TALK 2 (MONDAY, SEPT. 18, 2017): THE RELATIVE FARGUES-FONTAINE CURVE, PART I

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Contents

1. Geometerization of local Langlands

1.1. **The quotient by Frobenius-twisted conjugacy.** Let $\mathbf{K}$ be a local field. Classically (and very roughly speaking), local Langlands aims to match homomorphisms

$$\sigma : \mathrm{Gal}(\mathbf{K}) \to \check{G} \tag{1.1}$$

with (irreducible) representations of the locally compact group $G(\mathbf{K})$.

However, there are multiple indications that the RHS is not quite the ring thing to consider. Here is a very rough indication as to what one may want to replace it by.

1.1.1. First, we consider the case when $\mathbf{K}$ is of characteristic p, i.e.,

$$\mathbf{K} = \mathbb{F}_p((z)).$$

We consider the prestack quotient

$$G((z))/\operatorname{Ad}^{\varphi}_{G((z))},$$

i.e., the quotient of $G((z))$ by the action of $G((z))$ given by

$$g \star g' = g \cdot g' \cdot \varphi(g^{-1}), \tag{1.2}$$

where φ denotes the geometric Frobenius.

For a test-scheme S over $\mathrm{Spec}(\mathbb{F}_p)$, we can think of

$$\mathrm{Hom}(S, G((z))/\operatorname{Ad}^{\varphi}_{G((z))})$$

as the groupoid of *shtukas* on the formal punctures disc (with coordinate z), parameterized by S.

This is a quotient is an ind-scheme that is *not* of ind-finite type by a group ind-scheme, so the resulting algebro-geometric object is quite unwieldy.

1.1.2. Yet, suppose we can make sense of the derived category

$$D(G((z))/\operatorname{Ad}^{\varphi}_{G((z))})$$

of ℓ-adic sheaves.

The functor of taking the fiber at $1 \in G((z))$ defines a functor

$$D(G((z))/\operatorname{Ad}^{\varphi}_{G((z))}) \to G(\mathbf{K})\text{-mod}$$

(because $G(\mathbf{K})$ is the stabilizer of 1 under the action (1.2)).

But the datum of an object of $D(G((z))/\operatorname{Ad}^{\varphi}_{G((z))}) \to G(\mathbf{K})$-mod carries quite a bit more information.

1.1.3. The idea that $D(G((z))/\operatorname{Ad}^{\varphi}_{G((z))})$ is the "right" substitute of $G(\mathbf{K})$-mod was originally suggested by V. Lafforgue.

A conceptual but heuristic explanation of why this is indeed the natural thing to do may be found in [Ga, Sect. 4]. It follows into the paradigm "classical Langlands" is the (categorical) trace of Frobenius on geometric Langlands.

1.1.4. Points of $G((z))/\operatorname{Ad}^{\varphi}_{G((z))}$ with coefficients in $\overline{\mathbb{F}}_p$ are called *isocrystals*; the set of isomorphism classes of isocrystals is denoted $B(G)$.

Inside $B(G)$ one singles out a subset denoted $B(G)_{\text{basic}}$. For $b \in B(G)_{\text{basic}}$, the group J_b of its automorphisms, i.e.,

$$J_b := \{g \in G((z)) \,|\, g \cdot b = b \cdot \varphi(g)\}$$

is an inner form of G; in this way we obtain all *extended pure inner forms* of G.

Thus, taking the fiber at an $\overline{\mathbb{F}}_p$-point of $G((z))/\operatorname{Ad}^{\varphi}_{G((z))}$ corresponding to $b \in B(G)_{\text{basic}}$, we obtain a functor

$$D(G((z))/\operatorname{Ad}^{\varphi}_{G((z))}) \to J_b(\mathbf{K})\text{-mod}.$$

Thus, the datum of an object of $D(G((z))/\operatorname{Ad}^{\varphi}_{G((z))})$ remembers not only the representation of $G(\mathbf{K})$ but of $J(\mathbf{K})$ for all extended pure inner forms J of G.

