Foundations of perfectoid spaces

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Abstract

We give a reasonably detailed overview of the various tilting correspondences for perfectoid rings, the almost purity theorem, almost vanishing theorems, etc. We largely follow Scholze's *Perfectoid spaces* and *Etale cohomology of diamonds*, together with a tiny amount of Bhatt–M.–Scholze's *Integral p-adic Hodge theory* in order to formulate some results in terms of integral perfectoid rings rather than perfectoid Tate rings. I've also benefited a lot from Bhatt's *Lecture notes for a class on perfectoid spaces.*

Fix a prime number p forever.

1 Integral perfectoid rings

Let A be a complete topological ring. We say that A is *integral perfectoid* if and only if there exists a non-zero-divisor (for simplicity – otherwise condition (iii) should be modified, but for our goal of studying perfectoid Tate rings the non-zero-divisor case is sufficient) $\pi \in A$ such that

(a) the topology on A is the π-adic topology;

(b) $p \in \pi^p A$;

(c) $\Phi : A/\pi A \stackrel{\simeq}{\to} A/\pi^p A$, $a \mapsto a^p$

It is convenient, but not standard, to call any such element π a perfectoid pseudo-uniformiser (ppu).

Lemma 1.1. *Let A be integral perfectoid, and $\pi \in A$ a perfectoid pseudo-uniformiser. Then:*

(i) Every element of $A/\pi pA$ is a p^{th}-power (n.b., this is not a characteristic p ring).

(ii) If an element $a \in A[\frac{1}{\pi}]$ satisfies $a^p \in A$, then $a \in A$.

(iii) After multiplying π by a unit it admits a compatible sequence of p-power roots $\pi^{1/p}, \pi^{1/p^2}, \dots \in A$.

Proof. (i): Using the surjectivity of Φ, a simple induction lets us write any $a \in A$ as an infinite sum $a = \sum_{i\geq 0} a_i^p \pi^{pi}$ for some $a_i \in A$; but this is $\equiv (\sum_{i\geq 0} a_i \pi^i)^p$ mod $p\pi A$.

(ii): Let $l \geq 0$ be the smallest integer such that $\pi^l a \in A$. Assuming that $l > 0$, we get a contradiction by noting that $\pi^{pl} a^p \in \pi^{pl} A \subseteq \pi^p A$, whence $\pi^l a \in \pi A$ by condition (c), and so $\pi^{l-1} a \in A$.

(iii): Recall that if B is any p-adically complete ring, then the natural map $\varprojlim_{x\mapsto x^p} B \to \varprojlim_{x\mapsto x^p} B/pB$ is an isomorphism of monoids; applying this to the rings A and $A/\pi^p A$, we deduce that the natural map $\varprojlim_{x\mapsto x^p} A \to \varprojlim_{x\mapsto x^p} A/\pi^p A$ is an isomorphism. Since the Frobenius is surjective on A/π^p, it follows that there exists a compatible sequence $a, a^{1/p}, a^{1/p^s}, \dots \in A$ such that $a \equiv \pi$ mod $\pi^p A$; therefore $a = u\pi$ for some $u \in 1 + \pi^{p-1} A \subseteq A^\times$. $\square$

Lemma 1.2. *Let A be integral perfectoid, and $\varpi \in A$ any element satisfying (a) and (b). Then ϖ is a perfectoid pseudo-uniformiser.*

Proof. It follows from Lemma 1.1(i) that every element of A/pA is a p^{th}-power; hence every element of its quotient $A/\varpi^p A$ is a p^{th}-power. It remains to prove that $\Phi : A/\varpi A \to A/\varpi^p A$ is injective.

Let $\pi \in A$ be a perfectoid pseudo-unifomiser. The fact that π and ϖ define the same topology easily implies that ϖ is a non-zero-divisor and that $A[\frac{1}{\varpi}] = A[\frac{1}{\pi}]$. If $a \in A$ satisfies $a^p \in \varpi^p A$, then $(a/\varpi) \in A[\frac{1}{\pi}]$ satisfies $(a/\varpi)^p \in A$, and it then follows from Lemma 1.1(ii) that in fact $a \in \varpi A$ as desired. □

Lemma 1.3. *Suppose A is a complete topological ring such that $pA = 0$. Then A is integral perfectoid if and only if it is perfect and the topology is π-adic for some non-zero-divisor $\pi \in A$.*

Proof. Condition (b) is clearly vacuous, while (c) implies by a trivial inductive argument and taking the limit. □

1.1 The tilt of an integral perfectoid ring

Definition 1.4. The *tilt* of an integral perfectoid ring A is $A^\flat := \varprojlim_\varphi A/pA$, equipped with the inverse limit topology (A/pA is of course given the quotient topology).

Recalling again that the natural map $\varprojlim_{x \mapsto x^p} A \to \varprojlim_\varphi A/pA$ is an isomorphism of monoids, we define $\# : A^\flat \to A$ to be projection to the first factor of the left side; this map is multiplicative but not additive. Explicitly, the map $\#$ is given by $\varprojlim_\varphi A/pA \ni (a_0, a_1, \dots) \mapsto \lim_{i \to \infty} \widetilde{a}_i^{p^i}$, where $\widetilde{a}_i \in A$ are arbitrary lifts of the elements $a_i \in A/pA$.

Note that the composition $A^\flat \overset{\#}{\longrightarrow} \overset{\text{mod } p}{} A/pA$ is the surjective ring homomorphism given by projecting $A^\flat = \varprojlim_\varphi A/pA$ to the first factor.

Lemma 1.5. *Let A be an integral perfectoid ring. Then:*

(i) $\# : A^\flat \to A$, is continuous;

(ii) the usual isomorphisms of monoids $\varprojlim_{x \mapsto x^p} A \to A^\flat = \varprojlim_\varphi A/pA \to \varprojlim_\varphi A/\varpi A$ are homeomorphisms, where $\varpi \in A$ is any element defining the topology which divides p.

(iii) $A^\flat$ is also an integral perfectoid ring.

Proof. Given $(1, \dots, 1, a_{n+1}, a_{n+2}, \dots) \in \varprojlim_\varphi A/\varpi A$, any chosen lifts $\widetilde{a_i}$ satisfy $\widetilde{a_i}^{p^{i-n}} \equiv 1 \bmod \varpi A$ for $i > n$, whence $\widetilde{a_i}^{p^i} \equiv 1 \bmod \varpi^n A$; taking the limit shows that the sharp is $\equiv 1 \bmod \varpi^n A$. This proves that the $\#$ map $\varprojlim_\varphi A/\varpi A \to A$ is continuous (for the inverse limit of discrete topologies on the left), from which (i) and (ii) easily follow.

