TALK 2 (MONDAY, SEPT. 25, 2017): THE RELATIVE FARGUES-FONTAINE CURVE, PART II

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Contents

1. The relative curve in the mixed characteristic case

1.1. **Creating the curve.** We now take our local field $\mathbf{K}$ to be $\mathbb{Q}_p$, and we will explain the modifications of the constructions in the previous talk Sects. 2.2 and 2.3.

1.1.1. The datum of Y_S still takes as an input $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perfctd}^{\mathrm{aff}}_{/\mathbb{F}_p}$, but Y_S will be an analytic space that lives over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, in the same way as the equal characteristic version lived over $\mathrm{Spa}(\mathbb{F}_p((z)), \mathbb{F}_p[[z]])$.

The starting point is to create the corresponding ring $A_{\mathrm{inf},S}$. It is supposed to look like power series in p with coefficients in R^+. So we stipulate

$$A_{\mathrm{inf},S} := W(R^+),$$

the ring of Witt vectors, which has the expected power series shape because R^+ was perfect (this follows from the perfectoid assumption on the pair (R, R^+)).

For an element $r \in R^+$ we let $[r] \in A_{\mathrm{inf},S}$ denote its Teichmüller lift.

1.1.2. Let us emphasize the analogies:

The role of $z \in R^+[[z]]$ is played by $p \in W(R^+)$.

The role of $\varpi \in R^+ \subset R^+[[z]]$ is played by $[\varpi] \in W(R^+)$.

1.1.3. We define the analytic space Y_S by removing from $\mathrm{Spa}(A_{\mathrm{inf},S}, A_{\mathrm{inf},S})$ the closed subspace defined by the ideal generated by $p \cdot [\varpi]$, in the same sense as in the previous talk, Sect. 2.2.3.

As in the case of equal characteristics, the analytic space Y_S is a union of affinoids Y_S^I taken over closed intervals $I \subset (0,1)$, see below.

1.1.4. As in the equal characteristic case, the Frobenius ϕ_S on S defines an automorphism of $A_{\mathrm{inf},S}$, and the resulting automorphism of Y_S maps Y_S^I isomorphically to $Y_S^{I^{\frac{1}{p}}}$.

We set $X_S := Y_S/\phi_S$.

This is the relative over S Fargues-Fontaine curve.

1.1.5. The definition of

$$\mathrm{Bun}_G \in \mathrm{PreDiam}_{/\mathbb{F}_p}$$

proceeds as in the equal characteristic case.

1.2. **The analytic structure on Y_S: mixed characteristic case.** We will now explain what the (affinoid) annuli Y_S^I are. Our $S = \mathrm{Spa}(R, R^+)$ is the same as in the equal characteristic case, see Sect. 2.3.1 of the previous talk.

1.2.1. We start with the ring $A_{\mathrm{inf},S} := W(R^+)$. Consider the localization $(A_{\mathrm{inf},S})_p$; its elements can be written as expressions

$$\sum_{n \geq -\infty} p^n \cdot [r_n], \quad r_n \in R^+.$$

For $\rho \in (0,1) \subset \mathbb{R}$, we define a norm $|-|_\rho$ on $(A_{\mathrm{inf},S})_p$ by

$$|\sum_{n \geq -\infty} p^n \cdot [r_n]| = \sup_n \rho^n \cdot |r_n|.$$

This norm extends continuously to the further localization $(A_{\mathrm{inf},S})_{p \cdot [\varpi]}$.

1.2.2. For $I = [\rho_1, \rho_2] \subset (0,1)$ we define $\Gamma(Y_S^I, \mathcal{O}_{Y_S^I})$ to be the completion of $(A_{\mathrm{inf},S})_{p\cdot[\varpi]}$ with respect to the semi-norms $|-|_{\rho_1}$ and $|-|_{\rho_2}$ (or the entire family of semi-norms $|-|_\rho$ for $\rho \in I$).

We let $\Gamma(Y_S^I, \mathcal{O}^+_{Y_S^I}) \subset \Gamma(Y_S^I, \mathcal{O}^+_{Y_S^I})$ be the subring consisting of elements of norm ≤ 1 with respect to both $|-|_{\rho_1}$ and $|-|_{\rho_2}$.

Then Y_S^I defined in this way is an adic affinoid.

1.2.3. We set

$$Y_S := \bigcup_I Y_S^I.$$

The algebra of global functions on Y_S is the limit

$$\Gamma(Y_S, \mathcal{O}_{Y_S}) = \lim_I \Gamma(Y_S^I, \mathcal{O}_{Y_S^I}),$$

and similarly for $\Gamma(Y_S, \mathcal{O}^+_{Y_S})$.

The algebra $\Gamma(Y_S, \mathcal{O}_{Y_S})$ can also be defined as the completion of $(A_{\mathrm{inf},S})_{p\cdot[\varpi]}$ with respect to the family of semi-norms $|-|_\rho$ for $\rho \in (0,1)$ (or we can take the ρ's belonging to two sequences, one converging to 1 and another to 0).

