LECTURE XXX: KOSZUL DUALITY, PART III

Fix a prime number p and an integer $n > 0$. We have seen that there is a monadic adjunction

$$\mathrm{Sp}_{T(n)} \underset{\Phi}{\overset{\Theta}{\rightleftarrows}} \mathcal{S}_*^{v_n},$$

and that the associated monad $\Phi\Theta : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ is coanalytic, and can therefore be identified with an operad in $T(n)$-local spectra. In the last lecture, we proved that the Koszul dual $\mathrm{KD}_{T(n)}(\Phi\Theta)$ can be identified with the nonunital commutative operad $\mathrm{Sym}^*_{\mathrm{red}}$. Our goal in this lecture is to prove the following closely related result:

Theorem 1. *The Koszul biduality map*

$$\Phi\Theta \to \mathrm{KD}_{T(n)}(\mathrm{KD}_{T(n)}(\Phi\Theta)) \simeq \mathrm{KD}_{T(n)}(\mathrm{Sym}^*_{\mathrm{red}})$$

is an equivalence of operads in $T(n)$-local spectra. In other words, $\Phi\Theta$ is equivalent to the $T(n)$-local Lie operad (so that $\mathcal{S}_*^{v_n}$ is equivalent to the ∞-category of Lie algebras in $T(n)$-local spectra).

Theorem 1 is mostly a consequence of the following formal result:

Proposition 2. *Let $\mathcal{O} = \{\mathcal{O}(k)\}_{k\geq 0}$ be an augmented operad in $T(n)$-local spectra. Assume that:*

- *(a) Each $\mathcal{O}(k)$ is dualizable as a $T(n)$-local spectrum.*
- *(b) The $T(n)$-local spectrum $\mathcal{O}(0)$ vanishes.*
- *(c) The augmentation on $\mathcal{O}$ induces an equivalence $\mathcal{O}(1) \to L_{T(n)}S$.*

Then the biduality map $\mathcal{O} \to \mathrm{KD}_{T(n)}(\mathrm{KD}_{T(n)}(\mathcal{O}))$ is an equivalence.

Remark 3. In the statement of Proposition 2, there is nothing special about the setting of $T(n)$-local spectra: we could replace $\mathrm{Sp}_{T(n)}$ by any presentable symmetric monoidal stable ∞-category (such as spectra, or chain complexes over a field).

The proof of Proposition 2 will require some preliminaries.

Lemma 4. *Let $\mathcal{O}$ be an augmented operad in $T(n)$-local spectra satisfying the hypotheses of Proposition 2. Then the cooperad $\mathcal{O}' = \mathrm{Bar}(\mathcal{O})$ satisfies the analogous conditions:*

- *(a′) Each $\mathcal{O}'(k)$ is dualizable as a $T(n)$-local spectrum.*
- *(b′) The $T(n)$-local spectrum $\mathcal{O}'(0)$ vanishes.*
- *(c′) The augmentation of $\mathcal{O}'$ induces an equivalence $L_{T(n)}S \to \mathcal{O}'(1)$.*

Proof. Let us say that a symmetric sequence $\mathcal{E}$ is *concentrated in degrees* $\geq m$ if $\mathcal{E}(k) \simeq 0$ for $k < m$. It follows immediately from the definitions that if $\mathcal{E}$ and $\mathcal{E}'$ are concentrated in degrees $\geq m$ and $\geq m'$, respectively, then the composition product $\mathcal{E} \circ \mathcal{E}'$ is concentrated in degrees $\geq mm'$. For any right $\mathcal{O}$-module $\mathcal{E}$, let $\mathcal{E} \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}}$ denote the relative composition product of $\mathcal{E}$ with $\mathcal{O}_{\mathrm{triv}}$ over $\mathcal{O}$, given by the geometric realization of a simplicial object given in simplicial degree k by the iterated composition product

$$\mathcal{E} \circ \mathcal{O} \circ \cdots \circ \mathcal{O} \circ \mathcal{O}_{\mathrm{triv}}$$