(iii) Let $\pi \in A$ be a perfectoid pseudo-uniformiser admitting p-power roots; let $\pi^\flat = (\pi, \pi^{1/p}, \dots,) \in A^\flat$ be the corresponding element of $A^\flat$. Note that if $a \in A$ satisfies $a^p \in \pi A$ then $(a\pi^{(p-1)/p})^p \in \pi^p A$, whence $a\pi^{(p-1)/p} \in \pi A$ by (b) and so $a \in \pi^{1/p} A$; similarly, by induction, if $a^{p^n} \in \pi A$ then $a \in \pi^{1/p^n} A$. Therefore we have exact sequences

$$0 \longrightarrow \pi^{1-1/p^n} A/\pi A \longrightarrow A/\pi A \overset{\pi^{1/p^n}}{\longrightarrow} A/\pi A \overset{\varphi^n}{\longrightarrow} A/\pi A \longrightarrow 0$$

for each $n \geq 1$, which are compatible over different n (wrt. φ acting on the left three terms, and id on the right-most term). Taking $\varprojlim_n$ clearly kills the left-most term (since $\varphi(\pi^{1-1/p^n}) \in \pi A$) and so we obtain a short exact sequence

$$0 \longrightarrow A^\flat \xrightarrow{\pi^\flat} A^\flat \overset{\sharp \bmod \pi}{\longrightarrow} A/\pi A \longrightarrow 0.$$

Therefore $\pi^\flat$ is a non-zero-divisor of $A^\flat$.

Since $A^\flat \to \varprojlim_\varphi A/\pi A$ is a homeomorphism, a basis of open neighbourhoods of $0 \in A^\flat$ is given by the kernels of $(\# \bmod \pi) \circ \varphi^{-n}$, for $n \geq 1$; but these kernels are $\pi^{\flat p^n}$ (since $\pi^\flat$ is a non-zero-divisor, $A^\flat$ is perfect, and we have proved it when $n = 0$), thereby showing that the topology on $A^\flat$ is the $\pi^\flat$-adic topology.

By the previous lemma, $A^\flat$ is integral perfectoid. □

We point out that we showed in the previous proof that the kernel of the surjective ring homomorphism $A^\flat \overset{\# \bmod \pi}{\longrightarrow} A/\pi A$ (i.e., projection to the first factor of $A^\flat = \varprojlim_\varphi A/\pi A$) is $\pi^\flat A^\flat$. This will be used repeatedly.

1.2 Fontaine's map θ

Proposition 1.6 (Fontaine)**.** *Let A be an integral perfectoid ring.*

(i) There is a unique ring homomorphism

$$\theta : W(A^\flat) \longrightarrow A$$

which sends $[a]$ to $a^\#$ for each $a \in A^\flat$.

(ii) θ is surjective and its kernel is generated by a non-zero-divisor.

(iii) An element $\xi' \in \operatorname{Ker}\theta$ is a generator if and only if its Witt vector expansion $\xi = (\xi'_0, \xi'_1, \dots)$ has the property that ξ'_1 is a unit of $A^\flat$.

(iv) $W(A^\flat)$ is $\operatorname{Ker}\theta$-adically complete.

Proof. Since any element $\alpha \in W(A^\flat)$ may be written uniquely as $\alpha = \sum_{i\geq 0}[a_i]p^i = (a_0, a_1^p, a_2^p, \dots)$ where $a_i \in A^\flat$, the content of (i) is the assertion that $\alpha \mapsto \sum_{i \geq 0} a_i^\# p^i$ is a ring homomorphism. In principle this can be checked directly using the polynomial expressions for addition and multiplication of Witt vectors; alternatively see Lemma 3.2 of *Integral p-adic Hodge theory* (we do not reproduce the proof here).

(ii): Since $W(A)$ is p-adically complete and A is p-adically separated, to prove surjectivity it is enough to show that θ is surjective mod p. But this follows from the fact that $\#$ mod p is surjective.

Now we construct a possible generator of the kernel; let $\pi \in A$ be a perfectoid pseudo-uniformiser admitting p-power roots. Since $p \in \pi^p A$ and θ is surjective, we may write $p = \pi^p\theta(-x)$ for some $x \in W(A^\flat)$, whence $\xi := p + [\pi^\flat]^p x \in \operatorname{Ker}\theta$. Since $W(A^\flat)$ is $[\pi^\flat]$-adically complete[1] and A is $\theta([\pi^\flat]) = \pi$-torsion-free, one easily sees that $\theta : W(A^\flat)/\xi \to A$ is an isomorphism if and only if it becomes an isomorphism when we mod out by $[\pi^\flat]$; but then it

[1]See, for example, *Integral p-adic Hodge theory*, Lemma 3.2(ii), for some standard comments about topologies on Witt rings; in particular $W_r(A^\flat)$ is $[\pi^\flat]$-adically complete for each $r \geq 1$, and then the same is true for $W(A^\flat)$ by taking the limit.

identifies with $A^\flat/\pi^\flat A^\flat \overset{\#\bmod \pi}{\longrightarrow} A/\pi A$ (note that $([\pi^\flat],\xi)=([\pi^\flat],p)$), which we saw was an isomorphism in the proof of Lemma 1.5(iii).

We must also show that ξ is a non-zero-divisor, so suppose that $\alpha\in W(A^\flat)$ satisfies $\xi\alpha=0$. Since ξ divides $p^m+[\pi^\flat]^{pm}x^m$ (for any odd $m\geq 1$) we deduce $(p^m+[\pi^\flat]^{pm}x^m)\alpha=0$, and so $p^m\alpha\in[\pi^\flat]^{pm}W(A^\flat)$; writing $\alpha=(a_0,a_1,\dots)$ in Witt coordinates and expanding, this shows that $a_i^{p^m}\in\pi^{\flat p^{m+i-1}m}A^\flat$ for all $i\geq 0$. But $A^\flat$ is perfect so this means $a_i\in\pi^{\flat p^{i+1}m}$, and then letting $m\to\infty$ shows $a_i=0$ for all $i\geq 0$, i.e., $\alpha=0$. (Note: this argument actually showed that $W(C)$ is ξ-torsion-free, where C is any perfect $A^\flat$ algebra which is $\pi^\flat$-adically separated; this will be useful later.)

(iii): First note that the Witt vector expansion of our element ξ looks like

$$(\xi_0,\xi_1,\dots)=p+[\pi^\flat]^px=(0,1,0,0,\dots)+(\pi^{\flat p}x_0,\pi^{\flat p^2}x_1,\dots)=(\pi^{\flat p}x_0,1+\pi^{\flat p^2}x_1,\dots),$$

in particular ξ_1 is a unit. Now let $\xi'=(\xi_0',\xi_1',\dots)\in\operatorname{Ker}\theta$ be another element (whence $\xi_0'^{\#}\in pA$ and so $\xi_0'\in\pi^\flat A^\flat$ since we know $A^\flat/\pi^\flat A \overset{\#\bmod \pi}{\longrightarrow} A/\pi A$ is an isomorphism), and write

$$\xi'=\alpha\xi=(a_0,a_1,\dots)(\xi_0,\xi_1,\dots)=(a_0\xi_0,a_1\xi_0^p+a_0^p\xi_1,\dots).$$

Then ξ' is also a generator if and only if α is a unit, which is equivalent to a_0 being a unit, which is equivalent to $\xi_1'=a_1\xi_0^p+a_0^p\xi_1$ being a unit (since $\xi_0\in\pi^\flat A^\flat$, ξ_1 is a unit, and $A^\flat$ is $\pi^\flat$-adically complete).

