LECTURE 5: v_n-PERIODIC HOMOTOPY GROUPS

Throughout this lecture, we fix a prime number p, an integer $n \geq 0$, and a finite space A of type $(n+1)$ which can be written as ΣB, for some other space B. We let $d = \mathrm{cn}(A) + 1$ denote the smallest integer for which the group $\mathrm{H}_d(A; \mathbf{Z})$ is nonzero. In the previous lecture, we defined the ∞-category $L_n^f \mathcal{S}_*^{\langle d \rangle}$ whose objects are pointed spaces X which are P_A-local, p-local, and d-connected. This space is equipped with a functor

$$L_n^f : \mathcal{S}_* \to L_n^f \mathcal{S}_*^{\langle d \rangle},$$

given by the formula $L_n^f(X) = P_A(X\langle d \rangle_{(p)})$. Our first goal in this lecture is to address the following:

Question 1. What information about a space X is captured by the object $L_n^f(X) \in L_n^f \mathcal{S}_*^{\langle d \rangle}$?

Let us begin by addressing Question 1 when $n = 0$. In this case, we can identify $L_n^f(X)$ with the rationalization $(X\langle d \rangle)_{\mathbf{Q}}$. We therefore obtain the following:

Proposition 2. *Let $u : X \to Y$ be a map of pointed spaces. Then $L_0^f(u)$ is a homotopy equivalence if and only if, for each $m > d$, the induced map of rational homotopy groups $(\pi_m X)_{\mathbf{Q}} \to (\pi_m Y)_{\mathbf{Q}}$ is an isomorphism.*

We will regard Proposition 2 as an answer to Question 1: roughly speaking, it says that the information captured by $L_0^f(X)$ is exactly the set of rational homotopy groups $\{(\pi_m X)_{\mathbf{Q}}\}_{m>d}$. (Of course, this paraphrase is slightly misleading, because the space $L_0^f(X)$ cannot be reconstructed from the rational homotopy groups of X alone.) We would like to establish a "higher" version of Proposition 2, which articulates what "extra" information is captured by $L_n^f(X)$ for $n > 0$. First, we review a construction from Lecture 3. Suppose that V is a finite pointed space of type m, equipped with a v_m-self map $v : \Sigma^t V \to V$. For any pointed space X, the *v_m-periodic homotopy groups of X* are defined by the formula

$$v^{-1}\pi_d(X; V) = \varinjlim(\pi_d \operatorname{Map}_*(V, X) \xrightarrow{v} \pi_{d+t} \operatorname{Map}_*(V, X) \xrightarrow{v} \pi_{d+2t} \operatorname{Map}_*(V, X) \xrightarrow{v} \cdots).$$

Note that these groups are well-defined (and abelian) for every integer d. In fact, they can be regarded as the homotopy groups of a spectrum.

Remark 3 (Periodic Spectra)**.** Fix an integer $t > 0$. We can identify a spectrum E with a sequence of spaces $\{Z(n)\}_{n \in \mathbf{Z}}$, together with homotopy equivalences $Z(n) \simeq \Omega^t Z(n+1)$. Under this identification, we see the following data are equivalent:

(a) The datum of a spectrum E together with a homotopy equivalence of spectra $E \simeq \Omega^t E$.

(b) The datum of a pointed space Z together with a homotopy equivalence $Z \simeq \Omega^t Z$.

Example 4. The periodic complex K-theory spectrum can be obtained from the equivalence Remark 3, noting that Bott periodicity supplies a homotopy equivalence $\mathbf{Z} \times \mathrm{BU} \simeq \Omega^2(\mathbf{Z} \times \mathrm{BU})$.

Construction 5. Let V be a finite pointed space of type m, equipped with a v_m-self map $v : \Sigma^t V \to V$. For any pointed space X, the direct limit

$$\mathrm{Map}_*(V, X) \to \mathrm{Map}_*(\Sigma^t V, X) \to \mathrm{Map}_*(\Sigma^{2t} V, X) \to \cdots$$

can be identified with the t-fold loop space of itself. We can therefore regard this direct limit as the 0th space of a spectrum $\Phi_v(X)$, which is periodic of period t. Note that the homotopy groups of $\Phi_v(X)$ can be identified with the v_m-periodic homotopy $v^{-1}\pi_*(X; V)$.

