THE DERIVATIVES OF THE IDENTITY FUNCTOR

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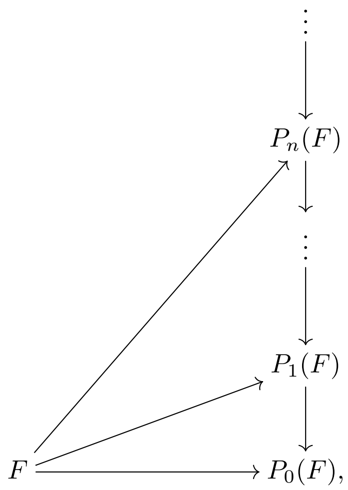
The goal of these notes is to give an outline of Johnson's calculation of the Goodwillie derivatives of the identity functor on pointed spaces [Joh95]. Recall that the theory of Goodwillie calculus associates to a reduced homotopy functor $F : \mathcal{T}\mathrm{op}_* \to \mathcal{T}\mathrm{op}_*$ a tower of functors



where $P_n(F)$ is the universal *n-excisive* or *n-polynomial* approximation to F [Goo03]. Setting $F = \mathrm{id}$, a theorem of Goodwillie asserts that the natural map

$$X \to \operatorname{holim} P_n(\mathrm{id})(X)$$

is a weak equivalence for X simply connected. Thus, it is of interest to understand the *layers*

$$D_n(F) := \mathrm{fib}\left(P_n(F) \to P_{n-1}(F)\right).$$

This functor is an *n-homogeneous* functor, and these are completely classified.

Theorem 1 (Goodwillie)**.** *The assignment $E \mapsto \Omega^\infty\left(E \wedge_{\Sigma_n} (-)^{\wedge n}\right)$ extends to an equivalence of ∞-categories between Σ_n-spectra and degree n homogeneous functors.*

This theorem motivates the following definition.

Definition 2. The nth *derivative* of F is the Σ_n-spectrum $\partial_n(F)$ such that

$$D_n(F) \simeq \Omega^\infty\left(\partial_n(F) \wedge_{\Sigma_n} (-)^{\wedge n}\right).$$

Our goal is to understand the symmetric sequence $\{\partial_n(\mathrm{id})\}_{n\geq 0}$. In order to do so, we require an algorithm for computing $\partial_n(F)$ *in terms of F*.

Definition 3. Let I be a finite set, and write $P(I) = \{0,1\}^I$ for the set of subsets of I, partially ordered by inclusion.

(1) A (pointed) *I-cube* is a functor $\mathcal{X} : P(I) \to \mathcal{T}\mathrm{op}_*$.

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(2) The *total fiber* of the I-cube $\mathcal{X}$ is

$$\mathrm{tfib}(\mathcal{X}) := \mathrm{fib}\left(\mathcal{X}(\varnothing) \to \lim_{\varnothing \neq S \in P(I)} \mathcal{X}(S)\right).$$

We write $[n] = \{1, \ldots, n\}$.

Example 4. A $[0]$-cube is a space, and the total fiber is the same space.

Example 5. A $[1]$-cube is a morphism, and the total fiber is its fiber.

Example 6. A $[2]$-cube is a commuting square, and the total fiber is the iterated fiber.

Definition 7. The nth *cross effect* of the functor $F : \mathcal{T}\mathrm{op}_* \to \mathcal{T}\mathrm{op}_*$ is the functor $\mathrm{cr}_n : \mathcal{T}\mathrm{op}_*^n \to \mathcal{T}\mathrm{op}_*$ defined by the formula

$$\mathrm{cr}_n(F)(X_1, \ldots, X_n) = \mathrm{tfib}\left(S \mapsto F\left(\bigvee_{i \notin S \subseteq [n]} X_i\right)\right).$$

With this definition in hand, we can state the following useful recipe.

