LECTURE 7: MONADICITY OF THE BOUSFIELD-KUHN FUNCTOR

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This talk is based on upcoming work of Heuts, the paper "Monadicity of the Bousfield-Kuhn functor" by Eldred, Heuts, Mathew, Meier, and "A short proof of telescopic Tate vanishing" by Clausen and Mathew. The reader is encouraged to consult the latter two sources, as much of these notes are just a slightly rearranged version of them.

Everything will implicitly be localized at a prime p and notations should be consistent from the other notes given for the seminar.

Recollections about the Bousfield-Kuhn functor. Let's continue our discussion of the Bousfield-Kuhn functor by briefly reviewing some highlights from last time.

We started with the telescopic functors; for each finite type n space V with a v_n-self map $v : \Sigma^t V \to V$, this is a functor $\Phi_V : \mathcal{S}_* \to \mathrm{Sp}$ defined by the formula

$$\Phi_V(X) = \operatorname{colim}\left(\Sigma^\infty \operatorname{Map}_*(V, X) \to \Sigma^{\infty-t} \operatorname{Map}_*(V, X) \to \cdots\right)$$

where each of the maps are induced by v. This turned out to be functorial in the category of finite *spectra* of type at least n, $\mathrm{Sp}^{\mathrm{fin}}_{\geq n}$. This allowed us to define the Bousfield-Kuhn functor

$$\Phi(X) = \lim_E \Phi_E(X),$$

where the limit is taken over the ∞-category of spectra $E \in \mathrm{Sp}^{\mathrm{fin}}_{\geq n}$ equipped with a map $E \to S^0$. We showed it had the following properties.

(1) Each Φ_V takes values in $T(n)$-local spectra, so $\Phi(X)$ is $T(n)$-local for any pointed space X.
(2) There are equivalences $\Phi_E(X) \simeq \Phi(X)^E$ depending functorially on $E \in \mathrm{Sp}^{\mathrm{fin}}_{\geq n}$ and $X \in \mathcal{S}_*$.
(3) The functor Φ takes v_n-periodic homotopy equivalences of spaces to equivalences of $T(n)$-local spectra, so it defines a functor $\Phi : \mathcal{S}_*^{v_n} \to \mathrm{Sp}_{T(n)}$.
(4) This new Φ then admits a left adjoint, $\Theta : \mathrm{Sp}_{T(n)} \to \mathcal{S}_*^{v_n}$.

There's one more fundamental property that we should add to this list.

Proposition 1. *Let X be a spectrum. Then there is a natural equivalence*

$$\Phi(\Omega^\infty X) \simeq L_{T(n)} X.$$

This says that the $T(n)$-localization of a spectrum depends only on the underlying space and not on the infinite loop space structure. As Mike explained, this was one of the original motivations for studying the Bousfield-Kuhn functor.

Proof. Let us first do the analogous computation for Φ_V, where V is a finite type n space with a v_n-self map $v : \Sigma^t V \to V$. Here,

$$\begin{aligned}\Omega^\infty \Phi_V(\Omega^\infty X) &= \operatorname{colim}\big(\operatorname{Map}_*(V, \Omega^\infty X) \to \Omega^t \operatorname{Map}_*(V, \Omega^\infty X) \to \cdots\big)\\ &= \operatorname{colim}\big(\operatorname{Map}(\Sigma^\infty V, X) \to \operatorname{Map}(\Sigma^{\infty - t} V, X) \to \cdots\big)\\ &= \operatorname{colim}\big(\Omega^\infty DV \wedge X \to \Omega^\infty \Sigma^{-t} DV \wedge X \to \cdots\big)\\ &= \Omega^\infty DV[v_n^{-1}] \wedge X\\ &= \Omega^\infty DV[v_n^{-1}] \wedge L_{T(n)} X.\end{aligned}$$

It follows that we're free to replace X by $L_{T(n)}X$ in the whole calculation. Now, consider the second line of the calculation: I claim that this sequence is constant. This is because the cofiber of a v_n-self map on a type n complex is of type $n+1$, and $L_{T(n)}X$ receives no maps from type $n+1$ complexes. This implies that we have the formula

$$\Phi_V(\Omega^\infty X) = (L_{T(n)}X)^V.$$

This certainly still holds replacing V with any type n finite spectrum E.