1.2.4. The discussion of the action of ϕ_S in the equal characteristic case carries over verbatim to the present situation: the corresponding action acts as "expansion" on Y_S (in terms of the presentation of the latter as a union of the Y_S^I).

2. The functor of tilt

2.1. **The Witt-tilt adjunction.** Above we have considered the functor of Witt vectors

$$\{\text{Perfect algebras over } \mathbb{F}_p\} \to \{p\text{-adically complete algebras over } \mathbb{Z}_p\},$$

which is a mixed characteristic analog of the functor

$$R \mapsto R[[z]].$$

The latter functor is the left adjoint to the tautological forgetful functor.

2.1.1. We now claim:

Proposition 2.1.2. *The functor $W(-)$ is the left adjoint of the functor of* tilt*:*

$$R \mapsto R^\flat := \lim_{\longleftarrow} R/pR,$$

where the transition maps are given by raising to the power p.

The proof of Proposition 2.1.2 uses the following (ubiquitous but elementary) lemma:

Lemma 2.1.3. *For a p-adically complete algebra R, the map*

$$\lim_{\longleftarrow} R \to \lim_{\longleftarrow} R/pR \tag{2.1}$$

is a multiplicative bijection.

Proof. Induction on n by considering R/p^nR. □

Remark 2.1.4. Applying the lemma to $R = W(R')$, where R' is a perfect $\mathbb{F}_p$-algebra, we obtain a multiplicative bijection

$$\lim_{\longleftarrow} W(R') \simeq \lim_{\longleftarrow} R'.$$

It is easy to see that we have a commutative diagram

$$\begin{array}{ccc} \lim\limits_{\longleftarrow} R' & \xrightarrow{\sim} & \lim\limits_{\longleftarrow} W(R') \\ \sim\downarrow & & \downarrow \\ R' & \xrightarrow{r \mapsto [r]} & W(R'), \end{array}$$

where the two vertical arrows are given by the projection on the last coordinate. This gives an explicit construction of the Teichmüller map.

Proof of Proposition 2.1.2. We will construct the unit and counit maps for this adjunction.

For a perfect $\mathbb{F}_p$-algebra R, we have $W(R)/pW(R) \simeq R$, and the projection on the last coordinate defines an isomorphism

$$\lim_{\longleftarrow} R \simeq R.$$

This defines the unit map

$$R \simeq (W(R))^\flat.$$

The counit map

$$W(R^\flat) \to R$$

is defined by sending

$$[r^\flat] \mapsto r_0 \text{ for } r^\flat = \{r_n,\, n \geq 0,\, r_n = r_{n+1}^p\} \in \underset{\longleftarrow}{\lim} R \simeq R^\flat.$$

This map extends continuously to all of $W(R^\flat)$ using the p-adic completeness property of R. One checks that this is indeed a ring homomorphism using the construction of Teichmüller representatives.

□

2.1.5. Let us say more explicitly how the adjunction of Proposition 2.1.2 works. Let R_1 be a perfect $\mathbb{F}_p$-algebra, and let us be given a homomorphism

$$\alpha : R_1 \to R^\flat.$$

We define a map

$$\beta : W(R_1) \to R$$

by sending $[r_1]$ to the image of $\alpha(r_1) \in R^\flat$ under the map

$$R^\flat \to \underset{\longleftarrow}{\lim} R \to R,$$

where the first arrow is the isomorphism inverse to (2.1), and the second arrow is the projection on the 0-component. We extend β to all of $W(R_1)$ by requiring that

$$\beta(\Sigma\, p^n \cdot [r_{1,n}]) = \Sigma\, p^n \cdot \beta(r_{1,n}),$$

where the sum makes sense due to the fact that R is p-adically complete.

2.2. **Tilts of perfectoid pairs.**

2.2.1. Let (R, R^+) be a perfectoid pair of $(\mathbb{Z}_p, \mathbb{Q}_p)$. Choose an element

$$\varpi^\flat \in \underset{\longleftarrow}{\lim} R^+ \simeq \underset{\longleftarrow}{\lim} R^+/pR^+ =: (R^+)^\flat$$

with first component ϖ.

Set

$$R^\flat := ((R^+)^\flat)_{\varpi^\flat}.$$

One shows that in this case $(R^\flat, (R^+)^\flat)$ is also a perfectoid pair. We will refer to it as the *tilt* of (R, R^+). We will refer to the original (R, R^+) as an *untilt* of $(R^\flat, (R^+)^\flat)$.

