The Dehn-Nielsen Theorem (Lecture 38)

May 13, 2009

In this lecture, we will complete our understanding of the homotopy types of diffeomorphism groups of hyperbolic surfaces by proving the following result:

Theorem 1 (Dehn-Nielsen)**.** *Let Σ be a compact oriented surface with $\chi(\Sigma) = -k < 0$. Then the map $\mathrm{Diff}_\partial(\Sigma) \to \mathrm{Out}_\partial(\Sigma)$ is surjective.*

Since $\mathrm{Out}_\partial(\Sigma)$ is the group of connected components of the space $\mathrm{Aut}_\partial(\Sigma)$ of self-homotopy equivalences of Σ which are fixed on the boundary, we can reformulate Theorem 1 as follows:

Theorem 2. *Let $f : \Sigma \to \Sigma'$ be a homotopy equivalence between compact oriented surfaces with $\chi(\Sigma) = \chi(\Sigma') = -k < 0$. Assume that f restricts to a diffeomorphism $\partial\,\Sigma \simeq \partial\,\Sigma'$. Then f is homotopic (relative to the boundary of Σ) to a diffeomorphism $\Sigma \simeq \Sigma'$.*

We may assume without loss of generality that f is a smooth map, and that $f^{-1}\,\partial\,\Sigma' = \partial\,\Sigma$. Choose a system of disjoint simple closed curves $C_1, C_2, \ldots, C_n$ in Σ' which cut Σ' into a union of finitely many pairs of pants (an Euler characteristic calculation shows that the number of pairs of pants must be exactly k, so that $\Sigma' = P_1 \cup \ldots \cup P_k$). Modifying f slightly, we may assume that f is transverse to the curves C_i. Let $T = f^{-1}(C_1 \cup \ldots \cup C_n)$, so that T is a smooth submanifold of Σ consisting of some finite number m of circles. We will assume that f has been chosen (in its homotopy class) so as to minimize m.

Let $Q_1, \ldots, Q_{k'}$ be the collection of components of the surface obtained by cutting Σ along T; we will identify each Q_i with a closed subset of Σ.

Claim 3. *Each Q_i has nonpositive Euler characteristic.*

Proof. If not, some Q_i must be a disk. Say f carries the boundary of Q_i into the circle C, and Q_i itself into a pair of pants P. Then f determines a class in the relative homotopy group $\pi_2(P, C)$, which is the fundamental group of the homotopy fiber of the inclusion $C \mapsto P$. Since $\pi_1 C \hookrightarrow \pi_1 P$, the relevant homotopy fiber is homotopy equivalent to the discrete space $\pi_1 P/\pi_1 C$, and has a trivial fundamental group. Consequently, $f|Q_i$ is homotopic to a map carrying Q_i into the circle C. Modifying this map by a small homotopy, we obtain a new map f' homotopic to the original f, such that ${f'}^{-1}(C_1 \cup \ldots \cup C_n)$ has fewer connected components than T. This contradicts the minimality of m. □

Claim 4. *Let T_i be a connected component of T, and suppose that f carries T_i into C_j. Then:*

(1) *The map $f|T_i : T_i \to C_j$ has degree ± 1.*

(2) *The loop T_i is not homotopic to any boundary loop of Σ.*

Proof. We first claim that T_i is not nullhomotopic in Σ. Otherwise, T_i would bound an embedded disk. Inside this disk we can find an "innermost" component $T_{i'}$ of T, which also bounds a disk, contradicting Claim 3. Thus $[T_i]$ is nontrivial in $\pi_1\Sigma$. Since f is a homotopy equivalence, $f_*[T_i] = [C_j]^d$ is nontrivial in $\pi_1\Sigma'$, where d is the degree of $f|T_i$. It follows that $d \neq 0$. If $|d| > 1$, then $f_*[T_i]$ is divisible in $\pi_1\Sigma'$, so that $[T_i]$ is divisible in $\pi_1\Sigma$; this contradicts our assumption that f is an embedded loop.

To prove (2), we note that if $[T_i]$ is conjugate to some boundary component of Σ, then $f_*[T_i] \simeq [C_j]^{\pm 1}$ is conjugate to some boundary component of Σ', which contradicts our choice of C_j. □

Adjusting f by a homotopy, we may assume that the restriction of f to each component of T is a diffeomorphism onto one of the circles C_i.

