

## ARONE-DWYER

The main result we will discuss today is an approximation result. The notation will be given later, but it says that, up to  $p$ -completion, for  $l$  odd there is a weak equivalence of spectra

$$\Sigma^\infty(S^{nl} \wedge \mathbb{P}_n^\diamond)_{\bar{h}\Sigma_n} \simeq_p \begin{cases} \Sigma^\infty(S^{nl} \wedge \mathbb{T}_{p,k}^\diamond)_{\bar{h}\text{Aff}_{p,k}} & \text{if } n = p^k, \\ * & \text{otherwise.} \end{cases}$$

Here it might be more accurate to write  $S^{l\rho}$  for  $S^{nl}$ , with  $\rho$  the standard  $n$ -dimensional representation of  $\Sigma_n$ , as it makes the action clear.

This equivalence connects to three other results. The first two were explained by Mike:

- on the left hand side, taking  $l = 1$  we get

$$\Sigma^\infty(S^n \wedge \mathbb{P}_n^\diamond)_{\bar{h}\Sigma_n} \simeq \text{SP}^n(\mathbb{S})/\text{SP}^{n-1}(\mathbb{S}),$$

where  $\mathbb{S} = \text{SP}^1(\mathbb{S}) \rightarrow \text{SP}^2(\mathbb{S}) \rightarrow \dots$  is the symmetric powers filtration of the Eilenberg-Mac Lane spectrum  $H\mathbb{Z}$ : the  $k$ th level of  $\text{SP}^n(\mathbb{S})$  is  $\text{SP}^n(S^k) := (S^k)^n/\Sigma_n$  (strict quotient).

- on the right hand side, using the fact the top homology of the Tits building is the Steinberg module (which is projective), we may identify the right hand side as one of the spectra in Mitchell-Priddy [MP83]:

$$\Sigma^\infty(S^{nl} \wedge \mathbb{T}_{p,k}^\diamond)_{\bar{h}\text{Aff}_{p,k}} \simeq e_{p,k}^{\text{St}} \cdot \text{Thom}(B\mathbb{F}_p^n; l\rho).$$

Today we will make the third and final connection: let  $D_n(S^l)$  denote the  $n$ th Goodwillie derivative of the odd sphere  $S^l$ :

$$D_n(S^l) \simeq_p \begin{cases} S^{-2(k-1)-1} \wedge \Sigma^\infty(S^{nl} \wedge \mathbb{P}_n^\diamond)_{\bar{h}\Sigma_n} & \text{if } n = p^k \\ * & \text{otherwise} \end{cases}$$

This connection shall go through  $\mathbb{T}_{p,k}^\diamond$  and follow from the above approximation result and a Spanier-Whitehead dual version of it.

**Convention 0.1.** A map that is  $G$ -equivariant and a weak equivalence will be called a  *$G$ -equivariant weak equivalence*. A map that is  $G$ -equivariant and a weak equivalence on all fixed point sets will be called a *genuine  $G$ -equivariant weak equivalence*.

**Convention 0.2.** All our spaces are really simplicial sets.

**Convention 0.3.** All our primes  $p$  are odd.

The main reference is of course [AD01], but the proof also depends on some homology computations in [Dwy98, Dwy97]. Our exposition of the Steinberg module is based on [Str], but see also [Kuh84].

## 1. APPROXIMATION RESULTS

From now on  $n > 1$ . The notation  $(-)^{\diamond}$  denotes the unreduced suspension, based at one of the two cone points. The *partition complex*  $\mathcal{P}_n$  is the geometric realization of the nerve of the poset of partitions of  $\{1, \dots, n\}$  which are neither indiscrete nor discrete, ordered by refinement. It has an action of  $\Sigma_n$ .

**Example 1.1.** The poset  $\mathcal{P}_2$  is empty, and  $\mathcal{P}_3$  is discrete with 3 elements: (1)(23), (2)(13) and (3)(12).

The *Tits building*  $\mathbb{T}_{p,k}$  is the geometric realization of the nerve of the poset of proper non-trivial subspaces of  $\mathbb{F}_p^k$ , ordered by inclusions. It has an action of the affine group  $\text{Aff}_{k,p} := \mathbb{F}_p^p \rtimes \text{GL}_k(\mathbb{F}_p)$ , which factors over  $\text{GL}_k(\mathbb{F}_p)$ .

**Example 1.2.** The poset  $\mathbb{T}_{p,1}$  is empty, and the poset of  $\mathbb{T}_{p,2}$  is discrete with points corresponding to the  $p+1$  lines in  $\mathbb{F}_p^2$ .

Our goal is to prove the following two results:

**Theorem 1.3.** *We have that*

$$\Sigma^\infty(S^{nl} \wedge \mathcal{P}_n^\diamond)_{\bar{h}\Sigma_n} \simeq_p \begin{cases} \Sigma^\infty(S^{nl} \wedge \mathbb{T}_{p,k}^\diamond)_{\bar{h}\text{Aff}_{p,k}} & \text{if } n = p^k \\ * & \text{otherwise.} \end{cases}$$

**Theorem 1.4.** *We have that*

$$(S^{nl} \wedge \mathbb{D}\mathcal{P}_n^\diamond)_{\bar{h}\Sigma_n} \simeq_p \begin{cases} (S^{nl} \wedge \mathbb{D}\mathbb{T}_{p,k}^\diamond)_{\bar{h}\text{Aff}_{p,k}} & \text{if } n = p^k \\ * & \text{otherwise.} \end{cases}$$

**1.1. Suspension version.** The proof of Theorem 1.3 involves a comparison of three collections of subgroups of  $\Sigma_n$  (these are posets of subgroups ordered by inclusion and closed under conjugation).

- $\mathcal{P}$  is the collection of *partition subgroups* of  $\Sigma_n$  that are not  $\{e\}$  or  $\Sigma_n$ , that is,  $H \in \mathcal{P}$  if there is a partition  $\lambda$  of  $\{1, \dots, n\}$  which is not indiscrete or discrete and such that  $H$  is the maximal subgroup of permutations such that  $\sigma(i) \sim_\lambda i$  for all  $i$ . Thus  $|\mathcal{P}| \cong \mathcal{P}_n$ . Note these are all non-transitive.
- $\mathcal{E}$  is the collection of the *elementary abelian  $p$ -subgroups* not equal to  $\{e\}$  (that is, those abelian subgroups where all elements have exponent  $p$ ). These are supposed to be related to  $\mathbb{F}_p$ -vector spaces eventually.
- $\mathcal{F}^\circ$  is the collection of *non-transitive subgroups* not equal to  $\{e\}$ . These will be used to connect  $\mathcal{P}$  and  $\mathcal{E}$ .

There is an obvious commutative diagram (there is nothing to verify for commutativity)

$$\begin{array}{ccc} |\mathcal{E} \cap \mathcal{F}^\circ| & \longrightarrow & |\mathcal{E}| \\ \downarrow & & \downarrow \\ |\mathcal{F}^\circ| & \longrightarrow & * \\ \uparrow & & \uparrow \\ |\mathcal{P}| & \longrightarrow & *. \end{array}$$

We will soon prove that all vertical maps are  $H_*^{\Sigma_n}(-; \mathbb{F}_p^\pm)$ -isomorphisms, where  $H_*^{\Sigma_n}(X; \mathbb{F}_p^\pm)$  is the homology of  $(E\Sigma_n \times X)/\Sigma_n$  with local coefficients in the sign representation  $\mathbb{F}_p^\pm$ . Then the induced vertical maps on homotopy cofibers are also  $\tilde{H}_*^{\Sigma_n}(-; \mathbb{F}_p^\pm)$ -isomorphisms.