1.1.5. So, a refinement of local Langlands would consist of associating to a Galois representation σ as in (1.1) an object

$$\mathcal{F}_\sigma \in D(G((z))/\operatorname{Ad}^{\varphi}_{G((z))}),$$

whose fiber at the unit isocrystal recovers the corresponding $G(\mathbf{K})$-representation, and whose fibers at other basic isocrystals would recover the representations of its inner forms.

Moreover, one can imagine that one could use the nearby cycles functor to formulate a Hecke eigen-property of $\mathcal{F}_\sigma$ with respect to σ. I.e., unlike the usual local Langlands, we will have a direct relation between σ and the automorphic object attached to it.

1.1.6. There are (at least) two major difficulties associated with realizing the above idea:

(I) It is really not clear how to work with $G((z))/\operatorname{Ad}^{\varphi}_{G((z))})$ so as to have a manageable category $D(G((z))/\operatorname{Ad}^{\varphi}_{G((z))})$, and especially the notion of nearby cycles (the latter in order to formulate the Hecke eigen-property).

(II) We would like a geometric theory also for $\mathbf{K}$ of characteristic 0. One can imagine using Witt vectors to define a prestack over $\mathbb{F}_p$ that would replace $G((z))/\operatorname{Ad}^{\varphi}_{G((z))}$ (so that its $\overline{\mathbb{F}}_p$-points will be $G(\mathbf{K}^{\text{unr}})/\operatorname{Ad}^{\varphi}_{G(\mathbf{K}^{\text{unr}})})$, where $\mathbf{K}^{\text{unr}} \supset \mathbf{K}$ is the maximal unramified extension); probably this can be done by the methods of [Zhu]. However, it is hard to imagine how one could make sense of nearby cycles, because the latter would have to combine the geometries of equal and mixed characteristics.

1.2. **Introducing analytic geometry.** The idea of the Fargues-Scholze can be interpreted as changing the paradigm as follows:

When working with prestacks over $\mathbb{F}_p$, we are led to considering the (parameterized) formal punctured disc

$$\mathcal{D}_S := S \widehat{\times} \mathcal{D}_z := \operatorname{Spec}(R((z))), \quad S = \operatorname{Spec}(R),\ \mathcal{D}_z = \operatorname{Spec}(\mathbb{F}_p((z))) \tag{1.3}$$

This formal disc is necessarily considered as a *scheme*, and as such it is not easy to manipulate. The problem is that S is too "skinny" to have a richer structure on the disc.

1.2.1. The idea is to replace *algebraic* geometry over $\mathbb{F}_p$ by some sort of *analytic* geometry, whereby our test objects are no longer affine schemes $\operatorname{Spec}(R)$ over $\mathbb{F}_p$, but *affinoid perfectoids* $\operatorname{Spa}(R, R^+)$ (whatever they are), and this allows to replace the parameterized formal punctured disc (1.3) by an appropriately defined *punctured open unit disc* $\mathbb{D}_S^{(0,1)} =: Y_S$.

This Y_S (and its quotient by the action of the Frobenius, denoted X_S) are actual geometric objects, and one can take bundles on them, consider Hecke correspondences, etc.

1.2.2. We will have a *pre-diamond* Bun_G, i.e., a functor

$$(\operatorname{Perfctd}^{\text{aff}}_{/\mathbb{F}_p})^{\text{op}} \to \text{Groupoids},$$

that assigns to S the groupoid of G-bundles on X_S.

Ultimately, the Fargues-Scholze program aims to construct

$$\mathcal{F}_\sigma \in D(\operatorname{Bun}_G),$$

that satisfies the Hecke property with respect to σ.

The main feature of the Fargues-Scholze program is that it is equally applicable when $\mathbf{K}$ is a local field of characteristic 0.

1.3. **How are the two approaches related?** Let us be very optimistic and imagine that both of the above programs have been carried. So, we have two versions of $\mathcal{F}_\sigma$: one is an object of $D(G((z))/\operatorname{Ad}^{\varphi}_{G((z))})$, and the other is an object of $D(\operatorname{Bun}_G)$.

How are they supposed to be related?

