# LECTURE XXX: KOSZUL DUALITY, PART III

Fix a prime number $p$ and an integer $n > 0$. We have seen that there is a monadic adjunction

$$\mathrm{Sp}_{T(n)} \underset{\Phi}{\overset{\Theta}{\rightleftarrows}} \mathcal{S}_*^{v_n},$$

and that the associated monad $\Phi\Theta : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ is coanalytic, and can therefore be identified with an operad in $T(n)$-local spectra. (Throughout this lecture, we will abuse notation by identifying the $\infty$-category $\mathrm{SSeq}_{T(n)}$ of symmetric sequences of $T(n)$-local spectra with the $\infty$-category of coanalytic functors from $\mathrm{Sp}_{T(n)}$ to itself). Let $\mathrm{Sym}^*_{\mathrm{red}} : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ denote the functor which associates to each $T(n)$-local spectrum $E$ the sum $\bigoplus_{n>0} E^{\otimes n}_{h\Sigma_n}$, which we identify with the free nonunital commutative algebra generated by $E$. In the previous lecture, we constructed comparison maps

$$\mu : \mathrm{Sym}^*_{\mathrm{red}} \to \mathrm{KD}_{T(n)}(\Phi\Theta),$$

$$\nu : \Phi\Theta \to \mathrm{KD}_{T(n)}(\mathrm{Sym}^*_{\mathrm{red}}).$$

We would like to prove the following:

**Theorem 1.** *The maps*

$$\mu : \mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}} \to \mathrm{KD}_{T(n)}(\Phi\Theta)$$

$$\nu : \Phi\Theta \to \mathrm{KD}_{T(n)}(\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}})$$

*are homotopy equivalences of operads in $T(n)$-local spectra.*

We will show in this lecture that $\mu$ is an equivalence; we defer the proof that $\nu$ is an equivalence (which is the statement we are really after) to the next lecture.

Recall that the Koszul dual $\mathrm{KD}_{T(n)}(\Phi\Theta)$ can be identified with the Spanier-Whitehead dual of the symmetric sequence corresponding to the coanalytic functor $\Sigma^\infty_{T(n)}\Omega^\infty_{T(n)} : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$. In other words, $\mathrm{KD}_{T(n)}(\Phi\Theta)$ is the best coanalytic approximation to the composition

$$\mathbf{D}_{T(n)} \circ \Sigma^\infty_{T(n)} \circ \Omega^\infty_{T(n)} \circ \mathbf{D}_{T(n)}.$$

The map $\mu$ can be constructed by combining the following observations:

(*i*) The functor $\mathbf{D}_{T(n)} \circ \Sigma^\infty_{T(n)} : (\mathcal{S}_*^{v_n}) \to \mathrm{Sp}_{T(n)}$ takes values in nonunital commutative algebras. In particular, there is a canonical map

$$a : \mathrm{Sym}^*_{\mathrm{red}} \circ \mathbf{D}_{T(n)} \circ \Sigma^\infty_{T(n)} \to \mathbf{D}_{T(n)} \circ \Sigma^\infty_{T(n)}.$$

$(ii)$ The counit map $v : \Sigma^\infty_{T(n)} \circ \Omega^\infty_{T(n)} \to \mathrm{id}$ induces a natural transformation

$$\mu_0 : \mathrm{id}_{\mathrm{Sp}_{T(n)}} \to \mathbf{D}_{T(n)} \circ \mathbf{D}_{T(n)} \xrightarrow{v} \mathbf{D}_{T(n)} \circ \Sigma^\infty_{T(n)} \circ \Omega^\infty_{T(n)} \circ \mathbf{D}_{T(n)}.$$

The map $\mu$ is then given by the composition

$$\begin{aligned} \mathrm{Sym}^*_{\mathrm{red}} &= \mathrm{Sym}^*_{\mathrm{red}} \circ \mathrm{id}_{\mathrm{Sp}_{T(n)}} \\ &\xrightarrow{\mu_0} \mathrm{Sym}^*_{\mathrm{red}} \circ \mathbf{D}_{T(n)} \circ \Sigma^\infty_{T(n)} \circ \Omega^\infty_{T(n)} \circ \mathbf{D}_{T(n)} \\ &\xrightarrow{a} \mathbf{D}_{T(n)} \circ \Sigma^\infty_{T(n)} \circ \Omega^\infty_{T(n)} \circ \mathbf{D}_{T(n)}. \end{aligned}$$