1.1.1. *The case  $n \neq p^k$ .* As  $|\mathcal{P}| \cong \mathbf{P}_n$ , the homotopy cofiber of the bottom map is the unreduced suspension  $\mathbf{P}_n^\diamond$ . The top map is an isomorphism if  $n \neq p^k$ , so has trivial homotopy cofiber. Thus when  $n \neq p^k$  we get

$$\begin{array}{ccccc} |\mathcal{E} \cap \mathcal{F}^\circ| & \longrightarrow & |\mathcal{E}| & \longrightarrow & * \\ \downarrow \simeq_{p,\pm} & & \downarrow \simeq_{p,\pm} & & \downarrow \simeq_{p,\pm} \\ |\mathcal{F}^\circ| & \longrightarrow & * & \longrightarrow & \Sigma|\mathcal{F}^\circ|_+ \\ \uparrow \simeq_{p,\pm} & & \uparrow \simeq_{p,\pm} & & \uparrow \simeq_{p,\pm} \\ |\mathcal{P}| & \longrightarrow & * & \longrightarrow & \mathbf{P}_n^\diamond \end{array}$$

and conclude that  $(S^{nl} \wedge \mathbf{P}_n^\diamond)_{\bar{h}\Sigma_n}$  has trivial  $\mathbb{F}_p$ -homology when  $l$  is odd.

1.1.2. *The case  $n = p^k$ .* Next we consider the case  $n = p^k$ . Then  $\mathcal{E}$  contains a single transitive group up to conjugacy; this is  $\mathbb{F}_p^k$  acting on itself, which we denote  $\Delta$  and which has stabilizer  $\text{Aff}_{k,p}$ . Let  $|\mathcal{E} \cap \mathcal{F}^\circ \downarrow \Delta|$  and  $|\mathcal{E} \downarrow \Delta|$  denote the geometric realization of the posets of subgroups contained in  $\Delta$ . This is worth considering because  $|\mathcal{E} \cap \mathcal{F}^\circ \downarrow \Delta|$  is just  $\mathbf{T}_{k,p}$ , as the non-transitive non-trivial elementary  $p$ -subgroups of  $\Delta$  are exactly its  $\mathbb{F}_p$ -subspaces, and  $|\mathcal{E} \downarrow \Delta|$  is contractible having a maximal element  $\Delta$ .

The inclusion  $|\mathcal{E} \cap \mathcal{F}^\circ \downarrow \Delta| \rightarrow |\mathcal{E} \cap \mathcal{F}^\circ|$  induces a map  $\Sigma_n \times_{\text{Aff}_{k,p}} |\mathcal{E} \cap \mathcal{F}^\circ|$ , and in fact we have a pushout diagram

$$\begin{array}{ccc} \Sigma_n \times_{\text{Aff}_{k,p}} |\mathcal{E} \cap \mathcal{F}^\circ \downarrow \Delta| & \longrightarrow & \Sigma_n \times_{\text{Aff}_{k,p}} |\mathcal{E} \downarrow \Delta| \\ \downarrow & & \downarrow \\ |\mathcal{E} \cap \mathcal{F}^\circ| & \longrightarrow & |\mathcal{E}|, \end{array}$$

which is a homotopy pushout diagram since the top map is a cofibration. Thus the homotopy cofiber of  $|\mathcal{E} \cap \mathcal{F}^\circ| \rightarrow |\mathcal{E}|$  is the homotopy cofiber of  $\Sigma_n \times_{\text{Aff}_{k,p}} |\mathcal{E} \cap \mathcal{F}^\circ \downarrow \Delta| \rightarrow \Sigma_n \times_{\text{Aff}_{k,p}} |\mathcal{E} \downarrow \Delta|$ , which is  $(\Sigma_n)_+ \wedge_{\text{Aff}_{k,p}} \mathbf{T}_{k,p}^\diamond$ . It is not so hard to check that the induced map is homotopic to the unreduced suspension of the map  $\mathbf{T}_{k,p} \rightarrow \mathbf{P}_n$ . Using a topological version of Shapiro's lemma we get for  $l$  odd a map

$$(S^{nl} \wedge \mathbf{T}_n^\diamond)_{\bar{h}\text{Aff}_{k,p}} \simeq (S^{nl} \wedge (\Sigma_n)_+ \wedge_{\text{Aff}_{k,p}} \mathbf{T}_{k,p}^\diamond)_{\bar{h}\Sigma_n} \rightarrow (S^{nl} \wedge \mathbf{P}_n^\diamond)_{\bar{\Sigma}_n}$$

which by a homotopy orbit spectral sequence argument induces an  $\tilde{H}_*(-; \mathbb{F}_p)$ -isomorphism. The conclusion is that the inclusion  $\mathbf{T}_{k,p} \hookrightarrow \mathbf{P}_n$  induces a  $p$ -complete equivalence

$$(S^{nl} \wedge \mathbf{P}_n^\diamond)_{\bar{\Sigma}_n} \simeq_p (S^{nl} \wedge \mathbf{T}_n^\diamond)_{\bar{h}\text{Aff}_{k,p}}.$$

This completes our proof of Theorem 1.3.

**1.2. Spanier-Whitehead dual version.** There is a dual diagram of Spanier-Whitehead duals:

$$\begin{array}{ccc}
\mathbb{D}|\mathcal{E} \cap \mathcal{F}^\circ|_+ & \longleftarrow & \mathbb{D}|\mathcal{E}|_+ \\
\uparrow & & \uparrow \\
\mathbb{D}|\mathcal{F}^\circ|_+ & \longleftarrow & \mathbb{D}*_+ \\
\downarrow & & \downarrow \\
\mathbb{D}|\mathcal{P}|_+ & \longleftarrow & \mathbb{D}*_+
\end{array}$$

Once again the vertical maps are  $H_*^{\Sigma_n}(-; \mathbb{F}_p^\pm)$ -isomorphisms and the fiber of the bottom map is  $\mathbb{D}(\mathbb{P}_n^\diamond)$ . We thus conclude  $\mathbb{D}(\mathbb{P}_n^\diamond)$  has trivial  $H_*^{\Sigma_n}(-; \mathbb{F}_p^\pm)$ -homology if  $n \neq p^k$ . If  $n = p^k$  it has the same  $H_*^{\Sigma_n}(-; \mathbb{F}_p^\pm)$ -homology as  $\mathbb{D}((\Sigma_n)_+ \wedge_{\text{Aff}_{k,p}} \mathbb{T}_{k,p}^\diamond)$ . We want to apply a version of Shapiro's lemma, which involves moving  $\Sigma_n$  outside of the Spanier-Whitehead dual.

**Lemma 1.5.** *If  $K \subset G$  with  $G$  finite and  $X$  is a  $K$ -space and  $X$  is a  $K$ -spectrum, then there is a  $G$ -equivariant weak equivalence*

$$G_+ \wedge_K \mathbb{D}X \rightarrow \mathbb{D}(G_+ \wedge_K X).$$

*Proof.* The  $\mathbb{Z}[G]$ -module isomorphism  $\lambda: \mathbb{Z}[G] \rightarrow \text{Hom}(\mathbb{Z}[G], \mathbb{Z})$  sending  $g$  to the indicator function  $\lambda_g$  on  $g$  this is 0 on  $x \neq g$  and 1 on  $x = g$ :  $g \cdot \phi$  is given by  $x \mapsto \phi(g^{-1}x)$ , so  $(h \cdot \lambda_g)(x) = \lambda_g(h^{-1}x)$ , which is 1 if  $h^{-1}x = g$ , i.e.  $x = hg$ , and zero otherwise, so equal to  $\lambda_{hg}$ .

