LIE ALGEBRAS IN HOMOTOPY THEORY

Let (X, x) be a pointed topological space. Among the most useful topological invariants of X are its homotopy groups $\pi_n(X, x)$. It is therefore natural to ask the following:

Question 1. What sort of structure does the collection of homotopy groups $\{\pi_n(X, x)\}_{n \geq 1}$ possess?

One way to interpret Question 1 is to look for *homotopy operations*: that is, maps

$$\pi_m(X, x) \to \pi_n(X, x)$$

that depend functorially on X. It follows from Yoneda's lemma that giving such an operation is equivalent to giving a homotopy class of pointed maps from S^n to S^m; that is, an element of $\pi_n(S^m)$. We therefore arrive at the following more precise version of Question 1:

Question 2. What are the homotopy groups $\pi_n(S^m)$ for $m, n \geq 1$?

Question 2 is famously difficult, so let's put it aside and ask a harder question: instead of trying to classify homotopy operations of *one* variable, we can ask for homotopy operations of several: that is, maps

$$\pi_{m_1}(X, x) \times \cdots \times \pi_{m_k}(X, x) \to \pi_n(X, x),$$

which we again require to depend functorially on X. Applying Yoneda's lemma again, we see that such operations are in bijection with homotopy classes of pointed maps from S^n to the wedge $S^{m_1} \vee S^{m_2} \vee \cdots \vee S^{m_k}$. We therefore arrive at the following:

Question 3. What are the homotopy groups $\pi_n(S^{m_1} \vee \cdots \vee S^{m_k})$ for $m_1, m_2, \cdots, m_k, n \geq 1$?

Perhaps surprisingly, Question 3 turns out to be only slightly more difficult than Question 2.

Construction 4. Let m and n be nonnegative integers. Regard the sphere S^{m+1} as a CW complex with a single 0-cell and a single $(m+1)$-cell, and regard S^{n+1} as a CW complex with a single 0-cell and a single $(n+1)$-cell. Then the product $S^{m+1} \times S^{n+1}$ inherits a CW decomposition, where the top cell of dimension $(m + n + 2)$ is attached by a map $\rho : S^{m+n+1} \to S^{m+1} \vee S^{n+1}$.

If (X, x) is a topological space, then composition with ρ determines a map

$$\pi_{m+1}(X, x) \times \pi_{n+1}(X, x) \to \pi_{m+n+1}(X, x).$$

We will denote the image of a pair (α, β) under this map by $[\alpha, \beta]$ and refer to it as the *Whitehead product* of a and b.

Example 5. When $m = n = 0$, the Whitehead product $\pi_1(X, x) \times \pi_1(X, x) \to \pi_1(X, x)$ is the commutator bracket on the (possibly non-commutative) group $\pi_1(X, x)$.

For $m = 0$ and $n > 0$, the action of $\pi_1(X, x)$ on $\pi_{n+1}(X, x)$ can be described in terms of the Whitehead product (and vice versa): every element $\alpha \in \pi_1(X, x)$, the induced automorphism of $\pi_{n+1}(X, x)$ is given by the construction $\beta \mapsto \beta + [\alpha, \beta]$.

For simplicity, let us restrict our attention to the Whitehead product in positive degrees (so that we do not need to worry about non-commutativity). In this case, we have the following:

Proposition 6. *Let (X, x) be a pointed topological space. Then the Whitehead product equips the collection of homotopy groups $\{\pi_{n+1}(X, x)\}_{n>0}$ with the structure of a graded Lie algebra: that is, we have identities*

$$[\alpha, \beta] + (-1)^{pq}[\beta, \alpha] = 0$$

for $\alpha \in \pi_{p+1}(X, x)$ and $\beta \in \pi_{q+1}(X, x)$, and

$$(-1)^{pr}[\alpha, [\beta, \gamma]] + (-1)^{pq}[\beta, [\gamma, \alpha]] + (-1)^{qr}[\gamma, [\alpha, \beta]] = 0$$

for $\alpha \in \pi_{p+1}(X, x), \beta \in \pi_{q+1}(X, x), \gamma \in \pi_{r+1}(X, x)$.