1.3.1. In order to streamline the exposition, let us replace the category $\mathrm{Sch}^{\mathrm{aff}}_{/\mathbb{F}_p}$ of affine schemes over $\mathbb{F}_q$ by the category $\mathrm{Sch}^{\mathrm{aff}}_{/\mathbb{F}_p,\mathrm{perf}}$ of perfect schemes (this is not supposed to have any effect on ℓ-adic sheaves. Correspondingly, we will replace the category

$$\mathrm{PreStk}_{/\mathbb{F}_p} = \mathrm{Funct}(\mathrm{Sch}^{\mathrm{aff}}_{/\mathbb{F}_p}, \mathrm{Groupoid})$$

by

$$\mathrm{PreStk}_{/\mathbb{F}_p,\mathrm{perf}} = \mathrm{Funct}(\mathrm{Sch}^{\mathrm{aff}}_{/\mathbb{F}_p,\mathrm{perf}}, \mathrm{Groupoid}).$$

1.3.2. The category $\mathrm{Sch}^{\mathrm{aff}}_{/\mathbb{F}_p,\mathrm{perf}}$ embeds into $\mathrm{PreDiam}_{/\mathbb{F}_p}$, because it makes sense to map an object of $\mathrm{Perfctd}^{\mathrm{aff}}_{/\mathbb{F}_p}$ to an object of $\mathrm{Sch}^{\mathrm{aff}}_{/\mathbb{F}_p,\mathrm{perf}}$ (just map the algebra of functions on affine the scheme to the algebra of functions on the affinoid perfectoid).

The operation of left Kan extension of the above functor

$$\mathrm{Sch}^{\mathrm{aff}}_{/\mathbb{F}_p,\mathrm{perf}} \to \mathrm{PreDiam}_{/\mathbb{F}_p}$$

along the Yoneda embedding $\mathrm{Sch}^{\mathrm{aff}}_{/\mathbb{F}_p,\mathrm{perf}} \hookrightarrow \mathrm{PreStk}_{/\mathbb{F}_p,\mathrm{perf}}$ defines a (fully faithful) functor

$$\mathrm{PreStk}_{/\mathbb{F}_p,\mathrm{perf}} \hookrightarrow \mathrm{PreDiam}_{/\mathbb{F}_p},$$

which admits a right adjoint (which is also a left inverse)

$$\mathrm{PreDiam}_{/\mathbb{F}_p} \to \mathrm{PreStk}_{/\mathbb{F}_p,\mathrm{perf}}, \tag{1.4}$$

given by restriction.

We call the above functor (1.4) the functor of *skeleton*, and denote it by Sk.

1.3.3. It will follow from the construction that there is a canonically defined map

$$G((z))/\,\mathrm{Ad}^{\varphi}_{G((z))} \to \mathrm{Sk}(\mathrm{Bun}_G);$$

probably it is an isomorphism, up to perfection.

In particular, we obtain a map in $\mathrm{PreDiam}_{/\mathbb{F}_p}$

$$G((z))/\,\mathrm{Ad}^{\varphi}_{G((z))} \to \mathrm{Bun}_G\,. \tag{1.5}$$

1.3.4. Therefore, if the original program of constructing $\mathcal{F}_\sigma$ as an object of $D(G((z))/\,\mathrm{Ad}^{\varphi}_{G((z))})$ is realized, one would want to compare it with the pullback of the Fargues-Scholze

$$\mathcal{F}_\sigma \in D(\mathrm{Bun}_G)$$

along (1.5).

Now, the conjecture would be that the former equals the pullback of the other.

2. The relative curve in the equal characteristic case

2.1. **Interlude: perfectoids.** Our analytic geometry looks a lot like algebraic geometry, but with a different category of test objects: the latter are affinoid perfectoids over $\mathbb{F}_p$. We will now make a short digression and explain what they are.

A good source for what follows is Chapters 1 and 4 in [Fe], and references therein.

2.1.1. Here are some definitions:

A *Huber* ring is a topological ring is a topological ring that contains an open subring A_0 with a finitely generated ideal $I \subset A_0$, such that the induced topology on A_0 is I-adic.

