LECTURE X: KOSZUL DUALITY

Fix a prime number p and an integer $n > 0$, and let $\mathcal{S}_*^{v_n}$ denote the ∞-category of v_n-periodic spaces. Last semester, we proved the following theorem of Heuts:

Theorem 1. *The Bousfield-Kuhn functor $\Phi : \mathcal{S}_*^{v_n} \to \mathrm{Sp}_{T(n)}$ induces an equivalence of $\mathcal{S}_*^{v_n}$ with the ∞-category of algebras over a monad $\Phi \circ \Theta : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$. Moreover, $\Phi \circ \Theta$ is a coanalytic functor: in other words, it corresponds to an operad $\{\mathcal{O}(d)\}_{d \geq 0}$ in the ∞-category of $T(n)$-local spectra.*

Our next goal is to identify the operad $\{\mathcal{O}(d)\}$ with the Lie operad. For this, we first need to *define* the Lie operad in the setting of spectra. Here, it will be convenient to use the language of Koszul duality: the (appropriately shifted) Lie operad can be described as the *Koszul dual* to the (nonunital) commutative operad.

Let us begin by reviewing the theory of Koszul duality in a simpler setting.

Notation 2. Let k be a field. We let Mod_k denote the ∞-category of k-module spectra and Alg_k the ∞-category of associative algebra objects of Mod_k. More concretely, the ∞-category Mod_k can be obtained from the ordinary category $\mathrm{Mod}_k^{\mathrm{dg}}$ of chain complexes of vector spaces over k by formally inverting quasi-isomorphisms, and the ∞-category Alg_k can be obtained from the ordinary category $\mathrm{Alg}_k^{\mathrm{dg}}$ of differential graded algebras over k by formally inverting quasi-isomorphisms.

Given an object $A \in \mathrm{Alg}_k$, we will refer to a morphism $\epsilon : A \to k$ in Alg_k as an *augmentation* on k. Given an augmentation ϵ, we can regard k as a left A-module object of Mod_k. We can then consider the *endomorphism algebra* $\underline{\mathrm{Map}}_A(k,k)$, which is characterized by the following universal property: for every object $M \in \mathrm{Mod}_k$, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Mod}_k}(M, \underline{\mathrm{Map}}_A(k,k)) = \mathrm{Map}_{\mathrm{LMod}_A}(M, k)$$

(where we regard $M \simeq k \otimes_k M$ and k as left A-modules via the augmentation ϵ). It follows from general nonsense that $\underline{\mathrm{Map}}_A(k,k)$ can be regarded as an object of Alg_k; we will denote this object by $\mathbf{D}(A)$ and refer to it as the *Koszul dual* of A. By construction, the Koszul dual $\mathbf{D}(A)$ is universal among those objects $B \in \mathrm{Alg}_k$ for which we can equip k with the structure of A-B bimodule.

Neglecting algebra structures, we have a canonical equivalence

$$\mathbf{D}(A) = \underline{\mathrm{Map}}_A(k,k) = \underline{\mathrm{Map}}_k(k \otimes_A k, k).$$

We can therefore understand Koszul duality as a two-step procedure: given an augmented algebra A, we first form the *bar construction* $\mathrm{Bar}(A) = k \otimes_A k$. The Koszul dual $\mathbf{D}(A)$ can then be identified with the k-linear dual of $\mathrm{Bar}(A)$. In this lecture, we will focus only on the first step; we'll consider the second (in the setting of operads) in the next lecture.

Since $\mathbf{D}(A)$ has the structure of an algebra object of Mod_k, it is natural to expect that the predual $\mathrm{Bar}(A)$ has the structure of a *coalgebra object* of Mod_k: that is, an algebra object of the opposite ∞-category $\mathrm{Mod}_k^{\mathrm{op}}$. In fact, there is an obvious comultiplication $\Delta : \mathrm{Bar}(A) \to \mathrm{Bar}(A) \otimes_k \mathrm{Bar}(A)$, given by the map

$$\mathrm{Bar}(A) = k \times_A k \simeq k \otimes_A A \otimes_k k \xrightarrow{\mathrm{id} \otimes \epsilon \otimes \mathrm{id}} k \otimes_A k \otimes_A k \simeq \mathrm{Bar}(A) \otimes_k \mathrm{Bar}(A).$$

This comultiplication turns out to be coherently associative, and to induce a coalgebra structure on $\mathrm{Bar}(A)$ which is predual to the algebra structure on $\mathbf{D}(A)$. To understand why, it will be convenient to describe this comultiplication in a different way. We also consider a more general situation.