Finally, this implies that

$$\begin{aligned}\Phi(\Omega^\infty X) &= \lim_E \Phi_E(\Omega^\infty X)\\ &= \lim_E (L_{T(n)}X)^E\\ &= L_{T(n)}X\end{aligned}$$

where the reader can refer to the remark below if the last equality is not clear. □

Remark 2. A friendly reminder of what happens in the stable case: there are localization functors L_n^f which kill exactly the finite spectra of type $\geq n+1$. Analogous to (but simpler than) what we have been discussing in the unstable case, the $T(n)$-local spectra are the layers between L_n^f and L_{n-1}^f. More precisely, there is an equivalence

$$L_{T(n)} : M_n^f(\mathrm{Sp}) \xrightarrow{\sim} \mathrm{Sp}_{T(n)}$$

where $M_n^f(\mathrm{Sp})$ is the category of L_n^f-local spectra X which satisfy $L_{n-1}^f X \simeq *$. The inverse equivalence is by taking the fiber of the L_{n-1}^f-localization map.

That means that any finite p-local spectrum X fits into a fiber sequence

$$C_{n-1}^f X \to X \to L_{n-1}^f X$$

where the last term is $T(n)$-acyclic and the fiber, $C_{n-1}^f X$, is a filtered colimit of finite spectra of type $\geq n$. In particular, this means that, $T(n)$-locally, the sphere is a filtered colimit of finite spectra of type $\geq n$.

The monad $\Phi\Theta$. At this point, let's remind ourselves where we're going with this story. We've produced an adjunction

$$\Phi : \mathcal{S}_*^{v_n} \rightleftarrows \mathrm{Sp}_{T(n)} : \Theta.$$

The category on the left is the unstable analog of the one on the right. The whole story was motivated by what happens at height $n = 0$; in that case, we're trying to understand the relationship between rational spaces and rational spectra. Quillen's theorem was that simply-connected rational pointed spaces are equivalent to connected differential graded Lie algebras over $\mathbb{Q}$. Rational chain complexes are the same as rational spectra, so I could suggestively write:

$$\mathcal{S}_*^{\mathbb{Q},sc} \xrightarrow{\sim} \mathrm{Lie}(\mathrm{Sp}_{\mathbb{Q}}^{\geq 1}).$$

Quillen's equivalence is implemented by taking homotopy; we've seen that the Bousfield-Kuhn functor is a spectrum-version of extracting v_n-periodic homotopy. Thus, we'd like to show the analogous statement:

$$\Phi : \mathcal{S}_*^{v_n} \xrightarrow{\sim} \mathrm{Lie}(\mathrm{Sp}_{T(n)}).$$

This is saying two things:

(1) $\mathcal{S}_*^{v_n}$ is the ∞-category of algebras for the monad $\Phi\Theta$.
(2) The monad $\Phi\Theta$ on $\mathrm{Sp}_{T(n)}$ is taking the "free Lie algebra."

The first of these is our goal for today; more precisely:

Theorem 3 (Eldred-Heuts-Mathew-Meier)**.** *The functor Φ lifts to an equivalence of ∞-categories*

$$\Phi : \mathcal{S}_*^{v_n} \simeq \mathrm{Alg}_{\Phi\Theta}(\mathrm{Sp}_{T(n)}).$$

i.e., the adjunction (Θ, Φ) is monadic.

This winds up being pretty straightforward given all the technology that we've been developing.

Fact (Barr-Beck, Lurie)**.** In this situation, it suffices to show that the right adjoint Φ is conservative and preserves geometric realizations of simplicial objects.

Remark. If this isn't familiar, think about the theory of monoids - the conservative condition is saying that a map of monoids is an equivalence iff it's an equivalence in the underlying category. On the other hand, a monoid structure is defined by products, and geometric realizations are precisely the sort of colimits that commute with products (this is why you put degeneracies in simplicial sets). The second condition is saying that geometric realizations in monoids are computed by computing them in the underlying category and giving it the natural monoid structure.