Warning 7. For $\alpha \in \pi_{n+1}(X, x)$ with n even, the first identity gives $2[\alpha, \alpha] = 0$. In general, the Whitehead product does not satisfy the stronger condition that $[\alpha, \alpha] = 0$.

We then have the following:

Theorem 8 (Hilton)**.** *Choose integers $m_1, m_2, \ldots, m_k \geq 2$, and let X denote the bouquet $S^{m_1} \vee \cdots \vee S^{m_k}$. For $1 \leq i \leq k$, let $\gamma_i \in \pi_{m_i}(X)$ denote the homotopy class of the inclusion of the ith summand. Then every element of $\pi_n(X)$ can be decomposed as a sum of elements of the form $\alpha \circ \beta$, where $\alpha \in \pi_n(S^t)$ and $\beta \in \pi_t(X)$ can be built from the γ_i using Whitehead products.*

Theorem 8 can be made more precise: by allowing only a restricted class of Whitehead products (indexed by a suitable basis for a free Lie algebra), one can arrange that the decomposition of Theorem 8 is unique. We can summarize the situation informally by the following heuristic equation:

$$\{\text{Structure of } \pi_*(X)\} = \{\text{Homotopy Groups of Spheres}\} + \{\text{Lie algebra structure } [\bullet, \bullet]\}$$

We can articulate this heuristic a little bit more clearly by working rationally. The rational homotopy groups of spheres are actually easy to describe: if we let

ι_n denote the tautological element of $\pi_n(S^n)$, we have

$$\pi_*(S^n)_{\mathbf{Q}} = \begin{cases} \mathbf{Q}\,\iota_n & \text{if } n \text{ is odd} \\ \mathbf{Q}\,\iota_n + \mathbf{Q}[\iota_n, \iota_n] & \text{if } n \text{ is even} \end{cases}$$

Our heuristic equation simplifies as

$$\{\text{Structure of } \pi_*(X)_{\mathbf{Q}}\} = \{\text{Lie algebra structure from Whitehead product}\}$$

Quillen's work on rational homotopy theory supplied a more precise articulation of this heuristic:

Theorem 9 (Quillen)**.** *There is an equivalence of homotopy theories*

$$\begin{array}{c} \{\textit{Simply connected pointed rational spaces}\} \\ \downarrow \sim \\ \{\textit{Connected differential graded Lie algebras over } \mathbf{Q}\}. \end{array}$$

Moreover, if X is a simply connected pointed rational space which corresponds to a Lie algebra $\mathfrak{g}_*$ under this equivalence, then the Lie algebra $(\pi_{*+1}(X), [\bullet, \bullet])$ can be identified with the homology of $\mathfrak{g}_*$.

If X is a simply connected pointed rational space, then the corresponding differential graded Lie algebra $\mathfrak{g}_*$ is only well-defined up to quasi-isomorphism. By making a cofibrant replacement, we can always arrange that $\mathfrak{g}_*$ is isomorphic, as a graded Lie algebra, to the *free* graded Lie algebra $\mathrm{Free}(V_*)$ on some positively graded vector space V_*. In this case, the graded vector space V_* is not canonically determined as a subspace of $\mathfrak{g}_*$. However, it can be realized canonically as a *quotient* of $\mathfrak{g}_*$: namely, it is given by the abelianization $\mathfrak{g}_*/[\mathfrak{g}_*, \mathfrak{g}_*]$. More generally, we can consider the lower central series filtration

$$\cdots \subseteq \mathfrak{g}_*^{(4)} \subseteq \mathfrak{g}_*^{(3)} \subseteq \mathfrak{g}_*^{(2)} \subseteq \mathfrak{g}_*^{(1)},$$

so that $\mathfrak{g}_*$ can be realized as the (homotopy) inverse limit of a tower of differential graded Lie algebras $\{\mathfrak{g}_*/\mathfrak{g}_*^{(n+1)}\}_{n\geq 0}$. Chasing this algebraic construction through the equivalence of Theorem 9, we obtain a realization of X as the inverse limit of a tower

$$\cdot \to X_4 \to X_3 \to X_2 \to X_1 \to X_0 \simeq *$$

This tower has the following features:

- Each of the homotopy fibers $F_n = \mathrm{fib}(X_n \to X_{n-1})$ corresponds, under Theorem 9, to the quotient $\mathfrak{g}_*^{(n)}/\mathfrak{g}_*^{(n+1)}$. This is an *abelian* Lie algebra, so that F_n is actually an infinite loop space.
- The map $X \to X_1$ can be identified with the unit map $X \to \Omega^\infty \Sigma^\infty X$. In particular, the homotopy groups of X_1 are just the (reduced) homology groups of X with coefficients in $\mathbf{Q}$ (note that all of the spaces under consideration are rational).

- Each quotient $\mathfrak{g}_*^{(n)}/\mathfrak{g}_*^{(n+1)}$ can be identified, as a chain complex, with $\mathrm{Free}_n(V_*)$, where $\mathrm{Free}_n(V_*)$ denotes the "degree n" part of the free graded Lie algebra generated by V_*. This has a differential inherited from the differential on V_*. (via the identification $V_* \simeq \mathrm{gr}^1(\mathfrak{g}_*) = \mathfrak{g}_*/[\mathfrak{g}_*, \mathfrak{g}_*]$). We therefore obtain isomorphisms
$$\pi_{*+1}(F_n) \simeq \mathrm{Free}_n(\pi_{*+1}(X_1)) \simeq \mathrm{Free}_n(\mathrm{H}^{\mathrm{red}}_{*+1}(X; \mathbf{Q})).$$

We can summarize the situation more informally as follows: every rational space X admits a canonical filtration, whose associated graded behaves like a free Lie algebra on the *stable* homotopy type of X.

Using the calculus of functors, Goodwillie introduced a refinement of this picture which works at the integral level:

Theorem 10 (Goodwillie). *Let X be a simply connected pointed space. Then X can be realized (in a canonical way) as the homotopy limit of a tower*
$$\cdots \to P_4(X) \to P_3(X) \to P_2(X) \to P_1(X)$$
with the following features:

(a) *The map $X \to P_1(X)$ agrees with the unit map $X \to \Omega^\infty \Sigma^\infty X$.*
(b) *Each of the homotopy fibers $D_n(X) = \mathrm{fib}(P_n(X) \to P_{n-1}(X))$ is an infinite loop space, given by a formula $D_n(X) = \Omega^\infty((\Sigma^\infty X)^{\wedge n} \wedge \mathcal{O}(n))_{h\Sigma_n}$, where $\mathcal{O}(n)$ is a certain spectrum equipped with an action of the symmetric group Σ_n.*

To connect this with the theory of Lie algebras, we have the following result of Ching:

Theorem 11 (Ching). *The spectra $\{\mathcal{O}(n)\}_{n>0}$ of Theorem 10 form an operad (in the category of spectra).*

The spectra $\mathcal{O}(n)$ of Theorem 10 are relatively simple: each can be described as bouquet of spheres of dimension $1-n$ (though the action of Σ_n is quite interesting). In particular, the homology of $\mathcal{O}(n)$ is free abelian, concentrated in degree $1-n$. Using Theorem 11, we can organize the collection $\{\mathrm{H}_{1-n}(\mathcal{O}(n); \mathbf{Z})\}_{n>0}$ into an operad in the category of abelian groups. In fact, this operad turns out to be familiar: it is just the Lie operad. We can therefore view $\{\mathcal{O}(n)\}_{n>0}$ as an incarnation of the Lie operad in the setting of stable homotopy theory. Theorem 10 can then be summarized more informally as follows:

($*$) Every simply connected space X admits a canonical filtration, whose associated graded can be written as $\Omega^{\infty-1}\,\mathrm{Free}(\Sigma^{\infty-1} X)$, where $\mathrm{Free}(\Sigma^{\infty-1} X)$ denotes the free Lie algebra generated by the shifted suspension spectrum $\Sigma^{\infty-1} X$.