Notation 3. Let $\mathcal{C}$ be a monoidal ∞-category, equipped with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object $\mathbf{1} \in \mathcal{C}$. In what follows, we let $\mathcal{M}$ denote an ∞-category equipped with a right action of $\mathcal{C}$, via a tensor product

$$\otimes : \mathcal{M} \times \mathcal{C} \to \mathcal{C}.$$

Construction 4 (Endomorphism Algebras)**.** Fix an object $M \in \mathcal{M}$. Given an object $A \in \mathcal{C}$, we will say that a morphism $f : M \otimes A \to M$ in $\mathcal{M}$ *exhibits A as an endomorphism object of M* if, for every object $X \in \mathcal{C}$, the composite map

$$\mathrm{Hom}_{\mathcal{C}}(X, A) \to \mathrm{Hom}_{\mathcal{M}}(M \otimes X, M \otimes A) \xrightarrow{f \circ} \mathrm{Hom}_{\mathcal{M}}(M \otimes X, M)$$

is a homotopy equivalence. In this case, the object A (and the morphism f) are uniquely determined up to equivalence; we denote A by $\mathrm{End}(M)$. It follows from general nonsense that $\mathrm{End}(M)$ has the structure of an associative algebra object of $\mathcal{C}$, which can be characterized by the following universal property: if B is any associative algebra object of $\mathcal{C}$, then there is a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Alg}(\mathcal{C})}(B, \mathrm{End}(M)) \simeq \mathrm{RMod}_B(\mathcal{M}) \times_{\mathcal{M}} \{M\}.$$

In other words, giving a map of associative algebras $B \to \mathrm{End}(M)$ is equivalent to giving a right action of B on M.

A right action of $\mathcal{C}$ on $\mathcal{M}$ induces a right action of the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$ on $\mathcal{M}^{\mathrm{op}}$. We therefore have the following dual notion:

Construction 5 (Coendomorphism Coalgebras)**.** Fix an object $M \in \mathcal{M}$. Given an object $C \in \mathcal{C}$, we will say that a morphism $g : M \to M \otimes C$ in $\mathcal{M}$ *exhibits C as a coendomorphism object of M* if, for every object $X \in \mathcal{C}$, the composite map

$$\mathrm{Hom}_{\mathcal{C}}(C, X) \to \mathrm{Hom}_{\mathcal{M}}(M \otimes C, M \otimes X) \xrightarrow{\circ g} \mathrm{Hom}_{\mathcal{M}}(M, M \otimes X)$$

is a homotopy equivalence. In this case, the object C (and the morphism g) are uniquely determined up to equivalence; we will denote C by $\mathrm{coEnd}(M)$.

Let $\mathrm{coAlg}(\mathcal{C})$ denote the category of (associative) coalgebra objects of $\mathcal{C}$, defined by $\mathrm{coAlg}(\mathcal{C}) = \mathrm{Alg}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. By the categorical dual of the general nonsense described in Construction 4, we can regard the coendomorphism object $\mathrm{coEnd}(M)$ (if it exists) as a coalgebra object of $\mathcal{C}$, which is characterized by the following universal property: for any coalgebra object B of $\mathcal{C}$, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{coAlg}(\mathcal{C})}(\mathrm{coEnd}(M), B) \simeq \mathrm{RMod}_B(\mathcal{M}^{\mathrm{op}}) \times_{\mathcal{M}^{\mathrm{op}}} \{M\}.$$

In other words, giving a map of coalgebras $\mathrm{coEnd}(M) \to B$ is equivalent to giving a right coaction of B on M.

Example 6. Fix a monoidal ∞-category $\mathcal{C}$ as above. Let A be an associative algebra object of $\mathcal{C}$ and let $\mathcal{M} = \mathrm{LMod}_A(\mathcal{C})$ be the ∞-category of left A-module objects of $\mathcal{C}$. Then the right action of $\mathcal{C}$ on itself induces a right action of $\mathcal{C}$ on $\mathcal{M}$.

