Chromatic Localization

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Origin story

Thm (Sullivan, Atiyah-Segal) After p-completion there is an equivalence of infinite loop spaces

$$BSU \xrightarrow{\sim} BSU^{\otimes}$$
$$BSO \xrightarrow{\sim} BSO^{\otimes}$$

Thm (Mahowald, Adams-Priddy) If X is a connected 2-complete spectrum with $\Omega^\infty X \sim BSO$ then $X \sim bso$. Similarly if X is a connected p-complete spectrum with $\Omega^\infty X \sim BSU_p$ then $X \sim bsu_p$.

Remark Not true for BU.

$$BU \sim \mathbb{CP}^\infty \times BSU$$
$$bu \not\sim \Sigma^2 H\mathbb{Z} \vee bsu$$

Remark This is not true without p-adic completion.

Thm (Madsen-Snaith-Tornehave) After p-completion the map

$$\Omega^\infty : [bsu, bsu] \to [BSU, BSU]$$

is a monomorphism with a describable image.

Bousfield gave an elegant, "non-computational" proof.

Bousfield's proof

1) Factorization

Construct a functor Φ sitting into a factorization

$$\begin{tikzcd}
\text{Spaces} \arrow[r, "\Phi"] & K(1)\text{-local spectra} \\
\text{Spectra} \arrow[u, "\Omega^\infty"] \arrow[ur, "L_{K(1)}"'] &
\end{tikzcd}$$

2) Connectivity

Suppose that E is a spectrum. Show that the fiber of the map

$$L_{K(1)}\Omega^\infty E \longrightarrow \Omega^\infty L_{K(1)} E$$

is 2 co-connected,* so that

$$L_{K(1)}\Omega^\infty E\langle 3\rangle \longrightarrow (\Omega^\infty L_{K(1)} E)\langle 3\rangle$$

is an equivalence.

Remark Actually one shows $\pi_i = 0 \quad i > 2$
π_2 is torsion free.

Proof of uniqueness of BSU_p:

$$\Omega^\infty X_p \xrightarrow{\sim} BSU_p \underset{\Phi}{\Longrightarrow} L_{K(1)}\Phi \xrightarrow{\sim} L_{K(1)}\, bsu$$

Connectivity: $\Omega^\infty \Phi \simeq L_{K(1)}\Omega^\infty X = L_{K(1)}\Omega^\infty \Phi\langle 3\rangle = \Omega^\infty L_{K(1)}\Phi\langle 3\rangle$

so the desired spectrum equivalence is $L_{K(1)}\Phi\langle 3\rangle \xrightarrow{\sim} L_{K(1)}\, bsu\langle 3\rangle$.

Remark There are higher chromatic analogues of the above. Here is an example of a much more general result of Jeremy Hahn.

Thm (Hahn) If X is a connective, p-complete spectrum with

$$\Omega^\infty X \sim \Omega^\infty \Sigma^8 BP\langle 2\rangle$$

then $X \sim BP\langle 2\rangle$.

Not provable by the analogue of Bousfied's argument.

Bousfield localization for spectra

$\mathcal{S}$ = category of spectra

$E \in \mathcal{S}$ a spectrum.

$\mathrm{Null}_E = \{ X \in \mathcal{S} \mid E \wedge X \sim * \}$

Definition $X \in \mathcal{S}$ is E-local if

$$Z \in \mathrm{Null}_E \Rightarrow [Z, X] = 0.$$

Theorem (Bousfield) There exists a functor

$$L_E : \mathcal{S} \to \mathcal{S}$$

and a natural transformation

$$\mathrm{Id} \to L_E$$

with the following properties

i) $L_E X$ is E-local

ii) The map $X \to L_E X$ is an E-equivalence.

Consequences i) The fiber $M_E X \to X \to L_E X$ is E-acyclic, so if Y is E-local then the restriction map

$$[L_E X, Y] \to [X, Y]$$

is an isomorphism, and

$$\mathrm{Map}(L_E X, Y) \to \mathrm{Map}(X, Y)$$

is a weak equivalence. So L_E can be regarded as a left adjoint to the inclusion $\mathcal{S}_E \subset \mathcal{S}$ of the full subcategory of E-local spectra.

ii) L_E is idempotent: the map $L_E X \to L_E L_E X$ induced by $1 \to L_E$ is a weak equivalence.

Bousfield equivalence

Note that $1 \to L_E$ depends only on Null_E.

