LECTURE XX: KOSZUL DUALITY, PART II

In the previous lecture, we discussed a *covariant* version of Koszul duality: given an augmented associative algebra A in a monoidal ∞-category $\mathcal{C}$, we can (under mild hypotheses) form the *bar construction* $\mathrm{Bar}(A) = \mathbf{1} \otimes_A \mathbf{1}$, and endow it with the structure of an associative coalgebra object of $\mathcal{C}$. When $\mathcal{C}$ is equipped with some notion of duality $\mathbf{D} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ which is lax monoidal, we can then take the dual $\mathbf{D}\,\mathrm{Bar}(A)$ and regard it as an associative algebra object of $\mathcal{C}$. We will be particularly interested in the special case where $\mathcal{C} = \mathrm{SSeq} = \mathrm{Fun}(\mathrm{Set}_{\mathrm{fin}}^{\simeq}, \mathrm{Sp})$ is the ∞-category of symmetric sequences of spectra.

Construction 1 (Spanier-Whitehead Duality for Symmetric Sequences)**.** For every spectrum E, we let $\mathbf{D}(E)$ denote the Spanier-Whitehead dual $\underline{\mathrm{Map}}(E, S)$. The construction $E \mapsto \mathbf{D}(E)$ induces a functor from the ∞-category Sp of spectra to $\mathrm{Sp}^{\mathrm{op}}$. Consequently, if $\mathcal{O} : \mathrm{Set}_{\mathrm{fin}}^{\simeq} \to \mathrm{Sp}$ is a symmetric sequence of spectra, then the composite functor

$$\mathrm{Set}_{\mathrm{fin}}^{\simeq} \simeq (\mathrm{Set}_{\mathrm{fin}}^{\simeq})^{\mathrm{op}} \xrightarrow{\mathcal{O}} \mathrm{Sp}^{\mathrm{op}} \xrightarrow{\mathbf{D}} \mathrm{Sp}$$

is also a symmetric sequence of spectra, which we will denote by $\mathbf{D}(\mathcal{O})$ and refer to as the *Spanier-Whitehead dual of* $\mathcal{O}$. Concretely, it is given by the formula

$$\mathbf{D}(\mathcal{O})(T) = \mathbf{D}(\mathcal{O}(T)).$$

Variant 2. Let $\mathrm{SSeq}_{T(n)}$ denote the ∞-category $\mathrm{Fun}(\mathrm{Set}_{\mathrm{fin}}^{\simeq}, \mathrm{Sp}_{T(n)})$ of symmetric sequences of $T(n)$-local spectra. There is a $T(n)$-local version of Spanier-Whitehead duality, which we will denote by $\mathbf{D}_{T(n)} : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}^{\mathrm{op}}$, given by

$$\mathbf{D}_{T(n)}(E) = \underline{\mathrm{Map}}(E, L_{T(n)} S).$$

Composition with $\mathbf{D}_{T(n)}$ induces an operation of $T(n)$-local Spanier-Whitehead duality on the ∞-category $\mathrm{SSeq}_{T(n)}$, given by the formula

$$\mathbf{D}_{T(n)}(\mathcal{O})(T) = \mathbf{D}_{T(n)}(\mathcal{O}(T)).$$

Let us regard the ∞-category SSeq as endowed with the monoidal structure given by the *composition product* of symmetric sequences. In this case, the Spanier-Whitehead duality functor $\mathbf{D} : \mathrm{SSeq}^{\mathrm{op}} \to \mathrm{SSeq}$ is lax monoidal: in particular, for every pair of symmetric sequences symmetric sequences $\mathcal{O}$ and $\mathcal{O}'$, there is a canonical map

$$\mathbf{D}(\mathcal{O}) \circ \mathbf{D}(\mathcal{O}') \to \mathbf{D}(\mathcal{O} \circ \mathcal{O}').$$

Remark 3 (Preliminary Explanation)**.** Let us suppose that the symmetric sequences $\mathcal{O}$ and $\mathcal{O}'$ are reduced. In this case, the composition products in question are given concretely by the formulae

$$(\mathbf{D}(\mathcal{O}) \circ \mathbf{D}(\mathcal{O}'))(T) = \bigoplus_{E \in \mathrm{Equiv}(T)} \mathbf{D}(\mathcal{O}(T/E)) \wedge \bigwedge_{T' \in T/E} \mathbf{D}(\mathcal{O}'(T'))$$

$$(\mathbf{D}(\mathcal{O} \circ \mathcal{O}'))(T) = \bigoplus_{E \in \mathrm{Equiv}(T)} \mathbf{D}(\mathcal{O}(T/E) \wedge \bigwedge_{T' \in T/E} \mathbf{D}(\mathcal{O}'(T'))).$$

There is a canonical map from the first spectrum to the second, given by the lax symmetric monoidal structure on the usual Spanier-Whitehead duality functor $\mathbf{D} : \mathrm{Sp}^{\mathrm{op}} \to \mathrm{Sp}$.