(in which the operad $\mathcal{O}$ appears k times). Suppose that $\mathcal{E}$ is concentrated in degrees $\geq m$. Since $\mathcal{O}$ and $\mathcal{O}_{\mathrm{triv}}$ are concentrated in degrees ≥ 1, it follows that each of these iterated composition products is concentrated in degrees $\geq m$, so that the geometric realization $\mathcal{E} \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}}$ is concentrated in degrees $\geq m$. Moreover, it follows from assumption (c) that the simplicial spectrum

$$[k] \mapsto (\mathcal{E} \circ \mathcal{O} \circ \cdots \circ \mathcal{O} \circ \mathcal{O}_{\mathrm{triv}})(m)$$

is constant with value $\mathcal{E}(m)$, so that the canonical map $\mathcal{E} \to \mathcal{E} \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}}$ is an equivalence in degree m. Taking $\mathcal{E} = \mathcal{O}_{\mathrm{triv}}$, we immediately deduce (b') and (c').

To deduce (a'), it will suffice to prove the following:

- $(*)$ Let $\mathcal{E}$ be a right $\mathcal{O}$-module and let $m \geq 0$ be an integer. Then the following conditions are equivalent:
  - (i) The $T(n)$-local spectra $\mathcal{E}(k)$ are dualizable for $k \leq m$.
  - (ii) The $T(n)$-local spectra $(\mathcal{E} \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}})$ are dualizable for $k \leq m$.

Let us regard m as fixed. We will show that $(*)$ holds in the case where $\mathcal{E}$ is concentrated in degrees $\geq m'$, using descending induction on m'. If $m' > m$, then $\mathcal{E}(k)$ and $(\mathcal{E} \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}})(k) \simeq 0$ for $k \leq m$, so there is nothing to prove. To carry out the inductive step, let $\mathcal{E}'$ be the symmetric sequence which agrees with $\mathcal{E}$ in degree m', and vanishes in all other degrees. Note that we have $\mathcal{E}'(m') \simeq \mathcal{E}(m') \simeq (\mathcal{E} \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}})(m')$, so that we may assume that $\mathcal{E}'$ is dualizable in each degree (as this follows from either (i) or (ii)). The inclusion $\mathcal{E}' \hookrightarrow \mathcal{E}$ induces a map of right $\mathcal{O}$-modules $\mathcal{E}' \circ \mathcal{O} \to \mathcal{E}$, whose cofiber is some right $\mathcal{O}$-module $\mathcal{E}''$. Using the equivalence $\mathcal{E}' \circ \mathcal{O} \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}} \simeq \mathcal{E}'$, we obtain a cofiber sequence of symmetric sequences

$$\mathcal{E}' \to \mathcal{E} \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}} \to \mathcal{E}'' \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}}.$$

If condition (i) is satisfied, then $\mathcal{E}''$ is dualizable in degrees $\leq m$ (since this is true for both $\mathcal{E}' \circ \mathcal{O}$ and $\mathcal{E}$). Applying our inductive hypothesis, we conclude that $\mathcal{E}'' \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}}$ is dualizable in degrees $\leq m$. The cofiber sequence then shows that $\mathcal{E} \circ \mathcal{O}\, \mathcal{O}_{\mathrm{triv}}$ is dualizable in degrees $\leq m$, proving (ii).

Conversely, if (ii) is satisfied, then the cofiber sequence shows that $\mathcal{E}'' \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}}$ is dualizable in degrees $\leq m$. Applying our inductive hypothesis, we conclude that $\mathcal{E}''$ is dualizable in degrees $\leq m$. Using the fiber sequence

$$\mathcal{E}' \circ \mathcal{O} \to \mathcal{E} \to \mathcal{E}'',$$

we conclude that $\mathcal{E}$ is dualizable in degrees $\leq m$. □

Corollary 5. *Let $\mathcal{O}$ be an augmented operad satisfying the hypotheses of Proposition 2. Then the Koszul dual $\mathrm{KD}_{T(n)}(\mathcal{O})$ also satisfies the hypotheses of Proposition 2.*