2.2.2. *Examples.* We have:

$$(\mathbb{Q}_p^{\text{cycl}}, \mathbb{Z}_p^{\text{cycl}})^\flat \simeq (\mathbb{F}_p((t^{\frac{1}{p^\infty}})), \mathbb{F}_p[[t^{\frac{1}{p^\infty}}]])$$

and

$$(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})^\flat \simeq (\overline{\mathbb{F}_p((t))}^\wedge, \mathcal{O}_{\overline{\mathbb{F}_p((t))}^\wedge}).$$

Explicitly, in the above examples, the corresponding element $t \in (\mathbb{Z}_p^{\text{cycl}})^\flat$ can be taken to be $\zeta^\flat - 1$, where

$$\zeta^\flat \in (\mathbb{Z}_p^{\text{cycl}})^\flat = \underset{\longleftarrow}{\lim} \mathbb{Z}_p^{\text{cycl}}/p\mathbb{Z}_p^{\text{cycl}} \simeq \underset{\longleftarrow}{\lim} \mathbb{Z}_p^{\text{cycl}}$$

corresponds to an element

$$(1, \zeta_p, ...) \in \mathbb{Z}_p^{\text{cycl}},$$

where ζ_p is a p-th primitive root of unity.

If $\mathbf{F}$ is a perfectoid field over $\mathbb{Q}_p$, then

$$(\mathbf{F}^0)^\flat \simeq (\mathbf{F}^\flat)^0$$

and

$$(\mathbf{F}^0\langle s_1^{\frac{1}{p^\infty}},...,s_m^{\frac{1}{p^\infty}}\rangle_\varpi, \mathbf{F}^0\langle s_1^{\frac{1}{p^\infty}},...,s_m^{\frac{1}{p^\infty}}\rangle)^\flat = ((\mathbf{F}^\flat)^0\langle s_1^{\frac{1}{p^\infty}},...,s_m^{\frac{1}{p^\infty}}\rangle_{\varpi^\flat}, (\mathbf{F}^\flat)^0\langle s_1^{\frac{1}{p^\infty}},...,s_m^{\frac{1}{p^\infty}}\rangle).$$

2.2.3. We have the following fundamental result of P. Scholze:

Theorem 2.2.4. *Given an affinoid perfectoid S, the tilting functor defines an equivalence*

$$\mathrm{Perfctd}^{\mathrm{aff}}_{/S} \to \mathrm{Perfctd}^{\mathrm{aff}}_{/S^\flat}.$$

Another way to formulate this theorem is that given $S \in \mathrm{Perfctd}^{\mathrm{aff}}_{/\mathbb{F}_p}$ and its untilt $S^\sharp$, any other affinoid perfectoid $S_1 \to S$ has a unique untilt $S_1^\sharp$ equipped with a compatible map to $S^\sharp$.

In particular, étale sites of S and $S^\sharp$ are naturally equivalent.

2.3. **The diamonization functor.**

2.3.1. The operation of left Kan extension along the tilting functor

$$(\mathrm{Perfctd}^{\mathrm{aff}}_{/\,\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)})^{\mathrm{op}} \to (\mathrm{Perfctd}^{\mathrm{aff}}_{/\mathbb{F}_p})^{\mathrm{op}}$$

defines a functor

$$\diamondsuit : \mathrm{PreDiam}_{/\,\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)} \to \mathrm{PreDiam}_{/\mathbb{F}_q},$$

where

$$\mathrm{PreDiam}_{/-} := \mathrm{Funct}((\mathrm{Perfctd}^{\mathrm{aff}}_{/-})^{\mathrm{op}}, \mathrm{Groupids}).$$

I.e., this is the unique functor that preserves colimits and makes the diagram

$$\begin{array}{ccc} \mathrm{Perfctd}^{\mathrm{aff}}_{/\,\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)} & \xrightarrow{S\mapsto S^\flat} & \mathrm{Perfctd}^{\mathrm{aff}}_{/\mathbb{F}_p} \\ \downarrow{\scriptstyle \mathrm{Yoneda}} & & \downarrow{\scriptstyle \mathrm{Yoneda}} \\ \mathrm{Funct}((\mathrm{Perfctd}^{\mathrm{aff}}_{/\,\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)})^{\mathrm{op}}, \mathrm{Groupids}) & \xrightarrow{\diamondsuit} & \mathrm{Funct}((\mathrm{Perfctd}^{\mathrm{aff}}_{/\mathbb{F}_p})^{\mathrm{op}}, \mathrm{Groupids}) \end{array}$$

2.3.2. We claim that the functor $\diamondsuit$ can be explicitly described as follows:

Lemma 2.3.3. *For $\mathcal{X} \in \mathrm{PreDiam}_{/\,\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)}$ and $S \in \mathrm{Perfctd}^{\mathrm{aff}}_{/\mathbb{F}_p}$, the groupoid $\mathrm{Hom}(S, \mathcal{X}^\diamondsuit)$ consists of pairs $(S^\sharp, x)$, where $S^\sharp$ is an untilt of S and $x \in \mathrm{Hom}(S^\sharp, \mathcal{X})$.*

Proof. For a fixed $S \in \mathrm{Perfctd}^{\mathrm{aff}}_{/\mathbb{F}_p}$, the assignment that takes an object $\mathcal{X}$ to the groupoid of pairs $(S^\sharp, x)$ preserves colimits.