(iv): $W(A^\flat)$ is $([\pi^\flat],p)$-adically complete, since each $W_m(A^\flat)=W(A^\flat)/p^m$ is $[\pi^\flat]$-adically complete; therefore it is a fortiori ξ-adicially complete. □

1.3 Tilting correspondence for integral perfectoid rings

We begin with some remarks on topologies. Given a topological ring R, we always equip $W_r(R)$ with the product topology having identified it with R^r (if the topology on R is π-adic, then the topology on $W_r(R)$ is the $[\pi]$-adic topology), and then $W(R)=\varprojlim_r W_r(R)$ with the inverse limit topology (if R is perfect then this is the $([\pi],p)$-adic topology). Note that if A is integral perfectoid then, under these topologies, the map $\theta:W(A^\flat)\to A$ is continuous and open, since $(\theta([\pi^\flat]),p)=\pi A$, i.e., $W(A^\flat)/\operatorname{Ker}\theta\to A$ is an isomorphism of topological rings.

Given an integral perfectoid ring A, when we speak of an "integral perfectoid A-algebra B" we mean that B is an A-algebra which is integral perfectoid and whose topology comes from A, i.e., is πB-adic for any $\pi\in A$ defining the topology on A (equivalently, by Lemma 1.2, any ppu of A is also a ppu of B).

Theorem 1.7 (Tilting correspondence – integral perfectoid rings)**.** *Fix an integral perfectoid ring A. Then tilting induces an equivalence of categories*

$$\textit{integral perfectoid } A\textit{-algerbras} \overset{\simeq}{\longrightarrow} \textit{integral perfectoid } A^\flat\textit{-algebras},\qquad B\mapsto B^\flat,$$

with inverse given by sending an integral perfectoid $A^\flat$-algebra C to $C^\#:=W(C)\otimes_{W(A^\flat),\theta}A$ (topologised as a quotient of $W(C)$ following the above conventions).

This equivalence is compatible with almost isomorphisms (i.e., kernel and cokernel killed by all topologically nilpotent elements of A, resp. $A^\flat$).

Proof. Let $\pi\in A$ be a perfectoid pseudo-uniformiser admitting p-power-roots, and $\pi^\flat$ the associated ppu of $A^\flat$; also let $\xi=p+[\pi^\flat]^px\in W(A^\flat)$ be the generator of $\operatorname{Ker}(\theta:W(A^\flat)\to A)$ constructed in the previous proposition.

Suppose first that B is an integral perfectoid A-algebra. Then the image of ξ in $W(B^\flat)$ certainly lands in the kernel of θ_B (i.e., the θ-map for the integral perfectoid B), and condition (iii) of the previous proposition immediately shows that it is a generator of this ideal (since $A \to B$ sends units to units); i.e., the diagram

$$\begin{array}{ccc} W(B^\flat) & \xrightarrow{\theta_B} & B \\ \uparrow & & \uparrow \\ W(A^\flat) & \xrightarrow[\theta_A]{} & A \end{array}$$

is a pushout and so $B^{\flat\#} = B$. For the moment this is only an identification of A-algebras, but the next paragraph will shows that the topologies coincide (both have the π-adic topology), i.e., $\#$ inverts tilting.

Now let C be an integral perfectoid $A^\flat$-algebra. As explained in the paragraph immediately above the theorem, Lemma 1.2 implies that $\pi^\flat$ is also a ppu of C, and thus the topology we have put on $C^\#$ is $(\theta([\pi^\flat]), p)$-adic topology, i.e., the π-adic topology. We must show that C is an integral perfectoid ring.

First we check that π is a non-zero-divisor of $C^\#$ π, i.e., that $W(C)/\xi W(C)$ has no $[\pi^\flat]$-torsion. As mentioned parenthetically in the proof of part (ii) of the previous proposition, the argument there shows that ξ is a non-zero-divisor of $W(C)$; since $[\pi^\flat]$ is also a non-zero-divisor, it enough show that $W(C)/[\pi^\flat]W(C)$ has no non-zero ξ-torsion. But $\xi \equiv p \bmod [\pi^\flat]$ and p is a non-zero-divisor of $W(C)$, so this is equivalent to $\pi^\flat$ being a non-zero-divisor of $W(C)/p = C$, which is indeed the case.

Next, noting that $([\pi^\flat]^p, \xi) = ([\pi^\flat]^p, p)$ and $([\pi^\flat], \xi) = ([\pi^\flat], p)$ in $W(C)$, the map $\Phi : C^\#/\pi C^\# \to C^\#/\pi^p C^\#$ identifies with $\Phi : C/\pi^\flat C \to C/\pi^{\flat p} C$, which is an isomorphism since C is perfectoid.

Thus the topological ring $C^\#$ satisfies all the conditions to be integral perfectoid, except we do not know yet that it is π-adically complete (it follows formally that it is derived π-adically complete, so it would be enough to show it is π-adically separated, but I do not see an explicit argument...). So let $\widehat{C^\#}$ be its π-adic completion, which is an integral perfectoid A-algebra. Then there are identifications $\widehat{C^\#}/\pi\widehat{C^\#} = C^\#/\pi C^\# = C/\pi^\flat C$ (the second of which comes from the previous paragraph), and taking $\varprojlim_{x \mapsto x^p}$ shows that $\widehat{C^\#}^\flat = C$ as integral perfectoid $A^\flat$-algebras. Since we have already proved that $\#$ inverts tilting, we apply $\#$ to deduce that $\widehat{C^\#} = C^\#$, i.e., $C^\#$ was already complete, and hence it is an integral perfectoid A-algebra. This argument has therefore also shown that $C^{\#\flat} = C$, i.e., tilting inverts $\#$. □

2 Perfectoid Tate rings

Recall that a complete topological ring R is called *Tate* if and only if it contains an open subring $R_0 \subseteq R$ whose topology is πR_0-adic for some $\pi \in R_0 \cap R^\times$; any such R_0 is called a subring of definition. We assume some familiarity with Tate rings (in particular that a given subring $R_0 \subseteq R$ is a subring of definition if and only if it is open and bounded) though we include some definitions in the footnotes.

Definition 2.1. A Tate ring R is called *perfectoid* if and only if the following equivalent conditions are satisfied:

(i) the topological ring R° is integral perfectoid;[2]

(ii) there exists a subring of definition R_0 which is integral perfectoid;

(iii) R is uniform[3] and every subring of integral elements[4] is integral perfectoid.

(iv) Fontaine's Bourbaki definition: R is uniform and there exists a topologically nilpotent unit $\pi \in R$ such that $p \in \pi^p R^\circ$ and $\Phi : R^\circ/\pi R^\circ \to R^\circ/\pi^p R^\circ$, $a \mapsto a^p$ is an isomorphism.

Proof of equivalences. (i)⇒(ii): If R° is integral perfectoid, then in particular its topology is adic for a finitely generated ideal, whence it is a subring of definition of R (hence bounded by standard theory).

(ii)⇒(iii): Suppose that R_0 is a subring of definition which is integral perfectoid, and let $\pi \in R_0$ be a perfectoid pseudo-uniformiser admitting p-power roots; note that π is a pseudo-uniformiser for the Tate ring R (i.e., a topologically nilpotent unit).

