

THE FUNDAMENTAL EXACT SEQUENCES AND THE SCHEMATIC CURVE

1. THE SPACE Y^{ad} AND X^{ad}

1.1. Notation.

In this talk we continue with the following notation from previous talks:

- (1) Let C^b be an algebraically closed field of characteristic p , complete under $|\cdot|_b : C^b \rightarrow \mathbb{R}_{\geq 0}$.
- (2) Let $\varpi^b \in C^b$ be a quasi-uniformizer, $0 < |\varpi^b| < 1$.
- (3) Let $A_{\text{inf}} = W(\mathcal{O}_{C^b})$ be Fontaine's period ring A_{inf} .¹
- (4) $\text{Un}(C^b)$ denotes the set of characteristic 0 untilts of C^b .

1.2. Review.

We have introduced

$$Y^{ad} = Y_{C^b}^{ad} = \text{Spa}(A_{\text{inf}}, A_{\text{inf}}) \setminus V(p \cdot [\varpi^b])$$

which parametrizes all characteristic 0 untilts of C^b . As notation suggests, Y^{ad} is an adic space (though it is not affinoid and the fact the adic structural presheaf is a sheaf is nontrivial). The adic Fargues-Fontaine curve will be a certain quotient $X^{ad} = Y^{ad}/\varphi^{\mathbb{Z}}$ which parametrizes $\text{Un}(C^b)/\varphi^{\mathbb{Z}}$.

We have already seen the heuristic picture of Y^{ad} : it can be thought of as a punctured open disc, where $C \in \text{Un}(C^b) \mapsto |p|_C$ gives the radial “coordinate”. The origin of the disc can be thought of as “ $p = 0$ ” while the boundary of the disc is “ $[\varpi^b] = 0$ ”. The untilts of C^b (which in our setup where C^b is algebraically closed can be identified with classical points of Y^{ad}) tries to degenerate to the characteristic p untilt C^b as they tend to the origin of the disc.

Remark. In fact, the global functions on Y^{ad} (which we recall in next subsection) have appeared in p -adic Hodge theory before the “curve” is introduced, but it is Fargues and Fontaine's point of view that such functions can be regarded as “holomorphic functions with variable p ”.

1.3. The ring B .

The ring B that Jacob defined in the previous lecture should be regarded as global functions on Y^{ad} , namely $B = \mathcal{O}(Y^{ad})$. To construct B , we consider a sequence of rings $R(k)$:

$$A_{\text{inf}} \subset \cdots \subset R(k) \subset R(k-1) \cdots \subset B_0 = A_{\text{inf}}\left[\frac{1}{[\varpi^b]}, \frac{1}{p}\right]$$

where $R(k) = A_{\text{inf}}\left[\frac{p^k}{[\varpi^b]}, \frac{[\varpi^b]^k}{p}\right]$. Then we take the \mathbb{Q}_p Banach space

$$R(k)_{\mathbb{Q}_p}^{\wedge} := \varprojlim_m (R(k)/p^m R(k))\left[\frac{1}{p}\right]$$

¹In the book of Fargues and Fontaine, they consider more generally $A_{\text{inf}, E} = W_{\mathcal{O}_E}(\mathcal{O}_F)$ where E/\mathbb{Q}_p is totally ramified finite extension of \mathbb{Q}_p and F is a perfectoid field of characteristic p . In this talk, we let $E = \mathbb{Q}_p$ and $F = C^b$ algebraically closed.

where $R(k)^\wedge = \varprojlim (R(k)/p^m)$ is bounded open in $R(k)^\wedge_{\mathbb{Q}_p}$. Then we take

$$B = \varprojlim_k R(k)^\wedge_{\mathbb{Q}_p}.$$

This is the same as taking the completion of B_0 with respect to the seminorms determined by $\{R(k)\}_{k \geq 1}$, and hence a Frechet algebra.

Let us further recall the universal property of this construction: to give a continuous ring homomorphism from $B \rightarrow A$ where A is a \mathbb{Q}_p Banach algebra, it is the same as giving a homomorphism $R(k) \rightarrow A$ with bounded image for some large enough k .

Finally, recall that

$$P := \bigoplus_{d \geq 0} B^{\varphi=p^d}$$

and

$$X := \text{Proj} P$$

is the schematic curve.

1.4. The ring B^+ .

Now we introduce a variant of the ring B , denoted by B^+ . For this, consider ²

$$A_{\text{inf}} \subset \cdots \subset R(k)^+ \subset R(k-1)^+ \cdots \subset B_0^+ = A_{\text{inf}}\left[\frac{1}{p}\right]$$

with $R(k)^+ := A_{\text{inf}}\left[\frac{[\varpi^b]^k}{p}\right]$ replacing $R(k)$. Then take the inverse limit as before we arrive at

Definition 1.1.

$$B^+ := \varprojlim_k (R(k)^+)^\wedge_{\mathbb{Q}_p}.$$

Remark.

- (1) Consider $\text{Un}(C^b)$ as points in the punctured disk described above. Then functions in B_0 and B_0^+ have finitely many ‘‘poles’’ at the origin (since only a bounded power of p can appear in the denominator); but B^+ and B could have ‘‘essential singularities’’. ³
- (2) The difference between B and B^+ is that one should heuristically think of elements in B^+ as ‘‘power expansions in variable p ’’ where coefficients are ‘‘integral’’. Though this description does makes sense (recall the next remark), it can be explained later (and in future talks) when we discuss Newton polygons. Another heuristic to keep in mind is that, if we interpret $B = \mathcal{O}(\mathbb{D}_{(0,1)})$ as functions on the open punctured disc, then $B^+ = \mathcal{O}(\mathbb{D}_{(0,1]})$ are functions on the closed punctured disc.
- (3) Elements in B or B^+ might not have the form of power series expansions, and when they do have such expansions they might not be unique.