<u>Definition</u> Two spectra E and F are <u>Bousfield equivalent</u> if $\mathrm{Null}_E = \mathrm{Null}_F$. The Bousfield class $\langle E \rangle$ of E is the equivalence class of E under this equivalence relation.

Relations among Bousfield classes:

$$\langle E \rangle \geq \langle F \rangle \iff \mathrm{Null}_E \subseteq \mathrm{Null}_F$$

$$\langle E \rangle \vee \langle F \rangle = \langle E \vee F \rangle$$

$$\langle F \rangle \wedge \langle F \rangle = \langle E \wedge F \rangle$$

There is a cool theorem which we won't need

<u>Thm (Okawa)</u> The collection of Bousfield classes forms a set.

Note that if $\langle E \rangle \geq \langle F \rangle$ then

$$X \text{ is } E\text{-local} \Rightarrow X \text{ is } F\text{-local}$$

and so there is a natural transformation

$$L_E \longrightarrow L_F$$

which may be regarded as being derived from $1 \to L_F$ by applying L_E.

Dror nullification (Bousfield colocalization)

Generalization of Bousfield localization gotten by replacing Null_E by something more general. First described by Bousfield in "A Boolean algebra of spectra". Later generalized by Dror/Farjoun.

We will consider a special case.

Suppose $A \in \mathcal{S}$.

natural transformati
on

Definition $X \in \mathcal{S}$ is A-null if for all t

$$[\Sigma^t A, X] = 0$$

or equivalently if the function spectrum X^A is contractible.

Thm (Bousfield, Farjoun) There is a functor N_A: Spectra $\to$ Spectra and a natural transformation $1 \to N_A$ with the properties

i) N_A is A-null

ii) For any Y which is A-null, the map

$$[N_A X, Y] \to [X, Y]$$

is an isomorphism.

Variation Given $f: A \to B$ say X is "f-null" if

$$X^B \to X^A$$

is a weak equivalence. There is also a functor N_f and a natural transformation $1 \to N_A$ as above.

Remark In spectra X is f-null iff X is C-null, where $C = B \cup_f CA$.

It follows that $N_f = N_C$.

Construction

In principle, every assertion about L_E and N_A follows from the universal property. Some things, however, are easier to prove using the construction.

The functor $N_A(X)$ is constructed (transfinite) inductively as follows. Choose an infinite ordinal κ for which A is κ-small:

$$\lim_{s<\kappa} \mathrm{Map}(A, Y_s) \xrightarrow{\sim} \mathrm{Map}(A, \varinjlim_{s<\kappa} Y_s)$$

(for example one can assume A is a CW spectrum and take κ to be the smallest infinite ordinal of cardinality greater than the number of cells of A). One constructs $N_A X$ inductively, starting with $X_0 = X$ and, having defined X_α for all $\alpha < \beta < \kappa$ defining $X_\beta = \lim_{\alpha<\beta} X_\alpha$, if β is a limit ordinal, and by the pushout diagram

$$\begin{tikzcd} \bigvee\limits_{t,\, \Sigma^t A \to X_{\beta-1}} \Sigma^t A \arrow[r] \arrow[d] & \bigvee\limits_{t,\, \Sigma^t A \to X_{\beta-1}} C\Sigma^t A \arrow[d] \\ X_{\beta-1} \arrow[r] & X_\beta \end{tikzcd} .$$

In our case of interest A will be a finite CW complex so we can take $\{\alpha < \kappa\}$ to be the ordered set $\{0 < 1 < 2 < \cdots\}$.

Here is an example of a theorem most easily seen from the construction

Theorem If $E \wedge A \sim *$ then for all X, the map

$$E \wedge X \to E \wedge N_A(X)$$

is an equivalence.

The example Theorem

Examples of Bousfield classes

C_0 = p-localization of finite spectra

$C_n = \{ X \in C_0 \mid K(n-1)_* X = 0 \}$ finite spectra of type n.

Thm (Landweber, Ravenel) $C_n \subset C_{n-1}$

Thm (Mitchell) $C_n \neq C_{n-1}$

Thm (H, Smith) If $C \subseteq C_0$ is thick then $C = C_n$ for some n.

