\begin{array}{ccc}
(\mathcal{A}_\partial \star \Delta^m) \coprod_{\mathcal{A}_\partial \star \partial \Delta^m} (\mathcal{A} \star \partial \Delta^m) & \longrightarrow & \mathcal{C}_\beta^1 \\
\downarrow & & \downarrow \\
\mathcal{A} \star \Delta^m & \longrightarrow & \mathcal{C}_{\beta+1}^1 \,.
\end{array}
$$

Since σ is nondegenerate, this diagram is a pushout square. The existence of $F_{\beta+1}$ is therefore equivalent to the solubility of the lifting problem

$$
\begin{array}{ccc}
(\mathcal{A}_\partial \star \Delta^m) \coprod_{\mathcal{A}_\partial \star \partial \Delta^m} (\mathcal{A} \star \partial \Delta^m) & \longrightarrow & \mathcal{D} \\
{\scriptstyle i} \downarrow & \nearrow & \downarrow {\scriptstyle p} \\
\mathcal{A} \star \Delta^m & \longrightarrow & \mathcal{E} \,.
\end{array}
$$

Since p is a categorical fibration, it will suffice to show that the map i is inner anodyne. By virtue of Lemma T.2.1.2.3, it will suffice to show that the inclusion $i' : \mathcal{A}_\partial \to \mathcal{A}$ is right anodyne. In fact, we prove the following more general claim:

($*$) For every simplex $\tau \subseteq \Delta^{\{0,\dots,n-2\}}$ and every simplicial subset $K \subseteq \tau$, the inclusion $\mathcal{A}_K \subseteq \mathcal{A}_\tau$ is right anodyne.

The proof proceeds by induction on the dimension of τ, and on the number of nondegenerate simplices of K. If $K = \tau$ there is nothing to prove. If $K \neq \tau$ is nonempty, then we can write K as a pushout $K' \coprod_{\partial \tau'} \tau'$, where τ' is a simplex of small dimension than τ. The inductive hypothesis shows that the inclusion $\mathcal{A}_{K'} \subseteq \mathcal{A}_\tau$ is right anodyne. By virtue of Proposition T.4.1.1.3, it will suffice to show that the inclusion $\mathcal{A}_{K'} \subseteq \mathcal{A}_K$ is right anodyne. For this, we consider the pushout diagram

$$
\begin{array}{ccc}
\mathcal{A}_{\partial \tau'} & \longrightarrow & \mathcal{A}_{\tau'} \\
\downarrow & & \downarrow \\
\mathcal{A}_{K'} & \longrightarrow & \mathcal{A}_K \,.
\end{array}
$$

Since the upper horizontal map is right anodyne by the inductive hypothesis, the lower horizontal map is right anodyne as well.

It remains to consider the case $K = \emptyset$. In this case, the inclusion $\mathcal{A}_K \subseteq \mathcal{A}_\tau$ is a pushout of the inclusion

$$
i'' : \mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/\sigma} \times_{\Delta^n} \tau \subseteq \mathcal{C}_{/\sigma} \times_{\Delta^n} \tau.
$$

It will therefore suffice to show that i'' is right anodyne, which is equivalent (Proposition T.4.1.1.3) to the assertion that i'' is left cofinal. Using Proposition T.4.1.1.3 again, we see that it suffices to produce an object $x \in \mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/\sigma} \times_{\Delta^n} \tau$ such that the corresponding maps

$$
\mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/\sigma} \times_{\Delta^n} \tau \hookleftarrow \Delta^0 \hookrightarrow \mathcal{C}_{/\sigma} \times_{\Delta^n} \tau
$$

are both left cofinal. In other words, it suffices to guarantee that the object x is final in $\mathcal{C}_{/\sigma} \times_{\Delta^n} \tau$. The existence of such an object follows from assumption (ii) (applied to the object $y \in \mathcal{C} \times_{\Delta^n} \{n-1\}$ given by the initial vertex of σ). This completes the construction of $F_{\beta+1}$ in the case where $\sigma = \sigma_\beta$ is nondegenerate.

Now suppose that $\sigma = \sigma_\beta : \Delta^m \to X$ is a degenerate simplex. We wish to show that the lifting problem

$$
\begin{array}{ccc}
\mathcal{C}_\beta^1 & \xrightarrow{F_\beta} & \mathcal{D} \\
\downarrow{\scriptstyle j} & \nearrow & \downarrow{\scriptstyle p} \\
\mathcal{C}_{\beta+1}^1 & \longrightarrow & \mathcal{E}
\end{array}
$$

admits a solution. Since p is a categorical fibration, it will suffice to show that j is a categorical equivalence. Let $\mathcal{A}$ and $\mathcal{A}_\partial$ be defined as above. Let $\mathcal{A}^1$ denote the simplicial subset of $\mathcal{A}$ spanned by those simplices $\tau : \Delta^k \to \mathcal{A}$ such that the induced map $\Delta^k \star \Delta^m \to \mathcal{C}$ factors through $\Delta^k \star \Delta^{m'}$ for some surjective map $\Delta^m \to \Delta^{m'}$ with $m' < m$. Let $\mathcal{A}_\partial^1 = \mathcal{A}^1 \cap \mathcal{A}_\partial$, and let $\mathcal{A}' = \mathcal{A}^1 \coprod_{\mathcal{A}_\partial^1} \mathcal{A}_\partial$. Unwinding the definitions, we have a pushout diagram of simplicial sets

$$
\begin{array}{ccc}
(\mathcal{A} \star \partial\,\Delta^m) \coprod_{\mathcal{A}' \star \partial\,\Delta^m} (\mathcal{A}' \star \Delta^m) & \longrightarrow & \mathcal{C}_\beta^1 \\
\downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} \\
\mathcal{A} \star \Delta^m & \longrightarrow & \mathcal{C}_{\beta+1}^1 .
\end{array}
$$

It will therefore suffice to show that j' is a categorical equivalence: that is, that the diagram

$$
\begin{array}{ccc}
\mathcal{A}' \star \partial\,\Delta^m & \longrightarrow & \mathcal{A} \star \Delta^m \\
\downarrow & & \downarrow \\
\mathcal{A}' \star \partial\,\Delta^m & \longrightarrow & \mathcal{A} \star \Delta^m
\end{array}
$$

is a homotopy pushout square (with respect to the Joyal model structure). In fact, we show that the vertical maps in this diagram are categorical equivalences. For this, it suffices to show that the inclusion $\mathcal{A}' \hookrightarrow \mathcal{A}$ is a categorical equivalence: that is, that the diagram

$$
\begin{array}{ccc}
\mathcal{A}_\partial^1 & \longrightarrow & \mathcal{A}_\partial \\
\downarrow & & \downarrow \\
\mathcal{A}^1 & \longrightarrow & \mathcal{A}
\end{array}
$$

is a homotopy pushout square. We will prove that the horizontal maps are categorical equivalences.