In our case, the conservative condition is immediate: let $f : X \to Y$ be a map in $\mathcal{S}_*^{v_n}$ which induces an equivalence

$$\Phi(f) : \Phi(X) \to \Phi(Y).$$

This implies that we get an equivalence

$$\Phi_V(f) : \Phi_V(X) \to \Phi_V(Y)$$

for any finite space V of type n with a v_n-self map, and so by definition, f induces an equivalence on v_n-periodic homotopy. Thus, f is an equivalence in $\mathcal{S}_*^{v_n}$.

Remark 4. In fact, a map $g : W \to Z$ is a $T(n)$-local equivalence if and only if the induced map $g' : V \wedge W \to V \wedge Z$ is an equivalence for some finite type n space V. This is because we may choose $v : \Sigma^{\infty+t} V \to \Sigma^\infty V$ a v_n-self map, and then $(\Sigma^\infty V)[v^{-1}]$ has the same Bousfield class as $T(n)$. We may therefore assume that $T(n) \simeq (\Sigma^\infty V)[v^{-1}]$. Consequently, an equivalence $g' : V \wedge W \to V \wedge Z$ yields an equivalence $T(n) \wedge W \to T(n) \wedge Z$, which means that g is a $T(n)$-local equivalence.

Commuting with geometric realizations. This might seem like a strange statement. The Bousfield-Kuhn functor is like forming a mapping space $\mathrm{Map}_*(V, X)$ - that's something that doesn't generally commute with geometric realizations. If the theorem is true, something has to save us...

Digression: Let V be a pointed space. To what extent does $\mathrm{Map}_*(V, -) : \mathcal{S}_* \to \mathcal{S}_*$ commute with geometric realizations? It certainly works for $V = S^0$, since that's just the identity functor.

Example. Let's look at the case $V = S^1$. We're asking if the functor $\Omega : \mathcal{S}_* \to \mathcal{S}_*$ commutes with geometric realization.

This is visibly false. Take the simplicial pointed space $T_\bullet$ where $T_0 = \{*, a\}$ and $T_1 = \{*, e_{*a}, e_{a*}\}$ are discrete pointed spaces with basepoint $*$ and the only non-degenerate simplices are the edges e_{*a} and e_{a*}, which go between $*$ and a. The problem is that Ω forgets about π_0.

However, if we restrict to connected spaces, things are fine. The reason is that the map $\Omega : \mathcal{S}_*^{\geq 1} \to \mathcal{S}_*$ lifts through an equivalence

$$\begin{array}{ccc} & & \text{Grouplike } A_\infty\text{-Spaces} \\ & \overset{\sim}{\nearrow} & \big\downarrow {\scriptstyle \text{forget}} \\ \mathcal{S}_*^{\geq 1} & \longrightarrow & \mathcal{S}_* \end{array}$$

and the forgetful functor from grouplike A_∞-spaces preserves geometric realizations. This kind of argument generalizes to show that $\Omega^n : \mathcal{S}_*^{\geq n} \to \mathcal{S}_*$ commutes with geometric realizations. This should motivate the following general statement:

Proposition 5. *Let $V \in \mathcal{S}_*$ be a finite space of dimension d. Then, the functor*

$$\mathrm{Map}_*(V, -) : \mathcal{S}_*^{\geq d} \to \mathcal{S}_*$$

commutes with geometric realizations of simplicial objects.

We've seen this is true for spheres (though it won't be logically necessary); the thing that isn't clear is that we can glue them together. For that, we'll need the following fact about commuting geometric realizations and pullbacks in spaces:

Lemma 6. *Let $X_\bullet, Y_\bullet, B_\bullet \in \mathcal{S}^{\Delta^{op}}$ be simplicial spaces, and assume that B_n is connected for all n. Then, the natural map*

$$|X_\bullet \times_{B_\bullet} Y_\bullet| \to |X_\bullet| \times_{|B_\bullet|} |Y_\bullet|$$

from the geometric realization of the levelwise homotopy pullback is an equivalence.