We now study the dependence of the construction $X \mapsto \Phi_v(X)$ on the datum of the map $v : \Sigma^t V \to V$.

Remark 6. Let V be a finite pointed space equipped with a v_m self-map $v : \Sigma^t V \to V$. Then the suspension ΣV is equipped with the v_m-self self-map $\Sigma(v); \Sigma^{t+1} V \to \Sigma V$. It follows easily from the definitions that for any pointed space X, we have canonical equivalence $\Phi_{\Sigma(v)}(X) = \Omega \Phi_v(X)$.

Remark 7. Let V be a finite pointed space equipped with a v_m self-map $v : \Sigma^t V \to V$. For any $k \geq 1$, we let v^k denote the kth iterate

$$\Sigma^{kt} V \xrightarrow{\Sigma^{(k-1)t}(v)} \Sigma^{(k-1)t} V \to \cdots \to \Sigma^t V \xrightarrow{v} V.$$

For any space X, we have a canonical homotopy equivalence $\Phi_{v^k}(X) \simeq \Phi_v(X)$.

Remark 8 (Uniqueness)**.** Let V be a finite pointed space of type m. Recall that, if $v : \Sigma^t V \to V$ and $v' : \Sigma^{t'} V \to V$ are v_m-self maps, then there exist integers k and k' such that v^k and $v'^{k'}$ are stably homotopic. Using Remarks 6 and 7, we obtain an equivalence between the functors Φ_v and $\Phi_{v'}$. In other words, for a pointed space X, the spectrum $\Phi_v(X)$ depends only on the space V. We will henceforth emphasize this dependence by denoting it by $\Phi_V(X)$, rather than $\Phi_v(X)$.

The notation of Remark 8 is *a priori* dangerous: it is not yet clear to what extent $\Phi_V(X)$ depends functorially on V, since its construction involves auxiliary choices. We will discuss this point in more detail in the next lecture, when we introduce the Bousfield-Kuhn functor. For the moment, we note the following weak form of functoriality. Suppose we are given spaces V and V' of type m, equipped with v_m-self maps $v : \Sigma^t V \to V$ and $v' : \Sigma^{t'} V' \to V$. For any map

$f : V \to V'$, we can arrange, after replacing V and V' by suitable suspensions and the maps v and v' by suitable powers, that $t = t'$ and the diagram

$$\begin{array}{ccc} \Sigma^t V' & \xrightarrow{\Sigma^t(f)} & \Sigma^t V \\ \downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle v} \\ V' & \xrightarrow{f} & V \end{array}$$

commutes up to homotopy. In this case, the above diagram (and the homotopy) induce a v_m-self map $v'' : \Sigma^t V'' \to V''$, where $V'' = \mathrm{cofib}(f)$. For any space X, this supplies a fiber sequence

$$\Phi_{v''}(X) \to \Phi_v(X) \to \Phi_{v'}(X).$$

Warning 9. In the situation above, it is possible that V'' has type $> m$ (this would happen, for example, if $V' = \Sigma^d V$ and $f : V' \to V$ was another v_m-self map). In this case, the spectrum $\Phi_{v''}(X)$ is still well-defined, but is automatically nullhomotopic (since some power of v'' is stably nullhomotopic). In what follows, it will be convenient to extend our definition of $\Phi_V(X)$ to the case where V has type $\geq m$, setting $\Phi_V(X) = 0$ when the type of V is strictly larger than m. This convention is somewhat dangerous (because $\Phi_V(X)$ might have another meaning, arising from a $v_{m'}$-self map of V where m' is the type of V), but hopefully will not result in any confusion.