Set $\mathcal{A}^0 = \mathcal{A} \times_{\mathcal{C}} \mathcal{C}^0$. For every simplicial subset $K \subseteq \Delta^{\{0,\dots,n-2\}}$, let $\mathcal{B}_K = \mathcal{A} \times_{\Delta^n} K$ and $\mathcal{A}_K^0 = \mathcal{A}^0 \times_{\Delta^n} K$, so we have an isomorphism $\mathcal{A}_K \simeq \mathcal{A}^0 \coprod_{\mathcal{A}_K^0} \mathcal{B}_K$. Let

$$
\mathcal{A}^{01} = \mathcal{A}^0 \cap \mathcal{A}^1 \qquad \mathcal{A}_K^{01} = \mathcal{A}_K^0 \cap \mathcal{A}^1 \qquad \mathcal{B}_K^1 = \mathcal{B}_K \cap \mathcal{A}^1 .
$$

Consider the map $u_K : \mathcal{A}^{01} \coprod_{\mathcal{A}_K^{01}} \mathcal{B}_K^1 \to \mathcal{A}^0 \coprod_{\mathcal{A}_K^0} \mathcal{B}_K$. We wish to prove that this map is an equivalence when $K = \Delta^{\{0,\dots,n-2\}}$ or $K = \partial\,\Delta^{\{0,\dots,n-2\}}$. For this, it will suffice to prove the following trio of assertions, for every $K \subseteq \Delta^{\{0,\dots,n-2\}}$.

(a) The inclusion $\mathcal{A}^{01} \hookrightarrow \mathcal{A}^0$ is a categorical equivalence.

(b) The inclusion $\mathcal{A}_K^{01} \hookrightarrow \mathcal{A}_K^0$ is a categorical equivalence.

(c) The inclusion $\mathcal{B}_K^1 \hookrightarrow \mathcal{B}_K$ is a categorical equivalence.

Note that (a) is a special case of (b) (namely, the special case where $K = \Delta^{\{0,\dots,n-2\}}$). We will prove (c); assertion (b) will follow from the same argument, replacing $\mathcal{C}$ by the full subcategory spanned by X and $\mathcal{C}^0$. Note that the constructions $K \mapsto \mathcal{B}_K$ and $K \mapsto \mathcal{B}^1_K$ commute with homotopy pushouts; we may therefore reduce to the case where K is the image of a simplex $\Delta^k \to \Delta^{\{0,\dots,n-2\}}$. Replacing $\mathcal{C}$ by the pullback $\mathcal{C} \times_{\Delta^n} (\Delta^k \star \Delta^{\{n-1,n\}})$, we may reduce to the case $K = \Delta^{\{0,\dots,n-2\}}$. That is, we are reduced to proving that the inclusion $\mathcal{A}^1 \hookrightarrow \mathcal{A}$ is a categorical equivalence.

Let $\mathcal{J}$ be the category whose objects are commutative diagrams

$$
\begin{array}{ccc}
\Delta^m & \xrightarrow{\ \epsilon\ } & \Delta^{m'} \\
 & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle\tau} & \\
 & \mathcal{C} &
\end{array}
$$

where ϵ is surjective and $m' < m$; we will abuse notation by identifying the objects of $\mathcal{J}$ with simplices $\tau : \Delta^{m'} \to \mathcal{C}$. Unwinding the definitions, we see that $\mathcal{A}^0$ can be identified with the colimit of the cofibrant diagram $\theta : \mathcal{J}^{op} \to \mathrm{Set}_\Delta$ given by $\tau \mapsto \Delta^{\{0,1,\dots,n-2\}} \times_{\Delta^n} \mathcal{C}_{/\tau}$. It will therefore suffice to show that $\mathcal{A}$ is a homotopy colimit of the diagram θ. Our assumption that σ is degenerate implies that $\mathcal{J}$ has a final object (given by the factorization of σ as $\Delta^m \xrightarrow{\epsilon} \Delta^{m'} \xrightarrow{\tau} \mathcal{C}$ where τ is nondegenerate), it will suffice to show that the diagram θ is weakly equivalent to the constant diagram with value $\mathcal{A}$. In other words, we must show that for each $\tau \in \mathcal{J}$, the canonical map $\theta(\tau) \to \mathcal{A}$ is a categorical equivalence. This follows from the two-out-of-three property, since both of the vertical maps in the diagram

$$
\begin{array}{ccc}
\Delta^{\{0,\dots,n-2\}} \times_{\Delta^n} \mathcal{C}_{/\tau} & \xrightarrow{\hspace{4cm}} & \Delta^{\{0,\dots,n-2\}} \times_{\Delta^n} \mathcal{C}_{/\sigma} \\
 & \searrow \qquad\qquad \swarrow & \\
 & \Delta^{\{0,\dots,n-2\}} \times_{\Delta^n} \mathcal{C}^0_{/C} &
\end{array}
$$

are trivial Kan fibrations, where $C = \tau(0) = \sigma(0) \in \mathcal{C}$. This completes the construction of $F_{\beta+1}$ in the case where σ is degenerate, and the construction of the map $f_1 : \mathcal{C}^1 \to \mathcal{D}$ extending f_0.

We now show that f_1 can be extended to the desired map $f : \mathcal{C} \to \mathcal{D}$. Let Y denote the full subcategory of $\mathcal{C}$ spanned by those vertices which do not belong to X. Let A' be the collection of all simplices $\sigma' : \Delta^m \to Y$ which are not contained in the intersection $Y \cap (\mathcal{C} \times_{\Delta^n} \partial \Delta^{\{0,\dots,n-1\}})$. For each $\sigma' \in A'$, let $d(\sigma')$ denote the dimension of the simplex σ'. Choose a well-ordering of A' such that if $d(\sigma') < d(\tau')$, then $\sigma' < \tau'$. For every nondegenerate simplex $\sigma' : \Delta^m \to \mathcal{C}$ which does not belong to $\mathcal{C}^1$, we let $r'(\sigma') \in A'$ denote the simplex of Y given by $\Delta^m \times_{\Delta^n} Y \to X$. Let α' be the order type of A', so that we have an order-preserving bijection $\beta \mapsto \sigma'_\beta$ where β ranges over the set of ordinals $\{\beta : \beta < \alpha'\}$. For each $\beta \le \alpha'$, we let $\mathcal{C}_\beta$ denote the simplicial subset of $\mathcal{C}$ given by the union of $\mathcal{C}^1$ with those simplices σ' such that $r'(\sigma') = \sigma'_\gamma$ for some $\gamma < \beta$. Then f_1 defined a map $F'_0 : \mathcal{C}_0 \to \mathcal{D}$. We will show that F'_0 can be extended to a compatible family of maps $F'_\beta : \mathcal{C}_\beta \to \mathcal{D}$ satisfying $p \circ F'_\beta = g|\,\mathcal{C}_\beta$. Taking $\beta = \alpha'$, we will obtain the desired extension $f : \mathcal{C} \to \mathcal{D}$ of f_0.

