LECTURE 4: FUNCTIONS ON Y

Fix an algebraically closed field $C^\flat$ of characteristic p which is complete with respect to a nontrivial absolute value $|\,| : C^\flat \to \mathbf{R}_{\geq 0}$. Let $\mathrm{Un}(C^\flat)$ denote the set of all (isomorphism classes of) characteristic zero untilts of $C^\flat$. In the previous lecture, we explained that the set $\mathrm{Un}(C^\flat)$ is in some sense "parametrized" by a certain geometric object $Y^{\mathrm{ad}}_{C^\flat}$. Moreover, the Frobenius on $C^\flat$ induces a (topologically) free action of $\mathbf{Z}$ on $Y^{\mathrm{ad}}_{C^\flat}$, which allows us to define a quotient object $X^{\mathrm{ad}}_{C^\flat} = Y^{\mathrm{ad}}_{C^\flat}$ parametrizing the set $\mathrm{Un}(C^\flat)/\varphi^{\mathbf{Z}}$ of Frobenius-equivalence classes of untilts of $C^\flat$, called the *adic Fargues-Fontaine curve*. The objects $X^{\mathrm{ad}}_{C^\flat}$ and $Y^{\mathrm{ad}}_{C^\flat}$ are not schemes: they live in the category of *adic spaces over* $\mathbf{Q}_p$. However, the adic Fargues-Fontaine curve $X^{\mathrm{ad}}_{C^\flat}$ also has an incarnation in the world of schemes, which we will denote by $X_{C^\flat}$ and refer to as the *schematic Fargues-Fontaine curve*. For the rest of this seminar, we will be interested in understanding $X_{C^\flat}$. It was defined in the previous lecture as the projective scheme

$$\mathrm{Proj} \bigoplus_{n \geq 0} \Gamma(X_{C^\flat}, \mathcal{O}(n)),$$

where $\mathcal{O}(n)$ denotes the line bundle on $X_{C^\flat}$ which is obtained by starting with the trivial line bundle $\mathcal{L}$ on $Y_{C^\flat}$ and equipping it with the descent data given by the isomorphism

$$\mathcal{L} = \mathcal{O}_{Y^{\mathrm{ad}}_{C^\flat}} \xrightarrow{p^n} \mathcal{O}_{Y^{\mathrm{ad}}_{C^\flat}} \simeq \varphi^* \mathcal{L}.$$

It follows that a global section of $\mathcal{O}(n)$ is equivalent to giving a global function f on $Y^{\mathrm{ad}}_{C^\flat}$ satisfying the equation $\varphi(f) = p^n f$. In other words, the schematic Fargues-Fontaine curve is defined by the formula

$$X_{C^\flat} = \mathrm{Proj} \bigoplus_{n \geq 0} B^{\varphi = p^n},$$

where $B = \Gamma(Y^{\mathrm{ad}}_{C^\flat}; \mathcal{O}_{Y^{\mathrm{ad}}_{C^\flat}})$ is the ring of global functions on $Y^{\mathrm{ad}}_{C^\flat}$.

In order to understand the schematic Fargues-Fontaine curve and to establish its basic properties, we will need to understand something about the ring B and the Frobenius action on B. Our goal in this lecture is to start down this road, beginning by reviewing the construction of B. We will take a more pedestrian approach than in the previous lecture, and will not assume that the reader is familiar with the theory of adic spaces. Instead, we will regard the equation

$$B = \Gamma(Y^{\mathrm{ad}}_{C^\flat}, \mathcal{O}_{Y^{\mathrm{ad}}_{C^\flat}})$$

as a heuristic guiding principle, rather than a definition. Roughly speaking, we would like B to be a ring whose elements are "global functions on the space of untilts of $C^\flat$." In particular, the ring B should have the property that there is a canonical map

$$B \to \prod_{C \in \mathrm{Un}(C^\flat} C,$$

which assigns to each element $x \in B$ a family $\{x_C \in C\}_{C \in \mathrm{Un}(C^\flat)}$ consisting of an element in each untilt of $C^\flat$.