²The ring B_0 and B_0^+ are called $B^b = B_0$ and $B^{b,+} = B_0^+$ respectively in Fargues and Fontaine’s article.

³The geometric interpretation I gave in the lecture was incorrect.

1.5. Preview of newton polygons.

The newton polygons of elements in B_0 are what we expect: for $x = \sum_n [x_n]p^n \in B_0$,

$$\text{Newt}(x) := \text{decreasing convex hull of } \{(n, v_{\flat}(x_n))_{n \in \mathbb{Z}}\}$$

where v_{\flat} is the (fixed) valuation on $\mathcal{O}_{C^{\flat}}$. From definition, one can see that

- (1) For any $x \in B_0$, $\text{Newt}(x)$ is bounded from below and $\text{Newt}(x) = +\infty$ for $x \leq c$ for some c ;
- (2) $B_0^+ \subset B_0$ are precisely those x such that $\text{Newt}(x) \geq 0$.

The upshot is, despite part (3) of the remark above, there will be a way (somewhat indirect, using Legendre transfer) to define Newton polygons for all $x \in B$, which satisfy the following properties:

Lemma 1.2.

- (1) $B_0 = \{x \in B : \text{Newt}(x) \text{ is bounded from below, and } \exists c \text{ s.t. } \text{Newt}(x)|_{[\infty, c]} = +\infty\}$
- (2) $B^+ = \{x \in B : \text{Newt}(x) \geq 0\} \subset B$.
- (3) $B^{\times} = \{x \in B : \text{Newt}(x) \text{ has } 0 \text{ as its only finite slope.}\}$

Proof. The construction of $\text{Newt}(x)$ and the proof of the lemma will be a future talk. \square

We will only need the following corollary today:

Corollary 1.3.

- (1) $B^{\times} = (B_0)^{\times}$
- (2)

$$\begin{cases} B^{\varphi=1} = (B^+)^{\varphi=1} = \mathbb{Q}_p \\ B^{\varphi^h=p^d} = (B^+)^{\varphi^h=p^d} & \text{for } h \geq 1, d \geq 0 \\ B^{\varphi^h=p^d} = 0 & \text{if } h \geq 1, d < 0 \end{cases}$$

Sketch. (1) follows from the Lemma. For (2), if $x \in B$ is such that $\varphi(x) = p^d x$, then

$$p \cdot \text{Newt}(x)(T) = \text{Newt}(x)(T - d).$$

We prove the first assertion and leave others as exercises. Suppose $x \neq 0$, then $\varphi(x) = x$ implies $\text{Newt}(x) = 0$, so $x \in (B_0^{\times})^{\varphi=1}$, but $(B_0)^{\times} \subset B_0^+$, so we may write $x = \sum_{n \gg -\infty} [x_n]p^n$ where $x_n \in \mathcal{O}_{C^{\flat}}$.

Now from $\varphi(x) = x$ we can see that $x \in \mathbb{Q}_p$. \square

Another corollary that we will need is the following:

Corollary 1.4. *Suppose $t \in B^{\varphi=p}$ is a nonzero element, then*

$$(B^+[\frac{1}{t}])^{\varphi=1} = (B[\frac{1}{t}])^{\varphi=1}$$

Proof. This follows from part (2) of the previous corollary, since for all $t \geq 1$

$$(t^{-k} B^+)^{\varphi=1} = t^{-k} (B^+)^{\varphi=p^k} = t^{-k} (B)^{\varphi=p^k} = (t^{-k} B)^{\varphi=1}$$

Now take union on both sides for all $k \geq 1$. \square

2. DIVISORS

2.1. Review of B_{dR}^+ .

Let $C \in \text{Un}(C^b)$ be an untilt, recall that we have the surjective map

$$\theta_C : A_{\text{inf}} \rightarrow \mathcal{O}_C$$

given by

$$\sum p^n [x_n] \mapsto \sum p^n x_n^\sharp.$$

$\ker(\theta_C)$ is principally generated, by any element $\alpha \in \ker(\theta_C)$ such that $\bar{\alpha} \in \mathcal{O}_{C^b}$ has absolute value $|\bar{\alpha}|_b = |p|_C$. There are many choices of this generator, we will choose

$$\xi = \frac{[\epsilon] - 1}{[\epsilon]^{1/p} - 1} = \sum_{i=0}^{p-1} [\epsilon]^{i/p}$$

where $\epsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}_{C^b}$. Then we form the discrete valuation ring

$$B_{dR}^+ = B_{dR,C}^+ := \varprojlim_n A_{\text{inf}}[\frac{1}{p}] / (\ker \theta_C[\frac{1}{p}])^n$$

by ξ -adically completing $A_{\text{inf}}[\frac{1}{p}]$. As usual, let $t = \log[\epsilon]$, which is a uniformizer for B_{dR}^+ .

2.2. Points on Y^{ad} .**Definition 2.1.**

- (1) $x = \sum_{n \geq 0} [x_n] p^n \in A_{\text{inf}}$ is *primitive* if $x_0 \neq 0$ and there exists k such that $x_k \in \mathcal{O}_{C^b}^\times$.
- (2) If $x = \sum_{n \geq 0} [x_n] p^n \in A_{\text{inf}}$ is primitive, then $\deg(x) := \min\{k : x_k \in \mathcal{O}_{C^b}^\times\}$.
- (3) A primitive element x is *irreducible* if $\deg(x) \geq 1$ and x cannot be written as a product of two primitive elements of strictly less degrees.

Note that primitive elements of $\deg 0$ are $(A_{\text{inf}})^\times$, and we define $|Y|$ as the set

$$|Y| := \{\text{primitive irreducible elements of } A_{\text{inf}}\} / (A_{\text{inf}})^\times$$

Inside $|Y|$, we have $|Y|^{\deg=1}$ consisting of degree 1 elements.

