Some Engulfing (Lecture 9)

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Our goal in this lecture is to complete the proof that every Whitehead triangulation of a smooth fiber bundle yields a PL fiber bundle. Recall that we had reduced ourself to the case where the smooth fiber bundle in question was the projection $p : M \times \mathbb{R} \times \Delta \to \Delta$, where M is a compact smooth manifold and Δ is a simplex. We assume that we are given a Whitehead compatible triangulation of $M \times \mathbb{R} \times \Delta$ such that the map p is piecewise linear.

One strategy for analyzing the map p is to try to compare it with the projection $p' : M \times S^1 \times \Delta \to \Delta$. The map p' is proper, so any Whitehead compatible triangulation of $M \times S^1 \times \Delta$ making p' piecewise linear will automatically exhibit $M \times S^1 \times \Delta$ as a fiber bundle over Δ in the PL category. Unfortunately, the obvious translation map $T : (m, r, v) \mapsto (m, r - 2\pi, v)$ is probably not a piecewise linear map from $M \times \mathbb{R} \times \Delta$ to itself, so it is not clear that our Whitehead compatible triangulation of $M \times \mathbb{R} \times \Delta$ descends to give a Whitehead compatible triangulation of $M \times S^1 \times \Delta$. Our goal will be to find a good surrogate for the translation map T. Namely, we will prove the following:

Proposition 1. *There exists a PL homeomorphism $H : M \times \mathbb{R} \times \Delta \to M \times \mathbb{R} \times \Delta$ such that $p \circ H = p$ and $H(M \times (-\infty, 1] \times \Delta) \subseteq M \times (-\infty, -1) \times \Delta$.*

Assume Proposition 1 for the moment, and let D (for "fundamental domain") denote the closed set $(M \times (-\infty, 1] \times \Delta) - H(M \times (-\infty, 1) \times \Delta)$. Then the union $E = \bigcup_{n \in \mathbf{Z}} H^n D$ is an open subset of $M \times \mathbb{R} \times \Delta$ which is acted on by freely by the cyclic group $\{H^n\}_{n \in \mathbf{Z}} \simeq \mathbf{Z}$. Since H is a PL map, the quotient $E/\mathbf{Z}$ inherits a PL structure, and is equipped with a proper PL submersion $E/\mathbf{Z} \to \Delta$. We saw last time that this proper submersion must be a fiber bundle. It follows that E, being a cyclic cover of a fiber bundle, is also a fiber bundle. Since E contains $M \times [-1, 1] \times \Delta$, this will prove the lemma that we needed from the last lecture.

We now turn to the proof of Proposition 1. The idea is to construct H locally near each point of the simplex Δ, and then glue the resulting homeomorphisms together. To carry this out, we will need a more refined version of the statement of Proposition 1. Recall that a *piecewise linear isotopy* from $M \times \mathbb{R} \times \Delta$ to itself is a PL homeomorphism $h : M \times \mathbb{R} \times \Delta \times [0, 1] \to M \times \mathbb{R} \times \Delta \times [0, 1]$ which commutes with the projection to $[0, 1]$; we think of h as a family of PL homeomorphisms $h_t : M \times \mathbb{R} \times \Delta \to M \times \mathbb{R} \times \Delta$ parametrized by $t \in [0, 1]$. In what follows, we will assume that all of these homeomorphisms commute with the projection map $p : M \times \mathbb{R} \times \Delta$. We say that h is *supported* in a closed subset $K \subseteq M \times \mathbb{R} \times \Delta$ if each h_t is the identity on $M \times \mathbb{R} \times \Delta - K$.

Choose a large integer N (to be determined later). For each closed subset C and each integer $d > 0$, consider the following assertion:

$(P_{C,d})$ For every pair of integers $a \leq b \in \mathbf{Z}$ such that $-N \leq a - d \leq b + d \leq N$, there exists a PL isotopy $\{h_t : M \times \mathbb{R} \times \Delta \to M \times \mathbb{R} \times \Delta\}_{t \in [0,1]}$ (compatible with the projection p) having the following properties:

(1) The isotopy h is supported in a compact subset of $M \times (a - d, b + d) \times \Delta$.