Let's begin by considering some examples of such families.

Example 1. Recall that if C is an untilt of $C^\flat$, then $C^\flat$ can be identified (as a set) with the inverse limit

$$\cdots \to C \xrightarrow{x \mapsto x^p} C \xrightarrow{x \mapsto x^p} C.$$

In particular, projection onto the last component yields a canonical map

$$C^\flat \to C \qquad y \mapsto y^\sharp.$$

Consequently, every element of $y \in C^\flat$ determines a family of elements $\{y^\sharp \in C\}_{C \in \mathrm{Un}(C^\flat)}$.

Example 1 suggests some candidate elements for the ring B: for each $y \in C^\flat$, we might want an element of B whose image in the product $\prod_{C \in \mathrm{Un}(C^\flat)}$ is given by $\{y^\sharp \in C\}_{C \in \mathrm{Un}(C^\flat)}$. But we cannot take B to consist *only* of such elements: beware that for a fixed untilt C of $C^\flat$, the construction $y \mapsto y^\sharp$ is multiplicative but not additive. Consequently, if we want B to be closed under addition, then we need some more elements. We can produce such elements by taking advantage of the fact that every untilt of $C^\flat$ is itself complete with respect to a non-archimedean absolute value. More precisely, if C is an untilt of $C^\flat$, then there is a unique absolute value

$$|\,| : C \to \mathbf{R}_{\geq 0}$$

having the property that $|y^\sharp| = |y|$ for each $y \in C^\flat$. In particular, this guarantees that if y belongs to the ring of integers $\mathcal{O}_C^\flat = \{y \in C^\flat : |y| \leq 1\}$, then $y^\sharp$ belongs to the ring of integers $\mathcal{O}_C = \{x \in C : |x| \leq 1\}$. Since C is complete and we have $|p| < 1$, it follows that the expression

$$a_0^\sharp + a_1^\sharp p + a_2^\sharp p^2 + \cdots$$

converges to an element of $\mathcal{O}_C$, for any sequence of elements $a_0, a_1, a_2, \ldots \in \mathcal{O}_C^\flat$.

Exercise 2. *Let $W(\mathcal{O}_C^\flat)$ denote the ring of Witt vectors of $\mathcal{O}_C^\flat$. For each element $y \in \mathcal{O}_C^\flat$, let $[y] \in W(\mathcal{O}_C^\flat)$ denote its Teichmüller representative. Since $\mathcal{O}_C^\flat$ is a perfect ring of characteristic p, every element of $W(\mathcal{O}_C^\flat)$ admits a unique Teichmüller expansion*

$$[a_0] + [a_1]p + [a_2]p^2 + \cdots$$

Show that, if C is an untilt of $C^\flat$, then the construction

$$\sum_{n \geq 0} [a_n] p^n \mapsto \sum_{n \geq 0} a_n^\sharp p^n$$

determines a ring homomorphism $W(\mathcal{O}_C^\flat) \to \mathcal{O}_C$ (this ring homomorphism appeared already in a previous lecture, from a more abstract point of view).

It follows from Exercise 2 that elements of $W(\mathcal{O}_C^\flat)$ also give rise to elements of the product $\prod_{C \in \mathrm{Un}\, C^\flat} C$, via the construction

$$(\sum [a_n] p^n \in W(\mathcal{O}_C^\flat)) \mapsto \{\sum a_n^\sharp p^n \in C\}_{C \in \mathrm{Un}(C^\flat)}.$$

This can be considered as an elaboration of Example 1, but differs in an important respect: the map described above is additive as well as multiplicative. In particular, the collection of families having the form $\{\sum a_n^\sharp p^n \in C\}_{C \in \mathrm{Un}(C^\flat)}$ is a subring of $\prod_{C \in \mathrm{Un}(C^\flat)} C$.