Suppose we are given an augmentation $\epsilon : A \to \mathbf{1}$, which endows the unit object $\mathbf{1}$ with the structure of a left A-module. By definition, a coendomorphism object of $\mathbf{1} \in \mathcal{M}$, if it exists, is characterized by the existence of a canonical homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}}(\mathrm{coEnd}(\mathbf{1}), X) \simeq \mathrm{Map}_{\mathrm{LMod}_A(\mathcal{C})}(\mathbf{1}, \mathbf{1} \otimes X) = \mathrm{Map}_{\mathrm{LMod}_A(\mathcal{C})}(\mathbf{1}, X).$$

Proposition 7. *Let $\mathcal{C}$ be a monoidal ∞-category and let $\epsilon : A \to \mathbf{1}$ be an augmented algebra object of $\mathcal{C}$. Suppose that $\mathcal{C}$ admits geometric realizations of simplicial objects. Then the $\mathbf{1} \in \mathrm{LMod}_A(\mathcal{C})$ admits a coendomorphism object* $\mathrm{coEnd}(\mathbf{1})$.

Remark 8. In the statement of Proposition 7, it is not necessary to assume that the tensor product on $\mathcal{C}$ is compatible with geometric realizations of simplicial objects.

Proof of Proposition 7. Note that composition with ϵ determines a forgetful functor $\mathcal{C} \simeq \mathrm{LMod}_{\mathbf{1}}(\mathcal{C}) \to \mathrm{LMod}_A(\mathcal{C})$. It will suffice to show that this functor admits a left adjoint λ: the coendomorphism object $\mathrm{coEnd}(\mathbf{1})$ will then be given by as $\lambda(\mathbf{1})$. Note that every left A-module M can be obtained as the geometric realization of a simplicial left A-module which is free in every degree, given by the *bar resolution* $[n] \mapsto A^{\otimes(n+1)} \otimes M$. It will therefore suffice to show that the functor λ is well-defined on every free A-module $A \otimes N$, which is clear (we have $\lambda(A \otimes N) \simeq N$). □

Remark 9. The proof of Proposition 7 shows that the functor λ carries a left A-module M to the geometric realization of the two-sided bar construction $[n] \mapsto A^{\otimes n} \otimes M$. In particular, the coendomorphism object $\mathrm{coEnd}(\mathbf{1})$ is given by the bar construction $\mathrm{Bar}(A)$ described above. (Beware, however, that viewing $\mathrm{Bar}(A)$ as a relative tensor product $\mathbf{1} \otimes_A \mathbf{1}$ can be a bit dangerous when the tensor product on $\mathcal{C}$ is not compatible with geometric realizations).

We now specialize the bar construction to the setting of operads.

Notation 10. Let $\mathrm{Set}_{\mathrm{fin}}^{\simeq}$ denote the category whose objects are finite sets and whose morphisms are bijections. A *symmetric sequence of spectra* is a functor $\mathcal{O} : \mathrm{Set}_{\mathrm{fin}}^{\simeq} \to \mathrm{Sp}$. Given such a functor, we let $\mathcal{O}(m)$ denote the value of $\mathcal{O}$ on the standard set $\{1, 2, \ldots, m\}$ of cardinality m, so that $\mathcal{O}(m)$ carries an action of the symmetric group $\Sigma_{,}$. We can identify a symmetric sequence with the collection of objects $\{\mathcal{O}(m)\}_{m \geq 0}$, each equipped with the associated Σ_m; however, this will be slightly inconvenient in what follows.

Let $\mathrm{SSeq} = \mathrm{Fun}(\mathrm{Set}_{\mathrm{fin}}^{\simeq}, \mathrm{Sp})$ denote the ∞-category of symmetric sequences. The ∞-category SSeq admits a symmetric monoidal structure given by Day convolution. We will denote the underlying tensor product by $\otimes$, given concretely by the formula

$$(\mathcal{O} \otimes \mathcal{O}')(T) = \bigoplus_{T = T_0 \amalg T_1} \mathcal{O}(T_0) \wedge \mathcal{O}'(T_1).$$

Note that the category $\mathrm{Set}_{\mathrm{fin}}^{\simeq}$ is characterized by a universal property: as a symmetric monoidal category (under disjoint union), it is freely generated by a single object (given by a one-element set). By formal nonsense, this yields a universal property of the ∞-category SSeq: in the setting of presentable symmetric monoidal stable ∞-categories (where the tensor product is required to distribute over colimits), it is freely generated by a single object: namely, the symmetric sequence $\mathcal{O}_{\mathrm{triv}}$ given by

$$\mathcal{O}_{\mathrm{triv}}(T) = \begin{cases} S & \text{if } |T| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