Let us now give another explanation for the lax monoidal structure on the Spanier-Whitehead duality functor $\mathbf{D} : \mathrm{SSeq}^{\mathrm{op}} \to \mathrm{SSeq}$. Recall that every symmetric sequence $\mathcal{O} \in \mathrm{SSeq}$ determines a functor $F_{\mathcal{O}} : \mathrm{Sp} \to \mathrm{Sp}$, given by the formula

$$F_{\mathcal{O}}(X) = \varinjlim_{T \in \mathrm{Set}^{\simeq}_{\mathrm{fin}}} \mathcal{O}(T) \wedge X^{\wedge T} \simeq \bigoplus_{n \geq 0} (\mathcal{O}(n) \wedge X^{\wedge n})_{h\Sigma_n}.$$

The construction $\mathcal{O} \mapsto F_{\mathcal{O}}$ determines a (monoidal) functor $\mathrm{SSeq} \to \mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})$. This functor has a right adjoint, given by the construction

$$(F \in \mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})) \mapsto \{\partial^n(F)\}_{n \geq 0};$$

here $\partial^n(F)$ denotes the nth coderivative of F in the sense of Lecture 8 (in that case, we were working with $T(n)$-local spectra, which guarantees that the construction $\mathcal{O} \mapsto F_{\mathcal{O}}$ is fully faithful; however, the right adjoint exists in general).

Let us say that a functor $\mathrm{Sp} \to \mathrm{Sp}$ is *coanalytic* if it has the form $F_{\mathcal{O}}$, for some symmetric sequence $\mathcal{O}$. In this case, we can form a new functor from Sp to Sp, given by the composition $\mathbf{D} \circ F_{\mathcal{O}} \circ \mathbf{D}$. This functor is usually not coanalytic. However, we can approximate it by a coanalytic functor, by applying the coderivative construction $\partial^{\bullet}$.

Proposition 4. *Let $\mathcal{O}$ be a symmetric sequence of spectra. Then the Spanier-Whitehead dual symmetric sequence $\mathbf{D}(\mathcal{O})$ is given by $\{\partial^n(\mathbf{D} \circ F_{\mathcal{O}} \circ \mathbf{D})\}_{n \geq 0}$.*

Proof. Unwinding the definitions, we have the formula

$$\begin{aligned}(\mathbf{D} \circ F_{\mathcal{O}} \circ \mathbf{D})(X) &= \mathbf{D}(\varinjlim_{T \in \mathrm{Set}^{\simeq}_{\mathrm{fin}}} (\mathcal{O}(T) \wedge \mathbf{D}(X)^{\wedge T}) \\ &= \varprojlim_{T \in \mathrm{Set}^{\simeq}_{\mathrm{fin}}} \mathbf{D}(\mathcal{O}(T) \wedge \mathbf{D}(X)^{\wedge T}).\end{aligned}$$

For each $n \geq 0$, the coderivative functor ∂^n commutes with inverse limits, so we have

$$\begin{aligned}\partial^n(\mathbf{D} \circ F_{\mathcal{O}} \circ \mathbf{D}) &= \varprojlim_{T \in \mathrm{Set}^{\simeq}_{\mathrm{fin}}} \partial^n(X \mapsto \mathbf{D}(\mathcal{O}(T) \wedge \mathbf{D}(X)^{\wedge T})) \\ &\simeq \varprojlim_{T \in \mathrm{Set}^{\simeq}_{\mathrm{fin}}} \partial^n(X \mapsto \mathbf{D}(\mathcal{O}(T)) \wedge X^{\wedge T})\end{aligned}$$

where the second equality follows because the canonical map

$$\mathbf{D}(\mathcal{O}(T)) \wedge X^{\wedge T} \to \mathbf{D}(\mathcal{O}(T) \wedge \mathbf{D}(X)^{\wedge T})$$