Since both  $\Sigma^\infty G_+$  and  $\mathbb{D}G_+$  are wedges of spheres, this determines a  $G$ -equivariant weak equivalence  $\Sigma^\infty G_+ \rightarrow \mathbb{D}G_+$ . Now take

$$\Sigma^\infty G_+ \wedge \mathbb{D}X \simeq \mathbb{D}G_+ \wedge \mathbb{D}X \simeq \mathbb{D}(\Sigma^\infty G_+ \wedge X) \simeq \mathbb{D}(G_+ \wedge_K X). \quad \square$$

Thus  $\mathbb{D}((\Sigma_n)_+ \wedge_{\text{Aff}_{k,p}} \mathbb{T}_{k,p}^\diamond) \simeq (\Sigma_n)_+ \wedge_{\text{Aff}_{k,p}} \mathbb{D}(\mathbb{T}_{k,p}^\diamond)$  as spectra with  $\Sigma_n$ -action, and we conclude that

$$(S^{nl} \wedge \mathbb{D}(\mathbb{P}_n^\diamond))_{\bar{h}\Sigma_n} \simeq_p (S^{nl} \wedge \mathbb{D}(\mathbb{T}_{k,p}^\diamond))_{\text{Aff}_{k,p}}.$$

This completes our proof of Theorem 1.4.

**1.3. Proving the approximation results.** It remains to show that various vertical maps are  $H_*^{\Sigma_n}(-; \mathbb{F}_p^\pm)$ -isomorphisms. We only prove the ordinary version, leaving the Spanier-Whitehead dual version to the interested reader. It is not significantly different in outline, though at the final steps (when we've reduced it down enough) it is different and slightly harder.

Let us go back to the commutative diagram

$$\begin{array}{ccc}
|\mathcal{E} \cap \mathcal{F}^\circ| & \longrightarrow & |\mathcal{E}| \\
\downarrow & & \downarrow \\
|\mathcal{F}^\circ| & \longrightarrow & * \\
\uparrow & & \uparrow \\
|\mathcal{P}| & \longrightarrow & *
\end{array}$$

Recall that if  $\mathcal{C}$  is a collection of subgroups (the only condition is that they are closed under conjugation), then  $\mathcal{O}(\mathcal{C})$  was the category of transitive  $G$ -sets  $(G/K)$  with  $K$  in  $\mathcal{C}$ . Let  $\mathcal{O}(\mathcal{C})_*$  be the Grothendieck construction of  $S: \mathcal{O}(\mathcal{C}) \rightarrow \text{Set}$  sending  $(G/K)$  to  $G/K$ , i.e. objects  $(G/K, c)$  for  $c \in G/K$ . There is a functor  $\mathcal{O}(\mathcal{C})_* \rightarrow \mathcal{C}$  given by sending  $c$  to  $G_c$  (the stabilizer). Then  $EC := |\mathcal{O}(\mathcal{C})_*|$  and this functor induced a  $G$ -equivariant map  $EC \rightarrow |\mathcal{C}|$ . There is a (non-equivariant) inverse up to natural transformation given by sending  $K \in \mathcal{C}$  to  $(G/K, e)$  and thus  $EC \rightarrow |\mathcal{C}|$  is a  $G$ -equivariant weak equivalence. Note that Thomason's theorem,  $|\mathcal{O}(\mathcal{C})_*|$  is a model for  $\text{hocolim}_{\mathcal{O}(\mathcal{C})} S$  [Tho79].

We add the corresponding approximations of the point to get

$$(1) \quad \begin{array}{ccccc} & & |\mathcal{E} \cap \mathcal{F}^\circ| & \xrightarrow{\quad} & |\mathcal{E}| \\ & \nearrow \simeq & \downarrow & & \downarrow \simeq \\ E(\mathcal{E} \cap \mathcal{F}^\circ) & \xrightarrow{\quad} & E(\mathcal{E}) & & E(\mathcal{E}) \\ & \downarrow & \downarrow & & \downarrow \text{Lemma 1.10} \\ \text{Lemma 1.8} & & |\mathcal{F}^\circ| & \xrightarrow{\quad} & * \\ & \nearrow \simeq & \downarrow & & \downarrow \simeq \\ E\mathcal{F}^\circ & \xrightarrow{\quad} & * & & * \\ & \downarrow & \downarrow & & \downarrow \simeq \\ \text{Lemma 1.6} & & |\mathcal{P}| & \xrightarrow{\quad} & * \\ & \nearrow \simeq & \downarrow & & \downarrow \simeq \\ E\mathcal{P} & \xrightarrow{\quad} & * & & * \end{array}$$

where  $E(\mathcal{C}) \rightarrow |\mathcal{C}|$  is always a  $\Sigma_n$ -equivariant weak equivalence and hence induces isomorphisms on  $H_*^{\Sigma_n}(-; M)$ -isomorphisms for any  $\Sigma_n$ -module  $M$ . Thus it suffices to prove that the vertical arrows of the front face are  $H_*^{\Sigma_n}(-; \mathbb{F}_p^\pm)$ -isomorphisms. Note that  $* \rightarrow *$  is trivially a  $H_*^{\Sigma_n}(-; \mathbb{F}_p^\pm)$ -isomorphism.

**Lemma 1.6.**  *$E\mathcal{P} \rightarrow E\mathcal{F}^\circ$  is a genuine  $\Sigma_n$ -equivariant weak equivalence.*

We need an observation: the fixed point set of  $EC$  is equivariantly weakly equivariant to  $|H \downarrow \mathcal{C}|$ , the geometric realization of the nerve of the poset for the collection of subgroups in  $\mathcal{C}$  which contain  $H$ . Indeed,  $EC = \text{hocolim}_{G/K \in \mathcal{O}(\mathcal{C})} G/K$ , so  $(EC)^H = \text{hocolim}_{G/K \in \mathcal{O}(\mathcal{C})} (G/K)^H$  and  $(G/K)^H$  is non-empty equal to  $G/K$  exactly when  $H \subset K$ . Now repeat the argument as before.

*Proof.* Since the isotropy groups are all contained in  $\mathcal{F}^\circ$  (since  $\mathcal{P} \subset \mathcal{F}^\circ$ ) it suffices to prove this on  $H$ -fixed points for  $H \in \mathcal{F}^\circ$ . The above discussion then tells us it suffices to prove that  $|H \downarrow \mathcal{P}| \rightarrow |H \downarrow \mathcal{F}^\circ|$  is a weak equivalence, and since  $H$  is minimal in  $|H \downarrow \mathcal{F}^\circ|$  this is equivalent to showing that  $|H \downarrow \mathcal{P}|$  is contractible. But the partition corresponding to the  $H$ -orbits of  $\{1, \dots, n\}$  gives a minimal element.  $\square$

**1.4.  $\mathcal{C}$ -approximation and its applications.** We study the  $\mathcal{C}$ -approximation functor  $(-)_\mathcal{C}: \text{Top}^G \rightarrow \text{Top}^G$  generalizing  $(*)_\mathcal{C} = EC$ , as described in [Dwy97]. This is given as

follows:

$$X \mapsto X_{\mathcal{C}} := \text{hocoend}_{\mathcal{O}(\mathcal{C})}(\text{Map}_G(-, X), \mathcal{S}(-)),$$

where we note that  $\text{Map}_G(G/K, X) = X^K$ . A few properties follow directly from this construction: (i) it takes maps that are weak equivalence on  $H$ -fixed points for  $H \in \mathcal{C}$  to genuine  $G$ -equivariant weak equivalences, (ii)  $\text{Iso}(X_{\mathcal{C}}) \subset \mathcal{C}$ , and (iii) for  $H \in \mathcal{C}$ ,  $(X_{\mathcal{C}})^H \rightarrow X^H$  is a weak equivalence.