This reduces the assertion of the lemma to the case when $\mathcal{X}$ is representable, i.e., corresponds to $T \in \mathrm{Perfctd}^{\mathrm{aff}}_{/\,\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)}$. In the latter case, it becomes a reformulation of Theorem 2.2.4. □

2.3.4. *Example.* The prediamond $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)^\diamondsuit$ classifies untilts: its value on $S \in \mathrm{Perfctd}^{\mathrm{aff}}_{/\mathbb{F}_p}$ is the groupoid of its untilts over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$.

Remark 2.3.5. The theory of ℓ-adic sheaves defines a functor

$$(\mathrm{Perfctd}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{DGCat}, \quad S \mapsto D(S)$$

(say, we use upper-* for pullback).

Applying the functor of right Kan extension along the Yoneda emnedding

$$(\mathrm{Perfctd}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreDiam})^{\mathrm{op}},$$

we obtain a functor

$$(\mathrm{PreDiam})^{\mathrm{op}} \to \mathrm{DGCat}, \quad \mathcal{X} \mapsto D(\mathcal{X}).$$

It follows from Theorem 2.2.4 that we have a canonical equivalence

$$D(\mathcal{X}) \simeq D(\mathcal{X}^\diamondsuit), \quad \mathcal{X} \in \text{PreDiam}_{/\operatorname{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)}.$$

3. Distinguished sections

3.1. **The equal characteristic case.** Whatever the precise definition of Y_S as an analytic space is, we will now describe a particular family of its codimension 1 subspaces. These play the role of loci of where Hecke modifications take place.

3.1.1. Namely, consider an "untilt" of (R, R^+) over $(\mathbb{F}_p((z)), \mathbb{F}_p[[z]])$, i.e., a homomorphism

$$\alpha : \mathbb{F}_p[[z]] \to R^+,$$

so that z is topologically nilpotent and invertible in R.

Then we have a map

$$\theta : A_{\text{inf},S} \to R^+,$$

whose ideal is generated by the element $\alpha(z) - z$.

3.1.2. The homomorphism θ extends to a homomorphism

$$(A_{\text{inf},S})_{z\cdot\varpi} \to R$$

and gives rise to a map of analytic spaces

$$\operatorname{Spa}(R, R^+) \to Y_S, \tag{3.1}$$

and this is the sought-for subspace of codimension 1.

3.1.3. Note that if we compose α with the Frobenius of R^+, the new map $\operatorname{Spa}(R, R^+) \to Y_S$ will be obtained from the original (3.1) by applying the automorphism ϕ_S of Y_S.

Hence, the composite map

$$\operatorname{Spa}(R, R^+) \to Y_S \to X_S$$

does not change when we modify the untilt α by composing it with the Frobenius on S.

3.2. **The mixed characteristic case.** We will now describe the mixed characteristic analog of the construction of distinguished sections of Y_S from Sect. 3.1.

3.2.1. Let $S = \operatorname{Spa}(R, R^+)$ be an object of $\text{Perfctd}^{\text{aff}}_{/\mathbb{F}_p}$, and let $S^\sharp = \operatorname{Spa}(R^\sharp, (R^+)^\sharp)$ be its untilt over $\operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$.

Then by Proposition 2.1.2, we obtain a map

$$\theta : A_{\text{inf},R^+} \to (R^+)^\sharp.$$

3.2.2. Let $\varpi \in (R^+)^\sharp$ be a quasi-uniformizer so that $\varpi = p \cdot u$ with u a unit in $(R^+)^\sharp$ and such that ϖ admits all p^n-roots (one can show that such ϖ always exists).

Let $\varpi^\flat$ be an element of R^+ represented by

$$\{\varpi^{\frac{1}{p^n}}\} \in \varprojlim (R^+)^\sharp \simeq ((R^+)^\sharp)^\flat = R^+.$$

(Note that this element depends on the untilt.)

We claim:

Proposition 3.2.3. *The ideal* $\ker(\theta)$ *is generated by the element* $\varpi - [\varpi^\flat]$.

Proof. First off, it is clear that $\varpi - [\varpi^\flat] \in \ker(\theta)$, as

$$\theta([\varpi^\flat]) = \varpi.$$

Vice versa, let

$$a = \Sigma\, \varpi^n \cdot [r_n] \in A_{\mathrm{inf},S}$$

be annihilated by θ. Since $(R^+)^\sharp$ has no ϖ-torsion, by induction, it suffices to find an element $a' \in A_{\mathrm{inf},S}$ such that

$$a - (\varpi - [\varpi^\flat]) \cdot a'$$

is ϖ-divisible.

The latter is equivalent to finding $r_0' \in R^+$ such that

$$[r_0] = [\varpi^\flat] \cdot [r_0'],$$

i.e., that r_0 is divisible by $\varpi^\flat$ in R^+.