More precisely, if $\mathcal{E}$ is any other symmetric monoidal presentable stable ∞-category for which the tensor product on $\mathcal{E}$ distributes over colimits, then evaluation on $\mathcal{O}_{\mathrm{triv}}$ induces an equivalence of ∞-categories

$$\mathrm{LFun}^{\otimes}(\mathrm{SSeq}, \mathcal{E}) \to \mathcal{E},$$

where $\mathrm{LFun}^{\otimes}(\mathrm{SSeq}, \mathcal{E})$ is the ∞-category of colimit-preserving symmetric monoidal functors from SSeq to $\mathcal{E}$.

Applying this observation when $\mathcal{E} = \mathrm{SSeq}$, we obtain an equivalence of ∞-categories

$$\mathrm{LFun}^{\otimes}(\mathrm{SSeq}, \mathrm{SSeq}) \to \mathrm{SSeq}$$

(given by evaluation on $\mathcal{O}_{\mathrm{triv}}$). Note that the left hand side has a (non-symmetric!) monoidal structure given by composition of functors, so that SSeq inherits a monoidal structure. We will denote the underlying tensor product by $\circ : \mathrm{SSeq} \times \mathrm{SSeq} \to \mathrm{SSeq}$ and refer to it as the *composition product of symmetric sequences.*

We will say that a symmetric sequence $\mathcal{O}$ is *reduced* if $\mathcal{O}(0) \simeq 0$. The composition product of *reduced* symmetric sequences is given concretely by the formula

$$(\mathcal{O} \circ \mathcal{O}')(T) = \bigoplus_{E \in \mathrm{Equiv}(T)} \mathcal{O}(T/E) \wedge \bigwedge_{T' \in T/E} \mathcal{O}'(T')$$

The unit object for this composition product is the object $\mathcal{O}_{\mathrm{triv}}$.

An *operad in spectra* is an associative algebra for the composition product on SSeq, and a *cooperad in spectra* is an associative coalgebra object of SSeq. Applying the general formalism of Koszul duality, we see that for every operad in spectra $\mathcal{O}$ equipped with an augmentation $\epsilon : \mathcal{O} \to \mathcal{O}_{\mathrm{triv}}$, we can form the bar construction $\mathrm{Bar}(\mathcal{O}) = \mathcal{O}_{\mathrm{triv}} \circ_{\mathcal{O}} \mathcal{O}_{\mathrm{triv}}$ and regard it as a cooperad in spectra.

Definition 11. Let $\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}}$ denote the *nonunital* commutative operad in spectra, given by

$$\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}}(T) = \begin{cases} S & \text{if } T \neq \varnothing \\ 0 & \text{if } T = \varnothing. \end{cases}$$

Then $\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}}$ admits an essentially unique augmentation $\epsilon : \mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}} \to \mathcal{O}_{\mathrm{triv}}$. We define the *Lie cooperad* to be the bar construction $\mathrm{Bar}(\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}})$.

Remark 12 (Partition Complexes)**.** For each integer $k \geq 0$, let B_k denote the kth stage in the bar construction for $\mathcal{O}_{\mathrm{triv}} \circ_{\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}}} \mathcal{O}_{\mathrm{triv}}$. This can be identified with the k-fold composition product of $\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}}$ with itself, but it will be more convenient to view it as a quotient of the $(k+2)$-fold composition product of $\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}}$ with itself.

For every finite set T, let $\mathrm{Equiv}(T)$ denote the poset of equivalence relations on T, let $\mathrm{Equiv}^+(T)$ denote the subset of $\mathrm{Equiv}(T)$ obtained by removing the smallest element, and let $\mathrm{Equiv}^-(T)$ the subset obtained by removing the largest element, and set $\mathrm{Equiv}^{\pm}(T) = \mathrm{Equiv}^+(T) \cap \mathrm{Equiv}^-(T)$. For simplicity, let us consider only the case where T has at least two elements. Unwinding the definitions above, we see that B_k is the symmetric sequence given by

$$\begin{aligned} B_k(T) &= \bigoplus_{(E_0 \subseteq E_1 \subseteq \cdots \subseteq E_k) \in \mathrm{Equiv}(T)} \begin{cases} 0 & \text{if } E_0 \in \mathrm{Equiv}^+(T) \text{ or } E_k \in \mathrm{Equiv}^-(T) \ \cup N^+(\mathrm{Equiv}(T)) \\ S & \text{otherwise} \end{cases} \\ &= \Sigma^\infty (N(\mathrm{Equiv}(T))_k / (N(\mathrm{Equiv}^-(T))_k \cup N(\mathrm{Equiv}^+(T))_k). \end{aligned}$$