It will actually be convenient to consider a slight variant, where we replace the functor $\Sigma^\infty_{T(n)}$ by an *unreduced* suspension functor $\Sigma^\infty_{T(n)+}$, given by the construction $\Sigma^\infty_{T(n)+}(X) = \Sigma^\infty_{T(n)}(X) \oplus L_{T(n)}(S)$. Then $\mu$ induces a natural transformation

$$\mu_+ : \mathrm{Sym}^* \to \mathbf{D}_{T(n)} \circ \Sigma^\infty_{T(n)+} \circ \Omega^\infty_{T(n)} \circ \mathbf{D}_{T(n)},$$

which arises from the fact that $\mathbf{D}_{T(n)} \circ \Sigma^\infty_{T(n)+}$ takes values in *unital* commutative algebra objects of $\mathrm{Sp}_{T(n)}$.

For any object $X \in \mathcal{S}^{v_n}_*$, the commutative algebra structure on $\mathbf{D}_{T(n)} \Sigma^\infty_{T(n)+}(X)$ arises from a commutative coalgebra structure on the predual $\Sigma^\infty_{T(n)+}(X)$. This coalgebra structure can be obtained formally from the following observations:

- The functor $\Sigma^\infty_{T(n)+}$ is symmetric monoidal: that is, it carries Cartesian products in $\mathcal{S}^{v_n}_*$ to smash products in $\mathrm{Sp}_{T(n)}$.
- Every object $X$ of the category $\mathcal{S}^{v_n}_*$ (or of any other $\infty$-category with finite products) can be regarded as a commutative coalgebra with respect to the Cartesian product in a unique way: the comultiplication is just given by the diagonal map $\delta : X \to X \times X$.

In practice, we will be interested in coalgebras of the form $\Sigma^\infty_{T(n)+} \Omega^\infty_{T(n)} E$, where $E$ is a $T(n)$-local spectrum. Note that objects of this form are actually *bialgebras* in the $\infty$-category $\mathrm{Sp}_{T(n)}$: they have a comultiplication coming from the diagonal on $\Omega^\infty_{T(n)}(E)$, and an algebra structure coming from the $E_\infty$-structure on $\Omega^\infty_{T(n)}(E)$. It will be useful to observe that we can formally extract the comultiplication from the multiplication:

**Proposition 2.** *Let $F : \mathrm{Sp}_{T(n)} \to \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$ be a functor which preserves finite coproducts: that is, it carries direct sums in $\mathrm{Sp}_{T(n)}$ to smash products in $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$. Then $F$ can be canonically promoted to a functor taking values in bialgebra objects of $\mathrm{Sp}_{T(n)}$.*

*Proof.* For any $T(n)$-local spectrum $E$, the diagonal map $E \to E \oplus E$ induces a comultiplication

$$F(E) \to F(E \oplus E) \to F(E) \otimes F(E)$$

in $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$. This comultiplication is (coherently) commutative and associative because the diagonal of $E$ has the same features. □

**Example 3.** If we regard the construction $E \mapsto \Sigma^\infty_{T(n)+}\Omega^\infty_{T(n)}(E)$ as a functor from $\mathrm{Sp}_{T(n)}$ to $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$, then it satisfies the requirement of Proposition 2. We therefore obtain a bialgebra structure on $\Sigma^\infty_{T(n)+}\Omega^\infty_{T(n)}(E)$. We note that the underlying coalgebra structure agrees with the natural coalgebra structure on $\Sigma^\infty_{T(n)+}(X)$ for any $X \in \mathcal{S}^{v_n}_*$.

**Example 4.** The free commutative algebra functor $E \mapsto \mathrm{Sym}^*(E)$ is a functor from $\mathrm{Sp}_{T(n)}$ to $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$ which satisfies the requirements of Proposition 2, and therefore also takes values in bialgebra objects of $\mathrm{Sp}_{T(n)}$.

Examples 3 and 4 are related.