Proposition 8. *Let $F : \mathcal{T}\mathrm{op}_* \to \mathcal{T}\mathrm{op}_*$ be a reduced homotopy functor. There are natural Σ_n-equivariant equivalences*

$$\Omega^\infty \partial_n(F) \simeq \operatorname*{colim}_{k_1, \ldots, k_n} \Omega^{k_1 + \cdots + k_n} \mathrm{cr}_n(F)(S^{k_1}, \ldots, S^{k_n}).$$

This formula should be compared to the usual formula for the linearizatio $\Omega^\infty G \Sigma^\infty$ of a functor G. Because of this parallel, we may at times refer to this construction as *multilinearization.*

Thus, our goal is to understand the multilinearization of the functor

$$\mathrm{cr}_n(\mathrm{id})(X_1, \ldots, X_n) = \mathrm{tfib}\left(S \mapsto \bigvee_{i \notin S \subseteq [n]} X_i\right).$$

In order to do so, it will be helpful to have a model for the total fiber.

Notation 9. Let I be a finite set. For $S \subseteq I$, we write

$$[0,1]^S := \left\{t \in [0,1]^I : t_i = 0 \text{ if } i \notin S\right\}.$$

We further write

$$\partial_1 [0,1]^S := \left\{(t \in [0,1]^S : t_i = 1 \text{ for some } i \in S\right\}.$$

Evidently, there is an inclusion $S \subseteq T$ of subsets of I if and only if there is an inclusion $[0,1]^S \subseteq [0,1]^T$ of subspaces of $[0,1]^I$; thus, we obtain a functor $[0,1]^\bullet : P(I) \to \mathcal{T}\mathrm{op}$. The same remarks apply to the subspaces $\partial_1[0,1]^S$, and we have the following generalization of the standard formula for the homotopy fiber of a map.

Lemma 10. *Let $\mathcal{X}$ be an I-cube. There is a pullback diagram*

$$\begin{array}{ccc} \mathrm{tfib}(\mathcal{X}) & \longrightarrow & \mathrm{Nat}\left([0,1]^\bullet, \mathcal{X}\right) \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \mathrm{Nat}\left(\partial_1[0,1]^\bullet, \mathcal{X}\right), \end{array}$$

where the bottom map is induced by the inclusion of the basepoint.

In other words, a point in the total fiber of $\mathcal{X}$ is a collection of maps $\{f_S : [0,1]^S \to \mathcal{X}(S)\}_{S \subseteq I}$ such that

(1) for each $S \subseteq T \subseteq I$, the diagram

$$\begin{array}{ccc} [0,1]^S & \xrightarrow{f_S} & \mathcal{X}(S) \\ \downarrow & & \downarrow \\ [0,1]^T & \xrightarrow{f_T} & \mathcal{X}(T) \end{array}$$

commutes, and

(2) $f_S(t)$ is the basepoint in $\mathcal{X}(S)$ whenever some $t_i = 1$.

Definition 11. For a nonempty finite set I and an element $i \in I$ and an I-cube $\mathcal{X}$, the *comparison map* for $\mathcal{X}$ is the map

$$\mathrm{tfib}(\mathcal{X}) \to \mathrm{Map}_*\left([0,1]^{|I|(|I|-1)}, \bigwedge_{i \in I} \mathcal{X}(I_i)\right)$$

defined by evaluation at $I \setminus \{i\}$ for $i \in I$ (note that $[0,1]^{I \setminus \{i\}}$ is an $(|I|-1)$-dimensional cube.

Applying this construction to the n-cube of Definition 7, we obtain the clockwise composite in the commuting diagram

$$\begin{array}{ccc} \mathrm{cr}_n(\mathrm{id})(X_1, \ldots, X_n) & \longrightarrow & \mathrm{Map}_*\left([0,1]^{n(n-1)}, \prod_{i=1}^n X_i\right) \\ \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow \\ \mathrm{Map}_*\left(\Delta_n, \bigwedge_{i=1}^n X_i\right) & \longrightarrow & \mathrm{Map}_*\left([0,1]^{n(n-1)}, \bigwedge_{i=1}^n X_i\right), \end{array}$$