Passing to the geometric realization and using the fact that the simplicial sets $N(\mathrm{Equiv}(T))$, $N(\mathrm{Equiv}^+(T))$, and $N(\mathrm{Equiv}^-(T))$ are contractible (for $|T| > 1$) we obtain

$$\begin{aligned} |B_\bullet(T)| &= \Sigma^\infty (N(\mathrm{Equiv}(T))/((\mathrm{Equiv}^+(T)) \cup N(\mathrm{Equiv}^-(T)))) \\ &= \Sigma^{\infty+1}((N(\mathrm{Equiv}^+(T)) \cup N(\mathrm{Equiv}^-(T)))) \\ &= \Sigma^{\infty+2} N(\mathrm{Equiv}^{\pm}(T)) \end{aligned}$$

where we adopt the convention that $\Sigma^{\infty+2}(\varnothing) = S^1$ in the case $|T| = 2$. It follows that the Lie cooperad $(\mathcal{O}_{\mathrm{triv}} \circ_{\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}}} \mathcal{O}_{\mathrm{triv}})(k)$ is given, in degrees $k \geq 2$, by the (shifted) suspension spectrum of the partition complex $N(\mathrm{Equiv}^{\pm}(\{1,\ldots,k\})$.

If we specialize to the $T(n)$-local setting, the theory of symmetric sequences can be simplified: we can identify symmetric sequences of $T(n)$-local spectra with coanalytic functors from the ∞-category $\mathrm{Sp}_{T(n)}$ to itself, and the composition product on symmetric sequences with the usual composition of (coanalytic) functors. Let us see what Koszul duality has to say in this case.

Construction 13. Let $\mathcal{C}$ be an ∞-category and let T be a monad on $\mathcal{C}$: that is, an algebra object in the monoidal ∞-category $\mathrm{Fun}(\mathcal{C},\mathcal{C})$. We let $\mathcal{D} = \mathrm{LMod}_T(\mathcal{C})$ denote the ∞-category of left T-module objects of $\mathcal{C}$ (that is, the ∞-category of *algebras* for the monad T). Then the monad T factors as a composition $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{C}$, where G is the forgetful functor and F is its left adjoint (given by the construction $T \mapsto TC$, where we regard TC as an object of $\mathrm{LMod}_T(\mathcal{C})$).

Suppose we are given another adjunction

$$\mathcal{D} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{C},$$

together with an identification $G \circ G' \simeq \mathrm{id}_{\mathcal{C}}$ (which induces a compatible identification $F' \circ F \simeq \mathrm{id}_{\mathcal{C}}$). In this case, we obtain a map of monads

$$\epsilon : T \simeq G \circ F \to (G \circ G') \circ (F' \circ F) \simeq \mathrm{id}_{\mathcal{C}},$$

which endows T with the structure of an *augmented* algebra object of $\mathrm{Fun}(\mathcal{C},\mathcal{C})$. Then we have a canonical equivalence $\mathrm{LMod}_T(\mathrm{Fun}(\mathcal{C},\mathcal{C})) = \mathrm{Fun}(\mathcal{C}, \mathrm{LMod}_T(\mathcal{C})) = \mathrm{Fun}(\mathcal{C},\mathcal{D})$. Under this equivalence, the left T-module structure on the identity functor $\mathrm{id}_{\mathcal{C}} \simeq G \circ G'$ corresponds to the functor $G' \in \mathrm{Fun}(\mathcal{C},\mathcal{D})$.

If $\mathcal{C}$ admits geometric realizations of simplicial objects, then so does $\mathrm{Fun}(\mathcal{C},\mathcal{C})$, so that the bar construction $\mathrm{Bar}(T)$ is well-defined and can be identified with the coendomorphism object of G'. In other words, $\mathrm{Bar}(T)$ is an endofunctor of $\mathcal{C}$ with the following universal property: for every functor $H : \mathcal{C} \to \mathcal{C}$, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{C})}(\mathrm{Bar}(T), H) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(G', G' \circ H).$$

Using the adjunction between F' and G', we see that $\mathrm{Bar}(T)$ is given by the composition $F' \circ G'$. Moreover, the coalgebra structure on $\mathrm{Bar}(T)$ agrees with the *comonad* structure $F' \circ G'$, arising from the adjunction between F' and G'. (The existence of geometric realizations in $\mathcal{C}$ is not even really needed here.)