In what follows, we let $\mathcal{A}$ denote the full subcategory of $\mathrm{Alg}^{\mathrm{aug}}(\mathrm{SSeq}_{T(n)})$ spanned by those augmented operads satisfying the hypotheses of Proposition 2. We wish to show that for each $\mathcal{O}$ in $\mathcal{A}$, the biduality map

$$u : \mathcal{O} \to \mathrm{KD}_{T(n)}(\mathrm{KD}_{T(n)}(\mathcal{O}))$$

is an equivalence. It follows from Corollary 5 that u is a morphism in $\mathcal{A}$. It will therefore suffice to show that for each object $\mathcal{O}' \in \mathcal{A}$, composition with u induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{A}}(\mathcal{O}', \mathcal{O}) \to \mathrm{Map}_{\mathcal{A}}(\mathcal{O}', \mathrm{KD}_{T(n)}(\mathrm{KD}_{T(n)}(\mathcal{O}))) \simeq \mathrm{Map}_{\mathcal{A}}(\mathrm{KD}_{T(n)}(\mathcal{O}), \mathrm{KD}_{T(n)}(\mathcal{O}')).$$

In other words, we are reduced to showing that the Koszul duality functor is fully faithful, when regarded as a contravariant functor from $\mathcal{A}$ to itself. Note that this functor factors as a composition

$$\mathcal{A} \xrightarrow{\mathrm{Bar}} \mathcal{A}' \xrightarrow{\mathbf{D}_{T(n)}} \mathcal{A}^{\mathrm{op}},$$

where $\mathcal{A}'$ is the full subcategory of $\mathrm{coAlg}^{\mathrm{aug}}(\mathrm{SSeq}_{T(n)})$ spanned by those augmented cooperads satisfying conditions (a'), (b'), and (c') of Lemma 4. It follows immediately from the definitions that the Spanier-Whitehead duality functor $\mathbf{D}_{T(n)} : \mathcal{A}' \to \mathcal{A}^{\mathrm{op}}$ is an equivalence of ∞-categories. Consequently, Proposition 2 is a consequence of the following general result about the comparison between operads and cooperads:

Proposition 6. *Let $\overline{\mathcal{A}} \subseteq \mathrm{Alg}^{\mathrm{aug}}(\mathrm{SSeq}_{T(n)})$ denote the full subcategory of $\mathrm{Alg}^{\mathrm{aug}}(\mathrm{SSeq}_{T(n)})$ spanned by those symmetric sequences satisfying conditions (b) and (c) of Proposition 2, and define $\overline{\mathcal{A}}' \subseteq \mathrm{coAlg}^{\mathrm{aug}}(\mathrm{SSeq}_{T(n)})$ similarly. Then the bar construction $\mathcal{O} \mapsto \mathrm{Bar}(\mathcal{O})$ induces an equivalence of ∞-categories $\overline{\mathcal{A}} \to \overline{\mathcal{A}}'$.*

Let us postpone the proof of Proposition 6 for the moment, and return to the proof of Theorem 1. By virtue of Proposition 2, it will suffice to prove the following:

Proposition 7. *Let $\mathcal{O}$ be the symmetric sequence of coderivatives of $\Phi \circ \Theta$. Then:*

- (a) *Each $\mathcal{O}(k)$ is dualizable as a $T(n)$-local spectrum.*
- (b) *The $T(n)$-local spectrum $\mathcal{O}(0)$ vanishes.*
- (c) *The unit map $L_{T(n)}S \to \mathcal{O}(1)$ is an equivalence.*

Proof. Assertion (b) is obvious (since $(\Phi \circ \Theta)(0) \simeq \Phi(*) \simeq 0$). To prove (a), it will suffice to show that the bar construction $\mathrm{Bar}(\mathcal{O})$ is dualizable in each degree. This can be identified with the symmetric sequence of coderivatives of $\Sigma^\infty_{T(n)} \circ \Omega^\infty_{T(n)}$, which we showed (in the previous lecture) to agree with the $T(n)$-local sphere in every positive degree.