The construction of the maps F'_β proceeds by induction on β. If β is a nonzero limit ordinal, we set $F'_\beta = \bigcup_{\gamma < \beta} F'_\gamma$. To handle successor stages, let us assume that F'_β has already been defined for some $\beta < \alpha'$, and let $\sigma' = \sigma'_\beta : \Delta^m \to Y$ be the corresponding simplex. We first treat the case where σ' is nondegenerate. Let $X_{\sigma'/}$ denote the fiber product $X \times_{\mathcal{C}} \mathcal{C}_{\sigma'/}$. Since σ' is nondegenerate, we have a pushout diagram

$$
\begin{array}{ccc}
\partial\,\Delta^m \star X_{\sigma'/} & \longrightarrow & \mathcal{C}_\beta \\
\downarrow & & \downarrow \\
\Delta^m \star X_{\sigma'/} & \longrightarrow & \mathcal{C}_{\beta+1}\,.
\end{array}
$$

To construct $F'_{\beta+1}$, it suffices to solve the lifting problem depicted in the diagram

$$
\begin{array}{ccc}
\partial\,\Delta^m \star X_{\sigma'/} & \overset{h}{\longrightarrow} & \mathcal{D} \\
\downarrow & & \downarrow{\scriptstyle p} \\
\Delta^m \star X_{\sigma'/} & \longrightarrow & \mathcal{E}\,.
\end{array}
$$

Let v denote the final vertex of σ'. It will suffice to show that h induces a p-limit diagram $h' : \{v\}\star X_{\sigma'/} \to \mathcal{D}$. Since the projection map $X_{\sigma'/} \to X \times_{\mathcal{C}} \mathcal{C}_{v/}$ is a trivial Kan fibration, it suffices to show that h exhibits $h(v)$ as a p-limit of the diagram $X \times_{\mathcal{C}} \mathcal{C}_{v/} \to \mathcal{D}$, which follows from assumption (iii).

We now treat the case where $\sigma' = \sigma'_\beta$ is a degenerate simplex of Y. Let $X'_{\sigma'/}$ denote the simplicial subset of $X_{\sigma'/}$ spanned by those simplices $\tau : \Delta^k \to X_{\sigma'/}$ such that the induced map $\Delta^m \star \Delta^k \to \mathcal{C}$ factors through $\Delta^{m'} \star \Delta^k$, for some surjective map $\Delta^m \to \Delta^{m'}$ with $m' < m$. We have a pushout diagram of simplicial sets

$$
\begin{array}{ccc}
(\partial\,\Delta^m \star X_{\sigma'/}) \star_{\partial\,\Delta^m \star X'_{\sigma'/}} (\Delta^m \star X'_{\sigma'/}) & \longrightarrow & \mathcal{C}_\beta \\
\downarrow & \nearrow & \downarrow \\
\Delta^m \star X_{\sigma'/} & \longrightarrow & \mathcal{C}_{\beta+1}\,.
\end{array}
$$

Consequently, to prove the existence of $F_{\beta+1}$, it suffices to solve a lifting problem of the form

$$
\begin{array}{ccc}
(\partial\,\Delta^m \star X_{\sigma'/}) \star_{\partial\,\Delta^m \star X'_{\sigma'/}} (\Delta^m \star X'_{\sigma'/}) & \longrightarrow & \mathcal{D} \\
\downarrow{\scriptstyle j'} & \nearrow & \downarrow{\scriptstyle p} \\
\Delta^m \star X_{\sigma'/} & \longrightarrow & \mathcal{E}\,.
\end{array}
$$

Since p is a categorical fibration, this lifting problem will admit a solution provided that j' is a categorical equivalence. We are therefore reduced to proving that the diagram

$$
\begin{array}{ccc}
\partial\,\Delta^m \star X'_{\sigma'/} & \longrightarrow & \partial\,\Delta^m \star X_{\sigma'/} \\
\downarrow & & \downarrow \\
\Delta^m \star X'_{\sigma'/} & \longrightarrow & \Delta^m \star X_{\sigma'/}.
\end{array}
$$

is a homotopy pushout square (with respect to the Joyal model structure). In fact, we claim that the horizontal maps in this diagram are categorical equivalences. To prove this, it suffices to show that the inclusion $X'_{\sigma'/} \hookrightarrow X_{\sigma'/}$ is a categorical equivalence.

Let $\mathcal{I}$ be the category whose objects are commutative diagrams

$$
\begin{array}{ccc}
\Delta^m & \overset{\epsilon}{\longrightarrow} & \Delta^{m'} \\
& {\scriptstyle \sigma'}\searrow \quad \swarrow{\scriptstyle \tau} & \\
& \mathcal{C} &
\end{array}
$$

where ϵ is surjective and $m' < m$; we will abuse notation by identifying the objects of $\mathcal{I}$ with the underlying simplices $\tau : \Delta^{m'} \to \mathcal{C}$. Unwinding the definitions, we see that $X'_{\sigma'/}$ can be identified with the colimit of the cofibrant diagram $\theta' : \mathcal{I}^{op} \to \mathcal{S}et_\Delta$ given by $\tau \mapsto X \times_{\mathcal{C}} \mathcal{C}_{\tau/}$. It will therefore suffice to show that $X_{\sigma'/}$ is a homotopy colimit of the diagram θ'. Our assumption that σ is degenerate implies that $\mathcal{I}$ has a final object

(given by the factorization of σ as $\Delta^M \xrightarrow{\epsilon} \Delta^{m'} \xrightarrow{\tau} \mathcal{C}$ where τ is nondegenerate), it will suffice to show that the diagram θ' is weakly equivalent to the constant diagram with value $X'_{\sigma'/}$. In other words, we must show that for each $\tau \in \mathcal{I}$, the canonical map

$$X \times_{\mathcal{C}} \mathcal{C}_{\tau/} \to X_{\sigma'/}$$

is a categorical equivalence. This follows from the two-out-of-three property, since we have a commutative diagram

where $D = \sigma'(m) \in \mathcal{C}$, and the vertical maps are trivial Kan fibrations. $\square$

Proposition B.4.9. *Suppose we are given a diagram of ∞-categories $X \xrightarrow{\phi} Y \xrightarrow{\pi} Z$ where ϕ is a categorical fibration and π is a flat categorical fibration. Let $Y' \subseteq Y$ be a full subcategory. Let $X' = Y' \times_Y X$, let $\pi' = \pi|Y'$, and let $\psi : \pi_* X \to \pi'_* X'$ denote the restriction map. Let K be another ∞-category, let $\overline{p} : K^\triangleleft \to \pi_* X$ be a diagram, and suppose that the following conditions are satisfied:*

(i) The full subcategory $Y' \times_Z K^\triangleleft \subseteq Y \times_Z K^\triangleleft$ is a cosieve on $Y \times_Z K^\triangleleft$.

(ii) For every object $y \in Y'$ and every morphism $f : z \to \pi(y)$ in Z, there exists a π-Cartesian morphism $\overline{f} : \overline{z} \to y$ in Y' with $\pi(\overline{f}) = f$.