When we insert  $*$ , we get  $\text{hocoend}_{\mathcal{O}(\mathcal{C})}(*, \mathcal{S}) = \text{hocolim}_{\mathcal{O}(\mathcal{C})} \mathcal{S}$ , which is given by the Grothendieck construction by Thomason's theorem [Tho79]. This recovers the construction of  $EC$  given above, and the properties discussed above.

There is a pairing  $\text{Map}_G(-, X) \times \mathcal{S}(-) \rightarrow X$  given on  $G/K$  by evaluation  $\text{Map}_G(G/K, X) \times G/L \rightarrow X$ . This induces a  $G$ -equivariant map  $X_{\mathcal{C}} \rightarrow X$  which is a genuine  $G$ -equivariant weak equivalence on  $X = G/H$  for  $H \in \mathcal{C}$ . Since the construction  $(-)_{\mathcal{C}}$  preserves coproducts, pushouts along monomorphisms and sequential colimits, when  $X$  is constructed from equivariant cells  $\Delta^k \times G/H$  with  $H \in \mathcal{C}$ , the map  $X_{\mathcal{C}} \rightarrow X$  is a genuine  $G$ -equivariant weak equivalence. As a consequence  $X_{\mathcal{C}} \rightarrow X$  is a  $G$ -equivariant weak equivalence when  $\text{Iso}(X) \subset \mathcal{C}$ . We shall use the following lemma later:

**Lemma 1.7.** *If  $f: X \rightarrow Y$  is a  $G$ -equivariant map, it is a genuine  $G$ -equivariant weak equivalence if and only if  $f^H: X^H \rightarrow Y^H$  is a weak equivalence for all  $H \in \text{Iso}(X) \cup \text{Iso}(Y)$ .*

*Proof.*  $\Leftarrow$  is obvious, so let's prove  $\Rightarrow$ . Consider

$$\begin{array}{ccc} X_{\mathcal{C}} & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y_{\mathcal{C}} & \longrightarrow & Y, \end{array}$$

for  $\mathcal{C} = \text{Iso}(X) \cup \text{Iso}(Y)$ . By construction  $X_{\mathcal{C}} \rightarrow X$  and  $Y_{\mathcal{C}} \rightarrow Y$  are genuine  $G$ -equivariant weak equivalences. The assumption that  $f^H: X^H \rightarrow Y^H$  is a weak equivalence for all  $H \in \mathcal{C}$  implies that the left vertical map is a genuine  $G$ -equivariant weak equivalence, by property (i) listed above.  $\square$

Thus the construction  $(-)_{\mathcal{C}}$  has the universal property that there is a unique natural map  $X_{\mathcal{C}} \rightarrow X$  up to homotopy such that  $\text{Iso}(X_{\mathcal{C}}) \subset \mathcal{C}$  and which is a weak equivalence on  $H$ -fixed points for  $H \in \mathcal{C}$ . We shall use this in our consideration of diagram (1).

**Lemma 1.8.** *The map  $E(\mathcal{E} \cap \mathcal{F}^{\circ}) \rightarrow E\mathcal{F}^{\circ}$  is an  $H_*^{\Sigma_n}(-; \mathbb{F}_p^{\pm})$ -isomorphism.*

*Proof.* We start by noting that  $E(\mathcal{E} \cap \mathcal{F}^{\circ}) \rightarrow E\mathcal{F}^{\circ}$  is the  $\mathcal{E}$ -approximation by the universal property. By the previous lemma it is thus also the  $\mathcal{E}$ -approximation to  $E\mathcal{P}$  in the sense that in the extended diagram

$$\begin{array}{ccc} E(\mathcal{E} \cap \mathcal{F}^{\circ}) & \longrightarrow & E\mathcal{F}^{\circ} \\ \simeq \uparrow & & \simeq \uparrow \\ (E\mathcal{P})_{\mathcal{E}} & \longrightarrow & E\mathcal{P} \end{array}$$

both vertical maps are  $\Sigma_n$ -equivariant weak equivalences. To prove the statement, it suffices to prove that  $(E\mathcal{P})_{\mathcal{E}} \rightarrow E\mathcal{P}$  is an  $H_*^{\Sigma_n}(-; \mathbb{F}_p^{\pm})$ -isomorphism. By a Mayer-Vietoris argument,

it suffices to prove that for all  $K \in \text{Iso}(E\mathcal{P}) = \mathcal{P}$  each  $(\Sigma_n/K)_\mathcal{E} \rightarrow \Sigma_n/K$  is a  $H_*^{\Sigma_n}(-; \mathbb{F}_p^\pm)$ -isomorphism, since  $(E\mathcal{P})_\mathcal{E} \rightarrow E\mathcal{P}$  is built by iterated pushouts of products of such maps with  $\Delta^k$ . Then we use that  $(\Sigma_n/K)_\mathcal{E} \cong G \times_K (*)_{\mathcal{E} \downarrow K}$  by the universal property, and Shapiro's lemma to reduce to checking that  $(*)_{\mathcal{E} \downarrow K} \rightarrow *$  is an  $H_*^K(-; \mathbb{F}_p^\pm)$ -isomorphism. But there is a weak equivalence  $(*)_{\mathcal{E} \downarrow K} \rightarrow |\mathcal{E} \downarrow K|$  which is  $K$ -equivariant, so we need to check that  $|\mathcal{E} \downarrow K| \rightarrow *$  is an  $H_*^K(-; \mathbb{F}_p^\pm)$ -isomorphism.  $\square$

**1.5. Bredon homology and its applications.** Before proving that  $|\mathcal{E} \downarrow K| \rightarrow *$  is an  $H_*^K(-; \mathbb{F}_p^\pm)$ -isomorphism, we show that  $|\mathcal{E}| \rightarrow *$  is an  $H_*^{\Sigma_n}(-; \mathbb{F}_p^\pm)$ -isomorphism, the top right vertical arrow in (1). This uses the techniques of transfer with discarded orbits, a version of the transfer argument.

Let us recall the basic transfer argument: if  $X \rightarrow Y$  is a map of  $G$ -spaces and  $M$  is an  $\mathbb{F}_p[G]$ -module, we consider the map  $f_*: H_*^G(X; M) \rightarrow H_*^G(Y; M)$  and for any subgroup  $K \subset G$  the map  $f_*: H_*^K(X; M) \rightarrow H_*^K(Y; M)$ . Using reduction  $\rho$  from  $K$  to  $G$ , there is of course a commutative diagram

$$\begin{array}{ccc} H_*^K(X; M) & \xrightarrow{f_*} & H_*^K(Y; M) \\ \downarrow \rho & & \downarrow \rho \\ H_*^G(X; M) & \longrightarrow & H_*^G(Y; M). \end{array}$$

but if the index  $[G : K]$  is finite we can sum over cosets to get transfer maps

$$\begin{array}{ccc} H_*^G(X; M) & \xrightarrow{f_*} & H_*^G(Y; M) \\ \downarrow \text{tr} & & \downarrow \text{tr} \\ H_*^K(X; M) & \longrightarrow & H_*^K(Y; M) \end{array}$$

such that the composite  $\rho \circ \text{tr}$  is given by multiplication with  $[G : K]$ . Thus if  $[G : K] \not\equiv 0 \pmod{p}$ , then  $f_*: H_*^G(X; M) \rightarrow H_*^G(Y; M)$  is a retract of  $f_*: H_*^K(X; M) \rightarrow H_*^K(Y; M)$ . In particular  $f_*: H_*^G(X; M) \rightarrow H_*^G(Y; M)$  is an isomorphism if  $f_*: H_*^K(X; M) \rightarrow H_*^K(Y; M)$  is.