Remark 10. Let V be a finite pointed space of type m equipped with a v_m-self map $v : \Sigma^t V \to V$. For any finite pointed space W, v induces another v_m-self map $(v \wedge \mathrm{id}_W) : \Sigma^t(V \wedge W) \to V \wedge W$. For any space X, the canonical equivalence $\mathrm{Map}_*(V \wedge W, X) \simeq \mathrm{Map}_*(V, X)^W$ induces an equivalence

$$\Phi_V(X)^W = \Phi_v(X)^W = \Phi_{v \wedge \mathrm{id}_W}(X) = \Phi_{V \wedge W}(X),$$

where the last term is defined using the convention of Warning 9 (that is, it vanishes if $V \wedge W$ has type $> m$).

Combining the preceding remarks with the thick subcategory theorem, we obtain the following:

Proposition 11. *Let $g : X \to Y$ be a map of pointed spaces and let m be a nonnegative integer. The following conditions are equivalent:*

- (a) *There exists a finite pointed space V of type m and a v_m-self map $v : \Sigma^t V \to V$ such that $\Phi_v(g) : \Phi_v(X) \to \Phi_v(Y)$ is a homotopy equivalence.*
- (b) *For every finite pointed space V of type m and every v_m-self map $v : \Sigma^t V \to V$, the map $\Phi_v(g) : \Phi_v(X) \to \Phi_v(Y)$ is a homotopy equivalence.*

Definition 12. We will say that a map of pointed spaces $f : X \to Y$ is a v_m-*periodic homotopy equivalence* if it satisfies the equivalent conditions of Proposition 11.

Proposition 13. *Let V be a finite pointed space of type m equipped with a v_m-self map $v : \Sigma^t V \to V$. For any pointed space X, the spectrum $\Phi_V(X)$ is $T(m)$-local.*

Proof. We first show that $\Phi_v(X)$ is L_m^f-local. Let W be a finite space of type $(m+1)$; we wish to show that the spectrum of maps from W into $\Phi_V(X)$ vanishes. This follows from Remark 10.

Now suppose that W is a finite space of type $k < m$, equipped with a v_k-self map $w : \Sigma^s W \to W$. We wish to show that the spectrum of maps from $\Sigma^\infty(W)[w^{-1}]$ into $\Phi_V(X)$ vanishes. Unwinding the definitions, we see that this spectrum is given as the inverse limit of a tower

$$\cdots \to \Sigma^{2s}\Phi_V(X)^W \to \Sigma^s \Phi_V(X)^W \to \Phi_V(X)^W,$$

where the transition maps are induced by w. Rewriting this sequence as

$$\cdots \to \Sigma^{2s}\Phi_{V\wedge W}(X) \to \Sigma^s \Phi_{V\wedge W}(X) \to \Phi_{V\wedge W}(X),$$

we see that the desired result follows from the observation that the map $\mathrm{id}_V \wedge w$ is stably nilpotent (since it is a v_k-self map of a pointed space of type $m > k$). □

We now study the relationship with the localization $L_n^f \mathcal{S}_*^{\langle d \rangle}$. We begin with the following:

Proposition 14. *Let V be a finite pointed space of type m, where $0 < m \leq n$, which is equipped with a v_m-self map $v : \Sigma^t V \to V$. Then, for any pointed space X, we have a canonical homotopy equivalence $\Phi_V(X) \simeq \Phi_V(L_n^f X)$.*

Proof. It will suffice to verify the following:

(1) For any pointed space X, the canonical map $X\langle d \rangle \to X$ induces a homotopy equivalence of spectra $\Phi_V(X\langle d\rangle) \to \Phi_V(X)$. This follows immediately from the formula

$$\pi_*\Phi_V(X) \simeq \varinjlim_k \pi_{*+tk} \operatorname{Map}_*(V, X) \simeq \varinjlim_k \pi_* \operatorname{Map}_*(V, \Omega^{tk} X)$$

which shows the v_m-periodic homotopy of X is insensitive to passing to highly connected covers.