Let $r_0 \in R^+$ correspond to an element

$$\{r_n^\sharp\} \in \varprojlim (R^+)^\sharp \simeq \varprojlim R^+ \simeq R^+.$$

We need to find an element

$$\{(r_n')^\sharp\} \in \varprojlim (R^+)^\sharp \simeq R^+$$

so that $r_n^\sharp = \varpi^{\frac{1}{p^n}} \cdot (r_n')^\sharp$ for all n. Since $(R^+)^\sharp$ has no ϖ-torsion, it suffices to show that $r_n^\sharp$ is $\varpi^{\frac{1}{p^n}}$-divisible in $(R^+)^\sharp$ for every n. We will do so by induction on n.

The assumption that $\theta(a) = 0$ implies that $r_0^\sharp$ is ϖ-divisible in $(R^+)^\sharp$. Suppose now that $r_{n-1}^\sharp$ is $\varpi^{\frac{1}{p^{n-1}}}$-divisible and consider the map

$$(R^+)^\sharp/\varpi^{\frac{1}{p^n}} \cdot (R^+)^\sharp \to (R^+)^\sharp/\varpi^{\frac{1}{p^{n-1}}} \cdot (R^+)^\sharp,$$

given by raising to the power p. This map is known to be injective (even without the perfectoid condition). Now, the image of $r_n^\sharp$ under

$$(R^+)^\sharp \to (R^+)^\sharp/\varpi^{\frac{1}{p^n}} \cdot (R^+)^\sharp$$

belongs to the kernel of the above map, by the induction hypothesis. Hence, $r_n^\sharp$ is $\varpi^{\frac{1}{p^n}}$-divisible, as required.

□

As a by-product we obtain:

Corollary 3.2.4. *The map* θ *induces an isomorphism*

$$R^+/\varpi^\flat \cdot R^+ \to (R^+)^\sharp/\varpi \cdot (R^+)^\sharp.$$

3.2.5. The map θ extends to a map

$$(A_{\mathrm{inf},R^+})_{p\cdot[\varpi]} \to R^\sharp,$$

denoted by the same symbol θ.

We have:

Lemma 3.2.6. *The above maps*

$$\theta : A_{\mathrm{inf},R^+} \to (R^+)^\sharp \text{ and } (A_{\mathrm{inf},R^+})_{p\cdot[\varpi]} \to R^\sharp$$

extend to a map of analytic spaces over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$

$$S^\sharp \to Y_S.$$

In addition, one shows:

Lemma 3.2.7. *For an untilt $S^\sharp$ of S over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, and the corresponding map $S^\sharp \to Y_S$, the composite map*

$$S^\sharp \to Y_S \to X_S$$

is also an embedding of a Cartier divisor. Moreover, the map induced by $Y_S \to X_S$ between the formal completions of $S^\sharp$ in Y_S and X_S, respectively, is an isomorphism.

3.2.8. Note that Lemma 3.2.6 admits the following interpretation in terms of the functor $\diamondsuit$. Let us regard Y_S as an object of $\mathrm{PreDiam}_{/\,\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)}$ by sending

$$(T \in \mathrm{Perfctd}^{\mathrm{aff}}_{/\,\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)}) \mapsto \mathrm{Hom}_{/\,\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)}(T, Y_S),$$

where Hom is understood as the set of maps of analytic spaces.

We have:

Corollary 3.2.9. *There exists a canonical isomorphism*

$$(Y_S)^\diamondsuit \simeq \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)^\diamondsuit \times S.$$

Proof. For $T \in \mathrm{Perfctd}^{\mathrm{aff}}_{/\mathbb{F}_p}$, we need to functorially compare the following two sets:

(i) Untilts $T^\sharp$ of T, equipped with a map $T^\sharp \to \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ and a map $\alpha : T \to S$.

(ii) Untilts $T^\sharp$ of T, equipped with a map of analytic spaces $\beta : T^\sharp \to Y_S$;

Given a datum in (i), the map α by functoriality gives rise to a map $Y_T \to Y_S$, and we set β to be the composition

$$T^\sharp \to Y_T \to Y_S,$$

where the first arrow is provided by Lemma 3.2.6.

Vice versa, given a datum in (ii), we define the corresponding map $T^\sharp \to \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ to be the composition of β and the projection $Y_S \to \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. In addition, if $T^\sharp = \mathrm{Spa}(R_1, R_1^+)$, the map β gives rise to a map

$$A_{\mathrm{inf},R^+} \to R_1^+,$$

which by Proposition 2.1.2 gives rise to a map

$$R^+ \to (R_1^+)^\flat,$$

which extends to a map $R \to R_1^\flat$. The resulting map of pairs

$$(R, R^+) \to (R_1^\flat, (R_1^+)^\flat)$$

is the desired map β.

□

4. The period rings

Fontaine's period rings B^+_{dR} and B_{dR} have a very natural interpretation in terms of Y_S (or X_S), as we we shall presently explain.

4.1. **Equal characteristic case.**

To motivate the definition in the mixed characteristic case, we will first consider the situation when our local field $\mathbf{K}$ is $\mathbb{F}_p((z))$.