(iii) The map $\overline{F} : K^\triangleleft \times_Z Y \to X$ classified by $\overline{p}$ is a ϕ-right Kan extension of

$$F = \overline{F}|((K \times_Z Y) \coprod_{K \times_Z Y'} (K^\triangleleft \times_Z Y')).$$

Then $\overline{p}$ is a ψ-limit diagram.

Proof. We must show that for $n \geq 2$, every lifting problem of the form

admits a solution, provided that $f_0|\{n-1\} \star K$ coincides with $\overline{p}$. Let $\mathcal{C} = (\Delta^{n-1} \star K) \times_Z Y$, and observe that $\mathcal{C}$ is equipped with a map $\mathcal{C} \to \Delta^n$. Let $\mathcal{C}^0 = \mathcal{C} \times_Y Y'$. Unwinding the definitions, we see that it suffices to solve a lifting problem of the form

The desired result now follows from Lemma B.4.8 and our hypothesis on $\overline{F}$. $\square$

We will typically apply Proposition B.4.9 in the special case where $Y' = \emptyset$, so that $\pi'_* X' \simeq Z$. In this cases, conditions (i) and (ii) are automatic.

Proposition B.4.10. *Suppose we are given a diagram of categorical fibrations $X \xrightarrow{\phi} Y \xrightarrow{\pi} Z$ where π is a Cartesian fibration and ϕ is a categorical fibration. Suppose that the following condition is satisfied:*

($*$) *For every vertex x of X and every π-Cartesian edge $f : \phi(x) \to y$ in Y, there exists a ϕ-coCartesian edge $\overline{f} : x \to \overline{y}$ such that $f = \phi(\overline{f})$.*

Then:

(1) *The map $\psi : \pi_* X \to Z$ is a coCartesian fibration.*

(2) *Let $\overline{e}$ be an edge of $\pi_* X$ lying over an edge $e : z \to z'$ in Z, corresponding to a map $F : \Delta^1 \times_Z Y \to X$. Then $\overline{e}$ is ψ-coCartesian if and only if the following condition is satisfied:*

   (a) *For every π-Cartesian edge $\widetilde{e}$ of Y lying over e, the image $F(e)$ is a ϕ-coCartesian edge of X.*

Proof. We use the same strategy as in the proof of Proposition T.3.2.2.12. Proposition B.4.5 guarantees that ψ is a categorical fibration, and in particular an inner fibration. Let us say that an edge of $\pi_* X$ is *special* if it satisfies condition (a). We will prove:

(i) For every vertex $A \in \pi_* X$ and every edge $e : \psi(A) \to z$ in Z, there exists a special edge $\overline{e} : A \to \overline{z}$ in $\pi_* X$ such that $\psi(\overline{e}) = e$.

(ii) Every special edge of $\pi_* X$ is ψ-coCartesian.

This will prove (1) and the "if" direction of (2). To prove the "only if" direction, we consider an arbitrary ψ-coCartesian edge $\overline{e} : A \to B$ in $\pi_* X$ covering an edge $e : z \to z'$ in Z. Using (i), we can choose a special edge $\overline{e}' : A \to C$ in $\pi_* X$ covering e. Using the assumption that $\overline{e}$ is ψ-coCartesian, we can choose a 2-simplex

$$
\begin{array}{ccc}
 & B & \\
\overline{e}' \nearrow & & \searrow \overline{e}'' \\
A & \xrightarrow{\quad \overline{e} \quad} & C
\end{array}
$$
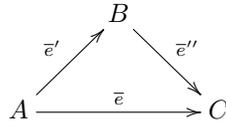
whose image in Z is degenerate. Since $\overline{e}'$ and $\overline{e}$ are both ψ-coCartesian (by (ii)), we conclude that $\overline{e}''$ is an equivalence in the ∞-category $(\pi_* X)_{z'}$. Since $\overline{e}'$ satisfies (a), we deduce that $\overline{e}$ satisfies (a), as desired.

We now prove (i). Without loss of generality, we may replace X and Y by their pullbacks along the edge $e : \Delta^1 \to Z$, and thereby reduce to the case $Z = \Delta^1$. We can identify A with a section of the projection map $X_0 \to Y_0$. To produce an edge $\overline{e} : A \to \overline{z}$ as in (i), we must solve the lifting problem depicted in the diagram

$$
\begin{array}{ccc}
Y_0 & \xrightarrow{\;A\;} & X \\
\downarrow & {\scriptstyle A'} \nearrow & \downarrow {\scriptstyle \phi} \\
Y & \longrightarrow & Y.
\end{array}
$$
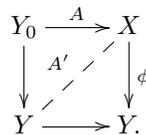
Moreover, $\overline{e}$ is special if and only if the map A' carries π-Cartesian morphisms of Y to ϕ-coCartesian morphisms in X. Using Proposition T.3.2.2.7, we can choose a functor $\chi : X_1 \to X_0$ and a quasi-equivalence $M(\phi) \to X$. Using Propositions T.A.2.3.1, we may reduce to the problem of providing a dotted arrow in the diagram

$$
\begin{array}{ccc}
X_0 & \longrightarrow & X \\
\cap \downarrow & \nearrow & \downarrow {\scriptstyle \phi} \\
M(\phi) & \longrightarrow & Y
\end{array}
$$
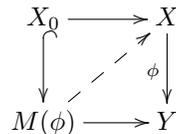
which carries the marked edges of $M^\natural(\phi)$ to ϕ-coCartesian edges of X. This follows from the fact that $\phi^{X_1} : \mathrm{Fun}(X_1, X) \to \mathrm{Fun}(X_1, Y)$ is a coCartesian fibration and the description of the ϕ^{X_1}-coCartesian morphisms (Proposition T.3.1.2.1).

The proof of (ii) is similar. We wish to prove that every lifting problem

$$
\begin{array}{ccc}
\Lambda_0^n & \longrightarrow & \pi_* X \\
\downarrow & \nearrow & \downarrow{\scriptstyle \psi} \\
\Delta^n & \longrightarrow & Z
\end{array}
$$

has a solution provided that $n \geq 2$ and the upper horizontal map carries $\Delta^{\{0,1\}} \subseteq \Lambda_0^n$ to a special edge of $\pi_* X$. Replacing X and Y by their pullbacks along $\Delta^n \to Z$, we can assume that the lower horizontal map is an isomorphism. Unwinding the definitions, we are reduced to solving the lifting problem

$$
\begin{array}{ccc}
Y \times_{\Delta^n} \Lambda_0^n & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow{\scriptstyle \phi} \\
Y & \longrightarrow & Y.
\end{array}
$$

Using Proposition T.3.2.2.7, we can choose a composable sequence of morphisms

$$
\chi : Y_0 \leftarrow \cdots \leftarrow Y_n
$$

and a quasi-equivalence $M(\chi) \to Y$. Invoking Propositions T.A.2.3.1, we may reduce to the associated mapping problem

$$
\begin{array}{ccc}
M(\psi) \times_{\Delta^n} \Lambda_0^n & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow{\scriptstyle \phi} \\
M(\psi) & \longrightarrow & Y.
\end{array}
$$