is an equivalence when X is a finite spectrum. It now suffices to observe that the functor $\mathbf{D}(\mathcal{O}(T)) \wedge X^{\wedge T})$ is cohomogeneous of degree $|T|$, so that the coderivative $\partial^n(X \mapsto \mathbf{D}(\mathcal{O}(T)) \wedge X^{\wedge T})$ vanishes for $|T| \neq n$. For $|T| = n$, the coderivative is given by the smash product $\mathbf{D}(\mathcal{O}(T)) \wedge (\Sigma_n)_+$. Passing to the inverse limit over T, we obtain the desired equivalence

$$\partial^n(\mathbf{D} \circ F_{\mathcal{O}} \circ \mathbf{D}) \simeq \mathbf{D}(\mathcal{O}(n)).$$

□

The $T(n)$-local version of Proposition 4 follows by the same argument, and is perhaps a bit easier to formulate.

Notation 5. Let $F : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ be a coanalytic functor, so that F has an essentially unique expression $F(X) = L_{T(n)} F_{\mathcal{O}}(X)$ for some $T(n)$-local symmetric sequence $\mathcal{O} \in \mathrm{SSeq}_{T(n)}$. We let $F^{\mathbf{D}}$ denote the coanalytic functor $L_{T(n)} \circ F_{\mathbf{D}_{T(n)} \mathcal{O}}$ given by the symmetric sequence which is $T(n)$-locally Spanier-Whitehead dual to $\mathcal{O}$. The proof of Proposition **??** shows that $F^{\mathbf{D}}$ is universal among coanalytic functors equipped with a natural transformation

$$F^{\mathbf{D}} \to \mathbf{D}_{T(n)} \circ F \circ \mathbf{D}_{T(n)}$$

(or, equivalently, with a natural transformation $F \circ \mathbf{D}_{T(n)} \to \mathbf{D}_{T(n)} \circ F^{\mathbf{D}}$).

Remark 6. Let $F, G : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ be coanalytic functors. Then the composition $F^{\mathbf{D}} \circ G^{\mathbf{D}}$ is also a coanalytic functor, equipped with a canonical map

$$F^{\mathbf{D}} \circ G^{\mathbf{D}} \to (\mathbf{D}_{T(n)} \circ F \circ \mathbf{D}_{T(n)}) \circ (\mathbf{D}_{T(n)} \circ G \circ \mathbf{D}_{T(n)}) \to \mathbf{D}_{T(n)} \circ F \circ G \circ \mathbf{D}_{T(n)}.$$

Invoking the universal property of $(F \circ G)^{\mathbf{D}}$, we obtain a comparison map

$$(F \circ G)^{\mathbf{D}} \to F^{\mathbf{D}} \circ G^{\mathbf{D}}.$$

It is not difficult to see that this map is given by the analogue of Remark 3, carried out in the $T(n)$-local setting (at least in the case where the functors F and G are reduced).

Let us describe a third way of understanding the lax compatibility of Spanier-Whitehead duality with composition.

Definition 7. Let $\mathcal{S}$ denote the ∞-category of spaces. We define a functor

$$\mu : \mathrm{Sp}^{\mathrm{op}} \times \mathrm{Sp}^{\mathrm{op}} \to \mathcal{S}$$

by the formula

$$\mu(X,Y) = \mathrm{Map}_{\mathrm{Sp}}(X \wedge Y, S) \simeq \mathrm{Map}_{\mathrm{Sp}}(X, \mathbf{D}(Y)) \simeq \mathrm{Map}_{\mathrm{Sp}}(Y, \mathbf{D}(X)).$$

The map μ classifies a *right fibration* of ∞-categories

$$\mathcal{E} \to \mathrm{Sp} \times \mathrm{Sp}.$$

More informally, $\mathcal{E}$ is the ∞-category whose objects are triples (X, Y, b), where X and Y are spectra and $b : X \wedge Y \to S$ is a map of spectra.