Cor (H, Smith) If X & Y are finite then

$$\langle X \rangle \leq \langle Y \rangle \iff \{ n \mid K(n)_* X = 0 \} \supseteq \{ n \mid K(n)_* Y = 0 \}$$

Class invariance

v_n self maps

Def Suppose $X \in C_0$. A map $f : \Sigma^d X \to X$ is a v_n self-map if

1) $K(m)_* f$ is nilpotent for $m \neq n$
2) $n = 0$ and $H_*(f;\mathbb{Q}) =$ mult by d for some d

or 3) $n > 0$ and $K(n)_* f$ is an isomorphism.

Remark There was a choice made in this definition. After raising to a power a v_n-self-map can be shown to satisfy

1') $K(m)_* f = 0$ for $m \neq n$
2') $K(n)_* f =$ mult by a power of v_n.

Thm (H,S) i) X admits a v_n self map $\iff X \in C_n$

ii) If v, v' are v_n self maps of X then $v^a = v'^b$ for some a, b.

iii) If $X, Y \in C_n$ $f : X \to Y$ then there exist v_n self maps

$$v_X : \Sigma^d X \to X \qquad\qquad v_Y : \Sigma^d Y \to Y$$

such that

$$\begin{tikzcd} \Sigma^d X \arrow[r] \arrow[d] & \Sigma^d Y \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

commutes up to homotopy.

Exercise i) If $X \in C_n \setminus C_{n+1}$ and $v : \Sigma^d X \to X$ is a v_n self-map then

$$X \cup_v C\Sigma^d X \in C_{n+1} - C_n .$$

v_n self-maps and Bousfield classes

Lemma (Ravenel) Suppose $f: \Sigma^d X \to X$ is a self map. Then

$$\langle X \rangle = \langle f^{-1} X \rangle \vee \langle X \cup C\Sigma^d X \rangle$$

proof An enjoyable exercise.

Start with $X_0 \in C_0 \setminus C_1$, (say $X_0 = S^0$.)

$$v_0 : X_0 \to X_0 \quad (\text{say } v_0 = p)$$

a v_0 self map

Set $X_1 = \text{cofiber } v_0$.

Having defined X_{n-1} choose a v_{n-1} self-map

$$v_{n-1} : \Sigma^{d_{n-1}} X_{n-1} \to X_{n-1}$$

and set

$$X_n = \text{cofiber } v_{n-1}.$$

By class invariance $\langle X_0 \rangle = \langle S^0 \rangle$. By Ravenel's lemma

$$\begin{aligned}
\langle S^0 \rangle &= \langle v_0^{-1} X_0 \rangle \vee \langle X_1 \rangle \\
&= \langle v_0^{-1} X_0 \rangle \vee \langle v_1^{-1} X_1 \rangle \vee \langle X_2 \rangle \\
&\vdots \\
&= \langle v_0^{-1} X_0 \rangle \vee \cdots \vee \langle v_n^{-1} X_n \rangle \vee \langle X_{n+1} \rangle.
\end{aligned}$$

Proposition The above decomposition of $\langle S^0\rangle$ is orthogonal in the sense that if $i<j$ then $v_i^{-1}X_i \wedge X_j$ is contractible and hence so is $(v_i^{-1}X_i)\wedge(v_j^{-1}X_j)$.

proof Some power of

$$v_i \wedge 1\!\!1 : \Sigma^{d_i} X_i \wedge X_j \longrightarrow X_i \wedge X_j$$

is zero in all Morava K-theories, hence contractible.

For simplicity write

$$E_n^f = v_0^{-1}X_0 \vee \cdots \vee v_n^{-1}X_n$$

(it is really the Bousfield class of E_n^f we care about). Note that

$$\langle S^0\rangle = \langle E_n^f\rangle \vee \langle X_{n+1}\rangle$$

and that

$$E_n^f \wedge X_{n+1} \simeq * \,.$$

We are interested in the localization functors

$$L_n^f = L_{E_n}$$
$$L_{T(n)} = L_{v_n^{-1}X_n}$$

Bousfield and Dror

Proposition $\quad L_n^f = N_{X_{n+1}}$

Bousfield $\qquad\qquad$ Dror

We need:

Lemma Both L_E and N_A commute with finite homotopy (co-) limits. In particular if S is finite (dualizable) then the natural maps

$$(L_E X)\wedge S \to L_E(X\wedge S)$$
$$(N_A X)\wedge S \to N_A(X\wedge S)$$

are weak equivalences.