We can try to compute  $H_*^G(-; M)$  as follows: there is an isomorphism  $C_*(EG \times_G X; M) \cong (C_*(EG) \otimes C_*(X) \otimes M)/G$ . If we filter by the second degree, we get a spectral sequence converging to  $H_{p+q}^G(X)$  whose  $E^1$ -page is given by the chain complex  $E_{p,q}^1 = \bigoplus_{\sigma \in X_p/G} H_q(G_\sigma; M)$ . Its homology, the  $E^2$ -page, is called Bredon homology  $H_*^{G, \text{Br}}(X; M)$ . (This spectral sequence is equal to the Leray spectral sequence for  $X // G \rightarrow X/G$ ). Thus a map induces an isomorphism on  $H_*^G(-; M)$  if it induces an isomorphism on  $H_*^{G, \text{Br}}(X; M)$ . The converse need not be true. A disadvantage of Bredon homology is that it is not invariant under  $G$ -equivariant weak equivalences, but it is under genuine  $G$ -equivariant weak equivalences.

Using this we can prove a fancier version of the transfer argument says that you can discard orbits that do not contribute to the transfer, Section 6.7 of [Dwy98]:

**Lemma 1.9** (Webb, Dwyer). *Let  $X$  be a  $G$ -space,  $P$  of index coprime to  $p$ ,  $Y \subset X$  preserved by  $K$  and suppose that for each  $x \in X \setminus Y$  the transfer  $H_*(G_x, M) \rightarrow H_*(P_x, M)$  vanishes,*

then  $X \rightarrow *$  is an  $H_*^{G, \text{Br}}(-; M)$ -isomorphism if and only if  $Y \rightarrow *$  is an  $H_*^{P, \text{Br}}(-; M)$ -isomorphism.

In particular, under the above assumptions, if  $Y \rightarrow *$  is an  $H_*^{P, \text{Br}}(-; M)$ -isomorphism then  $X \rightarrow *$  is an  $H_*^G(-; M)$ -isomorphism.

*Proof.* The transfer map lifts to Bredon homology, so that we have maps

$$C_*^{G, \text{Br}}(-; M) \xrightarrow{\text{tr}} C_*^{K, \text{Br}}(-; M) \xrightarrow{\rho} C_*^{G, \text{Br}}(-; M)$$

whose composition induces an isomorphism on homology. The assumption that for each  $x \in X \setminus Y$  the transfer  $H_*(G_x, M) \rightarrow H_*(P_x, M)$  vanishes, implies that there is a factorization

$$\begin{array}{ccccc} C_*^{G, \text{Br}}(X; M) & \longrightarrow & C_*^{K, \text{Br}}(X; M) & \longrightarrow & C_*^{G, \text{Br}}(X; M) \\ & \searrow & \uparrow & & \\ & & C_*^{K, \text{Br}}(Y; M) & & \end{array}$$

and comparing to the version with  $X$  and  $Y$  replaced by  $*$  shows that upon passing to homology,  $H_*^{G, \text{Br}}(X; M) \rightarrow H_*^{G, \text{Br}}(*; M)$  is a retract of  $H_*^{K, \text{Br}}(Y; M) \rightarrow H_*^{K, \text{Br}}(*; M)$ .  $\square$

A version of the following is proven in much larger generality in [Dwy98] ( $|\mathcal{C}|$  is there called the “normalizer decomposition”, but note that Section 3 of [Dwy97] proves that the other decompositions in that paper are equivariantly weakly equivalent to the normalizer decomposition).

**Lemma 1.10.** *The map  $|\mathcal{E}| \rightarrow *$  is an  $H_*^{\Sigma_n}(-; \mathbb{F}_p^\pm)$ -isomorphism.*

*Proof.* There are two cases:

(I)  $1 < n < p$ . In this case  $|\mathcal{E}| = \emptyset$ , so we need to show that  $H_*^{\Sigma_n}(*; \mathbb{F}_p^\pm) = 0$ . In this case  $|\Sigma_n|$  is coprime to  $p$  and we can transfer to  $\{e\}$ . This shows that  $H_*^{\Sigma_n}(*; \mathbb{F}_p^\pm)$  vanishes for  $* \neq 0$ . For  $* = 0$ , we use that there is an element  $g$  of sign  $-1$  and thus for any  $x \in \mathbb{F}_p^\pm$  we have  $x \sim g \cdot x = -x$  so  $2x = 0$  and since  $p$  is odd  $x = 0$ .

(II)  $n \geq p$ . We use Lemma 1.9 with  $X = |\mathcal{E}|$ ,  $P$  a  $p$ -Sylow of  $\Sigma_n$  and  $Y = \bigcup_{\{e\} \neq Q \subset \Sigma_n} |\mathcal{E}|^Q$ . This choice of  $Y$  has two reasons: (a) it guarantees that the isotropy of any point in  $Y$  is non-trivial, and (b) that the stabilizer of every point in  $X \setminus Y$  is trivial.

We first verify that  $Y \rightarrow *$  is an  $H_*^{P, \text{Br}}(-; M)$ -isomorphism for any  $M$  (including  $\mathbb{F}_p^\pm$ ). For this it suffices to prove that  $Y \rightarrow *$  is a genuine  $P$ -equivariant weak equivalence, and by Lemma 1.7 it suffices to verify the map is a weak equivalence on  $Q$ -fixed points for  $Q \subset P$  non-trivial.

Now we use that  $|\mathcal{E}|^Q$  is the geometric realization of the poset of those elementary abelian subgroups  $H$  such that  $Q \subset N(H)$ . We give a zigzag of natural transformations from  $H$  to the subgroup  $Z$  of exponent  $p$  elements in the center of  $Q$ . This is non-empty because  $Q$  is a non-trivial  $p$ -group (let  $Q$  acts on itself by conjugation and count orbits at length 1). This is where (a) was used. The idea is to take

$$H \supset H \cap H' \subset H'Z \supset Z$$



where  $H'$  is the subgroups of exponent  $p$  elements of  $QH$  (which is group because  $Q \subset N(H)$ ). The only non-obvious statement is that  $H \cap H'$  is non-empty;  $H$  is abelian and normalized by  $Q$  so that the action of  $Q$  on  $H$  must have at least  $p$  orbits of length 1.

Next we verify that  $H_*(G_x; \mathbb{F}_p^\pm) \rightarrow H_*(P_x; \mathbb{F}_p^\pm)$  is zero. By (b) the target is trivial in all degrees except 0. As  $p$  is odd, we shall show that  $H_0(G_x; \mathbb{F}_p^\pm) = 0$ . The stabilizer  $G_x$  are given by the intersection of normalizers  $N(E_0) \supset N(E_1) \supset \dots \supset N(E_k)$  of a sequence  $E_0 \subset E_1 \subset \dots \subset E_k$  of elementary abelian  $p$ -subgroups of  $\Sigma_n$ . If  $p$  is odd we take an element  $g$  of order  $p$  in  $E_0$ ; these always act trivially on  $\mathbb{F}_p^\pm$  (as  $1 = \epsilon(g^p) = \epsilon(g)^p$  and  $p$  is odd) and then  $x \sim g^p x = px = 0$ .  $\square$

We now do the slighter harder case needed for Lemma 1.8.