Suppose we are given symmetric sequences $\mathcal{O}, \mathcal{O}' \in \mathrm{SSeq}$. We define a *pairing of* $\mathcal{O}$ *with* $\mathcal{O}'$ to be an endomorphism of the right fibration $\mathcal{E} \to \mathrm{Sp} \times \mathrm{Sp}$ which restricts to the endomorphism of the base $\mathrm{Sp} \times \mathrm{Sp}$ given by

$$(X, Y) \mapsto (F_{\mathcal{O}}(X), F_{\mathcal{O}'}(Y)).$$

More informally: a pairing of $\mathcal{O}$ with $\mathcal{O}'$ is a rule which associates to each "bilinear form" $b : X \wedge Y \to S$ a new "bilinear form" $F_{\mathcal{O}}(X) \wedge F_{\mathcal{O}'}(Y) \to S$. We $\mathcal{P}$ denote the ∞-category whose objects are triples $(\mathcal{O}, \mathcal{O}', \phi)$, where $\mathcal{O}, \mathcal{O}' \in \mathrm{SSeq}$ and ϕ is a pairing of $\mathcal{O}$ with $\mathcal{O}'$. This is a monoidal ∞-category which acts compatibly on $\mathcal{E}$ and $\mathrm{Sp} \times \mathrm{Sp}$.

Remark 8. Let $\mathcal{O}$ and $\mathcal{O}'$ be symmetric sequences. To specify a pairing of $\mathcal{O}$ with $\mathcal{O}'$, we must associate to every map of spectra $b : X \to \mathbf{D}(Y)$ another map of spectra $F_{\mathcal{O}}(X) \to \mathbf{D}(F_{\mathcal{O}'}(Y))$. To supply such a map functorially in X and Y, it suffices to specify it in the universal case where $X = \mathbf{D}(Y)$ and b is the identity map. In this case, we are giving a map

$$(F_{\mathcal{O}} \circ \mathbf{D})(Y) \to (\mathbf{D} \circ F_{\mathcal{O}'})(Y)$$

which depends functorially on Y, or equivalently (by duality) a natural transformation of functors

$$F_{\mathcal{O}'} \to \mathbf{D} \circ F_{\mathcal{O}} \circ \mathbf{D}.$$

The calculation of Proposition 4 shows that this is the same data as a map of symmetric sequences $\mathcal{O}' \to \mathbf{D}(\mathcal{O})$. Or, by symmetry, the same data as a map of symmetric sequences $\mathcal{O} \to \mathbf{D}(\mathcal{O}')$.

The forgetful functor $\mathcal{P} \to \mathrm{SSeq} \times \mathrm{SSeq}$ is an example of a *pairing of monoidal* ∞*-categories*. Applying some general nonsense, one can deduce the following:

- The construction $\mathcal{O} \mapsto \mathbf{D}(\mathcal{O})$ determines a lax monoidal functor $\mathrm{SSeq}^{\mathrm{op}} \to \mathrm{SSeq}$ (concretely, this lax monoidal structure induces the comparison maps $\mathbf{D}(\mathcal{O}) \circ \mathbf{D}(\mathcal{O}') \to \mathbf{D}(\mathcal{O} \circ \mathcal{O}')$ of Remark 3. In particular, it carries (augmented) coalgebra objects of SSeq to (augmented) algebra objects of SSeq.
- The induced map $\mathrm{Alg}(\mathcal{P}) \to \mathrm{Alg}(\mathrm{SSeq}) \times \mathrm{Alg}(\mathrm{SSeq})$ is a right fibration.

- We can identify objects of $\mathrm{Alg}(\mathcal{P})$ with triples $(\mathcal{O}, \mathcal{O}', f)$ where $\mathcal{O}$ and $\mathcal{O}'$ are *augmented* algebra objects of $\mathrm{Alg}(\mathrm{SSeq})$ (that is, augmented operads) and f is a map of augmented operads $\mathcal{O}' \to \mathbf{D}(\mathrm{Bar}(\mathcal{O}))$ (or, by symmetry, a map of augmented operads $\mathcal{O} \to \mathbf{D}(\mathrm{Bar}(\mathcal{O}'))$.

Notation 9. Let $\mathcal{O}$ be an augmented operad. We let $\mathrm{KD}(\mathcal{O}) = \mathbf{D}(\mathrm{Bar}(\mathcal{O}))$. We will refer to $\mathrm{KD}(\mathcal{O})$ as the *Koszul dual* of $\mathcal{O}$.