proof For example, to show that

$$(L_E X)\wedge S \to L_E(X\wedge S)$$

is an equivalence it suffices to show that if W is E-local then $W\wedge S$ is E-local. For this, suppose $Z\in \mathrm{Null}_E$. Then

Spanier-Whitehead dual

$$E\wedge(Z\wedge DS)\simeq(E\wedge Z)\wedge DS \simeq *$$

implies that $Z\wedge DS\in \mathrm{Null}_E$. Now observe

$$[Z, W\wedge S] = [Z\wedge DS, W] = 0 \ .$$

The other assertions are similar. $\square$

proof of the Proposition: Since $E_n^f \wedge X_{n+1} \sim *$, the spectrum X_{n+1} is E_n^f-null and so for all X

$$(L_n^f X)^{X_{n+1}} \sim * .$$

By the universal property of $N_{X_{n+1}}$ this means that there is a unique natural transformation

$$1 \longrightarrow N_{X_{n+1}} \longrightarrow L_n^f$$

of functors (over 1). To show it is an equivalence it suffices to show that for all X, $N_{X_{n+1}} X$ is E_n^f-local. For this suppose that $Z \in \mathrm{Null}_{E_n^f}$. We must show that any map

$$Z \longrightarrow N_{X_{n+1}} X$$

is null. Since $N_{X_{n+1}}$ is idempotent, it suffices to show that

$$N_{X_{n+1}} Z$$

is contractible. By orthogonality, we know that for $i \leq n$

$$v_i^{-1} X_i \wedge X_{n+1} \sim * .$$

The example theorem then implies that for all $i \leq n$

$$(v_i^{-1} X_i) \wedge Z \longrightarrow (v_i^{-1} X_i) \wedge N_{X_{n+1}}(Z)$$

and so

$$(v_i^{-1} X_i) \wedge N_{X_{n+1}}(Z) \sim * .$$

But we also know, by definition, that

$$D\Sigma_{n+1} \wedge N_{\Sigma_{n+1}} Z \sim \left(N_{\Sigma_{n+1}}\right)^{\Sigma_{n+1}} \sim * .$$

By class invariance, $\langle D\Sigma_{n+1} \rangle = \langle \Sigma_{n+1} \rangle$. So after all of this we learn that

$$\left(N_{\Sigma_{n+1}} Z\right) \wedge v_i^{-1} \Sigma_i \sim * \quad i \leq n$$

$$\left(N_{\Sigma_{n+1}} Z\right) \wedge \Sigma_{n+1} \sim * \quad .$$

It follows from

$$\langle S^0 \rangle = \langle v_0^{-1} \Sigma_0 \rangle \vee \cdots \vee \langle v_n^{-1} \Sigma_n \rangle \vee \langle \Sigma_{n+1} \rangle$$

that

$$N_{\Sigma_{n+1}} Z \sim S^0 \wedge N_{\Sigma_{n+1}} Z \sim * . \qquad \square$$

Localization an nullification of spaces

The theory needs to be set up a little differently in Spaces.

Def A map $A \to B$ is an E-equivalence if $E_*A \to E_*B$ is an iso.

Def A space W is E-local if for all E-equivalences $A \to B$, the map

$$W^B \to W^A$$

is a weak equivalence.

Bousfield constructs an E-localization functor $L_E : \text{Spaces} \to \text{Spaces}$ and a natural transformation

$$X \to L_E X$$

characterized by the evident analogue of the properties characterizing the localization of spectra.

Let A be a space. A space X is A-null if the inclusion

$$X \to X^{A_+} \leftarrow \text{to indicate unpointed maps}$$

of the constant maps is a weak equivalence. Following Bousfield, Farjoun (Dror) constructs an A-nullification functor (and natural trans)

$$X \to N_A X$$

characterized by properties analogous to those characterizing the nullification of spectra.

Unstable L_n^f localization

(all spaces are pointed, all homology is reduced)

So we have several candidates for L_n^f on Spaces.

1) Localization with respect to E_n^f

2) Or nullification with respect to $X_{n+1} \to pt$ ← space of type $(n+1)$.