Remark 12. Recall that, in the previous semester, we proved that Bousfield-Kuhn functor determines a monadic adjunction

$$\mathrm{Sp}_{T(n)} \underset{\Phi}{\overset{\Theta}{\rightleftarrows}} \mathcal{S}_*^{v_n},$$

whose associated monad $U = \Phi \circ \Theta \in \mathrm{Fun}(\mathrm{Sp}_{T(n)}, \mathrm{Sp}_{T(n)})$ is coanalytic: that is, it can be identified with an operad in the ∞-category $\mathrm{Sp}_{T(n)}$. One of our objectives in this semester will be to show that this operad coincides with the operad $\{\mathcal{O}(k)\}_{n>0}$ of Theorem 11.

Returning to Theorem 10, one can ask if the tower $\{P_n(X)\}_{n>0}$ can be put to some computational use. Note that the filtration gives a spectral sequence

$$E_2^{s,t} : \pi_s(\mathcal{O}(t) \wedge \Sigma^\infty X^{\wedge t})_{h\Sigma_t} \Rightarrow \pi_s X.$$

In the case where X is a sphere of dimension k, this simplifies to

$$E_2^{s,t} \pi_s(\Sigma^{tk}\, \mathcal{O}(t))_{h\Sigma_t} \Rightarrow \pi_s(S^k).$$

In principle, this gives information about *unstable* homotopy groups in terms of *stable* homotopy groups.

If one only wants rational homotopy groups, this works out quite simply. Over $\mathbf{Q}$, the free graded Lie algebra on a single generator x is either one-dimensional (if the degree of x is even) or two-dimensional (if the degree of x is odd), and we recover the calculation

$$\pi_*(S^k)_{\mathbf{Q}} = \begin{cases} \mathbf{Q}\,\iota_k & \text{if } k \text{ is odd} \\ \mathbf{Q}\,\iota_k + \mathbf{Q}[\iota_k, \iota_k] & \text{if } k \text{ is even.} \end{cases}$$

Working integrally is much harder. However, we can try to follow a middle path, by applying the v_n-periodic homotopy theory of the previous semester. Recall that the *Bousfield-Kuhn functor*

$$\Phi : \{\text{ Pointed Spaces }\} \to \{T(n)\text{-Local Spectra}\}$$

has the following properties:

- The functor Φ commutes with finite homotopy limits. In particular, it commutes with the formation of homotopy fibers
- For every spectrum E, we have a canonical homotopy equivalence $\Phi\Omega^\infty E \simeq L_{T(n)} E$.

For any space X, we can apply Φ to the Goodwillie tower

$$\cdots \to P_4(X) \to P_3(X) \to P_2(X) \to P_1(X)$$

of Theorem 10 to obtain a tower of $T(n)$-local spectra

$$\cdots \to \Phi P_4(X) \to \Phi P_3(X) \to \Phi P_2(X) \to \Phi P_1(X) \simeq L_{T(n)} \Sigma^\infty X$$

with homotopy fibers given by

$$\Phi D_k(X) = L_{T(n)}(\mathcal{O}(k) \wedge (\Sigma^\infty X)^{\wedge k})_{h\Sigma_k}$$

In general, the homotopy limit of this tower need not be $\Phi(X)$, since the functor Φ does not commute with infinite homotopy limits. However, we have the following:

Theorem 13 (Arone-Mahowald)**.** *Let X be a sphere. Then ΦX can be realized as the homotopy limit of the tower $\{\Phi P_k(X)\}$. Moreover, the tower actually stabilizes (that is, we have $\Phi D_k(X) \simeq 0$ for all but finitely many k).*