Remark. We thank Ben Knudsen for pointing out that the following reasoning for this lemma may be circular as the proof of the equivalence between group-like A_∞ spaces and loop spaces relies on this statement. We offer it anyway as a useful way of thinking about this lemma. For a more in-depth exposition, the reader is encouraged to look at Charles Rezk's notes (https://faculty.math.illinois.edu/ rezk/i-hate-the-pi-star-kan-condition.pdf).

"Proof". This is powered by a relative version of the observation above, which one might call Koszul duality for spaces:

Observation. Let (A, a) be a connected pointed space. Then, consider the functor category $\mathrm{Fun}(A, \mathcal{S})$. On the one hand, it's giving you local systems on A, which is the category of spaces over A. On the other hand, it's $\mathrm{Fun}(B(\Omega A), \mathcal{S})$, which is spaces with an action of the group ΩA. As a result, there is an equivalence of ∞-categories

$$\{\text{Spaces } X \text{ with a map } X \to A\} \simeq \{\text{Spaces } Y \text{ with an action of } \Omega A\}$$

which is implemented by sending $f : X \to A$ to the fiber $Y = f^{-1}(a)$ together with the action of ΩA.

This equivalence is functorial in the connected pointed space (A, a), which means we actually can vary our base to get an equivalence of ∞-categories $\mathcal{C} \simeq \mathcal{C}^\vee$ where

$$\mathcal{C} := \{(X, A, f) | X \in \mathcal{S}, A \in \mathcal{S}_*^{\geq 1}, f : X \to A\},$$
$$\mathcal{C}^\vee := \{(G, X) | G \in \mathcal{S} \text{ a group}, X \in \mathcal{S}^{BG}\}.$$

As a consequence of this, suppose we have a simplicial object $X_\bullet \to B_\bullet$ of $\mathcal{C}$. We can compute its geometric realization in either $\mathcal{C}$ or $\mathcal{C}^\vee$, since they're equivalent. On the other hand, geometric realizations are computed pointwise in $\mathcal{C}^\vee$ because geometric realizations commute with products. This is saying that geometric realization commutes with taking the fiber in this scenario. Since geometric realizations also commute with taking products, it follows that they also commute with fiber products, as desired.

□

The proof of the proposition is now easy.

Proof of Proposition 5. Let $X_\bullet$ be a simplicial space such that each X_n is $(d-1)$-connected. Consider the collection $\mathcal{A}$ of all finite spaces V such that the natural map $|\operatorname{Map}_*(V, X_\bullet)| \to \operatorname{Map}_*(V, |X_\bullet|)$ is an equivalence. We argue by induction that $\mathcal{A}$ contains all finite pointed spaces of dimension at most d. For dimension 0, it clearly contains S^0, and since geometric realizations commute with finite products, it is closed under wedge sums.

Now suppose T is a finite space of dimension $m \leq d$. We may write T as a pushout:

$$\begin{array}{ccc} \bigvee S^{m-1} & \longrightarrow & \mathrm{sk}_{m-1} T \\ \downarrow & & \downarrow \\ * & \longrightarrow & T \end{array}$$

The inductive hypothesis tells us all but the bottom right corner are in $\mathcal{A}$. Since $m \leq d$ and each X_n is $(d-1)$-connected, we know that $\operatorname{Map}_*(\bigvee S^{m-1}, X_n)$ is connected. Lemma 6 then implies the result. □

In fact, Proposition 5 together with the flexibility in choosing the embedding $\mathcal{S}_*^{v_n} \hookrightarrow \mathcal{S}_*$ powers the rest of the proof, as we shall now see:

Proof of Theorem, continued. Let $X_\bullet$ be a simplicial diagram in $\mathcal{S}_*^{v_n}$. We want to show that the natural map

$$|\Phi(X_\bullet)| \to \Phi(|X_\bullet|) \tag{*}$$

is a $T(n)$-local equivalence. We'll do so by a series of reductions:

(i) Choose a finite type n space V with a v_n-self map $v : \Sigma^t V \to V$. By Remark 4, it suffices to prove (*) is an equivalence after smashing with the dual DV. By Spanier-Whitehead duality, this means it suffices to show that the map

$$|\Phi(X_\bullet)|^V \to (\Phi(|X_\bullet|))^V$$

is an equivalence.