We claim that $R^{\circ\circ} \subseteq R_0$: indeed, if $x \in R$ is topologically nilpotent then $x^N \in R_0$ for $N \gg 0$, and so $x \in R_0$ by Lemma 1.1(ii). In particular this shows that $\pi R^\circ \subseteq R_0$; therefore R° is bounded, i.e., R is uniform.

Next let $R^+ \subseteq R$ be a subring of integral elements; it is an open and bounded (since R° is bounded) subring containing π (since $\pi^N \in R^+$ for $N \gg 0$ and R^+ is integrally closed in R), whence general theory of Tate rings tells us that its topology is the πR^+-adic topology.

We claim that every element of R^+ is a p^{th}-power modulo π (resp. $\pi^{1/2}$ if $p = 2$); let $x \in R^+$. Then $\pi x \in R^{\circ\circ} \subseteq R_0$ and so there exist $y, z \in R_0$ such that $\pi x = y^p + \pi^p z$; after multiplying π by a unit we may assume it admits a p^{th}-root in R_0, and we then deduce that $(y\pi^{-1/p})^p = x + \pi^{p-1}z \in R^+$ (note that $\pi^{p-1}z$ is topologically nilpotent, hence in R^+ by integral closedness), whence $y\pi^{-1/p} \in R^+$ by integral closedness again. Since $\pi^{p-1}z \in \pi R^+$ (unless $p = 2$, in which case it is $\in \pi^{1/2}R^+$ since $\pi^{1/2}z \in R^+$), we have shown that x is a p^{th}-power modulo πR^+ (resp. $\pi^{1/2}R^+$), which proves the claim.

Next note that $p \in (\pi^{1/p})^p R^+$: indeed, we know that $p \in \pi^p R_0 \subseteq \pi R^{\circ\circ}$ and that $R^{\circ\circ} \subseteq R^+$. Since the integral closedness of R^+ in R easily implies that $\Phi : R^+/\pi^{1/p}R^+ \to R^+/\pi R^+$ (resp. $R^+/\pi^{1/4}R^+ \to R^+/\pi^{1/2}R^+$) is injective, we have indeed proved that R^+ is perfectoid (with pseudo-uniformiser $\pi^{1/p}$, resp. $\pi^{1/4}$).

(iii)⇒(iv) and (iv)⇒(i) are trivial. □

The arguments implicitly showed that, for any perfectoid Tate ring, one has the following inclusions between the various classes of interesting subrings:

Subrings of integral elements ⊆ integral perfectoid open subrings ⊆ subrings of definition.

We record explicitly two other consequences of the above arguments:

Corollary 2.2. *Let R be a perfectoid Tate ring and $R_0 \subseteq R$ any subring of definition.*

(i) If R_0 is integral perfectoid, then the inclusion $R_0 \subseteq R^\circ$ is an almost equality (we will say something more systematic about almost mathematics later; for the moment just take this to mean that $R_0 \supseteq R^{\circ\circ}$).

[2] Given an Tate ring R, then R° denotes its subring of power bounded elements, and $R^{\circ\circ}$ denotes the set of topologically nilpotent elements (it is an ideal of R°).

[3] This means that R° is bounded.

[4] A *subring of integral elements* of a Tate ring R is an open subring $R^+ \subseteq R$ which is integrally closed inside R and consists of power bounded elements; the maximal subring of integral elements is R° itself.

(ii) There exists a minimal integral perfectoid subring containing R_0, namely

$$\widetilde{R}_0 := \{f \in R : f^{p^N} \in R_0 + pR^\circ \text{ for } N \gg 0\}.$$

Proof. (i): The proof of (ii)⇒(iii) in the previous definition showed that R_0 contains $R^{\circ\circ}$.

(ii): Note that $\widetilde{R}_0$ really is a subring of R: it is the preimage from R°/pR° of the p-radical closure of the image of R_0 (or just check by hand!). Since $\widetilde{R}_0$ is sandwiched between R_0 and R° it is open and bounded in R, hence is a subring of definition. To prove it is integral perfectoid, it remains only to check that $\Phi : \widetilde{R}_0/\pi\widetilde{R}_0 \xrightarrow{\simeq} \widetilde{R}_0/\pi^p\widetilde{R}_0$, where π is a ppu of R° (note that $\pi^N \in R_0$ for $N \gg 0$, therefore $\pi \in \widetilde{R}_0$). For injectivity, suppose that $f \in \widetilde{R}_0$ satisfies $f^p = \pi^p g$ for some $g \in \widetilde{R}_0$; then the fact that R° is integral perfectoid implies that $f = \pi g'$ for some $g' \in R^\circ$, whence $g'^p = g \in R_0$ and so $g' \in \widetilde{R}_0$. For surjectivity, let $f \in \widetilde{R}_0$ and use the fact that R° is integral perfectoid to find $g, h \in R^\circ$ such that $g^p = f + p\pi h$ (Lemma 1.1(i)); but ph belongs to $\widetilde{R}_0$ since it is topologically nilpotent, whence also $f \in \widetilde{R}_0$. So we have shown that any element of $\widetilde{R}_0$ is a p^{th}-power modulo $\pi\widetilde{R}_0$, from which a trivial induction (using the fact that $\pi^{1/p} \in \widetilde{R}_0$) shows that any element is a p^{th}-power modulo $\pi^p\widetilde{R}_0$

Regarding the minimality claim, if $R_0' \subseteq R$ is any integral perfectoid subring of definition then we proved in (ii)⇒(iii) above that "$f^p \in R_0' \implies f \in R_0'$", and we proved in part (i) that R_0' contains $R^{\circ\circ} \supseteq pR_0$; so if R_0' contains R_0 then it contains $\widetilde{R}_0$. □

Corollary 2.3. *Let R be a perfectoid Tate ring. Then R contains a smallest integral perfectoid subring of definition; it also contains a smallest subring of integral elements.*

Proof. Let $R_- := \bigcap_{R_0} R_0$ be the intersection of all integral perfectoid subrings of definition. Then R_- is open (since it contains $R^{\circ\circ}$, in particular it contains pR°) and bounded (since it is contained in R^0). Moreover, R_- has the property that $f^p \in R_- \implies f \in R_-$ since each integral perfectoid subring of definition has this property (by Lemma 1.1(ii)). It follows that $\widetilde{R}_- = R_-$, i.e., R_- is integral perfectoid, as required.

To construct the smallest subring of integral elements, simply take the intersection of all subrings of integral elements (all of which we know are integral perfectoid) and make the same argument. □

Note that if A is an integral perfectoid ring, then its *generic fibre* $R := A[\frac{1}{\pi}]$[5], where $\pi \in A$ is any element defining the topology, is well-defined; indeed, it is equivalent to invert all elements $f \in A$ such that fA is open. We give R the unique linear topology for which A is open in R; then standard theory implies that R is a Tate ring with subring of definition A, and so part (ii) of the previous definition shows that R is perfectoid, and Corollary 2.2(i) shows that the inclusion $A \subseteq R^\circ$ is an almost equality.