Beware that the preceding construction is not quite a generalization of Example 1, because we require that each coefficient a_n belongs to $\mathcal{O}_C^\flat$. Let us now loosen this requirement. Fix a quasi-uniformizer $\pi \in C^\flat$: that is, an element of $C^\flat$ satisfying $0 < |\pi| < 1$. For each untilt C of $C^\flat$, the image $\pi^\sharp$ is a quasi-uniformizer in C: that is, the map

$$W(\mathcal{O}_C^\flat) \to \mathcal{O}_C$$

extends to a map of localizations

$$W(\mathcal{O}_C^\flat)[[\pi]^{-1}] \to \mathcal{O}_C[(\pi^\sharp)^{-1}] = C.$$

Note that the localization on the left hand side is independent of π. Concretely, each element of the localization $W(\mathcal{O}_C^\flat)[[\pi]^{-1}]$ admits a unique Teichmüller expansion

$$\sum_{n \geq 0} [a_n] p^n,$$

where each a_n belongs to $C^\flat$ and the set $\{|a_n|\}_{n \geq 0}$ is bounded; here $\sum_{n \geq 0} [a_n] p^n$ is defined as the fraction

$$\frac{\sum_{n \geq 0} [a_n \pi^m] p^n}{[\pi^m]}$$

for m sufficiently large.

Since we are considering only *characteristic zero* untilts of $C^\flat$, there is a further enlargement we can make: for each untilt C of $C^\flat$, the map

$$W(\mathcal{O}_C^\flat)[[\pi]^{-1}] \to C$$

factors through the localization $W(\mathcal{O}_C^\flat)[[\pi]^{-1}, p^{-1}]$. Once again, elements of this localization have a concrete description: they admit Teichmüller expansions

$$\sum_{-\infty < n < \infty} [a_n] p^n,$$

where we allow negative powers of p but require that $a_n = 0$ for $n \ll 0$ (and where the set $\{|a_n|\}$ is bounded and $a_n = 0$ for $n \ll 0$.

The ring $B_0 = W(\mathcal{O}_C^\flat)[[\pi]^{-1}, p^{-1}]$ has many of the features that we are looking for in the ring B: it can be interpreted as a "ring of functions" on the set $\mathrm{Un}(C^\flat)$ (via the homomorphism $W(\mathcal{O}_C^\flat)[[\pi]^{-1}] \to \prod_{C \in \mathrm{Un}(C^\flat)} C$), and is fairly large (for example, every individual untilt of $C^\flat$ can be regarded as a quotient of B_0). However, it is still not large enough for our purposes. Note that the Frobenius automorphism of $C^\flat$ induces an automorphism of B_0, given concretely by the formula

$$\varphi(\sum[a_n]p^n) = \sum[a_n^p]p^n.$$

For $y \in B_0$, we can rewrite the equation $\varphi(y) = p^k y$ as an equality

$$\sum_{n\in\mathbf{Z}}[a_n^p]p^n = \sum_{n\in\mathbf{Z}}[a_{n-k}]p^n,$$

which is satisfied if and only if $a_{n-k} = a_n^p$ for every integer n. For $k \neq 0$, this equation is incompatible with the requirement that $a_n = 0$ for $n \ll 0$, except in the trivial case where each coefficient a_i vanishes. In other words, the graded ring $\bigoplus B_0^{\varphi=p^k}$ vanishes in degrees different from zero.

To remedy the situation, we would like to further enlarge B_0: roughly speaking, we would like to allow Teichmüller expansions which are "infinite in both directions" (but see Warning 11 below). Before explaining how this enlargement works, let us review a bit of topological algebra.