(ii) This is happening in spectra, where geometric realizations commute with finite limits, so the left-hand side may be identified with $|\Phi(X_\bullet)^V|$.

(iii) By property (2) of the Bousfield-Kuhn functor from the beginning of the lecture, we have that $\Phi(X)^V \simeq \Phi_V(X)$ for any space X, so it suffices to show

$$|\Phi_V(X_\bullet)| \simeq \Phi_V(|X_\bullet|). \qquad (**)$$

(iv) Here, we're regarding Φ_V as a functor from $\mathcal{S}_*^{v_n}$. In particular, the geometric realization $|X_\bullet|$ in (**) is taken inside $\mathcal{S}_*^{v_n}$.

Question. How do you compute a colimit in $\mathcal{S}_*^{v_n}$?

This is a good time to recall how we constructed the category $\mathcal{S}_*^{v_n}$ – let A, B be finite $(d-1)$-connected spaces which are suspensions, and such that $\mathrm{type}(B) = n = \mathrm{type}(A) - 1$. The space A determined

$$L_n^f\,\mathcal{S}_*^{\langle d\rangle} := \{\text{the full subcategory of } \mathcal{S}_* \text{ which are } \mathcal{P}_A\text{-local and } d\text{-connected}\}$$

which could also be thought of as the localization of spaces at the collection of maps which are simultaneously v_k-periodic equivalences for $0 \le k \le n$. The space B then determined a further localization

$$\mathcal{P}_B : L_n^f\,\mathcal{S}_*^{\langle d\rangle} \to L_{n-1}^f\,\mathcal{S}_*^{\langle d\rangle}.$$

Finally, $\mathcal{S}_*^{v_n} \subset L_n^f\,\mathcal{S}_*^{\langle d\rangle}$ was the full subcategory where $\mathcal{P}_B$ vanishes.

Since localizations commute with colimits, it's clear that the subcategory $\mathcal{S}_*^{v_n} \subset L_n^f\,\mathcal{S}_*^{\langle d\rangle}$ is closed under colimits. Hence, colimits are computed simply in $L_n^f\,\mathcal{S}_*^{\langle d\rangle}$. But there, colimits are just computed by computing the corresponding colimit in the category $\mathcal{S}_*^{\langle d\rangle}$ of d-connected spaces, and then localizing. But the localization map $L_n^f : \mathcal{S}_*^{\langle d\rangle} \to L_n^f\,\mathcal{S}_*^{\langle d\rangle}$ induces an equivalence on v_n-periodic homotopy, and thus after applying Φ_V. Hence, it suffices to show that $\Phi_V : \mathcal{S}_*^{\langle d\rangle} \to \mathrm{Sp}_{T(n)}$ commutes with geometric realizations. Of course, colimits in $\mathcal{S}_*^{\langle d\rangle}$ are just computed in pointed spaces, but we will need to use the fact that our spaces are sufficiently connected to verify the statement.

(v) We now examine the formula

$$\Phi_V(X) = \mathrm{colim}(\Sigma^\infty \mathrm{Map}_*(V, X) \to \Sigma^{\infty - t} \mathrm{Map}_*(V, X) \to \cdots)$$

and note that Σ^∞, colim, and Σ^{-t} commute with all colimits. Thus, we only need to show that $\mathrm{Map}_*(V, -) : \mathcal{S}_*^{\langle d\rangle} \to \mathrm{Sp}_{T(n)}$ commutes with geometric realizations.

(vi) There are two moving parts here - the spaces A and B we chose to define $\mathcal{S}_*^{v_n}$ with, which were $(d-1)$-connected, and the finite type n space V that we chose to use to check that this particular map $\Phi(|X_\bullet|) \to |\Phi(X_\bullet)|$ was an equivalence. We saw that the category $\mathcal{S}_*^{v_n}$ is independent of which A and B we chose; we may therefore choose them so that $d \ge \dim(V)$. In this case, each of the $X_\bullet$ are by definition d-connected, and Proposition 5 implies the result.