This is equivalent to the mapping problem

$$
\begin{array}{ccc}
X_n \times \Lambda_i^n & \longrightarrow & X \\
\downarrow & & \downarrow{\scriptstyle \phi} \\
X_n \times \Delta^n & \longrightarrow & Y.
\end{array}
$$

which admits a solution by virtue of Proposition T.3.1.2.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Corollary B.4.11. *Suppose we are given a diagram of categorical fibrations $X \xrightarrow{\phi} Y \xrightarrow{\pi} Z$. Let M be a collection of edges of Z containing all degenerate edges and T a collection of 2-simplices of Z containing all degenerate 2-simplices. Suppose that the following conditions are satisfied:*

(a) *The categorical fibration π is flat.*

(b) *For every vertex $y \in Y$ and every edge $f : z \to \pi(y)$ of Z which belongs to M, there exists a locally π-Cartesian edge $\overline{f} : \overline{z} \to y$ such that $\pi(\overline{f}) = f$.*

(c) *For every vertex x of X and every locally π-Cartesian edge $f : \phi(x) \to y$ in Y such that $\pi(f) \in M$, there exists a locally ϕ-coCartesian edge $\overline{f} : x \to \overline{y}$ such that $f = \phi(\overline{f})$.*

(d) *Let $\overline{f}$ be a locally ϕ-coCartesian edge of X such that $f = \phi(\overline{f})$ is locally π-Cartesian and $\pi(f) \in M$, and suppose that $f : \Delta^{\{0,1\}} \to Y$ is extended to a 2-simplex $\sigma : \Delta^2 \to Y$ such that $\pi(\sigma) \in T$. Then $\overline{f}$ determines a ϕ'-coCartesian morphism of $X \times_Y \Delta^2$, where $\phi' : X \times_Y \Delta^2 \to \Delta^2$ denotes the projection.*

Then:

(1) *The map $\psi : \pi_* X \to Z$ is a categorical fibration.*

(2) *For every vertex $x \in \pi_* X$ and edge morphism $f : \psi(x) \to z$ of Z which belongs to M, there exists a locally ψ-coCartesian edge $\overline{f} : x \to \overline{z}$ of $\pi_* X$ with $\psi(\overline{f}) = f$.*

(3) *Let $\overline{f}$ be an edge of $\pi_* X$ lying over an edge $\psi(\overline{f}) = f : z \to z'$ which belongs to M, corresponding to a map $F : \Delta^1 \times_Z Y \to X$. Then $\overline{f}$ is locally ψ-coCartesian if and only if the following condition is satisfied:*

   (∗) *For every locally π-Cartesian edge $\widetilde{f}$ of Y lying over f, the image $F(\widetilde{f})$ is a locally ϕ-coCartesian edge of X.*

(4) *Let $\sigma : \Delta^2 \to Z$ be 2-simplex belonging to T such that the edge $f = \sigma|\Delta^{\{0,1\}}$ belongs to M, and let $\overline{f}$ be a locally ψ-coCartesian edge of $\pi_* X$ lying over f. Then $\overline{f}$ determines a ψ'-coCartesian edge of $\pi_* X \times_Z \Delta^2$, where ψ' denotes the projection $\pi_* \times_Z \Delta^2 \to \Delta^2$.*

Proof. Assertion (1) follows from Proposition B.4.5, and assertions (2) and (3) follow by applying Proposition B.4.10 to the diagram $X \times_Z \Delta^1 \to Y \times_Z \Delta^1 \to \Delta^1$. To prove (4), we are free to replace Z by Δ^2 and thereby reduce to the case where σ is an isomorphism. Let $\overline{f}$ be a locally ψ-coCartesian edge of $\pi_* X$ lying over $\Delta^{\{0,1\}}$, which we can identify with a functor $F : Y \times_{\Delta^2} \Delta^{\{0,1\}} \to X \times_{\Delta^2} \Delta^{\{0,1\}}$. We wish to show that f is ψ-coCartesian. By virtue of (the dual of) Proposition B.4.9, it will suffice to show that the functor F is a ψ-left Kan extension of $F|Y_0$, where $Y_0 = Y \times_{\Delta^2} \{0\}$. Unwinding the definitions, we must show that for each object $y \in Y \times_{\Delta^2} \{1\}$, the map F induces a ϕ-colimit diagram

$$\theta : (Y_0 \times_Y Y_{/y})^{\triangleright} \to (Y \times_{\Delta^2} \Delta^{\{0,1\}})^{\triangleright}_{/y} \to Y \times_{\Delta^2} \Delta^{\{0,1\}} \xrightarrow{F} X.$$

Condition (*b*) guarantees that the projection $Y \times_{\Delta^2} \Delta^{\{0,1\}} \to \Delta^{\{0,1\}}$ is a Cartesian fibration, so the ∞-category $Y_0 \times_Y Y_{/y}$ has a final object, given by a locally π-Cartesian morphism $f : y' \to y$. It follows that θ is a ϕ-colimit diagram if and only if $F(f)$ is a ϕ-coCartesian morphism in X. Criterion (3) guarantees that $F(f)$ is locally ψ-coCartesian, which implies that $F(f)$ is ψ-coCartesian by virtue of assumption (*d*). $\square$

Proposition B.4.12. *Suppose we are given a diagram of ∞-categories $X \xrightarrow{\phi} Y \xrightarrow{\pi} Z$ where π is a flat categorical fibration and ϕ is a categorical fibration. Let $\psi : \pi_* X \to Z$ denote the projection, let K be an ∞-category, let $\overline{p}_0 : K^{\triangleleft} \to Z$ be a diagram, and assume that the induced map $\pi' : K^{\triangleleft} \times_Z Y \to K^{\triangleleft}$ is a coCartesian fibration. Let v denote the cone point of $K^{\triangleleft}$, let $\mathcal{C} = \pi'^{-1}\{v\}$, and choose a map $K^{\triangleleft} \times \mathcal{C} \to K^{\triangleleft} \times_Z Y$ which is the identity on $\{v\} \times \mathcal{C}$ and carries $e \times \{C\}$ to a π'-coCartesian edge of $K^{\triangleleft} \times_Z Y$, for each edge e of $K^{\triangleleft}$ and each object C of $\mathcal{C}$. Then:*

(1) *Let $\overline{p} : K^{\triangleleft} \to \pi_* X$ be a map lifting $\overline{p}_0$, and suppose that for each $C \in \mathcal{C}$ the induced map*

$$K^{\triangleleft} \times \{C\} \hookrightarrow K^{\triangleleft} \times \mathcal{C} \to K^{\triangleleft} \times_Z Y \to X$$

   is a ϕ-limit diagram. Then $\overline{p}$ is a ψ-limit diagram.