Let $\mathfrak{p} = (x) \subset A_{\text{inf}}$ be a principal ideal generated by a deg 1 element, then $A_{\text{inf}}/\mathfrak{p}$ is an untilt of \mathcal{O}_{C^b} . Conversely, if $C \in C^b$, then $\ker(\theta_C) = ([p^b] - p)$ for some degree 1 element $([p^b] - p) \in A_{\text{inf}}$.⁴

Moreover, we have

Theorem 2.2 (Fargues-Fontaine). *Suppose C^b is algebraically closed, then primitive irreducible elements have degree 1. Namely $|Y|^{\deg=1} = |Y|$.*

Remark. This is really a factorization statement, namely, for any $x \in A_{\text{inf}}$ primitive of degree d , one can write (in a non-unique(!) way):

$$x = u(p - [a_1]) \cdots (p - [a_d])$$

where $a_i \in \mathcal{O}_{C^b}$ and $a_i \neq 0$.

⁴The choice of $[p^b]$ in this expression is not unique.

To summarize, we now have

$$\mathrm{Un}(C^b) = |Y|^{\mathrm{deg}=1} = |Y|.$$

Remark. Let $|X|$ be the set of φ -orbits in $|Y|^{\mathrm{deg}=1} = |Y|$, which can be identified with $\mathrm{Un}(C^b)/\varphi^{\mathbb{Z}}$. Later we will see that these are precisely the closed points of the schematic curve X , so the notation $|X|$ is justified.

2.3. Divisors on Y^{ad} .

Let $C = C_{\mathbf{m}} \in \mathrm{Un}(C^b)$ be an untilt (corresponding to the point $\mathbf{m} \in |Y|$), then the map $A_{\mathrm{inf}} \rightarrow \mathcal{O}_C \rightarrow C$ sends $[\varpi^b]$ to $\varpi := (\varpi^b)^{\sharp}$ a quasi-uniformizer of C , therefore for large enough k , the image of $R(k)$ is a bounded subalgebra of C . By the universal property of B , $\theta_C[\frac{1}{p}] : A_{\mathrm{inf}}[\frac{1}{p}] \rightarrow C$ extends to $A_{\mathrm{inf}}[\frac{1}{p}] \rightarrow B \xrightarrow{\tilde{\theta}_C} C$.

Remark. $\ker(\tilde{\theta}_C)$ is again principal and can be generated by any generator of $\ker(\theta_C)$, and if we complete B along $\ker(\tilde{\theta}_C)$ we get $B_{\mathrm{dR},\mathbf{m}}^+$, which is (canonically) isomorphic to $B_{\mathrm{dR},C}^+$.

In particular, we can view $\mathrm{Un}(C^b)$ as a subset of the maximal ideals of B .⁵ Moreover, for each $\mathbf{m} \in |Y|$, we have $\mathrm{ord}_{\mathbf{m}} : B_{\mathrm{dR},\mathbf{m}}^+ \rightarrow \mathbb{N} \cup \{\infty\}$.

Definition 2.3. Define the set (or monoid) of effective divisors of Y^{ad} to be

$$\mathrm{Div}^+(Y^{ad}) = \left\{ \sum_{\mathbf{m} \in |Y|} a_{\mathbf{m}}[\mathbf{m}] \mid \forall \text{ cpt } I \subset (0, 1), \{\mathbf{m} \in |Y| : a_{\mathbf{m}} \neq 0\} \text{ is finite} \right\}$$

where $|Y_I| = \{\mathbf{m} : |C_{\mathbf{m}}| \in I\} \subset |Y|$ is the closed annulus consisting of points of radius $r \in I$ inside the punctured open disc $|Y|$.

Namely for a divisor $D \in \mathrm{Div}^+(Y^{ad})$, we allow only finitely many nonzero terms on any closed annulus, but they could accumulate on the boundary (or origin). Now for $f \in B \setminus \{0\}$, define

$$\mathrm{div}(f) = \sum_{\mathbf{m} \in |Y|} \mathrm{ord}_{\mathbf{m}}(f)[\mathbf{m}]$$

It remains to be justified⁶ in a future lecture that

$$\mathrm{div}(f) \in \mathrm{Div}^+(Y^{ad}).$$

Comparing zeroes of functions in B gives a way to factorize in B . More precisely, we state the following theorem, which we need as a blackbox in this talk:

Theorem 2.4 (Blackbox 1). *Let x, y be two nonzero elements of B , then $x = yz$ for some $z \in B$ if and only if $\mathrm{div}(x) \geq \mathrm{div}(y)$.*

⁵In a latter draft, I will try to explain closed maximal ideals of B .

⁶It is clear that f only has “zeroes” and no poles on $|Y|$, what we need to prove is that f has only finitely many zeroes on each $|Y_I|$

2.4. Divisors on X^{ad} .

Similarly, we define

$$\mathrm{Div}^+(X^{ad}) := \{D \in \mathrm{Div}^+(Y^{ad}) \mid \varphi^* D = D\}.$$

Remark.

(1) Naturally, we have

$$\mathrm{Un}(C^b)/\varphi^{\mathbb{Z}} = |X| = |Y|/\varphi \hookrightarrow \mathrm{Div}^+(X^{ad})$$

$$C_{\mathfrak{m}} \longleftrightarrow \mathfrak{m} \longmapsto \sum_{n \in \mathbb{Z}} [\varphi^n(\mathfrak{m})]$$

(2) If $f \neq 0 \in B^{\varphi=p^d}$ where $d \geq 0$, then $\mathrm{div}(f) \in \mathrm{Div}^+(X^{ad})$, so we have a morphism of monoids:

$$\mathrm{div} : \left(\bigcup_{d \geq 0} B^{\varphi=p^d} \setminus \{0\} \right) / \mathbb{Q}_p^\times \rightarrow \mathrm{Div}^+(X^{ad})$$

Proposition 2.5. *The morphism*

$$\mathrm{div} : \left(\bigcup_{d \geq 0} B^{\varphi=p^d} \setminus \{0\} \right) / \mathbb{Q}_p^\times \xrightarrow{\sim} \mathrm{Div}^+(X^{ad})$$

is an isomorphism.