It remains to prove (c). Let $\mathrm{Sp}(\mathcal{S}^{v_n}_*)$ denote the stabilization of the ∞-category $\mathcal{S}^{v_n}_*$, and let $\Sigma^\infty : \mathcal{S}^{v_n}_* \to \mathrm{Sp}(\mathcal{S}^{v_n}_*)$ denote the left adjoint to the 0th space functor. Then Σ^∞ is *universal* among colimit-preserving functors from $\mathcal{S}^{v_n}_*$ to a presentable stable ∞-category. In particular, the functor $\Sigma^\infty_{T(n)} : \mathcal{S}^{v_n}_* \to \mathrm{Sp}_{T(n)}$ admits an essentially unique factorization

$$\mathcal{S}^{v_n}_* \xrightarrow{\Sigma^\infty} \mathrm{Sp}(\mathcal{S}^{v_n}_*) \xrightarrow{F} \mathrm{Sp}_{T(n)}.$$

Since $\mathcal{S}^{v_n}_*$ can be identified with the ∞-category of left modules over the (reduced) monad $\Phi \circ \Theta$ on $\mathrm{Sp}_{T(n)}$, it follows that the stabilization $\mathrm{Sp}(\mathcal{S}^{v_n}_*)$ can be identified with the ∞-category of left modules over the first derivative $\partial_1(\Phi \circ \Theta) : \mathrm{Sp}(\mathrm{Sp}_{T(n)}) \to \mathrm{Sp}(\mathrm{Sp}_{T(n)})$. In other words, we obtain an equivalence of ∞-categories $\mathrm{Sp}(\mathcal{S}^{v_n}_*) \simeq \mathrm{LMod}_{\mathcal{O}(1)}(\mathrm{Sp}_{T(n)})$, where we regard $\mathcal{O}(1)$ as an associative ring spectrum (using the operad structure on $\mathcal{O}$). Under this equivalence, the functor $F : \mathrm{Sp}(\mathcal{S}^{v_n}_*) \to \mathrm{Sp}_{T(n)}$ is given by extension of scalars along the augmentation $\epsilon : \mathcal{O}(1) \to L_{T(n)}(S)$ determined by the augmentation of $\mathcal{O}$. We may therefore reformulate (c) as follows:

- (c') The functor $F : \mathrm{Sp}(\mathcal{S}^{v_n}_*) \to \mathrm{Sp}_{T(n)}$ is an equivalence of ∞-categories. In other words, the functor $\Sigma^\infty_{T(n)} : \mathcal{S}^{v_n}_* \to \mathrm{Sp}_{T(n)}$ exhibits $\mathrm{Sp}_{T(n)}$ as the stabilization of $\mathcal{S}^{v_n}_*$.

Let us prove (c'). If $\mathcal{C}$ is a pointed ∞-category which admits finite limits, we let $\mathrm{Sp}(\mathcal{C})$ denote the ∞-category of spectrum objects of $\mathcal{C}$, given by the inverse limit

$$\cdots \to \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}.$$

The usual ∞-category of spectra is obtained by applying this construction in the case where $\mathcal{C} = \mathcal{S}_*$ is the ∞-category of pointed spaces. However, it can just as well be obtained by applying the same construction to the ∞-category $\mathcal{S}^{\langle d \rangle}_*$ of d-connected pointed spaces, for any $d \geq 0$. Let A be a finite (p)-local suspension space of type $(n+1)$, and let d be the smallest integer such that $\mathrm{H}_d(A; \mathbf{F}_p) \neq 0$. Recall that $L^f_n \mathcal{S}^{\langle d \rangle}_*$ denotes the full subcategory of $\mathcal{S}^{\langle d \rangle}_*$ spanned by those d-connected pointed spaces which are P_A-local and p-local. Then the identification $\mathrm{Sp} \simeq \mathrm{Sp}(\mathcal{S}^{\langle d \rangle}_*$ restricts to an identification $\mathcal{X} \simeq \mathrm{Sp}(L^f_n \mathcal{S}^{\langle d \rangle}_*)$, where $\mathcal{X} \subseteq \mathrm{Sp}$ is the full subcategory spanned by those spectra X such that each of the spaces $(\Omega^{\infty+k} X)\langle d \rangle$ is P_A-local and p-local. This is equivalent to the requirement that X is p-local and the mapping spectrum X^A is contractible; that is, that X is an L^f_n-local spectrum.