where Δ_n is defined as a quotient of the form

$$\Delta_n := [0,1]^{n(n-1)} \Big/ Z \cup \bigcup_{1 \le i < j \le n} W_{ij}.$$

In order to describe the subspaces in question, it will be covenient to think of $[0,1]^{n(n-1)}$ as the space of matrices $t = (t_{ij})_{1 \le i,j \le n}$ with $t_{ij} \in [0,1]$ and $t_{ii} = 0$; here, the ith row $(t_{ij})_{1 \le j \le n}$ contains the coordinates of the ith $(n-1)$-dimensional cube $[0,1]^{\{1,\ldots,\hat{\imath},\ldots,n\}}$. With this notation in mind, we define

$$Z = \left\{ t \in [0,1]^{n(n-1)} : t_{ij} = 1 \text{ for some } 1 \le i,j \le n \right\}$$
$$W_{ij} = \left\{ t \in [0,1]^{n(n-1)} : t_{ik} = t_{jk} \text{ for all } 1 \le k \le n \right\}.$$

Since it is immediate from Lemma 10 that any $f : [0,1]^{n(n-1)} \to \bigwedge_{i=1}^n X_i$ in the image of the comparison map sends Z to the basepoint, all that remains in constructing the map φ is to check the following.

Lemma 12. *If $f : [0,1]^{n(n-1)} \to \bigwedge_{i=1}^n X_i$ lies in the image of the comparison map, then f sends W_{ij} to the basepoint for any $1 \le i < j \le k$.*

Proof. In the solid diagram

$$\begin{array}{ccccc} & & [0,1]^{n(n-1)} & & \\ & \swarrow & \uparrow & \searrow & \\ & & W_{ij} & & \\ & & \downarrow & & \\ [0,1]^{n-1} & \longleftarrow & [0,1]^{n-2} & \longrightarrow & [0,1]^{n-1} \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{ij}} & & \downarrow{\scriptstyle f_j} \\ X_i & \longleftarrow & X_i \vee X_j & \longrightarrow & X_j \end{array}$$

the squares commute by the assumption that f lies in the image of the comparison map. The dashed filler exists by the definition of W_{ij}, and it follows that, for $t \in W_{ij}$, the points $f_i(t) \in X_i$ and $f_j(t) \in X_j$ are retracts of the same point in $X_i \vee X_j$, so each is the respective basepoint. □

Thus, the map $\varphi : \mathrm{cr}_n(\mathrm{id})(X_1, \ldots, X_n) \to \mathrm{Map}_*\left(\Delta_n, \bigwedge_{i=1}^n X_i\right)$ is defined. Note, moreover, that Δ_n is closed in $[0,1]^{n(n-1)}$ under the action of Σ_n on the rows, and the map φ is Σ_n-equivariant.

On the face of it, this map would seem to discard a great deal of information about the cross effect, but it turns out that only the "first order" information it captures can influence the respective multilinearizations.

Theorem 13 (Johnson)**.** *The map φ induces an equivalence after multilinearization.*

Corollary 14. *There is an equivalence of Σ_n-spectra*

$$\partial_n(\mathrm{id}) \simeq \mathbb{D}\left(\Sigma^\infty \Delta_n\right),$$

where $\mathbb{D} = \mathcal{S}\mathrm{p}(-, \mathbb{S})$ denotes the Spanier-Whitehead dual.

Proof. Since φ is Σ_n-equivariant, Theorem 13 and Proposition 8 supply the equivariant equivalence of infinite loop spaces

$$\begin{aligned} \Omega^\infty \partial_n(\mathrm{id}) &\simeq \operatorname*{colim}_{k_1,\ldots,k_n} \Omega^{k_1+\cdots+k_n} \mathrm{Map}_*\left(\Delta_n, S^{k_1+\cdots+k_n}\right) \\ &\simeq \operatorname*{colim}_{k} \mathrm{Map}_*\left(\Sigma^k \Delta_n, \Sigma^k S^0\right) \\ &\simeq \Omega^\infty \mathcal{S}\mathrm{p}(\Sigma^\infty \Delta_n, \mathbb{S}). \end{aligned}$$

□

In order to prove this theorem, we require a criterion for recognizing such maps.