4.1.1. Let $S = \mathrm{Spa}(R, R^+)$ be an object of $\mathrm{Perfctd}^{\mathrm{aff}}_{/\mathbb{F}_p}$, and consider the corresponding ring

$$A_{\mathrm{inf},S} := R^+[[z]].$$

Fix an "untilt" of (R, R^+), i.e., a homomorphism

$$\alpha : \mathbb{F}_p[[z]] \to R^+,$$

so that z is topologically nilpotent and invertible in R.

Consider the corresponding homomorphism

$$\theta : (A_{\mathrm{inf},S})_\varpi \to R.$$

We define $B^+_{\mathrm{dR}}(R, \alpha)$ as the completion of $(A_{\mathrm{inf},S})_\varpi$ with respect to $\ker(\theta)$.

Remark 4.1.2. Note that $B^+_{\mathrm{dR}}(R, \alpha)$ would be the same if instead of $(A_{\mathrm{inf},S})_\varpi$ we took $(A_{\mathrm{inf},S})_{\varpi\cdot z}$. This is because z is invertible in R.

4.1.3. Recall that $\ker(\theta)$ is generated by the element $\xi := z - \alpha(z)$. From here it is not difficult to see that $B^+_{\mathrm{dR}}(R, \alpha)$ is isomorphic to $R[[\xi]]$.

We define $B_{\mathrm{dR}}(R, \alpha)$ as the localization

$$(B^+_{\mathrm{dR}}(R, \alpha))_\xi.$$

It is easy to see that this definition is independent of the choice of the particular generator of $\ker(\theta)$.

4.1.4. Consider the section

$$\mathrm{Spa}(R, R^+) \to Y_S,$$

corresponding to α.

We can think of $B^+_{\mathrm{dR}}(R, \alpha)$ as the ring of functions on the completion of Y_S along this section.

4.1.5. Note that the Frobenius of S defines an isomorphism

$$B^+_{\mathrm{dR}}(R, \alpha) \simeq B^+_{\mathrm{dR}}(R, \varphi_S \circ \alpha).$$

4.2. **Mixed characteristic case.**

We will now adapt the above discussion to the mixed characteristic case.

4.2.1. Let $S = \mathrm{Spa}(R, R^+)$ be an object of $\mathrm{Perfctd}^{\mathrm{aff}}_{/\mathbb{F}_p}$, and consider the corresponding ring

$$A_{\mathrm{inf},S} := W(R^+).$$

Fix an untilt $(R^\sharp, (R^+)^\sharp)$ of (R, R^+). Consider the corresponding homomorphism

$$\theta : (A_{\mathrm{inf},S})_{[\varpi]} \to R^\sharp.$$

We define $B^+_{\mathrm{dR}}(R, R^\sharp)$ as the completion of $(A_{\mathrm{inf},S})_{[\varpi]}$ with respect to $\ker(\theta)$.

Remark 4.2.2. As in the equal characteristic case, $B^+_{\mathrm{dR}}(R, R^\sharp)$ would be the same if instead of $(A_{\mathrm{inf},S})_{[\varpi]}$ we took $(A_{\mathrm{inf},S})_{[\varpi]\cdot p}$. This is because p is invertible in $R^\sharp$.

4.2.3. Recall (see Proposition 3.2.3) that $\ker(\theta)$ is uni-generated; let ξ denote a generator. The ring $B^+_{\mathrm{dR}}(R, R^\sharp)$ comes equipped with filtration generated by powers of ξ. We have:

Lemma 4.2.4. *The n-associated graded quotient $\mathrm{Fil}^n(B^+_{\mathrm{dR}}(R, R^\sharp))/\mathrm{Fil}^{n+1}(B^+_{\mathrm{dR}}(R, R^\sharp))$ is free as an $R^\sharp$-module on the generator ξ^n.*

We define $B_{\mathrm{dR}}(R, R^\sharp)$ as the localization

$$(B^+_{\mathrm{dR}}(R, R^\sharp))_\xi.$$

4.2.5. Again, we can think of $B^+_{\mathrm{dR}}(R, R^\sharp)$ as the ring of functions on the completion of Y_S along the section

$$S^\sharp \to Y_S$$

corresponding to the untilt $S^\sharp$.

Using Lemma 3.2.7, we obtain that $B^+_{\mathrm{dR}}(R, R^\sharp)$ identifies also with the ring of functions on the completion of X_S along the composite map

$$S^\sharp \to Y_S \to X_S.$$

The action of φ_S on $A_{\mathrm{inf},S}$ induces an isomorphism

$$B^+_{\mathrm{dR}}(R, R^\sharp) \simeq B^+_{\mathrm{dR}}(R, R^\sharp_\varphi),$$

where the untilt $R^\sharp_\varphi$ corresponds to the same $S^\sharp \in \mathrm{PreDiam}_{/\,\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)}$ but the new isomorphism $(S^\sharp)^\flat \simeq S$ is obtained from the old one by composing with φ_S.