For a subset of a Huber It make sense to ask whether it is *bounded.* We say that $a \in A$ is *power-bounded* if the set $\{a^n, n \in \mathbb{N}\}$ is bounded. We let

$$A^{\text{p-bdd}} \subset A$$

denote the set of power-bounded elements. We let

$$A^{\text{t-nilp}} \subset A^{\text{p-bdd}}$$

denoted the subset of topologically nilpotent elements.

A Huber pair is (A, A^+), where A is a Huber ring, and $A^+ \subset A$ is an open subring, contained and integrally closed in $A^{\text{p-bdd}}$.

2.1.2. We will take a minimalist approach to analytic geometry. We define the category of *affinoid adic spaces* to be the opposite to that of Huber pairs. We will use the notation

$$(A, A^+) \rightsquigarrow \text{Spa}(A, A^+).$$

For us, an analytic space will be just a functor

$$\{\text{Affinoid adic spaces}\}^{\text{op}} \to \text{Groupoids}\,.$$

2.1.3. A *Tate* ring is a Huber ring for which there exists an element $\varpi \in A^{\text{t-nilp}} \cap A^\times$. (Such an element if called a *pseudo-uniformizer.*)

The existence of a pseudo-uniformizer is what makes the ring R not skinny.

2.1.4. When working over $\mathbb{F}_p$, we define a *perfectoid* ring to be a topological ring R such that

- R is a complete Tate ring;
- the set $A^{\text{p-bdd}}$ is itself bounded;
- R is perfect (i.e., Frobenius is bijective).

A typical example of a perfectoid ring is

$$\overline{\mathbb{F}}_p((t^{\frac{1}{p^\infty}})) := \left(\left(\underset{n}{\cup}\, \overline{\mathbb{F}}_p[[t^{\frac{1}{p^n}}]]\right)^\wedge\right)_t,$$

where the subscript t means localization with respect to t.

A quasi-uniformizer is given by $\varpi = t$. Another example, containing the previous one, is $\overline{\mathbb{F}_p((t))}^\wedge$.

In this talk we will only need perfectoid rings *over* $\mathbb{F}_p$.

2.1.5. When R is not over $\mathbb{F}_p$, the definition is a bit more elaborate. One replaces the last condition by the following: there exists a quasi-uniformizer $\varpi \in R$ such that

- $\frac{p}{\varpi^p} \in R^{\text{p-bdd}}$;
- Raising to the power p defines an isomorphism $R/\varpi \to R/\varpi^p$.

Here are the most typical examples:

(a) $R = \mathbb{Q}_p^{\text{cycl}}$, i.e., $(\mathbb{Z}_p(\mu_{p^\infty})^\wedge)_p$, where the subscript p means localization with respect to p;

(b) $R = \mathbb{C}_p$, i.e., $(\overline{\mathbb{Z}}_p{}^\wedge)_p$.

In both these examples, a quasi-uniformizer can be taken to be $\varpi = p$.

2.1.6. The examples of perfectoid rings that we have are above are actually perfectoid *fields*.

If $\mathbf{F}$ is a perfectoid field, we let $\mathbf{F}^0$ denote the subring $\mathbf{F}^{\text{p-bdd}}$. We have $\mathbf{F} = (\mathbf{F}^0)_\varpi$.

For example, for $\mathbf{F} = \mathbb{F}_p((t^{\frac{1}{p^\infty}}))$, we have $\mathbf{F}^0 = \mathbb{F}_p[[t^{\frac{1}{p^\infty}}]]$ and for $\mathbf{F} = \mathbb{Q}_p^{\text{cycl}}$, we have $\mathbf{F}^0 = \mathbb{Z}_p^{\text{cycl}}$.