To put ourselves in the situation of Construction 13, we need a bit more information about the relationship between stable and unstable v_n-periodic homotopy theory. Fix finite pointed spaces A and B, of types $(n+1)$ and n, respectively.

Assume that A and B are both suspensions, and that their first nonvanishing homology group (with $\mathbf{F}_p$-coefficients) appears in the same degree d. Recall that $\mathcal{S}_*^{v_n}$ can be realized as the full subcategory of $\mathcal{S}_*$ spanned by those spaces which are d-connected, P_A-local, and P_B-acyclic. We let

$$\Sigma^\infty_{T(n)} : \mathcal{S}_*^{v_n} \to \mathrm{Sp}_{T(n)}$$

denote the functor given by the composition

$$\mathcal{S}_*^{v_n} \hookrightarrow \mathcal{S}_* \xrightarrow{\Sigma^\infty} \mathrm{Sp} \xrightarrow{L_{T(n)}} \mathrm{Sp}_{T(n)}.$$

Proposition 14. *The functor $\Sigma^\infty_{T(n)} : \mathcal{S}_*^{v_n} \to \mathrm{Sp}_{T(n)}$ is independent of choices.*

Proof. By virtue of the results of Bousfield that we proved last semester, the only possible dependence is on the integer d. Let us show that replacing A and B by their suspensions does not change the functor $\Sigma^\infty_{T(n)}$. For this, it will suffice to show that if X is a d-connected, P_A-local, and P_B-acyclic, then the canonical map $f : X\langle d+1\rangle \to X$ induces an equivalence $L_{T(n)}(X\langle d+1\rangle) L_{T(n)} X$. Note that the homotopy fibers of f have the form $K(G,d)$, where $G = \pi_{d+1}(X)$. It will therefore suffice to show that the spectrum $L_{T(n)}\Sigma^\infty K(G,d)$ vanishes. Since X is P_B-acyclic, the group G is p-power torsion. It follows that $K(G,d)$ is P_A-acyclic (Proposition 12 of Lecture 4). Since A has type $> n$, we have $L_{T(n)}\Sigma^\infty A \simeq 0$, and therefore $L_{T(n)}\Sigma^\infty K(G,d) \simeq 0$. □

Proposition 15. *The functor $\Sigma^\infty_{T(n)} : \mathcal{S}_*^{v_n} \to \mathrm{Sp}_{T(n)}$ has a right adjoint $\Omega^\infty_{T(n)}$ given by the construction $E \mapsto \mathrm{fib}((\Omega^\infty E)\langle d\rangle \to P_B((\Omega^\infty E)\langle d\rangle))$.*

Proof. If E is $T(n)$-local, then the 0th space $\Omega^\infty E$ is automatically p-local and P_A-local. □

Corollary 16. *The composition*

$$\mathrm{Sp}_{T(n)} \xrightarrow{\Omega^\infty_{T(n)}} \mathcal{S}_*^{v_n} \xrightarrow{\Phi} \mathrm{Sp}_{T(n)}$$

is (canonically) homotopic to the identity.

Proof. For a $T(n)$-local spectrum E, the canonical map $\Omega^\infty_{T(n)} E \to \Omega^\infty E$ is a v_n-periodic homotopy equivalence, and therefore induces an equivalence

$$\Phi\Omega^\infty_{T(n)} E \simeq (\Phi \circ \Omega^\infty)(E) \simeq L_{T(n)} E \simeq E.$$

□

Corollary 17. *Let $\Phi \circ \Theta : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ be our monad. Then there is an augmentation $\epsilon : \Phi \circ \Theta \to \mathrm{id}_{\mathrm{Sp}_{T(n)}}$ for which the bar construction $\overline{(}\Phi \circ \Theta)$ can be identified with $\Sigma^\infty_{T(n)} \circ \Omega^\infty_{T(n)}$, as a coalgebra in the ∞-category $\mathrm{Fun}(\mathrm{Sp}_{T(n)}, \mathrm{Sp}_{T(n)})$.*

Proof. Combine Corollary 16 with Construction 13. □