4.3. **The B_{dR}-Grassmannian.** Let G be an algebraic group over $\mathbf{K} = \mathbb{Q}_p$. We will now introduce the B_{dR} version of the affine Grassmannian, which controls modifications of the trivial G-bundle.

4.3.1. By definition, $\mathrm{Gr}_{G,B_{\mathrm{dR}}}$ is a prediamond over $\mathbb{F}_q$, i.e., a functor

$$(\mathrm{Perfctd}^{\mathrm{aff}}_{/\mathbb{F}_p})^{\mathrm{op}} \to \mathrm{Groupoid}$$

that attaches to $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perfctd}^{\mathrm{aff}}_{/\mathbb{F}_p}$ the data of

$$(S^\sharp, \mathcal{F}_G, \gamma),$$

where $S^\sharp$ is an untilt of S over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, $\mathcal{F}_G$ is a G-bundle on $\mathrm{Spec}(B^+_{\mathrm{dR}}(R, R^\sharp_\varphi))$ and γ is a trivialization of $\mathcal{F}_G$ over $\mathrm{Spec}(B_{\mathrm{dR}}(R, R^\sharp_\varphi))$.

4.3.2. As in the case of the usual affine Grassmannian, one has the Beauville-Laszlo type theorem that says that restriction defines an isomorphism to $\mathrm{Gr}_{G,B_{\mathrm{dR}}}$ from a prediamond over $\mathbb{F}_q$ that attaches to $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perfctd}^{\mathrm{aff}}_{/\mathbb{F}_p}$ the data of

$$(S^\sharp, \mathcal{F}_G, \gamma),$$

where $S^\sharp$ as above, but where $\mathcal{F}_G$ is now a G-bundle on X_S and and γ is a trivialization of $\mathcal{F}_G$ over $X_S - S^\sharp$.

4.3.3. Recall the prediamond Bun_G that attaches to $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perfctd}^{\mathrm{aff}}_{/\mathbb{F}_p}$ the data of G-bundle on X_S.

The above version of the Beauville-Laszlo theorem defines a map

$$\mathrm{Gr}_{G,B_{\mathrm{dR}}} \to \mathrm{Bun}_G.$$

4.3.4. As in the case of the usual affine Grassmannian, we can define a stratification of $\mathrm{Gr}_{G,B_{\mathrm{dR}}}$ by Schubert cells, parameterized by dominant coweights of G (say, for G split).

4.4. **The basic Schubert cell.** Let us consider the particular case when $G = GL_n$ and consider the basic Schubert cell

$$\mathrm{Gr}^{(1,\dots,0)}_{GL_n,B_{\mathrm{dR}}} \subset \mathrm{Gr}_{GL_n,B_{\mathrm{dR}}}.$$

We will describe the corresponding prediamond more explicitly.

4.4.1. When we think of the data of $(\mathcal{F}_G, \gamma)$ as $B^+_{\mathrm{dR}}(R, R^\sharp_\varphi)$-lattices

$$\mathcal{M} \subset B_{\mathrm{dR}}(R, R^\sharp_\varphi)^{\oplus n},$$

then $\mathrm{Gr}^{(1,\dots,0)}_{GL_n,B_{\mathrm{dR}}}$ corresponds to the condition that

$$\mathcal{M}_0 \subset \mathcal{M} \subset \xi^{-1} \cdot \mathcal{M}_0, \quad \mathcal{M}_0 := B^+_{\mathrm{dR}}(R, R^\sharp_\varphi)^{\oplus n},$$

and $\mathcal{M}/\mathcal{M}_0$ is a line bundle over $S^\sharp \subset X_S$.

4.4.2. Note that the above set identifies with the subset of line sub-bundles in $\mathcal{O}^{\oplus n}_{S^\sharp}$, i.e., with the set of maps

$$S^\sharp \to \mathbb{P}^{n-1}_{\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)},$$

where $\mathbb{P}^{n-1}_{\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)}$ is the appropriate version of the projective space, perceived as an object of $\mathrm{PreDiam}_{/\,\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)}$.

4.4.3. Thus, we obtain:

$$\mathrm{Gr}^{(1,\dots,0)}_{GL_n,B_{\mathrm{dR}}} \simeq (\mathbb{P}^{n-1}_{\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)})^\diamond.$$

5. Vector bundles and the algebro-geometric version of the curve

5.1. **Construction of vector/line bundles.**

5.1.1. Let us take S to be an object of $\mathrm{Perfctd}^{\mathrm{aff}}_{/\overline{\mathbb{F}}_p}$ (i.e., we are replacing $\mathbb{F}_p$ by its algebraic closure). Note that in this case Y_S maps to

$$\mathrm{Spa}(W(\overline{\mathbb{F}}_p)_p, W(\overline{\mathbb{F}}_p)) = \mathrm{Spa}(\mathbb{Q}^{\mathrm{unr}}_p, \mathbb{Z}^{\mathrm{unr}}_p),$$

in a way compatible with the action of the Frobenius.