Lemma 15. *Let $\psi : F \to G$ be a natural transformation between functors of n variables. If $\psi_{(X_1,\ldots,X_n)}$ is $((n+1)k - c)$-connected whenever each X_i is k-connected, then ψ induces a weak equivalence after multilinarization.*

Proof. The hypothesis on the X_i implies that each $\Sigma^\ell X_i$ is $(k+\ell)$-connected, which implies, using the hypothesis on ψ, that $\Omega^{n\ell}\psi_{(\Sigma^\ell X_1,\ldots,\Sigma^\ell X_n)}$ is $((n+1)(k+\ell) - c - n\ell)$-connected. Since this number tends to infinity with n, and since spheres are compact, it follows that the induced map on colimits is an equivalence. □

Corollary 16. *If $\Omega\psi_{(\Sigma X_1,\ldots,\Sigma X_n)}$ is $((n+1)k - c)$-connected whenever each X_i is k-connected, then ψ induces a weak equivalence after multilinarization.*

Proof. The hypothesis implies that $\Omega^n \varphi_{(\Sigma X_1,\dots,\Sigma X_n)}$ is $((n+1)k-(n-1+c))$-connected. Viewing this map as a natural transformation

$$\psi : \Omega^n F\left(\Sigma(-),\dots,\Sigma(-)\right) \to \Omega^n G\left(\Sigma(-),\dots,\Sigma(-)\right),$$

Lemma 15 implies that ψ induces a weak equivalence on multilinearizations. Since the multilinearizations of these functors coincide with the respective multilinearizations of F and G, the claim follows. □

Now, we have the equivalences

$$\begin{aligned}
\Omega\,\mathrm{cr}_n(\mathrm{id})(\Sigma X_1,\dots,\Sigma X_n) &\simeq \Omega\,\mathrm{tfib}\left(S \mapsto \bigvee_{i\notin S\subseteq[n-1]} \Sigma X_i\right)\\
&\simeq \Omega\,\mathrm{tfib}\left(S \mapsto \Sigma\left(\bigvee_{i\notin S\subseteq[n-1]} X_i\right)\right)\\
&\simeq \mathrm{tfib}\left(S \mapsto \Omega\Sigma\left(\bigvee_{i\notin S\subseteq[n-1]} X_i\right)\right)\\
&\simeq \mathrm{cr}_n(\Omega\Sigma)(X_1,\dots,X_n).
\end{aligned}$$

Thus, in order to obtain the kind of connectivity estimate required by Corollary 16, we may instead study the cross effects of the functor $\Omega\Sigma$. We will be aided in this task by the following classical result.

Theorem 17 (Hilton-Milnor)**.** *Let $\{X_i\}_{i=1}^n$ be pointed, connected spaces. There is a canonical natural weak equivalence*

$$\prod_{w\in L_n}{}' \Omega\Sigma(w(X_1,\dots,X_n)) \xrightarrow{\sim} \Omega\Sigma(X_1\vee\cdots\vee X_n).$$

We pause to explain some of the terms of this theorem.

1. The symbol $\prod'$ denotes the *weak infinite product*, which is defined as the colimit of products over all finite subsets of the indexing set.
2. The set L_n is an additive basis for the free Lie algebra on generators $\{x_1,\dots,x_n\}$. These basis elements are often called *basic products.*
3. The space $w(X_1,\dots,X_n)$ is obtained by substituting X_i for x_i and $\wedge$ for $[-,-]$ in the expression for a basic product. For example, $[x_1,[x_2,x_3]] = X_1\wedge X_2\wedge X_3$.
4. The map is given by taking products of nested Samelson brackets patterned after the basic products w.

See [Whi78] for a proof of this theorem. It should be emphasized that the Hilton-Milnor map is not Σ_n-equivariant, since the set of basic products is not closed under the action of Σ_n.

Corollary 18. *If X_i is k-connected for every $1\le i\le n$, then*

$$\pi_m(\mathrm{cr}_n(\Omega\Sigma)(X_1,\dots,X_n)) \cong \pi_m\left(\bigwedge_{i=1}^n X_i\right)^{(n-1)!}$$

for $0\le m\le (n+1)(k+1)-1$.