Proof. Injectivity follows from our blackbox theorem 1 (2.4), if $x \in B^{\varphi=p^d}$ and $y \in B^{\varphi=p^e}$ such that $\mathrm{div}(x) = \mathrm{div}(y)$ then $y = ux$ where $u \in B^\times$ is a unit. Moreover $u \in (B^\times)^{\varphi=p^{d-e}}$, by corollary 1.3, $(B^\times)^{\varphi=p^k}$ is either \emptyset (if $d \neq e$) or \mathbb{Q}_p^\times (if $d = e$), hence injectivity follows. Surjectivity is not too difficult, but it was omitted in the talk. I will insert a proof for a second draft. \square

Corollary 2.6. *(Assuming C^b algebraically closed). The graded algebra $P := \bigoplus_{d \geq 0} B^{\varphi=p^d}$ is graded factorial, more precisely, if $x \in P_d := B^{\varphi=p^d}$ is nonzero, then there exists a unique (up to \mathbb{Q}_p^\times scaling) factorization*

$$x = t_1 \cdots t_d$$

where $t_i \in P_1 = B^{\varphi=p}$.

Proof. This is clear by induction on d . For $x \in P_d$ nonzero, take any point $[\mathfrak{m}] \in |Y|$ in the support of $\mathrm{div}(x)$, then $\mathrm{div}(x) \geq \sum_{n \in \mathbb{Z}} [\varphi^n \mathfrak{m}] = \mathrm{div} t$ for some $t \in P_1$ (unique up to \mathbb{Q}_p^\times), now by blackbox theorem 1 (2.4), $x = y \cdot t$ where $y \in P_{d-1}$. \square

3. THE FUNDAMENTAL EXACT SEQUENCE

Our first major goal is to show that $X = \mathrm{Proj} P$ is a ‘‘curve’’, namely a noetherian integral scheme of dimension 1. The key ingredient is the following fundamental exact sequence in p -adic Hodge theory.

3.1. The statement.

Theorem 3.1. *Let $t \in B^{\varphi=p}$ be a nonzero element, corresponding to the divisor $\text{div}(t) = \sum_{n \in \mathbb{Z}} [\varphi^n \mathfrak{m}]$ for some point $\mathfrak{m} \in |Y|$, then we have the following exact sequence*

$$0 \rightarrow B^{\varphi=p^n} t^k \rightarrow B^{\varphi=p^{n+k}} \rightarrow B_{dR, \mathfrak{m}}^+ / \mathfrak{m}^k B_{dR, \mathfrak{m}}^+$$

Remark.

- (1) In the statement we do not distinguish the point $\mathfrak{m} \in |Y|$ and the maximal ideal $\mathfrak{m} = \ker(B \rightarrow C_{\mathfrak{m}})$
- (2) $\text{div}(t) = \sum_n [\varphi^n \mathfrak{m}]$, so there \mathbb{Z} -ambiguity of choices for \mathfrak{m} . The ring $B_{dR, \mathfrak{m}}^+$ and the map $B \rightarrow B_{dR, \mathfrak{m}}^+$ both depend on \mathfrak{m} . It does not matter which one we choose.

Remark. More generally, Fargues and Fontaine prove that, for t_1, \dots, t_r such that $t_i \notin \mathbb{Q}_p t_j$ and $a_1, \dots, a_r \geq 1$, we have

$$0 \rightarrow B^{\varphi=p^n} \prod_{i=1}^r t_i^{a_i} \rightarrow B^{\varphi=p^{n+k}} \rightarrow \prod_{i=1}^r B_{dR, \mathfrak{m}}^+ / \mathfrak{m}_i^{a_i} B_{dR, \mathfrak{m}}^+$$

We immediately have the following corollary, which is almost what we want to prove:

Corollary 3.2. *Let $t \in B^{\varphi=p}$ be a nonzero element, then $(P[\frac{1}{t}])_{(0)} = (B[\frac{1}{t}])^{\varphi=1}$ is a PID.*

Proof. We know already that $(B[\frac{1}{t}])^{\varphi=1}$ is factorial, so it suffices to show that each irreducible element $t'/t \in (B[\frac{1}{t}])^{\varphi=1}$ where $t' \notin \mathbb{Q}_p t$ generates a maximal ideal. Take the fundamental exact sequence for t' and divide by t we get

$$0 \rightarrow \mathbb{Q}_p \frac{t'}{t} \rightarrow (t^{-1}B)^{\varphi=1} \rightarrow B_{dR, \mathfrak{m}'}^+ / t' B_{dR, \mathfrak{m}'}^+ = C_{\mathfrak{m}'} \rightarrow 0$$

The surjective map $(t^{-1}B)^{\varphi=1} \rightarrow C_{\mathfrak{m}'}$ factors through

$$(t^{-1}B)^{\varphi=1} \rightarrow (B[\frac{1}{t}])^{\varphi=1} \hookrightarrow \bigcup (t^{-k}B)^{\varphi=1} \rightarrow C_{\mathfrak{m}'},$$

therefore we get a surjection $(B[\frac{1}{t}])^{\varphi=1} \rightarrow C_{\mathfrak{m}'}$ whose kernel is generated by t'/t : suppose b/t^n lies in this kernel, then applying the fundamental exact sequence again we know that $b = b't'$ where $b' \in (B)^{\varphi=p^{n-1}}$, so $b/t^n = (t'/t)(b'/t^{n-1})$. \square

3.2. A digression to p -adic Hodge theory.

Before we prove the result above, let us recall the classical fundamental exact sequence in p -adic Hodge theory and a variant. For this we fix an untilt C of C^{\flat} and retain notations from 2.1.