Construction 3. Let M be an abelian group with no p-torsion. We set $M^\wedge_{\mathbf{Q}_p} = (\varprojlim M/p^k M)[p^{-1}]$. Then $M^\wedge_{\mathbf{Q}_p}$ admits the structure of a Banach space over $\mathbf{Q}_p$, whose topology is uniquely determined by the requirement that the completion $M^\wedge = \varprojlim M/p^k M$ is an open an bounded subset of $M^\wedge_{\mathbf{Q}_p}$. There is a canonical map $u : M \to M^\wedge_{\mathbf{Q}_p}$, which enjoys the following universal property

- $(*)$ Let V be any Banach space over $\mathbf{Q}_p$. Then composition with u determines a bijection

$$\{\text{Continuous homomorphisms } M^\wedge_{\mathbf{Q}_p} \to V\} \to \{\text{Homomorphisms } M \to V \text{ with bounded image}\}.$$

In other words, $M^\wedge_{\mathbf{Q}_p}$ is universal among Banach spaces over $\mathbf{Q}_p$ equipped with a map $M \to M^\wedge_{\mathbf{Q}_p}$ having bounded image.

If R is a commutative ring with no p-torsion, then $R^\wedge_{\mathbf{Q}_p}$ has the structure of a (commutative) *Banach algebra* over $\mathbf{Q}_p$. It enjoys the same universal property: for any Banach algebra A over $\mathbf{Q}_p$, giving a continuous $\mathbf{Q}_p$-algebra homomorphism $R^\wedge_{\mathbf{Q}_p} \to A$ is equivalent to giving a ring homomorphism $R \to A$ having bounded image.

Remark 4. In the situation of Construction 3, the map $R[p^{-1}] \to R^\wedge_{\mathbf{Q}_p}$ has dense image. We can therefore regard $R^\wedge_{\mathbf{Q}_p}$ as the completion of $R[p^{-1}]$ with respect to a suitable semi-norm: for example, the semi-norm given by

$$|x| = \inf\{p^n : p^n x \in R\}.$$

Beware that the map $R[p^{-1}] \to R^{\wedge}_{\mathbf{Q}_p}$ is not necessarily injective (in other words, the preceding semi-norm might not be a norm, because some elements of R might be infinitely p-divisible).

We now apply Construction 3 to our situation.

Construction 5. Fix some positive integer k. We let $R(k)$ denote the subring of $W(\mathcal{O}_C^\flat)[[\pi]^{-1}, p^{-1}]$ generated by $W(\mathcal{O}_C^\flat)$ and the elements $p^k/[\pi]$ and $[\pi^k]/p$. Note that we have inclusions

$$\cdots \subseteq R(5) \subseteq R(4) \subseteq R(3) \subseteq R(2) \subseteq R(1).$$

Moreover, we have $R(k)[p^{-1}] \simeq W(\mathcal{O}_C^\flat)[[\pi]^{-1}, p^{-1}]$ for each k.

We define B to be the inverse limit of the tower

$$\cdots \to R(5)^{\wedge}_{\mathbf{Q}_p} \to R(4)^{\wedge}_{\mathbf{Q}_p} \to R(3)^{\wedge}_{\mathbf{Q}_p} \to R(2)^{\wedge}_{\mathbf{Q}_p} \to R(1)^{\wedge}_{\mathbf{Q}_p}.$$

Then B inherits the structure of a *Frechet algebra* over $\mathbf{Q}_p$: it can be viewed as the completion of $W(\mathcal{O}_C^\flat)[[\pi]^{-1}, p^{-1}]$ with respect to the countable family of (semi)-norms determined by the lattices $R(k) \subseteq W(\mathcal{O}_C^\flat)[[\pi]^{-1}, p^{-1}]$.

Warning 6. The individual subalgebras $R(k) \subseteq W(\mathcal{O}_C^\flat)[[\pi]^{-1}, p^{-1}]$ depend on the choice of quasi-uniformizer $\pi \in C^\flat$. But the completion B (and its topology) are independent of this choice.