(2) For any simply connected pointed space X, the canonical map $\Phi_V(X) \to \Phi_V(X_{(p)})$ is a homotopy equivalence of spectra. To prove this, we note that there is a homotopy pullback square

$$\begin{array}{ccc} X & \longrightarrow & X_{(p)} \\ \downarrow & & \downarrow \\ X[p^{-1}] & \longrightarrow & X_{\mathbf{Q}}, \end{array}$$

hence another homotopy pullback square

$$\begin{array}{ccc} \mathrm{Map}_*(V, X) & \longrightarrow & \mathrm{Map}_*(V, X_{(p)}) \\ \downarrow & & \downarrow \\ \mathrm{Map}_*(V, X[p^{-1}]) & \longrightarrow & \mathrm{Map}_*(V, X_{\mathbf{Q}}). \end{array}$$

The spaces at the bottom of this diagram are contractible (since V has type $m > 0$, so that $V[p^{-1}]$ is contractible), so the upper horizontal map is a homotopy equivalence, which immediately implies that $\Phi_V(X) \to \Phi_V(X_{(p)})$ is also a homotopy equivalence.

(3) For any d-connected space X, the canonical map $\Phi_V(X) \to \Phi_V(P_A(X))$ is a homotopy equivalence of spectra. Here, we must work a little bit harder. Since both sides are periodic with period t, it will suffice to show that the induced map of 0th spaces is a homotopy equivalence after passing to d-connected covers. Unwinding the definitions, we can write this map as

$$\rho : \varinjlim_k \mathrm{Map}_*(\Sigma^{kt} V, X)\langle d\rangle \to \varinjlim_k \mathrm{Map}_*(\Sigma^{kt} V, P_A(X))\langle d\rangle.$$

Since A has type $> n$, the suspension spectrum $\Sigma^\infty(A)$ vanishes in the $T(m)$-local category of spectra. Proposition 13 shows that $\Phi_V(X)$ is $T(m)$-local, so that $\mathrm{Map}_*(A, \Omega^\infty \Phi_V(X))$ is contractible: that is, $\Omega^\infty \Phi_V(X)$ is P_A-local. It follows that the d-connected cover $\Omega^\infty \Phi_V(X)\langle d\rangle$ is also P_A-local: that is, the domain of ρ does not change when we apply the functor P_A. Since A is finite, the functor P_A commutes with filtered homotopy colimits. We may therefore rewrite ρ as a map

$$\varinjlim_k P_A(\mathrm{Map}_*(\Sigma^{kt} V, X)\langle d\rangle) \to \varinjlim_k \mathrm{Map}_*(\Sigma^{kt} V, P_A(X))\langle d\rangle.$$

To prove that this map is a homotopy equivalence, it will suffice to show that each of the individual maps

$$\rho_k : P_A(\mathrm{Map}_*(\Sigma^{kt} V, X)\langle d\rangle) \to \mathrm{Map}_*(\Sigma^{kt} V, P_A(X))\langle d\rangle$$

is a homotopy equivalence. This is a special case of the assertion that the functor $P_A : \mathcal{S}_*^{\langle d\rangle} \to \mathcal{S}_*^{\langle d\rangle}$ commutes with finite homotopy limits, which we established in the last lecture.

□

Proposition 14 asserts that replacing a space X by $L_n^f X$ does not change its v_m-periodic homotopy groups for $m \leq n$. However, it does make the v_n-periodic homotopy groups easier to compute:

Proposition 15. *Let X be a pointed space which is P_A-local, and let V be a finite type n space equipped with a v_n-self map $v : \Sigma^t V \to V$. Then the canonical map $\mathrm{Map}_*(V, X) \to \Omega^\infty \Phi_V(X)$ induces a homotopy equivalence after passing to d-connected covers. In other words, the canonical map $\pi_* \mathrm{Map}_*(V, X) \to v^{-1}\pi_*(X; V)$ is an isomorphism for $* > d$.*