Proof. We calculate that

$$\begin{aligned}\mathrm{cr}_n(\Omega\Sigma)(X_1,\ldots,X_n) &= \mathrm{tfib}\left(S\mapsto \Omega\Sigma\left(\bigvee_{i\notin S\subseteq[n]} X_i\right)\right)\\ &\simeq \mathrm{tfib}\left(S\mapsto \prod_{L_{n-|S|}}{}' \,\Omega\Sigma(w(X_1,\ldots,X_n))\right)\\ &\simeq \prod_{L_n^\circ}\Omega\Sigma(w(X_1,\ldots,X_n)),\end{aligned}$$

where L_n° is the set of basic products involving each x_i at least once. The claim now follows from the fact that there are $(n-1)!$ basic products involving each x_i *exactly* once, together with the fact that the factor pertaining to any longer basic product is $((n+1)(k+1)-1)$-connected. □

On the other hand, by the Freudenthal suspension theorem, we compute that, in the range of interest,

$$\begin{aligned}\pi_m\left(\bigwedge_{i=1}^n X_i\right)^{(n-1)!} &\cong \pi_{m+n}\left(\bigwedge_{i=1}^n \Sigma X_i\right)^{(n-1)!}\\ &\cong \pi_m\left(\Omega\,\mathrm{Map}_*\left(\bigvee_{(n-1)!} S^{n-1}, \bigwedge_{i=1}^n \Sigma X_i\right)\right).\end{aligned}$$

This calculation, together with the criterion of Corollary 16, leads us to hope for the following result.

Proposition 19. *There is a canonical weak equivalence*

$$\bigvee_{(n-1)!} S^{n-1} \xrightarrow{\sim} \Delta_n$$

The proof proceeds through the intermediary complex $\widetilde{\Delta}_n := \{t\in\Delta_n : t_{ij}=0 \text{ for } j>1\}$. Concerning this space, we have the following.

Lemma 20. *There is a canonical homeomorphism*

$$\widetilde{\Delta}_n \cong \bigvee_{(n-1)!} S^{n-1}$$

Proof. The space in question is obtained as a quotient of an $(n-1)$-dimensional cube by the "fat diagonal," which is the subspace where any two coordinates agree, together with the subspace where any coordinate is either 0 or 1. Thus,

$$\begin{aligned}\widetilde{\Delta}_n &\cong \mathrm{Conf}_{n-1}((0,1))^+\\ &\cong \left(\coprod_{\sigma\in\Sigma_{n-1}}\left\{(s_1,\ldots,s_{n-1}) : 0 < s_{\sigma(1)} < \cdots < s_{\sigma(n-1)} < 1\right\}\right)^+\\ &\cong \bigvee_{(n-1)!} (\mathring{\Delta}^{n-1})^+.\end{aligned}$$

Using the standard identification $S^{n-1}\cong \Delta^{n-1}/\partial\Delta^{n-1}$, the proof is complete. □

Thus, in order to prove Proposition 19, it suffices to check that the inclusion $\widetilde{\Delta}_n \to \Delta_n$ is a weak equivalence. Setting $W = \bigcup_{1\le i<j\le n} W_{ij}$, the (homotopy) pushout squares

$$\begin{array}{ccc} \widetilde{Z}\cup\widetilde{W} & \longrightarrow & I^{n-1} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \widetilde{\Delta}_n \end{array} \qquad\qquad \begin{array}{ccc} Z\cup W & \longrightarrow & I^{n(n-1)} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \Delta_n \end{array}$$

show that it suffices to check that the inclusion $\widetilde{Z}\cup\widetilde{W} \to Z\cup W$ is a weak equivalence, and the (homotopy) pushout squares

$$\begin{array}{ccc} \widetilde{Z}\cap\widetilde{W} & \longrightarrow & \widetilde{Z} \\ \downarrow & & \downarrow \\ \widetilde{W} & \longrightarrow & \widetilde{Z}\cup\widetilde{W} \end{array} \qquad\qquad \begin{array}{ccc} Z\cap W & \longrightarrow & Z \\ \downarrow & & \downarrow \\ W & \longrightarrow & Z\cup W \end{array}$$

show that it suffices to check that the inclusion $\widetilde{Z}\cap\widetilde{W} \to Z\cap W$ is a weak equivalence, since W and Z are contractible. In order to verify this equivalence, we make use of the following result, which is called the "Nerve Theorem."