The construction $\mathcal{O} \mapsto \mathrm{KD}(\mathcal{O})$ induces a functor

$$\mathrm{KD} : \mathrm{Alg}^{\mathrm{nu}}(\mathrm{SSeq})^{\mathrm{op}} \to \mathrm{Alg}^{\mathrm{nu}}(\mathrm{SSeq}),$$

which we will refer to as *Koszul duality for operads*. By construction, it is self-adjoint: that is, we have canonical homotopy equivalences

$$\mathrm{Map}_{\mathrm{Alg}^{\mathrm{nu}}(\mathrm{SSeq})^{\mathrm{op}}}(\mathcal{O}, \mathrm{KD}(\mathcal{O}')) \simeq \mathrm{Map}_{\mathrm{Alg}^{\mathrm{nu}}(\mathrm{SSeq})^{\mathrm{op}}}(\mathcal{O}', \mathrm{KD}(\mathcal{O})).$$

We now explain a less symmetric way to think about Koszul duality. In what follows, let us regard the ∞-category Sp of spectra as equipped with a *left* action of the ∞-category SSeq, via the construction

$$\mathrm{SSeq} \times \mathrm{Sp} \to \mathrm{Sp}$$

$$(\mathcal{O}, X) \mapsto F_{\mathcal{O}}(X) = \varinjlim_{T \in \mathrm{Set}^{\simeq}_{\mathrm{fin}}} \mathcal{O}(T) \wedge X^{\wedge T}.$$

Warning 10. The construction $(\mathcal{O}, X) \mapsto F_{\mathcal{O}}(X)$ determines a left action of SSeq on Sp with respect to the convention where the composition product on reduced symmetric sequences is given by the formula

$$(\mathcal{O} \circ \mathcal{O}')(T) = \bigoplus_{E \in \mathrm{Equiv}(T)} (\mathcal{O}(T/E) \wedge \bigwedge_{T' \in T/E} \mathcal{O}'(T')).$$

Beware that this is actually the *opposite* of the multiplication induced by the identification $\mathrm{SSeq} \simeq \mathrm{LFun}^{\otimes}(\mathrm{SSeq}, \mathrm{SSeq})$ of the previous lecture.

Let $\mathcal{O}$ be an operad, and let $\mathcal{A} = \mathrm{LMod}_{\mathcal{O}}(\mathrm{Sp})$ denote the ∞-category of spectra equipped with an $\mathcal{O}$-algebra structure. There is a forgetful functor $G : \mathcal{A} \to \mathrm{Sp}$ with a left adjoint $F : \mathrm{Sp} \to \mathcal{A}$ (given by functor $F_{\mathcal{O}}$). An augmentation $\epsilon : \mathcal{O} \to \mathcal{O}_{\mathrm{triv}}$ induces another functor $G' : \mathrm{Sp} \to \mathcal{A}$, which admits a left adjoint $F' : \mathcal{A} \to \mathrm{Sp}$. Composing F' with Spanier-Whitehead duality, we obtain a functor

$$\mathcal{A}^{\mathrm{op}} \xrightarrow{F'} \mathrm{Sp}^{\mathrm{op}} \xrightarrow{\mathbf{D}} \mathrm{Sp}.$$

Proposition 11. *Let $\mathcal{O}$ be an augmented operad, and let $\mathcal{A} = \mathrm{LMod}_{\mathcal{O}}(\mathrm{Sp})$ be as above. Then the Koszul dual $\mathrm{KD}(\mathcal{O})$ can be identified with the endomorphism algebra of the object $(\mathbf{D} \circ F') \in \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathrm{Sp})$ (which we regard as equipped with a left action of SSeq).*

Proof. For every symmetric sequence $\mathcal{O}'$, let $F_{\mathcal{O}'}$ denote the associated endofunctor of Sp. By definition, an endomorphism object $\mathrm{End}(\mathbf{D}\circ F') \in \mathrm{SSeq}$, if it exists, is a symmetric sequence with the universal property

$$\begin{aligned}
\mathrm{Map}_{\mathrm{SSeq}}(\mathcal{O}', \mathrm{End}(\mathbf{D}\circ F')) &= \mathrm{Map}_{\mathrm{Fun}(\mathcal{A}^{\mathrm{op}},\mathrm{Sp})}(F_{\mathcal{O}'}\circ \mathbf{D}\circ F', \mathbf{D}\circ F') \\
&\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp}^{\mathrm{op}},\mathrm{Sp})}(F_{\mathcal{O}'}\circ \mathbf{D}, \mathbf{D}\circ F'\circ G') \\
&\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp},\mathrm{Sp})}(F_{\mathcal{O}'}, \mathbf{D}\circ F'\circ G'\circ \mathbf{D}) \\
&\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp},\mathrm{Sp})}(F_{\mathcal{O}'}, \mathbf{D}\circ \mathrm{Bar}(G\circ F)\circ \mathbf{D}) \\
&\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp},\mathrm{Sp})}(F_{\mathcal{O}'}, \mathbf{D}\circ F_{\mathrm{Bar}(\mathcal{O})}\circ \mathbf{D}) \\
&\simeq \mathrm{Map}_{\mathrm{SSeq}}(\mathcal{O}', \mathbf{D}(\mathrm{Bar}(\mathcal{O}))).
\end{aligned}$$