**Proposition 5.** *The functors $\Sigma^\infty_{T(n)+}\Omega^\infty$ and $\mathrm{Sym}^*$ are canonically equivalent when regarded as functors from $\mathrm{Sp}_{T(n)}$ to $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$, and therefore also when regarded as functors from $\mathrm{Sp}_{T(n)}$ to bialgebra objects of $\mathrm{Sp}_{T(n)}$ (or coalgebra objects of $\mathrm{Sp}_{T(n)}$).*

*Proof.* This essentially equivalent to the theorem of Kuhn that we proved last semester. We begin with some notation. Let $L^f_n, L^f_{n-1} : \mathrm{Sp} \to \mathrm{Sp}$ denote the localizations obtained by $(p)$-localizing and killing finite spectra of type $> n$ and $\geq n$, respectively. Recall that a spectrum $X$ is *monochromatic of height $n$* if it is $L^f_n$-local and $L^f_{n-1}$ acyclic. The collection of such spectra span a full subcategory $\mathrm{Sp}_{M(n)} \subseteq \mathrm{Sp}$, and the localization functor $L_{T(n)}$ induces an equivalence

$$L_{T(n)} : \mathrm{Sp}_{M(n)} \simeq \mathrm{Sp}_{T(n)},$$

with inverse given by the construction $E \mapsto \mathrm{fib}(E \to L^f_{n-1}E)$.

Fix finite pointed spaces $A$ and $B$, of types $(n+1)$ and $n$, respectively. Assume that $A$ and $B$ are both suspensions, and that their first nonvanishing homology group (with $\mathbf{F}_p$-coefficients) appears in the same degree $d$. Recall that $\mathcal{S}^{v_n}_*$ can be realized as the full subcategory of $\mathcal{S}_*$ spanned by those spaces which are $d$-connected, $P_A$-local, and $P_B$-acyclic.

**Lemma 6.** (1) *For $X \in \mathcal{S}^{v_n}_*$, the spectrum $L^f_n\Sigma^\infty X$ belongs to $\mathrm{Sp}_{M(n)}$.*
(2) *For $E \in \mathrm{Sp}_{M(n)}$, the space $(\Omega^\infty E)\langle d\rangle$ belongs to $\mathcal{S}^{v_n}_*$.*

*Proof.* To prove (1), we note that $L^f_n\Sigma^\infty X$ is automatically $L^f_n$-local. It will therefore suffice to show that

$$\mathrm{Map}_{\mathrm{Sp}}(L^f_n\Sigma^\infty X, E) \simeq \mathrm{Map}_{\mathcal{S}_*}(X, \Omega^\infty E)$$

is contractible when $E$ is $L^f_{n-1}$-local. This is clear: the space $X$ is $P_B$-acyclic, and the space $\Omega^\infty E$ is $P_B$-local (since $\Sigma^\infty B$ is a finite spectrum of type $\geq n$).

To prove (2), we note that $E \in \mathrm{Sp}_{M(n)}$ is $L_n^f$-local, so that $\Omega^\infty E$ is automatically $(p)$-local and $P_A$-local. Moreover, since $E$ is rationally trivial, the space $\Omega^\infty E$ is also rationally trivial. It will therefore suffice to show that the $v_m$-periodic homotopy groups of $\Omega^\infty E$ vanish for $0 < m < n$. This follows from the fact that the $T(m)$-local Bousfield-Kuhn functor carries $\Omega^\infty E$ to $L_{T(m)} E \simeq 0$. □

It follows from Lemma 6 that the functor $\Sigma^\infty_{T(n)}$ is given by composing $L_n^f \Sigma^\infty$ with the equivalence $L_{T(n)} : \mathrm{Sp}_{M(n)} \to \mathrm{Sp}_{T(n)}$, and that $\Omega^\infty_{T(n)}$ is given by composing the functor $\Omega^\infty \langle d \rangle$ with the inverse equivalence $\mathrm{Sp}_{T(n)} \simeq \mathrm{Sp}_{M(n)}$. It follows that the composite functor $\Sigma^\infty_{T(n)} \circ \Omega^\infty_{T(n)}$ is given by composing the equivalence $\mathrm{Sp}_{T(n)} \simeq \mathrm{Sp}_{M(n)}$ with the functor

$$\mathrm{Sp}_{M(n)} \to \mathrm{Sp}_{T(n)} \qquad E \mapsto L_{T(n)} \Sigma^\infty \Omega^\infty E \langle d \rangle.$$

Similarly, $\Sigma^\infty_{T(n)+} \circ \Omega^\infty_{T(n)}$ is given by composing the equivalence $\mathrm{Sp}_{T(n)} \simeq \mathrm{Sp}_{M(n)}$ with the functor

$$\mathrm{Sp}_{M(n)} \to \mathrm{Sp}_{T(n)} \qquad E \mapsto L_{T(n)} \Sigma^\infty_+ \Omega^\infty E \langle d \rangle.$$