Theorem 21 (Borsuk)**.** *If X is covered by subcomplexes K_i such that every nonempty finite intersection $K_{i_1}\cap\cdots\cap K_{i_r}$ is contractible, then X is weakly equivalent to the nerve of the partially ordered set of finite intersections of elements of $\{K_i\}$.*

Proof of Proposition 19. We will show that both $\widetilde{Z}\cap\widetilde{W}$ and $Z\cap W$ admit a cover by subcomplexes whose associated poset is isomorphic to the poset of nontrivial partitions λ of the set $\{1,\dots,n\}$ (recall that a partition is simply an equivalence relation, and a partition is trivial if either $i\sim_\lambda j$ for all i and j or $i\not\sim_\lambda j$ unless $i=j$).

The cover is by the subcomplexes $\{Z\cap W_{ij}\}_{1\le i<j\le n}$ (resp. $\widetilde{W_{ij}}$). The finite intersections corresponding to λ is the subspace of matrices with the following properties:

(1) all diagonal entries vanish;
(2) some entry is 1; and
(3) the ith row and the jth row coincide if $i\sim_\lambda j$.

(in the "tilde" case, there is the further condition that all columns but the first vanish). Each of these is contractible by a coordinatewise straight line homotopy sending t_{ij} to 0 if $i\sim_\lambda j$ and sending t_{ij} to 1 otherwise (in the "tilde" case we perform the homotopy only on the first column). □

Remark 22. From this calculation and Theorem 13, it follows that $\partial_n(\mathrm{id}) = \mathbb{D}(\Sigma^{\infty+2}N(P_n))$, where P_n is the category of nontrivial partitions of $\{1,\dots,n\}$ under refinement. This reformulation due to Arone-Mahowald [AM99] has borne much subsequent fruit—see [Chi05], for example.

To summarize the results of our investigation so far, the domain and codomain of the map

$$\varphi : \mathrm{cr}_n(\mathrm{id})(X_1,\dots,X_n) \to \mathrm{Map}_*\left(\Delta_n, \bigwedge_{i=1}^{n} X_i\right)$$

constructed above have homotopy groups that are abstractly isomorphic in the desired range. Thus, all that remains is to verify that φ induces this isomorphism. The key fact in this verification is the construction of a family of maps with the following properties.

Proposition 23. *For each $\sigma \in \Sigma_{n-1}$ there is a map C_σ fitting into a commuting diagram*

$$\begin{array}{ccccc}
\prod_{i=1}^n X_i & \xrightarrow{C_\sigma} \Omega \operatorname{cr}_n(\mathrm{id})(\Sigma X_1, \ldots, \Sigma X_n) & \xrightarrow{\Omega\varphi} & \Omega \operatorname{Map}_*\left(\Delta_n, \bigwedge_{i=1}^n \Sigma X_i\right) \\
& & & \downarrow \lambda_\tau^* \\
\Big\downarrow q & & & \Omega \operatorname{Map}_*\left(S^{n-1}, \bigwedge_{i=1}^n \Sigma X_i\right) \\
& & & \Big\| \wr \\
\bigwedge_{i=1}^n X_i & \xrightarrow{\Gamma_{\sigma\tau} \wedge (-)} & & \Omega^n \Sigma^n \left(\bigwedge_{i=1}^n X_i\right)
\end{array}$$

for each $\tau \in \Sigma_{n-1}$, where $\lambda_\tau : S^{n-1} \to \bigvee_{\Sigma_{n-1}} S^{n-1} \to \Delta_n$ is the inclusion of the τ factor. Moreover, $\deg(\Gamma_{\sigma\tau}) = \delta_{\sigma\tau}$.

Given these maps, a diagram chase in reduced homology implies that $\Omega\varphi$ induces the desired isomorphism. The maps C_σ are constructed by modifying the iterated commutators appearing in the Hilton-Milnor map with explicit homotopies in order to map into the total fiber.

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