**Lemma 1.11.** *For  $K \in \mathcal{P}$ , the map  $|\mathcal{E} \downarrow K| \rightarrow *$  is an  $H_*^K(-; \mathbb{F}_p^\pm)$ -isomorphism.*

*Proof.* This is the same argument as in Lemma 1.10. There are again two cases:

(I)  $p$  does not divide the order of  $K$ . Then  $\mathcal{E} \downarrow K$  is empty, so we must show that  $H_*^K(*; \mathbb{F}_p^\pm) = 0$ . The assumption that  $p$  does not divide  $|K|$ , means that the transfer map  $H_*^K(*; \mathbb{F}_p^\pm) \rightarrow H_*(*; \mathbb{F}_p)$  is injective, so vanishes in all positive degrees. For vanishing in degree 0, we use there is an element  $g$  of sign  $-1$  and thus for any  $x \in \mathbb{F}_p^\pm$  we have  $x \sim g \cdot x = -x$  so  $2x = 0$  and since  $p$  is odd  $x = 0$ .

(II)  $p$  divides the order of  $K$ . We again use Lemma 1.9:  $X = |\mathcal{E} \downarrow K|$ ,  $P$  is a  $p$ -Sylow of  $K$ , and  $Y = \bigcup_{\{e\} \neq Q \subset K} |\mathcal{E} \downarrow K|^Q$ . As in Lemma 1.10, the map  $Y \rightarrow *$  is a genuine  $K$ -equivariant weak equivalence, so it remains to show that for the complement of  $Y$  all stabilizers  $K_x$  have transfer to  $P_x$  which is 0. As before by construction  $P_x = \{e\}$ , so it suffices to prove that the  $G_x$ -coinvariants of  $H_0(K_x; \mathbb{F}_p^\pm)$  vanish. For this it suffices to prove that there exists an element  $g$  of order  $p$  in  $K_x$  which acts trivially on  $\mathbb{F}_p^\pm$ , which exist as in Lemma 1.10.  $\square$

## 2. THE STEINBERG MODULE

The goal is to explain why  $\mathbb{T}_{p,k}$  is  $(k-3)$ -connected and  $(k-2)$ -dimensional, and its top reduced homology  $\text{St}_{p,k} := \tilde{H}_{k-1}(\mathbb{T}_{p,k}; \mathbb{F}_p)$  is a projective self-dual  $\mathbb{F}_p[\text{GL}_n(\mathbb{F}_p)]$ -module. We loosely follow [Str]. A spectral lift of the projectivity was used by Mike for the connection to Mitchell-Priddy, and we will need a spectral lift of the self-duality to study the Goodwillie derivatives of the identity on odd spheres.

Recall that  $\mathbb{T}_{p,k}$  is the poset of proper non-trivial linear subspaces of  $\mathbb{F}_p^n$  and thus has a left  $\text{GL}_{p,k}$ -action.

**Lemma 2.1.** *We have that  $\mathbb{T}_{p,k}$  is homotopy equivalent to the subcomplex of the full simplex of all hyperplanes in  $\mathbb{F}_p^k$  consisting of those simplices whose intersection is not  $\{0\}$ . In particular it is  $(k-3)$ -connected.*

*Proof.* We have cover  $\mathbb{T}_{p,k}$  by the sub-posets  $T_H$  of flags contained in the hyperplane  $H$ . These are all contractible since they have a maximal element. In fact, their  $j$ -fold intersections  $j \leq k-2$  are contractible for the same reason. The higher intersections may either be empty or contractible, depending on the intersection pattern of the hyperplanes.

Since any  $\leq (k-1)$  hyperplanes must have empty intersection, it contains the full  $(k-2)$ -skeleton of the full simplex on all hyperplanes, which is  $(k-3)$ -connected.  $\square$

**Example 2.2.** If  $k = 2$  this is saying that  $\mathbb{T}_{p,k}$  is non-empty, and there are indeed hyperplanes (lines) in  $\mathbb{F}_p^2$ . There are no higher-dimensional simplices, so this connectivity result can not be improved.

An alternative proof gives us more information: let  $\mathbb{U}_{k,p}$  denote the simplicial complex of partial bases (also called unimodular sequences). It is a theorem of Maazen-van der Kallen that this is  $(k - 2)$ -connected [vdK80].

**Example 2.3.** If  $k = 2$ , then this is saying that  $\mathbb{U}_{k,p}$  is path-connected. Indeed, any basis element  $\alpha\vec{e}_1 + \beta\vec{e}_2$  can be connected to either  $\vec{e}_1$  or  $\vec{e}_2$ , and these can be connected to each other.

Let  $\text{Simp}(\mathbb{U}_{k,p}^{(k-2)})$  be the poset of simplices of its  $(k - 2)$ -skeleton. This maps to  $\mathbb{T}_{k,p}$  by sending  $\sigma_0 \subset \dots \subset \sigma_j$  to  $\text{span}(\sigma_0) \subset \dots \subset \text{span}(\sigma_j)$ .

**Lemma 2.4.** *The map  $\text{span}: \text{Simp}(\mathbb{U}_{k,p}^{(k-2)}) \rightarrow \mathbb{T}_{k,p}$  is  $(k - 2)$ -connected.*

*Proof.* Cover  $\text{Simp}(\mathbb{U}_{k,p}^{(k-2)})$  by subsets  $Y_H$  of partial basis containing in a hyperplane. This maps to  $T_H$  under  $\text{span}$ . The subcomplex of partial bases contained in  $H$  is  $(k - 3)$ -connected by Maazen-van der Kallen and thus the map to the contractible space  $T_H$  is  $(k - 2)$ -connected, similarly the partial bases contained in a  $j$ -fold intersection to a point is  $(k - 1 - j)$ -connected. This means that the map from  $\mathbb{U}_{k,p}^{(k-2)}$  to the subcomplex of the full simplex of all hyperplanes consisting of those simplices whose intersection is not  $\{0\}$  (which is homotopy equivalent to  $\mathbb{T}_{p,k}$  by the previous lemma) is  $(k - 2)$ -connected.  $\square$

Thus we see that there is a commutative diagram of left  $\text{GL}_{k,p}$ -modules

$$\begin{array}{ccc}
\mathbb{F}_p[\text{ordered bases}] & & \\
\downarrow & & \\
\mathbb{F}_p[(\mathbb{U}_{k,p})_{k-1}] \cong & & \\
\mathbb{F}_p[\text{unordered bases}] & & \\
\downarrow & & \\
H_{k-2}(\text{Simp}(\mathbb{U}_{k,p}^{(k-2)})) & \twoheadrightarrow & H_{k-2}(\mathbb{T}_{k,p}) = \text{St}_{p,k} \hookrightarrow \frac{\mathbb{F}_p[(\mathbb{T}_{k,p})_{k-2}] \cong}{\mathbb{F}_p[\text{full flags}]}
\end{array}$$

where the bottom left vertical map is the surjection induced by the isomorphism  $\text{im}(d) \cong \ker(d)$  which follows the vanishing of  $H_{k-2}(\mathbb{U}_{k,p})$  composed with the quotient from  $\ker(d)$  to  $H_{k-2}$ , and the right horizontal map is the inclusion of  $\ker(d)$  into all  $(k - 2)$ -simplices.