First recall Fontaine's period rings.

$$A_{\text{cris}} := \left(A_{\text{inf}} \left[\frac{\xi^k}{k!} \right]_{k \geq 1} \right)_{(p)}^{\wedge}$$

is the p -adic completion of the PD envelope of A_{inf} with respect to $\ker(\theta : A_{\text{inf}} \rightarrow \mathcal{O}_C)$. This is in fact a complicated ring, which has the following properties:

(1) There exists a commutative diagram

$$\begin{array}{ccc} A_{\text{inf}} \left[\frac{\xi^k}{k!} \right]_{k \geq 1} & \longrightarrow & A_{\text{inf}} \left[\frac{1}{p} \right] \\ \downarrow & & \downarrow \\ A_{\text{cris}} & \xrightarrow{j} & B_{\text{dR}}^+ \end{array}$$

(2) The map j above is injective (which is a nontrivial fact), in particular, A_{cris} is a domain.

(3) The image of j has the following description:

$$A_{\text{cris}} \xrightarrow{\sim} \text{im}(j) = \left\{ \sum_{n \geq 0} a_n \frac{\xi^n}{n!} \mid a_n \in A_{\text{inf}}, a_n \rightarrow 0 \text{ } p\text{-adically} \right\}$$

[Warning: an element in $A_{\text{cris}} \cong \text{im}(j)$ may not have a unique such expansion.]

One can easily check that $t \in A_{\text{cris}}$ and more generally $\alpha^k/k! \in A_{\text{cris}}$ for any $\alpha \in \ker(\theta_C)$ and $k \geq 1$.⁷ Finally we make the following definitions

$$B_{\text{cris}}^+ = A_{\text{cris}} \left[\frac{1}{p} \right], \quad B_{\text{cris}} = B_{\text{cris}}^+ \left[\frac{1}{t} \right]$$

The point here is that we have created $B_{\text{cris}} \subset B_{\text{dR}}$ which admits a Frobenius endomorphism φ , extending the Frobenius on $A_{\text{inf}} \left[\frac{1}{p} \right]$. One first shows that $A_{\text{inf}} \left[\frac{\xi^k}{k!} \right]_{k \geq 1}$ is stable under the Frobenius of $A_{\text{inf}} \left[\frac{1}{p} \right]$, which then extends uniquely by continuity to A_{cris} and B_{cris}^+ . With some care⁸ it is not difficult to check that $\varphi(t) = pt$, therefore the Frobenius on B_{cris}^+ uniquely extends to B_{cris} .

Remark (Some further properties of A_{cris}).

(1) $\varphi : A_{\text{cris}} \rightarrow A_{\text{cris}}$ is injective but not surjective.

(2) The filtration on B_{cris} is given by $\text{Fil}^i B_{\text{cris}} = \text{Fil}^i B_{\text{dR}} \cap B_{\text{cris}}$, note that $\text{Fil}^0 B_{\text{cris}} \supsetneq B_{\text{cris}}^+$.

For example, the element $\frac{[\epsilon]^{1/p^2} - 1}{[\epsilon]^{1/p} - 1} \in \text{Fil}^0 B_{\text{cris}}$ does not lie in B_{cris}^+ .

Now we state the fundamental exact sequences for B_{cris} :

Theorem 3.3.

(1)

$$0 \rightarrow \mathbb{Q}_p t^k \rightarrow (B_{\text{cris}}^+)^{\varphi=p^k} \rightarrow B_{\text{dR},C}^+ / t^k B_{\text{dR},C}^+ \rightarrow 0$$

(2)

$$0 \rightarrow \mathbb{Q}_p \rightarrow \text{Fil}^0 B_{\text{cris}} \xrightarrow{\varphi-1} B_{\text{cris}} \rightarrow 0$$

Remark.

(1) By dividing t^k in the first short exact sequence and let $k \rightarrow \infty$, we easily obtain

$$0 \rightarrow \mathbb{Q}_p \rightarrow B_{\text{cris}}^{\varphi=1} \rightarrow B_{\text{dR},C} / B_{\text{dR},C}^+ \rightarrow 0$$

⁷more precisely $t^k/k! \in \text{im}(j) \subset B_{\text{dR}}^+$, but from now on we will not distinguish these two descriptions of A_{cris} .

⁸this is not surprising since one would expect “ $\varphi \log[\epsilon] = \log[\epsilon^p] = p \log[\epsilon]$ ”, but some additional work is needed to make sense of this.

- (2) The exactness in the middle of the second exact sequence formally follows from the exactness (in the middle) of the sequence above:

$$(\mathrm{Fil}^0 \mathbf{B}_{\mathrm{cris}})^{\varphi=1} = (\mathbf{B}_{\mathrm{cris}})^{\varphi=1} \cap \mathbf{B}_{\mathrm{dR}}^+ = \ker(\mathbf{B}_{\mathrm{cris}} \xrightarrow{\varphi-1} \mathbf{B}_{\mathrm{dR}} / \mathbf{B}_{\mathrm{dR}}^+) = \mathbb{Q}_p.$$

- (3) The crucial observation here is that we can extract \mathbb{Q}_p from $\mathbf{B}_{\mathrm{cris}}$ via its filtration and φ action, and this is the key to prove the Dieudonné functor

$$\mathrm{D}_{\mathrm{cris}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_K) \xrightarrow{((K \otimes_{K_0} \mathbf{B}_{\mathrm{cris}}) \otimes -)^{G_K}} \mathrm{MF}_K^\phi = \{\text{Filtered } \phi\text{-modules}\} / K$$

is *fully* faithful.