□

Examples. This result means that we can think of Θ as taking a free algebra over a monad. In fact, it's possible to compute Θ explicitly in some cases.

Let W be a finite spectrum of type n. We might try to compute Θ on it by mapping into a v_n-periodic space X:

$$\begin{aligned}\mathrm{Map}_{\mathcal{S}_*^{v_n}}(\Theta(L_{T(n)}W), X) &\simeq \mathrm{Map}_{\mathrm{Sp}}(L_{T(n)}W, \Phi(X)) \\ &\simeq \Omega^\infty \Phi_W(X).\end{aligned}$$

One of the ideas we have seen in the previous lectures was that the v_n-periodic homotopy of a space which has no v_k-periodic homotopy for $k > n$ (such as X) is easy to compute. This came from observing that the latter space $\Omega^\infty \Phi_W(X)$ was defined by a colimit which was eventually constant, equal to $\mathrm{Map}(\Sigma^j W, X)$ for some j. The following lemma captures the conditions under which the colimit is *actually* constant, and gives a computation of Θ in this case:

Lemma 7. *Let V be a $(d-2)$-connected finite space of type n with a v_n-self map $v : \Sigma^t V \to V$. Then, there is a canonical equivalence*

$$\Theta(L_{T(n)}\Sigma^\infty \Sigma^2 V) \simeq L_n^f \Sigma^2 V.$$

Remark. As a sanity check, one should see that $L_n^f \Sigma^2 V \in \mathcal{S}_*^{v_n}$. It is certainly d-connected and p-local, so it suffices to see that it is $\mathcal{P}_B$-acyclic. But $L_{n-1}^f L_n^f \Sigma^2 V = L_{n-1}^f \Sigma^2 V = *$, but since $L_n^f \Sigma^2 V$ is d-connected, that means $\mathcal{P}_B$ vanishes on it.

Proof. Consider the cofiber sequence

$$\Sigma^t V \xrightarrow{v} V \to \mathrm{cof}(v).$$

The space $\mathrm{cof}(v)$ has type $n+1$, but it may not be a suspension, so $\mathcal{P}_A$ may not vanish on it. On the other hand, $\Sigma\mathrm{cof}(v)$ *is* a suspension space which is $(d-1)$-connected and type $n+1$. Bousfield's theorem then implies that $\mathcal{P}_A \Sigma^i \mathrm{cof}(v) \simeq *$ for any $i \geq 1$.

Let $X \in \mathcal{S}_*^{v_n}$ and consider the cofiber sequence

$$\Sigma\mathrm{cof}(v) \to \Sigma^{2+t} V \to \Sigma^2 V \to \Sigma^2 \mathrm{cof}(v).$$

The previous observation implies that

$$\mathrm{Map}_*(\Sigma^i \mathrm{cof}(v), X) \simeq \mathrm{Map}_*(L_n^f \Sigma^i \mathrm{cof}(v), X) \simeq *$$

for any $i \geq 1$ because X is L_n^f-local. This means that

$$\mathrm{Map}_*(\Sigma^2 V, X) \simeq \mathrm{Map}_*(\Sigma^{2+t} V, X) \simeq \mathrm{Map}_*(\Sigma^{2+2t} V, X) \simeq \cdots .$$

Thus, for any $X \in \mathcal{S}_*^{v_n}$, we have:

$$\begin{aligned}
\mathrm{Map}(\Theta(L_{T(n)}\Sigma^\infty \Sigma^2 V), X) &\simeq \mathrm{Map}(L_{T(n)}\Sigma^\infty \Sigma^2 V, \Phi(X)) \\
&\simeq \Omega^\infty \Phi_{\Sigma^2 V}(X) \\
&\simeq \mathrm{Map}_{\mathcal{S}_*}(\Sigma^2 V, X) \\
&\simeq \mathrm{Map}_{\mathcal{S}_*^{v_n}}(L_n^f \Sigma^2 V, X).
\end{aligned}$$