Remark 7. The Frechet algebra B also has a universal mapping property. For every Banach algebra A over $\mathbf{Q}_p$, the following data are equivalent:

- (a) Continuous $\mathbf{Q}_p$-algebra homomorphisms $B \to A$.
- (b) Ring homomorphisms $f : W(\mathcal{O}_C^\flat) \to A$ such that $f([\pi])$ is invertible, and there exists some $k \gg 0$ such that $\frac{p^k}{f([\pi])}$, $\frac{f([\pi^k])}{p}$, and $\mathrm{im}(f)$ generate a bounded subalgebra of A.

Example 8. Let C be an untilt of $C^\flat$ and let $f : W(\mathcal{O}_C^\flat) \to \mathcal{O}_C \subseteq C$ be the homomorphism of Exercise 2, given by

$$f(\sum [a_n] p^n) = \sum a_n^\sharp p^n.$$

Then f satisfies requirement (b) of Remark 7, and therefore uniquely extends to a continuous homomorphism $B \to C$. To see this, it suffices to observe that $\frac{p^k}{\pi^\sharp}$ and $\frac{(\pi^\sharp)^k}{p}$ belong to $\mathcal{O}_C$ for some $k \gg 0$ (since p and $\pi^\sharp$ are both quasi-uniformizers of C). Beware that k depends on C in general (which is why we need to consider a Frechet topology, rather than a Banach space topology).

Allowing C to vary, we obtain a map

$$B \to \prod_{C \in \mathrm{Un}(C^\flat)} C.$$

The definition of B as a completion allows us to make sense of certain infinite sums in B. Namely, if we are given collection of elements $\{y_i\}_{i\in I}$ in $W(\mathcal{O}_C^\flat)[[\pi]^{-1}, p^{-1}]$, then the sum $\sum_{i\in I} y_i$ converges to an element of B if and only if it converges with respect to the norms defined by each $R(k)$: in other words, for every pair of integers $k, m > 0$, we have $y_i \in p^m R(k)$ for almost every $i \in I$.

Example 9 (Infinite Teichmüller Expansions)**.** Suppose we are given a sequence $\{a_n\}_{n\in\mathbf{Z}}$ of elements of $C^\flat$. Unwinding the definitions, we see that the infinite sum $\sum [a_n]p^n$ converges in B if and only if $\{a_n\}_{n\in\mathbf{Z}}$ satisfies the following pair of conditions:

$$\limsup_{n>0}\{\sqrt[n]{|a_n|}\} \leq 1 \qquad \lim_{n\to\infty} \sqrt[n]{|a_{-n}|} = 0.$$

Remark 10 (Complex-Analytic Analogue)**.** Let f be a holomorphic function defined on the punctured unit disk $D^\times = \{z \in \mathbf{C} : 0 < |z| < 1\}$. Then f admits a Laurent series expansion

$$f(z) = \sum c_n z^n,$$

where the coefficients c_n are complex numbers satisfying the conditions

$$\limsup_{n>0}\{\sqrt[n]{|c_n|}\} \leq 1 \qquad \lim_{n\to\infty} \sqrt[n]{|c_{-n}|} = 0.$$

Conversely, for any sequence $\{c_n\}_{n\in\mathbf{Z}}$ of complex numbers satisfying these conditions, the sum $\sum c_n z^n$ determines a holomorphic function on $D^\times$. The same phenomenon occurs for the punctured (open) unit disk in rigid analytic geometry (which is the equicharacteristic analogue of the adic space $Y_{C^\flat}$).

By virtue of Remark 10, it may be helpful to think of elements of the ring B as "holomorphic functions of the variable p, defined for $0 < |p| < 1$." This heuristic is supported by the observation that the construction

$$(C \in \mathrm{Un}(C^\flat)) \mapsto |p|_C$$

determines a function $\mathrm{Un}(C^\flat) \to (0,1)$. This actually extends to a *continuous* $(0,1)$-valued function on the underlying topological space of the adic space $Y_{C^\flat}^{\mathrm{ad}}$, analogous to the absolute value function on the punctured unit disk $D^\times$.