Here is a typical example of a perfectoid ring that is not a field. Start with a perfectoid field $\mathbf{F}$. Set

$$R^+ := \mathbf{F}^0\langle s_1^{\frac{1}{p^\infty}}, ..., s_m^{\frac{1}{p^\infty}}\rangle,$$

i.e., the completion in the topology coming from $\mathbf{F}^0$ of

$$\mathbf{F}^0[s_1^{\frac{1}{p^\infty}}, ..., s_m^{\frac{1}{p^\infty}}] = \underset{n}{\cup}\, \mathbf{F}^0[s_1^{\frac{1}{n}}, ..., s_m^{\frac{1}{n}}],$$

and set

$$R := (R^+)_\varpi.$$

2.1.7. The category of of affinoid perfectoids, denoted $\text{Perfctd}^{\text{aff}}$ is by definition the full subcategory of affinoid adic spaces (R, R^+) with R perfectoid ring.

2.2. **Creating the relative curve (equal characteristic case).** As was mentioned above, the sought-for prediamond Bun_G is supposed to assign to

$$S = \text{Spa}(R, R^+) \in \text{Perfctd}^{\text{aff}}_{/\mathbb{F}_p}$$

the groupoid of G-bundles on a certain geometric object X_S.

We will now indicate what this X_S (for the actual definition, see Sect. 2.3 below). We will first consider the case of the local field $\mathbf{K} = \mathbb{F}_p((z))$, which would later motivate the construction for $\mathbf{K} = \mathbb{Q}_p$.

A good reference for what follows is Chapter 10 in [Fe].

2.2.1. We start with a perfectoid affinoid $S = \text{Spa}(R, R^+)$ over $\mathbb{F}_p$. Consider the topological ring

$$A_{\text{inf},S} := R^+[[z]]$$

over $\mathbb{F}_p[[z]]$.

Tautologically, $A_{\text{inf},S}$ has the following universal property: for a ring R' over $\mathbb{F}_p[[z]]$, compete in the z-adic topology, the datum of a map of $\mathbb{F}_p[[z]]$-algebras

$$A_{\text{inf},S} \to R'$$

is equivalent to the datum of a homomorphism of $\mathbb{F}_p$-algebras

$$R^+ \to R'.$$

Remark 2.2.2. The above forgetful functor

$$\{\mathbb{F}_p[[z]]\text{-algebras complete in the } z\text{-adic topology}\} \to \{\mathbb{F}_p\text{-algebras}\}$$

in the equal characteristic analog of the functor of tilt:

$$\{\mathbb{Z}_p\text{-algebra complete in the } p\text{-adic topology}\} \to \{\text{perfect } \mathbb{F}_p\text{-algebras}\},$$

to be discussed later.

In the equal characteristic case, the above functor has an obvious left adjoint

$$R^+ \mapsto R^+[[z]].$$

We will soon see what replaces this left adjoint in the mixed characteristic case.

2.2.3. We consider the analytic space

$$\mathrm{Spa}(A_{\mathrm{inf},S}, A_{\mathrm{inf},S})$$

and we will remove from it the closed subspace defined by the ideal generated by $z \cdot \varpi$.

By definition, this means that we are considering a subfunctor on affinoid adic spaces represented by $\mathrm{Spa}(A_{\mathrm{inf},S}, A_{\mathrm{inf},S})$ that sends $\mathrm{Spa}(R_1, R_1^+)$ to the set of maps

$$A_{\mathrm{inf},S} \to R_1^+$$

such that the images of z and ϖ are invertible in R_1.

The resulting analytic space is the sought-for Y_S.

Note that Y_S is *not affinoid*. Rather, in Sect. 2.3 we will describe Y_S as the union of affinoid spaces Y_S^I, taken over closed intervals

$$I \subset (0,1),$$

to be thought of closed (relative over S) annuli corresponding to points $|z| \in I$.

2.2.4. By transport of structure, the Frobenius automorphism φ_S of the pair (R, R^+) induces an endomorphism of A_{inf}, and hence on Y_S. In fact, it will map the affinoid Y_S^I isomorphically to the affinoid $Y_S^{I^{\frac{1}{p}}}$, where for $I = (a, b)$ we have $I^{\frac{1}{p}} := (a^{\frac{1}{p}}, b^{\frac{1}{p}})$.