For any spectrum $Z$, we can apply the Bousfield-Kuhn functor $\Phi$ to the unit map $\Omega^\infty Z \to \Omega^\infty \Sigma^\infty_+ \Omega^\infty Z$ to obtain a map

$$L_{T(n)} Z \to L_{T(n)} \Sigma^\infty_+ \Omega^\infty Z,$$

which we can extend to a map of commutative algebras $\mathrm{Sym}^*(L_{T(n)} Z) \to L_{T(n)} \Sigma^\infty_+ \Omega^\infty Z$. Kuhn's theorem asserts that this map is an equivalence when $Z$ is $L_{n-1}^f$-acyclic and $d$-connected. Applying this observation in the case where $Z = (\mathrm{fib} : E \to L_{n-1}^f E) \langle d \rangle$ for $E \in \mathrm{Sp}_{T(n)}$, we obtain the desired equivalence

$$\mathrm{Sym}^*(E) \to L_{T(n)} \Sigma^\infty_+ \Omega^\infty Z \simeq \Sigma^\infty_{T(n)+} \Omega^\infty_{T(n)} E.$$

□

**Remark 7.** In the proof of Proposition 5, the unit map $\Omega^\infty Z \to \Omega^\infty \Sigma^\infty_+ \Omega^\infty Z$ is a right inverse to the map $\Omega^\infty(v')$, where $v' : \Sigma^\infty_+ \Omega^\infty Z \to Z$ is the counit. Taking $Z = (\mathrm{fib} : E \to L_{n-1}^f E) \langle d \rangle$ for $E \in \mathrm{Sp}_{T(n)}$ and applying the Bousfield-Kuhn functor, we deduce that the composition

$$E \to \Sigma^\infty_{T(n)+} \Omega^\infty_{T(n)} E \xrightarrow{v} E$$

is the identity on $E$ (where $v$ is the counit map). It follows that the composition of the equivalence $\mathrm{Sym}^*(E) \simeq \Sigma^\infty_{T(n)+} \Omega^\infty_{T(n)} E$ with the counit map $\Sigma^\infty_{T(n)+} \Omega^\infty_{T(n)} E \to E$ is the identity on $\mathrm{Sym}^1(E)$ (and automatically vanishes on $\mathrm{Sym}^n(E)$ for $n \neq 1$, since $\mathrm{Sym}^n(E)$ is $n$-homogeneous).

Using Proposition 5 (and Remark 7), we can reformulate the first part of Theorem 1 as follows:

**Proposition 8.** *Let $E$ be a $T(n)$-local spectrum, and regard* $\mathrm{Sym}^*(\mathbf{D}_{T(n)}E)$ *as a commutative coalgebra via Example 4, so that the dual* $\mathbf{D}_{T(n)}\mathrm{Sym}^*(\mathbf{D}_{T(n)}E)$ *inherits a commutative algebra structure, and therefore the map*

$$E \to \mathbf{D}_{T(n)}\mathrm{Sym}^1(\mathbf{D}_{T(n)}E) \to \mathbf{D}_{T(n)}\mathrm{Sym}^*(\mathbf{D}_{T(n)}E)$$

*extends uniquely to a morphism of commutative algebras*

$$\mathrm{Sym}^*(E) \to \mathbf{D}_{T(n)}\mathrm{Sym}^*(\mathbf{D}_{T(n)}E).$$

*These maps exhibit* $\mathrm{Sym}^*$ *as the best coanalytic approximation to the functor* $\mathbf{D}_{T(n)} \circ \mathrm{Sym}^* \circ \mathbf{D}_{T(n)}$.

*Proof.* Unwinding the definitions, we see that when $E$ is dualizable, this map is given by

$$\bigoplus_{n\geq 0}(E^{\otimes n})_{h\Sigma_n} \to \prod_{n\geq 0}(E^{\otimes n})^{h\Sigma_n})$$

given by the norm map $(E^{\otimes n})_{h\Sigma_n} \to (E^{\otimes n})^{h\Sigma_n}$ in each degree. Using the vanishing of the $T(n)$-local Tate construction, we can identify this with the map from the sum $\bigoplus \mathrm{Sym}^n(E)$ to the product $\prod \mathrm{Sym}^n(E)$. □