(2) *Suppose that $p : K^{\triangleleft} \to \pi_* X$ is a map lifting $p_0 = \overline{p}_0|K$, and suppose that for each $C \in \mathcal{C}$ the induced map*

$$K \times \{C\} \hookrightarrow K \times \mathcal{C} \to K \times_Z Y \to X$$

   admits a ψ-limit diagram lifting the map

$$K^{\triangleleft} \times \{C\} \hookrightarrow K^{\triangleleft} \times \mathcal{C} \to K^{\triangleleft} \times_Z Y \to Y.$$

   Then there exists an extension $\overline{p} : K^{\triangleleft} \to \pi_* X$ of p lifting $\overline{p}_0$ which satisfies condition (1).

Proof. Let $\overline{p} : K^\triangleleft \to \pi_* X$ satisfy the condition described in (1). We can identify $\overline{p}$ with a map $\overline{F} : K^\triangleleft \times_Z Y \to X$. In view of Proposition B.4.9, it will suffice to show that $\overline{F}$ is a ϕ-right Kan extension of $\overline{F} | K \times_Z Y$. Pick an object $C \in \mathcal{C}$; we wish to show that $\overline{F}$ is a ϕ-right Kan extension of F at C. In other words, we wish to show that the map

$$(K \times_Z Y)^\triangleleft_{C/} \to K^\triangleleft \times_Z Y \xrightarrow{\overline{F}} X$$

is a ϕ-limit diagram. Since $\overline{p}$ satisfies (1), it suffices to show that the map

$$s : K \times \{C\} \to (K \times_Z Y)_{C/}$$

is right cofinal. It follows from Proposition T.2.4.3.2 that the projection $q : (K^\triangleleft \times_Z Y)_{C/} \to K^\triangleleft$ is a coCartesian fibration, and that s is a coCartesian section of q. To show that s is right cofinal, it will suffice to show that s admits right adjoint (this follows from Corollary T.4.1.3.1). In fact, we will show that the identity map $\mathrm{id}_{K^\triangleleft} \to q \circ s$ exhibits q as a right adjoint to s. For this, we must show that for every object $a \in (K \times_Z Y)_{C/}$ and every object $b \in K$, the map

$$\mathrm{Map}_{(K \times_Z Y)_{C/}}(s(b), a) \to \mathrm{Map}_K(b, q(a))$$

is a homotopy equivalence. Let $c = \mathrm{id}_C$ denote the initial object of $(K^\triangleleft \times_Z Y)_{C/}$, and let η denote the unique map from the cone point $v \in K^\triangleleft$ to b in $K^\triangleleft$. Using Proposition T.2.4.4.3, we obtain a homotopy pullback diagram

$$
\begin{array}{ccc}
\mathrm{Map}_{(K^\triangleleft \times_Z Y)_{C/}}(s(b), a) & \longrightarrow & \mathrm{Map}_{(K^\triangleleft \times_Z Y)_{C/}}(c, a) \\
\downarrow & & \downarrow{\scriptstyle \theta} \\
\mathrm{Map}_{K^\triangleleft}(b, q(a)) & \longrightarrow & \mathrm{Map}_{K^\triangleleft}(v, q(a)).
\end{array}
$$

It therefore suffices to show that the θ is a homotopy equivalence, which is clear (both the domain and the codomain of θ are contractible). This completes the proof of (1).

We now prove (2). The diagram p gives rise to a map $F : K \times_Z Y \to X$ fitting into a commutative diagram

$$
\begin{array}{ccc}
K \times_Z Y & \xrightarrow{F} & X \\
\downarrow & \nearrow{\scriptstyle \overline{F}} & \downarrow{\scriptstyle \phi} \\
K^\triangleleft \times_Z Y & \longrightarrow & Y.
\end{array}
$$

The above argument shows that a dotted arrow $\overline{F}$ as indicated will correspond to a map $\overline{p} : K^\triangleleft \to \pi_* X$ satisfying (1) if and only if $\overline{F}$ is a ϕ-right Kan extension of F. In view of Lemma T.4.3.2.13, the existence of such an extension is equivalent to the requirement that for each $C \in \mathcal{C}$, the diagram

$$(K \times_Z Y)_{C/} \to K \times_Z Y \xrightarrow{F} X$$

can be extended to a ϕ-limit diagram lifting the map

$$(K \times_Z Y)^\triangleleft_{C/} \to K^\triangleleft \times_Z Y \to Y.$$

This follows from the hypothesis of part (2) together with the fact (established above) that s is right cofinal. $\square$

Proof of Proposition B.4.1. The functor π^* admits a right adjoint $\overline{\pi}_*$, given by the formula $\overline{\pi}_*(X', M') = (X, M)$, where:

   (a) The simplicial set X is the full simplicial subset of $\pi_* X'$ spanned by those vertices lying over objects $s \in S$ which classify maps $S'_s \to X'_s$ which carry edges of $M_{S'}$ (which belong to S'_s) into M'.

(*b*) An edge e of X belongs to M if and only if its image in S belongs to M_S, and e classifies a map $S' \times_S \Delta^1 \to X'$ which carries the inverse image of $M_{S'}$ in $S' \times_S \Delta^1$ into M'.

We wish to prove that the adjoint functors $(\pi^*, \overline{\pi}_*)$ give a Quillen adjunction between $(\mathcal{S}et^+_\Delta)_{/\mathfrak{P}}$ and $(\mathcal{S}et^+_\Delta)_{/\mathfrak{P}'}$. To prove this, it will suffice to show that π^* preserves cofibrations and weak equivalences. The case of cofibrations is obvious. To prove that π^* preserves weak equivalences, consider an arbitrary $\mathfrak{P}$-equivalence $\overline{Y} \to \overline{Z}$. We wish to prove that for every $\mathfrak{P}$-fibered object $\overline{X} \in (\mathcal{S}et^+_\Delta)_{/\mathfrak{P}'}$, the induced map

$$\mathrm{Map}'^\sharp_S(\pi^*\overline{Z}, \overline{X}') \to \mathrm{Map}'^\sharp_S(\pi^*\overline{Y}, \overline{X})$$

is a homotopy equivalence. We can identify this with the canonical map

$$\mathrm{Map}^\sharp_S(\overline{Z}, \overline{\pi}_*\overline{X}') \to \mathrm{Map}^\sharp_S(\overline{Y}, \overline{\pi}_*\overline{X}').$$

It will therefore suffice to show that $\overline{\pi}_*\overline{X}'$ is $\mathfrak{P}$-fibered.

Write $\overline{X}' = (X', M')$, and let $\overline{\pi}_*\overline{X}' = (X, M)$ where (X, M) is described by (*a*) and (*b*), and let $p : X' \to S'$ denote the projection. Set $W = \pi_* X'$, so that X can be identified with a full simplicial subset of W. Let M_W denote the collection of edges $e : \Delta^1 \to W$ satisfying the following condition:

(∗) The image of e in S belongs to M_S, and the edge e classifies a map $S' \times_S \Delta^1 \to X'$ which carries π_{Δ^1}-Cartesian edges of $S' \times_S \Delta^1$ into M', where $\pi_{\Delta^1} : S' \times_S \Delta^1 \to \Delta^1$ denotes the projection.