(2) The map h_0 is the identity.

(3) The homeomorphism h_1 carries $M \times (-\infty, b] \times C$ into $M \times (-\infty, a) \times C$.

Remark 2. Note that there are only finitely many pairs of integers a, b satisfying the condition $-N \leq a - d \leq b + d \leq N$, so $P_{C,d}$ asserts the existence of only finitely many isotopies; this is the reason for introducing the parameter N.

We will prove that there exists an integer d such that $P_{\Delta,d}$ holds, where d does not depend on N. Then, if $N \geq d + 1$, we can apply $P_{C,d}$ in the case $a = -1$, $b = 1$ to obtain an PL homeomorphism h_1 satisfying the requirements of Proposition 1.

The basic observation is the following:

Lemma 3. *If $P_{C,d}$ and $P_{C',d'}$ hold, then $P_{C \cup C', d+d'}$ holds.*

Proof. Assume that $-N \leq a - d - d' \leq b + d + d' \leq N$. Applying $P_{C,d}$, we can choose a PL isotopy h_t supported in $M \times (a-d, b+d+d') \times \Delta$ such that h_1 $M \times (-\infty, b+d'] \times C$ into $M \times (\infty, a) \times C$. Applying $P_{D,d'}$, we can choose a PL isotopy h'_t supported in $M \times (a-d-d', b+d') \times \Delta$ such that h'_1 carries $M \times (\infty, b] \times D$ into $M \times (\infty, a-d) \times D$. We claim that $h''_t = h_t \circ h'_t$ is an isotopy which verifies the conditions of $P_{C \cup D, d+d}$. □

Remark 4. In assertion $P_{C,d}$, we can assume that the isotopy h_t is supported in a compact subset $p^{-1}(U)$ for any fixed open neighborhood U of C: to achieve this, choose a PL function χ such that $\chi = 1$ on C and χ is supported in a compact subset of U, and replace $h_t(m, r, v)$ by $h_{\chi(v)t}(m, r, v)$.

It follows that if C is a union of closed subsets C_i with disjoint open neighborhoods U_i and $P_{C_i,d}$ holds for each i, then $P_{C,d}$ holds: we can define an isotopy $h_t(m, r, v)$ by the formula

$$h_t(m, r, v) = \begin{cases} h^i_t(m, r, v) & \text{if } v \in U_i \\ (m, r, v) & \text{otherwise,} \end{cases}$$

where each h^i_t is an isotopy verifying $P_{C_i,d}$ supported in $p^{-1}U_i$.

We will prove the following:

Lemma 5. *For point $v \in \Delta$, there is a closed neighborhood C of v such that $P_{C,1}$ holds.*

Assuming Lemma 5 for a moment, we can complete the proof. Note that the simplex Δ is homeomorphic to a cube $[0, 1]^n$ for some integer n. Fix $k > 0$, and decompose this cube into smaller cubes

$$C_{i_1,i_2,\ldots,i_n} = \prod_{1 \leq j \leq n} [\frac{i_j}{k}, \frac{i_j + 1}{k}]$$

where $0 \leq i_1, \ldots, i_n < k$. For $k \gg 0$, Lemma 5 guarantees that for each cube $C = C_{i_1,\ldots,i_n}$, condition $P_{C,1}$ is satisfied. For any sequence of bits $b_1, \ldots, b_n \in \{0, 1\}$, let

$$C'_{b_1,\ldots,b_n} = \bigcup C_{i_1,\ldots,i_n}$$

where the union is taken over all sequences $(i_1, \ldots, i_n)$ such that i_j is congruent to b_j modulo 2 for each j. Applying Remark 4, we deduce that $P_{C',1}$ holds for each of the closed subsets $C' = C'_{b_1,\ldots,b_n}$. Applying Lemma 3, we deduce that $P_{\Delta,2^n}$ holds, and the proof is complete.