The tilt of a perfectoid Tate ring R is defined to be

$$R^\flat := \text{generic fibre of } R_0^\flat,$$

where $R_0 \subseteq R$ is any integral perfectoid subring of definition; according to the previous corollary $R_0^\flat \subseteq R^\circ$ is an almost equality, whence $R_0^\flat \subseteq R^{\circ\flat}$ is also an almost equality by Theorem 1.7 and so they have the same generic fibre; therefore the above definition of $R^\flat$ does not depend on the choice of R_0 (and we can just take R° if we want a manifestly natural choice). Note that the multiplicative map $\# : R_0^\flat \to R_0$ admits a unique extension to a multiplicative map $\# : R^\flat \to R$: indeed, picking a perfectoid pseudo-uniformiser $\pi \in R_0$, it is given by $a\pi^{\flat s} \mapsto a^\# \pi^s$ where $a \in R_0^\flat$ and $s \in \mathbb{Z}$.

[5]It would be clearer to introduce a notation like A_{gen} or A_η, but this is not standard practice so we avoid doing it.

Theorem 2.4 (Titling correspondence – perfectoid Tate rings)**.** *Let R be a perfectoid Tate ring. Then tilting $S \mapsto S^\flat$ defines an equivalence of categories between perfectoid Tate rings S over R and perfectoid Tate rings $S^\flat$ over $R^\flat$.*

Proof. Inverse functor is given by sending a perfectoid Tate ring T over $R^\flat$ to the generic fibre of the integral perfectoid R°-algebra given by untilting T°. This inverts the above process since $S^{\circ\flat}$ is almost the subring of bounded elements of $S^\flat$, and we stated in Theorem 1.7 that tilting intergral perfectoid preserves almost isomorphisms. There is nothing more to prove but we leave it to the reader to write down the details. □

It is also helpful to note the following for later:

Theorem 2.5 (Titling correspondence – lattice of integral perfectoid subrings of definition)**.** *Let R be a perfectoid Tate ring. Then tilting $R_0 \mapsto R_0^\flat \subseteq R^\flat$ defines an inclusion-preserving equivalence between integral perfectoid subrings of definition of R, resp. of $R^\flat$; it also preserves integral closures of one integral perfectoid subring of definition inside another. In particular, it restricts to an inclusion-preserving equivalence between subrings of integral elements of R, resp. of $R^\flat$.*

Proof. Let R_- be the smallest integral perfectoid subring of definition of R. Then the category of integral perfectoid subrings of definition of R is the same as the category of integral perfectoid R_--algebras B for which $R_- \to B$ is an almost isomorphism. Also, the tilt of R_- must be the smallest integral perfectoid subring of definition of $R^\flat$, since otherwise untilting back to R would yield a contradiction. So the desired correspondence of lattices of integral subrings of definition of R (resp. of $R^\flat$) follows from Theorem 1.7.

Let $R_0 \subseteq R_0'$ be integral perfectoid subrings of definition of R, and let R_0'' be the integral closure of R_0 in R_0'. Then R_0'' contains pR° (since this is contained in R_0) and is closed under taking p^{th}-roots of elements; it follows that $R_0'' = \widetilde{R_0''}$ (in the notation of Corollary 2.2), i.e., R_0'' is integral perfectoid. Moreover, to check whether an element $x \in R_0'$ is integral over R_0, it is enough to check mod π since $\pi R_0' \subseteq R_0$, and analgously for the tilts; it easily follows from this that $(R_0'')^\flat$ is the integral closure of $R_0^\flat$ in $R_0'^\flat$. □

3 Tilting perfectoid spaces

We begin with a short reminder on the building blocks of adic spaces, namely the adic spectra of Huber pairs. A *Huber pair*[6] (R, R^+) is a Tate ring R together with a choice of subring of integral elements $R^+ \subseteq R$. To this data we associate its *adic spectrum* $\mathrm{Spa}(R, R^+)$, which is the set of all valuations[7] $|\cdot| : R \to \Gamma \cup \{0\}$ where

- Γ is a totally ordered, multiplicatively written, abelian group;
- $|\cdot|$ is a non-archimedean absolute value, i.e., $|0| = 0$, $|1| = 1$, $|fg| = |f||g|$, and $|f+g| \leq \max(|f|, |g|)$.
- $|\cdot|$ is continuous in the sense that, for any $\gamma \in \Gamma$, the set $\{f \in R : |f| < \gamma\}$ is open.

[6]More precisely, here we define a Tate–Huber pair; we do not need the more general theory of non-Tate Huber pairs.

[7]Modulo equivalence: valuations $|\cdot| : R \to \Gamma \cup \{0\}$ and $|\cdot|' : R \to \Gamma' \cup \{0\}$ are considered equivalent if and only if for all $f, g \in R$ one has $|f| \leq |g| \Leftrightarrow |f|' \leq |g|'$.

A *rational subset* of $\mathrm{Spa}(R, R^+)$ is any subset which can be written as

$$\mathrm{Spa}(R, R^+)(\tfrac{f_1,\dots,f_n}{g}) := \{|\cdot| \in \mathrm{Spa}(R, R^+) : |f_i| \leq |g| \text{ for all } i = 1, \dots, n\}$$

for some $f_1, \dots, f_n \in R$ generating the unit ideal and some $g \in R$. Note: letting $\pi \in R$ be a topologically nilpotent unit, we can always rescale the f_i and g so that they belong to

We topologise $\mathrm{Spa}(R, R^+)$ by declaring a basis of open sets to the rational subsets (exercise: check that the rational subsets are closed under finite intersection.)

A Huber pair (R, R^+) is *perfectoid* if and only if R is a perfectoid Tate algebra (or, equivalently, R^+ is an integral perfectoid ring). In this section we fix a perfectoid Huber pair (R, R^+), denote by $(R^\flat, R^{+\flat})$ its tilt, and let

$$X := \mathrm{Spa}(R, R^+), \qquad X^\flat := \mathrm{Spa}(R^\flat, R^{+\flat})$$

be the corresponding adic spectra. Define the *tilting map*

$$\flat : X \to X^\flat, \qquad |\cdot| \mapsto |\cdot|^\flat$$

as follows: given a continuous valuation $|\,| : R \to \Gamma \cup \{0\}$, then define $|\,|^\flat : R^\flat \to \Gamma \cap \{0\}$ by $|f|^\flat := |f^\#|$.

Lemma 3.1. *The tilting map is well-defined, i.e.,* $|\,|^\flat : R^\flat \to \Gamma \cap \{0\}$ *really is a continuous absolute value, and continuous.*

Proof. Since $f \mapsto f^\#$ is multiplicative and preserves $0, 1$, the only axiom for an absolute value which is not obviously satisfied for $|\,|^\flat$ is the non-archimedean inequality. But, if $f, g \in R^\flat$, then

$$\begin{aligned}|f+g|^\flat = |(f+g)^\#| = |\lim_{n\to\infty}(f^{1/p^n\#} + g^{1/p^n\#})^{p^n}| &= \lim_{n\to\infty}|(f^{1/p^n\#} + g^{1/p^n\#})|^{p^n} \\ &\leq \lim_{n\to\infty}\max(|f^{1/p^n\#}|^{p^n}, |g^{1/p^n\#}|^{p^n}) = \max(|f^\#|, |g^\#|),\end{aligned}$$

which is exactly $\max(|f|^\flat, |g|^\flat)$ by definition. The fact that $\# : R^\flat \to R$ is continuous implies that $|\,|$ is continuous.