We claim that M is the inverse image of M_W in X. To see that M is contained in this inverse image, it suffices to observe that every locally π-Cartesian edge of $\pi^{-1} M_S$ belongs to $M_{S'}$, which follows from (*v*). Conversely, suppose that $e : x \to x'$ is an edge of X belonging to M_W, and let e classify a map $E : S' \times_S \Delta^1 \to X'$. We wish to prove that if f is an edge of $S' \times_S \Delta^1$ whose image in S' belongs to $M_{S'}$, then $E(f) \in M'$. If the composite map $\Delta^1 \xrightarrow{f} S' \times_S \Delta^1 \to \Delta^1$ is not the identity, then the inclusion $E(f) \in M'$ follows from the assumption that the vertices x and x' belong to X. Otherwise, we can factor f as a composition $f' \circ f''$, where f'' is a morphism in $S' \times_S \{0\}$ and f' is π_{Δ^1}-Cartesian. Using (*v*), we see that the image of f'' in S' belongs to $M_{S'}$, so that $E(f'') \in M'$ by virtue of our assumption that $x \in X$. Condition (∗) guarantees that $E(f') \in M'$. Using the assumption that $\overline{X}'$ is $\mathfrak{P}'$-fibered, we deduce that $E(f) \in M'$ as desired.

We wish to prove that the pair (X, M) is $\mathfrak{P}$-fibered. For this, we will verify that the map $q : X \to S$ satisfies conditions (1), (2), (3), (4), and (6) of Definition B.0.18, together with condition (5′) of Remark B.0.25.

(1) We must show that the map $q : X \to S$ is an inner fibration. It follows from Proposition B.2.7 (together with conditions (*iv*) and (*vi*)) that $X' \to S'$ is a categorical fibration. Proposition B.4.5 and assumption (*ii*) guarantee that the map $q' : W \to S$ is a categorical fibration, and therefore an inner fibration. Since X is a full simplicial subset of W, it follows also that $X \to S$ is an inner fibration.

(2) For each edge $\Delta^1 \to S$ belonging to M_S, the induced map $q_{\Delta^1} : X \times_S \Delta^1 \to \Delta^1$ is a coCartesian fibration. It follows from Corollary B.4.11 that the map $q'_{\Delta^1} : W \times_S \Delta^1 \to \Delta^1$ is a coCartesian fibration, and that an edge of $W \times_S \Delta^1$ is q'_{Δ^1}-coCartesian if and only if its image in W belongs to M_W. To complete the proof, it will suffice to show that if $f : x \to y$ is a q'_{Δ^1}-coCartesian morphism in $W \times_S \Delta^1$ with nondegenerate image in Δ^1 and $x \in X \times_S \{0\}$, then $y \in X \times_S \{1\}$. We can identify f with a map $F : S' \times_S \Delta^1 \to X'$. To prove that $y \in X \times_S \{1\}$, we must show that for every morphism $\alpha : t \to t'$ in $S' \times_S \{1\}$ whose image in S' belongs to $M_{S'}$, we have $F(\alpha) \in M'$. Form a commutative diagram

$$
\begin{array}{ccc}
s & \xrightarrow{\ \beta\ } & t \\
{\scriptstyle \alpha'}\downarrow & {\scriptstyle \gamma}\searrow & \downarrow{\scriptstyle \alpha} \\
s' & \xrightarrow[\ \beta'\ ]{} & t'
\end{array}
$$

in $S' \times_S \Delta^1$, where $s, s' \in S' \times_S \{0\}$ and the horizontal maps are π_{Δ^1}-Cartesian. Condition (v) guarantees that the images of β and β' in S' belong to $M_{S'}$. Invoking (iv), we deduce that the image of γ in S' belongs to $M_{S'}$. Invoking (v) again, the image of α' in S belongs to $M_{S'}$. Since the image of f in Y belongs to M_W, and $x \in X \times_S \{0\}$, we conclude that F carries α', β, and β' into M'.

Let $\sigma : \Delta^2 \to X'$ be the 2-simplex

$$
\begin{array}{ccc}
 & F(t) & \\
 \overset{F(\beta)}{\nearrow} & & \overset{F(\alpha)}{\searrow} \\
F(s) & \xrightarrow{\quad F(\gamma) \quad} & F(t').
\end{array}
$$

Note that since (X', M') is $\mathfrak{P}'$-fibered and $F(\alpha'), F(\beta') \in M'$, we have $F(\gamma) \in M'$. Since the image of this 2-simplex in S is degenerate, condition (vi) guarantees that its image in S' belongs to T'. Because (X', M') is $\mathfrak{P}'$-fibered, we conclude that the induced map $p' : X' \times_{S'} \Delta^2 \to \Delta^2$ is a coCartesian fibration. To prove that $F(\alpha) \in M'$, it suffices to show that $F(\alpha)$ is locally p-coCartesian, which is equivalent to the requirement that it is p'-coCartesian when regarded as a morphism of $X' \times_{S'} \Delta^2$. This follows from Proposition T.2.4.1.7, since $F(\beta), F(\gamma) \in M'$ implies that $F(\beta)$ and $F(\gamma)$ determine p'-coCartesian morphisms in $X' \times_{S'} \Delta^2$.

(3) A morphism f of X belongs to M if and only if $q(f)$ belongs to M_S and f is locally q-coCartesian. This follows from the proof of (2), since both conditions are equivalent to the requirement that $f \in M_W$.

(4) Given a commutative diagram

$$
\begin{array}{ccc}
\Delta^{\{0,1\}} & \xrightarrow{\ f\ } & X \\
\downarrow & & \downarrow{\scriptstyle q} \\
\Delta^2 & \xrightarrow{\ \sigma\ } & S,
\end{array}
$$

if $f \in M$ and $\sigma \in T$, then f determines an q_{Δ^2}-coCartesian edge of $X \times_S \Delta^2$, where $q_{\Delta^2} : X \times_S \Delta^2 \to \Delta^2$ denotes the projection map. In fact, f determines a q'_{Δ^2}-coCartesian edge of $W \times_S \Delta^2$, where $q'_{\Delta^2} : W \times_S \Delta^2 \to \Delta^2$ denotes the projection: this follows from Corollary B.4.11.