3.3. A variant of $\mathbf{B}_{\mathrm{cris}}$.

Due to the technical nature of the ring $\mathbf{B}_{\mathrm{cris}}$, in this talk we prove something much less technical to give some ideas of how the arguments work⁹, namely we introduce a variant of $\mathbf{B}_{\mathrm{cris}}$.

$$\mathbf{A}_{\mathrm{max}} := \left\{ \sum_{k \geq 0} a_k \frac{\xi^k}{p^k} \mid a_k \in \mathbf{A}_{\mathrm{inf}}, a_k \rightarrow 0 \text{ } p\text{-adically} \right\}$$

$$\mathbf{B}_{\mathrm{max}}^+ = \mathbf{A}_{\mathrm{max}}\left[\frac{1}{p}\right], \quad \mathbf{B}_{\mathrm{max}} = \mathbf{B}_{\mathrm{max}}^+\left[\frac{1}{t}\right]$$

$\mathbf{A}_{\mathrm{max}} \subset \mathbf{A}_{\mathrm{inf}}\left[\frac{1}{p}\right]$ is stable under Frobenius.

First we state a lemma which replaces our blackbox theorem in the case of $\mathbf{B}_{\mathrm{max}}$

Lemma 3.4.

- (1) If $x \in \mathbf{A}_{\mathrm{max}} \subset \mathbf{B}_{\mathrm{dR}}^+$ is such that $\theta(\varphi^n(x)) = 0$ for all $n \geq 0$, then $\varphi(x) = \frac{[\epsilon]-1}{p} \cdot a$ for some $a \in \mathbf{A}_{\mathrm{max}}$.
- (2) If $x \in \mathbf{B}_{\mathrm{max}}^+$ is an element such that $\theta(\varphi^n(x)) = 0$ for all $n \geq 0$, then $t|\varphi(x)$ in $\mathbf{B}_{\mathrm{max}}^+$
- (3) $(\mathbf{B}_{\mathrm{max}}^+)^{\varphi=1} = \mathbb{Q}_p$.

Proof.

- (1). Note that in $\mathbf{A}_{\mathrm{max}}$,

$$\varphi\left(\frac{\xi}{p}\right) = 1 + \sum_{i=0}^{p-1} \frac{[\epsilon^i] - 1}{p} \equiv 1 \pmod{\left(\frac{[\epsilon^i] - 1}{p}\right)}$$

Now write $x = \sum_{i \geq 0} a_i \frac{\xi^i}{p^i}$ where $a_i \in \mathbf{A}_{\mathrm{inf}}$, then the above implies

$$\varphi^n(x) \equiv \varphi^n\left(\sum_{i \geq 0} a_i\right) \pmod{\left(\frac{[\epsilon^i] - 1}{p}\right)}$$

which implies that

$$\varphi^n(x) = \varphi^n\left(\sum_{i \geq 0} a_i\right) + \frac{[\epsilon^i] - 1}{p} \cdot b_n$$

⁹The ring $\mathbf{B}_{\mathrm{max}}$ has a nicer topology and the Frobenius action is easier to describe. It gives the same notion of admissibility as $\mathbf{B}_{\mathrm{cris}}$. For arguments of the corresponding statements for $\mathbf{B}_{\mathrm{cris}}$, see Fontaine's original paper and his asterisque article

where $b_n \in A_{\max}$. By assumption,

$$\theta(\varphi^n(\sum_{i \geq 0} a_i)) = 0 \quad \text{for all } n \geq 0.$$

Now the lemma follows from the following exercise ¹⁰:

Exercise. If $\alpha \in A_{\inf}$ is such that $\varphi^n(\alpha) \in \ker(\theta)$ for all $n \geq 0$, then $\varphi(\alpha) \in ([\epsilon] - 1)A_{\inf}$.

(2). Without loss of generality assume that $x \in A_{\max}$, which means $([\epsilon] - 1)|\varphi(x)$ in B_{\max}^+ , but $([\epsilon] - 1)/t$ is a unit since $t/([\epsilon] - 1) = 1 + \sum_{n \geq 2} \frac{(1-[\epsilon])^{n-1}}{n}$.

(3). This is the same as $(A_{\max})^{\varphi=1} = \mathbb{Z}_p$, which is an exercise. \square

Now we are ready to prove

Theorem 3.5. *There is a short exact sequence:*

$$0 \rightarrow \mathbb{Q}_p t^k \rightarrow (B_{\max}^+)^{\varphi=p^k} \rightarrow B_{dR}^+/t^k B_{dR}^+ \rightarrow 0.$$

Proof.

(1). We first prove the exactness in the middle by induction on k . The case $k = 0$ simply says $(B_{\max}^+)^{\varphi=1} = \mathbb{Q}_p$. Now suppose $x \in (B_{\max}^+)^{\varphi=p^k} \cap t^k B_{dR}^+$, therefore,

$$\theta(\varphi^n(x)) = \theta(p^{nk}x) = 0$$

By lemma 3.4, $t|\varphi(x) = p^k x$ in B_{\max}^+ , therefore $x = t \cdot y$ where $y \in (B_{\max}^+)^{\varphi=p^{k-1}} \cap t^{k-1} B_{dR}^+$, so $y \in \mathbb{Q}_p t^{k-1}$ by induction.

Remark. The upshot here is that we want to factor x as $y \cdot t$, and we can check that x contains t as a factor by showing that $\varphi^n(x)$ vanishes in \mathcal{O}_C , i.e., “ x has zeroes at $\varphi^n(\mathfrak{m}_C)$ ”, this is precisely why we need the blackbox theorem for the ring B .