To complete it, we note that geometrically passing to the poset of simplices is taking the barycentric subdivision. Hence a basis is sent to the linear combination of full flags obtained by applying  $\text{span}$  to the boundary of a subdivided  $(k - 1)$ -simplex. A universal computation of the  $(k - 2)$ -simplices in the subdivided  $(k - 1)$ -simplex tells us this is given by

$$\sum_{\sigma \in \Sigma_k} \epsilon(\sigma) \cdot \text{span}(\{e_{\sigma(1)}\} \subset \{e_{\sigma(1)}, e_{\sigma(2)}\} \subset \dots)$$

In other words, it is obtained by applying the map  $\omega$ :

$$\omega(e_1, \dots, e_k) = \sum_{\sigma \in \Sigma_k} \epsilon(\sigma)(e_{\sigma(1)}, \dots, e_{\sigma(k)}) = \sum_{\sigma \in \Sigma_k} \epsilon(\sigma)\sigma^*$$

followed by the projection  $\pi: \mathbb{F}_p[\text{ordered bases}] \rightarrow \mathbb{F}_p[\text{full flags}]$ . The composite

$$\mu := \pi\omega: \mathbb{F}_p[\text{ordered bases}] \rightarrow \mathbb{F}_p[\text{full flags}]$$

is the *apartment class map*. By construction it factors over the Steinberg:

$$\begin{array}{ccc} \mathbb{F}_p[\text{ordered bases}] & \xrightarrow{\omega} & \mathbb{F}_p[\text{ordered bases}] \\ \downarrow & & \downarrow \pi \\ \mathbb{F}_p[(\mathbf{U}_{k,p})_{k-1}] & & \\ \downarrow & \swarrow & \\ H_{k-2}(\text{Simp}(\mathbf{U}_{k,p}^{(k-2)})) & \twoheadrightarrow H_{k-2}(\mathbf{T}_{k,p}) = \text{St}_{p,k} & \hookrightarrow \mathbb{F}_p[\text{full flags}] \end{array}$$

Since both  $\mathbb{F}_p[\text{ordered bases}]$  and  $\mathbb{F}_p[\text{full flags}]$  are permutation modules they are isomorphic to their dual representations. We may thus include the isomorphisms in a larger commutative diagram, defining (\*):

$$\begin{array}{ccccc} \mathbb{F}_p[\text{ordered bases}] & & & & \mathbb{F}_p[\text{ordered bases}] \\ \downarrow \omega & \searrow & & & \omega^t \uparrow \\ \mathbb{F}_p[\text{ordered bases}] & \twoheadrightarrow & \text{St}_{p,k} & \xrightarrow{(*)} & \text{St}_{p,k}^* & \twoheadrightarrow & \mathbb{F}_p[\text{ordered bases}] \\ \downarrow \pi & \swarrow & & & & \swarrow & \pi^t \uparrow \\ \mathbb{F}_p[\text{full flags}] & \xrightarrow{\cong} & & & & \xrightarrow{\cong} & \mathbb{F}_p[\text{full flags}] \end{array}$$

We may then define a map

$$e_{p,k}^{\text{St}} := \pi^t \mu: \mathbb{F}_p[\text{ordered bases}] \rightarrow \mathbb{F}_p[\text{ordered bases}].$$

This is the *Steinberg idempotent* (up to a invertible scaling factor which becomes relevant working over  $\mathbb{Z}_{(p)}$  instead of  $\mathbb{F}_p$ ).

It is useful to define a closely related map

$$e := \mu\pi^t: \mathbb{F}_p[\text{full flags}] \rightarrow \mathbb{F}_p[\text{full flags}].$$

The reason is that this is a  $\text{GL}_n(\mathbb{F}_p)$ -equivariant endomorphism of  $\mathbb{F}_p[\text{full flags}]$ , the *modular Hecke algebra*  $\mathcal{H}$ , a well-understood object.

We can define some elements of  $\mathcal{H}$  in terms of the terms of the differential

$$\mathbb{F}_p[\text{full flags}] \rightarrow \mathbb{F}_p[\text{length } k-2 \text{ flags}]$$

and its dual. There are terms  $d_i$  deleting or merging flag steps with transposes  $d^i$ . In terms of a choice of Borel this is  $\mathbb{F}_p[\text{full flags}] \cong \mathbb{F}_p[\text{GL}_{k,p}/B_k] \rightarrow \mathbb{F}_p[\text{length } k-2 \text{ flags}] \cong \mathbb{F}_p[\text{GL}_{k,p}/P_i]$  and thus  $\hat{e}_i := d^i d_i$  is projection down to  $\text{GL}_{k,p}/P_i$  and transfer back up. This is an element of the *modular Hecke algebra*. As  $B_n$  has index  $(p+1)$  in  $P_i$ , we have that  $\hat{e}_i^2 = (p+1)\hat{e}_i \equiv \hat{e}_i$ .

We can also obtain this from the correspondence

$$\mathbb{F}_p[\text{full flags}] \leftarrow \mathbb{F}_p[\text{pairs of full flags with Jordan permutation } s_i \text{ or id}] \rightarrow \mathbb{F}_p[\text{full flags}],$$

and thus can write  $\hat{e}_i = 1 + T_{s_i}$  with  $T_{s_i}$  is obtained by the correspondence

$$\mathbb{F}_p[\text{full flags}] \leftarrow \mathbb{F}_p[\text{pair full flags with Jordan permutation } s_i] \rightarrow \mathbb{F}_p[\text{full flags}].$$

**Theorem 2.5.** *The modular Hecke algebra is generated by  $T_i := T_{s_i}$  for  $1 \leq i \leq k-1$ , subject only to the relations*

- (1)  $T_i^2 = -T_i$ .
- (2)  $T_i T_j = T_j T_i$  if  $|i - j| \geq 2$ .
- (3)  $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ .

*Sketch of proof.* To obtain generation, one also defines  $T_\sigma$  for  $\sigma \in \Sigma_k$  and proves that if  $\sigma = s_{i_1} \cdots s_{i_r}$  is a reduced word then  $T_\sigma = T_{i_1} \cdots T_{i_r}$ . Pairs  $(\bar{U}, \bar{V})$  of complete flags are up to the action of  $\text{GL}_{k,p}$  classified by their Jordan permutation. This means that for a  $\text{GL}_{k,p}$ -equivariant endomorphism  $f$ , the coefficient of a full flag  $[\bar{V}]$  in  $f(\bar{U})$  only depends on  $\sigma(\bar{U}, \bar{V})$ , and that  $f$  is a sum of  $T_\sigma$ 's.

It is clear that  $T_i^2 = -T_i$  from  $\hat{e}_i^2 = \hat{e}_i$ . Since  $s_i s_j$  and  $s_j s_i$  or  $s_i s_{i+1} s_i$  or  $s_{i+1} s_i s_{i+1}$  are both reduced words, the above also implies (2) and (3).  $\square$

Furthermore  $T_i[\bar{U}]$  only depends on  $U_j$  for  $i \neq j$ . Thus we see that  $(1 + T_i)\pi = (1 + T_i)\pi s_i^*$ . Applying this to  $\omega$  using the fact that  $s_i^* \omega = -\omega$  we see that  $(1 + T_i)\mu = -(1 + T_i)\mu$ , so that  $(1 + T_i)\mu = 0$ . This implies that for all elements of the Hecke algebra (such as  $e$ ), we have that  $e\mu = \epsilon(e)\mu$  where  $\epsilon: \mathcal{H} \rightarrow \mathbb{F}_p$  is the homomorphism determined by  $\epsilon(T_i) = -1$ .