It remains to show that $\flat : X \to X^\flat$ is continuous, for which it is enough to check that the pre-image of any rational subset $U \subseteq X^\flat$ is a rational subset of X. Pick elements $f_1, \dots, f_n, g \in R^{+\flat}$, where f_n is a power of $\pi^\flat$ (we have implicitly picked a pseudo-uniformiser for R^+ having p-power roots), such that $U = X^\flat(\frac{f_1,\dots,f_n}{g})$ (it is standard that any rational subset may be described by such elements). The untilts $f_1^\#, \dots, f_n^\#, g^\# \in R^+$ define a corresponding rational subset $V := X(\frac{f_1^\#,\dots,f_n^\#}{g^\#})$ of X (n.b., this really is a rational subset since $f_n^\#$ is a power of π), and it is tautological from the definition of the tiling map that $\flat^{-1}(U) = V$. $\square$

The goal of the section is to prove the following:

Theorem 3.2 (Tilting correspondence – analytic topology of perfectoid spaces)**.** *The tilting map* $\flat : X \to X^\flat$ *is a homeomorphism which identifies rational subsets. If* $V \subseteq X$ *and* $U \subseteq X^\flat$ *are corresponding rational subsets, then:*

(i) $\mathcal{O}_X(V)$ *is a perfectoid Tate R-algebra;*

(ii) $\mathcal{O}_{X^\flat}(U)$ *is a perfectoid Tate* $R^\flat$*-algebra;*

(iii) there is a unique continuous map of R-algebras $\mathcal{O}_X(V)^\flat \to \mathcal{O}_{X^\flat}(U)$*; it is an isomorphism and restricts to an isomorphism of integral perfectoid* $A^\flat$*-algebras* $\mathcal{O}_X^+(V)^\flat \xrightarrow{\simeq} \mathcal{O}_{X^\flat}^+(U)$*.*

The proof of the theorem requires two key steps; the first of these is a careful study of rational subsets of $X^\flat$ and their pull-backs to X:

Proposition 3.3. *Let A be an integral perfectoid ring and $\pi \in A$ a ppu with p-power roots; let $f_1, \dots, f_n, g \in A^\flat$ where f_n is a power of $\pi^\flat$. Let C be the $A^\flat$-subalgebra of $A^\flat[1/g]$ generated by $f_i^{1/p^k}/g^{1/p^k}$ for $i = 1, \dots, n$ and $k \geq 1$; similarly, let B be the A-subalgebra of $A[1/g^\#]$ generated by $f_i^{\#1/p^k}/g^{\#1/p^k}$ for $i = 1, \dots, n$ and $k \geq 1$. Then:*

(i) The kernel of the surjection

$$\psi^\flat : A^\flat[\underline{X}^{1/p^\infty}] \longrightarrow C, \qquad X_i^{1/p^k} \mapsto f_i^{1/p^k}/g_i^{1/p^k}$$

is almost the ideal generated by $g_i^{1/p^k} X_i^{1/p^k} - f_i^{1/p^k}$ for $i = 1, \dots, n$ and $k \geq 1$.

(ii) Similarly, the kernel of the surjection

$$\psi : A[\underline{X}^{1/p^\infty}] \longrightarrow B, \qquad X_i^{1/p^k} \mapsto f_i^{\#1/p^k}/g_i^{\#1/p^k}$$

is almost the ideal generated by $g_i^{\#1/p^k} X_i^{1/p^k} - f_i^{\#1/p^k}$ for $i = 1, \dots, n$ and $k \geq 1$.

(iii) The $\pi^\flat$-adic completion $\widehat{C}$ is an integral perfectoid $A^\flat$-algebra; let $\widehat{C}^\#$ be its untilt (an integral perfectoid A-algebra), with corresponding generic fibre $\widehat{C}^\#[\frac{1}{\pi}]$ (a perfectoid Tate R-algebra).

(iv) There exists a unique continuous map of R-algebras $\widehat{B}[\frac{1}{\pi}] \to \widehat{C}^\#[\frac{1}{\pi}]$; it is an isomorphism (therefore $\widehat{B}[\frac{1}{\pi}]$ is a perfectoid Tate algebra, which is not obvious; in particular we cannot show that $\widehat{B}$ is an integral perfectoid A-algebra), it restricts to an injective almost surjection $B \hookrightarrow \widehat{C}^\#$, and it sends sends $f_i^{\#1/p^k}/g^{\#1/p^k}$ to $(f_i^{1/p^k}/g^{1/p^k})^\#$ for all $i = 1, \dots, n$ and $k \geq 1$.

(v) $\widehat{C}^\#$ is the smallest integral perfectoid subring of $\widehat{B}[\frac{1}{\pi}] = \widehat{C}^\#[\frac{1}{\pi}]$ which contains B; moreover, for any $x \in \widehat{C}^\#$ then $x^{p^N} \in \widehat{B}$ for $N \gg 1$.

(vi) $\widehat{C}$ is integral over its subring $\widehat{A^\flat[f_i/g]}$, and the difference between the rings is killed by a power of $\pi^\flat$; similarly B is integral over its subring $\widehat{A[f_i^\#/g^\#]}$ and the difference between the rings is killed by a power of π.

Proof. Although the above seems to be the clearest order in which the state the results, the proof proceeds differently:

(iii): C is obviously perfect and $\pi^\flat$ is a non-zero-divisor of it; therefore its completion $\widehat{C}$, equipped with the $\pi^\flat$-adic topology, is an integral perfectoid $A^\flat$-algebra by Lemma 1.3.

(i): Let $J \subseteq A[\underline{X}^{1/p^\infty}]$ be the ideal generated by the given elements; we obviously have an inclusion $J \subseteq \operatorname{Ker}\psi$, and we want to show that it is an almost equality. It is clear that the Frobenius acts isomorphically on both J and $\operatorname{Ker}\psi$, so it is enough to show that $\operatorname{Ker}\psi/J$ vanishes after inverting $\pi^\flat$, i.e., that

$$A^\flat[\tfrac{1}{\pi^\flat}][\underline{X}^{1/p^\infty}]/\langle g^{1/p^k} X_i^{1/p^k} - f_i^{1/p^k} : i, k\rangle \longrightarrow B[\tfrac{1}{\pi^\flat}]$$

is injective. We have assumed that f_N is a power of $\pi^\flat$, whence f_N is invertible on the left side, and so g is also invertible on both sides thanks to the relation $gX_N - f$. By rescalling the relations it is therefore equivalent to show that

$$A[\tfrac{1}{\pi^\flat},\tfrac{1}{g}][\underline{X}^{1/p^\infty}]/\langle X_i^{1/p^k} - f_i^{1/p^k}/g^{1/p^k} : i,k\rangle \longrightarrow B[\tfrac{1}{\pi^\flat},\tfrac{1}{g}] = A[\tfrac{1}{\pi^\flat},\tfrac{1}{g}]$$

is injective; but it is clearly an isomorphism by elementary algebra.