(2). The surjectivity was essentially proven last time by Jacob. We claim that the following diagram (of sets) commutes

$$\begin{array}{ccc} 1 + \mathfrak{m}_{C^b} & \xrightarrow{\log[\cdot]} & B_{dR}^+ \\ \downarrow & & \downarrow \theta \\ 1 + \mathfrak{m}_C & \xrightarrow{\log} & C \end{array}$$

where on the left, the map is given by $x \mapsto x^\sharp$. First we check that if $x = 1 + y \in \mathfrak{m}_{C^b}$, namely $y = (y_{(0)}, y_{(1)}, \dots) \in \mathfrak{m}_{C^b}$ (here lower indices means each $y_{(i)} \in \mathcal{O}_C/p$ and $y_{(i+1)}^p = y_{(i)}$), then

$$(1 + y)^\sharp = \lim_{n \rightarrow \infty} (1 + \tilde{y}_{(n)})^{p^n} \in 1 + \mathfrak{m}_C.$$

Remark. (1) To check commutativity, we need to justify

$$\theta(\log[x]) = \theta\left(\sum_{n \geq 1} \frac{([x] - 1)^n}{n}\right) = \sum_{n \geq 1} \frac{(x^\sharp - 1)^n}{n} = \log(x^\sharp).$$

We need to make sense of $\log[x]$, namely it converges to an element. For this we need the assumption $|x - 1|_b < 1$ and the convergence is shown in the previous lecture.

¹⁰for example, one get this by looking at Fontaine’s I^r filtration on A_{\inf}

- (2) Last time we have shown already that $1 + \mathfrak{m}_{C^b} \xrightarrow{\theta(\log[\cdot])} C$ is surjective, this is equivalent to $\log : 1 + \mathfrak{m}_C \rightarrow C$ being surjective. This is a standard fact in p -adic analysis, usually proven by looking at Newton polygons.

Now we induct on k to prove the surjectivity of

$$(\mathbb{B}_{\max}^+)^{\varphi=p^k} \rightarrow \mathbb{B}_{\mathrm{dR}}^+ / t^k \mathbb{B}_{\mathrm{dR}}^+.$$

Let $x \in \mathbb{B}_{\mathrm{dR}}^+$ and let $\alpha \in 1 + \mathfrak{m}_{C^b}$ be an element such that $\theta(\log[\alpha])^k = \theta(x)$. It is easy to check (similar to showing that $\log[\cdot]$ converges) that for $z \in 1 + \mathfrak{m}_{C^b}$, $\log[z] \in \mathbb{B}_{\max}^+$. Furthermore, since $\varphi(\log[\alpha]) = p \log[\alpha]$, we know that

$$(\log[\alpha])^k \in (\mathbb{B}_{\max}^+)^{\varphi=p^k}.$$

Now since $x - (\log[\alpha])^k \in \ker(\theta)$ we conclude that $x = t \cdot y + (\log[\alpha])^k$ where $y \in \mathbb{B}_{\mathrm{dR}}^+$, by induction $y = t^{k-1}z + b$ where $z \in \mathbb{B}_{\mathrm{dR}}^+$ and $(\mathbb{B}_{\max}^+)^{\varphi=p^{k-1}}$, therefore,

$$x = t^k \cdot z + t \cdot b + (\log[\alpha])^k \in t^k \mathbb{B}_{\mathrm{dR}}^+ + (\mathbb{B}_{\max}^+)^{\varphi=p^k}.$$

□

3.4. Back to the curve.

Now we prove the fundamental exact sequence for B (assuming blackbox 2.4).

Proof of Theorem 3.1. We prove the exactness of the following sequence from which the general case follows (for example see the proof of 3.5).

$$0 \rightarrow B^{\varphi=1}t \rightarrow B^{\varphi=p} \rightarrow C_{\mathfrak{m}} \rightarrow 0$$

(1). For the exactness in the middle, we need to factor t out in the ring B . Suppose $x \in \mathfrak{m}B_{\mathrm{dR},\mathfrak{m}}^+$, which is another way to say that $\mathrm{div}(x) \geq [\mathfrak{m}]$. Since $\mathrm{div}(x)$ is invariant under φ , we know that

$$\mathrm{div}(x) \geq \sum_{n \in \mathbb{Z}} [\varphi^n(\mathfrak{m})] = \mathrm{div}(t)$$

Now by theorem 2.4 (this is where we use the blackbox theorem) we know that $x = y \cdot t$ where $y \in B^{\varphi=1}$.

(2). In the previous lecture, we established the following commutative diagram

$$\begin{array}{ccc} 1 + \mathfrak{m}_{C^b} & \xrightarrow[\cong]{\log[\cdot]} & (\mathbb{B})^{\varphi=p} \\ \downarrow & \searrow & \downarrow \theta \\ 1 + \mathfrak{m} & \xrightarrow{\log} & C_{\mathfrak{m}} \end{array}$$

The surjectivity of $\log([\cdot])$ (or the surjectivity of $\log : 1 + \mathfrak{m} \rightarrow C_{\mathfrak{m}}$ which is more classical) forces $\theta : (\mathbb{B})^{\varphi=p} \rightarrow C_{\mathfrak{m}}$ to be surjective. □

4. THE SCHEMATIC CURVE

The goal of this subsection is to show that

- (1) The nonzero element $t \in B^{\varphi=p}$ correspond to a point in X , with local ring $B_{\text{dR},\mathfrak{m}}^+$.
- (2) There is an isomorphism $(B[\frac{1}{t}])^{\varphi=1} \xrightarrow{\sim} (B_{\text{cris}})^{\varphi=1}$. Note that this means X can be obtained by gluing $\text{Spec}(B_{\text{cris}})^{\varphi=1}$ and $\text{Spec} B_{\text{dR}}^+$ along $\text{Spec} B_{\text{dR}}$, as stated in the first lecture by Jacob.

4.1. Local ring at t .

Now we have shown that away from $t \in B^{\varphi=p}$, the affine scheme $(B[\frac{1}{t}])^{\varphi=1}$ is a PID, and it follows that X is a regular noetherian integral scheme of dimension 1. We can say a little more about the local ring at t .

Exercise. $V(t) = \text{Proj}(P/tP)$ consists of a closed point in X .