Thus we have that  $(e^{\text{St}})^2 = \pi^t \mu \pi^t \mu = \pi^t \epsilon(e) \mu = \epsilon(e) e^{\text{St}}$ . Once we compute that  $\epsilon(e) \neq 0$  we get one idempotent up to scaling. To compute it we must express it terms of  $T_\sigma$ . It turns out that  $e \equiv (-1)^n \epsilon(\rho) T_\rho$  for  $\rho(i) = k + 1 - i$ , so that  $\epsilon(e)$  is  $\pm 1$ , see Proposition 9.3 of [Str]. We may adjust the sign of  $e^{\text{St}}$  and  $e$  if necessary so that this sign is 1.

We now show that  $\text{St}_{p,k}$  is isomorphic to both  $\text{im}(e^{\text{St}})$  and  $\text{im}(e)$ . First of all, the restriction of  $\mu$  and  $\pi^t$  induce isomorphisms between  $\text{im}(e^{\text{St}})$  and  $\text{im}(e)$ . We thus have a factorization

$$\begin{array}{ccc} \text{im}(e^{\text{St}}) & \xrightarrow[\cong]{\mu} & \text{im}(e) \\ & \searrow & \nearrow \\ & \text{St}_{p,k} & \end{array}$$

so that the left diagonal map is injective. The left diagonal is also surjective as  $\mu$  is surjective onto  $\text{St}_{p,k}$ , and  $\mu e^{\text{St}} = \mu \pi^t \mu = \mu$ . Thus  $\text{St}_{p,k}$  is a summand of the free  $\mathbb{F}_p[\text{GL}_{p,k}] = \mathbb{F}_p[\text{ordered bases}]$ . *Upshot:  $\text{St}_{p,k}$  is projective.*

Finally, we need to verify that  $\text{St}_{p,k} \cong \text{St}_{p,k}^*$ . The above argument tells us that  $\mu \pi^t$  is the identity on  $\text{St}_{p,k}$ , but this is  $\pi \omega \pi^t$  and since  $\omega^t = \omega$  we see that it factors over the map  $\text{St}_{p,k} \rightarrow \text{St}_{p,k}^*$ . Thus this is injective and since both sides have the same dimension they are isomorphic. *Upshot:  $\text{St}_{p,k}$  is self-dual.*

We next explain how to obtain the spectral lift of the self-duality isomorphism. The construction of the isomorphism  $\text{St}_{p,k} \rightarrow \text{St}_{p,k}^*$  given by

$$(2) \quad \text{St}_{p,k} \hookrightarrow \mathbb{F}_p[\text{full flags}] \cong \mathbb{F}_p[\text{full flags}]^* \rightarrow \text{St}_{p,k}^*$$

may be upgraded to a map of spectra. First there is the  $\mathrm{GL}_{k,p}$ -equivariant map

$$\Sigma^\infty \mathbb{T}_{k,p}^\diamond \rightarrow \Sigma^\infty(S^{k-1} \wedge \{\text{full flags}\})$$

induced by collapsing the  $(k-2)$ -skeleton of  $\mathbb{T}_{k,p}^\diamond$ . On homology in degree  $(k-1)$  it induces the first map of (2). The third map of (2) is just its Spanier-Whitehead dual shifted by  $2(k-1)$ :

$$S^{2(k-1)} \wedge \mathbb{D}(S^{k-1} \wedge \{\text{full flags}\}) \rightarrow S^{2(k-1)} \wedge \mathbb{D}\mathbb{T}_{k,p}^\diamond.$$

It remains to produce the map

$$\Sigma^\infty(S^{k-1} \wedge \{\text{full flags}\}) \rightarrow S^{2(k-1)} \wedge \mathbb{D}(S^{k-1} \wedge \{\text{full flags}\}),$$

but this is just the  $2(k-1)$ -fold suspension of the map  $G_+ \wedge_K \mathbb{D}X \rightarrow \mathbb{D}(G_+ \wedge_K X)$  with  $G = \mathrm{GL}_{k,p}$ ,  $K$  a Borel and  $X = \mathbb{S}$ . Thus there is a  $\mathrm{Aff}_{k,p}$ -equivariant map

$$\Sigma^\infty \mathbb{T}_{k,p}^\diamond \rightarrow S^{2(k-1)} \wedge \mathbb{D}\mathbb{T}_{k,p}^\diamond$$

which is a  $p$ -complete equivalence.

### 3. THE GOODWILLIE DERIVATIVES

Our next goal is to finally connect all of this to the Goodwillie derivatives of the identity functor evaluated on an odd sphere. Then  $D_n(S^l)$  denote these derivatives.

**Theorem 3.1.** *Let  $l$  be odd, then  $D_n(S^l) \simeq_p *$  is  $n \neq p^k$  and if  $n = p^k$  we have that*

$$S^{2(k-1)+1} \wedge D_n(S^l) \simeq_p \Sigma^\infty(S^{nl} \wedge \mathbb{P}_n^\diamond)_{\bar{h}\Sigma_n}.$$

*Proof.* Johnson proved that [Joh95]

$$D_n(S^l) \simeq \mathrm{Map}_*(\Sigma \mathbb{P}_n^\diamond, \Sigma^\infty S^{nl})_{\bar{h}\Sigma_n}.$$

Moving the suspension in the domain outside we get

$$D_n(S^l) \simeq S^{-1} \wedge \mathrm{Map}_*(\mathbb{P}_n^\diamond, \Sigma^\infty S^{nl})_{\bar{h}\Sigma_n}$$

and the right hand side may be rewritten as  $S^{-1} \wedge (S^{nl} \wedge \mathbb{D}\mathbb{P}_n^\diamond)_{\bar{h}\Sigma_n}$ , which vanishes at  $p$  unless  $n = p^k$ .

So assume  $n = p^k$  and suspend by  $S^{2(k-1)+1}$  to get

$$S^{2(k-1)} \wedge \mathrm{Map}_*(\mathbb{P}_n^\diamond, \Sigma^\infty S^{nl})_{\bar{h}\Sigma_n} \simeq S^{2(k-1)} \wedge (S^{nl} \wedge \mathbb{D}\mathbb{P}_n^\diamond)_{\bar{h}\Sigma_n}.$$

We use the Spanier-Whitehead dual approximation result to rewrite it terms of the Tits building up to  $p$ -completion

$$S^{2(k-1)} \wedge (S^{nl} \wedge \mathbb{D}\mathbb{P}_n^\diamond)_{\bar{h}\Sigma_n} \simeq_p S^{2(k-1)} \wedge (S^{nl} \wedge \mathbb{D}\mathbb{T}_{p,k}^\diamond)_{\bar{h}\mathrm{Aff}_{k,p}}.$$

The spectral self-duality of the Tits building (coming from self-duality of the Steinberg), gives us a further  $p$ -complete equivalence

$$S^{2(k-1)} \wedge (S^{nl} \wedge \mathbb{D}\mathbb{T}_{p,k}^\diamond)_{\bar{h}\mathrm{Aff}_{k,p}} \simeq_p \Sigma^\infty(S^{nl} \wedge \mathbb{T}_{p,k}^\diamond)_{\bar{h}\mathrm{Aff}_{k,p}}.$$

We finish by an application of the ordinary approximation theorem to get

$$\Sigma^\infty(S^{nl} \wedge \mathbb{T}_{p,k}^\diamond)_{\bar{h}\mathrm{Aff}_{k,p}} \simeq \Sigma^\infty(S^{nl} \wedge \mathbb{P}_n^\diamond)_{\bar{h}\Sigma_n}. \quad \square$$

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