Warning 11. The preceding heuristic suggests a number questions, whose answers are (to my knowledge) not known:

- Does every of B admit an expansion $\sum_{n\in\mathbf{Z}}[a_n]p^n$ as in Example 9?
- Suppose that we are given two sequences $\{a_n\}_{n\in\mathbf{Z}}, \{a'_n\}_{n\in\mathbf{Z}}$ of elements of $C^\flat$, both satisfying the requirements of Example 9, and that we have an equality

  $$\sum_{n\in\mathbf{Z}}[a_n]p^n = \sum_{n\in\mathbf{Z}}[a'_n]p^n$$

  in the ring B. Does it follow that $a_n = a'_n$ for each n?

- Let $B' \subseteq B$ be the subset consisting of all elements of B which can be realized as a sum $\sum_{n\in\mathbf{Z}}[a_n]p^n$, where $\{a_n\}_{n\in\mathbf{Z}}$ satisfies the requirements of Example 9. Is B' closed under addition? Under multiplication?

Note that an essential difference from the complex-analytic analogues of these questions is that we are asking aout Teichmüller expansions, rather than a power series expansions. The former are inherently more difficult to work with, because one cannot simply add them pointwise.

Let us now take advantage of our ability to form infinite sums to produce some elements of the subset $B^{\varphi=p} \subseteq B$. We begin with a general remark.

Construction 12. Let A be a Banach algebra over $\mathbf{Q}_p$, and let $x \in A$ be an element with the property that $x-1$ is topologically nilpotent in A. We define $\log(x)$ by the formula

$$\log(x) = \sum_{k>0} \frac{(-1)^{k+1}}{k}(x-1)^k.$$

The assumption that $x-1$ is topologically nilpotent guarantees that this expression converges (the absolute value of $(x-1)^k$ decays exponentially in k, while the absolute value of $1/k$ grows only linearly in k).

Exercise 13. *Let A be a Banach algebra over $\mathbf{Q}_p$, and suppose we are given a pair of elements $x, y \in A$ such that $x-1$ and $y-1$ are both topologically nilpotent. Show that* $\log(xy) = \log(x) + \log(y)$.

Variant 14. In the setting of Construction 12, we can replace the Banach algebra A with a Frechet algebra: note that an infinite sum in A converges if and only if it converges with respect to every continuous norm on A.

Proposition 15. *Let x be an element of $C^\flat$ satisfying $|x-1| < 1$. Then $[x]-1$ is topologically nilpotent in B.*

Proof. We must show that $[x]-1$ is topologically nilpotent in each $R(k)^\wedge_{\mathbf{Q}_p}$. Note that $[x]-1 \equiv [x-1] \pmod{p}$, so it will suffice to show that $[x-1]$ is topologically nilpotent in $R(k)^\wedge_{\mathbf{Q}_p}$. This is clear: for m sufficiently large, we have $|(x-1)|^m \leq |\pi|$, so that $[x-1]^m$ is divisible by $[\pi]$ and therefore $[x-1]^{km}$ is divisible by $[\pi^k]$, and hence by p, in the ring $R(k)$. □

Construction 16 (The Logarithm)**.** Let $\mathfrak{m}$ denote the maximal ideal of the valuation ring $\mathcal{O}_C^\flat$. It follows from Proposition 15 that the construction

$$x \mapsto \log[x]$$

determines a map

$$\log : 1 + \mathfrak{m} \to B,$$

and from Exercise 13 that this map is a group homomorphism (where we regard $1+\mathfrak{m}$ as an abelian group under multiplication). Note that, for any $x \in 1+\mathfrak{m}$, we have

$$\begin{aligned} \varphi \log([x]) &= \log(\varphi([x])) \\ &= \log([x^p]) \\ &= p \log([x]), \end{aligned}$$

so that the logarithm takes values in the subgroup $B^{\varphi=p} \subseteq B$.