Claim 5. *Each Q_i has negative Euler characteristic.*

Proof. Assume that $\chi(Q_i) \geq 0$. It follows from Claim 3 that $\chi(Q_i) = 0$, so that Q_i is an annulus. Using Claim 4, we deduce that both boundary components of Q_i belong to T. Let us denote these boundary components by B and B'. Let P be the pair of pants containing $f(Q_i)$. Then $f(B)$ and $f(B')$ are boundary components of P. Since $f(B)$ and $f(B')$ are freely homotopic in P, they must be the same boundary component $P_0 \subseteq P$. Consider the map

$$\phi : \mathrm{Map}(S^1, P_0) \to \mathrm{Map}(S^1, P).$$

If we restrict attention to the connected component containing the isomorphism $S^1 \simeq P_0$, then ϕ is a homotopy equivalence: this follows from the observation that the centralizer of $[P_0]$ in $\pi_1 P$ is isomorphic to its centralizer in $\pi_0 P_0 \simeq \mathbf{Z}$. Consequently, the map $Q_i \to P$ is homotopic (relative to its boundary) to a map $Q_i \to P_0$. Modifying this map by a small homotopy, we obtain a new map $f' : \Sigma' \to \Sigma$ such that ${f'}^{-1}(C_1, \ldots, C_n)$ has fewer than m components, which is a contradiction. $\square$

Since the map f has degree ± 1 (being a homotopy equivalence) it must be surjective. Consequently, the inverse image of each P_i is a finite union of Q_j's. According to Claim 5, each of these components has negative Euler characteristic. It follows that $\chi(f^{-1}(P_i)) \leq -1$. We have

$$-k = \chi(\Sigma) = \chi(f^{-1}P_1) + \ldots + \chi(f^{-1}P_k) \leq -1 + \ldots + -1 = -k.$$

It follows that each $f^{-1}P_i$ must consist of exactly one connected component (which we will denote by Q_i) having Euler characteristic -1. Since the map f is surjective, $Q_i \to P_i$ is surjective, so that Q_i has at least three boundary components. It follows that Q_i is also a pair of pants, and that f restricts to a map $f_i : Q_i \to P_i$ which is a diffeomorphism between their boundaries. To complete the proof, it will suffice to show that each f_i is homotopic to a diffeomorphism.

Choose disjoint smooth arcs D_1, D_2, D_3 which join the boundary components of P_i. We may assume without loss of generality that f_i is transverse to the arcs D_j, so that $f_i^{-1} D_j$ is a smooth submanifold (with boundary) of Q_i. The boundary of $f_i^{-1}(D_1 \cup D_2 \cup D_3)$ is $f_i^{-1}((D_1 \cup D_2 \cup D_3) \cap \partial P_i)$ which consists of six points (since f_i is a diffeomorphism on the boundaries). It follows that $f_i^{-1}(D_1 \cup D_2 \cup D_3)$ consists of three arcs together with m' circles, for some $m' \geq 0$. Let us denote these arcs by D'_1, D'_2, and D'_3; modifying f_i by a homotopy we may assume that D'_j maps homeomorphically onto D_j for $j \in \{1, 2, 3\}$.

We will assume that f_i has been chosen (in its homotopy class) to minimize m'. Cutting Q_i along the arcs D'_1, D'_2, and D'_3, we obtain a decomposition $Q_i \simeq Q_i^+ \cup Q_i^-$, where Q_i^+ and Q_i^- are disks.

Claim 6. *The integer m' is equal to zero.*

Proof. Let $\widetilde{C}$ be a circle component of $f_i^{-1}(D_1 \cup D_2 \cup D_3)$. Without loss of generality, $\widetilde{C} \subseteq Q_i^+$. Choosing a different circle component if necessary, we may assume that $\widetilde{C}$ is innermost: that is, $\widetilde{C}$ bounds a disk E such that $E \cap f^{-1}(D_1 \cup D_2 \cup D_3) = \partial E = \widetilde{C}$. Without loss of generality, the map f carries ∂E to the arc D_1. Then f determines a class in the relative homotopy group $\pi_2(E', D_1)$, where E' is one of the disks obtained by cutting P_i along $D_1 \cup D_2 \cup D_3$). Since E' and D_1 are both contractible, this homotopy group is trivial. It follows that $f_i|E$ is homotopic (relative to its boundary) to a map carrying E into D_1. Modifying this map by a small homotopy, we obtain a new map f'_i such that ${f'_i}^{-1}(D_1 \cup D_2 \cup D_3)$ has fewer circle components, contradicting the minimality of m'. $\square$

The arcs $D_1 \cup D_2 \cup D_3$ cut P_i into two components, which we will denote by P_i^+ and P_i^-. Using Claim 6, we may assume without loss of generality that f_i restricts to a pair of maps

$$f_i^+ : Q_i^+ \to P_i^+ \qquad f_i^- : Q_i^- \to P_i^-,$$

each of which is a diffeomorphism on the boundary. Using the Alexander trick (remember that there is no essential difference between the smooth and PL categories in dimensison 2), we can assume that f_i^+ and f_i^- are diffeomorphisms.