(ii) & (iv): The untilts $(f_i^{1/p^k}/g^{1/p^k})^\# \in \widehat{C}^\#$ satisfy

$$g^{\#1/p^k}(f_i^{1/p^k}/g^{1/p^k})^\# = (g^{1/p^k}(f_i^{1/p^k}/g^{1/p^k}))^\# = f_i^{1/p^k\#} = f_i^{\#1/p^k},$$

and $g^\#$ is a non-zero-divisor in $\widehat{C}^\#$ (since it divides f_n, thanks to the previous line with $i=n$ and $k=1$, which is a power of π). So there is a unique map of A-algebras $e : B \to \widehat{C}^\#[\frac{1}{\pi}]$; it sends $f_i^{\#1/p^k}/g^{\#1/p^k}$ to $(f_i^{1/p^k}/g^{1/p^k})^\#$ and has image in $\widehat{C}^\#$. Taking π-adic completion extends this map to $\widehat{e} : \widehat{B} \to \widehat{C}^\#$, and then we may invert π to obtain the desired map; conversely, since any continuous map of A-algebras $\widehat{B}[\frac{1}{\pi}] \to \widehat{C}^\#[\frac{1}{\pi}]$ is determined by its restriction to B, we have also proved uniqueness.

We next consider the composition

$$e\psi : A[\underline{X}^{1/p^\infty}]/J \longrightarrow B \longrightarrow \widehat{C}^\#,$$

whose reduction modulo π identifies with the reduction modulo $\pi^\flat$ of $\psi^\flat$. This latter map is an almost isomorphism by part (i), whence $e\psi$ is an almost isomorphism modulo π. So ψ is almost injective modulo π and surjective, whence it formally follows that it is almost injective (since its domain, resp. codomain, is π-adically separated, resp. π-torsion-free); this proves (ii).

Since $e\psi$ is an almost isomorphism modulo π and ψ is surjective, we also deduce that e is an almost isomorphism modulo π, hence also modulo any power of π by a trivial induction; taking the limit we deduce that $\widehat{e} : \widehat{B} \to \widehat{C}^\#$ is an almost isomorphism. But $\widehat{B}$ has no π-torsion, so the almost zero kernel is actually zero. This completes the proof of (iv).

(v): Suppose that B' is an integral perfectoid subring sandwiched between $\widehat{B}$ and $\widehat{C}^\#$; then B' contains $f_i^{\#1/p^k}/g^{\#1/p^k}$ for all $k \geq 0$, whence its tilt $B'^\flat$ (which is an integral perfectoid $A^\flat$-subalgebra of $\widehat{C}$) contains $f_i^{1/p^k}/g^{1/p^k}$ for all $k \geq 0$. But therefore $B'^\flat$ contains C, whence its completeness forces $B'^\flat = \widehat{C}$; the tilting correspondence now tells us that in fact $B' = \widehat{C}^\#$, as desired.

Moreover, Corollary 2.2(ii) now implies that

$$\widehat{C}^\# = \{f \in \widehat{C}^\# : f^{p^N} \in \widehat{B} + p\widehat{C}^\# \text{ for } N \gg 0\}.$$

But $\widehat{B} \subseteq \widehat{C}^\#$ is an almost equality, so $p\widehat{C}^\# \subseteq \widehat{B}$ and we deduce that any $x \in \widehat{C}^\#$ satisfies $x^{p^N} \in \widehat{B}$ for $N \gg 0$.

(vi): There is an obvious inclusion $A^\flat[f_i/g] \subseteq C$; we claim that the difference is killed by a power of $\pi^\flat$. Since f_n equals a power of $\pi^\flat$, it is enough to show that the difference is killed by f_n^n (the n both upstairs and downstairs is not a typo!); this follows from the nice observation that the fractional powers can be treated with the argument

$$f_n^n \prod_{i=1}^n \frac{f_i^{1/p^{k_i}}}{g^{1/p^{k_i}}} = \prod_{i=1}^n((f_i^{1/p^{k_i}} g_i^{1-1/p^{k_i}} \frac{f_n}{g}) \in A[f_i/g : i]$$

for any $k_1, \ldots, k_n \geq 0$. Having proved the claim, taking $\pi^\flat$-adic completions therefore yields an inclusion $\widehat{A^\flat[f_i/g]} \subseteq \widehat{C}$ such that $\widehat{A^\flat[f_i/g]} \supseteq \pi^{\flat N}\widehat{C}$ for $N \gg 0$ (this is the openness assertion); combined with the fact that C is clearly integral over $A^\flat[f_i/g]$, this shows that $\widehat{C}$ is integral over $\widehat{A^\flat[f_i/g]}$.

The verbatim argument works on the untilted side. □

Corollary 3.4. *Let $X := \mathrm{Spa}(R, R^+)$ and $X^\flat := \mathrm{Spa}(R^\flat, R^{+\flat})$ be as at the start of the section, and let $U \subseteq X^\flat$ be a rational subset. Then $V := \flat^{-1}(U)$ is a rational subset of X and the assertions of Theorem 3.2 are true in this case.*

Proof. Let $\pi \in R^+$ be a ppu with p-power roots, and pick elements $f_1, \ldots, f_n, g \in R^{+\flat}$, where f_n is a power of $\pi^\flat$, such that $U = X^\flat(\frac{f_1,\ldots,f_n}{g})$. As we already explained in the proof of Lemma 3.1, it follows that $V := \flat^{-1}(U)$ is the rational subset $X(\frac{f_1^\#,\ldots,f_n^\#}{g^\#})$.

Let $B, \widehat{B}, C, \widehat{C}$ be as in the statement of the previous proposition. Part (vi) of the lemma shows gives equality of Tate algebras $\widehat{C}[\frac{1}{\pi^\flat}] = \mathcal{O}_{X^\flat}(V)$ and $\widehat{B}[\frac{1}{\pi}] = \mathcal{O}_X(U)$. Therefore parts (iii) and (iv) of the previous proposition show that $\mathcal{O}_{X^\flat}(V)$ and $\mathcal{O}_X(V)$ are perfectoid Tate algebras over $R^\flat$ and R respectively, that there is a unique continuous map of R-algebras $\mathcal{O}_X(V) \to \mathcal{O}_{X^\flat}(U)^\#$, and that this map is an isomorphism. Identifying these perfectoid R-algebras, we then have

$$\begin{aligned}
\mathcal{O}_X^+(V) &= \text{integral closure of } \widehat{R^+[f_i^\#/g^\#]} \text{ in } \mathcal{O}_X(U) && \text{(by def.)}\\
&= \text{integral closure of } \widehat{B} \text{ in } \widehat{B}[\tfrac{1}{\pi}] && \text{(by (vi) of prev. prop.)}\\
&= \text{integral closure of } \widehat{C}^\# \text{ in } \widehat{C}^\#[\tfrac{1}{\pi}] && \text{(by (v) of prev. prop.)}\\
&= (\text{integral closure of } \widehat{C} \text{ in } \widehat{C}[\tfrac{1}{\pi^\flat}])^\# &&\\
&= (\text{integral closure of } R^{+\flat}[f_i/g] \text{ in } \mathcal{O}_{X^\flat}(U))^\# && \text{(by (vi) of prev. prop.)}\\
&= \mathcal{O}_{X^\flat}^+(U)^\# && \text{(by def.)}
\end{aligned}$$