Hence, any $\mathbb{Q}^{\mathrm{unr}}_p$-isocrystal, i.e., a pair (M, φ_M), where M is a finite-dimensional $\mathbb{Q}^{\mathrm{unr}}_p$-vector space and φ_M is a φ-linear automorphism of M gives rise to a ϕ_S-equivariant vector bundle on Y_S.

By descent, we obtain a functor

$$\{\mathbb{Q}^{\mathrm{unr}}_p - \text{isocrystals}\} \to \{\text{Vector bundles on } X_S\}.$$

5.1.2. Here is a particular family of isocrystals that we will use (in fact, the Dieudonné-Manin theorem implies that category of isocrystals is semi-simple and and the family below describes all the irreducible ones):

For a rational number λ written as an irreducible fraction $\frac{d}{h}$, we define an isocrystal $M(\lambda)$ to be h-dimensional with basis

$$e_0, \dots, e_{h-1}$$

with ϕ_M acting as follows:

$$\phi_M(e_{i-1}) = \begin{cases} e_i \text{ if } i < h, \\ p^d \cdot e_0 \text{ if } i = h. \end{cases}$$

5.1.3. We will denote by $\mathcal{O}(\lambda)$ the corresponding vector bundle on X_S.

In particular, taking $h=1$, we obtain the line bundles $\mathcal{O}(d)$.

By construction,

$$\Gamma(X_S, \mathcal{O}(d)) = \Gamma(Y_S, \mathcal{O}_{Y_S})^{\phi=p^d} := \{f \in \Gamma(Y_S, \mathcal{O}_{Y_S}),\, \phi_S(f) = p^d \cdot f\}.$$

5.2. **The algebraic curve of Fargues-Fontaine.** We are finally ready to define the algebraic version of the Fargues-Fontaine curve.

5.2.1. Take $S = \mathrm{Spa}(R^+, R)$, where R is the perfectoid field $\mathbf{F} := \overline{\mathbb{F}_p((t))}^{\wedge}$ and R^+ its subring of integral elements.

Consider the $\mathbb{Z}^{\geq 0}$-graded algebra A with

$$A^d := \Gamma(X_S, \mathcal{O}(d)).$$

We set

$$X^{\mathrm{alg}} := \mathrm{Proj}(A).$$

Note that by construction, the vector bundles $\mathcal{O}(\lambda)$, defined on X_S, give rise to quasi-coherent sheaves on X^{alg}; denote them by $\mathcal{O}(\lambda)^{\mathrm{alg}}$.

5.2.2. Here is the first set of assertions that we will need to prove about X^{alg}:

Theorem 5.2.3.
(a) *The map* $\mathbb{Q}_p \to \Gamma(X^{\mathrm{alg}}, \mathcal{O}_{X^{\mathrm{alg}}}) := \Gamma(Y_S, \mathcal{O}_{Y_S})^{\varphi=1}$ *is an isomorphism.*
(b) *Every* $\mathcal{O}(\lambda)^{\mathrm{alg}}$ *is a vector bundle of rank equal to the rank of* $\mathcal{O}(\lambda)$*.*
(c) *For* $d \geq 0$*, the map*

$$\Gamma(X_S, \mathcal{O}(d)) \to \Gamma(X^{\mathrm{alg}}, \mathcal{O}(d)^{\mathrm{alg}})$$

is an isomorphism.

5.2.4. Let $S^\sharp$ be an untilt of S (i.e., we have an untilt $\mathbf{F}^\sharp$ of $\mathbf{F}$). It defines a map

$$S^\sharp \to X_S.$$

This map gives rise to a closed subscheme, denoted x, of X^{alg}, corresponding to the homogeneous ideal in A given by sections of $\mathcal{O}(d)$ that vanish when pulled back to $S^\sharp$.

Theorem 5.2.5.
(a) *The above subscheme* x *of* X^{alg} *is a Cartier divisor.*
(b) *We have* $\mathcal{O}_X(x) \simeq \mathcal{O}(1)$*.*
(c) *The open subscheme* $X^{\mathrm{alg}} - x$ *is affine and its algebra of functions is a Dedekind domain.*

Here is a reformulation of what we will learn as the *fundamental exact sequence* of p-adic Hodge theory:

Theorem 5.2.6. *Let* $S^\sharp$ *and* x *be as above. Then the map*

$$\Gamma(X^{\mathrm{alg}}, \mathcal{O}(1)) \to \mathrm{Fil}^{-1}(B_{\mathrm{dR}}(\mathbf{F}, \mathbf{F}^\sharp))/B^+_{\mathrm{dR}}(\mathbf{F}, \mathbf{F}^\sharp)$$

is surjective.

5.2.7. One of the goals in this seminar will be to prove:

Theorem 5.2.8.
(a) *Every vector bundle on* X^{alg} *is a direct sum of vector bundles* $\mathcal{O}(\lambda)$ *for* $\lambda \in \mathbb{Q}$.

(b) *The functor of direct image along* $X \to X^{\mathrm{alg}}$ *defines an equivalence between the corresponding categories of vector bundles.*