Remark 17. Let us mention one immediate consequence of Construction 16, which we will elaborate on in the next lecture. Suppose we are given an untilt $C \in \mathrm{Un}(C^\flat)$, and let $\epsilon = (\cdots, \zeta_{p^2}, \zeta_p, 1)$ be a compatible system of p-power roots of unity in C. Then we can regard ϵ as an element of $C^\flat$ satisfying $\epsilon^\sharp = 1$. It follows that $\epsilon \in 1+\mathfrak{m}$, and that $\log[\epsilon]$ is an element of $B^{\varphi=p}$ whose image vanishes in C. In terms of the Fargues-Fontaine curve $X_{C^\flat}$, this is a global section of the line bundle $\mathcal{O}(1)$ which vanishes at the point determined by C.

Theorem 18. *The map*

$$1+\mathfrak{m} \to B^{\varphi=p} \qquad x \mapsto \log[x]$$

is an isomorphism.

We defer the proof of Theorem 18 to a future lecture. However, we want to give a heuristic for why one might already expect Theorem 18 to be true. Namely, if one could show that every element of B has a unique Teichmüller expansion, then Theorem 18 would follow formally.

Construction 19. Fix an integer k. Suppose that we wish to construct an element of $B^{\varphi=p^k}$ as a Teichmüller expansion $y = \sum_{n\in\mathbf{Z}}[a_n]p^n$, where the sequence $\{a_n\}$ satisfies the requirements of Example 9. In this case, we can rewrite the equation $\varphi(y) = p^k y$ as an equality

$$\sum_{n\in\mathbf{Z}}[a_n^p]p^n = \sum_{n\in\mathbf{Z}}[a_{n-k}]p^n.$$

This equality is obviously satisfied if we have $a_{n-k} = a_n^p$ for all n: in this case, we will refer to $\sum_{n\in\mathbf{Z}}[a_n]p^n$ as an "obvious" element of $B^{\varphi=p^k}$.

Example 20. Let us first consider the case $k = 0$. A Teichmüller expansion $\sum[a_n]p^n$ is "obviously" a fixed point under φ if each a_n is fixed by the Frobenius: that is, each a_n belongs to the prime field $\mathbf{F}_p \subseteq C$. In this case, the conditions of Example 9 guarantee that $a_n = 0$ for $n \ll 0$. Such sequences arise precisely as the Teichmüller expansions of elements of $\mathbf{Q}_p$.

Example 21. Now suppose that $k < 0$. The "obvious" elements of $B^{\varphi=p^k}$ are those given by Teichmüller expansions $\sum[a_n]p^n$ satisfying $a_{n+k} = \sqrt[p]{a_n}$. This condition is incompatible the requirement that the sequence $\{\sqrt[n]{|a_{-n}|}\}$ converges to zero, unless the sequence $\{a_n\}$ is identically zero. Consequently, the only "obvious" element of $B^{\varphi=p^k}$ is the zero element.

Let us now focus on the case $k=1$. In this case, we are interested in Teichmüller expansions $\sum[a_n]p^n$ whose coefficients satisfy the condition $a_{n-1}=a_n^p$. Taking $a=a_0$, we can rewrite this expansion as

$$\sum_{n\in\mathbf{Z}}\frac{[a^{p^n}]}{p^n}.$$

Note that such an expression satisfies the requirements of Example 9 if and only if $|a|<1$: that is, if and only if a belongs to the maximal ideal $\mathfrak{m}\subseteq\mathcal{O}_C^\flat$. We therefore obtain a map of sets

$$U:\mathfrak{m}\to B^{\varphi=p}$$
$$U(a)=\sum_{n\in\mathbf{Z}}\frac{[a^{p^n}]}{p^n}.$$

If we could affirmatively answer the questions posed in Warning 11, then this map would be bijective. In this case, Theorem 18 would follow immediately from the following:

Theorem 22. *There exists a commutative diagram of sets*

$$\begin{array}{ccccc} & & 1+\mathfrak{m} & & \\ & \overset{E}{\nearrow} & & \overset{\log[\bullet]}{\searrow} & \\ \mathfrak{m} & & \xrightarrow{\quad U \quad} & & B^{\varphi=p}, \end{array}$$

where E is bijective.