Proof. We claim that each of the transition maps

$$\mathrm{Map}_*(V, X) \to \mathrm{Map}_*(\Sigma^t V, X) \to \mathrm{Map}_*(\Sigma^{2t} V, X) \to \cdots$$

induces a homotopy equivalence after passing to d-connective covers; the desired result then follows by passing to the limit. Replacing V by a suitable suspension, we are reduced to proving that the map $\mathrm{Map}_*(V, X) \to \mathrm{Map}_*(\Sigma^t V, X)$ induces an equivalence of d-connected covers. In fact, we claim that the identity component of the homotopy fiber $\mathrm{Map}_*(\mathrm{cofib}(v), X)$ is $(d-1)$-truncated. To prove this, it suffices to show that $\Omega^{d-1} \mathrm{Map}_*(\mathrm{cofib}(v), X) \simeq \mathrm{Map}_*(\Sigma^{d-1} \mathrm{cofib}(v), X)$ is contractible: that is, X is P_B-local for $B = \Sigma^{d-1} \mathrm{cofib}(v)$. This follows from Bousfield's theorem from the last lecture, since X is assumed to be P_A-local and we have $\mathrm{tp}(B) \geq n + 1 = \mathrm{tp}(A)$ and $\mathrm{cn}(B) \geq d - 1 = \mathrm{cn}(A)$. □

We are now in a position to answer Question 1.

Proposition 16. *Let $u : X \to Y$ be a map of pointed spaces. Then $L_n^f(u)$ is a homotopy equivalence if and only if the following conditions are satisfied:*

- *(a) For $0 < m \leq n$, the map u is a v_m-periodic homotopy equivalence.*
- *(b) The induced map of rational homotopy groups $(\pi_* X)_{\mathbf{Q}} \to (\pi_* Y)_{\mathbf{Q}}$ is an isomorphism for $* > d$.*

Proof. The necessity of (b) is obvious, and the necessity of (a) follows from Proposition 14. We will show that (a) and (b) are sufficient. The proof proceeds by induction on n, where the base case $n = 0$ was treated at the beginning of this lecture. Let us therefore suppose that $u : X \to Y$ is a map of pointed spaces satisfying (a) and (b). Without loss of generality, we may assume that $X, Y \in L_n^f \mathcal{S}_*^{\langle d \rangle}$: that is, they are p-local, P_A-local, and d-connected. Let F denote the homotopy fiber of u. It follows from (a) and (b) that F is rationally acyclic and v_m-homotopy equivalent to a point, for $0 < m \leq n$.

Let V be a finite pointed space of type n, equipped with a v_n-self map $v : \Sigma^t V \to V$. Proposition 15 supplies a homotopy equivalence $\mathrm{Map}_*(V, F)\langle d \rangle \simeq \Omega^\infty \Phi_V(F)\langle d \rangle \simeq *$: that is, the mapping space $\mathrm{Map}_*(V, F)$ is d-truncated. Set $d' = \mathrm{cn}(V) + 1$. Replacing V by a suitable suspension if necessary, we may assume that $d' \geq d$ and that $\mathrm{Map}_*(V, F)$ is contractible: that is, F is P_V-local. Then $F\langle d' \rangle$ is also P_V-local and can therefore be regarded as an object of $L_{n-1}^f \mathcal{S}_*^{\langle d' \rangle}$. Applying our inductive hypothesis, we deduce that $F\langle d' \rangle$ is contractible. Since $F\langle d \rangle$ is rationally trivial and P_A-local, we proved in the last lecture that the canonical map $P_A F\langle d' \rangle \to F\langle d \rangle$ is an equivalence. That is, F is d-truncated. As a fiber of a map of d-connected spaces, it must also be $(d-1)$-connected: that is, we have $F \simeq K(G,d)$ for some abelian group G. Our assumption that u is a rational homotopy equivalence guarantees that G is a torsion group, and since everything is p-local it is a p-power torsion group. Then $K(G,d)$ is P_A-acyclic, so the map u becomes an equivalence after applying the functor P_A. Since X and Y are both P_A-local, we conclude that u is a homotopy equivalence. □