□

This tells you what Θ is on (the $T(n)$-localization of) a general type n finite spectrum E. Choose a v_n-self map $v : \Sigma^t E \to E$ and choose an integer k such that $\Sigma^{kt} E = \Sigma^2 V$ for V satisfying the conditions of the lemma above. Then, since $v^k : \Sigma^{kt} E \to E$ becomes an equivalence $T(n)$-locally (in fact, L_n^f-locally), we have

$$\Theta(L_{T(n)} E) \simeq \Theta(L_{T(n)}\Sigma^{kt} E) \simeq L_n^f \Sigma^2 V.$$

This gives a general procedure for computing $\Theta(X)$ for a $T(n)$-local finite spectrum X:

(1) First, construct a sequence $F(0) \to F(1) \to F(2) \to \cdots \to X$ of finite type n spectra $F(i)$ such that the natural map $\mathrm{colim}\, F(i) \to X$ is an equivalence after $T(n)$-localization. This can be done explicitly.
(2) Then, apply the procedure of the previous paragraph to find, for each i, an appropriate finite type n space $V(i)$.
(3) These $V(i)$'s have maps between them induced by the maps on the $F(i)$ (up to possibly composing with v_n-self maps; one should be careful here).

(4) Since Θ is a left adjoint, it commutes with filtered colimits, and we may compute

$$\Theta(S^0) \simeq \operatorname{colim} \Theta(F(k)) \simeq \operatorname{colim} L_n^f(\Sigma^2 V(k)),$$

where again, one should be careful that the $V(k)$ only map to each other up to a v_n-self map, which becomes an equivalence after localization.

This also gives a procedure for "computing" the Bousfield-Kuhn functor in certain cases: the point is that we can apply Φ to both sides of Lemma 7. Hence, if V is a finite $(d-2)$-connected type n space with a v_n-self map, we find that

$$\Phi(\Sigma^2 V) = \Phi(L_n^f \Sigma^2 V) = \Phi\Theta(L_{T(n)} \Sigma^\infty \Sigma^2 V).$$

In fact, there's a trick to see that the connectivity hypothesis is not necessary: both sides of this equation are independent of our choice of A. In particular, we could have chosen A to be the type $n+1$ complex $\Sigma \mathrm{cof}(v)$ and everything goes through as before.

Tate Vanishing. In this brief section, I'll explain a proof by Clausen and Mathew of Tate vanishing in the telescopically localized category. For simplicity, we will just do the case $G = C_2$ and work at the prime 2. Before stating the theorem, let me remind you of some of the ingredients:

(1) Associated to the covering $EC_2 \to BC_2$, there is a stable transfer map $\mathrm{tr} : \Sigma^\infty_+ BC_2 \to S^0$. For instance, one could define this by considering the diagonal map

$$S^0 \xrightarrow{(1,1)} S^0 \vee S^0$$

with the left-hand side given the trivial action and the right-hand side given the swap action, and then taking homotopy orbits.

(2) This fits into a diagram with the norm map

$$\begin{array}{ccccc} (S^0)_{hC_2} & \xrightarrow{N} & (S^0)^{hC_2} & \longrightarrow & (S^0)^{tC_2} \\ & \searrow^{\mathrm{tr}} & \downarrow{\scriptstyle r} & & \\ & & S^0 & & \end{array}$$

where the downward map r is given by restriction to the basepoint of BC_{2+}.

(3) We'll need the interaction of this with the monoidal structure. To illustrate this, consider the situation in classical algebra: let R be a ring with C_2-action. Then, the fixed points R^{C_2} form a ring. Moreover, the norms from the orbits, $N(R_{C_2})$, form an ideal. Consequently, the map $R \to R/N(R_{C_2})$ is a map of rings. An analogous thing happens in topology, which can be summarized by the following statement:

Proposition 8. *There is a canonical lax monoidal natural transformation*

$$(-)^{hC_2} \to (-)^{tC_2}$$

between lax monoidal functors $\mathrm{Sp}^{BC_2} \to \mathrm{Sp}$.