Clearly, this universal property is enjoyed by the Koszul dual $\mathrm{KD}(\mathcal{O}) = \mathbf{D}(\mathrm{Bar}(\mathcal{O}))$. (Technically this argument only establishes an equivalence $\mathrm{End}(\mathcal{D}\circ F') \simeq \mathrm{KD}(\mathcal{O})$ as objects of the ∞-category SSeq, but a more careful analysis gives an identification as *algebra* objects of SSeq.) □

Let us now return to the case of interest. Let $\mathcal{S}_*^{v_n}$ denote the ∞-category of v_n-periodic spaces, so that we can identify $\mathcal{S}_*^{v_n}$ with the ∞-category of left modules over the coanalytic monad $\Phi\circ\Theta : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$. In the previous lecture, we described an augmentation on the monad $\Phi\circ\Theta$ for which the associated map

$$\mathcal{S}_*^{v_n} \simeq \mathrm{LMod}_{\Phi\circ\Theta}(\mathrm{Sp}_{T(n)}) \to \mathrm{Sp}_{T(n)}$$

is given by $\Sigma^\infty_{T(n)}$. Composing with the $T(n)$-local Spanier-Whitehead duality functor $\mathbf{D}_{T(n)}$, we obtain the map

$$\mathbf{D}_{T(n)}\circ \Sigma^\infty_{T(n)} : (\mathcal{S}_*^{v_n})^{\mathrm{op}} \to \mathrm{Sp}_{T(n)}\,.$$

$$X \mapsto \underline{\mathrm{Map}}(\Sigma^\infty X, L_{T(n)}S).$$

We can identify $\Phi\circ\Theta$ with an augmented operad in the ∞-category $\mathrm{Sp}_{T(n)}$ of $T(n)$-local spectra. This operad has a $T(n)$-local Koszul dual, given by

$$\mathrm{KD}_{T(n)}(\Phi\circ\Theta) = \mathrm{Bar}(\Phi\circ\Theta)^{\mathbf{D}}.$$

Using the $T(n)$-local analogue of Proposition 11, we can identify $\mathrm{KD}_{T(n)}(\Phi\circ\Theta)$ with the endomorphism algebra of the functor $\mathbf{D}_{T(n)}\circ\Sigma^\infty_{T(n)}$.

Note that for any object $X \in \mathcal{S}_*^{v_n}$, the $T(n)$-local Spanier-Whitehead dual

$$(\mathbf{D}_{T(n)}\circ\Sigma^\infty_{T(n)})(X) = \underline{\mathrm{Map}}(\Sigma^\infty X, L_{T(n)}S)$$

can be regarded as a nonunital E_∞-ring spectrum (with the E_∞ structure induced from the E_∞-structure on $L_{T(n)}S$). In other words, the nonunital commutative operad $\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}}$ (in its $T(n)$-local incarnation) acts on the functor $\mathbf{D}_{T(n)}\circ\Sigma^\infty_{T(n)}$. This action is then encoded by a canonical map

$$\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}} \to \mathrm{End}(\mathbf{D}_{T(n)}\circ\Sigma^\infty_{T(n)}) = \mathrm{KD}_{T(n)}(\Phi\circ\Theta),$$

or equivalently by the adjoint map $\Phi\circ\Theta \to \mathrm{KD}_{T(n)}(\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}})$. We can now precisely formulate the end of the story:

Theorem 12. *The maps*

$$\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}} \to \mathrm{KD}_{T(n)}(\Phi \circ \Theta)$$
$$\Phi \circ \Theta \to \mathrm{KD}_{T(n)}(\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}})$$

are homotopy equivalences of operads in $T(n)$-local spectra. In particular, $\Phi \circ \Theta$ is the Lie operad in $T(n)$-local spectra.

We postpone the proof to the next lecture.