Lemma 4.1. *Let $\mathcal{O}_{X,t}$ be the local ring at the point defined by t , then there is an isomorphism (depending on the choice of \mathfrak{m}):*

$$\widehat{\mathcal{O}}_{X,t} \xrightarrow{\sim} B_{\text{dR},\mathfrak{m}}^+$$

Proof.

$$\mathcal{O}_{X,t} = \left\{ \frac{x}{y} \in \text{Frac}(P) : x \in P_d = B^{\varphi=p^d}, y \in P_d \setminus tP_{d-1}, \text{ where } d \geq 1 \right\}$$

with uniformizer of the form t/t' where $t' \in P_1 \setminus \mathbb{Q}_p t$. Consider an element $x/y \in \mathcal{O}_{X,t}$, where $y \in P_d \setminus tP_{d-1}$, by the fundamental exact sequence

$$0 \rightarrow P_{d-1}t \rightarrow P_d \rightarrow B_{\text{dR},\mathfrak{m}}^+ / \mathfrak{m}B_{\text{dR},\mathfrak{m}}^+ \rightarrow 0$$

we know that $y \in (B_{\text{dR},\mathfrak{m}}^+)^{\times}$, so $x/y \in B_{\text{dR},\mathfrak{m}}^+$. This gives a map

$$\mathcal{O}_{X,t} \rightarrow B_{\text{dR},\mathfrak{m}}^+.$$

Note that, any uniformizer t/t' of $\mathcal{O}_{X,t}$ is a uniformizer of $B_{\text{dR},\mathfrak{m}}^+$ since $t' \notin \mathfrak{m}B_{\text{dR},\mathfrak{m}}^+$, and it induces an isomorphism on residue fields, so after completion we get the lemma. \square

Finally, let us summarize what we have shown so far (modulo the theory of Newton polygons and the blackboxed theorem):

- (1) We have shown that X is a regular noetherian integral scheme of $\dim = 1$.
- (2) For a closed point $V(t)$ defined by $t \in P_1 \setminus \{0\}$ in X , the completed local ring is (non-canonically) isomorphic to B_{dR}^+ ; removing $V(t)$ we get an affine scheme $(B[\frac{1}{t}])^{\varphi=1}$ which is a PID.
- (3) It is not hard to show that $\mathbb{Q}_p^{\times} t \mapsto V(t)$ gives a bijection between

$$(P_1 \setminus \{0\}) / \mathbb{Q}_p^{\times} \rightarrow \{\text{closed points in } X\}.$$

Combined with 2.5 and the discussions above it, we have identifications

$$\text{Un}(C^b) / \varphi^{\mathbb{Z}} = |X| = (P_1 \setminus \{0\}) / \mathbb{Q}_p^{\times} = \{\text{closed points in } X\},$$

where $|X| = |Y| / \varphi^{\mathbb{Z}}$ by definition is the φ -orbits of (degree 1 primitive elements) $|Y|$.

4.2. $(\mathbf{B}_{\text{cris}})^{\varphi=1}$.

Finally we connect this talk with lecture 1.

Consider the quasi-uniformizer ϖ^b , and let $\varpi = (\varpi^b)^\sharp \in \mathcal{O}_C$, note that $k \geq 1$, one has

$$\frac{([\varpi^b] - \varpi)^k}{k!} \in \mathbf{A}_{\text{cris}}$$

In particular, for k large enough, $[\varpi^b]^k \in p \mathbf{A}_{\text{cris}}$. From the discussion in 3.2, we have a natural map $\mathbf{A}_{\text{inf}}[\frac{1}{p}] \hookrightarrow \mathbf{B}_{\text{cris}}^+$, and for k large enough, the image of $R(k)^+ = \mathbf{A}_{\text{inf}}[\frac{[\varpi^b]^k}{p}]$ lies in \mathbf{A}_{cris} , therefore it extends to a continuous map

$$B^+ \rightarrow \mathbf{B}_{\text{cris}}^+$$

which is compatible with Frobenius. Inverting t ¹¹, this induces a map

$$B^+[\frac{1}{t}] \rightarrow \mathbf{B}_{\text{cris}}$$

Lemma 4.2.

$$(B^+[\frac{1}{t}])^{\varphi=1} \rightarrow (\mathbf{B}_{\text{cris}})^{\varphi=1}$$

Remark. Once we know $B^+ \rightarrow \mathbf{B}_{\text{cris}}^+$ (in fact $B^+ \subset \mathbf{B}_{\text{cris}}^+$ is a subring), one can in fact show that

$$B^+ = \bigcap_{n \geq 0} \varphi^n(\mathbf{B}_{\text{cris}}^+)$$

then the lemma also follows. We give the following amusing argument.

Sketch. By the remark immediately after theorem 3.3, we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Q}_p t & \longrightarrow & (B[\frac{1}{t}])^{\varphi=1} & \longrightarrow & \mathbf{B}_{\text{dR}}/\mathbf{B}_{\text{dR}}^+ \longrightarrow 0 \\ & & \parallel & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathbb{Q}_p t & \longrightarrow & (\mathbf{B}_{\text{cris}})^{\varphi=1} & \longrightarrow & \mathbf{B}_{\text{dR}}/\mathbf{B}_{\text{dR}}^+ \longrightarrow 0 \end{array}$$

□

Finally, by corollary 1.4, we know that

$$(B[\frac{1}{t}])^{\varphi=1} \xrightarrow{\sim} (\mathbf{B}_{\text{cris}})^{\varphi=1}$$

Therefore, indeed, the schematic curve X can be regarded as $\text{Spec}(\mathbf{B}_{\text{cris}})^{\varphi=1}$ and $\text{Spec} \mathbf{B}_{\text{dR}}^+$ glued together along $\text{Spec} \mathbf{B}_{\text{dR}}$.

¹¹We need to be careful about what t means on the right hand side. I will make this precise in a later draft