Corollary 17. *The functor $L_n^f : \mathcal{S}_* \to L_n^f \mathcal{S}_*^{\langle d \rangle}$ exhibits the ∞-category $L_n^f \mathcal{S}_*^{\langle d \rangle}$ as the localization of $\mathcal{S}_*$ with respect to the collection of all maps which satisfy conditions (a) and (b) of Proposition 16. More precisely, for any ∞-category $\mathcal{C}$, composition with L_n^f induces a fully faithful embedding*

$$\phi : \mathrm{Fun}(L_n^f \mathcal{S}_*^{\langle d \rangle}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{S}_*, \mathcal{C}),$$

whose essential image is spanned by those functors $F : \mathcal{S}_* \to \mathcal{C}$ having the property that, for each morphism of pointed spaces $u : X \to Y$ satisfying (a) and (b), the image $F(u)$ is an equivalence in $\mathcal{C}$.

Proof. Let $\mathcal{E} \subseteq \mathrm{Fun}(\mathcal{S}_*, \mathcal{C})$ be the full subcategory spanned by those functors $F : \mathcal{S}_* \to \mathcal{C}$ having the property that, for each morphism of pointed spaces $u : X \to Y$ satisfying (a) and (b), the image $F(u)$ is an equivalence in $\mathcal{C}$. It follows from Proposition 16 that we can regard ϕ as a functor from $\mathrm{Fun}(L_n^f \mathcal{S}_*^{\langle d \rangle}, \mathcal{C})$ to $\mathcal{E}$. We also have a functor

$$\psi : \mathcal{E} \to \mathrm{Fun}(L_n^f \mathcal{S}_*^{\langle d \rangle}, \mathcal{C}),$$

which carries a functor $F : \mathcal{S}_* \to \mathcal{C}$ to the restriction $F|_{L_n^f \mathcal{S}_*^{\langle d \rangle}}$. It follows immediately from the definitions that ψ is a left homotopy inverse to ϕ, and Proposition 14 guarantees that it is also a right homotopy inverse to ϕ. □

It follows from Corollary 17 that we can regard the ∞-category $L_n^f \mathcal{S}_*^{\langle d \rangle}$ as containing information about v_m-periodic homotopy theory for $0 < m \leq n$, as well as rational homotopy theory. We now define a variant which captures *only* v_n-periodic information. First, we need a variant of Proposition 15.

Proposition 18. *Let V be a finite pointed space of type $m > n$, equipped with a v_m-self map $v : \Sigma^t V \to V$. If X is a P_A-local space, then $\Phi_V(X)$ is contractible.*

Proof. Replacing V by a suspension, we may suppose that V is a suspension and that the connectivity of V is at least as large as the connectivity of A. It follows from Bousfield's theorem from the last lecture that every P_A-local space is also P_V-local. In particular, X is P_V-local: that is, the mapping space $\mathrm{Map}_*(V, X)$ is contractible. It then follows immediately from the construction that $\Phi_V(X)$ is contractible. □

Let us assume now that we have *two* finite spaces A and B, having types $(n+1)$ and n, of the same connectivity $d = \mathrm{cn}(A) + 1 = \mathrm{cn}(B) + 1$ (this can always be achieved by taking d sufficiently large). In this case, Bousfield's theorem from the previous lecture implies that every B-local space is also A-local, so that the functor P_B carries $L_n^f \mathcal{S}_*^{\langle d \rangle}$ to the subcategory $L_{n-1}^f \mathcal{S}_*^{\langle d \rangle}$.

Definition 19. We let $\mathcal{S}_*^{v_n}$ denote the full subcategory of $L_n^f \mathcal{S}_*^{\langle d \rangle}$ spanned by those objects X satisfying $P_B(X) = *$. In other words, $\mathcal{S}_*^{v_n}$ is the ∞-category of spaces which are d-connected, p-local, P_A-local, and P_B-acyclic.