(6) For every index $\alpha \in A$ and every coCartesian section s of the map $q_\alpha : X \times_S K_\alpha^\triangleleft \to K_\alpha^\triangleleft$, the map s is a q-limit diagram in X. To prove this, it will suffice to show that s is a q'-limit diagram in W. We will prove this by applying Proposition B.4.12. Let s classify a map $F : K_\alpha^\triangleleft \times_S S' \to X'$, and note that the map F carries the inverse image of $M_{S'}$ into M'. Let $\mathcal{C}$ denote the fiber of the map $\pi : S' \to S$ over the image of the cone point of $K_\alpha^\triangleleft$. Choose a map $\mathcal{C} \times K_\alpha^\triangleleft \to K_\alpha^\triangleleft \times_S S' \to X'$ as in the statement of Proposition B.4.12. We wish to show that, for each $C \in \mathcal{C}$, the induced map

$$
\theta : K_\alpha^\triangleleft \times \{C\} \xrightarrow{\theta_0} K_\alpha^\triangleleft \times \mathcal{C} \xrightarrow{\theta_1} K_\alpha^\triangleleft \times_S S' \xrightarrow{F} X'
$$

is a p-limit diagram in X'. Let $\pi_\alpha : K_\alpha^\triangleleft \times_S S' \to S'$ denote the restriction of π. Since $\theta_1 \circ \theta_0$ can be identified with a π_α-coCartesian section of π_α, condition $(viii)$ and the fact that F carries the inverse image of $M_{S'}$ into M' guarantee that θ carries every edge of $K_\alpha^\triangleleft$ into M'. Using the assumption that (X', M') is $\mathfrak{P}$-fibered and condition (ix), we conclude that θ is a p-limit diagram as desired.

($5'$) For each $\alpha \in A$ and every coCartesian section s_0 of the projection $X \times_S K_\alpha \to K_\alpha$, there exists a coCartesian section s of $X \times_S K_\alpha^\triangleleft \to K_\alpha^\triangleleft$ extending s_0. The construction of s amounts to solving a lifting problem

$$
\begin{array}{ccc}
S' \times_S K_\alpha & \xrightarrow{\ F\ } & X' \\
\downarrow & \nearrow^{\overline{F}} & \downarrow{\scriptstyle p} \\
S' \times_S K_\alpha^\triangleleft & \longrightarrow & S'.
\end{array}
$$

Since s_0 is a coCartesian section, we have $F(e) \in M'$ for every edge e of $S' \times_S K_\alpha$ whose image in S' belongs to $M_{S'}$. As in the proof of (6), we let $\mathcal{C}$ denote the fiber of the map π over the image of the cone point of $K_\alpha^\triangleleft$, and choose a map $\mathcal{C} \times K_\alpha^\triangleleft \to S' \times_S K_\triangleleft$ as in the statement of Proposition B.4.12. Using condition (ix), the assumption that (X', M') is $\mathfrak{P}'$-fibered, and Proposition B.4.12, we conclude that there exists an extension $\overline{F}$ of F such that for each $C \in \mathcal{C}$, the composite map

$$\overline{F}_C : \{C\} \times K_\alpha^\triangleleft \hookrightarrow \mathcal{C} \times K^\triangleleft \to S' \times_S K_\alpha^\triangleleft \xrightarrow{\overline{F}} X'$$

is a p-limit diagram. Invoking again our assumption that (X', M') is $\mathfrak{P}$-fibered (and that $\overline{F}_C$ carries each edge of $\{C\} \times K_\alpha$ into M', by virtue of $(viii)$), we deduce that $\overline{F}_C$ carries each edge of $K_\alpha^\triangleleft$ into M'. The map $\overline{F}$ corresponds to a section s of the projection $X \times_S K_\alpha^\triangleleft \to K_\alpha^\triangleleft$ extending s_0. To complete the verification of $(6')$, it will suffice to show that s is a coCartesian: in other words, we must show that $\overline{F}(e) \in M'$ whenever $e : x \to y$ is an morphism of $S' \times_S K_\alpha^\triangleleft$ whose image in S' belongs to $M_{S'}$.

If $x \notin \mathcal{C}$, then e is a morphism of $S' \times_S K_\alpha$ so that $\overline{F}(e) = F(e) \in M'$ as desired. We may therefore assume that $x \in \mathcal{C}$. Suppose for the moment that $y \notin \mathcal{C}$, so that the image of y in $K_\alpha^\triangleleft$ is a vertex $y_0 \in K_\alpha$. We can factor e as a composition $e' \circ e''$, where e'' is a π_α-coCartesian morphism lying in the image of the map $\{x\} \times K_\alpha^\triangleleft \to S' \times_S K_\alpha^\triangleleft$ and e' is a morphism in the fiber $\{y_0\} \times_S S'$. Invoking assumption $(viii)$, we deduce that $e' \in M_{S'}$, so that $\overline{F}(e') = F(e') \in M'$. Since $\overline{F}(e'')$ lies in the image of $\overline{F}_x$, we conclude that $\overline{F}(e'') \in M'$. Using the fact that (X', M') is $\mathfrak{P}$-fibered, we conclude that $\overline{F}(e) \in M'$, as desired.

We now treat the case where $x, y \in \mathcal{C}$. Let ψ denote the projection map $X' \times_S K_\alpha^\triangleleft \to S' \times_S K_\alpha^\triangleleft$. Applying $\overline{F}$ to e yields a morphism $\overline{e} : \overline{x} \to \overline{y}$ of $X' \times_S K_\alpha^\triangleleft$ with $\psi(\overline{e}) = e$. Since the image in S' of e belongs to $M_{S'}$, we can factor $\overline{e}$ as a composition $\overline{e}' \circ \overline{e}''$, where $\overline{e}''$ is locally ψ-coCartesian and $\overline{e}'$ is a morphism belonging to $\psi^{-1}\{x\}$. Using assumption (vi) and Lemma B.1.7, we deduce that every locally ψ-coCartesian morphism is ψ-coCartesian provided that its image in S' belongs to $M_{S'}$; in particular, $\overline{e}''$ is ψ-coCartesian. We wish to prove that $\overline{e}$ is locally ψ-coCartesian, which is equivalent to the assertion that $\overline{e}'$ is an equivalence. Choose a π_α-coCartesian section θ of the projection π_α which carries the cone point of $K_\alpha^\triangleleft$ to y. Since (X', M') is $\mathfrak{P}$-fibered, the coCartesian fibration

$$X' \times_{S'} K_\alpha^\triangleleft \to K_\alpha^\triangleleft$$

is classified by a limit diagram $\chi : K_\alpha^\triangleleft \to \mathcal{C}\mathrm{at}_\infty$, so that $\overline{e}'$ is an equivalence if and only if $\gamma_! \overline{e}'$ is an equivalence in $\psi^{-1}\{y'\}$ for every morphism $\gamma : y \to y'$ lying in the image of θ. We have a commutative diagram in $X' \times_S K_\alpha^\triangleleft$

$$
\begin{array}{ccc}
\overline{x} \xrightarrow{\overline{e}'} & \overline{z} \longrightarrow & \overline{z}' \\
{\scriptstyle\overline{e}}\searrow \quad {\scriptstyle\overline{e}}\downarrow & & \downarrow{\scriptstyle\gamma_!\overline{e}'} \\
\overline{y} & \longrightarrow & \overline{y}'
\end{array}
$$

where the horizontal maps are ψ-coCartesian. Moreover, the argument of the preceding paragraph shows that the map $\overline{x} \to \overline{y}'$ is ψ-coCartesian. Applying Proposition T.2.4.1.7, we deduce that $\gamma_! \overline{e}'$ is locally ψ-coCartesian, and therefore an equivalence because it belongs to a fiber of ψ.

$\square$

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