This implies that the resulting action of $\mathbb{Z}$ on Y_S is *properly discontinuous* in a suitable sense, and we can form the analytic space quotient

$$X_S := Y_S/\varphi_S.$$

It is covered by affinoid adic spaces Y_S^I for I such that $I \cap I^p = \emptyset$.

This is the desired analytic quotient of the (relative over S) punctured open unit disc by the action of the Frobenius.

2.2.5. We have a theory of vector bundles over analytic spaces. Using the Tannakian formalism, we can thus make sense of G-bundles on X_S. Thus, we obtain the sought-for groupoid

$$\mathrm{Hom}(S, \mathrm{Bun}_G).$$

2.3. **The (punctured open) unit disc.** In the equal characteristic case, Y_S is just the punctured open unit disc over S, denoted below by $\mathbb{D}_S^{(0,1)}$ We will now explain what it is.

2.3.1. Let (R, R^+) be a Tate pair, i.e., R is a Tate ring, and $R^+ \subset R$ an open subring contained in $R^{\text{p-bdd}}$. For motivational purposes, we might as well take R to be a local field $\mathbf{F}$ (not to be confused with "our" local field $\mathbf{K}$!) and $R^+ := \mathcal{O}_{\mathbf{F}}$).

In any case, we will assume that the topology on R comes from a multiplicative norm in which R is complete (i.e., R is a Banach algebra), and R^+ is the unit ball in R.

Denote $S = \mathrm{Spa}(R, R^+)$.

2.3.2. Perhaps, the most basic disc-like object associated to S is the *closed unit disc* over S of radius 1, denoted $\mathbb{D}_S^{[0,1]}$. By definition.

$$\mathbb{D}^{[0,1]} = \operatorname{Spa}(R\langle z\rangle, R^+\langle z\rangle),$$

where $R^+\langle z\rangle$ is the completion of $R^+[z]$, i.e., the set of

$$\sum_{n\geq 0} r_n \cdot z^n, \quad r_n \to 0 \text{ in } R^+,$$

and $R\langle z\rangle = (R^+\langle z\rangle)_\varpi$.

To get a feel for why $\mathbb{D}_S^{[0,1]}$ is indeed the unit disc, let us describe its points with coefficients in some non-archimedean valued field $\mathbf{C}$. Unwinding the definitions, a datum of such a point is a homomorphism $R \to \mathbf{C}$ (so that R^+ automatically maps to $\mathcal{O}_\mathbf{C}$), and an element $c \in \mathbf{C}$ with $|c| \leq 1$. The latter condition is what ensures the convergence of the series

$$\sum_{n\geq 0} r_n \cdot c^n \in \mathbf{C} \text{ for } \sum_{n\geq 0} r_n \cdot z^n \in R\langle z\rangle.$$

2.3.3. In a similar way one defines the closed disc $\mathbb{D}_S^{[0,\rho]}$ of radius ρ for any $\rho \in \mathbb{R}^{>0}$.

Namely, it is defined in the same way, modulo replacing $R^+\langle z\rangle$ by $R^+\langle z\rangle^\rho$, where the latter consists of

$$\sum_{n\geq 0} r_n \cdot z^n, \quad |r_n| \cdot \rho^n \to 0,$$

where $r \mapsto |r|$ is the norm on R.

Note that any $\rho' \leq \rho$ defines a semi-norm $|-|_\rho$ on $R\langle z\rangle^\rho$:

$$|\sum_{n\geq 0} r_n \cdot z^n|_\rho := \sup_n |r_n| \cdot (\rho')^n.$$

2.3.4. Similar definitions apply when we have several variables $z_1, ..., z_k$. I.e., we can create the closed multi-disc with radii $(\rho_1, ..., \rho_k)$:

$$\operatorname{Spa}(R\langle z_1, ..., z_k\rangle^{\rho_1,...,\rho_k}, R^+\langle z_1, ..., z_k\rangle^{\rho_1,...,\rho_k}).$$

2.3.5. For a closed interval $I = [\rho_1, \rho_2] \subset \mathbb{R}^{>0}$ one defines the closed annulus $\mathbb{D}_S^I$ to be $\operatorname{Spa}(R\langle z\rangle_I, R^+\langle z\rangle^I)$, where

$$R\langle z\rangle^I = R\langle z_1, z_2\rangle_{\rho_1^{-1},\rho_2}/z_1 \cdot z_2 - 1,$$

and similarly for $R^+\langle z\rangle_I$.