Let us identify $\mathcal{S}_*^{v_n}$ with the full subcategory of $L_n^f \mathcal{S}_*^{\langle d \rangle}$ spanned by those objects whose v_m-local homotopy groups vanish for $0 \leq m < n$. Then the equivalence $\mathrm{Sp}(L_n^f \mathcal{S}_*^{\langle d \rangle}) \simeq \mathcal{X}$ restricts to an equivalence $\mathrm{Sp}(\mathcal{S}_*^{v_n}) \simeq \mathcal{Y}$, where $\mathcal{Y}$ is the full subcategory of Sp spanned by those spectra X which are L_n^f-local and have the property that each of the spaces $(\Omega^{\infty+k} X)\langle d \rangle$ has trivial v_m-periodic homotopy for $0 \leq m < n$. This is equivalent to the requirement that X is rationally trivial and that the Bousfield-Kuhn functors $\Phi_{T(m)}(\Omega^{\infty+k} X\langle d \rangle) \simeq \Omega^k L_{T(m)} X$ vanish for $0 < m < n$. In other words, the functor $\Omega^\infty \langle d \rangle$ induces an identification $\mathrm{Sp}_{M(n)} \to \mathrm{Sp}(\mathcal{S}_*^{v_n})$. Arguing as in the previous lecture, we see that this identification carries the functor $F : \mathrm{Sp}(\mathcal{S}_*^{v_n}) \to \mathrm{Sp}_{T(n)}$ to the $T(n)$-localization functor $L_{T(n)} : \mathrm{Sp}_{M(n)} \to \mathrm{Sp}_{T(n)}$, which is an equivalence as desired. □

We conclude this lecture by sketching a proof of Proposition 6. The bar construction

$$\mathrm{Bar} : \mathrm{Alg}^{\mathrm{aug}}(\mathrm{SSeq}_{T(n)}) \to \mathrm{coAlg}^{\mathrm{aug}}(\mathrm{SSeq}_{T(n)})$$

has a right adjoint, given by the *cobar constructon* $\mathrm{Cobar} : \mathrm{coAlg}^{\mathrm{aug}}(\mathrm{SSeq}_{T(n)}) \to \mathrm{Alg}^{\mathrm{aug}}(\mathrm{SSeq}_{T(n)})$. (that is, applying the bar construction in the opposite category $\mathrm{SSeq}_{T(n)}^{\mathrm{op}}$: beware however that the composition product does not preserve geometric realizations in $\mathrm{SSeq}_{T(n)}^{\mathrm{op}}$). To show that the bar construction is fully faithful, it will suffice to show that if $\mathcal{O}$ satisfies conditions (b) and (c) of Proposition 2, then the unit map

$$u_{\mathcal{O}} : \mathcal{O} \to \mathrm{Cobar}(\mathrm{Bar}(\mathcal{O}))$$

is an equivalence (one can use essentially the same argument to show that the cobar construction is fully faithful on objects of $\overline{\mathcal{A}}'$; we leave the details to the reader).

As in the proof of Lemma 4, it will be convenient to prove a more general result concerning *right modules* over the operad $\mathcal{O}$. Note that if $\mathcal{E}$ is a right module over $\mathcal{O}$, then the relative composition product $\mathcal{E} \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}}$ can be regarded as a right comodule over $\mathrm{Bar}(\mathcal{O})$, with structure map given by

$$\begin{aligned} \mathcal{E} \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}} &\simeq \mathcal{E} \circ_{\mathcal{O}} \mathcal{O} \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}} \\ &\to \mathcal{E} \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}} \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}} \\ &\simeq (\mathcal{E} \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}}) \circ \mathrm{Bar}(\mathcal{O}) \end{aligned}$$

Let us denote $\mathcal{E} \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}}$ by $F(\mathcal{E})$, so that F determines a functor $\mathrm{RMod}_{\mathcal{O}}(\mathrm{SSeq}_{T(n)}) \to \mathrm{coRMod}_{\mathrm{Bar}(\mathcal{O})}(\mathrm{SSeq}_{T(n)})$. This functor has a right adjoint $G : \mathrm{coRMod}_{\mathrm{Bar}(\mathcal{O})}(\mathrm{SSeq}_{T(n)}) \to \mathrm{RMod}_{\mathcal{O}}(\mathrm{SSeq}_{T(n)})$ (given by applying a similar construction in the opposite category) which we will denote by $G(\mathcal{F}) = \mathcal{F} \circ^{\mathrm{Bar}(\mathcal{O})} \mathcal{O}_{\mathrm{triv}}$.