Corollary 23. *The collection of elements of B of the form $\sum_{n\in\mathbf{Z}}\frac{[a^{p^n}]}{p^n}$ is closed under addition.*

Proof Sketch of Theorem 22. The expression $\sum_{n\in\mathbf{Z}}\frac{[a^{p^n}]}{p^n}$ should remind the reader of the *Artin-Hasse exponential*, namely, the power series with rational coefficients given by the formula

$$x\mapsto\exp\Big(\sum_{n\geq 0}\frac{x^{p^n}}{p^n}\Big).$$

One can show that this is a power series in x with coefficients in $\mathbf{Z}_{(p)}$: that is, the denominators of its coefficients are not divisible by p. More precisely, this power series is given by the expression

$$\prod_{(d,p)=1}(1-x^d)^{\frac{-\mu(d)}{d}},$$

where μ denotes the Möbius function. Evaluating this power series determines a bijection $E : \mathfrak{m} \to 1 + \mathfrak{m}$, and we claim that this map has the properties required by Theorem 22. In other words, we claim that for each element $a \in \mathfrak{m}$, we have an equality

$$\sum_{n \in \mathbf{Z}} \frac{[a^{p^n}]}{p^n} = \log[\prod_{(d,p)=1} (1 - a^d)^{\frac{-\mu(d)}{d}}]$$

in the ring B. We will establish this identity by manipulation of formal series, and leave it to the reader to justify that our manipulations are legal (that is, that all of the relevant expressions actually converge in B).

We first recall that for any element $y \in \mathcal{O}_C^\flat$, the Teichmüller representative $[y] \in W(\mathcal{O}_C^\flat)$ can be computed as the limit $\varinjlim_{k \to \infty} \widetilde{y}_k^{p^k}$, where $\widetilde{y}_k$ is any element of $W(\mathcal{O}_C^\flat)$ lying over $y^{p^{-k}}$. In particular, for $x \in \mathfrak{m}$, we have

$$[1 - x] = \lim_{k \to \infty} (1 - [x^{p^{-k}}])^{p^k}$$

$$\begin{aligned} \log \frac{1}{[1-x]} &= \lim_{k \to \infty} p^k \log(\frac{1}{1 - [x^{p^{-k}}]}) \\ &= \lim_{k \to \infty} p^k \sum_{m > 0} \frac{[x^{mp^{-k}}]}{m} \\ &= \sum_{\alpha \in \mathbf{Z}[1/p], \alpha > 0} \frac{[x^\alpha]}{\alpha}. \end{aligned}$$

We now write

$$\begin{aligned} \log[\prod_{(d,p)=1} (1 - a^d)^{\frac{-\mu(d)}{d}}] &= \sum_{(d,p)=1} \frac{\mu(d)}{d} \log \frac{1}{[1 - a^d]} \\ &= \sum_{(d,p)=1} \sum_{\alpha \in \mathbf{Z}[1/p], \alpha > 0} \mu(d) \frac{[a^{d\alpha}]}{d\alpha} \\ &= \sum_{\beta \in \mathbf{Z}[1/p], \beta > 0} \sum_d \mu(d) \frac{[a^\beta]}{\beta}, \end{aligned}$$

where, in the final expression, we write $\beta = p^n k$ for $(k, p) = 1$ and d ranges over all divisors of k. It follows from the Möbius inversion formula that this inner sum vanishes for $k \neq 1$: that is, we can neglect all values of β which are not powers of p. Doing so, we obtain the expression

$$\sum_{n \in \mathbf{Z}} \frac{[a^{p^n}]}{p^n},$$

as desired. □