In short, the functors $(-)^{hC_2}$ and $(-)^{tC_2}$ take rings to rings and the natural transformation is a ring homomorphism.

With this out of the way, we're ready to state the theorem. It is due to Greenlees, Sadofsky, and Hovey in the $K(n)$-local case and was extended by Kuhn to the $T(n)$-local category.

Theorem 9. *Let $X \in \mathrm{Sp}_{T(n)}$ be a spectrum with C_2 action. Then there is an equivalence*

$$L_{T(n)}(X^{tC_2}) \simeq *.$$

Proof. For this proof, we will let $L : \mathrm{Sp} \to \mathrm{Sp}_{T(n)}$ denote $T(n)$-localization. Recall that L is symmetric monoidal, where the symmetric monoidal structure on $\mathrm{Sp}_{T(n)}$ is given by smashing and then localizing.

Any spectrum X is a module over S^0. By Proposition 8 in the previous remarks, this means that $L(X^{tC_2})$ is a module over $L((S^0)^{tC_2})$, and so it suffices to consider the case $X = LS^0$. For this case, we examine the diagram:

$$\begin{array}{ccccc} L((S^0)_{hC_2}) & \xrightarrow{N} & L((S^0)^{hC_2}) & \longrightarrow & L((S^0)^{tC_2}) \\ & \searrow^{\mathrm{tr}} & \downarrow{\scriptstyle r} & & \\ & & LS^0. & & \end{array}$$

Since the norm is an ideal in homotopy, it suffices to show that $1 \in \pi_*(L((S^0)^{hC_2}))$ is in the image of the norm. In fact, it suffices to show that $1 \in \pi_*(LS^0)$ is in the image of the transfer [1].

Example 10. Let's try to see why this is true in an example. Suppose we are $K(1)$-local. Then, we are working in K-theory mod p, and we can just calculate everything. We have

$$\begin{aligned} K(1)^*(BC_2) &\simeq K(1)^*[[z]]/([2](z)) \\ &\simeq K(1)^*[[z]]/(2z - z^2) \\ &\simeq K(1)^* \cdot 1 \oplus K(1)^* \cdot z. \end{aligned}$$

For any complex oriented cohomology theory E, the transfer map $E^* \to E^*(BC_2)$ is given by the formula $\mathrm{tr}(1) = [2](z)/z$. In this case, this says that $\mathrm{tr}(1) = 2 - z = z$. As a result, we see that the reduced transfer map $\Sigma^\infty BC_2 \to \Sigma^\infty_+ BC_2 \xrightarrow{\mathrm{tr}} S^0$ is a $K(1)$-local equivalence.

In general, our life isn't quite so easy. The input that we need is the following classical theorem, which may be covered later in the seminar:

Theorem 11 (Kahn-Priddy)**.** *Let $\Omega^\infty_0 S^0$ denote the identity component of $\Omega^\infty S^0$. Then, the stable transfer induces a map*

$$\Omega^\infty \mathrm{tr} : \Omega^\infty \Sigma^\infty BC_2 \to \Omega^\infty_0 S^0$$

which admits a homotopy section.

In particular, this implies that the transfer map

$$\Omega^\infty \Sigma^\infty_+ BC_2 \to \Omega^\infty S^0 \tag{1}$$

admits a section. Recall that the existence of the Bousfield-Kuhn functor implies that the $T(n)$-localized picture depends only on the underlying spaces (Proposition 1). Thus, we may apply Φ to the above equation to find that the transfer $L((S^0)_{hC_2}) \to LS^0$ admits a section. Hence, it is surjective on homotopy and so 1 is in the image of tr, as desired. $\square$

[1] Intuitively, the reason is that $\pi_*(L((S^0)^{hC_2}))$ is augmented over $\pi_*(LS^0)$ and it's the cohomology of a space (BC_2, where the cohomology theory is LS^0) so elements in the augmentation ideal are at least topologically nilpotent (see Lemma 2.1 from the paper of Clausen-Mathew).