□

The second key step is a subtle approximation argument which will imply that all rational subsets of X are obtained by pulling back rational subsets of $X^\flat$:

Proposition 3.5 (Approximation lemma)**.** *Fix a perfectoid Tate K (I hope that it does not need to be a field!) and set $R := K\langle \underline{T}^{1/p^\infty}\rangle$; let $f \in R^\circ$ be homogeneous of degree $d \in \mathbb{Z}[\frac{1}{p}]$*[8] *and fix rational numbers $c \geq 0$ and $\varepsilon > 0$. Then there exists $g_{c,\varepsilon} \in R^{\flat\circ}$ homogeneous of degree d such that, for all $x \in \mathrm{Spa}(R, R^\circ)$,*

$$|f(x) - g_{c,\varepsilon}^\#(x)| \leq |\pi|^{1-\varepsilon} \max(|f(x)|, |\pi|^c)$$

Proof. To do. □

The previous special case easily implies the approximation lemma in general:

Corollary 3.6. *R a perfectoid Tate algebra; let $f \in R$ and fix rational numbers $c \geq 0$ and $\varepsilon > 0$. Then there exists $g_{c,\varepsilon} \in R^\flat$ such that*

$$|f(x) - g_{c,\varepsilon}^\#(x)| \leq |\pi|^{1-\varepsilon} \max(|f(x)|, |\pi|^c)$$

for all $x \in \mathrm{Spa}(R, R^\circ)$.

[8] i.e., in the completion of $\bigoplus_{k_1,\ldots,k_n \in \mathbb{Z}[\frac{1}{p}] : \sum_i k_i = d} \mathcal{O}_K T_1^{k_1} \cdots T_n^{k_n}$.

Proof. To do. □

Corollary 3.7. *Any rational subset of X has the form $\flat^{-1}(U)$ for some rational subset U of $X^\flat$.*

Proof. Let $V \subseteq X$ be a rational subset; pick $f_1, \dots, f_n, g \in R^+$, where $f_n = \pi^N$ is a power of π, such that $U = X(\frac{f_1,\dots,f_n}{g})$. Then $U = \bigcap_{i=1}^{n-1} X(\frac{f_i,\pi^N}{g})$, so it is enough to show that each $X(\frac{f_i,\pi^N}{g})$ is the preimage of a rational subset of $X^\flat$ (note that rational subsets are closed under finite intersections).

By two applications of the approximation lemma (first with $f = f_i$, $c = N$, and any $\varepsilon \in (0,1)$; secondly with $f = g$, $c = N$, and $\varepsilon = 1$), there exist $a, b \in R^\flat$ such that

$$\max(|f(x)|, |\pi(x)^N|) = \max(|a^\#(x)|, |\pi(x)^N|)$$

$$|g(x) - b^\#(x)| \le \max(|g(x)|, |\pi(x)|^N).$$

Straightforward arguments with non-archimedean inequalities shows that $X(\frac{f,\pi^N}{g}) = X(\frac{a^\#,\pi^N}{b^\#})$, which is we already know is $\flat^{-1}(X^\flat(\frac{a,\pi^{\flat N}}{b}))$. □

We can now complete the proof of Theorem 3.2:

Proof of Theorem 3.2. We have proved that X has a basis of opens (namely the rational subsets) which are pull-backs from $X^\flat$; it follows from topology that $\flat : X \to X^\flat$ is injective (since X is T_0) and then that $\flat$ is a homeomorphism onto its image.

For surjecitivity argue through perfectoid valuation rings – to do, using the following lemma: □

Lemma 3.8 (Valuation rings)**.** *Recall that a valuation ring is a ring in which, given any two elements a, b, either $a|b$ or $b|a$. Let $\mathcal{O}$ be an integral perfectoid ring; then $\mathcal{O}$ is a valuation ring if and only if $\mathcal{O}^\flat$ is a valuation ring.*

Then $\mathcal{O}[\frac{1}{\pi}]$ is a field.

Proof. Since being a valuation ring is a monoid-theoretic property, it is easy to see that if $\mathcal{O}$ is a valuation ring then so is $\mathcal{O}^\flat = \varprojlim_{x\mapsto x^p} \mathcal{O}$.

Conversely, now assume that $\mathcal{O}^\flat$ is a valuation ring. Let $\pi \in \mathcal{O}$ be a ppu with p-power roots, and let $\pi^\flat \in \mathcal{O}^\flat$ be the usual corresponding element. We begin by proving that, given any element $a \in \mathcal{O}$, either $a|\pi$ or $\pi|a$. Let $\alpha \in \mathcal{O}^\flat$ be an element satisfying $\alpha^\# \equiv a \bmod \pi^p\mathcal{O}$, and assume that $a \notin \pi\mathcal{O}$ (else we are done); then $\alpha^\# \notin \pi\mathcal{O}$ and so $\alpha \notin \pi^\flat\mathcal{O}^\flat$. Since $\mathcal{O}^\flat$ is a valuation ring it follows that $\pi^\flat \in \alpha\mathcal{O}^\flat$, and then untilting implies that $\pi = \alpha^\# x^\#$ for some $x \in \mathcal{O}^\flat$. But $\alpha^\# = a + \pi^p y$ for some $y \in \mathcal{O}$, and so in conclusion $\pi = ax^\# + \pi^p y x^\#$, i.e., $\pi(1 + \pi^{p-1} y x^\#) \in a\mathcal{O}$; but the bracketed term is a unit by π-adic completeness and so $\pi \in a\mathcal{O}$, as desired.

Now let $a, b \in \mathcal{O}$ be arbitrary; we must prove that $a|b$ or $b|a$. After cancelling the same power of π from both a and b, we may assume that either $a \notin \pi\mathcal{O}$ or $b \notin \pi\mathcal{O}$; for concreteness wlog let's suppose $a \notin \pi\mathcal{O}$, whence $\pi \in a\mathcal{O}$ by the previous paragraph. If $b \in \pi\mathcal{O}$ then clearly $b \in a\mathcal{O}$ we are done, so suppose instead that also $b \notin \pi\mathcal{O}$, whence $\pi \in b\mathcal{O}$ by the previous paragraph.

We have reduced to the case in which π belongs to both $a\mathcal{O}$ and $b\mathcal{O}$. Since $\mathcal{O}/\pi\mathcal{O} = \mathcal{O}^\flat/\pi^\flat\mathcal{O}$ is a valuation ring, we can suppose wlog that b divides a modulo π, i.e., $a = bx + \pi y$ for some $x, y \in \mathcal{O}$. But $\pi \in b\mathcal{O}$ so this implies that $a \in b\mathcal{O}$ and completes the proof.

For the field assertion, note that if a is non-zero then $\pi^N \in a\mathcal{O}$ for $N \gg 0$. □