Proposition Suppose that V and V' are finite CW complexes whose suspension spectra are in $C_{n+1} \setminus C_{n+2}$. Choose an integer $m > 0$ with the property that $\exists\, d, d' \leq m$ with $H_d(V; \mathbb{Z}/p) \neq 0$ and $H_{d'}(V'; \mathbb{Z}/p) \neq 0$. There is a natural weak equivalence

$$N_V X\langle m \rangle \sim N_{V'} X \langle m \rangle$$

(m-1) - connected cover

Proposition If V and d are as above, and W is a spectrum then the maps

$$(L_n^f \Omega^\infty W)\langle d \rangle \longrightarrow (\Omega^\infty L_n^f W)\langle d \rangle$$

$$\cong$$

$$(N_V \Omega^\infty W)\langle d \rangle \xrightarrow{\sim} (\Omega^\infty N_V W)\langle d \rangle$$

are weak equivalences.

I will establish one key point in the argument for these results now and return to the rest of the proof in a later talk.

<u>Proposition</u> Suppose that X is a space, and $d>0$ is an integer and

i) $(E_n^f)_* X = 0$

ii) $H_d(X;\mathbb{Z}/p) \neq 0$. $\qquad$ $K(\mathbb{Z}/p, d)$ Proposition

Then for all $d' \geq d$,

$$(E_n^f)_*(K(\mathbb{Z}/p, d')) = 0.$$

Note that for any non-contractible finite spectrum of type $(n+1)$, some suspension of V is the suspension spectrum of a space which we might as well call V. Since V is not contractible, there is a d for which $H_d(V;\mathbb{Z}/p) \neq 0$. Thus a d exists with the property that for $d' \geq d$,

$$L_n^f K(\mathbb{Z}/p, d') \sim * \ .$$

The smallest possible d is one greater than the minimum possible connectivity of an E_n^f-acyclic space. The theorem shows that this is attained on an Eilenberg-MacLane space $K(\mathbb{Z}/p, d)$ (and not on a finite complex). The precise value of d is not known. One does know that

$$d \geq n+1$$

and the conjecture is that $d = n+1$. There is some progress toward proving this.

The proof of the proposition requires:

Lemma Suppose that E is a homology theory. If $X \to B$ is a map having the property that for each $b \in B$ the map

Fibration Lemma

$$F_b \to b$$

is an E equivalence, where F_b is the homotopy fiber over b, then

$$X \to B$$

is an E-equivalence.

proof One can use the usual argument inducting over a covering of B over which the fibration is a product, or, equivalently write

$$\begin{array}{ccc} X & = & \underrightarrow{\mathrm{holim}}_B\, F_b \\ \downarrow & & \\ B & = & \underrightarrow{\mathrm{holim}}_B\, pt \end{array}$$

and appeal to the fact that E-equivalences are preserved under homotopy colimits. $\square$

Lemma Suppose E is a spectrum. If $X \to pt$ is an E-equivalence then $QX = \Omega^\infty \Sigma^\infty X \to pt$ is an E-equivalence.

proof If E is contractible there is nothing to prove. If E is not contractible then X must be connected. In this case we can use the Snaith splitting of the May model

$$\Sigma^\infty QX \simeq \bigvee E\Sigma_{n+} \underset{\Sigma_n}{\wedge} X^{\wedge n}$$

to reduce to showing that $E\Sigma_{n+} \underset{\Sigma_n}{\wedge} X^{\wedge n}$ is E-acyclic. But as in the proof of the previous proposition the map

$$E\Sigma_{n+} \underset{\Sigma_n}{\wedge} X^{\wedge n} \longrightarrow E\Sigma_{n+} \underset{\Sigma_n}{\wedge} pt \simeq *$$

is an E-equivalence.

Corollary With E and X as above, If Z is a (-1)-connected spectrum then

$$\Omega^\infty Z \wedge X$$

is E-acyclic.

proof: Working through a cell decomposition of Z one reduces to the assertion that if $S^{n-1} \to Z_1 \to Z_2$ is a cofibration sequence and $n \geq 1$ then the map

$$\Omega^\infty Z_1 \wedge \underline{X} \to \Omega^\infty Z_2 \wedge \underline{X}$$

is an E-equivalence. But this is a principal fibration with fiber

$$\Omega^\infty \Sigma^\infty S^{n-1} \wedge \underline{X}$$

so the result follows from the fibration lemma, and the above lemma with $S^{n-1} \wedge \underline{X}$ playing the role of $\underline{X}$. $\square$

Proof of the $K(\mathbb{Z}/p, d)$ proposition: Since $H_d(\underline{X}; \mathbb{Z}/p) \neq 0$ the spectrum $\Sigma^d H\mathbb{Z}/p$ is a retract of $H\mathbb{Z}/p \wedge \underline{X}$, and so $K(\mathbb{Z}/p, d)$ is a retract of $\Omega^\infty H\mathbb{Z}/p \wedge \underline{X}$. The latter is E_n^f-acyclic by the corollary above. This implies that $K(\mathbb{Z}/p, d)$ is E_n^f-acyclic. Replacing $\underline{X}$ with $\Sigma^{d'-d}\underline{X}$ gives the result for $K(\mathbb{Z}/p, d')$. $\square$

Nullification analogue

The above results hold for the nullification N_V with respect to a finite space V of type n.