For every right $\mathcal{O}$-module $\mathcal{E}$, we have a map of symmetric sequences

$$u_{\mathcal{E}} : \mathcal{E} \to (G \circ F)(\mathcal{E}) = (\mathcal{E} \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}}) \circ^{\mathrm{Bar}(\mathcal{O})} \mathcal{O}_{\mathrm{triv}} .$$

In the special case where $\mathcal{E} = \mathcal{O}$ (regarded as a right module over itself), the map $u_{\mathcal{E}}$ underlies the map of operads $\mathcal{O} \to \mathrm{Cobar}(\mathrm{Bar}(\mathcal{O}))$ given by the adjointness of the bar and cobar constructions. It will therefore suffice to prove the following:

($*$) For any right $\mathcal{O}$-module $\mathcal{E}$, the unit map $u_{\mathcal{E}} : \mathcal{E} \to (G \circ F)(\mathcal{E})$ is an equivalence.

It will suffice to prove the following assertion for each $m \geq 0$:

($*_m$) For any right $\mathcal{O}'$-module $\mathcal{E}$, the unit map $u_{\mathcal{E}} : \mathcal{E} \to (G \circ F)(\mathcal{E})$ is an equivalence in degrees $\leq m$.

We will show that ($*_m$) holds for every right $\mathcal{O}$-module $\mathcal{E}$ which is concentrated in degrees $\geq m'$. This is clear for $m' > m$ (since then both sides vanish in degrees $\leq m$). We handle the remaining cases by descending induction on m'. Assume that $\mathcal{E}$ is concentrated in degrees $\geq m'$. Then $\mathcal{E}$ fits into a fiber sequence of right $\mathcal{O}$-modules

$$\mathcal{E}' \to \mathcal{E} \to \mathcal{E}'',$$

where $\mathcal{E}'$ is concentrated in degrees $> m'$, and the action of $\mathcal{O}$ on $\mathcal{E}''$ is trivial (here we invoke our assumption that $\mathcal{O}$ satisfies (b) and (c) of Proposition 2). Using the exactness of the functors F and G, we obtain a commutative diagram of fiber sequences

$$\begin{array}{ccccc} \mathcal{E}' & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}'' \\ \downarrow{\scriptstyle u_{\mathcal{E}'}} & & \downarrow{\scriptstyle u_{\mathcal{E}}} & & \downarrow{\scriptstyle u_{\mathcal{E}''}} \\ (G \circ F)(\mathcal{E}') & \longrightarrow & (G \circ F)(\mathcal{E}) & \longrightarrow & (G \circ F)(\mathcal{E}''). \end{array}$$

Our inductive hypothesis guarantees that $u_{\mathcal{E}'}$ is an equivalence in degrees $\leq m$. Consequently, to prove ($*_m$), it will suffice to show that the map $u_{\mathcal{E}''}$ is an equivalence. We are therefore reduced to proving assertion ($*$) in the special case where the action of $\mathcal{O}$ on $\mathcal{E}$ is *trivial*: that is, it factors through the augmentation $\mathcal{O} \to \mathcal{O}_{\mathrm{triv}}$. In this case, we compute

$$\begin{aligned} (G \circ F)(\mathcal{E}) &\simeq G(\mathcal{E} \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}}) \\ &\simeq G(\mathcal{E} \circ \mathcal{O}_{\mathrm{triv}} \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}}) \\ &\simeq G(\mathcal{E} \circ \mathrm{Bar}(\mathcal{O})) \\ &\simeq \mathcal{E}, \end{aligned}$$

where the final equivalence follows from the observation that $\mathcal{E}' = \mathcal{E} \circ \mathrm{Bar}(\mathcal{O})$ is *cofree* as a right comodule over $\mathrm{Bar}(\mathcal{O})$ (so that the cobar construction on $\mathcal{E}'$ splits).