Note that the functor $X \mapsto P_B X$ does not change the v_m-periodic homotopy groups for $0 < m < n$ (Proposition 14). Consequently, if a pointed space X is P_B-acyclic, then X is rationally acyclic and the v_m-periodic homotopy groups of X vanish for $0 < m < n$. Combining this observation with Proposition 16, we obtain the following:

Proposition 20. *Let $f : X \to Y$ be a morphism in $\mathcal{S}_*^{v_n}$. Then f is a homotopy equivalence if and only if it is a v_n-periodic homotopy equivalence.*

Note that for any space $X \in L_n^f \mathcal{S}_*^{\langle d \rangle}$, the canonical map $u : X \to P_B X$ has fiber which is P_A-local (since the source and target are P_A-local), (p)-local, and P_B-acyclic. In the last lecture, we showed that this implies that the d-connected cover $\mathrm{fib}(u)\langle d \rangle$ is again P_B-acyclic (and even $P_{\Sigma B}$-acyclic), and therefore belongs to $\mathcal{S}_*^{v_n}$. More generally, the construction $X \mapsto \mathrm{fib}(P_A(X\langle d \rangle_{(p)}) \to P_B(X\langle d \rangle_{(p)}))\langle d \rangle$ determines a functor

$$M_n^f : \mathcal{S}_* \to \mathcal{S}_*^{v_n}.$$

This functor satisfies the following analogue of Proposition 14:

Proposition 21. *Let V be a finite space of type n equipped with a v_n-self map $v : \Sigma^t V \to V$. Then there is a canonical equivalence of functors $\Phi_V \simeq \Phi_V \simeq M_n^f$. In other words, the functor $X \mapsto M_n^f$ does not change the v_n-periodic homotopy of X.*

Proof. We saw in the proof of Proposition 21 that the functors $X \mapsto P_A X$, $X \mapsto X_{(p)}$, and $X \mapsto X\langle d \rangle$ do not change the v_n-periodic homotopy of X. It therefore suffices to show that when X is P_A-local, the construction $X \mapsto \mathrm{fib}(X \to P_B(X))$ does not change the v_n-periodic homotopy of X. This is clear, since the v_n-periodic homotopy of $P_B(X)$ vanishes (Proposition 18). □

Corollary 22. *The functor $M_n^f : \mathcal{S}_* \to \mathcal{S}_*^{v_n}$ exhibits the ∞-category $\mathcal{S}_*^{v_n}$ as the localization of $\mathcal{S}_*$ with respect to the collection of all v_n-periodic homotopy equivalences. More precisely, for any ∞-category $\mathcal{C}$, composition with M_n^f induces a fully faithful embedding*

$$\phi : \mathrm{Fun}(\mathcal{S}_*^{v_n}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{S}_*, \mathcal{C}),$$

whose essential image is spanned by those functors $F : \mathcal{S}_* \to \mathcal{C}$ which carry v_n-periodic homotopy equivalences to equivalences in $\mathcal{C}$.

Proof. Let $\mathcal{E}' \subseteq \mathrm{Fun}(\mathcal{S}_*, \mathcal{C})$ be the full subcategory spanned by those functors $F : \mathcal{S}_* \to \mathcal{C}$ which carry v_n-periodic homotopy equivalences to equivalences in $\mathcal{C}$. It follows from Proposition 20 that we can regard ϕ as a functor from $\mathrm{Fun}(\mathcal{S}_*^{v_n}, \mathcal{C})$ to $\mathcal{E}$. We also have a functor

$$\psi : \mathcal{E} \to \mathrm{Fun}(\mathcal{S}_*^{v_n}, \mathcal{C}),$$

which carries a functor $F : \mathcal{S}_* \to \mathcal{C}$ to the restriction $F|_{\mathcal{S}_*^{v_n}}$. It follows immediately from the definitions that ψ is a left homotopy inverse to ϕ, and Proposition 21 guarantees that it is also a right homotopy inverse to ϕ. □

Warning 23. It follows from Corollary 22 that the abstract ∞-category $\mathcal{S}_*^{v_n}$ depends only n, and not on the integer $d \gg 0$. Beware, however, that the realization of $\mathcal{S}_*^{v_n}$ as a full subcategory of $\mathcal{S}_*$ *does* depend on d (by definition, every object of $\mathcal{S}_*^{v_n}$ is d-connected when regarded as a pointed space).