We start with some general facts. First

Definition A space X is A-null if the inclusion of the constant maps $X \to X^{A_+}$ is an equivalence.

A space B is A-periodic (A co-null) if for all A-null spaces X the inclusion of the constant maps $X \to X^{B_+}$ is an equivalence.

Lemma Suppose $Z \to W$ has the property that for all A-null spaces D, the map

$$D^{W_+} \longrightarrow D^{Z_+}$$

is an equivalence. Then $N_A Z \to N_A W$ is a weak equivalence.

pf: Immediate from the universal property.

Lemma If $X \to B$ is a map having the property that for each $b \in B$ the homotopy fiber F_b is A-periodic then

(Fibration Lemma)

$$F_b \to b$$

then $N_A X \longrightarrow N_A B$ is a weak equivalence.

proof As in the proof of the previous fibration lemma, the assumptions imply that for each A-null space D, the map $D^{B_+} \to D^{X_+}$ is a weak equivalence. The claim follows from the lemma above.

Proposition If A is connected then $QA = \Omega^\infty \Sigma^\infty A$ is A-periodic.

proof We will use the May model $QA = \lim\limits_{n \to \infty} \mathrm{Fil}_n$ in which $\mathrm{Fil}_1 = A$ and for $n>1$, Fil_n is defined inductively by the pushout square

$$\begin{array}{ccc} C_n'(\mathbb{R}^\infty; A) & \longrightarrow & C_n(\mathbb{R}^\infty; A) \\ \downarrow & & \downarrow \\ \mathrm{Fil}_{n-1} & \longrightarrow & \mathrm{Fil}_n \end{array}$$

in which

$$C_n(\mathbb{R}^\infty; A) = \{ S \subset \mathbb{R}^\infty,\ \ell : S \to A \mid |S| = n \}$$

is the configuration space of n points in $\mathbb{R}^\infty$ labeled by elements of A, the subspace $C_n'(\mathbb{R}^\infty; A)$ is the subspace of (S, ℓ) such that $\ell(s) = *$ for some $s \in S$ and the left map sends (S, ℓ) to (S', ℓ') where S' is the complement of some $s \in S$ with $\ell(s) = *$ and ℓ' is the restriction of ℓ. The result follows once one shows that for all A-null spaces D, the map

$$D^{(\mathrm{Fil}_n)_+} \longrightarrow D^{(\mathrm{Fil}_{n-1})_+}$$

is a weak equivalence. From the pushout square it is enough to show that

$$D^{C_n(\mathbb{R}^\infty; A)_+} \longrightarrow D^{C_n'(\mathbb{R}^\infty; A)_+}$$

is a weak equivalence. Now the maps

$$C_n(\mathbb{R}^\infty; A) \longrightarrow C_n(\mathbb{R}^\infty; pt)$$
$$C_n'(\mathbb{R}^\infty; A) \longrightarrow C_n(\mathbb{R}^\infty; pt)$$

are fibrations with fibers A^n and $T^n(A) = \{(a_1, \dots, a_n) \in A^n \mid a_i = *, \text{some } i\}$.
One easily checks that A^n and $T^n A$ are A-periodic, so the fibration lemma implies that in the diagram,

$$\begin{tikzcd}
D^{C_n(\mathbb{R}^\infty; pt)_+} \arrow[r, equal] \arrow[d] & D^{C_n(\mathbb{R}^\infty; pt)_+} \arrow[d] \\
D^{C_n(\mathbb{R}^\infty; A)_+} \arrow[r] & D^{C_n'(\mathbb{R}^\infty; A)_+}
\end{tikzcd}$$

the vertical maps are weak equivalences. $\blacksquare$

Now one can imitate the argument in the stable case to show

Proposition If $H_d(A; \mathbb{Z}/p) \neq 0$ then $K(\mathbb{Z}/p, d)$ is A-periodic. $\blacksquare$