It remains to prove Lemma 5. Since there are only finitely many pairs of integers a, b such that $-N \leq a-1 \leq b+1 \leq N$ (and since an finite intersection of closed neighborhoods of v is again a closed neighborhood of v), it will suffice to prove the existence of an isotopy h_t as in assertion $P_{C,1}$ for each pair (a, b) satisfying $-N \leq a - 1 \leq b + 1 \leq N$. We do this in a sequence of steps:

(1) Suppose first that Δ consists of a single point, and that $M \times \mathbb{R}$ is given the product PL structure (for some fixed Whitehead compatible triangulation on M). Then the existence of the desired isotopies is obvious: we can take $h_t(m, r) = (m, f_t(r))$, where $f_t : \mathbb{R} \to \mathbb{R}$ is a PL isotopy supported in a compact subset of $(a - 1, b + 1)$ which carries $(-\infty, b]$ into $(-\infty, a)$.

(2) Suppose again that Δ consists of a single point, but that the PL structure on $M \times \mathbb{R}$ is arbitrary. Choose a PD homeomorphism $f : K \to M$, and endow $K \times \mathbb{R}$ with the product PL structure. Our uniqueness result from Lecture 5 asserts that there exists a PL homeomorphism $g : K \times \mathbb{R} \to M \times \mathbb{R}$ which is arbitrarily close to the map $f \times \mathrm{id}_{\mathbb{R}}$. Step (1) shows the existence of a PL isotopy h_t of $K \times \mathbb{R}$ with the desired properties. We define a PL isotopy h'_t of $M \times \mathbb{R}$ by the formula $h'_t = g \circ h_t \circ g^{-1}$. It is easy to see that if g is close enough to $f \times \mathrm{id}_{\mathbb{R}}$, then h'_t will satisfy the requirements of $P_{\Delta,1}$.

(3) We now suppose that Δ is arbitrary. Since $M \times [a-1, b+1] \times \{v\}$ is a compact subset of the fiber $p^{-1}\{v\}$, it is contained in a finite polyhedron. Since p is a submersion, the results of the previous lecture show that $M \times [a-1, b+1] \times \{v\}$ has an open neighborhood which is PL homeomorphic to $U \times V$, where $U \subseteq M \times \mathbb{R}$ and $V \subseteq \Delta$ are open subsets containing $M \times [a-1, b+1]$ and v, respectively. Let h'_t be the isotopy constructed in (2). Since h'_t is supported in a compact subset K of $M \times (a-1, b+1)$, it restricts to an isotopy of U and therefore defines an isotopy h''_t of $U \times V$. Choose a compact neighborhood K' of v in V such that the map

$$K \times K' \to U \times V \to M \times \mathbb{R} \times \Delta \to \mathbb{R}$$

has image contained in a compact subset of $(a-1, b+1)$. Let $\chi : \Delta \to [0,1]$ be a PL map such that $\chi = 0$ outside of K and $\chi = 1$ in a neighborhood of v, and define an isotopy k_t by the formula

$$k_t(m,r,v) = \begin{cases} h''_{\chi(v)t}(m,r,v) & \text{if } (m,r,v) \in U \times V \\ (m,r,v) & \text{otherwise.} \end{cases}$$

Then k_t is an isotopy of $M \times \mathbb{R} \times \Delta$ which is supported in a compact subset of $M \times (a-1, b+1) \times \Delta$, with $k_0 = \mathrm{id}$. We observe that k_1 carries $M \times (-\infty, b] \times \{v\}$ into $M \times (-\infty, a) \times \Delta$. It therefore does the same for $M \times (-\infty, b] \times C$ where C is any sufficiently small neighborhood of v, which completes the proof.