2.3.6. However, we can also define $R\langle z\rangle^I$ differently, and this will be of use for us later:

Say for simplicity that $\rho_2 \leq 1$. Then $R\langle z\rangle^I$ is the completion of $(R\langle z\rangle)_z$ with respect to the semi-norms $|-|_{\rho_1}$ and $|-|_{\rho_2}$ (equivalently, we can take the completion with respect to the family of semi-norms $|-|_{\rho'}$ for all $\rho_1 \leq \rho' \leq \rho_2$).

Note also that if $\rho_2 < 1$, we have a natural map

$$(R^+((z)))_\varpi = (R^+[[z]])_{z\cdot\varpi} \to R\langle z\rangle^I,$$

and $R\langle z\rangle^I$ identifies the completion of $(R^+((z)))_\varpi$ with respect to the above semi-norms. (Note that for $\rho < 1$, the corresponding semi-norm $|-|_\rho$ continuously extends from $(R\langle z\rangle)_z$ to $(R^+((z)))_\varpi$.)

The subring $R^+\langle z\rangle^I \subset R\langle z\rangle^I$ is the subset of elements with norm ≤ 1 with respect to both $|-|_{\rho_1}$ and $|-|_{\rho_2}$.

2.3.7. We can now introduce other versions of the unit disc:

The open unit disc $\mathbb{D}_S^{[0,1)}$ (resp., punctured unit disc $\mathbb{D}_S^{(0,1]}$, open punctured unit disc $\mathbb{D}_S^{(0,1)}$) is defined as the colimit of the closed annuli $\mathbb{D}_S^I$ over $I \subset [0,1)$ (resp., $I \subset (0,1]$, $I \subset (0,1)$).

2.3.8. The corresponding spaces of global functions are defined to be the limits of functions on the $\mathbb{D}_S^I$'s that comprise our version of the disc.

Note that we have maps

$$(R^+[[z]])_\varpi \mapsto \Gamma(\mathbb{D}_S^{[0,1)}, \mathcal{O}_{\mathbb{D}_S^{[0,1)}}),\ (R\langle z\rangle)_z \mapsto \Gamma(\mathbb{D}_S^{(0,1]}, \mathcal{O}_{\mathbb{D}_S^{(0,1]}}),$$

and what is of most of relevance for us, the map

$$(R^+((z)))_\varpi = (R^+[[z]])_{z\cdot\varpi} \to \Gamma(\mathbb{D}_S^{(0,1)}, \mathcal{O}_{\mathbb{D}_S^{(0,1)}}).$$

2.3.9. It also follows that $\Gamma(\mathbb{D}_S^{(0,1)}, \mathcal{O}_{\mathbb{D}_S^{(0,1)}})$ can be described as the completion of $(R^+((z)))_\varpi$ with respect to the family of semi-norms $|-|_\rho$ for $0 < \rho < 1$ introduced above (or we can take the ρ's belonging to two sequences, one converging to 1 and another to 0).

2.3.10. *Action of the Frobenius.* Assume that S is over $\mathbb{F}_p$, and let us comment on the action of the Frobenius ϕ_S on Y_S.

The action of ϕ_S on an element

$$\sum_{n\geq 0} r_n \cdot z^n \in (R^+[[z]])_{z\cdot\varpi}$$

is given by

$$\sum_{n\geq 0} (r_n)^p \cdot z^n.$$

I.e.,

$$|\phi_S(f)|_\rho = (|\phi_S(f)|_{\rho^{\frac{1}{p}}})^p.$$

It follows that the action of ϕ_S on Y_S is such that it maps each

$$Y_S^I := \mathbb{D}_S^I$$

to the corresponding $Y_S^{I^{\frac{1}{p}}}$. I.e., this action "expands" the radii.

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